Revenues and Welfare in Auctions with Information Release*

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Abstract

Auctions are the allocation-mechanisms of choice whenever goods and information in a market are scarce. Therefore, understanding how information in these markets affects welfare and revenues is of fundamental interest. We introduce new mathematical concepts, k- and k-m-dispersion, for understanding the impact of information. With these tools, we study the comparative statics of welfare versus revenues for markets with one or more objects and varying numbers of bidders. Depending on which parts of a distribution of valuations are most affected by release of information, welfare may react stronger than revenues, or vice versa.

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1 Introduction

Auctions are the mechanism of choice whenever goods and information in the respective market are scarce. Auctions allow to allocate goods in rather efficient or profitable ways without precise knowledge of bidders' valuations. They are therefore not only applied in highly capitalized spectrum and timber markets, but also in selling various items from used cars up to fine arts. In addition, auctions can serve as a stylized model for other situations of competition for a scarce good such as college admissions, electoral competition, or R&D races. As it is a crucial feature of an auction that it allocates goods well despite a fundamental lack of information on valuations, it is surprising how little we know about the interaction of information, welfare and revenues in the auction context. This is the starting point of our paper.

Generating information is typically a costly endeavor. For a welfare maximizer, the incentives to provide information on the goods for sale may be very different from the incentives a revenue-maximizing seller faces. The reason behind is that a welfare maximizer incorporates bidders' aggregated rents into his calculation, while a revenue-maximizing seller focuses on the the selling price. A priori, releasing information could increase competition at the top such that bidders' rents become smaller. This may affect selling prices a lot, but increase overall efficiency of allocation and thus welfare only marginally. Yet information release could also lead to a further differentiation of the bidders with the highest valuations, thus affecting and increasing bidders' rents and welfare more strongly than the seller's revenue.

Understanding how welfare and revenue incentives relate to each other in these contexts requires a thorough understanding of the behavior of order statistics. In case of a one object auction, the first and second order statistics, i.e. the highest and the second highest bids submitted, and the expected difference between the two, are crucial. Looking at multi-object auctions, more of the highest order statistics become relevant. For example, assume several prizes, like grants or promotions, are "auctioned off" to applicants in order to reward those who exerted the highest efforts (bids). Then, not only the best and second best applicant, but also other applicants scratching the top will matter for overall efforts exerted as well as efficiency. For example, Harvard University selected out of 34,000 applicants its 2,000 students for its class of 2018. In such a situation, many order statistics of high rank matter.[1]

[1]See https://college.harvard.edu/admissions/admissions-statistics.

The literature so far focuses on one-object auctions and has imposed conditions on the effects of information release which guarantee that welfare reacts more sensitively towards information than revenue (compare Ganuza and Penalva, 2010). There are however examples in which the opposite conclusion holds true. Bidders' rents decrease in response to information release due to increased competition, implying that a revenue maximizer has stronger incentives to release information than a welfare maximizer. Essentially, this is the case when information release affects bidders with intermediate valuations more strongly than bidders with high valuations. Information release can then induce a fiercer competition at the top.

In this paper, we develop a model of information release in multi-object auctions which is flexible enough to accommodate both directions of the effect of information on bidders' rents. In particular, we develop novel criteria which allow to conclude whether information release strengthens or weakens competition between bidders in an auction. The previous literature has modeled the effect of information release as increasing the variability of valuations in the sense of the dispersive order (Ganuza and Penalva, 2010). This assumption – which is violated in many examples – implies that information weakens competition by increasing the differences between all order statistics. Releasing information can never decrease variability in the sense of the dispersive order.

We introduce two new classes of stochastic orders which allow for a more flexible and directed control of the behavior of order statistics. For this purpose, we develop the k- and the k-m-dispersion order. Increased variability in the sense of the k-dispersion order implies that the k highest order statistics move further apart through information release. Increased variability in the sense of the k-m-dispersion order implies that the same conclusion holds true when the overall number of bidders is greater than $k + m$. Both orders are weaker than the dispersive order. In particular, information release can either increase or decrease the variability of valuations in the sense of the k-m-order, implying either a weakening or a softening of competition. Consequently, a welfare maximizer may have either a stronger or a weaker incentive to release information than a welfare maximizer. k-m-dispersion provides a criterion to decide which of the two is the case.

We demonstrate the power of our new tools by investigating an auction in which information release is modeled in terms of information partitions. Bidders do not know their true valuations, yet they know which interval of the distribution contains their valuation. Information release renders these intervals finer. This is a prominent model of information release in economic theory (see Bergemann and Pesendorfer, 2007) which is not tractable with the dispersive order. Via k-m-dispersion, we can draw clear conclusions about multi-object auctions with sufficiently many bidders. Information release decreases bidders' rents if and only if information affects the bidders with the highest possible valuations.

Our techniques have various implications beyond auction theory. In Section 5, we discuss applications in other fields of economics, as well as theoretical properties of the k-dispersion order beyond spacings of order statistics. Our results transfer to worst realizations of a distribution as needed in reliability theory and risk management. Further applications in which differences between order statistics matter are matching markets. Considering expected matches between firms and workers, or men and women, requires to control distances between order statistics not only at the top, but also on lower levels of a distribution. Our techniques allow to generalize prominent results on overall welfare and revenues in these contexts due to Hoppe, Moldovanu and Sela (2009). Another field of application beyond the scope of this paper may be measurements of inequality, where distances from the poorest (or the richest) to the middle income quantiles of a population may be of specific interest. For example, recent developments in Western countries such as the US suggest that a greater focus on the distances between the richest 400 families and the middle class may be of specific interest for defining educational goals for the next decades.[2]

Related Literature

This paper is related to several contributions in the literatures on auctions and on stochastic orders.[3] Our auction-theoretic applications generalize results of Ganuza and Penalva (2010) and thus contribute to the literature on information in auctions and mechanism design.[4] Jia, Harstad and Rothkopf (2010) study information release in auctions when bidders know parts of their valuations and the other additive part can be disclosed. They illustrate that the comparative statistics of bidders' revenues are intricate and conclude that "no illuminating necessary condition seems possible". This is the problem we address. Stochastic orders, especially the dispersive order, have also been applied to study other questions concerning auctions and related

[2]See "America's elite. An hereditary meritocracy", The Economist, 01/24/2015.

[3]For introductions to these two fields, see Krishna (2002) and Shaked and Shanthikumar (2007).

[4]For a survey, see Bergemann and Välimäki (2006).

contexts, see, for instance, Johnson and Myatt (2006) and, for a recent contribution with many references, Kirkegaard (2014).

In the literature on stochastic orders, our analysis builds on a result of Li and Shaked (2004) who proved one of the main properties of the k-dispersion order[5] without explicitly introducing this order. We provide several new of insights on k-dispersion and introduce the generalized k-m-dispersion orders. As the k-dispersion order coincides with the excess wealth order in the case $k = 1$, our results are also related to two contributions from the operations research literature which apply the excess wealth order in auction theory, Li (2005) and Xu and Li (2008). Analyzing the case $k > 1$ allows us to address many questions which are not tractable under the excess wealth order. Paul and Gutierrez (2004) provide several results related to ours based on the star order. Yet their results stating that differences of order statistics can be controlled in terms of the star order is incorrect as is shown in Xu and Li (2008).[6] The same remark applies to Rösler (2014) whose work partially builds on the same incorrect result. Our results can help to alleviate these issues.

Outline

Section 2 introduces our model and discusses the scope and limitations of modeling information release in terms of the dispersive order. Section 3 introduces our new stochastic orders as well as their key properties. Section 4 presents our main results on information release in multi-object auctions, first in the general case and then in the case of information partitions. Section 5 sketches further economic applications of our methods and presents additional properties of k-dispersion. All proofs are in the appendix.

2 The Setting

2.1 Auction Model with Information Release

We study a symmetric independent private values auction model with information release. Our techniques will allow us to handle one object as well as k object auctions. We therefore introduce the broader setting straight away.

[5] This result is Proposition 2 below.

[6] This incorrect result is also included in Shaked and Shanthikumar (2007) as Theorem 4.B.19 and as Lemma 4 in Rösler (2014).

A risk-neutral seller auctions off k identical objects in a $(k+1)^{th}$ price auction. The $n > k$ bidders are all risk-neutral. Those who submit the k highest bids will receive an object and each of them will pay the $(k+1)^{th}$ highest bid.

Before the auction takes place, the seller decides whether he wants to release information to the bidders. If he opts against information release, the bidders will stick to their initial private estimates Y_i of their valuations. The Y_i are nonnegative and each distributed according to a commonly known cumulative distribution function G with finite mean. If the seller opts for information release, each bidder receives an independent signal that reveals more about his valuation for winning an object. We denote by X_i the accordingly updated estimates of valuations. The random variables X_i are again nonnegative, independent and identically distributed with finite mean and we denote their cumulative distribution function by F. We denote by F^{-1} and G^{-1} the generalized inverse (quantile) functions of F and G.

Throughout we assume that the bidders follow their weakly dominant strategy of bidding their best estimate of their valuation in the auction. Thus, bidder i bids X_i if information was released and Y_i otherwise. We denote by $X_{i:n}$ the i^{th} order statistic, i.e., the i^{th}-largest out of $X_1, \ldots, X_n$, and define $Y_{i:n}$ analogously. Lemma 1 summarizes the main equilibrium properties of the bidding game.

Lemma 1 *Set $Z = X$ if information was released and $Z = Y$ if no information was released.*

(i) The expected selling price in the auction is given by $E[Z_{k+1:n}]$.

(ii) The seller's expected payoff is given by $k\, E[Z_{k+1:n}]$.

(iii) Bidders' aggregate rents are given by

$$\sum_{j=1}^{k} E[Z_{j:n} - Z_{k+1:n}].$$

(iv) Total welfare amounts to

$$\sum_{j=1}^{k} E[Z_{j:n}].$$

In the sequel, we call the seller a welfare maximizer if he is interested in maximizing total welfare, and we call him a revenue maximizer if his objective is his expected payoff.

An alternative interpretation of the model is that F denotes a finer information structure compared to G, and the seller decides whether to release a signal implementing G or F. In the context of information release with Bayesian updating, it is plausible to assume that F and G share the same mean. Our analysis, however, does not rely on this assumption, thus incorporating the possibility of non-Bayesian updating by the bidders. As a final interpretation, the seller could decide between running the auction in one or another country, with bidders from completely different populations F versus G.

We do not impose that F and G are continuous. This allows us to provide results for models of information release such as information partitions that would violate a continuity requirement. The additional structures introduced in Ganuza and Penalva (2010) in the one object case – a prior distribution of valuations, a continuous family of signals with an associated cost function, and a continuous family of (posterior) distributions of valuations – directly translate to our setting.

2.2 Information Release and the Dispersive Order

Our main interest lies in the comparative statics of the model. How does information release affect welfare and revenues? What can we say about the interplay of information release and the number of bidders? Who releases more information, a revenue maximizer or a welfare maximizer? Providing bidders with more information should increase the variability in their estimated valuations. F should thus be more variable (or "dispersed") than G. In their analysis of information release, Ganuza and Penalva (2010) study two notions of dispersion, an ordering between F and G in the convex order, and an ordering of F and G in the dispersive order. These are defined as follows.[7]

Definition 1

(i) F is more variable than G in the convex order[8]*, $F \succeq_{conv} G$, if $E[X_1] = E[Y_1]$ and*

$$E[(X_1 - k)^+] \;\geq\; E[(Y_1 - k)^+] \;\; \textit{for all } k \in \mathbb{R}.$$

[7]For background on these two orders, see Chapters 3.A and 3.B of Shaked and Shanthikumar (2007). Our definitions follow their Theorem 3.A.1 and Formula 3.B.1.

[8]For our purposes it proves to be more convenient to formulate stochastic orders on the level of distribution functions and not on the level of random variables as is done, e.g., in Shaked and Shanthikumar (2007).

(ii) F is more variable than G in the dispersive order, $F \succeq_{disp} G$, if

$$F^{-1}(p) - F^{-1}(q) \geq G^{-1}(p) - G^{-1}(q) \quad \textit{for all } 0 < q < p < 1. \tag{1}$$

An ordering in the convex order is a weak requirement closely related to second-order stochastic dominance. It is satisfied in many models of information release. Under the assumption that $F \succeq_{conv} G$, Ganuza and Penalva show that releasing information increases expected welfare and, with sufficiently many bidders, the expected revenue in the auction.[9] Both results follow from the intuition that increasing the variability of valuations tends to increase the highest valuations.

In order to control differences between overall welfare and seller's revenues, stronger orderings need to be imposed. Ganuza and Penalva rely on the dispersive order. F dominates G in the dispersive order if all pairs of quantiles lie further apart under F than under G. As we will see below, this is a rather rigid requirement which is violated in many models of information release. The next lemma summarizes their results on information release in auctions under the assumption that $F \succeq_{disp} G$.[10]

Lemma 2 *Assume $F \succeq_{disp} G$ and $k = 1$.*

(i) Bidders' aggregate rents increase when information is released,

$$E[X_{1:n} - X_{2:n}] \geq E[Y_{1:n} - Y_{2:n}].$$

(ii) A welfare maximizing seller has a stronger incentive to release information than a revenue maximizing seller,

$$E[X_{1:n} - Y_{1:n}] \geq E[X_{2:n} - Y_{2:n}].$$

(iii) The expected welfare generated by the auction increases more strongly when the number of bidders increases under information release than when no information is released,

$$E[X_{1:n} - X_{1:n-1}] \geq E[Y_{1:n} - Y_{1:n-1}].$$

[9]These results are their Theorems 3 and 5. A generalization to the k object case, relying on their techniques and results by de la Cal and Cárcamo (2006) on majorization of order statistics is straightforward and omitted here.

[10]The four parts of Lemma 2 correspond to Proposition 6, Theorem 7, Theorem 4 and Theorem 6 of Ganuza and Penalva (2010).

(iv) The seller's expected payoff increase more strongly when the number of bidders increases under information release than when no information is released

$$E[X_{2:n} - X_{2:n-1}] \geq E[Y_{2:n} - Y_{2:n-1}].$$

All four results rely on comparisons of differences of order statistics, so-called spacings. Technically, they stem from the following fact about the dispersive order.[11]

Lemma 3 *Let $F \succeq_{disp} G$. Then for all $k < n$*

$$E[X_{k:n} - X_{k+1:n}] \geq E[Y_{k:n} - Y_{k+1:n}]$$

and

$$E[X_{k:n} - X_{k:n-1}] \geq E[Y_{k:n} - Y_{k:n-1}].$$

In the remainder of this section, we illustrate that many economic contexts do not fall under Lemma 2 and lead to the opposite implications.

Example 1

Assume that bidders' true valuations are distributed uniformly on $[0, 1]$. Bidders do not know their true valuations. They only know whether their valuation is below 2/3 or not. By releasing information, the seller can furnish bidders with the additional information whether their valuations lie below or above 1/3. Consequently, the a priori distribution G puts a mass of 2/3 on the value 1/3 and the remaining mass on 5/6.[12] The a posteriori distribution F is a uniform distribution on 1/6, 1/2 and 5/6. Notice first that F and G are not comparable in the dispersive order. When moving from G to F the lowest third of probability mass moves downwards from 1/3 to 1/6 while the middle third moves upwards from 1/3 to 1/2. The upper quantiles do not react to the information release. Therefore, the lower quantiles are indeed more dispersed under F than under G. Yet the upper quantiles lie closer together. When working with information partitions, information release will always lead to such ambiguous effects and thus preclude a direct application of the dispersive order.

As Lemma 2 is not applicable in our example, we compare welfare and seller's

[11] The first claim of Lemma 3 follows from Theorem 3.B.41 of Shaked and Shanthikumar (2007). The second claim follows from the first and formula (6) below.

[12] For a more formal introduction of this model, see Section 4.2.

revenues by a direct calculation.

$$E[X_{1:n} - X_{2:n}] = \frac{1}{9}n\left(\frac{2}{3}\right)^{n-1}\left(1 + \left(\frac{1}{2}\right)^{n-2}\right) \quad \text{and} \quad E[Y_{1:n} - Y_{2:n}] = \frac{1}{6}n\left(\frac{2}{3}\right)^{n-1}.$$

For $n = 2$, we obtain results similar to parts (i) and (ii) of Lemma 2. For $n = 3$, welfare and seller's revenues react equally strongly. With four or more bidders, the results are reversed. Bidders' aggregate rents decrease when information is released. Thus a revenue maximizing seller has a stronger incentive to release information than a welfare maximizing one.[13]

In our example information affects bidders with intermediate valuations more strongly than bidders with high valuations. This renders the auction more competitive. In particular, information release does not increase the differences between high order statistics.

If we look at restrictions of F and G to sufficiently high quantiles, we see that information release reduces dispersion in this example.

Definition 2 *For $p \in (0, 1)$ define the restriction of F to its quantiles higher than p as the cumulative distribution function*

$$F_{>p}(x) = \begin{cases} \frac{F(x)-p}{1-p} & x \geq F^{-1}(p) \\ 0 & x < F^{-1}(p) \end{cases}$$

and define $G_{>p}(x)$ analogously.[14]

Now consider the distribution functions $F_{>1/3}$ and $G_{>1/3}$. $F_{>1/3}$ is the uniform distribution on $\{1/2, 5/6\}$ while $G_{>1/3}$ is the uniform distribution on $\{1/3, 5/6\}$. Unlike F and G themselves, these restrictions can be compared in the dispersive order. Yet it is the distribution without information release that is more dispersed, $G_{>\frac{1}{3}} \succeq_{disp} F_{>\frac{1}{3}}$.[15] Since higher quantiles dominate the behavior of higher order statistics with sufficiently many bidders, this observation explains the reversal of Lemma 2. Indeed, we will see in Proposition 6 and Theorem 1 that a dispersive

[13] As we will see in greater generality in Section 4.2, parts (iii) and (iv) are also reversed with sufficiently many bidders.

[14] Notice that the definition is such that if F has an atom on $F^{-1}(p)$, i.e., $F(F^{-1}(p)) = q > p$ then $F_{>p}(x)$ has an atom of size $(q-p)/(1-p)$ on $F^{-1}(p)$.

[15] Notice that releasing information cannot have the effect that *all* quantiles move more closely together, $G \succeq_{disp} F$.

ordering of F and G above some quantile is essentially a sufficient condition for whether Lemma 2 holds or whether it is reversed.

3 Dispersion Criteria for Order Statistics

As seen in Section 2, the dispersive order implies a control over *all* spacings of order statistics while the outcomes of auctions depend only on the highest few. This motivates us to introduce the k-dispersion orders which are specifically designed to control spacings of the k highest order statistics. Example 1 shows that a clear monotonicity behavior of these spacings may only develop with sufficiently many bidders. To capture these situations, we introduce the weaker k-m-dispersion orders. These allow to control the behavior of the k highest order statistics in auctions with more than $k + m$ bidders. The goal of both families of orders is to focus on the parts of a distribution which are most relevant for the auctions' outcomes, and not to impose more restrictions than needed.

3.1 The k-Dispersion Orders

This section introduces the family of k-dispersion orders, compares them with other orders, and develops their implications.

Definition 3 (k-Dispersion) *For an integer $k \geq 1$, F is more dispersed than G in the k-dispersion order, $F \succeq_k G$, if*

$$\int_p^1 (1-u)^k dF^{-1}(u) \geq \int_p^1 (1-u)^k \, dG^{-1}(u) \tag{2}$$

for all $p \in (0, 1)$.

This definition is inspired by the following representation of spacings in terms of the quantile function, see, e.g., Kadane (1971),

$$E[X_{k:n} - X_{k+1:n}] = \binom{n}{k} \int_0^1 u^{n-k}(1-u)^k dF^{-1}(u). \tag{3}$$

The k-dispersion order $\succeq_k$ allows to control spacings of neighboring order statistics independently of n. In Section 5, we demonstrate its applicability to order statistics that lie further apart as well as to normalized spacings.

$\succeq_k$ is a genuine stochastic order in that it is transitive:[16] For three distribution functions F, G, and H, $F \succeq_k G$ and $G \succeq_k H$ imply $F \succeq_k H$. While the 1-dispersion order coincides with the excess wealth order,[17] the k-dispersion orders for $k > 1$ appear to be novel.[18] Like the excess wealth order, all k-dispersion orders are location independent, i.e., $F \succeq_k G$ remains fulfilled if either of the two distributions is shifted by a constant.

Proposition 1 sets k-dispersion into context. The dispersive order is stronger (and thus less broadly applicable) than all k-dispersion orders. For instance, it is a necessary condition for the dispersive order that F^{-1} and G^{-1} cross only once. k-dispersion does not rely on such a single-crossing condition.

Within the family of k-dispersion orders, $(k + 1)$-dispersion implies k-dispersion. The convex order can generally not be compared to k-dispersion and the dispersive order as it is not location independent: $F \succeq_{conv} G$ can only hold if F and G have the same mean. Under the assumption that F and G share the same mean, the convex order is implied by each of the other orderings. Yet the convex order itself is not strong enough to control spacings of order statistics.

Proposition 1

(i) If $F \succeq_{disp} G$ then $F \succeq_k G$ for all $k \geq 1$.

(ii) If $F \succeq_{k+1} G$ then $F \succeq_k G$ for all $k \geq 1$.

(iii) If $E[X_1] = E[Y_1]$ and $F \succeq_k G$ then $F \succeq_{conv} G$ for all $k \geq 1$.

Proposition 2 demonstrates the suitability of k-dispersion for controlling spacings of high order statistics. The result combines Proposition 1 (ii) with Proposition 3.4 of Li and Shaked (2004).

Proposition 2 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$*

$$E[X_{i:n} - X_{i+1:n}] \geq E[Y_{i:n} - Y_{i+1:n}].$$

Next, we extend this result to the other class of spacings of order statistics where we vary n while keeping i fixed. The key observation is that the two types of spacings differ only by a combinatorial factor which is not distribution-dependent.

[16] This separates k-dispersion from some single-crossing criteria for dispersion such as the rotation criterion of Johnson and Myatt (2006).

[17] See Shaked and Shanthikumar (2007) for background on the excess wealth order.

[18] The definition is motivated by an observation of Li and Shaked (2004), see Proposition 2 below.

Proposition 3 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$*

$$E[X_{i:n} - X_{i:n-1}] \geq E[Y_{i:n} - Y_{i:n-1}].$$

Since k-dispersion orders are location independent, we cannot expect to obtain results comparing $E[X_{k:n}]$ and $E[Y_{k:n}]$ without further assumptions. Under the additional requirement that F and G share the same mean, results of this type can be derived from the fact that k-dispersion implies the convex order.

3.2 The k-m-Dispersion Orders

While the k-dispersion orders are conceptually much weaker than the dispersive order, there are still many economically interesting examples for which they are too rigid. For instance, in Example 1 monotonicity of spacings sets in only with sufficiently many bidders. Building on k-dispersion we therefore introduce the weaker class of k-m-dispersion orders. These imply the results of Propositions 2 and 3 under the additional restriction that the number of bidders is sufficiently large, namely $n > k + m$.

Definition 4 (k-m-Dispersion) *For integers $k \geq 1$ and $m \geq 0$, F is more dispersed than G in the k-m-dispersion order, $F \succeq_{k,m} G$, if*

$$\int_p^1 u^m (1-u)^k dF^{-1}(u) \geq \int_p^1 u^m (1-u)^k \, dG^{-1}(u) \tag{4}$$

for all $p \in (0, 1)$.

The k-0-dispersion order coincides with our previous k-dispersion order. Compared to k-dispersion, the increasing function u^m in the integrand shifts attention into the right tail of the distribution. With many bidders, the behavior at this tail is crucial for an auction's outcomes. Proposition 4 summarizes the central properties of k-m-dispersion. The proposition generalizes Propositions 2 and 3.

Proposition 4

(i) If $F \succeq_{k,m} G$ for some k and m with $k + m < n$ then for all $i \leq k$

$$E[X_{i:n} - X_{i+1:n}] \geq E[Y_{i:n} - Y_{i+1:n}]$$

(ii) If $F \succeq_{k,m} G$ for some k and m with $k+m<n$ then for all $i \leq k$

$$E[X_{i:n} - X_{i:n-1}] \geq E[Y_{i:n} - Y_{i:n-1}]$$

To put the k-m-dispersion orders into context we add the following result in the spirit of Proposition 1.

Proposition 5

(i) If $F \succeq_{k,m} G$ then $F \succeq_{k,m+1} G$ for all $k \geq 1$ and for all $m \geq 0$.

(ii) If $F \succeq_{k+1,m} G$ then $F \succeq_{k,m} G$ for all $k \geq 1$ and for all $m \geq 0$.

Accordingly, increasing m renders the k-m-dispersion order less rigid. All k-m-dispersion orders are weaker than the k-dispersion order and, consequently, the dispersive order. This allows to apply them to information partitions as demonstrated in Section 4.2.

The following alternative sufficient condition for k-m-dispersion is useful, e.g. when working with discrete distributions. The condition relies on the dispersive order between restrictions of F and G to high quantiles as introduced in Definition 2.

Proposition 6 *Suppose there exists $p \in (0,1)$ such that $F_{>p} \succeq_{disp} G_{>p}$ and there exist $q_1, q_2 \in (p,1)$ with $F^{-1}(q_2) - F^{-1}(q_1) > G^{-1}(q_2) - G^{-1}(q_1)$. Then for any k there exists m such that $F \succeq_{k,m} G$.*

In the proposition, the condition involving q_1 and q_2 ensures that the comparison in the dispersive order holds, in a sense, strictly.

4 Information Release in Multi-Object Auctions

4.1 The General Case

This section applies k- and k-m-dispersion to information release in k object auctions. Theorem 1 generalizes Lemma 2. It provides conditions for welfare reacting stronger to information than seller's revenues, as well as conditions for the opposite situation. Further, it covers the cases in which sufficiently many bidders need to take part in order to arrive at clearcut results.

Theorem 1

(i) If $F \succeq_{k,m} G$ *and* $n > k+m$*, then bidders' aggregate rents increase when information is released.*

(ii) If $F \succeq_{k,m} G$ *and* $n > k+m$*, then a welfare maximizing seller has a stronger incentive to release information than a revenue maximizing seller.*

(iii) If $F \succeq_{k,m} G$ *and* $n > k+m$*, then the welfare generated by the auction increases more strongly when the number of bidders increases under information release than when no information is released.*

(iv) If $F \succeq_{k+1,m} G$ *and* $n > k+1+m$*, then the expected selling price and the seller's payoff increase more strongly when the number of bidders increases under information release than when no information is released.*

(v) The conclusions of (i-iii) are reversed if $G \succeq_{k,m} F$ *and* $n > k+m$*. The conclusion of (iv) is reversed if* $G \succeq_{k+1,m} F$ *and* $n > k+1+m$*.*

Thus, in the setting $k = 1$ and $m = 0$ of Ganuza and Penalva, the excess wealth order is sufficient for (i) to (iii) while the stronger 2-dispersion order is needed for (iv). Note also that we need to require stronger dispersion orders when the number of objects k increases. An immediate consequence of (ii) is that if information release is costly then for intermediate cost levels a welfare maximizer will release information while a revenue maximizer will not. Finally, while the results may only hold with sufficiently many bidders, they are more than asymptotic results. In particular, k-m-dispersion provides an explicit criterion for determining the value of m when k, F and G are given.

4.2 Information Partitions

When information release takes the form of increasingly finer information partitions, Theorem 1 yields a complete characterization of information release with sufficiently many bidders. As we are going to see, if information release increases the highest valuation estimate, the requirements of claims (i) to (iv) of the theorem are fulfilled. If the highest valuation estimate is unaffected by information release, the four claims are reversed.

Assume that bidders' true valuations are distributed according to a continuous distribution function H with a strictly positive density h on an interval $[a, b]$ with $a \geq 0$

and $a < b \leq \infty$. Denote by $(\beta_i)_i$ an ordered and strictly increasing subsequence of (a, b) with $B > 0$ elements. Thus, β_1 and β_B are the lowest and highest values in the sequence. Without information release, bidders only know for each of the values β_i whether their valuations lie above or below. Accordingly, the distribution G of valuation estimates assigns probability

$$H(\beta_i) - H(\beta_{i-1}) \quad \text{to the estimate} \quad \frac{\int_{\beta_{i-1}}^{\beta_i} xh(x)dx}{H(\beta_i) - H(\beta_{i-1})} \tag{5}$$

with the obvious modifications for β_1 and β_B.

Information release is modeled such that the seller increases the number of values for which bidders know whether their valuation lies above or below. The sequence $(\beta_i)_i$ is thus replaced by another ordered and strictly increasing sequence $(\alpha_i)_i$ with $A > B$ elements. $(\beta_i)_i$ is a subsequence of $(\alpha_i)_i$. The distribution F of posterior valuation estimates is derived from $(\alpha_i)_i$ analogously to (5).

Proposition 7 shows that for any k, F and G are always comparable in the k-m-dispersion order for sufficiently large m.

Proposition 7

(i) If $\alpha_A = \beta_B$, then for any k there exists an m such that $G \succeq_{k,m} F$.

(ii) If $\alpha_A > \beta_B$, then for any k there exists an m such that $F \succeq_{k,m} G$.

Whether F or G is more dispersed thus depends on whether information release affects the highest valuation estimates or not. If $\alpha_A = \beta_B$, the bidders with the highest valuation estimates are not affected by information release. The auction thus becomes more competitive such that the reverses of claims (i-iv) of Theorem 1 hold with sufficiently many bidders. If $\alpha_A > \beta_B$, information release further differentiates the valuation estimates of the highest valuation bidders. Consequently, the auction becomes less competitive and the four claims of Theorem 1 hold with sufficiently many bidders.

5 Further Applications

This section extends our analysis to other economic contexts, like matching markets, and the control of differences in low realizations which is important for risk management and reliability theory. Via k-dispersion, we can compare increments of expected order statistics $E[X_{k:n}]$ that are next to each other with regard to k or n. In this section, we show that k-dispersion serves as a tool for controlling differences of order statistics that lie further apart as well, and describe where this control can be applied.

Proposition 8 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$ and all $l > i$*

$$E[X_{i:n} - X_{l:n}] \geq E[Y_{i:n} - Y_{l:n}]$$

Proposition 8 characterizes which degree of k-dispersion is needed in order to compare specific differences of order statistics. For example, the 1-dispersion order allows to contrast differences between first and third order statistics across distributions. A similar comparison of the second and third order statistics requires the stronger 2-dispersion order. The proposition generalizes the main result of Kochar, Li and Xu (2007)[19] which treats the case $k = 1$.

A direct consequence of Proposition 8 is that it allows to compare sums of spacings of order statistics: $F \succeq_k G$ implies that

$$\sum_{j=i}^{l-1} E[X_{j:n} - X_{j+1:n}] \geq \sum_{j=i}^{l-1} E[Y_{j:n} - Y_{j+1:n}]$$

for all $i \leq k$ and all $n > l > i$. Proposition 9 provides similar results for the so-called normalized spacings of order statistics.

Proposition 9 *If $F \succeq_k G$ for some $k < n$ then for all $i \leq k$ and all $l > i$*

$$\sum_{l=i}^{m} lE[X_{l:n} - X_{l+1:n}] \geq \sum_{l=i}^{m} lE[Y_{l:n} - Y_{l+1:n}].$$

The case $k = 1$ generalizes a result of Barlow and Proschan (1966) which is a key ingredient of Hoppe, Moldovanu and Sela (2009)'s analysis of matching markets. In the latter paper, women and men can invest into costly presents in order to improve their matching outcomes (and thus, e.g., match with a partner that is ranked better than the partner they would obtain otherwise). The inequality of

[19]Kochar, Li and Xu apply their result to the study of one object k^{th}-price auctions. This part of their analysis is problematic from the viewpoint of game-theoretic auction theory since it relies on the assumption that bidders truthfully bid their valuations independently of the auction format.

Proposition 9 allows to study the comparative statics of signaling costs and welfare in this marriage market. Barlow and Proschan (1966) rely on the convex transform order which is stronger than the excess wealth order at least when F and G have the same mean.[20] Proposition 9 shows that the results of Hoppe, Moldovanu, and Sela hold under weaker requirements on the distributions.

Regarding the spacings of the k lowest order statistics, one can define the family of $\overline{k}$-dispersion orders given by

$$F \succeq_{\overline{k}} G \quad \Leftrightarrow \quad \int_0^p u^k dF^{-1}(u) \geq \int_0^p u^k \, dG^{-1}(u) \quad \forall p \in (0,1).$$

For example, expected differences in quality for the worst, second to worst, third to worst, etc. product out of a production series can be compared through these orders. All arguments for this family of orders are parallel to those we obtained for the k-dispersion orders. Like the 1-dispersion order, the $\overline{1}$-dispersion order coincides with a familiar stochastic order, namely, with the location independent risk order of Jewitt (1989).

A Proofs

Proof of Proposition 1

To see (i), notice that $F \succeq_{disp} G$ implies that the measure ν given by $d\nu(u) = d(F^{-1}(u) - G^{-1}(u))$ is non-negative, so that integrals of non-negative functions against ν are non-negative. Thus (2) holds for all p. (ii) is shown as follows: Lemma 7.1(a) of Chapter 4 of Barlow and Proschan (1981) states that for any signed measure ν on $\mathbb{R}^+$ and any non-decreasing, non-negative function h

$$\int_p^\infty d\nu(u) \geq 0 \quad \forall p > 0 \Rightarrow \int_0^\infty h(u) d\nu(u) \geq 0.$$

Applying this result with $d\nu(u) = (1-u)^{k+1} d(F^{-1}(u) - G^{-1}(u))$ shows that $F \succeq_{k+1} G$ implies

$$\int_0^1 h(u)(1-u)^{k+1} dF^{-1}(u) \geq \int_0^1 h(u)(1-u)^{k+1} \, dG^{-1}(u)$$

[20] Shaked and Shanthikumar (2007), formula (4.B.3) shows that the convex transform order implies the star order. Li (2005), Remark 2.7, shows that the star order implies the excess wealth order if F and G share the same mean.

for any non-decreasing, non-negative h. Applying this inequality to all members of the family of non-decreasing functions $(h_q)_{q\in(0,1)}$ defined by $h_q(u) = (1-u)^{-1}1_{\{u\geq q\}}$ yields

$$\int_q^1 (1-u)^k dF^{-1}(u) \geq \int_q^1 (1-u)^k \, dG^{-1}(u) \quad \forall q \in (0,1)$$

and thus $F \succeq_k G$. (iii) follows from the fact that $F \succeq_k G$ implies $F \succeq_1 G$ by (ii), and from the fact that $\succeq_1$ is the excess wealth order so that we can apply Formula 3.C.8 of Shaked and Shanthikumar (2007). □

Proof of Proposition 2

By Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. In light of formula (3), it is sufficient to show that

$$\int_0^1 u^{n-k}(1-u)^k dF^{-1}(u) \geq \int_0^1 u^{n-k}(1-u)^k dG^{-1}(u).$$

This inequality follows from the definition (2) of the k-dispersion order by applying - like in the proof of Proposition 1 - Lemma 7.1(a) of Chapter 4 of Barlow and Proschan (1981) to the signed measure ν given by $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and to the non-decreasing function $h(u) = u^{n-k}$. □

Proof of Proposition 3

Again, by Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. Rewriting Relation 1 from David (1970, p. 45) into our notation yields

$$(n-k)E[X_{k:n}] + kE[X_{k+1:n}] = nE[X_{k:n-1}]$$

which can be written as

$$E[X_{k:n}] - E[X_{k:n-1}] = \frac{k}{n}\left(E[X_{k:n}] - E[X_{k+1:n}]\right). \qquad (6)$$

Thus we can apply Proposition 2 and conclude that $F \succeq_k G$ implies

$$\begin{aligned} E[X_{k:n}] - E[X_{k:n-1}] &= \frac{k}{n}(E[X_{k:n}] - E[X_{k+1:n}]) \\ &\geq \frac{k}{n}(E[Y_{k:n}] - E[Y_{k+1:n}]) = E[Y_{k:n}] - E[Y_{k:n-1}] \end{aligned}$$

□

Proof of Proposition 4

By Proposition 5 (ii) we can focus on the case $i = k$. The proof of (i) is entirely parallel to the one of Proposition 2 except that we choose $d\nu(u) = u^m(1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and $h(u) = u^{n-k-m}$. (ii) follows from (i) with the same argument that deduced Proposition 3 from Proposition 2. □

Proof of Proposition 5

The proof of (i) is entirely parallel to the one of Proposition 1 (ii) except that we choose $d\nu(u) = u^m(1-u)^{k+1} d(F^{-1}(u) - G^{-1}(u))$. The same is true for the proof of (ii) where we choose $d\nu(u) = u^m(1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and $h_q(u) = u1_{\{u \geq q\}}$.□

Proof of Proposition 6

Choose the measure ν as $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$. We have to show that there exists m such that

$$L(r) = \int_r^1 u^m d\nu(u)$$

is non-negative for all $r \in (0, 1)$. By assumption, the measure ν is nonnegative over $[p, 1]$. This proves the claim for $r \geq p$. For $r < p$ consider the decomposition

$$L(r) = \int_r^p u^m d\nu(u) + \int_p^{q_1} u^m d\nu(u) + \int_{q_1}^{q_2} u^m d\nu(u) + \int_{q_2}^1 u^m d\nu(u).$$

The second and fourth integrals are non-negative by assumption. The first integral we can bound from below by replacing ν by its negative part ν^- which exists by the Hahn decomposition:

$$L(r) \geq \int_r^p u^m d\nu^-(u) + \int_{q_1}^{q_2} u^m d\nu(u).$$

Since these are integrals with respect to a negative and a positive measure, we can further bound them from below as follows:

$$L(r) \geq p^m \left(\int_0^p d\nu^-(u) \right) + q_1^m \left(\int_{q_1}^{q_2} d\nu(u) \right).$$

Since the second term in brackets is strictly positive by our assumption on q_1 and q_2, and since $q_1 > p$, it follows that $L(r) \geq 0$ for sufficiently large m. Since the final lower bound is independent of r, this choice of m is the same for all r. □

Proof of Theorem 1

Observe that we can write bidders' aggregate rents after information release as

$$\sum_{j=1}^{k} E[X_{j:n} - X_{k+1:n}] = \sum_{j=1}^{k} jE[X_{j:n} - X_{j+1:n}]$$

To the expression on the right hand side we can apply Proposition 4 and conclude

$$\sum_{j=1}^{k} E[X_{j:n} - X_{k+1:n}] \geq \sum_{j=1}^{k} E[Y_{j:n} - Y_{k+1:n}]$$

which is (i). Rearranging this inequality yields

$$\sum_{j=1}^{k} E[X_{j:n} - Y_{j:n}] \geq E[kX_{k+1:n} - kY_{k+1:n}]$$

which proves (ii) since the welfare generated by the auction is given by the sum of the k highest valuations. The gains from adding an additional bidder when releasing information are given by

$$\sum_{j=1}^{k} E[X_{j:n} - X_{j:n-1}].$$

This is greater than the corresponding quantity with Y in place of X by Proposition 4 provided that $F \succeq_k G$. This shows (iii). The statement about the expected selling price in (iv) follows immediately from observing that Proposition 4 yields

$$E[X_{k+1:n} - X_{k+1:n-1}] \geq E[Y_{k+1:n} - Y_{k+1:n-1}]$$

provided that $F \succeq_{k+1} G$. The statement about the seller's payoff follows by multiplying this inequality with k. (v) follows by exchanging the roles of F and G. $\square$

Proof of Proposition 7

Denote by α^* the largest element of $(\alpha_i)_i$ which is not included in $(\beta_i)_i$ and set $p = H(\alpha^*)$. We prove (i) by showing that $G_{>p} \succeq_{disp} F_{>p}$ and then invoking Proposition 6. Denote by $\beta_+^* > \beta_-^*$ the upper and lower neighbors of α^* in the sequence $(\beta_i)_i$. Observe that the distributions $F_{>p}$ and $G_{>p}$ are both discrete distributions concentrated on a finite number of values. In particular, since the two partitions are identical from $\beta_+^* \in (\alpha_i)_i$ on, the two distributions are identical except for the lowest value. For $F_{>p}$, the lowest possible realization l_F is the conditional mean of H over the set $[\alpha^*, \beta_+^*]$, while for $G_{>p}$ this lowest realization is the conditional mean l_G over $[\beta_-^*, \beta_+^*]$. Both occur with the same positive probability $(H(\beta_+^*) - H(\alpha^*))/(1-p)$. Clearly, we have $l_F > l_G$. Since this difference between the lowest realizations is the only difference of $F_{>p}$ and $G_{>p}$, it follows directly that $G_{>p} \succeq_{disp} F_{>p}$. Since all probabilities are strictly positive, we can also guarantee existence of q_1 and q_2 as required by Proposition 6.

The proof of (i) proceeds similarly by showing that $F_{>p} \succeq_{disp} G_{>p}$. We set $p = H(\beta_B)$. Then $G_{>p}$ is a degenerate distribution which takes as its only value the conditional mean of H over $[\beta_B, b]$. $F_{>p}$ takes at least two values with positive probability, since the sequence (α_i) contains at least one value which is greater than β_B. Thus, $F_{>p}$ is easily seen to be strictly more dispersed than $G_{>p}$. □

Proof of Propositions 8 and 9

It is convenient to give a combined proof of the two propositions. By Assertion (ii) of Proposition 1, it is sufficient to consider the case $k = i$. From (3) we obtain that

$$E[X_{k:n} - X_{l:n}] = \int_0^1 \sum_{j=k}^{l-1} \binom{n}{j} u^{n-j}(1-u)^j dF^{-1}(u)$$

and

$$\sum_{j=k}^{l} jE[X_{j:n} - X_{j+1:n}] = \int_0^1 \sum_{j=k}^{l-1} j \binom{n}{j} u^{n-j}(1-u)^j dF^{-1}(u).$$

Obviously, the right hand sides coincide up to the factor j in the second sum. In the following, we denote this factor by $\varphi(j)$ and consider the choices $\varphi(j) = 1$ and $\varphi(j) = j$. Now we claim the following:

Claim: For both, $\varphi(j) = 1$ and $\varphi(j) = j$, there exists a non-decreasing function h such that we can write

$$\sum_{j=k}^{l-1} \varphi(j) \binom{n}{j} u^{n-j}(1-u)^j = h(u)(1-u)^k$$

Provided that this claim is true, the desired inequality follows from the definition (2) of the k-dispersion order by applying - like in the proof of Proposition 1 - Lemma 7.1(a) of Chapter 4 of Barlow and Proschan (1981) to the signed measure ν given by $d\nu(u) = (1-u)^k d(F^{-1}(u) - G^{-1}(u))$ and to the non-decreasing function h identified in the claim: We obtain

$$\int_0^1 h(u)d\nu(u) \geq 0.$$

and thus

$$\int_0^1 \sum_{j=k}^{l-1} \varphi(j)\binom{n}{j}u^{n-j}(1-u)^j dF^{-1}(u) \geq \int_0^1 \sum_{j=k}^{l-1} \varphi(j)\binom{n}{j}u^{n-j}(1-u)^j dG^{-1}(u).$$

Thus it remains to prove the claim. Since we can write

$$\sum_{j=k}^{l-1} \varphi(j)\binom{n}{j}u^{n-j}(1-u)^j = (1-u)^k \sum_{j=k}^{l-1} \varphi(j)\binom{n}{j}u^{n-j}(1-u)^{j-k},$$

this amounts to proving that

$$h(u) = \sum_{j=k}^{l-1} \varphi(j)\binom{n}{j}u^{n-j}(1-u)^{j-k}$$

is increasing in u for our two choices of $\varphi(j)$. The key idea is to rewrite h in terms of a Binomial$(n-k, 1-u)$ distribution. We can write

$$h(u) = \sum_{j=0}^{l-k-1} \varphi(k+j)\binom{n}{k+j}u^{n-k-j}(1-u)^j = \sum_{j=0}^{n-k} \Psi(j)\binom{n-k}{j}u^{n-k-j}(1-u)^j$$

where

$$\Psi(j) = \varphi(k+j)\frac{\binom{n}{k+j}}{\binom{n-k}{j}}1_{\{j<l-k\}} = \varphi(k+j)\frac{n\cdot\ldots\cdot(n-k+1)}{(j+k)\cdot\ldots\cdot(j+1)}1_{\{j<l-k\}}.$$

For our two choices of φ which yield, respectively $\varphi(k+j) = 1$ and $\varphi(k+j) = j+k$, $\Psi(j)$ is clearly a non-negative, non-increasing function. Now denote by $Z_{n-k,1-u}$ a random variable distributed according to the Binomial$(n-k, 1-u)$ distribution. From writing h as

$$h(u) = E[\Psi(Z_{n-k,1-u})]$$

we can see that h is non-decreasing in u since Ψ is non-increasing. □

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