Semiparametric Identification of Panel Data Discrete Choice Demand Models*

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- Preliminary and Incomplete -

Abstract

Most of the papers dealing with identification of panel data discrete choice models with fixed effects, use the binary case as an example, while their results are also extended to the multinomial case. This paper extends the work on semi parametric identification of static panel data binary models to dynamic binary and static ordered panel data models and gives clear identification results. Set identification is achieved by differencing out the additively separable unobserved heterogeneity as in the case of linear panel data models. We review the work that has been done in semiparametric identification of static binary panel data models and provide valid identification bounds for the objects of interest in semi-parametric dynamic binary and static ordered panel data models. We find two main results. Firstly, in contrast to the static binary panel data model which is complete, the dynamic binary panel data model we study is incomplete and incoherent. Also, we find that in contrast with the binary panel data model where information is only provided from individuals who switch from one period to the next, in the ordered panel data model individuals who decide to stay with the "in-between" option are a useful source for identification.

Keywords: Panel Data, Static Discrete Choice, Dynamic Discrete Choice, Binary Response Models, Ordered Response Models

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1 Introduction

Most of the papers dealing with identification of discrete choice panel data models, use the binary case as an example, while their results are also extended to the multinomial case. This paper extends this work and studies semiparametric identification of panel data binary and ordered choice models. The implications that different assumptions have for the semiparametric identification of static and dynamic binary, as well as ordered choice models are examined and clear identification results are given.

It is often noted that individuals who were observed making a specific purchase decision in the past are more likely to make the same purchase decision in the future. In panel dara settings, when consumers are observed making a specific choice for a number of periods, it is often the case that future choices are functions of past decisions. As pointed out by Heckman (1981) there are two possible explanations for this effect. The first one is that consumers differ by some unobserved factor that is correlated over time, also referred to as spurious state dependence. The second explanation is that past choices enter the utility function directly and therefore affect future decisions. This dependence is referred to as the true state dependence. Panel data models where only spurious state dependence is present are called static panel data models, whereas when lagged dependent variables are also allowed to enter the utility function the panel data models are called dynamic.

Identification and estimation of panel data discrete choice models have been the centre of attention in theoretical and applied econometrics for some decades now. Panel data models where individuals are being observed for more than two periods, can be seen as solving an endogeneity problem, arising from the presence of a time-invariant unobserved component in the utility function. Identification and estimation of these models heavily rely on the assumptions econometricians place on this unobserved heterogeneity and the challenges these models face have been well established in the literature. Choosing between random and fixed effects, how to deal with initial conditions and lagged dependent variables, as well as the incidental parameters problem and the calculation of marginal effects, have all been extensively studied by a number of authors. A detailed summary of developments can be found in Arellano and Honoré (2001) and more recently in Arellano and Bonhomme (2011).

In linear panel data models where the dependent outcome is continuous, identification of the regression parameters is achieved by differencing out the additively separable unobserved heterogeneity. This paper uses the same approach in linear-in parameters discrete panel data models and studies identification when this individual effect is eliminated. As the model is non-linear, it is shown that differencing out the individual effect generally leads to set identification of the parameters of interest. The question this paper then aims to answer is:

> What bounds can be achieved when limited assumptions are being imposed, and under what conditions (distributional and support) are these bounds sharp?

Several papers, including Chamberlain (1982, 2010), Honoré (1992, 2002) and Honoré and Kyriazidou (2000), have shown that in linear index panel data models with binary outcomes, parametric point-identification of the regression parameters when regressors have bounded support can only be achieved under the assumption of logistically distributed time-varying unobservables. When at least one regressor is allowed to have unbounded support, Manski (1987) has shown that under the conditional median restriction, point identification is achieved when individuals are observed changing behaviour from one period to the next. These results rely on strong assumptions and thus are not robust to misspecifications. In an attempt to overcome these limitations, several papers, including Chernozhukov, Hahn, and Newey (2005), Honoré and Tamer (2006), Rosen (2012),Chernozhukov, Fernández-Val, Hahn, and Newey (2013) and Rosen and Weidner (2013,WP), study identification of static binary panel data models under fairly weak conditions. Even though, point identification of all the parameters and objects of interest usually fails in these cases, identification of (sharp) sets can be achieved and estimation can be done using linear programming. This paper provides results on semi-parametric identification of dynamic binary choice models.

This paper also studies identification in static panel data ordered choice models. Following the work by Honoré (1992) that shows how to consistently estimate the parameters in the truncated/censored panel data model, this paper will focus in the "in-between" case of ordered outcomes. As already mentioned, panel data models are sometimes seen as solving an endogeneity problem. Chesher (2010) and Chesher and Smolinski (2012) derive sharp identification bounds in non-parametric ordered response models in the presence of endogenous variables. They show that these bounds can shrink at a relatively faster rate as the relevance of the instruments increases and as the number of ordered outcomes becomes larger. In a more general setting a series of papers, Chesher, Rosen, and Smolinski (2013), Chesher and Rosen (2012, 2013), study identification of instrumental variable models with discrete dependent variables.

Another motivation for studying ordered choice panel data models is the intention to apply the models to study consumer behaviour in differentiated product markets. Different products are vertically (in terms of quality) differentiated as in Berry (1994), Bresnahan (1987) and Shaked and Sutton (1983) and consumers are observed purchasing a particular product for a number of periods.

Valid identification bounds in the binary and the ordered choice panel data model are provided. In addition, two important results are found. Firstly, in contrast to the static binary panel data which is complete, the dynamic binary panel data model studied in this paper is incomplete and incoherent. Also, in contrast to the binary panel data model where information is only provided from individuals who switch from one period to the next, in the ordered panel data model individuals who decide to stay with the "in-between" option are a useful source for identification of the regression coefficients.

The rest of the paper is structured as follows. Section 2 introduces the general model and the assumptions imposed. Most of the results are given in Section 3. In this section the assumption that the unobserved heterogeneity appears additively separable in the utility function is imposed and the approach used in linear panel data models, which tries to eliminate it from the utility function is used. Section 3.1 reviews the results on identification of static binary panel data model studied in Rosen and Weidner (2013,WP) and studies identification of dynamic binary panel data models. Section 3.2 studies identification in the static and dynamic ordered panel data model and extends the one-period vertically differentiation model into a multi-period one in a monopoly setting. The main results of this paper is that point identification fails in this set-ups, however the derivation of valid identification bounds might be possible under fairly weak assumption. Section 4 gives some concluding remarks.

2 The Discrete Choice Panel Data Model

Consider a panel data set where each individual is being observed for $T = 2$ periods. Each individual in the population is then characterized by a set of observables (Y, X) such that $Y \equiv \{Y_1, Y_2\}$, $X \equiv \{X_1, X_2\}$, and a set of unobservables (V, α), where $V = \{V_1, V_2\}$.

In every period $t = 1$ and $t = 2$, each individual chooses one option from the set of vertically differentiated, in terms of quality, ordered alternatives $Y_t \in \mathcal{Y}_t$, where $\mathcal{Y}_t = \{0, 1, \ldots, \overline{y}_t\}$. The options in $\mathcal{Y}_t$ are vertically differentiated such that if all the options were offered at the same price, $(Y_t = 1) \preceq (Y_t = 2) \preceq \cdots \preceq (Y_t = \overline{y}_t)$, while $Y_t = 0$ denotes the outside option. Furthermore, define period $t = 0$ as the initial period, in which the outcome $y_0 \in \mathcal{Y}_0$ is assumed to be known, however the model is not specified and the explanatory variables do not need to be known.

Following the work in Aristodemou and Rosen (2013,WP), who develop a cross-sectional model of horizontally and vertically differentiated products, define the indirect utility an individual receives from any choice y_t given the covariates x_t, y_{t-1} and the unobservables α, v_t by,

$$U_{yt} = u(y_t, x_t, y_{t-1}, v_t, \alpha, p_{yt}) = y_t[g(x_t, y_{t-1}, \alpha, v_t)] - p_{yt} \tag{1}$$

where, y_t is the quality level chosen in period t, x_t are observed individual characteristics, y_{t-1} is the quality level chosen last period, α is the unobserved to the econometrician time-invariant individual heterogeneity, v_t is the unobserved to the econometrician time-varying component, p_{yt} is a common to all individuals "cost" or perceived price variable that an individual has to incur in period t to obtain quality level y and $g()$ is a non-parametric function. The utility of the outside option $Y_t = 0$ is normalized such that,

$$U_{0t} = 0$$

As in McFadden (1974) individuals choose the quality level y_t^*, each period that maximizes their utility in (1), such that

$$y_t^* = arg \max_{y_t \in \mathcal{Y}_t} U_{yt}$$

and the conditional probabilities are given by,

$$P(Y_t = y^* | X_t = x_t, Y_{t-1} = y_{t-1}, p_{yt}) = P(U_{y^*t} \geq \max_{y_t \in \mathcal{Y}_t} U_{yt} | X_t = x_t, Y_{t-1} = y_{t-1}, p_{yt})$$

2.1 Model Assumptions

Assumption 1. *The observed data comprise a random sample of N individuals from the population. For each individual $\{(Y, Y_0, X, V, \alpha)\}$ are defined on the probability space $(\Omega, \mathbb{P}_V, \mathbb{P}_\alpha)$. The support of (X, V, α) is $(\mathcal{X} \times \mathcal{V} \times \mathcal{A})$ where $\mathcal{V} \subseteq \mathbb{R}^T$ and $\mathcal{A} \subseteq \mathbb{R}$.*

Assumption 2. *For each value of $x \in \mathcal{X}$ and $y_0 \in \mathcal{Y}_0$ there is a proper conditional distribution of $\mathcal{Y}_1 \times \mathcal{Y}_2$ given $X = x, Y_0 = y_0$ and*

$$P^0(y_1, y_2 | x, y_0) \equiv \mathbb{P}(Y_1 = y_1 \cap Y_2 = y_2 | X = x, Y_0 = y_0)$$

the conditional probability of purchase of each quality pair (y_1, y_2), is point identified over the support of $\mathcal{Y}_1 \times \mathcal{Y}_2$ for almost every $x \in \mathcal{X}$ and $y_0 \in \mathcal{Y}_0$.

Assumption 3. *The conditional distribution of V given $(X = x, Y_0 = y_0)$, $G_{V|X,Y_0}$ is absolutely continuous with respect to Lebesgue measure with everywhere positive density and the marginal distribution of $\Delta V|X = x, Y_0 = y_0$ is given by $G_{\Delta V|X,Y_0}$, with $\Delta V = V_2 - V_1$ and $V_2 \perp V_1$.*

Assumption 4. *The conditional distribution of α given $X = x, Y_0 = y_0$ is absolutely continuous with respect to Lebesgue measure with everywhere positive density on $\mathbb{R}$ and marginal distribution $H_{\alpha|X,Y_0}$.*

Assumption 5. *X and V are stochastically independent. α is allowed to be correlated with both V and X in an arbitrary way. The joint distribution of (V, α) conditional on $(X = x, Y_0 = y_0)$ is given by $F_{(V,\alpha)|X,Y_0}$.*

Assumption 6. *The parameters p_{yt} are known, exogenously determined and constant to all individuals (market) parameters, such as prices. They are also non-negative, strictly increasing and convex in y. The support of $p \equiv \{p_1, \ldots, p_T\}$, with, $p_1 \equiv \{p_{1t}, \ldots, p_{\overline{y}t}\}$, is $\mathcal{P} \subseteq \mathbb{R}^{\sum_{t=1}^{T} \overline{y}_t}$.*

Assumption 7. *The function $g(\cdot)$ is an element of some parameter space $\mathcal{B}$, $\{G_{\Delta V|X,Y_0}(\cdot|x, y_0) : x \in \mathcal{X}, y_0 \in \mathcal{Y}_0\}$ is an element of $\mathcal{G}_{\Delta V|X,Y_0}$, $\{H_{\alpha|X,Y_0}(\cdot|x, y_0) : x \in \mathcal{X}, y_0 \in \mathcal{Y}_0\}$ is an element of $\mathcal{H}_{\alpha|X,Y_0}$ and $\{F_{(V,\alpha)|X,Y_0}(\cdot|x, y_0) : x \in \mathcal{X}, y_0 \in \mathcal{Y}_0\}$ is an element of $\mathcal{F}_{(V,\alpha)|X,Y_0}$.*

Assumption 8. *$S \equiv (g; F_{(V,\alpha)|X,Y_0}) \in \mathcal{S}$, a specified collection of utility functions and distributions of the time-varying unobservable and the unobserved heterogeneity. Such a S is called a **structure**.*

Assumption 9. *In every period t, Y_t is chosen to maximize $u(y_t, x_t, y_{t-1}, v_t, \alpha, p_{yt})$, where u belongs to a known class of functions $\mathcal{U}$ satisfying (i) $u(0, x_t, y_{t-1}, v_t, \alpha, p_{yt}) = 0$ for all $(x_t, y_{t-1}, v_t, \alpha)$, (ii) $u(y_t, x_t, y_{t-1}, v_t, \alpha, p_{yt})$ is strictly increasing and continuous in v_t for all $(y_t, x_t, , y_{t-1}, \alpha, p_{yt})$, and (iii) for each $(y_{t-1}, x_t, \alpha, p_{yt}) \in \mathcal{Y} \times \mathcal{X} \times \mathcal{A} \times \mathcal{P}$, $u(y_t, x_t, y_{t-1}, v_t, \alpha, p_{yt}) : v_t \in \mathbb{R}$ satisfies the single-crossing property in (y_t, v_t) namely that if $v_t^{'} > v_t$ and $y_t^{'} > y_t$, then*

$$
\begin{aligned}
u(y_t^{'}, x_t, y_{t-1}, v_t, \alpha, p_{yt}) \quad &- \quad u(y_t, x_t, y_{t-1}, v_t, \alpha, p_{yt}) \geq (>) 0 \\
&\Rightarrow \\
u(y_t^{'}, x_t, y_{t-1}, v_t^{'}, \alpha, p_{yt}) \quad &- \quad u(y_t, x_t, y_{t-1}, v_t^{'}, \alpha, p_{yt}) \geq (>) 0
\end{aligned}
$$

Assumption 10. *The distribution of X_t is discrete with everywhere a positive density on $\mathbb{R}^k$ and marginal distribution G_X, such that for all $x \in \mathcal{X}$, $Pr(x_t \neq x_{t-1}) > 0$.*

2.2 Identified Set

The identified set is characterized by the set of structures $(g, F_{(V,\alpha)|X,Y_0})$, such that:

$$S(\mathcal{X}) = \left\{ \begin{array}{c} (g, F_{(V,\alpha)|X,Y_0}) \in S : \forall (y_1, y_2) \in (\mathcal{Y}_1 \times \mathcal{Y}_2),\ s.t. \\ F(\mathcal{R}_{(y_1,y_2)}(x, y_0; g, p_{11}, p_{12}, p_{21}, p_{22})) = P^0(y_1, y_2|x, y_0) \\ a.e.\ x \in \mathcal{X}\ and\ y_0 \in \mathcal{Y}_0 \end{array} \right\}$$

where

$$\begin{array}{l} \mathcal{R}_{(y_1,y_2)}(x, y_0; g, p_{11}, p_{12}, p_{21}, p_{22}) \equiv \\ \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : (y_1, y_2) = arg \max_{(\tilde{y}_1,\tilde{y}_2) \in (\mathcal{Y}_1 \times \mathcal{Y}_2)} u(\tilde{y}_1, \tilde{y}_2|(x, y_0, \alpha, v_1, v_2; g, p_{11}, p_{12}, p_{21}, p_{22})\} \end{array}$$

Assumption 11. *$\exists S^0 \in \mathcal{S}$, $S^0 \equiv (g^0; F^0_{(V,\alpha)|X,Y_0})$ such that $\forall (y_1, y_2) \in (\mathcal{Y}_1 \times \mathcal{Y}_2)$ $F^0(\mathcal{R}_{(y_1,y_2)}(x, y_0; g^0, p_{11}, p_{12}, p_{21}, p_{22})) = P^0(y_1, y_2|x, y_0)$ a.e. $x \in \mathcal{X}$ and $y_0 \in \mathcal{Y}_0$.*

3 Eliminating α from the utility function

The presence of the time invariant unobserved heterogeneity, α, which is allowed to be correlated with the explanatory variables can be seen as an endogeneity problem, that needs to be addressed for identification and consistent estimation of the parameters of interest. This section examines identification in non-linear panel data models and mimic the approach used in linear utility panel data models to solve the problem of the unobserved heterogeneity. In contrast to the linear panel data models where differencing out the unobserved heterogeneity is sufficient to guarantee point identification of the regression parameters, in discrete panel data models removing the unobserved heterogeneity will (in general) lead to identification bounds.

Assumption 12. *The individual specific part, $g(\cdot)$, of the General Utility function* (1) *is linear in the variables such that,*

$$g(X_t, Y_{t-1}, \alpha, V_t) = X_t\beta + l(Y_{t-1})\gamma + \alpha + V_t$$

with $\beta, \gamma \in \mathcal{B}$.

Under Assumption 12 the General Utility model in (1) is given by:

$$U_t = Y_t(X_t\beta + l(Y_{t-1})\gamma + \alpha + V_t) - p_{yt} \tag{2}$$

To account for last period's choice, a different setting than the usual one will be used in which Y_{t-1} enters as an additional regressor. In the dynamic setting described in this paper, $l(Y_{t-1}) = 1(Y_t = Y_{t-1})$ where,

$$1(Y_t = Y_{t-1}) = \left\{ \begin{array}{cc} 1 & if\ Y_t = Y_{t-1} \\ 0 & otherwise \end{array} \right\} \tag{3}$$

The rationale behind this setting is that the options in the models considered here have a qualitative rather than quantitative meaning. Therefore, the parameter γ captures the effect of changing options from one period to the next, rather than "a unit change" effect.

The identified set of structures satisfying Assumptions 1-12 admitted by the model in (2) is characterized under several conditions imposed on the parameters of the utility function and bounds on the identified sets are provided.

3.1 Binary Panel Data Model

Point identification of the regression coefficients in discrete panel data models, has been shown to be difficult even when the model is linear in the parameters of interest and strong distributional assumptions are imposed. The seminal work by Chamberlain (1982, 2010), and work by Honoré (2002) and Honoré and Kyriazidou (2000) showed that if regressors have bounded support, then unless the time dependent unobservables are independently and identically distributed following a logistic distribution, point identification in fully parametric binary panel data models fails. Work by Manski (1987) proves point identification using only median restrictions and at least one regressor having unbounded support, however the rate of convergence is slower than $\sqrt{n}$. Identification of the regression coefficients under both the logistic distribution assumption and the median assumption comes by observing individuals who change behaviour from one period to the next. This behaviour gives rise to features of the distribution that do not depend on the unobserved heterogeneity.

This section uses exactly the same logic, and finds features of the distribution that do not depend on α, without imposing these strong assumption on the time-varying unobservables.

3.1.1 Static binary panel data model

This section starts with the simplest version of the model in (2) and assumes that $Y_t \in \mathcal{Y}_t \equiv \{0, 1\}$, $\gamma = 0$ and $p_{yt} = 0$[1]. The static binary panel data model can then be

[1] Note that for the y_0 does not appear anywhere in the static model.

expressed as,

$$\begin{aligned} U_t &= X_t\beta + \alpha + V_t \\ Y_t &= 1(U_t > 0) \end{aligned} \tag{4}$$

This model has been studied in Rosen and Weidner (2013,WP). From the assumption that individuals choose Y_t to maximise (4), they obtain the regions $\mathcal{R}^{SB}_{(y_1,y_2)}(x;\beta)$ that partition the *supp*(V,α) for all $(Y_1,Y_2) = (y_1^*, y_2^*)$ and $(X_1,X_2) = (x_1,x_2)$ such that:

$$\begin{aligned} \mathcal{R}^{SB}_{(0,0)}(x;\beta) &= \{(V,\alpha) \in (\mathcal{V},\mathcal{A}) : x_1\beta + \alpha + V_1 \leq 0 \text{ and } x_2\beta + \alpha + V_2 \leq 0\} \\ \mathcal{R}^{SB}_{(0,1)}(x;\beta) &= \{(V,\alpha) \in (\mathcal{V},\mathcal{A}) : x_2\beta + \alpha + V_2 > 0 \geq x_1\beta + \alpha + V_1\} \\ \mathcal{R}^{SB}_{(1,0)}(x;\beta) &= \{(V,\alpha) \in (\mathcal{V},\mathcal{A}) : x_2\beta + \alpha + V_2 \leq 0 < x_1\beta + \alpha + V_1\} \\ \mathcal{R}^{SB}_{(1,1)}(x;\beta) &= \{(V,\alpha) \in (\mathcal{V},\mathcal{A}) : x_1\beta + \alpha + V_1 \geq 0 \text{ and } x_2\beta + \alpha + V_2 \geq 0\} \end{aligned} \tag{5}$$

and the conditional joint distribution of $(V,\alpha)|X = x$ is given by,

$$\begin{aligned} P(0,0|x) &= F_{(V,\alpha)|X}(\mathcal{R}^{SB}_{(0,0)}(x;\beta)|X = x) \\ P(0,1|x) &= F_{(V,\alpha)|X}(\mathcal{R}^{SB}_{(0,1)}(x;\beta)|X = x) \\ P(1,0|x) &= F_{(V,\alpha)|X}(\mathcal{R}^{SB}_{(1,0)}(x;\beta)|X = x) \\ P(1,1|x) &= F_{(V,\alpha)|X}(\mathcal{R}^{SB}_{(1,1)}(x;\beta)|X = x) \end{aligned}$$

where $P(y_1,y_2|x) = P(Y_1 = y_1 \cap Y_2 = y_2|X = x)$.

Theorem 1. *Let $S^{SB} = (\beta^{SB}, F^{SB}_{(V,\alpha)|X})$ be a structure admitted by model (4). If S^{SB} is an observationally equivalent structure to S^0, then β^{SB} satisfies*

$$P(1,0|x) \leq G_{\Delta V|X}[-\Delta X\beta^{SB}|X = x] \leq [1 - P(0,1|x)] \tag{6}$$

where $\Delta X = X_2 - X_1$, such that $(X_1,X_2) \in \mathcal{X}\times\mathcal{X}$, $\Delta V = V_2 - V_1$, such that $(V_1,V_2) \in \mathcal{V}\times\mathcal{V}$.

Proof. The proof follows the work in Rosen and Weidner (2013,WP). Consider individuals who change from one period to the next:

$$\{Y_1 = 0 \cap Y_2 = 1\} \text{ and } \{Y_1 = 1 \cap Y_2 = 0\}$$

The Choice Probabilities of this event are given by:

$$\begin{aligned} P(0,1|x) &= P_{(V,\alpha)|X}[\{0 \geq X_1\beta + \alpha + V_1\} \cap \{X_2\beta + \alpha + V_2 \geq 0\}|X = x] \\ P(1,0|x) &= P_{(V,\alpha)|X}[\{X_1\beta + \alpha + V_1 \geq 0\} \cap \{0 \geq X_2\beta + \alpha + V_2\}|X = x] \end{aligned} \tag{7}$$

From (7) it is clear:

$$\begin{aligned}
P(0,1|x) &= P_{(V,\alpha)|X}[\{0 \geq X_1\beta + \alpha + V_1\} \cap \{X_2\beta + \alpha + V_2 \geq 0\}|X = x] \\
&\leq P_{V|X}[\{0 \geq (X_1 - X_2)\beta + (V_1 - V_2)\}|X = x] \\
P(1,0|x) &= P_{(V,\alpha)|X}[\{X_1\beta + \alpha + V_1 \geq 0\} \cap \{0 \geq X_2\beta + \alpha + V_2\}|X = x] \\
&\leq P_{V|X}[\{0 \geq (X_2 - X_1)\beta + (V_2 - V_1)\}|X = x] \qquad (8)
\end{aligned}$$

Then

$$\begin{aligned}
P(0,1|x) &\leq P_{\Delta V|X}[\Delta V \geq -\Delta X\beta|X = x] \\
P(1,0|x) &\leq P_{\Delta V|X}[-\Delta X\beta \geq \Delta V|X = x] \qquad (9)
\end{aligned}$$

$$\begin{aligned}
1 - P(0,1|x) &\geq P_{\Delta V|X}[\Delta V \leq -\Delta X\beta|X = x] \\
P(1,0|x) &\leq P_{\Delta V|X}[-\Delta X\beta \geq \Delta V|X = x] \qquad (10)
\end{aligned}$$

$\Rightarrow$

$$P(1,0|x) \leq P_{\Delta V|X}[\Delta V \leq -\Delta X\beta|X = x] \leq [1 - P(0,1|x)]$$

For any given $X = x$ and using Assumptions 5 and 10 ,

$$P(1,0|x) \leq G_{\Delta V}[\Delta V \leq -\Delta x\beta] \leq [1 - P(0,1|x)] \qquad (11)$$

□

The above relations provide restrictions on the distribution of $\Delta V|X$ for any realization of $x \in \mathcal{X}$ that does not depend on the fixed effect α. The events $Y_1 = 0 \cap Y_2 = 0$ and $Y_1 = 1 \cap Y_2 = 1$ provide no restrictions on ΔV and can not be used to difference out the fixed effect α. The distribution of $\Delta V \sim G_{\Delta V|X,Y_0}$ is equivalent to $\Delta V \sim G_{\Delta V}$ by Assumption 5 and by assuming $\gamma = 0$. This distribution is well defined, since by Assumption 3 V_t are independently and identically distributed across t.

Notice that in order for the bounds in Theorem 1 to be informative, $X_t \neq X_{t+1}$ with positive probability, a requirement satisfied by Assumption 10. Then, it can be clearly seen that the bounds change according to the different values of $x \in \mathcal{X}$. Suppose that $\mathcal{X} \in \{0,1\}$. There are two distinct cases where x_t changes from $t = 1$ to $t = 2$, $x = (x_1, x_2) \in \{(0,1),(1,0)\}$. The bounds in (11) are given by:

$$\begin{aligned}
P(1,0|0,1) \leq G_{\Delta V}[\Delta V \leq -\beta] \leq [1 - P(0,1|0,1)] \\
P(1,0|1,0) \leq G_{\Delta V}[\Delta V \leq +\beta] \leq [1 - P(0,1|1,0)]
\end{aligned}$$

Now suppose that $\mathcal{X} \in \{0,1,2\}$. There are 6 distinct cases where both x_t and y_t change from period $t=1$ to period $t=2$, $x=(x_1,x_2) \in \{(0,1),(1,0),(0,2),(2,0),(1,2),(2,1)\}$. Then the bounds in (11) are given by:

$$\begin{aligned}
P(1,0|0,1) \leq G_{\Delta V}[\Delta V \leq -\beta] &\leq [1-P(0,1|0,1)] \\
P(1,0|1,0) \leq G_{\Delta V}[\Delta V \leq \beta] &\leq [1-P(0,1|1,0)] \\
P(1,0|0,2) \leq G_{\Delta V}[\Delta V \leq -2\beta] &\leq [1-P(0,1|0,2)] \\
P(1,0|2,0) \leq G_{\Delta V}[\Delta V \leq 2\beta] &\leq [1-P(0,1|2,0)] \\
P(1,0|1,2) \leq G_{\Delta V}[\Delta V \leq -\beta] &\leq [1-P(0,1|1,2)] \\
P(1,0|2,1) \leq G_{\Delta V}[\Delta V \leq \beta] &\leq [1-P(0,1|2,1)]
\end{aligned}$$

From the above it is clear that different values of x_1, x_2 lead to different identified distributions for (Y_1, Y_2). To characterize therefore the sharp identified set, the greatest lower bound and the smallest upper bound of (11) need to be specified.

Rosen and Weidner (2013,WP) define the outer region within which S^{SB} lies. Take an arbitrary constant $\omega \in \mathbb{R}$ and notice that:

$$\begin{aligned}
(Y_1,Y_2)=(1,0) \cap -\Delta X\beta \leq \omega \quad &\Rightarrow \quad \Delta V \leq \omega \\
(Y_1,Y_2)=(0,1) \cap -\Delta X\beta \geq \omega \quad &\Rightarrow \quad \Delta V \geq \omega
\end{aligned} \tag{12}$$

Therefore $\forall \omega \in \mathbb{R}$

$$P[(Y_1=1 \cap Y_2=0) \cap -\Delta X\beta \leq \omega | X=x] \leq G_{\Delta V}(\omega) \leq 1-P[(Y_1=0 \cap Y_2=1) \cap -\Delta X\beta \geq \omega | X=x]$$

The upper and lower bounds will then be defined as:

$$\sup_{x\in\mathcal{X}} P[(Y_1=1 \cap Y_2=0) \cap -\Delta X\beta \leq \omega | X=x] \leq G_{\Delta V}(\omega) \leq \inf_{x\in\mathcal{X}} 1-P[(Y_1=0 \cap Y_2=1) \cap -\Delta X\beta \geq \omega | X=x] \tag{13}$$

Furthermore notice that depending on the value of ω the lower and the upper bounds will change. When $-\Delta X\beta \leq \omega$,

$$\begin{aligned}
1-P[(Y_1=0 \cap Y_2=1) \cap -\Delta X\beta \geq \omega | X=x] &= 1 \\
P[(Y_1=1 \cap Y_2=0) \cap -\Delta X\beta \leq \omega | X=x] &= P(1,0|x)
\end{aligned}$$

while when $-\Delta X\beta \geq \omega$,

$$\begin{aligned}
1-P[(Y_1=0 \cap Y_2=1) \cap -\Delta X\beta \geq \omega | X=x] &= 1-P(0,1|x) \\
P[(Y_1=1 \cap Y_2=0) \cap -\Delta X\beta \leq \omega | X=x] &= 0
\end{aligned}$$

Therefore this implies that (13) is equivalent to

$$\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x) \leq G_{\Delta V}(\omega) \leq \inf_{x:-\Delta x\beta\geq\omega}[1-P(0,1|x)] \tag{14}$$

Theorem 2. *Let Assumptions 1-12. The identified set for β is given by:*

$$\mathcal{B}^{SB} = \left\{\beta\in\mathcal{B}:\forall\omega\in\mathbb{R},\ \sup_{x:-\Delta x\beta\leq\omega}[P(1,0|x)] \leq \inf_{x:-\Delta x\beta\geq\omega}[1-P(0,1|x)]\ a.e.\ x\in\mathcal{X}\right\} \tag{15}$$

Theorem 3. *Let Assumptions 1-12. Then if $\beta\in\mathcal{B}^{SB}$, there exists a structure $S^{SB} = (\beta^{SB}, F^{SB}_{(V,\alpha)|X})$ that satisfies the restrictions of the model and is observationally equivalent to structure S^0 that generates $P^0(y_1,y_2|x)$. The sharp identified set for β and $F_{(V,\alpha)|X}$ is given by:*

$$S^{SB} = \left\{\begin{array}{c} (\beta, F_{(V,\alpha)|X}) \in (\mathcal{B},\mathcal{F}_{(V,\alpha)|X}) : \\ \forall\omega\in\mathbb{R}, s.t. \sup_{x:-\Delta x\beta\leq\omega}[P(1,0|x)] \leq G_{\Delta V}(\omega) \leq \inf_{x:-\Delta x\beta\geq\omega}[1-P(0,1|x)], \\ with\ G_{\Delta V}(\omega) = \int_{-\infty}^{\infty} F_{V|\alpha,X}(V_2-V_1<\omega|\alpha,x)dF_{\alpha|X}(\alpha|x)\ a.e.\ x\in\mathcal{X} \end{array}\right\}$$

Proof. The proof of Theorem 3 can be found in Rosen and Weidner (2013,WP). □

Corollary 1. *Let Assumptions 1-12. Then if $(\beta^{SB}, F^{SB}_{(V,\alpha)|X})\in S^{SB}$, the identified set for β and $G_{\Delta V}$ is defined as,*

$$S^{SB}_{\Delta} = \left\{\begin{array}{c} (\beta, G_{\Delta V}) \in (\mathcal{B},\mathcal{G}_{(\Delta V)|X}) : \forall\omega\in\mathbb{R}, \exists H_{\alpha|X}\in\mathcal{H}_{\alpha|X}\ s.t. \\ \sup_{x:-\Delta x\beta\leq\omega}[P(1,0|x)] \leq G_{\Delta V}(\omega) \leq \inf_{x:-\Delta x\beta\geq\omega}[1-P(0,1|x)]\ a.e.\ x\in\mathcal{X} \end{array}\right\}$$

Proof. From Theorem 1 the existence of the conditional joint distribution of $(V,\alpha)|X$ for every $\beta\in\mathcal{B}^{SB}$ implies the existence of the marginal distribution $H_{\alpha|X}$ for every $\beta\in\mathcal{B}^{SB}$ and a marginal distribution G_V with an associated marginal distribution of the difference $G_{\Delta V}$. □

3.1.2 Dynamic binary panel data model

Consider now the extension of model (4) that includes last period's choice. The utility function in (2) can be written as:

$$\begin{aligned} U_t &= X_t\beta + 1(Y_t = Y_{t-1})\gamma + \alpha + V_t \\ Y_t &= 1(U_t > 0) \end{aligned} \tag{16}$$

Notice that in the dynamic set up, last period's choice directly affects the utility function and the choice in period $t-1$ needs to be taken into account. Consider now the two cases where $Y_{t-1} \neq Y_t$ and $Y_{t-1} = Y_t$. In the first case the utility function defined in (16) becomes:

$$\begin{aligned} U_t &= X_t\beta + \alpha + V_t \\ U_{t+1} &= X_{t+1}\beta + \alpha + V_{t+1} \end{aligned} \tag{17}$$

which is identical to (4). In the latter case the utility function becomes:

$$\begin{aligned} U_t &= X_t\beta + \gamma + \alpha + V_t \\ U_{t+1} &= X_{t+1}\beta + \alpha + V_{t+1} \end{aligned} \tag{18}$$

and γ appears in the equation determining U_t.

The regions $\mathcal{R}^{DB}_{(y_1,y_2)}(x, y_0; \beta, \gamma)$ such that for all $(V, \alpha) \in (\mathcal{V}, \mathcal{A})$, $Y = (y_1^*, y_2^*)$ when $X = x$ and $Y_0 = y_0$ are given by:

$$\begin{aligned}
\mathcal{R}^{DB}_{(0,0)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \gamma + \alpha + V_1 \leq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \leq 0\} \\
\mathcal{R}^{DB}_{(0,1)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \alpha + V_2 > 0 \geq x_1\beta + \gamma + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,0)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \alpha + V_2 \leq 0 < x_1\beta + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,1)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \alpha + V_1 \geq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \geq 0\} \\
\mathcal{R}^{DB}_{(0,0)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \alpha + V_1 \leq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \leq 0\} \\
\mathcal{R}^{DB}_{(0,1)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \alpha + V_2 > 0 \geq x_1\beta + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,0)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \alpha + V_2 \leq 0 < x_1\beta + \gamma + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,1)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \gamma + \alpha + V_1 \geq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \geq 0\}
\end{aligned} \tag{19}$$

Proposition 1. *From the regions defined in* (19) *it can be seen that the dynamic model is not complete, in the sense that* $\exists (x, y_0) \in \mathcal{X} \times \mathcal{Y}_0$ *conditional on which there is not a unique solution to model* (16) *with probability 1.*

Proof. Consider the regions defined in (19) and take for example the regions,

$$\begin{aligned}
\mathcal{R}^{DB}_{(1,0)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \alpha + V_2 \leq 0 < x_1\beta + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,1)}(x, 0; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \alpha + V_1 \geq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \geq 0\} \\
\mathcal{R}^{DB}_{(1,0)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_2\beta + \gamma + \alpha + V_2 \leq 0 < x_1\beta + \alpha + V_1\} \\
\mathcal{R}^{DB}_{(1,1)}(x, 1; \beta, \gamma) &= \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : x_1\beta + \gamma + \alpha + V_1 \geq 0 \textit{ and } x_2\beta + \gamma + \alpha + V_2 \geq 0\}
\end{aligned}$$

Suppose also $\gamma > 0$ and consider any $(V, \alpha) \in \mathcal{V}^*$ and any $(V, \alpha) \in \mathcal{V}^{**}$ such that

$$\mathcal{V}^* = \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : -x_2\beta - \gamma \leq V_2 + \alpha \leq -x_2\beta \textit{ and } -x_1\beta \leq V_1 + \alpha\}$$
$$\mathcal{V}^{**} = \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : -x_1\beta - \gamma \leq V_1 + \alpha \leq -x_1\beta \textit{ and } -x_2\beta \leq V_2 + \alpha\}$$

Then it can be shown that:

$$(V, \alpha) \in \mathcal{V}^* \Rightarrow (V, \alpha) \in \mathcal{R}^{DB}_{(1,0)}(x, 0; \beta, \gamma)$$
$$(V, \alpha) \in \mathcal{V}^* \Rightarrow (V, \alpha) \in \mathcal{R}^{DB}_{(1,1)}(x, 0; \beta, \gamma)$$

and

$$(V, \alpha) \in \mathcal{V}^{**} \Rightarrow (V, \alpha) \in \mathcal{R}^{DB}_{(1,0)}(x, 1; \beta, \gamma)$$
$$(V, \alpha) \in \mathcal{V}^{**} \Rightarrow (V, \alpha) \in \mathcal{R}^{DB}_{(1,1)}(x, 1; \beta, \gamma)$$

This implies that the model in (16) is incomplete in the sense that there exists a pair of unobservables (V, α) such that conditional on X and Y_0 there are multiple solutions to the individual's problem with probability one. □

Proposition 2. *From the regions defined in* (19) *it can be seen that the dynamic model is incoherent, in the sense that* $\exists(x, y_0) \in \mathcal{X} \times \mathcal{Y}_0$ *conditional on which there is no solution to model* (16)*.*

Proof. Suppose $\gamma < 0$ and $Y_0 = 0$. Consider any $(V, \alpha) \in \mathcal{V}^{**}$, where

$$\mathcal{V}^{***} = \{(V, \alpha) \in (\mathcal{V}, \mathcal{A}) : -x_1\beta - \gamma \leq V_1 + \alpha \textit{ and } -x_2\beta < V_2 + \alpha < -x_2\beta - \gamma\}$$

If $-x_1\beta - \gamma \leq V_1 + \alpha$ then $Y_1 = 1$ and $Y_0 = 1$, which contradicts the conditioning on $Y_0 = 0$. However, $-x_1\beta - \gamma \leq V_1 + \alpha \Rightarrow -x_1\beta \leq V_1 + \alpha$. This implies that conditioning on $Y_0 = 0$, $-x_1\beta - \gamma \leq V_1 + \alpha \Rightarrow Y_1 = 1$.

For $(V, \alpha) \in \mathcal{V}^{**}$ it also means that $-x_2\beta < V_2 + \alpha < -x_2\beta - \gamma$. This can correspond to $-x_2\beta < V_2 + \alpha$ and/or $V_2 + \alpha < -x_2\beta - \gamma$. However, conditioning on $Y_1 = 1$ and $Y_0 = 0$ none of the two inequalities can hold. First take, $-x_2\beta < V_2 + \alpha$. This corresponds to the event $Y_2 = 1|Y_1 = 0$, which contradicts the $Y_1 = 1$. Then the event $V_2 + \alpha < -x_2\beta - \gamma$ corresponds to $Y_2 = 0|Y_1 = 0$ which again contradicts $Y_1 = 1$.

Therefore, $\mathcal{V}^{***} = \emptyset$ and the dynamic model in (16) is incoherent. □

The incompleteness and incoherency of the model in (16) generally results to failure of point identification of the regression parameters. Valid bounds can still be derived, by applying the same approach as in Section 3.1.1 that removes α. Notice now that the outcome in period $t = 0$ appears in the determination of the outcome in period $t = 1$. Therefore, all the 3 periods $0, 1, 2$ will be used. From (19) it is clear that in order to be able to identify both the parameters γ and β individuals should change behaviour between periods 1 and 2. The sequence of events that will allow for both β and γ to be identified through the elimination of α are:

$$\begin{aligned} A &= \{Y_0 = 0, Y_1 = 0, Y_2 = 1\} \\ B &= \{Y_0 = 0, Y_1 = 1, Y_2 = 0\} \\ C &= \{Y_0 = 1, Y_1 = 0, Y_2 = 1\} \\ D &= \{Y_0 = 1, Y_1 = 1, Y_2 = 0\} \end{aligned} \tag{20}$$

Theorem 4. *Let $S^{DB} = (\beta^{DB}, \gamma^{DB}, F^{DB}_{(V,\alpha)|X,Y_0})$ be a structure admitted by model (16) that is observationally equivalent to S^0, then $(\beta^{DB}, \gamma^{DB})$ satisfies*

$$\begin{aligned} 1 - P(0,1|x,0) &\geq P_{\Delta V|X,Y_0}[\Delta V < -\Delta X \beta^{DB} + \gamma^{DB} | X = x, Y_0 = 0] \\ P(1,0|x,0) &\leq P_{\Delta V|X,Y_0}[\Delta V < -\Delta X \beta^{DB} | X = x, Y_0 = 0] \\ 1 - P(0,1|x,1) &\geq P_{\Delta V|X,Y_0}[\Delta V < -\Delta X \beta^{DB} | X = x, Y_0 = 1] \\ P(1,0|x,1) &\leq P_{\Delta V|X,Y_0}[\Delta V < -\Delta X \beta^{DB} + \gamma^{DB} | X = x, Y_0 = 1] \end{aligned}$$

where $P(y_1, y_2|x, y_0) = P(Y_1 = y_1 \cap Y_2 = y_2 | X = x, Y_0 = y_0)$, $\Delta X = X_2 - X_1$ such that $(X_1, X_2) = (x_1, x_{2,})$.

Proof. Consider individuals with the consumption sequence as described in (20). Conditioning on $Y_0 = 0$ event A implies the following conditional probabilities for the event $(Y_1, Y_2) = (0, 1)$

$$\begin{aligned} P(0,1|x,0) &= P_{(V,\alpha)|X,Y_0}[\{X_1\beta + \gamma + \alpha + V_1 \leq 0\} \cap \{X_2\beta + \alpha + V_2 > 0\} | X = x, Y_0 = 0] \\ &\leq P_{V|X,Y_0}[(X_2 - X_1)\beta - \gamma + V_2 - V_1 > 0 | X = x, Y_0 = 0] \\ &= P_{\Delta V|X,Y_0}[\Delta V > -\Delta X\beta + \gamma | X = x, Y_0 = 0] \end{aligned}$$

Conditioning on $Y_0 = 0$ event B implies the following conditional probabilities for the event $(Y_1, Y_2) = (1, 0)$

$$\begin{aligned} P(1,0|x,0) &= P_{(V,\alpha)|X,Y_0}[\{X_1\beta + \alpha + V_1 \geq 0\} \cap \{X_2\beta + \alpha + V_2 < 0\} | X = x, Y_0 = 0] \\ &\leq P_{V|X,Y_0}[(X_2 - X_1)\beta + V_2 - V_1 < 0 | X = x, Y_0 = 0] \\ &= P_{\Delta V|X,Y_0}[\Delta V < -\Delta X\beta | X = x, Y_0 = 0] \end{aligned}$$

Conditioning on $Y_0 = 1$ event C implies that the event $(Y_1, Y_2) = (0, 1)$ is equivalent to

$$\begin{aligned} P(0,1|x,1) &= P_{(V,\alpha)|X,Y_0}[\{X_1\beta + \alpha + V_1 \leq 0\} \cap \{X_2\beta + \alpha + V_2 > 0\}|X = x, Y_0 = 1] \\ &\leq P_{V|X,Y_0}[(X_2 - X_1)\beta + V_2 - V_1 > 0|X = x, Y_0 = 1] \\ &= P_{\Delta V|X,Y_0}[\Delta V > -\Delta X\beta|X = x, Y_0 = 1] \end{aligned}$$

Conditioning on $Y_0 = 1$ event D implies that the event $(Y_1, Y_2) = (1, 0)$ is equivalent to

$$\begin{aligned} P(1,0|x,1) &= P_{(V,\alpha)|X,Y_0}[\{X_1\beta + \gamma + \alpha + V_1 \geq 0\} \cap \{X_2\beta + \alpha + V_2 < 0\}|X = x, Y_0 = 1] \\ &\leq P_{V|X,Y_0}[(X_2 - X_1)\beta - \gamma + V_2 - V_1 < 0|X = x, Y_0 = 1] \\ &= P_{\Delta V|X,Y_0}[\Delta V < -\Delta X\beta + \gamma|X = x, Y_0 = 1] \end{aligned}$$

For any fixed $X = x$, then

$$\begin{aligned} 1 - P(0,1|x,0) &\geq P_{\Delta V|X,Y_0}[\Delta V < -\Delta x\beta + \gamma|Y_0 = 0] \\ P(1,0|x,0) &\leq P_{\Delta V|X,Y_0}[\Delta V < -\Delta x\beta|Y_0 = 0] \\ 1 - P(0,1|x,1) &\geq P_{\Delta V|X,Y_0}[\Delta V < -\Delta x\beta|Y_0 = 1] \\ P(1,0|x,1) &\leq P_{\Delta V|X,Y_0}[\Delta V < -\Delta x\beta + \gamma|Y_0 = 1] \end{aligned}$$

□

Using arguments similar to the ones in Section 3.1 and in Chesher (2013) and Rosen and Weidner (2013,WP), it can be shown that for any constant $\omega \in \mathbb{R}$, conditioning on $Y_0 = 0$,

$$\begin{aligned} (Y_1, Y_2) = (0, 1) \cap -\Delta X\beta + \gamma \geq \omega &\Rightarrow \Delta V > \omega \\ (Y_1, Y_2) = (1, 0) \cap -\Delta X\beta \leq \omega &\Rightarrow \Delta V < \omega \end{aligned} \tag{21}$$

and conditioning on $Y_0 = 1$,

$$\begin{aligned} (Y_1, Y_2) = (0, 1) \cap -\Delta X\beta \geq \omega &\Rightarrow \Delta V > \omega \\ (Y_1, Y_2) = (1, 0) \cap -\Delta X\beta + \gamma \leq \omega &\Rightarrow \Delta V < \omega \end{aligned} \tag{22}$$

The relations in (21) and (22) imply that, $\forall \omega \in \mathbb{R}$:

$$\begin{aligned} 1 - P[(Y_1, Y_2) = (0, 1) \cap -\Delta X\beta + \gamma \geq \omega|X = x, Y_0 = 0] &\geq P[\Delta V < \omega|X = x, Y_0 = 0] \\ P[(Y_1, Y_2) = (1, 0) \cap -\Delta X\beta \leq \omega|X = x, Y_0 = 0] &\leq P[\Delta V < \omega|X = x, Y_0 = 0] \\ 1 - P[(Y_1, Y_2) = (0, 1) \cap -\Delta X\beta \geq \omega|X = x, Y_0 = 1] &\geq P[\Delta V < \omega|X = x, Y_0 = 1] \\ P[(Y_1, Y_2) = (1, 0) \cap -\Delta X\beta + \gamma \leq \omega|X = x, Y_0 = 1] &\leq P[\Delta V < \omega|X = x, Y_0 = 1] \end{aligned}$$

$$\Longleftrightarrow$$

$$\begin{array}{rcl}
\inf_{x\in\mathcal{X}} 1 - P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta+\gamma\geq\omega|X=x,Y_0=0] & \geq & P[\Delta V<\omega|X=x,Y_0=0] \\
\sup_{x\in\mathcal{X}} P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta\leq\omega|X=x,Y_0=0] & \leq & P[\Delta V<\omega|X=x,Y_0=0] \\
\inf_{x\in\mathcal{X}} 1 - P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta\geq\omega|X=x,Y_0=1] & \geq & P[\Delta V<\omega|X=x,Y_0=1] \\
\sup_{x\in\mathcal{X}} P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta+\gamma\leq\omega|X=x,Y_0=1] & \leq & P[\Delta V<\omega|X=x,Y_0=1]
\end{array} \tag{23}$$

When $-\Delta X\beta+\gamma\geq\omega$

$$P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta+\gamma\geq\omega|X=x,Y_0=0]=P(0,1|x,0)$$

otherwise

$$P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta+\gamma\geq\omega|X=x,Y_0=0]=0$$

When $-\Delta X\beta\leq\omega$

$$P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta\leq\omega|X=x,Y_0=0]=P(1,0|x,0)$$

otherwise

$$P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta\leq\omega|X=x,Y_0=0]=0$$

When $-\Delta X\beta\geq\omega$

$$P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta\geq\omega|X=x,Y_0=1]=P(0,1|x,1)$$

otherwise

$$P[(Y_1,Y_2)=(0,1)\cap -\Delta X\beta\geq\omega|X=x,Y_0=1]=0$$

When $-\Delta X\beta+\gamma\leq\omega$

$$P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta+\gamma\leq\omega|X=x,Y_0=1]=P(1,0|x,1)$$

otherwise

$$P[(Y_1,Y_2)=(1,0)\cap -\Delta X\beta+\gamma\leq\omega|X=x,Y_0=1]=0$$

These inequalities in combination with (21), (22) imply that (23) is equivalent to:

$$\begin{array}{rcl}
\inf_{x:-\Delta x\beta+\gamma\geq\omega} 1-P(0,1|x,0) & \geq & G_{\Delta V|X,Y_0}(\omega|x,0) \\
\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0) & \leq & G_{\Delta V|X,Y_0}(\omega|x,0) \\
\inf_{x:-\Delta x\beta\geq\omega} 1-P(0,1|x,1) & \geq & G_{\Delta V|X,Y_0}(\omega|x,1) \\
\sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1) & \leq & G_{\Delta V|X,Y_0}(\omega|x,1)
\end{array}$$

$$\Longleftrightarrow$$

$$\begin{aligned}
\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0) &\leq G_{\Delta V|X,Y_0}(\omega|x,0)\leq \inf_{x:-\Delta x\beta+\gamma\geq\omega} 1-P(0,1|x,0) \\
\sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1) &\leq G_{\Delta V|X,Y_0}(\omega|x,1)\leq \inf_{x:-\Delta x\beta\geq\omega} 1-P(0,1|x,1) \qquad (24)
\end{aligned}$$

In dynamic models this period's choice depends on last periods's choice. This implies that the choice in period $t=1$ will be affected by the choice in period $t=0$, which is the first period observed in the sample. Unless this period coincides with the first period of the process, it will depend on previous (not observed) periods, the exogenous variables in period $t=0$ and the joint distribution of the outcome in the first period and the unobserved heterogeneity. This joint distribution is (in general) different than the joint distribution of future outcomes and the unobserved heterogeneity. Therefore, since $V\not\perp\alpha$, Assumption 5 of $V\perp X$ does not imply (in general) $V\perp(X,Y_0)$. This would have been the case if in addition $V\perp X|Y_0$ was assumed, however such assumption seems implausible in practise.

Theorem 5. *Let Assumptions 1-12. Bounds for β,γ are given by the set:*

$$\mathcal{B}^{DB}=\left\{(\beta,\gamma)\in\mathcal{B}:\forall\omega\in\mathbb{R},\quad \begin{matrix}\max\left[\sup_{x:-\Delta x\beta\leq\omega}P(1,0|x,0),\ \sup_{x:-\Delta x\beta+\gamma\leq\omega}P(1,0|x,1)\right],\\ \min\left[\inf_{x:-\Delta x\beta+\gamma\geq\omega}1-P(0,1|x,0),\ \inf_{x:-\Delta x\beta\geq\omega}1-P(0,1|x,1)\right]\end{matrix}\right\}$$

Theorem 6. *Let Assumptions 1-12. Then if $(\beta,\gamma)\in\mathcal{B}^{DB}$, then there exists a structure $S^{DB}=(\beta^{DB},\gamma^{DB},F^{DB}_{(V,\alpha)|X,Y_0})$ that satisfies the restrictions of the model and is observationally equivalent to structure S^0 that generates $P^0(y_1,y_2|x,y_0)$. The identified set for β,γ and $F_{(V,\alpha)|X,Y_0}$ is given by:*

$$S^{DB}=\left\{\begin{matrix}(\beta,\gamma,F_{(V,\alpha)|X,Y_0})\in(\mathcal{B},\mathcal{F}_{(V,\alpha)|X,Y_0}):\forall\omega\in\mathbb{R},s.t.\\ \sup_{x:-\Delta x\beta\leq\omega}P(1,0|x,0)\leq G_{\Delta V|X,Y_0}(\omega|x,0)\leq\inf_{x:-\Delta x\beta+\gamma\geq\omega}1-P(0,1|x,0)\\ \textit{and}\\ \sup_{x:-\Delta x\beta+\gamma\leq\omega}P(1,0|x,1)\leq G_{\Delta V|X,Y_0}(\omega|x,1)\leq\inf_{x:-\Delta x\beta\geq\omega}1-P(0,1|x,1)\\ \textit{with } G_{\Delta V|X,Y_0}(\omega|x,y_0)=\int_{-\infty}^{\infty}F_{V|\alpha,X,Y_0}(V_2-V_1<\omega|\alpha,x,y_0)dF_{\alpha|X,Y_0}(\alpha|x,y_0)\ a.e.\ x\in\mathcal{X}\end{matrix}\right\}$$

Proof. The proof follows closely the proof of Rosen and Weidner (2013,WP) for the static binary model. Define

$$\begin{aligned}
\underline{G}(\omega|x,0) &= \sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0)\\
\underline{G}(\omega|x,1) &= \sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1)\\
\overline{G}(\omega|x,0) &= \inf_{x:-\Delta x\beta+\gamma\geq\omega} 1-P(0,1|x,0)\\
\overline{G}(\omega|x,1) &= \inf_{x:-\Delta x\beta\geq\omega} 1-P(0,1|x,1)
\end{aligned}$$

Notice from (19) that

$$(V,\alpha)\in R^{DB}_{(1,0)}(x,0;\beta)\Rightarrow(V,\alpha)\notin R^{DB}_{(0,1)}(x,0;\beta)\quad and\quad (V,\alpha)\in R^{DB}_{(0,1)}(x,0;\beta)\Rightarrow(V,\alpha)\notin R^{DB}_{(1,0)}(x,0;\beta)$$

$$(V,\alpha)\in R^{DB}_{(1,0)}(x,1;\beta)\Rightarrow(V,\alpha)\notin R^{DB}_{(0,1)}(x,1;\beta)\quad and\quad (V,\alpha)\in R^{DB}_{(0,1)}(x,1;\beta)\Rightarrow(V,\alpha)\notin R^{DB}_{(1,0)}(x,1;\beta)$$

This implies that:

$$\begin{aligned}
P(1,0|x,0)+P(0,1|x,0) &\leq 1\\
P(1,0|x,1)+P(0,1|x,1) &\leq 1 \qquad (25)
\end{aligned}$$

Let $(\beta,\gamma)\in\mathcal{B}^{DB}$. The identified set in Theorem 5 can be expressed as:

$$\mathcal{B}^{DB}=\left\{(\beta,\gamma)\in\mathcal{B}:\forall\omega\in\mathbb{R},\ \begin{array}{c}\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0)\leq \inf_{x:-\Delta x\beta+\gamma\geq\omega} 1-P(0,1|x,0)\\ \cap\\ \sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1)\leq \inf_{x:-\Delta x\beta\geq\omega} 1-P(0,1|x,1)\end{array}\right\} \qquad (26)$$

which is equivalent to

$$\mathcal{B}^{DB}=\left\{(\beta,\gamma)\in\mathcal{B}:\forall\omega\in\mathbb{R}\ \{\underline{G}(\omega|x,0)\leq\overline{G}(\omega|x,0)\}\cap\{\underline{G}(\omega|x,1)\leq\overline{G}(\omega|x,1)\}\ \right\}$$

and

$$\mathcal{B}^{DB}=\left\{(\beta,\gamma)\in\mathcal{B}:\forall\omega\in\mathbb{R},\max[\underline{G}(\omega|x,0),\underline{G}(\omega|x,1)],\min[\overline{G}(\omega|x,0),\overline{G}(\omega|x,1)]\right\}$$

From relation (25) it can be shown that $\forall\omega\in\mathbb{R}$,

$$\begin{aligned}
\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0) &+ \sup_{x:-\Delta x\beta+\gamma\geq\omega} P(0,1|x,0)\leq 1\\
\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0) &\leq 1-\sup_{x:-\Delta x\beta+\gamma\geq\omega} P(0,1|x,0)\\
\sup_{x:-\Delta x\beta\leq\omega} P(1,0|x,0) &\leq \inf_{x:-\Delta x\beta+\gamma\geq\omega} 1-P(0,1|x,0)\\
\underline{G}(\omega|x,0) &\leq \overline{G}(\omega|x,0) \qquad (27)
\end{aligned}$$

and

$$
\begin{aligned}
\sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1) &+ \sup_{x:-\Delta x\beta\geq\omega} P(0,1|x,1) \leq 1 \\
\sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1) &\leq 1-\sup_{x:-\Delta x\beta\geq\omega} P(0,1|x,1) \\
\sup_{x:-\Delta x\beta+\gamma\leq\omega} P(1,0|x,1) &\leq \inf_{x:-\Delta x\beta\geq\omega} 1-P(0,1|x,1) \\
\underline{G}(\omega|x,1) &\leq \overline{G}(\omega|x,1) \qquad (28)
\end{aligned}
$$

Notice that when $Y_0=0$ and when $Y_0=1$, $\underline{G}(\omega|x,0),\overline{G}(\omega|x,0)$ and $\underline{G}(\omega|x,1),\overline{G}(\omega|x,1)$ are all weakly increasing in ω and take values in the unit interval

$$
\begin{aligned}
\lim_{\omega\to\infty}\underline{G}(\omega|x,0) = \lim_{\omega\to\infty}\overline{G}(\omega|x,0) = 1 \quad \textit{and} \quad \lim_{\omega\to-\infty}\underline{G}(\omega|x,0) = \lim_{\omega\to-\infty}\overline{G}(\omega|x,0) = 0 \\
\lim_{\omega\to\infty}\underline{G}(\omega|x,1) = \lim_{\omega\to\infty}\overline{G}(\omega|x,1) = 1 \quad \textit{and} \quad \lim_{\omega\to-\infty}\underline{G}(\omega|x,1) = \lim_{\omega\to-\infty}\overline{G}(\omega|x,1) = 0
\end{aligned} \qquad (29)
$$

Define the CDF of $(\omega|X=x,Y_0=y_0)$ as $G(\omega|x,y_0)$ such that $y_0\in\{0,1\}$. The events $Y_0=0$ and $Y_0=1$ are mutually exclusive and therefore $G(\omega|x,y_0)\in\{G(\omega|x,0),G(\omega|x,1)\}$. Then following (27), (28) and (29), any function $G(\cdot|x,y_0):\mathbb{R}\to[0,1]$ such that $\forall\omega\in\mathbb{R}$

$$
\underline{G}(\omega|x,y_0)\leq G(\omega|x,y_0)\leq\overline{G}(\omega|x,y_0) \qquad (30)
$$

is a CDF that exists when $\mathcal{B}^{DB}$ is not empty. □

Theorem 6 gives the identified bounds for $(\beta^{DB},\gamma^{DB},F^{DB}_{(V,\alpha)|X,Y_0})$. Notice that the identified sets are not specified as the sharp identified sets. The incompleteness and incoherency of the model creates additional difficulties and therefore the proof for sharpness still remains an open question.

The identified set is defined in terms of the conditional probability of a sequence of events on the initial condition. Following the inequalities in Theorem 4 the identified set for beta can be expressed in terms of the unconditional probabilities such that,

Corollary 2. *Let Assumptions 1-12. Suppose $(\beta,\gamma)\in\mathcal{B}^{DB}$. Then Theorem 5 implies that $\forall\omega\in\mathbb{R}$ the (unconditional) identified set for (β,γ) is given by:*

$$
\mathcal{B}^{DB}_U=\{(\beta,\gamma)\in\mathcal{B}:\forall\omega\in\mathbb{R},\underline{G}(\omega|x,0)P_0(x)+\underline{G}(\omega|x,1)P_1(x)\leq\overline{G}(\omega|x,0)P_0(x)+\overline{G}(\omega|x,1)P_1(x)\}
$$

Proof. The proof follows by applying the Total Law of Probabilities to (24). First notice that

$$G_{\Delta V|X} = G_{\Delta V|X,Y_0}(\omega|x,0)P(Y_0=0|x) + G_{\Delta V|X,Y_0}(\omega|x,1)P(Y_0=1|x)$$

The probabilities $P(Y_0 = 0|x)$ and $P(Y_0 = 1|x)$ are fully observed and therefore from (24)

$$\begin{aligned}\underline{G}(\omega|x,0)P(Y_0=0|x) &\leq G_{\Delta V|X,Y_0}(\omega|x,0)P(Y_0=0|x) \leq \overline{G}(\omega|x,0)P(Y_0=0|x)\\ \underline{G}(\omega|x,1)P(Y_0=1)|x) &\leq G_{\Delta V|X,Y_0}(\omega|x,1)P(Y_0=1|x) \leq \overline{G}(\omega|x,1)P(Y_0=1|x)\end{aligned}$$

$$\Longrightarrow$$

$$\underline{G}(\omega|x,0)P(Y_0=0|x)+\underline{G}(\omega|x,1)P(Y_0=1|x) \leq G_{\Delta V|X} \leq \overline{G}(\omega|x,0)P(Y_0=0|x)+\overline{G}(\omega|x,1)P(Y_0=1|x)$$

$$\Longleftrightarrow$$

$$\underline{G}(\omega|x,0)P(Y_0=0|x)+\underline{G}(\omega|x,1)P(Y_0=1|x) \leq G_{\Delta V} \leq \overline{G}(\omega|x,0)P(Y_0=0|x)+\overline{G}(\omega|x,1)P(Y_0=1|x)$$

where the last result follows from Assumption 5 of $V \perp X$. □

3.2 Ordered Choice Panel Data Models

3.2.1 Static ordered panel data models

This section studies identification in static ordered panel data models. The assumption that $\gamma = 0$ will still be imposed, as in Section 3.1.1 however now individuals can choose from the set $\mathcal{Y}_t \equiv \{0,1,2\}$. The General Utility function in (1) is then given by:

$$\begin{aligned}U_{yt} &= Y_t(X_t\beta + \alpha + V_t) - p_{yt}\\ Y_t^* &= arg \max_{Y_t \in \mathcal{Y}_t} U_{yt}\end{aligned} \tag{31}$$

Notice that the parameters p_{yt} are now included in the utility function, although they are assumed to be known. Under Assumptions 6 and 9 for every period $t \in \{1,2\}$ the model (31) matches typical ordered response models as in Aristodemou and Rosen (2013,WP) for the cross-sectional model.

$$Y_t = \begin{cases} 0\ if & \alpha + V_t < -X_t\beta + p_{1t}\\ 1\ if & -X_t\beta + p_{1t} < \alpha + V_t < -X_t\beta + p_{2t} - p_{1t}\\ 2\ if & -X_t\beta + p_{2t} - p_{1t} < \alpha + V_t\end{cases}$$

The regions $\mathcal{R}^{SO}_{(y_1,y_2)}(x,p;\beta)$ that partition the $supp(V,\alpha)$ such that for all $(V,\alpha) \in (\mathcal{V},\mathcal{A})$, $Y = (y_1^*, y_2^*)$ when $X = x$ are given by:

$$
\begin{aligned}
\mathcal{R}^{SO}_{(0,0)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} x_1\beta+\alpha+V_1-p_{11}<0 \\ and \\ x_2\beta+\alpha+V_2-p_{12}<0 \end{array} \right\} \\
\mathcal{R}^{SO}_{(0,1)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} x_1\beta+\alpha+V_1-p_{11}<0 \\ and \\ -x_2\beta+p_{12}<\alpha+V_2<-x_2\beta+p_{22}-p_{12} \end{array} \right\} \\
\mathcal{R}^{SO}_{(0,2)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} x_1\beta+\alpha+V_1-p_{11}<0 \\ and \\ 0<x_2\beta+\alpha+V_2-p_{22}+p_{12} \end{array} \right\} \\
\mathcal{R}^{SO}_{(1,0)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} -x_1\beta+p_{11}<\alpha+V_1<-x_1\beta+p_{21}-p_{11} \\ and \\ x_2\beta+\alpha+V_2-p_{12}<0 \end{array} \right\} \\
\mathcal{R}^{SO}_{(1,1)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} -x_1\beta+p_{11}<\alpha+V_1<-x_1\beta+p_{21}-p_{11} \\ and \\ -x_2\beta+p_{12}<\alpha+V_2<-x_2\beta+p_{22}-p_{12} \end{array} \right\} \\
\mathcal{R}^{SO}_{(1,2)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} -x_1\beta+p_{11}<\alpha+V_1<-x_1\beta+p_{21}-p_{11} \\ and \\ 0<x_2\beta+\alpha+V_2-p_{22}+p_{12} \end{array} \right\} \\
\mathcal{R}^{SO}_{(2,0)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} 0<x_1\beta+\alpha+V_1-p_{21}+p_{11} \\ and \\ x_2\beta+\alpha+V_2-p_{12}<0 \end{array} \right\} \\
\mathcal{R}^{SO}_{(2,1)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} 0<x_1\beta+\alpha+V_1-p_{21}+p_{11} \\ and \\ -x_2\beta+p_{12}<\alpha+V_2<-X_2\beta+p_{22}-p_{12} \end{array} \right\} \\
\mathcal{R}^{SO}_{(2,2)}(x,p;\beta) &= \left\{ (V,\alpha)\in(\mathcal{V},\mathcal{A}) : \begin{array}{c} 0<x_1\beta+\alpha+V_1-p_{21}+p_{11} \\ and \\ 0<x_2\beta+\alpha+V_2-p_{22}+p_{12} \end{array} \right\}
\end{aligned}
\tag{32}
$$

and the conditional joint probabilities of $(V,\alpha)|X=x,p$, $F_{(V,\alpha)|X,p}$, are given by

$$\begin{aligned}
P(0,0|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(0,0)}(x,p;\beta)|X=x,p)\\
P(0,1|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(0,1)}(x,p;\beta)|X=x,p)\\
P(0,2|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(0,2)}(x,p;\beta)|X=x,p)\\
P(1,0|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(1,0)}(x,p;\beta)|X=x,p)\\
P(1,1|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(1,1)}(x,p;\beta)|X=x,p)\\
P(1,2|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(1,2)}(x,p;\beta)|X=x,p)\\
P(2,0|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(2,0)}(x,p;\beta)|X=x,p)\\
P(2,1|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(2,1)}(x,p;\beta)|X=x,p)\\
P(2,2|x,p) &= F_{(V,\alpha)|X,p}(\mathcal{R}^{SO}_{(2,2)}(x,p;\beta)|X=x,p)
\end{aligned}$$

where $P(y_1,y_2|x,p)=P(Y_1=y_1\cap Y_2=y_2|X=x,p)$.

From (32) it is easy to see that the model in complete in the sense that conditional on any value of exogenous variables, there is a unique solution to the individual choice problem with probability one. As in the case of the binary panel data models, in order to be able to set identify the parameters β individuals who change their choice from period $t=1$ to $t=2$ are used. This implies the following sequence of events:

$$\begin{aligned}
&\{Y_1=0 \;\cap\; Y_2=1\}\\
&\{Y_1=1 \;\cap\; Y_2=0\}\\
&\{Y_1=0 \;\cap\; Y_2=2\}\\
&\{Y_1=2 \;\cap\; Y_2=0\}\\
&\{Y_1=1 \;\cap\; Y_2=2\}\\
&\{Y_1=2 \;\cap\; Y_2=1\}
\end{aligned} \tag{33}$$

In addition to the individuals who change from period $t=1$ to $t=2$ it can be shown that information that is independent of α is also provided by considering individuals who choose that same option $Y=1$ in periods $t=1$ and $t=2$ implying the following sequence of events,

$$\{Y_1=1\cap Y_2=1\} \tag{34}$$

Assumption 13.

$$\forall x\in\mathcal{X} \textit{ and } \forall p\in\mathcal{P},\; P(Y_1=1\cap Y_2=1|X=x,p)<0.5$$

Theorem 7. *Let $S^{SO} = (\beta^{SO}, F^{SO}_{(V,\alpha)|X})$ be a structure admitted by model* (31)*. If S^{SO} is an observationally equivalent structure to S^0, then β^{SO} satisfies*

$$
\begin{aligned}
P(1,0|x,p) &\leq G_{\Delta V|X,p}[-\Delta X\beta - p_{11} + p_{12}|X = x, p] \leq 1 - P(0,1|x,p) \\
P(2,0|x,p) &\leq G_{\Delta V|X,p}[-\Delta X\beta + p_{12} + (p_{11} - p_{21})|X = x, p)] \\
& \quad G_{\Delta V|X,p}[-\Delta X\beta - p_{11} - (p_{12} - p_{22})|X = x, p] \leq 1 - P(0,2|x,p) \\
P(2,1|x,p) &\leq G_{\Delta V|X,p}[-\Delta X\beta - (p_{21} - p_{22}) + p_{11} - p_{12}|X = x, p] \leq 1 - P(1,2|x,p) \\
P(1,1|x,p) &\leq P_{V|X=x,p}[0 \geq (X_2 - X_1)\beta + (V_2 - V_1) - p_{22} + p_{12} + p_{11} \\
&\cap \ (X_2 - X_1)\beta + (V_2 - V_1) - p_{12} + p_{21} - p_{11} \geq 0|X = x, p]
\end{aligned}
$$

where $\Delta X = X_2 - X_1 \in \mathcal{X} \times \mathcal{X}$, $p = (p_{11}, p_{21}, p_{12}, p_{22})$.

Proof. The conditional probabilities $P(Y_1 = y_1 \cap Y_2 = y_2|X = x, p) = P(y_1, y_2|x, p)$ of the events given in (33) are then given by:

$$
\begin{aligned}
P(0,1|x,p)) &= P_{(V,\alpha)|X,p}[\{0 \geq X_1\beta + \alpha + V_1 - p_{11}\} \\
&\cap \ \{X_2\beta + \alpha + V_2 - p_{12} \geq 0 \cap 0 \geq X_2\beta + \alpha + V_2 - p_{22} + p_{12}\}|X = x, p] \\
P(1,0|x,p) &= P_{(V,\alpha)|X,p}[\{X_1\beta + \alpha + V_1 - p_{11} \geq 0 \cap 0 \geq X_1\beta + \alpha + V_1 - p_{21} + p_{11}\} \\
&\cap \ \{0 \geq X_2\beta + \alpha + V_2 - p_{12}\}|X = x, p] \\
P(0,2|x,p) &= P_{(V,\alpha)|X,p}[\{0 \geq X_1\beta + \alpha + V_1 - p_{11}\} \cap \{X_2\beta + \alpha + V_2 - p_{22} + p_{12} \geq 0\}|X = x, p] \\
P(2,0|x,p) &= P_{(V,\alpha)|X,p}[\{X_1\beta + \alpha + V_1 - p_{21} + p_{11} \geq 0\} \cap \{0 \geq X_2\beta + \alpha + V_2 - p_{12}\}|X = x, p] \\
P(1,2|x,p) &= P_{(V,\alpha)|X,p}[\{X_1\beta + \alpha + V_1 - p_{11} \geq 0 \cap 0 \geq X_1\beta + \alpha + V_1 - p_{21} + p_{11}\} \\
&\cap \ \{X_2\beta + \alpha + V_2 - p_{22} + p_{12} \geq 0\}|X = x, p] \\
P(2,1|x,p) &= P_{(V,\alpha)|X,p}[\{X_1\beta + \alpha + V_1 - p_{21} + p_{11} \geq 0\} \\
&\cap \ \{X_2\beta + \alpha + V_2 - p_{12} \geq 0 \cap 0 \geq X_2\beta + \alpha + V_2 - p_{22} + p_{12}\}|X = x, p] \qquad (35)
\end{aligned}
$$

From (35) it can be shown that:

$$\begin{array}{rcl} P(0,1|x,p) & \leq & P_{(V,\alpha)|X,p}(\{0 \geq X_1\beta+\alpha+V_1-p_{11}\}\cap\{X_2\beta+\alpha+V_2-p_{12}\geq 0\}|X=x,p) \\ & \leq & P_{(V,\alpha)|X,p}(0\geq (X_1-X_2)\beta+(V_1-V_2)-p_{11}+p_{12}|X=x,p) \\ P(1,0|x,p) & \leq & P_{(V,\alpha)|X,p}[\{X_1\beta+\alpha+V_1-p_{11}\geq 0\}\cap\{0\geq X_2\beta+\alpha+V_2-p_{12}\}|X=x,p] \\ & \leq & P_{(V,\alpha)|X,p}(0\geq (X_2-X_1)\beta+(V_2-V_1)-p_{12}+p_{11}|X=x,p) \\ P(0,2|x,p) & \leq & P_{(V,\alpha)|X,p}(0\geq (X_1-X_2)\beta+(V_1-V_2)-p_{11}-(p_{12}-p_{22})|X=x,p) \\ P(2,0|x,p) & \leq & P_{(V,\alpha)|X,p}(0\geq (X_2-X_1)\beta+(V_2-V_1)-p_{12}-(p_{11}-p_{21})|X=x,p) \\ P(1,2|x,p) & \leq & P_{(V,\alpha)|X,p}(\{0\geq X_1\beta+\alpha+V_1-p_{21}+p_{11}\}\cap\{X_2\beta+\alpha+V_2-p_{22}+p_{12}\geq 0\}|X=x,p) \\ & \leq & P_{(V,\alpha)|X,p}(0\geq (X_1-X_2)\beta+V_1-V_2+(p_{22}-p_{21})-p_{12}+p_{11}|X=x,p) \\ P(2,1|x,p)) & \leq & P_{(V,\alpha)|X,p}[\{X_1\beta+\alpha+V_1-p_{21}+p_{11}\geq 0\}\cap\{0\geq X_2\beta+\alpha+V_2-p_{22}+p_{12}\}|X=x,p] \\ & \leq & P_{(V,\alpha)|X,p}(0\geq (X_2-X_1)\beta+V_2-V_1+(p_{21}-p_{22})-p_{11}+p_{12}|X=x,p) \end{array} \tag{36}$$

Then using the same notation as in Section 3.1 and Assumptions 5 and 6, (36) can be expressed as:

$$\begin{array}{rcl} P(0,1|x,p) & \leq & P_{V|X,p}(V_2-V_1\geq -\Delta X\beta-p_{11}+p_{12}\geq |X=x,p) \\ & = & 1-G_{\Delta V}[-\Delta x\beta-p_{11}+p_{12}] \\ P(1,0|x,p) & \leq & P_{V|X,p}(V_2-V_1\leq -\Delta X\beta+p_{12}-p_{11}|X=x,p) \\ & = & G_{\Delta V}[-\Delta x\beta-p_{11}+p_{12}] \\ P(0,2|x,p) & \leq & P_{V|X,p}(-\Delta X\beta-p_{11}-(p_{12}-p_{22})\leq V_2-V_1|X=x,p) \\ & = & 1-G_{\Delta V}[-\Delta x\beta-p_{11}-(p_{12}-p_{22})] \\ P(2,0|x,p) & \leq & P_{V|X,p}(V_2-V_1\leq -\Delta X\beta+p_{12}+(p_{11}-p_{21})|X=x,p) \\ & = & G_{\Delta V}[-\Delta x\beta+p_{12}+(p_{11}-p_{21})] \\ P(1,2|x,p) & \leq & P_{V|X,p}(-\Delta X\beta+(p_{22}-p_{21})-p_{12}+p_{11})\leq V_2-V_1|X=x,p) \\ & = & 1-G_{\Delta V}[-\Delta x\beta+(p_{22}-p_{21})-p_{12}+p_{11}] \\ P(2,1|x,p) & \leq & P_{V|X,p}(V_2-V_1\leq -\Delta X\beta-(p_{21}-p_{22})+p_{11}-p_{12}|X=x,p) \\ & = & G_{\Delta V}[-\Delta x\beta-(p_{21}-p_{22})+p_{11}-p_{12}] \end{array} \tag{37}$$

The inequalities in (37) and the condition that $G_{\Delta V}$ is strictly increasing, lead to bounds for β.

$$\begin{array}{rcl}
G_{\Delta V}[-\Delta x\beta - p_{11} + p_{12}] & \leq & 1 - P(0,1|x,p) \\
P(1,0|x,p) & \leq & G_{\Delta V}[-\Delta x\beta - p_{11} + p_{12}] \\
P(2,0|x,p) & \leq & G_{\Delta V}[-\Delta x\beta + p_{11} + (p_{11} - p_{21})] \\
G_{\Delta V}[-\Delta x\beta - p_{11} - (p_{12} - p_{22})] & \leq & 1 - P(0,2|x,p) \\
G_{\Delta V}[-\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11}] & \leq & 1 - P(1,2|x,p) \\
P(2,1|x,p) & \leq & G_{\Delta V}[-\Delta x\beta - (p_{21} - p_{22}) + p_{11} - p_{12}]
\end{array} \tag{38}$$

For any given $X = x$ and using Assumptions 5 and 10 ,

$$\begin{array}{rcl}
P(1,0|x,p) & \leq & G_{\Delta V}[-\Delta x\beta - p_{11} + p_{12}] \leq 1 - P(0,1|x,p) \\
P(2,0|x,p) & \leq & G_{\Delta V}[-\Delta x\beta + p_{12} + (p_{11} - p_{21})] \\
 & & G_{\Delta V}[-\Delta x\beta - p_{11} - (p_{12} - p_{22})] \leq 1 - P(0,2|x,p) \\
P(2,1|x,p) & \leq & G_{\Delta V}[-\Delta x\beta - (p_{21} - p_{22}) + p_{11} - p_{12}] \leq 1 - P(1,2|x,p)
\end{array} \tag{39}$$

So far only the implications on the choice probabilities from those individuals who decide to change from one period to the next were considered. In the binary case discussed in Section 3.1 individuals who chose $(y_1, y_2) = (0,0)$ *or* $(1,1)$ gave no information on β since the behaviour for these "extreme" cases can be matched by either driving the α to $-\infty$ or ∞. This is also true for the static ordered model for those individuals choosing $(0,0)$ or $(2,2)$. However, in the static ordered model there is an "in-between" category, those individuals choosing $(1,1)$, that provides information on the β without involving α, even if they do not change. To see that consider the joint probability of choosing the event (34):

$$\begin{array}{rcl}
P(1,1|x,p) & = & P_{(V,\alpha)|X,p}[\{X_1\beta + \alpha + V_1 - p_{11} \geq 0 \cap 0 \geq X_1\beta + \alpha + V_1 - p_{21} + p_{11}\} \\
 & \cap & \{X_2\beta + \alpha + V_2 - p_{12} \geq 0 \cap 0 \geq X_2\beta + \alpha + V_2 - p_{22} + p_{12}\}|X = x, p] \\
 & = & P_{(V,\alpha)|X,p}[X_1\beta + \alpha + V_1 - p_{11} \geq 0 \cap 0 \geq X_1\beta + \alpha + V_1 - p_{21} + p_{11} \\
 & \cap & X_2\beta + \alpha + V_2 - p_{12} \geq 0 \cap 0 \geq X_2\beta + \alpha + V_2 - p_{22} + p_{12}|X = x, p]
\end{array} \tag{40}$$

By combining

$$X_1\beta + \alpha + V_1 - p_{11} \geq 0 \textit{ and } 0 \geq X_2\beta + \alpha + V_2 - p_{22} + p_{12}$$

and

$$0 \geq X_1\beta + \alpha + V_1 - p_{21} + p_{11} \; and \; X_2\beta + \alpha + V_2 - p_{12} \geq 0$$

it can be shown that:

$$\begin{aligned} P(1,1|x,p) &\leq P_{V|X,p}[0 \geq (X_2 - X_1)\beta + (V_2 - V_1) - p_{22} + p_{12} + p_{11} \\ &\cap \; (X_2 - X_1)\beta + (V_2 - V_1) - p_{12} + p_{21} - p_{11} \geq 0|X = x, p] \\ P(1,1|x,p) &\leq P_{V|X,p}[-\Delta X\beta + p_{22} - p_{12} - p_{11} \geq \Delta V \geq -\Delta X\beta + p_{12} - p_{21} + p_{11}|X = x, p] \end{aligned} \tag{41}$$

which doesn't depend on α. □

The above inequalities come from the observation that specific choice patterns have implications on the distribution of the ΔV. Using arguments as in Chesher (2013) and Rosen and Weidner (2013,WP) the following implication can be shown,

$$\begin{aligned} (Y_1, Y_2) = (0,1) &\Rightarrow \Delta V > -\Delta X\beta - p_{11} + p_{12} \\ (Y_1, Y_2) = (1,0) &\Rightarrow \Delta V < -\Delta X\beta - p_{11} + p_{12} \\ (Y_1, Y_2) = (0,2) &\Rightarrow \Delta V > -\Delta X\beta - p_{11} - (p_{12} - p_{22}) \\ (Y_1, Y_2) = (2,0) &\Rightarrow \Delta V < \Delta X\beta + p_{12} + (p_{11} - p_{21}) \\ (Y_1, Y_2) = (1,2) &\Rightarrow \Delta V > -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \\ (Y_1, Y_2) = (2,1) &\Rightarrow \Delta V < -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \\ (Y_1, Y_2) = (1,1) &\Rightarrow -\Delta X\beta + p_{22} - p_{12} - p_{11} \geq \Delta V \geq -\Delta X\beta + p_{12} - p_{21} + p_{11} \end{aligned}$$

Consider a set of arbitrary constants $\omega, \omega^{'}, \omega^{''} \in \mathbb{R}$:

$$\begin{aligned} (y_1, y_2) = (0,1) &\cap \; -\Delta X\beta - p_{11} + p_{12} \geq \omega \Rightarrow \Delta V > \omega \\ (y_1, y_2) = (1,0) &\cap \; -\Delta X\beta - p_{11} + p_{12} \leq \omega \Rightarrow \Delta V < \omega \\ (y_1, y_2) = (0,2) &\cap \; -\Delta X\beta - p_{11} - (p_{12} - p_{22}) \geq \omega \Rightarrow \Delta V > \omega \\ (y_1, y_2) = (2,0) &\cap \; -\Delta X\beta + p_{12} + (p_{11} - p_{21}) \leq \omega \Rightarrow \Delta V < \omega \\ (y_1, y_2) = (1,2) &\cap \; -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega \Rightarrow \Delta V > \omega \\ (y_1, y_2) = (2,1) &\cap \; -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega \Rightarrow \Delta V < \omega \\ (y_1, y_2) = (1,1) &\cap \; \omega^{'} > -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{1t} \geq \omega \Rightarrow \omega^{'} > \Delta V \geq \omega \\ (y_1, y_2) = (1,1) &\cap \; \omega \geq -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} > \omega^{''} \Rightarrow \omega \geq \Delta V > \omega^{''} \end{aligned} \tag{42}$$

The relations in (42) and Assumptions 5 and 6 imply the following, $\forall \omega, \omega^{'}, \omega^{''} \in \mathbb{R}$:

$$\begin{aligned}
1 - P[(y_1, y_2) = (0,1) \cap -\Delta X\beta - p_{11} + p_{12} \geq \omega | X = x, p] &\geq P[\Delta V < \omega] \\
P[(y_1, y_2) = (1,0) \cap -\Delta X\beta - p_{11} + p_{12} \leq \omega | X = x, p] &\leq P[\Delta V < \omega] \\
1 - P[(y_1, y_2) = (0,2) \cap -\Delta X\beta - p_{11} - (p_{12} - p_{22}) \geq \omega | X = x, p] &\geq P[\Delta V < \omega] \\
P[(y_1, y_2) = (2,0) \cap -\Delta X\beta + p_{12} + (p_{11} - p_{21}) \leq \omega | X = x, p] &\leq P[\Delta V < \omega] \\
1 - P[(y_1, y_2) = (1,2) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega | X = x, p] &\geq P[\Delta V < \omega] \\
P[(y_1, y_2) = (2,1) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega | X = x, p] &\leq P[\Delta V < \omega] \\
P[(y_1, y_2) = (1,1) \cap \{\omega^{'} > -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} \geq \omega\} | X = x, p] & \\
&\leq P[\omega \leq \Delta V < \omega^{'}] \\
P[(y_1, y_2) = (1,1) \cap \{\omega \geq -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} > \omega^{''}\} | X = x, p] & \\
&\leq P[\omega^{''} < \Delta V \leq \omega]
\end{aligned} \tag{43}$$

Following the same arguments as in proving Theorem 3 it can be shown that:

When $-\Delta X\beta - p_{11} + p_{12} \geq \omega$

$$1 - P[(y_1, y_2) = (0,1) \cap -\Delta X\beta - p_{11} + p_{12} \geq \omega | X = x, p] = 1 - P(0,1|x,p)$$

otherwise

$$1 - P[(y_1, y_2) = (0,1) \cap -\Delta X\beta - p_{11} + p_{12} \geq \omega | X = x, p] = 1$$

When $-\Delta X\beta - p_{11} + p_{12} \leq \omega$

$$P[(y_1, y_2) = (1,0) \cap -\Delta X\beta - p_{11} + p_{12} \leq \omega | X = x, p] = P(1,0|x,p)$$

otherwise

$$P[(y_1, y_2) = (1,0) \cap -\Delta X\beta - p_{11} + p_{12} \leq \omega | X = x, p] = 0$$

When $-\Delta X\beta - p_{11} - (p_{12} - p_{22}) \geq \omega$

$$1 - P[(y_1, y_2) = (0,2) \cap -\Delta X\beta - p_{11} - (p_{12} - p_{22}) \geq \omega | X = x, p] = 1 - P(0,2|x,p)$$

otherwise

$$1 - P[(y_1, y_2) = (0,2) \cap -\Delta X\beta - p_{11} - (p_{12} - p_{22}) \geq \omega | X = x, p] = 1$$

When $-\Delta X\beta + p_{12} + (p_{11} - p_{21}) \leq \omega$

$$P[(y_t, y_1) = (2,0) \cap -\Delta X\beta + p_{12} + (p_{11} - p_{21}) \leq \omega | X = x, p] = P(2,0|x,p)$$

otherwise

$$P[(y_1, y_2) = (2,0) \cap -\Delta X\beta + p_{12} + (p_{11} - p_{21}) \leq \omega | X = x, p] = 0$$

When $-\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega$

$$1 - P[(y_1, y_2) = (1,2) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega | X = x, p] = 1 - P(1, 2|x, p)$$

otherwise

$$1 - P[(y_1, y_2) = (1,2) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega | X = x, p] = 1$$

When $-\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega$

$$P[(y_1, y_2) = (2,1) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega | X = x, p] = P(2, 1|x, p)$$

otherwise

$$P[(y_1, y_2) = (2,1) \cap -\Delta X\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega | X = x, p] = 0$$

The above relationships in combination with (43) imply that distribution of the ΔV:

$$\begin{aligned}
\sup_{x: -\Delta x\beta - p_{11} + p_{12} \leq \omega} P(1,0|x,p) &\leq G_{\Delta V}(\omega) \\
G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta - p_{11} + p_{12} \geq \omega} [1 - P(0,1|x,p)] \\
G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} [1 - P(0,2|x,p)] \\
\sup_{x: -\Delta x\beta + p_{12} + (p_{11} - p_{21}) \leq \omega} P(2,0|x,p) &\leq G_{\Delta V}(\omega) \\
\sup_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega} P(2,1|x,p) &\leq G_{\Delta V}(\omega) \\
G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} [1 - P(1,2|x,p)]
\end{aligned} \tag{44}$$

As shown in (41) in the ordered choice panel data model information comes also from $(y_1, y_2) = (1, 1)$. Following same arguments as above it can be shown that for any arbitrary constant $\omega, \omega^{'}, \omega^{''} \in \mathbb{R}$.

When $\omega^{'} > -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} \geq \omega$,

$$P[(y_1, y_2) = (1,1) \cap \{\omega^{'} > -\Delta X\beta + p_{22} - p_{12} - p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} \geq \omega\} | X = x, p] = P(1, 1|x, p)$$

otherwise

$$P[(y_1, y_2) = (1,1) \cap \{\omega^{'} > -\Delta X\beta + p_{22} - p_{12} + p_{1t} \cap -\Delta X\beta + p_{12} - p_{21} + p_{1t} \geq \omega\} | X = x, p] = 0$$

When $\omega \geq -\Delta X\beta + p_{22} - p_{12} + p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} > \omega^{''}$

$P[(y_1, y_2) = (1,1) \cap \{\omega \geq -\Delta X\beta + p_{22} - p_{12} + p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} > \omega^{''}\} | X = x, p] = P(1,1|x,p)$

otherwise

$P[(y_1, y_2) = (1,1) \cap \{\omega \geq -\Delta X\beta + p_{22} - p_{12} + p_{11} \cap -\Delta X\beta + p_{12} - p_{21} + p_{11} > \omega^{''}\} | X = x, p] = 0$

Combining the above relations with (43):

$$\sup_{x \in X^*} P(1,1|x,p) \leq G_{\Delta V}(\omega^{'}) - G_{\Delta V}(\omega)$$

and

$$\sup_{x \in X^{**}} P(1,1|x,p) \leq G_{\Delta V}(\omega) - G_{\Delta V}(\omega^{''})$$

with,

$$\begin{aligned} X^* &= \{x : \omega^{'} > -\Delta x\beta + p_{22} - p_{12} - p_{11} \cap -\Delta x\beta + p_{12} - p_{21} + p_{11} \geq \omega\} \\ X^{**} &= \{x : \omega \geq -\Delta x\beta + p_{22} - p_{12} - p_{11} \cap -\Delta x\beta + p_{12} - p_{21} + p_{11} > \omega^{''}\} \end{aligned} \tag{45}$$

for any fixed $\omega, \omega^{'}, \omega^{''} \in \mathbb{R}$. Combining (39) and (45):

$$\begin{aligned} \sup_{x: -\Delta x\beta - p_{11} + p_{12} \leq \omega} P(1,0|x,p) &\leq G_{\Delta V}(\omega) \\ G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta - p_{11} + p_{12} \geq \omega} [1 - P(0,1|x,p)] \\ \sup_{x: -\Delta x\beta + p_{12} + (p_{11} - p_{21}) \leq \omega} P(2,0|x,p) &\leq G_{\Delta V}(\omega) \\ G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} [1 - P(0,2|x,p)] \\ \sup_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega} P(2,1|x,p) &\leq G_{\Delta V}(\omega) \\ G_{\Delta V}(\omega) &\leq \inf_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} [1 - P(1,2|x,p)] \\ \sup_{x \in X^*} P(1,1|x,p) &\leq G_{\Delta V}(\omega^{'}) - G_{\Delta V}(\omega) \\ \sup_{x \in X^{**}} P(1,1|x,p) &\leq G_{\Delta V}(\omega) - G_{\Delta V}(\omega^{''}) \end{aligned} \tag{46}$$

Notice that at the limit $\omega^{'} \to \infty$ and $\omega^{''} \to -\infty$:

$$\begin{aligned} P(1,1|x,p) &\leq 1 - G_{\Delta V}(\omega) \Leftrightarrow G_{\Delta V}(\omega) \leq 1 - P(1,1|x), \ when \ x \in X^* \\ P(1,1|x,p) &\leq G_{\Delta V}(\omega), \ when \ x \in X^{**} \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} G_{\Delta V}(\omega) &\leq \inf_{x \in X^*} 1 - P(1,1|x,p) \\ \sup_{x \in X^{**}} P(1,1|x,p) &\leq G_{\Delta V}(\omega) \end{aligned} \tag{47}$$

Define

$$\begin{aligned} s_{(1,0)}(\omega) &= \sup_{x: -\Delta x\beta - p_{11} + p_{12} \leq \omega} P(1,0|x,p) \\ s_{(2,0)}(\omega) &= \sup_{x: -\Delta x\beta + p_{12} + (p_{11} - p_{21}) \leq \omega} P(2,0|x,p) \\ s_{(2,1)}(\omega) &= \sup_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega} P(2,1|x,p) \\ s_{(1,1)}(\omega) &= \sup_{x \in X^{**}} P(1,1|x,p) \\ i_{(0,1)}(\omega) &= \inf_{x: -\Delta x\beta - p_{11} + p_{12} \geq \omega} [1 - P(0,1|x,p)] \\ i_{(0,2)}(\omega) &= \inf_{x: -\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} [1 - P(0,2|x,p)] \\ i_{(1,2)}(\omega) &= \inf_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} [1 - P(1,2|x,p)] \\ i_{(1,1)}(\omega) &= \inf_{x \in X*} [1 - P(1,1|x,p)] \end{aligned} \tag{48}$$

Theorem 8. *Let Assumptions 1-13. Using the definitions in* (48)*, an outer region for* β *is given by the set:*

$$\mathcal{B}^{SO} = \left\{ \begin{array}{c} \beta \in \mathcal{B} : \forall \omega, \omega', \omega'' \in \mathbb{R} \ s.t. \\ \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] \leq \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)] \\ a.e. \ x \in \mathcal{X} \end{array} \right\}$$

Theorem 9. *Let Assumptions 1-13 hold. Then if* $\beta \in \mathcal{B}^{SO}$*, then there exists a structure* $S^{SO} = (\beta^{SO}, F^{SO}_{(V,\alpha)|X})$ *that satisfies the restrictions of the model and is observationally equivalent to structure* S^0 *that generates* $P^0(y_1, y_2|x)$*. The identified set for* β *and* $F_{(V,\alpha)|X}$ *is given by:*

$$S^{SO} = \left\{ \begin{array}{c} (\beta, F_{(V,\alpha)|X} \in (\mathcal{B}, \mathcal{F}_{(V,\alpha)|X}) : \forall \omega, \omega', \omega'' \in \mathbb{R}, s.t. \text{ the relations in (46) hold} \\ \text{with } G_{\Delta V|X}(\omega|x) = \int_{-\infty}^{\infty} F_{V|\alpha,X}(V_2 - V_1 < \omega|\alpha, x) dF_{\alpha|X}(\alpha|x) \ a.e. \ x \in \mathcal{X} \end{array} \right\}$$

Proof. The proof follows the proof in Section 3.1 and in Rosen and Weidner (2013,WP) and is shown in Appendix A. □

Notice that Theorem 9 doesn't characterize the sharp identified set. The question of sharpness of the identified set still remains an open question, to be addressed in future work.

4 Conclusion

This paper studies identification of panel data discrete choice models. Under fairly weak conditions, identified sets in static and dynamic binary choice models as well as static ordered response models are derived, even without assuming distributional assumptions for the time-varying unobservables and unobserved heterogeneity. The bounds are achieved by removing the unobserved heterogeneity from the underlying utility function and assuming that individuals are observed for at least two periods.

Unlike the static binary panel data model, the dynamic binary panel data model is incomplete and incoherent, nevertheless it is still possible to derive valid bounds for the object of interest. Furthermore, unlike the binary panel data models, in the ordered one, individuals who do not change and purchase the "in-between" choice for two consequent periods, provide also information for identification of the object of interest.

Such models are useful in examining individual behaviour when purchases are observed for a number of periods and demand is linked intertemporaly. The binary choice panel data model can be used when the choice in each period involves only the decision of purchasing or not a product, while the ordered panel data model can be used when choice is between vertically differentiated alternatives.

A Proof of Theorem 9

This section proves the sharpness of the identified set in Theorem 9.

Proof. Consider any $\beta \in \mathcal{B}^{SO}$ and fix ω' and ω''. Define $\forall \omega \in \mathbb{R}$

$$\begin{aligned}\underline{G}(\omega|x,p) &= \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] \\ \overline{G}(\omega|x,p) &= \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)]\end{aligned}$$

The model is complete which implies that, $\forall x \in \mathcal{X}$ and any fixed $p \in \mathcal{P}$

$$\sum_{(y_1,y_2)} P_{(y_1,y_2)}(x,p) = 1$$

Also define,

$$\begin{aligned}
s_{(0,1)}(\omega) &= \sup_{x:-\Delta x\beta - p_{11} + p_{12} \geq \omega} P(0,1|x,p) \\
s_{(0,2)}(\omega) &= \sup_{x:-\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} P(0,2|x,p) \\
s_{(1,2)}(\omega) &= \sup_{x:-\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} P(1,2|x,p) \\
s^*_{(1,1)}(\omega) &= \sup_{x \in X^*} P(1,1|x,p)
\end{aligned} \tag{49}$$

The completeness, together with Assumption 13, imply that $\forall \omega \in \mathbb{R}$,

$$\begin{aligned}
\max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] &+ \max[s_{(0,1)}(\omega), s_{(0,2)}(\omega), s_{(1,2)}(\omega), s^*_{(1,1)}(\omega)] \leq 1 \\
\max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] &\leq 1 - \max[s_{(0,1)}(\omega), s_{(0,2)}(\omega), s_{(1,2)}(\omega), s^*_{(1,1)}(\omega)] \\
\max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] &\leq 1 + \min[-s_{(0,1)}(\omega), -s_{(0,2)}(\omega), -s_{(1,2)}(\omega), -s^*_{(1,1)}(\omega)] \\
\max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] &\leq \min[1 - s_{(0,1)}(\omega), 1 - s_{(0,2)}(\omega), 1 - s_{(1,2)}(\omega), 1 - s^*_{(1,1)}(\omega)] \\
\max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] &\leq \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)] \\
\underline{G}(\omega|x,p) &\leq \overline{G}(\omega|x,p)
\end{aligned}$$

where the last result comes from the fact that,

$$\begin{aligned}
-\sup_{x:-\Delta x\beta - p_{11} + p_{12} \geq \omega} P(0,1|x,p) &= \inf_{x:-\Delta x\beta - p_{11} + p_{12} \geq \omega} [-P(0,1|x,p)] \\
-\sup_{x:-\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} P(0,2|x,p) &= \inf_{x:-\Delta x\beta - p_{11} - (p_{12} - p_{22}) \geq \omega} [-P(0,2|x,p)] \\
-\sup_{x:-\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} P(1,2|x,p) &= \inf_{x:-\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \geq \omega} [-P_{(1,2)}(x,p)] \\
-\sup_{x \in X^*} P(1,1|x,p) &= \inf_{x \in X^*} [-P(1,1|x,p)]
\end{aligned}$$

Note also that the functions $\underline{G}(\omega|x,p)$ and $\overline{G}(\omega|x,p)$ are both weakly increasing in ω and take values in the unit internal such that,

$$\begin{aligned}
\lim_{\omega \to \infty} \underline{G}(\omega|x,p) = \lim_{\omega \to \infty} \overline{G}(\omega|x,p) &= 1 \\
\lim_{\omega \to -\infty} \underline{G}(\omega|x,p) = \lim_{\omega \to -\infty} \overline{G}(\omega|x,p) &= 0
\end{aligned} \tag{50}$$

First define $\forall \omega, \omega', \omega'' \in \mathbb{R}$,

$$\begin{aligned}
\underline{G}(\omega|x,p) &= \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] \\
\overline{G}(\omega|x,p) &= \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)]
\end{aligned}$$

Part 1: Show that $\underline{G}(\omega|x,p)$ and $\overline{G}(\omega|x,p)$ are both weakly increasing in ω.

First start from $\underline{G}(\omega|x,p)$. Recall that,

$$\begin{aligned}
s_{(1,0)}(\omega) &= \sup_{x:-\Delta x\beta - p_{11}+p_{12}\leq\omega} P(1,0|x,p)\\
s_{(2,0)}(\omega) &= \sup_{x:-\Delta x\beta + p_{12}+(p_{11}-p_{21})\leq\omega} P(2,0|x,p)\\
s_{(2,1)}(\omega) &= \sup_{x:-\Delta x\beta + (p_{22}-p_{21})-p_{12}+p_{11}\leq\omega} P(2,1|x,p)\\
s_{(1,1)}(\omega) &= \sup_{x\in X^{**}} P(1,1|x,p)
\end{aligned}$$

To show that $\underline{G}(\omega|x,p)$ is weakly increasing in ω, all $s_{(1,0)}(\omega)$, $s_{(2,0)}(\omega)$, $s_{(2,1)}(\omega)$ and $s_{(1,1)}(\omega)$ should be weakly increasing in ω. Consider the choice $(1,0)$.

Let $X' = \{x : -\Delta x\beta - p_{11} + p_{12} \leq \omega'\}$, $X'' = \{x : -\Delta x\beta - p_{11} + p_{12} \leq \omega''\}$ and $\omega' < \omega''$. This implies,

$$x : -\Delta x\beta - p_{11} + p_{12} \leq \omega' < \omega'' \;\Rightarrow\; X' \subseteq X''$$

If

$$\sup_{x\in X'} P(1,0|x,p) \geq \sup_{x\in X''} P(1,0|x,p) \Leftrightarrow \sup_{x\in X'} P(1,0|x,p) = \sup_{x\in X''} P(1,0|x,p)$$

and $\omega' \to \omega''$, the supremum remains the same.

If

$$\sup_{x\in X''} P(1,0|x,p) > \sup_{x\in X'} P(1,0|x,p)$$

and $\omega' \to \omega''$, the supremum increases.

The same holds for the choice sequences $(2,0), (2,1)$ and $(1,1)$. Then since $s_{(1,0)}(\omega)$, $s_{(2,0)}(\omega)$, $s_{(2,1)}(\omega)$ and $s_{(1,1)}(\omega)$ are all weakly increasing functions in ω, $\underline{G}(\omega|x,p) = \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)]$ is also weakly increasing.

Now consider $\overline{G}(\omega|x,p)$. Recall that,

$$\begin{aligned}
i_{(0,1)}(\omega) &= \inf_{x:-\Delta x\beta - p_{11}+p_{12}\geq\omega} [1 - P(0,1|x,p)]\\
i_{(0,2)}(\omega) &= \inf_{x:-\Delta x\beta - p_{11}-(p_{12}-p_{22})\geq\omega} [1 - P(0,2|x,p)]\\
i_{(1,2)}(\omega) &= \inf_{x:-\Delta x\beta + (p_{22}-p_{21})-p_{12}+p_{11}\geq\omega} [1 - P(1,2|x,p)]\\
i_{(1,1)}(\omega) &= \inf_{x\in X*} [1 - P(1,1|x,p)]
\end{aligned}$$

To show that $\overline{G}(\omega|x,p)$ is weakly increasing in ω, all $i_{(0,1)}(\omega)$, $i_{(0,2)}(\omega)$, $i_{(1,2)}(\omega)$ and $i_{(1,1)}(\omega)$ should be weakly increasing in ω. Consider the choice $(0,1)$.

Let $X' = \{x : -\Delta x\beta - p_{11} + p_{12} \geq \omega'\}$, $X'' = \{x : -\Delta x\beta - p_{11} + p_{12} \geq \omega''\}$ and $\omega' < \omega''$. This implies,

$$x : -\Delta x\beta - p_{11} + p_{12} \geq \omega'' > \omega' \;\Rightarrow\; X'' \subseteq X'$$

If

$$\inf_{x \in X''} P(0,1|x,p) \leq \inf_{x \in X'} P(0,1|x,p) \Leftrightarrow \inf_{x \in X''} P(0,1|x,p) = \inf_{x \in X'} P(0,1|x,p)$$

and $\omega' \to \omega''$, the infimum remains the same.

If

$$\inf_{x \in X'} P(0,1|x,p) < \inf_{x \in X''} P(0,1|x,p)$$

and $\omega' \to \omega''$, the infimum increases.

The same holds for the choice sequences $(0,2), (1,2)$ and $(1,1)$. Then since $i_{(0,1)}(\omega)$, $i_{(0,2)}(\omega)$, $i_{(1,2)}(\omega)$ and $i_{(1,1)}(\omega)$ are all weakly increasing functions in ω, $\overline{G}(\omega|x,p) = \min[i_{(0,1)}(\omega),\ i_{(0,2)}(\omega),\ i_{(1,2)}(\omega), i_{(1,1)}(\omega)]$ is also weakly increasing.

Part 2: Show that $\underline{G}(\omega|x,p) \in [0,1]$ and $\overline{G}(\omega|x,p) \in [0,1]$.

Start from the $\underline{G}(\omega|x,p)$ and consider the following limits,

$$\lim_{\omega \to \infty} \underline{G}(\omega|x,p) \text{ and } \lim_{\omega \to -\infty} \underline{G}(\omega|x,p)$$

For $\omega \to \infty$, what is the $\lim_{\omega \to \infty} \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)]$? Consider every element separately:

$$\begin{aligned}
\lim_{\omega \to \infty} s_{(1,0)}(\omega) &= \lim_{\omega \to \infty} \sup_{x: -\Delta x\beta - p_{11} + p_{12} \leq \omega} P(1,0|x,p) = 1 \\
\lim_{\omega \to \infty} s_{(2,0)}(\omega) &= \lim_{\omega \to \infty} \sup_{x: -\Delta x\beta + p_{12} + (p_{11} - p_{21}) \leq \omega} P(2,0|x,p) = 1 \\
\lim_{\omega \to \infty} s_{(2,1)}(\omega) &= \lim_{\omega \to \infty} \sup_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega} P(2,1|x,p) = 1 \\
\lim_{\omega \to \infty} s_{(1,1)}(\omega) &= \lim_{\omega \to \infty} \sup_{x: \omega \geq -\Delta x\beta + p_{22} - p_{12} + p_{11} \cap -\Delta x\beta + p_{12} - p_{21} + p_{11} > \omega''} P(1,1|x,p) \leq 1
\end{aligned}$$

This implies that $\lim_{\omega \to \infty} \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] = 1$.

For $\omega \to -\infty$, what is the $\lim_{\omega \to -\infty} \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)]$? Consider every element separately:

$$\begin{aligned}
\lim_{\omega \to -\infty} s_{(1,0)}(\omega) &= \lim_{\omega \to -\infty} \sup_{x: -\Delta x\beta - p_{11} + p_{12} \leq \omega} P(1,0|x,p) = 0 \\
\lim_{\omega \to -\infty} s_{(2,0)}(\omega) &= \lim_{\omega \to -\infty} \sup_{x: -\Delta x\beta + p_{12} + (p_{11} - p_{21}) \leq \omega} P_{(2,0)}(x,p) = 0 \\
\lim_{\omega \to -\infty} s_{(2,1)}(\omega) &= \lim_{\omega \to -\infty} \sup_{x: -\Delta x\beta + (p_{22} - p_{21}) - p_{12} + p_{11} \leq \omega} P(2,1|x,p) = 0 \\
\lim_{\omega \to -\infty} s_{(1,1)}(\omega) &= \lim_{\omega \to -\infty} \sup_{x: \omega \geq -\Delta x\beta + p_{22} - p_{12} + p_{11} \cap -\Delta x\beta + p_{12} - p_{21} + p_{11} > \omega''} P(1,1|x,p) = 0
\end{aligned}$$

This implies that $\lim_{\omega\to-\infty} \max[s_{(1,0)}(\omega), s_{(2,0)}(\omega), s_{(2,1)}(\omega), s_{(1,1)}(\omega)] = 0.$

$\therefore \underline{G}(\omega|x,p) \in [0,1].$

Using the same arguments it can be shown that $\overline{G}(\omega|x,p) \in [0,1]$. Consider

$$\lim_{\omega\to\infty} \overline{G}(\omega|x,p) \; and \; \lim_{\omega\to-\infty} \overline{G}(\omega|x,p)$$

For $\omega \to \infty$, what is the $\lim_{\omega\to\infty} \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)]$? Consider every element separately:

$$\begin{aligned}
\lim_{\omega\to\infty} i_{(0,1)}(\omega) &= \lim_{\omega\to\infty} \inf_{x:-\Delta x\beta-p_{11}+p_{12}\geq\omega} [1-P(0,1|x,p)] = 1 \\
\lim_{\omega\to\infty} i_{(0,2)}(\omega) &= \lim_{\omega\to\infty} \inf_{x:-\Delta x\beta-p_{11}-(p_{12}-p_{22})\geq\omega} [1-P(0,2|x,p)] = 1 \\
\lim_{\omega\to\infty} i_{(1,2)}(\omega) &= \lim_{\omega\to\infty} \inf_{x:-\Delta x\beta+(p_{22}-p_{21})-p_{12}+p_{11}\geq\omega} [1-P(1,2|x,p)] = 1 \\
\lim_{\omega\to\infty} i_{(1,1)}(\omega) &= \lim_{\omega\to\infty} \inf_{x:\omega'>-\Delta x\beta+p_{22}-p_{12}+p_{11}\cap-\Delta x\beta+p_{12}-p_{21}+p_{11}\geq\omega} [1-P(1,1|x,p)] = 1
\end{aligned}$$

This implies that $\lim_{\omega\to\infty} \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)] = 1$

For $\omega \to -\infty$, what is the $\lim_{\omega\to-\infty} \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)]$? Consider every element separately:

$$\begin{aligned}
\lim_{\omega\to-\infty} i_{(0,1)}(\omega) &= \lim_{\omega\to-\infty} \inf_{x:-\Delta x\beta-p_{11}+p_{12}\geq\omega} [1-P(0,1|x,p)] = 0 \\
\lim_{\omega\to-\infty} i_{(0,2)}(\omega) &= \lim_{\omega\to-\infty} \inf_{x:-\Delta x\beta-p_{11}-(p_{12}-p_{22})\geq\omega} [1-P(0,2|x,p)] = 0 \\
\lim_{\omega\to-\infty} i_{(1,2)}(\omega) &= \lim_{\omega\to-\infty} \inf_{x:-\Delta x\beta+(p_{22}-p_{21})-p_{12}+p_{11}\geq\omega} [1-P(1,2|x,p)] = 0 \\
\lim_{\omega\to-\infty} i_{(1,1)}(\omega) &= \lim_{\omega\to-\infty} \inf_{x:\omega'>-\Delta x\beta+p_{22}-p_{12}+p_{11}\cap-\Delta x\beta+p_{12}-p_{21}+p_{11}\geq\omega} [1-P(1,1|x,p)] \geq 0
\end{aligned}$$

This implies that $\lim_{\omega\to-\infty} \min[i_{(0,1)}(\omega), i_{(0,2)}(\omega), i_{(1,2)}(\omega), i_{(1,1)}(\omega)] = 0$

$\therefore \overline{G}(\omega|x,p) \in [0,1].$

Therefore any such function $G(\cdot|x,p) : \mathbb{R} \to [0,1]$ such that $\forall\omega \in \mathbb{R}$

$$\underline{G}(\omega|x,p) \leq G(\omega|x,p) \leq \overline{G}(\omega|x,p) \tag{51}$$

is a CDF that exists when $\mathcal{B}^{SO}$ is not empty. This completes the proof. □

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