Endogenous Capital- and Labor-Augmenting Technical Change in the Neoclassical Growth Model

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Abstract: The determinants of the direction of technical change and their implications for economic growth and economic policy are studied in the one-sector neoclassical growth model of Ramsey, Cass, and Koopmans extended to allow for endogenous capital- and labor-augmenting technical change. We develop a novel micro-foundation for the competitive production sector that rests on the idea that the fabrication of output requires tasks to be performed by capital and labor. Firms may engage in innovation investments that increase the productivity of capital and labor in the performance of their respective tasks. These investments are associated with new technological knowledge that accumulates over time and sustains long-run growth. We show that the equilibrium allocation is not Pareto-efficient since both forms of technical change give rise to an inter-temporal knowledge externality. An appropriate policy of investment subsidies may implement the efficient allocation.

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1 Introduction

This paper studies the determinants of the direction of technical change and highlights their implications for economic growth and economic policy. More precisely, we focus on endogenous capital- and labor-augmenting technical change and embed this phenomenon into the one-sector neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965). To accomplish this, we develop a novel micro-foundation of the competitive production sector that allows for endogenous innovation investments.[1] It rests on the idea that the fabrication of output requires tasks to be performed. Some tasks are carried out by capital, others by labor. Innovation investments increase the productivity of capital and labor in the performance of their respective tasks. These investments are associated with new technological knowledge that accumulates over time so that economic growth becomes sustainable in the long-run.

Our main findings may be summarized as follows. First, a key determinant of the direction of technical change is the relative scarcity of "efficient capital" with respect to "efficient labor" measured by the ratio of these two factors of production at the beginning of each period. This ratio determines relative factor prices and the relative profitability of innovation investments. If a factor of production becomes scarcer, then it also becomes more expensive, i. e., its price increases relative to the price of the other factor. Accordingly, an investment enhancing the productivity of this factor is more advantageous and the direction of technical change shifts towards this factor. It is in this sense that our analysis provides a formal interpretation of Hicks' famous assertion according to which technical change is directed to economizing the use of a factor that has become relatively more expensive (see, Hicks (1932), pp. 124-125).

Second, along the transition towards the steady state, the growth rate of the economy reflects both capital- and labor-augmenting technical progress. However, in steady state capital-augmenting technical progress vanishes. Hence, in the long run, the growth rate of per-capita variables reflects only labor-augmenting technical change.[2] The reason for this finding is closely related to the extension of Uzawa's steady-state growth theorem devised in Irmen (2013b). Roughly speaking, in steady state technical progress must be labor-augmenting since capital accumulates and the economy's net output function

[1] By now it is well understood that endogenous technical change that results from innovation investments may arise in a competitive economy even in steady state (see, e. g., Boldrin and Levine (2008), Hellwig and Irmen (2001), or Zeira (1998)). However, none of these contributions addresses the issue of endogenous capital- and labor-augmenting technical change.

[2] See, e. g., Klump, McAdam, and Willman (2007) for an empirical study of the US economy that confirms this pattern of technical change for the period 1953 to 1998.

exhibits constant returns to scale in capital and labor. Moreover, steady-state technical progress has not even a representation as labor-augmenting because the net output function cannot be written in the Cobb-Douglas form.

The third set of results relates to the steady state and its comparative-static properties. We show that the steady state is a "balanced growth path" that satisfies all of Kaldor's famous stylized facts (see, Kaldor (1961)). Moreover, the steady-state growth rate of all per-capita variables is predicted to increase in parameters capturing the positive effect of institutions, technical infrastructure, or geography on the efficiency of the production sector. However, other parameters that often bring about steady-state growth effects such as the discount factor of the representative household, the size of the population or its growth rate have no impact on the steady-state growth rate.[3] The mere feasibility of capital-augmenting technical change is shown to be the reason for this. Due to its presence, the steady-state growth rates of capital- and labor-augmenting technical change are determined by the properties of the production sector alone.

Fourth, we analyze the local stability properties of the steady state and establish saddle-path stability in the state space. Interestingly, this finding does not hinge on the elasticity of substitution between efficient capital and efficient labor. The relative scarcity measured by the ratio of efficient capital with respect to efficient labor is a key stabilizing force. In steady state, this ratio and, therewith, the direction of technical change are constant. A small shock that lowers this ratio renders efficient capital relatively scarcer and shifts the direction of technical change towards more capital-augmenting and less labor-augmenting technical progress. This adjustment and the concomitant effect of capital accumulation tend to reduce the relative scarcity of efficient capital and to move the economy back towards its steady state.

The fifth set of results concerns the determinants of the functional income distribution. We first characterize the equilibrium elasticity of substitution (EES) that determines the qualitative effect of changes in the capital-labor ratio on the (relative) factor shares that accrue to capital owners and wage earners. In principle this elasticity is expected to reflect induced technical change. However, we establish that technical change has no first-order effect on equilibrium factor prices at a given capital-labor ratio.[4] Second, in line with

[3]In contrast to many other endogenous growth models the economy studied in the present paper exhibits steady-state growth without scale effects. Indeed, in the terminology of Jones (2005) and unlike the first-generation models with endogenous technical change authored by Romer (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), there are no "strong" scale effects as the steady-state growth rate is independent of the size of the population. Moreover, unlike the "semi-endogenous" technical change models of Jones (1995), Kortum (1997), or Segerstrom (1998) there are no "weak" scale effects either since the level of steady-state per-capita income does not depend on population size.

[4]In the parlance of Acemoglu (see, Acemoglu (2007), Definitions 5 and 6) technical change has no *relative*

common intuition, we show that a value of the EES equal to unity is critical. For greater (smaller) values of the EES the labor share will fall (increase) in response to an increase in the capital-labor ratio. However, our findings emphasize that the EES depends on the state of the economy even if the production function of the final good exhibits a constant elasticity of substitution. Moreover, depending on the economy's state, the EES may fall short of or exceed unity.

Our analysis also highlights the important distinction between factor shares and the functional income distribution where the latter accounts for the depreciation of capital.[5] This distinction matters since parameters like the household's discount factor do not affect the former but the latter. More precisely, we find that due to the presence of capital-augmenting technical change the steady-state factor shares are independent of household characteristics. However, once the depreciation of capital is taken into account household characteristics matter for the functional income distribution. In particular, we show that, ceteris paribus, the relative income of capital owners in steady state is lower in a more patient economy. This reflects adjustments in the steady-state level of capital-augmenting technological knowledge that increase the depreciation rate per efficiency unit of capital.

Sixth, we deal with three different fiscal policies and study their implications for the steady-state growth rate of the economy and its functional income distribution. First, we consider a linear tax on capital. We find that the steady-state growth rate of the economy is unaffected by the tax. This reflects the fact that this growth rate is determined within the production sector. The tax on capital also reduces the relative income of capital owners. However, the direct effect of the tax is shown to be mitigated by a decline in the depreciation rate per efficiency unit of capital. Second, we study the consequences of a policy that pays a subsidy to innovation investments that increase the productivity of capital. This policy is shown to increase the steady-state growth rate of the economy whereas its impact on the functional income distribution is indeterminate in general. Finally, we turn to a policy that subsidizes innovation investments that increase the productivity of labor. We show that this policy increases the steady-state growth rate of the economy. Moreover, independently of the EES, this policy moves the functional income distribution in favor of capital owners.

Finally, we conduct a welfare analysis. We find that the equilibrium allocation is not

bias and, therefore, there is no *weak relative equilibrium bias* either. In Acemoglu's framework this possibility arises only if the production function of firms is Cobb-Douglas. In our framework, it is shown to arise as an equilibrium phenomenon.

[5]See, Bridgman (2014) and Karabarbounis and Neiman (2014a) for two recent analyses inquiring the role of capital depreciation for the functional income distribution. The theoretical arguments are, however, not based on models where technical change arises endogenously.

Pareto efficient. Innovation investments in a given period increase the stock of knowledge that is available for subsequent innovative activity and create inter-temporal knowledge spill-overs which are not taken into account by private firms. A benevolent social planner may choose a policy of investment subsidies that implements the Pareto efficient allocation. We fully characterize the policy that implements the efficient steady state. It involves a subsidization of both kinds of innovation investments at different rates.

The remainder of this paper is organized as follows. Section 2 relates the paper to the relevant literature. Section 3 presents the model. In particular, we detail the novel micro-foundation of the competitive production sector. Section 4 studies the dynamic competitive equilibrium, i. e., it defines the equilibrium (Section 4.1), sets up the canonical dynamical system (Section 4.2), provides the steady state analysis (Section 4.3), and clarifies the role of capital-augmenting technical change for our findings (Section 4.4). The focus of Section 5 is on the positive implications of endogenous capital- and labor-augmenting technical change. Section 5.1 studies the determinants of the functional income distribution in this context. Section 5.2 derives the implications of fiscal policy for long-run growth and for the functional income distribution. Policies considered involve a linear taxation of capital (Section 5.2.1), a subsidy to capital-augmenting innovation investments (Section 5.2.2), and a subsidy to labor-augmenting innovation investments (Section 5.2.3). Section 6 discusses the normative implications of endogenous capital- and labor-augmenting technical change. Here, we study the choices of a benevolent planner (Section 6.1), solve for the optimal steady-state allocation (Section 6.2), and show that it can be implemented with an appropriate choice of subsidies to innovation investments (Section 6.3). Section 7 concludes. All proofs are contained in Section 8, the Appendix.

2 Related Literature

The present paper builds on and contributes to several strands of the modern literature on endogenous technical change. Since its inception in the 1980s this literature has largely focussed on models where - by design - endogenous technical change is confined to be labor-augmenting. With the exception of Funk (2002), Acemoglu (2003b), and our own work (Irmen (2011) and Irmen (2013a)), the analysis of the determinants and the consequences of endogenous capital- and labor-augmenting technical change has received little attention. These recent studies have their roots in the so-called induced innovations literature of the 1960s. This literature constitutes the first attempt to systematically address the question of endogenous capital- and labor-augmenting technical change.[6] In

[6]Important articles of this literature include von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Samuelson (1966), or Drandakis and Phelps (1966).

light of these contributions it is the purpose of this section to clarify the main conceptual differences of our approach and to highlight the new results we derive in the present paper.

Arguably, there are three main reasons for which the induced innovations literature of the 1960s has been criticized: (i) the arbitrary optimization problem solved by firms to determine the endogenous growth rate, (ii) the ad hoc assumption of an exogenous Kennedy-von Weizsäcker Innovation Possibilities Frontier (IPF), and (iii) the fact that technical progress is costless.[7] Our approach addresses all of these concerns while maintaining the assumption of a competitive economy.

First, in our setting, capital- and labor-augmenting technical change results from a well-defined profit-maximization problem solved by price-taking firms. Second, the resulting choices give rise to an equilibrium innovation possibility frontier (EIPF). Hence, in equilibrium our economy faces a similar trade-off as in the literature of the 1960s. However, unlike the exogenous and concave IPF stipulated in the old literature the EIPF is endogenous and typically a convex function.

Third, in our framework technical change has a cost in terms of current output. This begs the question about how these costs are paid for in a competitive equilibrium. To clarify this issue in a heuristic way consider a competitive representative firm with access to an aggregate production function $F(BK, AL)$ with constant returns to scale in both arguments and strictly positive, yet declining marginal products. Here, B and A denote the stocks of capital-augmenting and of labor-augmenting technological knowledge, K is the capital stock, and L is employed labor. In addition, let R denote the price of capital, w the price of labor, p_B the price of capital-augmenting technological knowledge, and p_A the price of labor-augmenting technological knowledge. Then, for the competitive representative firm that takes $(R, w, p_B, p_A) \in \mathbb{R}^4_{++}$ as given the problem

$$\max_{(K,L,B,A)\in\mathbb{R}^4_+} \quad V = F(BK, AL) - RK - wL - p_B B - p_A A \tag{2.1}$$

has no solution: with constant returns to scale, the remuneration of more than two of the four factors of production will exceed aggregate output. The usual way to address

[7] See, e. g., Nordhaus (1973), Burmeister and Dobell (1970), Funk (2002), or Acemoglu (2003a). Funk (2002) addresses the first concern. This author enriches the analytical framework of the induced innovations literature by providing a micro-foundation for the innovation process. Acemoglu (2003b) broadens the analytical framework of Romer (1990) to allow for endogenous capital- and labor-augmenting technical change. This approach addresses concern (i) with a micro-foundation of firm behavior involving imperfect competition. As to concern (ii), the allocation of researchers is still subject to an exogenous innovation possibility frontier. Moreover, - concerning (iii) - technical progress is only costly in terms of foregone current output if the manufacturing sector and the research sector compete for labor which is not the case in the main setting studied by Acemoglu.

this problem involves monopolistic rents that redistribute total output in a way that all factors of production receive a strictly positive remuneration (see, e. g., Romer (1990) or Acemoglu (2003b)). In contrast, in the present paper we rely on a rent-sharing mechanism that works under perfect competition.

The intuition for this mechanism comes in two steps. The first step derives the representative firm's total cost for the use of the respective inputs in efficiency units, the second step explains how an endogenous technology choice becomes feasible in the competitive economy. To simplify we focus on efficient capital, BK. Mutatis mutandis, a similar line of reasoning may be spelled out for efficient labor, AL.

As to step one, think of K as the total number of machines in the economy and of B as an index of the quality of each of these machines. Then, the price of using BK reflects two different components. First, there is the rental price, R, that the competitive firm pays per machine, second there is the price p_B per unit of B. The product $p_B B$ has then an interpretation as the remuneration of the technological knowledge embodied in each machine. Accordingly, the cost per machine is equal to $R + p_B B$, and the total cost for the firm working with K machines is $(R + p_B B)\,K$. Similarly, the total cost of employing AL units of efficient labor amounts to $(w + p_A A)\,L$. Observe that these expressions of total costs take prices and the level of B and A as exogenously given.

The second step adds features allowing for B and A to be treated as endogenous variables. Suppose the firm chooses B from a menu $B \in [1, \infty)$ with corresponding prices $p_B(B)$ where $p_B(1) = 0$, $p'_B(B) > 0$, and $p''_B(B) > 0$. Similarly, A may be chosen from a menu $A \in [1, \infty)$ with corresponding prices $p_A(A)$ where $p_A(1) = 0$, $p'_A(A) > 0$, and $p''_A(A) > 0$. Then, the firm's problem (2.1) becomes

$$\max_{(K,L,B,A)\in\mathbb{R}^4_+} \quad V = F(BK, AL) - [R + p_B(B)B]\,K - [w + p_A(A)A]\,L, \tag{2.2}$$

where V is jointly concave in (K, L, B, A) and strictly concave in (B, A). The corresponding first-order (sufficient) conditions for an interior solution are

$$\frac{\partial V}{\partial K} = BF_1 - [R + p_B(B)B] = 0, \tag{2.3}$$

$$\frac{\partial V}{\partial L} = AF_2 - [w + p_A(A)A] = 0, \tag{2.4}$$

$$\frac{\partial V}{\partial B} = F_1 - \left[p'_B(B) + p_B(B)\right] = 0, \tag{2.5}$$

$$\frac{\partial V}{\partial A} = F_2 - \left[p'_A(A) + p_A(A)\right] = 0, \tag{2.6}$$

where F is evaluated at (BK, AL). Each condition equalizes the marginal benefit to the marginal cost. The system (2.3) - (2.6) determines a unique solution involving $R^* =$

$R(K,L)$, $w^* = w(K,L)$, $B^* = B(K,L)$, and $A^* = A(K,L)$. To see that this solution shifts rents from capital and labor to remunerate capital- and labor-augmenting technological knowledge use (2.3) and (2.4) to find

$$R^* = B^* \left[F_1 - p_B(B^*)\right] \quad \text{and} \quad w^* = A^* \left[F_2 - p_A(A^*)\right], \tag{2.7}$$

where here (and below) F is evaluated at (B^*K, A^*L). Hence, R^* and w^* are smaller than the respective marginal product of capital and labor. Using (2.7) in (2.2) gives

$$\begin{aligned} V^* &= F - R^*K - w^*L - p_B(B^*)\,B^*K - p_A(A^*)\,A^*L \\ &= F - B^*\left[F_1 - p_B(B^*)\right]K - A^*\left[F_2 - p_A(A^*)\right]L - p_B(B^*)\,B^*K - p_A(A^*)\,A^*L \\ &= F - F_1B^*K - F_2A^*L = 0. \end{aligned} \tag{2.8}$$

The second line reveals that the remuneration of capital (labor) is reduced by the same amount that capital-augmenting (labor-augmenting) technical knowledge earns. Therefore, by Euler's law, aggregate output is equal to the sum of all factor payments and $V^* = 0$. Hence, an endogenous choice of capital- and labor-augmenting technological knowledge involving a cost of knowledge creation is feasible in a competitive environment. Section 3 below develops a complete micro-foundation along these lines.

Next, we relate our main findings to the contributions of the induced innovations literature that involve capital accumulation (e. g., von Weizsäcker (1962), Drandakis and Phelps (1966), Samuelson (1966)), and to the more recent analysis of Acemoglu (2003b).

First, it is worth mentioning that these contributions exhibit a mechanism that links the ratio of efficient capital to efficient labor to the direction of technical change. Moreover, this link depends crucially on whether the elasticity of substitution between efficient capital and efficient labor is smaller or greater than unity. In particular, a decline of (increase in) this ratio induces faster (slower) capital-augmenting technical progress only if the elasticity of substitution falls short of unity. In contrast, in the present paper this response of the direction of technical change obtains irrespective of the elasticity of substitution.

This difference can be traced back to the fact that in our setting the direction of technical change is linked to the relative factor price ratio, R/w, whereas in the literature mentioned above the direction hinges on the ratio of the share of capital to the share of labor. This distinction matters in the following way. If the relative scarcity of efficient capital - measured by the ratio BK/AL - increases then the direction in which the ratio of the factor shares moves hinges on the elasticity of substitution between efficient capital and efficient labor. The same increase will, however, unequivocally lower the factor-price ratio, R/w. In our analytical framework, this movement induces less capital- and more labor-augmenting technical change. To see this combine (2.3) with (2.5) and (2.4) with (2.6). This gives

$$B \cdot p'_B(B) = R \quad \text{and} \quad A \cdot p'_A(A) = w,$$

where the left-hand sides increase in B and A, respectively. Hence, the conditions for a profit-maximum deliver a functional relationship saying that an increase in R/w will always increases the ratio B/A. Hence, in our framework the direction of technical change is linked to relative factor prices rather than to relative factor shares.

This latter property is not only intuitively appealing but also contributes to the local stability of the steady state of our dynamical system. This contrasts with the steady state in the above-mentioned literature which is unstable if the elasticity of substitution exceeds unity. Moreover, their off-steady-state dynamics exhibit fairly unreasonable properties.

Second, in line with the above-mentioned literature the steady state of our model exhibits only labor-augmenting technical change. This is just as much due to the incentives that competitive firms face as it is an artefact of the reduced form of the underlying model.[8] Observe also that unlike the steady state in Acemoglu (2003b), there are no scale effects in the present paper.

Third, we turn to the relationship between economic growth and the income distribution. We find that steady-state factor shares are invariant with respect to policies such as the taxation of capital income. The corresponding finding in Acemoglu (2003b) is the constancy of the ratio of factor shares that accrue in the production sector of his model. However, it should be noted that - unlike Acemoglu (2003b) - our analysis incorporates all incomes that accrue in the economy.[9] Moreover, we emphasize that unlike factor shares, the functional income distribution responds to policy changes and to changes in inter-temporal preferences once the depreciation of capital is taken into account. We establish this finding for a linear tax on capital incomes and for a subsidy to innovation investments that increases the productivity of capital or of labor.

Finally, it is worth mentioning that we contribute the first welfare analysis to the literature on endogenous capital- and labor-augmenting technical change. Due to two intertemporal externalities the competitive equilibrium allocation is not Pareto-efficient. In

[8] The same remark is true for the induced innovations literature and for Acemoglu (2003b). Indeed, as shown in Irmen (2013b), the reduced form of each of these models satisfies the assumptions of Uzawa's steady-state growth theorem (Uzawa (1961)). Hence, steady-state growth must be labor-augmenting.

[9] In fact, the analysis of the income distribution in Acemoglu (2003b) neglects the wage incomes of researchers. Moreover, it treats the dividends paid by the monopolists of labor-intensive intermediates as labor income and not as capital income. Finally, the distinction between factor shares and the functional income distribution does not arise here since capital does not depreciate.

spite of the intricate equilibrium interdependency between capital- and labor-augmenting technical change, we establish the subsidy rates that implement the Pareto-efficient steady state.

3 The Model

We consider a competitive closed economy in an infinite sequence of periods $t = 0, 1, 2, \ldots$. The economy consists of a household sector and a production sector. In each period there is a single *final good* that can be consumed or invested. If invested, it may either become future capital or an input in contemporaneous innovation investments that raise the productivity of capital or labor. Households supply *labor* and *capital*. Each period has a market for all three objects of exchange. The final good serves as numéraire.

3.1 The Household Sector

The household sector is populated by a single representative household comprising one member.[10] In each period, the household cares about the level of consumption, C_t, and supplies inelastically the labor endowment, L.

The per-period utility function is logarithmic, i. e., $u(C_t) = \ln C_t$. Moreover, the household evaluates sequences of consumption $\{C_t\}_{t=0}^{\infty}$ according to

$$\sum_{t=0}^{\infty} \beta^t \ln C_t. \tag{3.1}$$

The household owns all firms and the capital stock. Since profits, i. e., dividends, vanish in equilibrium, we do not explicitly account for the profit distribution. Capital is the only asset in the economy. Capital at t, K_t, is installed at $t-1$, and firms pay a real rental rate, R_t, per unit of capital they work with. After use the capital stock decays at rate $\delta^K \in [0, 1]$. Accordingly, the household's flow budget constraint at t is given by

$$K_{t+1} = \left(R_t + 1 - \delta^K\right) K_t + w_t L - C_t, \tag{3.2}$$

where w_t is the real wage.

[10] To simplify the exposition we make a few assumptions that are innocuous with respect to our main qualitative results. They include a constant household size, i. e., there is no population growth, an inelastic labor supply, and an inter-temporal elasticity of substitution equal to unity. We discuss the detailed role of these simplifying assumptions for our findings in the concluding Section 7.

Given $K_0 > 0$ and $L > 0$, the representative household maximizes (3.1) subject to (3.2), $C_t \geq 0$, $K_{t+1} \geq 0$, and an appropriate No-Ponzi Game condition by choosing a sequence $\{C_t\}_{t=0}^{\infty}$. By standard arguments, the solution to this problem satisfies for all t the flow budget constraint (3.2), the Euler condition,

$$\frac{C_{t+1}}{C_t} = \beta \left(R_{t+1} + 1 - \delta^K \right), \tag{3.3}$$

and the transversality condition

$$\lim_{t \to \infty} \beta^t C_t^{-1} K_{t+1} = 0. \tag{3.4}$$

3.2 The Production Sector

3.2.1 Technology

The production sector has a continuum of identical, competitive firms of measure one. Without loss of generality, the analysis proceeds through the lens of a single representative firm.

To produce output two types of tasks need to be performed. The first type needs capital, the second labor as the only input. Denote by $m \in \mathbb{R}_+$ a task performed by capital, and let $n \in \mathbb{R}_+$ be a task performed by labor. Further, let M_t and N_t denote the measure of all tasks of the respective type performed at time t so that $m \in [0, M_t]$ and $n \in [0, N_t]$. Tasks of the respective type are identical. Therefore, total output hinges only on M_t and N_t. The production function $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ assigns the maximum output, Y_t, to each pair $(M_t, N_t) \in \mathbb{R}_+^2$, i. e.,

$$Y_t = F(M_t, N_t), \tag{3.5}$$

where F has constant returns to scale in its arguments and is $\mathcal{C}^2$ on $\mathbb{R}_{++}$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$.[11] Let κ_t denote the period-t task intensity,

$$\kappa_t = \frac{M_t}{N_t}. \tag{3.6}$$

Then, output in intensive form is $F(\kappa_t, 1) \equiv f(\kappa_t)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f'(\kappa_t) > 0 > f''(\kappa_t)$ for all $\kappa_t > 0$.

[11] Throughout, we denote derivatives of functions with a single argument either by a prime or by a subscript. Hence, the first derivative of $f(x)$ is either $f'(x)$ or $f_x(x)$, its second derivative is either $f''(x)$ or $f_{xx}(x)$, and so on.

At t, a task m requires $k_t(m) = 1/b_t(m)$ units of capital whereas a task n needs $l_t(n) = 1/a_t(n)$ units of labor. Hence, $b_t(m)$ and $a_t(n)$ denote the productivity of capital and labor in the respective task. The levels of productivity are given by

$$b_t(m) = B_{t-1}\left(1-\delta^B\right)(1+q_t^B(m)) \quad \text{and} \quad a_t(n) = A_{t-1}\left(1-\delta^A\right)(1+q_t^A(n)). \tag{3.7}$$

Here, $B_{t-1}\left(1-\delta^B\right)$ and $A_{t-1}\left(1-\delta^A\right)$ represent stocks of "surviving technological knowledge" that the firm inherits from the previous period. More precisely, B_{t-1} and A_{t-1} denote the respective stocks of technological knowledge used in the production sector at period $t-1$, and $\delta^j \in (0,1)$, $j = A, B$, is the rate of depreciation of the respective knowledge stock.[12] Finally, $\left(q_t^B(m), q_t^A(n)\right) \in \mathbb{R}_+^2$ are indicators of productivity growth at t associated with task m and n, respectively.

To achieve positive productivity growth, i. e., $q^j > 0$, $j = A, B$, the firm must engage in an innovation investment. More precisely, at t it must invest $i\left(q_t^B(m)\right) > 0$ units of the final good to achieve $q_t^B(m) > 0$ and, similarly, $i\left(q_t^A(n)\right) > 0$ units of the final good to obtain $q_t^A(n) > 0$.

The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for all tasks, time invariant, $\mathcal{C}^2$ on $\mathbb{R}_{++}$, strictly increasing and strictly convex in q. Moreover, it satisfies the following regularity conditions for $j = A, B$:

$$i(0) = 0, \quad \lim_{q^j \to 0} i'\left(q^j\right) = 0, \quad \lim_{q^j \to \infty} i'\left(q^j\right) = \lim_{q^j \to \infty} i\left(q^j\right) = \infty. \tag{3.8}$$

Any new piece of technological knowledge is proprietary knowledge of a particular firm only in the period when it occurs. Subsequently, it becomes public and embodied in aggregate economy-wide productivity indicators $(A_t, B_t), (A_{t+1}, B_{t+1}), \ldots$. The details will be specified below.[13]

3.2.2 Firm's Optimization

The representative firm takes the sequence $\{R_t, w_t, A_{t-1}, B_{t-1}\}_{t=0}^{\infty}$ of real wages, real rental rates of capital, and aggregate productivity indicators as given. Its choice involves a pro-

[12]Suppose we represent the switch from sending and receiving printed pages or images via fax machines to modern transfer via e-mail by an increase in the stock of labor-augmenting technological knowledge. Then, $\delta^A > 0$ may capture the fact that some specific knowledge required to master a fax machine will no longer be useful once the e-mail technology is in place.

[13]If at t the firm makes no investment in a productivity enhancing technology, it nevertheless has access to the economy-wide technology represented by $B_{t-1}\left(1-\delta^B\right)$ and $A_{t-1}\left(1-\delta^A\right)$. Then, its task-specific productivity of capital and labor is given by $b_t(m) = B_{t-1}\left(1-\delta^B\right)$ and $a_t(n) = A_{t-1}\left(1-\delta^A\right)$. However, since $\lim_{q^j \to 0} i'\left(q^j\right) = 0$ the option not to invest will not be chosen in equilibrium.

duction plan comprising a sequence

$$\left\{M_t, N_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n)\right\}_{t=0}^{\infty}$$

for $m \in [0, M_t]$ and $n \in [0, N_t]$, respectively. This plan maximizes the sum of the present discounted values of profits in all periods. Because an innovation investment generates proprietary knowledge only in the period when it is made, the inter-temporal profit maximization problem of the firm boils down to the maximization of per-period profits given by

$$F(M_t, N_t) - TC_t, \tag{3.9}$$

where TC_t is the firm's total cost, comprising factor costs and investment outlays for all performed tasks, i. e.,

$$TC_t = \int_0^{M_t} \left[R_t k_t(m) + i\left(q_t^B(m)\right)\right] dm + \int_0^{N_t} \left[w_t l_t(n) + i\left(q_t^A(n)\right)\right] dn, \tag{3.10}$$

where,

$$k_t(m) = \frac{1}{B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B(m)\right)} \quad \text{and} \quad l_t(n) = \frac{1}{A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A(n)\right)}$$

are the respective input coefficients.

The maximization of (3.9) can be split up into two parts. First, for each $m \in [0, M_t]$ and $n \in [0, N_t]$ a choice of $q_t^B(m)$ and $q_t^A(n)$ must minimize TC_t. This leads to the first-order (sufficient) conditions

$$q_t^B(m): \quad \frac{-R_t}{B_{t-1}(1-\delta^B)(1+q_t^B(m))^2} + i'(q_t^B(m)) = 0, \tag{3.11}$$

$$q_t^A(n): \quad \frac{-w_t}{A_{t-1}(1-\delta^A)(1+q_t^A(n))^2} + i'(q_t^A(n)) = 0. \tag{3.12}$$

For each task, faster productivity growth means lower factor costs. At the margin, this advantage is equal to the required additional investment outlays. In light of (3.8), and assuming $R_t > 0$ and $w_t > 0$, the conditions (3.11) and (3.12) determine a unique $q_t^B(m) = q_t^B > 0$ and $q_t^A(n) = q_t^A > 0$. In what follows we refer to $\left(q_t^B, q_t^A\right)$ as the cost-minimizing growth rates of productivity, and use $b_t \equiv B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B\right)$ and $a_t \equiv A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A\right)$ to denote the corresponding cost-minimizing productivity levels of capital and labor.

At $\left(q_t^B, q_t^A\right)$ the costs per task are minimized and the same for all tasks of the same type. Let $c\left(q_t^B\right)$ and $c\left(q_t^A\right)$ denote these costs. Then, total cost, TC_t, of (3.10) boils down to

$$TC_t = M_t c\left(q_t^B\right) + N_t c\left(q_t^A\right). \tag{3.13}$$

Second, the firm determines how many tasks of either type to perform. Using (3.13), it solves

$$\max_{(M_t,N_t)\in\mathbb{R}^2_+} F(M_t,N_t) - M_t c\left(q_t^B\right) - N_t c\left(q_t^A\right). \tag{3.14}$$

The respective first-order (sufficient) conditions are

$$M_t \quad : \qquad f'(\kappa_t) = c\left(q_t^B\right), \tag{3.15}$$

$$N_t \quad : \qquad f(\kappa_t) - \kappa_t f'(\kappa_t) = c\left(q_t^A\right). \tag{3.16}$$

Hence, for the marginal task of each type the marginal value product is equal to the minimized marginal cost (= investment cost plus wage, respectively, capital cost).

Equations (3.11), (3.12), (3.15), and (3.16) fully characterize the equilibrium behavior of the representative firm at all t. Observe that profits of (3.14) exhibit constant returns to scale in (M_t, N_t). This has two (related) implications. First, equations (3.15) and (3.16) will only pin down the task intensity $\kappa_t = M_t/N_t$. The number of tasks performed in equilibrium will be determined by market clearing conditions. Second, by Euler's law, firm profits are zero.

The following proposition shows that the profit-maximizing conditions allow us to express in an intuitive way the productivity growth rates $\left(q_t^B, q_t^A\right)$ and the factor prices (R_t, w_t) in terms of κ_t, B_{t-1}, and A_{t-1}.

Proposition 1 *Suppose equations (3.11), (3.12), (3.15), and (3.16) are satisfied. Then, the following holds:*

1. *There are maps $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $g^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that for all $\kappa_t > 0$*

$$q_t^B = g^B(\kappa_t), \quad \text{with} \quad g^B_\kappa(\kappa_t) < 0, \tag{3.17}$$

$$q_t^A = g^A(\kappa_t), \quad \text{with} \quad g^A_\kappa(\kappa_t) > 0. \tag{3.18}$$

2. *There are maps $w : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ and $R : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ such that the real rental rate of capital and the real wage satisfy*

$$R_t = R(\kappa_t, B_{t-1}) \quad \text{with} \quad R_\kappa(\kappa_t, B_{t-1}) < 0 \quad \text{and} \quad R_B(\kappa_t, B_{t-1}) > 0, \tag{3.19}$$

$$w_t = w(\kappa_t, A_{t-1}) \quad \text{with} \quad w_\kappa(\kappa_t, A_{t-1}) > 0 \quad \text{and} \quad w_A(\kappa_t, A_{t-1}) > 0. \tag{3.20}$$

To gain intuition for these findings consider conditions (3.11) and (3.15) for tasks performed by capital. The latter condition requires $f'(\kappa_t)$ to equal the minimized costs of all M_t tasks. Then, the necessary adjustments for some $\kappa_t' > \kappa_t$ are as follows. Obviously, we have $f'(\kappa_t) > f'(\kappa_t')$ so that the cost minimum must fall. Given B_{t-1}, this requires $R_t' < R_t$, hence $R_\kappa(\kappa_t, B_{t-1}) < 0$. Moreover, at R_t' the marginal advantage of faster productivity growth is smaller. Therefore, the cost-minimizing productivity growth rate is reached at some $\left(q_t^B\right)' < q_t^B$, hence, $g_\kappa^B(\kappa_t) < 0$. Finally, to support some given $q_t^B = g^B(\kappa_t)$, (3.11) requires R_t to increase in B_{t-1}, hence $R_B(\kappa_t, B_{t-1}) > 0$. Mutatis mutandis, the same arguments apply to (3.12) and (3.16) and show that, indeed, $g_\kappa^A(\kappa_t) > 0$, $w_\kappa(\kappa_t, A_{t-1}) > 0$, and $w_A(\kappa_t, A_{t-1}) > 0$.

Proposition 1 will play a crucial role in the setup and the analysis of the dynamical system below. The following proposition highlights an important second perspective on the firm's optimal behavior that leads to the notion of an *equilibrium innovation possibility frontier* and of an *equilibrium factor price frontier*.

Proposition 2 *Suppose equations (3.11), (3.12), (3.15), and (3.16) are satisfied. Then, the following holds:*

1. *There is an* ***equilibrium innovation possibility frontier*** $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *such that*

$$q_t^A = g\left(q_t^B\right), \quad \textit{with} \quad g'\left(q_t^B\right) < 0. \tag{3.21}$$

2. *There is an* ***equilibrium factor price frontier*** $h : \mathbb{R}_{++}^3 \to \mathbb{R}_{++}$ *such that*

$$R_t = h(w_t, A_{t-1}, B_{t-1}) \quad \textit{with} \quad h_w(w_t, A_{t-1}, B_{t-1}) < 0. \tag{3.22}$$

According to Statement 1 of Proposition 2 profit-maximization implies a functional relationship between both cost-minimizing productivity growth rates. We refer to this relationship as the equilibrium innovation possibility frontier (EIPF). The qualification "equilibrium" distinguishes this concept from the exogenous innovation possibility frontier stipulated by the induced innovations literature of the 1960s: in the present model, this frontier is endogenous and results from the profit-maximizing choices of firms.

As illustrated in Example 1 below, the position and the shape of the EIPF will be determined by parameters that capture geographical, technical, or institutional properties of the economy in which firms operate. Moreover, in the example the EIPF is convex (see Figure 3.1 for an illustration). This also contrasts with the (strictly) concave innovation possibility frontier stipulated by the induced innovation literature of the 1960s.[14]

[14]Concavity is necessary in this literature to turn the frontier into a suitable constraint for the maximization of the instantaneous rate of output growth (see, e. g., Burmeister and Dobell (1970), Chapter 3), for details.

To develop an intuition for the existence of the EIPF it proves useful to take a look at the dual of the profit-maximization problem (3.14). Accordingly, suppose the firm seeks to find the pair (M_t, N_t) that minimizes total costs of (3.13) for a given quantity of output, Y_t, i. e., let the firm solve

$$\min_{(M_t,N_t)\in\mathbb{R}^2_+} TC_t = M_t c\left(q_t^B\right) + N_t c\left(q_t^A\right) \quad \text{s.t.} \quad F\left(M_t, N_t\right) \geq Y_t. \tag{3.23}$$

This problem delivers "conditional demand functions", $M_t = M\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t$ and $N_t = N\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t$. Linearity in Y_t follows since F has constant returns to scale. Plugging these functions into the objective function of (3.23) delivers the cost function

$$TC\left(c\left(q_t^B\right), c\left(q_t^A\right), Y_t\right) = tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) Y_t, \tag{3.24}$$

where $tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right)$ is the minimum cost per unit of Y_t. From Euler's law, we know that firms earn zero-profits. Hence, it must hold that

$$tc\left(c\left(q_t^B\right), c\left(q_t^A\right)\right) = 1. \tag{3.25}$$

The latter condition defines the EIPF implicitly. Hence, the EIPF may be seen as an equilibrium constraint on $\left(q_t^B, q_t^A\right)$ resulting from the zero-profit condition of a cost-efficient firm operating under constant returns to scale. As $tc(\cdot,\cdot)$ as well as $c\left(q_t^B\right)$ and $c\left(q_t^A\right)$ are time-invariant, so is the EIPF. Moreover, as $tc(\cdot,\cdot)$ is strictly increasing in both arguments, $c'\left(q_t^B\right) > 0$, and $c'\left(q_t^A\right) > 0$, it follows that the slope of the EIPF is negative, i. e., $g'\left(q_t^B\right) < 0$.

The flip side of the EIPF is the equilibrium factor price frontier (EFPF) of Statement 2 of Proposition 2, $R_t = h\left(w_t, A_{t-1}, B_{t-1}\right)$. It also reflects the firm's zero-profit condition (3.25), however, now in the space of factor prices.[15] To see this, consider condition (3.11) which pins down q_t^B as a function of $\left(R_t, B_{t-1}\left(1-\delta^B\right)\right)$. Let us call this function $z_t^B = z^B\left(R_t, B_{t-1}\left(1-\delta^B\right)\right)$. Similarly, condition (3.12) pins down q_t^A as a function of $\left(w_t, A_{t-1}\left(1-\delta^A\right)\right)$, and we call this relationship $z_t^A = z^A\left(w_t, A_{t-1}\left(1-\delta^A\right)\right)$. Substitution of q_t^B by z_t^B and of q_t^A by z_t^A in (3.25) delivers the EFPF. As the relevant partial derivatives are strictly positive, i. e., $z_R^B\left(R_t, B_{t-1}\left(1-\delta^B\right)\right) > 0$ and $z_w^A\left(w_t, A_{t-1}\left(1-\delta^A\right)\right) > 0$, it follows that $h_w\left(w_t, A_{t-1}, B_{t-1}\right) < 0$. Pairs of factor prices consistent with the EFPF hinge on A_{t-1} and B_{t-1} and, therefore, will change over time.

To strengthen our intuition about the determinants of the equilibrium innovation possibility frontier consider the following example.

[15]This line of reasoning is familiar from the factor price frontier introduced by Samuelson (1962).

Example 1 *Suppress time arguments and let*

$$F(M,N) = \Gamma \cdot M^{\alpha} \cdot N^{1-\alpha} \quad and \quad i^j\left(q^j\right) = \gamma^j \cdot \frac{\left(q^j\right)^2}{2}, \quad \gamma^j > 0, \quad j = A, B.$$

Here, $\Gamma > 0$ may reflect cross-country differences in geography, technical and social infrastructure that affect the transformation of tasks into the final good. In this example, we allow for innovation outlays to differ across task types, i. e., to achieve $q^B(m) > 0$ the firm must invest $\gamma^B\left(q^B(m)\right)^2/2$ units of the final good whereas to achieve $q^A(n) > 0$ the firm must invest $\gamma^A\left(q^A(n)\right)^2/2$ units and $\gamma^A \neq \gamma^B$ is permissible.

If we transcend the analytical framework of the model and admit that the final good and the resources needed for innovation investments are different goods then the parameters γ^A and γ^B have an interpretation as the price of the respective investment good in terms of the final good. Moreover, a fall in either price may be the consequence of some exogenous technical progress.

For this setting, the cost-minimizing productivity growth rates (3.17) and (3.18) of Proposition 1 are equal to

$$q^B = g^B(\kappa) = \frac{1}{3}\left(-1 + \sqrt{1 + \frac{6\Gamma\alpha}{\gamma^B}\kappa^{\alpha-1}}\right),$$

$$q^A = g^A(\kappa) = \frac{1}{3}\left(-1 + \sqrt{1 + \frac{6\Gamma(1-\alpha)}{\gamma^A}\kappa^{\alpha}}\right).$$

As expected, $g^A_\kappa(\kappa) > 0 > g^B_\kappa(\kappa)$. Moreover, both productivity growth rates increase in Γ and decline in γ_B or γ_A.

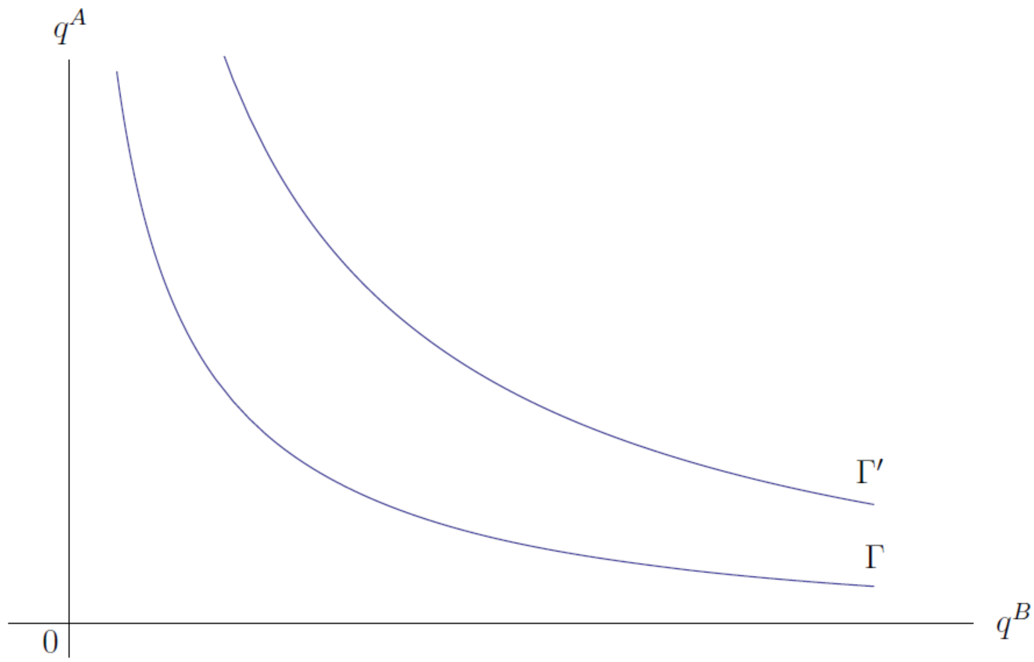
The time-invariant equilibrium innovation possibility frontier is given by

$$q^A = \frac{1}{3}\left(-1 + \sqrt{1 + \frac{6\Gamma(1-\alpha)}{\gamma^A}\left(\frac{2\Gamma\alpha}{\gamma^B q^B\left(3q^B+2\right)}\right)^{\frac{\alpha}{1-\alpha}}}\right).$$

Hence, given q^B, a greater Γ and lower values for γ_B and γ_A imply a greater q^A. Figure 3.1 shows that the EIPF is indeed convex.[16]

[16] Figure 3.1 uses the following values for the parameters that show up in Example 1: $\alpha = 1/2$, $\gamma^A = \gamma^B = \Gamma = 1$, and $\Gamma' = 2$. The *mathematica* notebook is available upon request.

Figure 3.1: Two Convex Equilibrium Innovation Possibility Frontiers for $\Gamma' > \Gamma$.



3.3 The Evolution of Technological Knowledge

The evolution of the technological knowledge to which firms have access is given by the evolution of the aggregate task-specific productivity indicators A_t and B_t. We stipulate that A_t and B_t correspond to the highest level of labor productivity and capital productivity attained across all tasks of the respective type at t, i. e.,

$$
\begin{aligned}
A_t &= \max \left\{ a_t(n) = A_{t-1} \left(1 - \delta^A\right) \left(1 + q_t^A(n)\right) \mid n \in [0, n_t] \right\}, \\
B_t &= \max \left\{ b_t(m) = B_{t-1} \left(1 - \delta^B\right) \left(1 + q_t^B(m)\right) \mid m \in [0, m_t] \right\}.
\end{aligned} \tag{3.26}
$$

Firm's optimization implies $q_t^B(m) = q_t^B$ and $q_t^A(n) = q_t^A$, as well as $a_t(n) = a_t$ and $b_t(m) = b_t$ so that

$$
\begin{aligned}
A_t &= a_t = A_{t-1} \left(1 - \delta^A\right) \left(1 + q_t^A\right), \\
B_t &= b_t = B_{t-1} \left(1 - \delta^B\right) \left(1 + q_t^B\right),
\end{aligned} \tag{3.27}
$$

for all $t = 0, 1, 2, \cdots$ with $A_{-1} > 0$ and $B_{-1} > 0$ given.

4 Dynamic Competitive Equilibrium

4.1 Definition

The dynamic competitive equilibrium is defined as follows.

Definition 1 *Given $L_t = L$, initial values of the capital stock, $K_0 > 0$, and of technological knowledge, $A_{-1} > 0$ and $B_{-1} > 0$, a dynamic competitive equilibrium is a sequence*

$$\left\{M_t, N_t, k_t(m), l_t(n), q_t^B(m), q_t^A(n), A_t, B_t, w_t, R_t, C_t, K_{t+1}, Y_t\right\}_{t=0}^{\infty},$$

for all $m \in [0, M_t]$ and $n \in [0, N_t]$, such that

(E1) the behavior of the representative household is described by (3.2), (3.3), (3.4), and $K_0 > 0$.

(E2) the production sector satisfies Proposition 1,

(E3) for all t, both factor markets clear, i. e.,

$$\int_0^{M_t} k_t(m)dm \leq K_t \quad \text{and} \quad \int_0^{N_t} l_t(n)dn \leq L,$$

each holding as equality if the corresponding factor price is strictly positive,

(E4) for all t, the market for the final good clears,

(E5) the productivity indicators A_t and B_t evolve according to equation (3.27).

Condition $(E1)$ requires household optimization while $(E2)$ ensures optimal behavior of firms and zero profits. Since profit-maximization implies $k_t(m) = k_t = 1/b_t = 1/B_t$ and $l_t(n) = l_t = 1/a_t = 1/A_t$, and equilibrium factor prices will be strictly positive $(E3)$ requires full employment of capital and labor. This condition determines the total number of each task type to be equal to the amount of the respective production factor in efficiency units, i. e.,

$$M_t = B_t K_t \quad \text{and} \quad N_t = A_t L_t. \tag{4.1}$$

Market clearing in the market for the final good, (E4), requires

$$K_{t+1} = F(M_t, N_t) - M_t i\left(q_t^B\right) - N_t i\left(q_t^A\right) - C_t + \left(1 - \delta^K\right) K_t, \tag{4.2}$$

i. e., next period's capital stock is equal to the surviving capital and the difference between total output of the final good and the investment outlays for all tasks performed by capital, $M_t i\left(q_t^B\right)$, the investment outlays for all tasks performed by labor, $N_t i\left(q_t^A\right)$, and consumption, C_t.

For further reference, let us define the notion of the economy's *net output* at t as the difference between total output of the final good and total investment outlays for all tasks and denote it by V_t. Depending on where this difference is evaluated we distinguish the following refinements. First, *net output at cost-minimizing productivity growth rates* evaluates this difference at $\left(q_t^B, q_t^A\right)$, i. e.,

$$V_t = V\left(M_t, N_t, q_t^B, q_t^A\right) \equiv F(M_t, N_t) - M_t i\left(q_t^B\right) - N_t i\left(q_t^A\right), \quad (4.3)$$

where $V : \mathbb{R}_{++}^4 \to \mathbb{R}_{++}$. Second, *net output at clearing factor markets* adds the two factor market clearing conditions of (4.1) so that

$$V_t = V\left(B_t K_t, A_t L, q_t^B, q_t^A\right) \equiv F\left(B_t K_t, A_t L\right) - B_t K_t i\left(q_t^B\right) - A_t L i\left(q_t^A\right), \quad (4.4)$$

where $V : \mathbb{R}_{++}^4 \to \mathbb{R}_{++}$. Third, there is the notion of *equilibrium net output* where (4.4) is evaluated at $\left(q_t^B, q_t^A\right) = \left(g^B(\kappa_t), g^A(\kappa_t)\right)$. In other words, equilibrium net output evaluates the economy's net output at Proposition 1 and (4.1). This gives

$$V_t = V\left(B_t K_t, A_t L, \kappa_t\right) \equiv F\left(B_t K_t, A_t L\right) - B_t K_t i\left(g^B(\kappa_t)\right) - A_t L i\left(g^A(\kappa_t)\right), \quad (4.5)$$

where $V : \mathbb{R}_{++}^3 \to \mathbb{R}_{++}$. Since $V\left(B_t K_t, A_t L, \kappa_t\right)$ exhibits constant returns to scale in $\left(B_t K_t, A_t L\right)$, we may also define the *equilibrium net output per unit of efficient labor*, $v_t = V_t/(A_t L_t)$, as

$$v_t = v(\kappa_t) \equiv f(\kappa_t) - \kappa_t i\left(g_t^B(\kappa_t)\right) - i\left(g_t^A(\kappa_t)\right), \quad (4.6)$$

where $v(\kappa_t) = V(\kappa_t, 1, \kappa_t)$ and $v : \mathbb{R}_{++} \to \mathbb{R}_{++}$.

Finally, observe that $(E2)$, $(E3)$, and $(E5)$, imply that in equilibrium the task intensity of equation (3.6), may be expressed as

$$\kappa_t = \frac{B_t K_t}{A_t L} = \frac{B_{t-1}\left(1-\delta^B\right)\left(1+g^B(\kappa_t)\right)K_t}{A_{t-1}\left(1-\delta^A\right)\left(1+g^A(\kappa_t)\right)L}. \quad (4.7)$$

Thus, in equilibrium the task intensity is equal to the ratio of efficient capital to efficient labor, or, for short, to the *efficient capital intensity*. Since innovations are induced, the respective efficiency units depend on the task intensity. The following proposition establishes that there is a unique value $\kappa_t > 0$ that satisfies (4.7). We refer to this value as the *equilibrium task intensity*.

Proposition 3 *There is a unique* equilibrium task intensity $\kappa_t > 0$ *that satisfies (4.7). Moreover, there is a function* $\kappa : \mathbb{R}_{++} \to \mathbb{R}_{++}$ *such that*

$$\kappa_t = \kappa\left(\frac{B_{t-1}\left(1-\delta^B\right)K_t}{A_{t-1}\left(1-\delta^A\right)L}\right) \quad \textit{with} \quad \kappa'\left(\frac{B_{t-1}\left(1-\delta^B\right)K_t}{A_{t-1}\left(1-\delta^A\right)L}\right) > 0.$$

The ratio $B_{t-1}\left(1-\delta^B\right) K_t / \left(A_{t-1}\left(1-\delta^A\right) L\right)$ has an interpretation as the efficient capital intensity of period t *before any investment activity is undertaken*. It is the correct measure of the relative scarcity of factors of production at t to which firms' investment behavior responds. In line with Hicks' famous assertion (Hicks (1932), p. 124) this ratio induces the degree to which firms will engage in capital- and labor-augmenting technical change. To see this, suppose the economy enters period t with $B_{t-1}\left(1-\delta^B\right) K_t / A_{t-1}\left(1-\delta^A\right) L > B_{t-2}\left(1-\delta^B\right) K_{t-1} / A_{t-2}\left(1-\delta^A\right) L$. Then, (efficient) labor has become scarcer between period $t-1$ and t. Moreover, as $\kappa'(\cdot) > 0$ we have $\kappa_t > \kappa_{t-1}$, and, in accordance with Proposition 1, $q_t^A > q_{t-1}^A$, $q_t^B < q_{t-1}^B$, $w_t > w_{t-1}$, and $R_t < R_{t-1}$.

4.2 The Canonical Dynamical System

The canonical dynamical system comprises two state variables, the equilibrium task intensity, κ_t, and the stock of capital-augmenting technological knowledge, B_t, and one control variable, the level of consumption per unit of efficient labor, $c_t \equiv C_t / (A_t L)$. The following proposition has the complete description of this system.

Proposition 4 *(Canonical Dynamical System)*

Given $L > 0$ and initial conditions $(A_{-1}, B_{-1}, K_0) > 0$, the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$ that satisfies

$$\kappa_{t+1} = \frac{\left(1-\delta^B\right)\left(1+g^B(\kappa_{t+1})\right)}{\left(1-\delta^A\right)\left(1+g^A(\kappa_{t+1})\right)} \cdot \left[B_t\left(v\left(\kappa_t\right)-c_t\right) + \left(1-\delta^K\right)\kappa_t\right], \tag{4.8}$$

$$\frac{c_{t+1}}{c_t} = \beta \cdot \frac{B_t\left(1-\delta^B\right)\left(1+g^B(\kappa_{t+1})\right)\left(f'\left(\kappa_{t+1}\right) - i\left(g^B\left(\kappa_{t+1}\right)\right)\right) + \left(1-\delta^K\right)}{\left(1-\delta^A\right)\left(1+g^A(\kappa_{t+1})\right)}, \tag{4.9}$$

$$B_t = B_{t-1}\left(1-\delta^B\right)\left(1+g^B\left(\kappa_t\right)\right), \tag{4.10}$$

the transversality condition

$$\lim_{t\to\infty} \beta^t \frac{\kappa_{t+1}\left(1+g^A\left(\kappa_{t+1}\right)\right)}{c_t B_{t+1}} = 0, \tag{4.11}$$

and for $t = 0$,

$$\kappa_0 = \kappa\left(\frac{B_{-1}\left(1-\delta^B\right) K_0}{A_{-1}\left(1-\delta^A\right) L}\right). \tag{4.12}$$

Intuitively, equation (4.8) reflects the economy's resource constraint. It is obtained from the household's budget constraint, (3.2), using (3.15), (3.16), and Claim 1 of Proposition 1. Equation (4.9) is the Euler equation. It results from (3.3) in conjunction with Claim 1 of Proposition 1 and (3.15). Finally, equation (4.10) states the evolution of capital-augmenting technological knowledge. It obtains from (3.27) and Claim 1 of Proposition 1. In conjunction with the transversality condition and the set of initial conditions, these equations form a three-dimensional system of first-order, non-linear difference equations and characterize a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$.[17]

4.3 Steady State Analysis

Definition 2 *A steady state is a path along which all variables grow at constant, but possibly different rates.*

We use an asterisk to denote steady-state variables, e. g., g^* is the steady-state growth rate of the economy. To guarantee the existence of a steady state with strictly positive and finite state variables we make the following two assumptions:

Assumption 1 *It holds that*

$$\lim_{\kappa \to 0} f'(\kappa) > c\left(\frac{\delta^B}{1-\delta^B}\right) > \lim_{\kappa \to \infty} f'(\kappa). \tag{4.13}$$

Assumption 2 *It holds that*

$$1 - \delta^A > \beta\left(1 - \delta^K\right). \tag{4.14}$$

The discussion of the following proposition will reveal the significance of these assumptions.

Proposition 5 *(Steady State)*

[17] To develop an intuitive understanding of the mechanics of the dynamical system start with the initial conditions. They deliver κ_0 from equation (4.12). Using this and $B_{-1} > 0$ in (4.10) gives a unique $B_0 > 0$. The resource constraint describes a relation between c_0 and κ_1 for any given pair $(\kappa_0, B_0) \in \mathbb{R}^2_{++}$. For any such pair the transversality condition pins down the initial choice of consumption c_0. Then, the resource constraint delivers a unique $\kappa_1 > 0$, whereas the Euler equation determines a unique c_1. Mutatis mutandis, the same reasoning applies to all periods $t > 0$.

1. *The dynamical system of Proposition 4 has a unique steady state involving* $(\kappa^*, B^*, c^*) \in \mathbb{R}^3_{++}$ *if Assumptions 1 and 2 hold. The steady state is a solution to*

$$c^* = v(\kappa^*) - \frac{\kappa^*}{B^*}\left(g^* + \delta^K\right), \tag{4.15}$$

$$B^* = \frac{(1+g^*) - \beta\left(1-\delta^K\right)}{\beta\left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)}, \tag{4.16}$$

$$g^B(\kappa^*) = \frac{\delta^B}{1-\delta^B}. \tag{4.17}$$

2. *The steady-state growth rate of the economy is*

$$g^* \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1-\delta^A\right)\left(1+g^A(\kappa^*)\right) - 1.$$

Moreover, along the steady-state path, it holds that

a) $$\frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{M_{t+1}}{M_t} = \frac{N_{t+1}}{N_t} = \frac{w_{t+1}}{w_t} = 1 + g^*,$$

b) $$R^* = B^*\left(f'(\kappa^*) - i\left(\frac{\delta^B}{1-\delta^B}\right)\right), \quad k^* = \frac{1}{B^*}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*}.$$

According to Statement 1 of Proposition 5 there is a unique steady state if Assumption 1 and 2 hold. To develop the explanation for this finding, start with the evolution of capital-augmenting technological knowledge of (4.10). Accordingly, any trajectory with $B_t/B_{t-1} - 1 = const.$ requires $\kappa_t = \kappa_{t+1} = \kappa^*$. Intuitively, the level of κ^* must be such that profit-maximizing firms undertake innovation investments that generate new capital-augmenting technological knowledge just enough to offset its depreciation. Invoking Proposition 1 this is the case if $\left(q^B\right)^* = g^B(\kappa^*) = \delta^B/(1-\delta^B)$ as stated in (4.17).

Assumption 1 assures that (4.17) has a solution $\kappa^* \in \mathbb{R}_{++}$. To see this, observe that a choice of $\left(q^B\right)^*$ means that (minimized) costs per task performed by capital are equal to $c\left(\left(q^B\right)^*\right) = c\left(\delta^B/(1-\delta^B)\right)$. Hence, for tasks $m < M_t$ the marginal value product, $f'(\kappa)$, must exceed, for $m > M_t$ it must fall short of these costs. This is what condition (4.13) guarantees.[18]

In steady state, the Euler equation makes sure that the household's desired consumption growth rate is equal to the growth rate of the economy. As discussed below, the latter satisfies $(1+g^*) = \left(1-\delta^A\right)\left(1+g^A(\kappa^*)\right)$. Then, using (3.19) and the fact that κ^* is pinned down by (4.17), the Euler equation determines B^* as a solution to $(1+g^*) =$

[18]Notice that a value $\kappa^* \in (0,\infty)$ would always exist if we had imposed the usual Inada conditions on F.

$\beta\left(R\left(\kappa^*, B\right)+1-\delta^K\right)$. Assumption 2 provides a sufficient condition for a solution $B^* > 0$ to exist. Intuitively, it assures that the numerator in (4.16) is strictly positive even if κ^* and $g^A\left(\kappa^*\right)$ are very small which is the case if δ^B is very large.

Finally, given $\left(\kappa^*, B^*\right)$, the resource constraint (4.15) determines a finite $c^* > 0$ as the difference between net output per unit of efficient labor and the required capital investment per unit of efficient labor needed to keep κ^* constant.

Statement 2 of Proposition 5 establishes that the steady-state growth rate of the economy is given by the growth rate of labor-augmenting technological knowledge. Absent population growth, this is the growth rate of final-good output, net output, consumption, capital, the respective total number of tasks, and of the real wage. The steady-state rental rate of capital is constant. Moreover, the input coefficients of capital are constant, whereas the one of labor hinges on g^*.

Why is it that in steady state only the stock of labor-augmenting technological knowledge evolves whereas the one of capital-augmenting technological knowledge stagnates? The conceptual answer to this question provides the "Generalized Steady-State Growth Theorem" developed in Irmen (2013b). This theorem generalizes Uzawa's Theorem (Uzawa (1961)) to settings where technical change has a cost in terms of current final-good output. Roughly speaking, it says that an economy where capital accumulates and the equilibrium net output function has constant returns to scale in capital and labor, technical change must be labor-augmenting in steady state whereas capital-augmenting technical change must vanish. We have seen that the equilibrium net output function of (4.5) has this property. Hence, in steady state, $B_t = B^*$ and A_t evolves at rate g^*.

Observe that the steady-state growth rate of the economy may be negative, i. e., $g^* \leq 0$. Intuitively, this is the case if $g^A\left(\kappa^*\right) \leq \delta^A / \left(1-\delta^A\right)$ which may be satisfied if κ^* is small due to a large δ^B. However, as $B^* > 0$ is required, the Euler equation (4.16) implies a lower bound on steady-state growth rate, i. e., $g^* > \beta(1-\delta^K)-1$.

Next, we turn to the comparative static properties of the steady state.

Proposition 6 *(Comparative Statics of the Steady State)*

1. *Consider two economies that differ only with respect to their discount factor such that $\beta' > \beta$. Then, their steady states satisfy*

$$\left(\kappa^*\right)' = \kappa^*, \quad \left(g^*\right)' = g^*, \quad \left(B^*\right)' < B^*, \quad \left(R^*\right)' < R^*, \quad \textit{and} \quad \left(c^*\right)' < c^*. \tag{4.18}$$

2. *Consider two economies that differ only with respect to their depreciation rate of the stock of capital-augmenting technological knowledge such that* $(\delta^B)' > \delta^B$. *Then, their steady states satisfy*

$$(\kappa^*)' < \kappa^*, \quad (g^*)' < g^*, \quad (B^*)' < B^*, \quad (R^*)' < R^*, \quad \textit{and} \quad (c^*)' \lesseqgtr c^*. \tag{4.19}$$

3. *Consider two economies that differ only with respect to their depreciation rate of the stock of labor-augmenting technological knowledge such that* $(\delta^A)' > \delta^A$. *Then, their steady states satisfy*

$$(\kappa^*)' = \kappa^*, \quad (g^*)' < g^*, \quad (B^*)' < B^*, \quad (R^*)' < R^*, \quad \textit{and} \quad (c^*)' > c^*. \tag{4.20}$$

Arguably, Statement 1 has the most important result of Proposition 5. The steady-state growth rate of the economy, g^*, does not hinge on characteristics of the household sector. Intuitively, this is so since the discount factor neither interferes with the incentives to engage in innovation investments as summarized by the function g^B nor does it directly affect the evolution of the stocks of technological knowledge. However, differences in the discount factor have level effects. The more patient economy is predicted to have the lower steady-state level of capital-augmenting technological knowledge, or, equivalently, a lower rental rate of capital. This follows from the steady-state Euler equation (4.16) that requires both economies to have consumption grow at the same rate $(g^*)' = g^*$. Then, a greater discount factor must be entirely offset by a decline in the rental rate of capital. This is accomplished through a decline in capital-augmenting technological knowledge. Finally, with $(B^*)' < B^*$ the resource constraint (4.15) requires the more patient economy to reduce its consumption per unit of efficient labor. Intuitively, since κ^* is the same in both economies it must be that $\left((K_t/A_tL)^*\right)' > (K_t/A_tL)^*$, i. e., capital per efficient labor is greater in the more patient economy. Therefore, more current output is needed to keep this ratio constant. Accordingly, $(c^*)' < c^*$.

Statements 2 and 3 highlight that differences in the depreciation rate of the stock of factor-augmenting technological knowledge generate growth effects. First, consider the steady-state effects of changing δ^B. A higher depreciation rate of capital-augmenting technological knowledge requires stronger private incentives to engage in innovation investments that raise the productivity of capital. Hence, in line with Proposition 1 and (4.17) the efficient capital intensity must fall, i. e., $(\kappa^*)' < \kappa^*$. This weakens the private incentives to engage in innovation investments that raise the productivity of labor so that $(g^*)' < g^*$. As consumption must grow at the latter rate the Euler condition requires $(R^*)' < R^*$. This adjustment implies $(B^*)' < B^*$. Finally, the effect on the steady-state consumption per unit of efficient labor remains indeterminate in general.

Second, consider changes in δ^A of Statement 3. As these changes leave (4.17) unaffected, we have $(\kappa^*)' = \kappa^*$, and $(g^*)' < g^*$ is due to faster depreciation of the stock of labor-augmenting technological knowledge. As above, slower growth of consumption requires adjustments in the Euler equation and the resource constraint. They lead to $(B^*)' < B^*$, $(R^*)' < R^*$, and $(c^*)' > c^*$.

Additional and intuitive comparative static results are obtained if we impose more structure.

Example 2 *Reconsider the setup of Example 1, and suppose that Assumption 2 holds. Then, a unique steady state exists and involves*

$$\kappa^* = \left(\frac{2\Gamma \alpha \tilde{\delta}^B}{\gamma^B} \right)^{\frac{1}{1-\alpha}}, \tag{4.21}$$

where $\tilde{\delta}^B = \left(1 - \delta^B\right)^2 / \left(\delta^B \left(2 + \delta^B\right)\right)$.

To interpret this equation recall that κ^* is the task intensity necessary to sustain innovation investments such that $\left(q^B\right)^* = \delta^B / \left(1 - \delta^B\right)$. Moreover, from Proposition 1 a higher κ reduces q^B. Then, it is straightforward to see why κ^* increases in the productivity parameter Γ and decreases in the cost parameter γ^B or in the depreciation rate δ^B. Since $g^* = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa^*\right)\right) - 1$ we arrive at

$$g^* = \frac{\left(1 - \delta^A\right)}{3} \left(2 + \sqrt{1 + \frac{6\Gamma(1-\alpha)}{\gamma^A} \left(\frac{2\Gamma \alpha \tilde{\delta}^B}{\gamma^B} \right)^{\frac{\alpha}{1-\alpha}}} \right) - 1. \tag{4.22}$$

Here, Γ exerts two positive effects on g^*. First, there is a direct effect since innovation incentives are higher the more productive the aggregate production function is. Second, there is a general equilibrium effect since κ^* also increases. A greater γ^A or δ^A has a direct negative effect on g^*, greater values for γ^B or δ^B reduce g^* through negative general equilibrium effects on κ^*.

Finally, consider the numerical reference case where $\Gamma = \gamma^A = \gamma^B = 1$, $\alpha = 1/2$, $\delta^B = 1/4$, and $\delta^A = 1/4 - \varepsilon$. Then, $\tilde{\delta}^B = 1$, $\left(q^A\right)^* = \left(q^B\right)^* = 1/3$, $\kappa^* = 1$ and

$$g^* = \frac{4}{3}\varepsilon. \tag{4.23}$$

Hence, if $\varepsilon = 0.015$ then $g^* = 0.02$.

Finally, let us note that the steady state of Proposition 5 is consistent with Kaldor's facts if $g^* > 0$ (see, Kaldor (1961)). Indeed, one readily verifies that the productivity of labor, measured either by a_t, V_t/L, or Y_t/L and capital per worker, K_t/L, grow at rate $g^* > 0$. Moreover, the capital coefficient in aggregate output, K_t/Y_t, or in aggregate net output,

K_t/V_t, and the return on capital are stable. What remains to be shown is that the functional income distribution is also stable in steady state. We defer this proof to Section 5 and turn now to the local stability properties of the steady state.

Proposition 7 *(Local Stability of the Steady State)*

The steady-state equilibrium of Proposition 5 is asymptotically locally stable in the state space.

To establish Proposition 7 the nonlinear dynamical system of Proposition 4 is linearly approximated around its steady-state equilibrium (κ^*, c^*, B^*). Given $L > 0$ and initial values $(A_{-1}, B_{-1}, K_0) > 0$ the state of the economy in period t is fully described by the two state variables κ_t and B_t, so that c_t is the only control variable. Hence, given the two state variables we need a stable eigenspace of dimension two for the dynamical system to exhibit a unique convergent path toward the steady state. The proof of Proposition 7 shows that the linearized dynamical system has two stable eigenvalues and one unstable eigenvalue. Observe that the convergence toward the steady state may be monotonic or oscillatory.

To gain intuition for Proposition 7 consider the case of a possible monotonous convergence. Initially, the economy is in its steady-state equilibrium (κ^*, c^*, B^*). At the beginning of some period t an exogenous event destroys a part of its capital-augmenting technological knowledge so that $B'_{t-1} < B^*$. Then, compared to the steady state the efficient capital intensity before any investment activity is undertaken at t falls. In other words, at the beginning of period t efficient capital is relatively scarcer than along the steady-state path. Accordingly, form Proposition 3 we have $\kappa'_t < \kappa^*$, and with Proposition 1 it follows that $\left(q_t^B\right)' > \left(q^B\right)^*$ and $\left(q_t^A\right)' < \left(q^A\right)^*$. This leads to $B'_{t-1} < B'_t < B^*$ and $A'_t < A^*_t$. Hence, the immediate effect of induced technical change is to partly offset the initial loss of capital-augmenting technical knowledge. For the periods that follow, $\kappa'_t < \kappa^*$ triggers a process of capital accumulation so that the sequences $\left\{\kappa'_{t+i}\right\}_{i=1}^{\infty}$ and $\left\{B'_{t+i}\right\}_{i=1}^{\infty}$ monotonically converge to κ^* and B^*, respectively.

Contrary to the existing literature on endogenous capital- and labor-augmenting technical change, the local stability of the steady-state equilibrium does not require the elasticity of substitution of the production function to be less than unity. Indeed, Proposition 7 holds irrespective of the elasticity of substitution.[19] To highlight this point consider the following numerical example. Notice in passing that here the elasticity of substitution

[19]We show in Section 5.1.1 that there is an important distinction between the elasticity of substitution of the production function and the one of the net output function. None of these two concepts bears on Proposition 7.

plays not only a role for the type of convergence to the steady-state equilibrium but also for the steady-state growth rate.

Example 3 *Reconsider the economy described in Example 1. We now choose the following parameter values:* $\Gamma = \gamma^A = \gamma^B = 1$, $\alpha = 1/3$, $\delta^B = 1/4$, $\delta^A = 1/4$, $\beta = 0.99$, *and* $\delta^K = 0.06$. *Furthermore, we allow for the production function to be of the CES-type, i. e.,*

$$F(M,N) = \begin{cases} \Gamma\left(\alpha M^{\frac{\sigma-1}{\sigma}} + (1-\alpha)N^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} & \textit{if} \quad \sigma \neq 1, \\ \\ \Gamma \cdot M^{\alpha} \cdot N^{1-\alpha} & \textit{if} \quad \sigma = 1, \end{cases}$$

where $\sigma > 0$ *is the elasticity of substitution between the two types of tasks.*

Table 1 shows the eigenvalues of the dynamical system of Proposition 4 linearized around the steady state of Proposition 5 for various values of the elasticity of substitution. For different values of σ *the convergence to the steady-state equilibrium may be monotonic or oscillatory. Notice also that the variables do not need to converge at the same speed to their steady-state levels.*

Finally, observe that the steady-state growth rate increases with the elasticity of substitution thus confirming the analytical finding established in Irmen (2011).We emphasize that the aim of our numerical example is not to calibrate the model, but rather to show the qualitative results.

Table 1: Elasticity of Substitution of the Production Function, Local Stability of the Steady State, and Steady-State Growth.

σ	eigenvalues			$g^*(\%)$
0.25	1.07941,	0.949732,	0.561591	0.51
0.5	1.09029,	0.939364,	0.71533	0.94
0.75	1.10048,	0.926904,	0.790202	1.31
1	1.11007,	0.908942,	0.841108	1.64
1.25	1.11917,	0.884550 ± 0.0280296i		1.92
1.5	1.12791,	0.890272 ± 0.0448385i		2.18
1.75	1.13637,	0.893694 ± 0.0535632i		2.41
2	1.14464,	0.895618 ± 0.0593213i		2.61

4.4 The Role of Capital-Augmenting Technical Change

What is the role of capital-augmenting technical change for the results derived so far? To address this question we contrast the model of the previous sections with a version that altogether dispenses with capital-augmenting technical change. To accomplish this, reconsider the model of Section 3 for $B_t = b_t = k_t = 1$, $q_t^B = \delta^B = i\left(q_t^B\right) = 0$, and $M_t = K_t$. Then, the "efficient capital intensity" of (4.7) becomes $\kappa_t = K_t / \left(A_t L\right)$, and we refer to it as the *capital-labor ratio in efficiency units*. Mutatis mutandis, the latter ratio also satisfies Proposition 3.

With these changes, the dynamical system of Proposition 4 reduces to a two-dimensional system of non-linear first-order difference equations involving one state variable, κ_t, and one control variable, c_t. These difference equations include the resource constraint and the Euler condition.

Proposition 8 *(Canonical Dynamical System without Capital-Augmenting Technical Change)*

Given $L > 0$ and initial conditions $(A_{-1}, K_0) > 0$, the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t\}_{t=0}^{\infty}$ that satisfies

$$\kappa_{t+1} = \frac{v(\kappa_t) - c_t + \left(1 - \delta^K\right)\kappa_t}{\left(1 - \delta^A\right)\left(1 + g^A(\kappa_{t+1})\right)}, \tag{4.24}$$

$$\frac{c_{t+1}}{c_t} = \beta \cdot \frac{f'(\kappa_{t+1}) + 1 - \delta^K}{\left(1 - \delta^A\right)\left(1 + g^A(\kappa_{t+1})\right)}, \tag{4.25}$$

the transversality condition

$$\lim_{t\to\infty} \beta^t \frac{\kappa_{t+1}\left(1 + g^A\left(\kappa_{t+1}\right)\right)}{c_t} = 0, \tag{4.26}$$

and for $t = 0$,

$$\kappa_0 = \kappa\left(\frac{K_0}{A_{-1}\left(1 - \delta^A\right)L}\right). \tag{4.27}$$

From Definition 2 a steady state involves $A_{t+1}/A_t - 1 = const$. Hence, with (3.27) and Proposition 1 we have $\kappa_t = \kappa_{t+1} = \kappa^*$ and $\left(q^A\right)^* = g^A\left(\kappa^*\right)$. Moreover, from (4.24) we obtain $c_t = c_{t+1} = c^*$. To guarantee the existence of a steady state with $\kappa^* \in \mathbb{R}_{++}$, Assumptions 1 and 2 must be replaced by

Assumption 3 *It holds that*

$$\lim_{\kappa\to 0} f'(\kappa) > \frac{1 - \delta^A}{\beta} - \left(1 - \delta^K\right) > \lim_{\kappa\to\infty} f'(\kappa). \tag{4.28}$$

The significance of Assumption 3 will become clear below.

Proposition 9 *(Steady State without Capital-Augmenting Technical Change)*

1. *The dynamical system of Proposition 8 has a unique steady state involving* $(\kappa^*, c^*) \in \mathbb{R}^2_{++}$ *if Assumption 3 holds. The steady state is a solution to*

$$c^* = v(\kappa^*) - \kappa^* \left(g^* + \delta^K\right), \tag{4.29}$$

$$1 + g^* = \beta \left(f'(\kappa^*) + 1 - \delta^K\right). \tag{4.30}$$

2. *The steady-state growth rate of the economy is*

$$g^* \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1 - \delta^A\right)\left(1 + g^A(\kappa^*)\right) - 1.$$

Moreover, along the steady-state path, it holds that

a) $$\frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{N_{t+1}}{N_t} = \frac{w_{t+1}}{w_t} = 1 + g^*,$$

b) $$R^* = f'(\kappa^*), \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^*}.$$

The comparison of the steady state with and without capital-augmenting technical change (Proposition 5 versus Proposition 9) reveals two important structural differences. First, in a world without capital-augmenting technical change the steady-state capital-labor ratio in efficiency units is pinned down by the Euler condition (4.30). Hence, κ^* is such that the desired growth rate of consumption (and of the economy as a whole), g^*, is supported by the steady-state rental rate of capital, $R^* = f'(\kappa^*)$. This alignment gives rise to a solution $\kappa^* > 0$ if Assumption 3 holds and will naturally depend on β.

On the contrary, in the world with capital-augmenting technical change, the role of κ^* is to induce innovation investments so that the stock of capital-augmenting technological knowledge remains constant over time. The necessary adjustments to meet this requirement reflect only the characteristics of the production sector including the way how technological knowledge accumulates. As a consequence condition (4.17) is independent of β.

Second, the comparative statics with respect to β and δ^A of the steady state of Proposition 9 involve adjustments in κ^*. In contrast, for the steady state with capital-augmenting technical change of Proposition 5 changes in the same parameters induce adjustments in B^*. The following proposition documents the resulting differences.

Proposition 10 *(Comparative Statics of the Steady State without Capital-Augmenting Technical Change)*

1. *Consider two economies that differ only with respect to their discount factor such that $\beta' > \beta$. Then, their steady states satisfy*

$$(\kappa^*)' > \kappa^*, \quad (g^*)' > g^*, \quad (R^*)' < R^*, \quad \text{and} \quad (c^*)' \gtreqless c^*. \tag{4.31}$$

2. *Consider two economies that differ only with respect to their depreciation rate of the stock of capital-augmenting technological knowledge such that $\left(\delta^A\right)' > \delta^A$. Then, their steady states satisfy*

$$(\kappa^*)' > \kappa^*, \quad (g^*)' < g^*, \quad (R^*)' < R^*, \quad \text{and} \quad (c^*)' \gtreqless c^*. \tag{4.32}$$

Hence, compared to Statements 1 and 3 of Proposition 6 there are two main differences. First, the more patient economy grows faster in steady state. Second, a the negative effect on the steady-state growth rate of the economy associated with a greater depreciation rate of labor-augmenting technological knowledge is partly offset by an increase in κ^*.[20]

5 Positive Implications: Functional Income Distribution and Fiscal Policy

This section establishes important positive implications of fiscal policy measures on the functional income distribution and on economic growth. The focus is on three different policy measures. First, we study the effects of a linear tax on the return to capital. Second, we analyze the role of subsidies for innovation investments that increase the productivity of capital. Finally, we turn to subsidies for innovation investments that increase the productivity of labor. For all three policies, the focus is on the long run, i. e., on the steady state. Throughout, we assume that the government redistributes its tax revenues in a lump-sum fashion to balance its budget. With the same purpose, the government finances its subsidies through a lump-sum tax.

5.1 Functional Income Distribution

5.1.1 Preliminary Remark on the Elasticity of Substitution

It is well known that the elasticity of substitution is a key variable in the analysis of changes in the functional income distribution. In a neoclassical competitive environment

[20]It is not difficult to see that the steady state of Proposition 9 satisfies all Kaldor's facts including the stability of the functional income distribution.

with a given technology the marginal rate of technical substitution (MRTS) is equal to the ratio of factor prices. Accordingly, if relative factor endowments change the concomitant change in the MRTS gives the effect on the factor price ratio, and the total effect on the relative factor shares, RK/wL, depends on the elasticity of substitution.

Since the technology is endogenous in our framework we need to clarify both the notion of the marginal rate of technical substitution and the notion of the elasticity of substitution that applies in the analysis of the functional income distribution. To accomplish this consider *net output at clearing factor markets* as defined in equation (4.4). Then, it is not difficult to see that in equilibrium factor prices are equal to the respective marginal net output at clearing factor markets

$$\begin{aligned} R_t &= \frac{\partial V\left(B_t K_t, A_t L, q_t^B, q_t^A\right)}{\partial K_t} = B_t\left(f'\left(\kappa_t\right) - i\left(q_t^B\right)\right), \\ w_t &= \frac{\partial V\left(B_t K_t, A_t L, q_t^B, q_t^A\right)}{\partial L} = A_t\left(f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right) - i\left(q_t^A\right)\right). \end{aligned}$$

Accordingly, the marginal rate of technical substitution is

$$MRTS_{K,L}\left(\kappa_t, q_t^B, q_t^A\right) \equiv \frac{B_{t-1}\left(1-\delta^B\right)\left(1+q_t^B\right)\left[f'(\kappa_t) - i(q_t^B)\right]}{A_{t-1}\left(1-\delta^A\right)\left(1+q_t^A\right)\left[f(\kappa_t) - \kappa_t f'(\kappa_t) - i(q_t^A)\right]} = \frac{R_t}{w_t}. \quad (5.1)$$

To capture the overall effect of changing factor endowments on (relative) factor prices we have to account for induced technical change. This requires two additional steps. First, we use Proposition 1 to express the MRTS as a function of κ_t, i. e.,

$$MRTS_{K,L}\left(\kappa_t\right) \equiv MRTS_{K,L}\left(\kappa_t, q_t^B, q_t^A\right)\Big|_{\left(q_t^B, q_t^A\right) = \left(g^B(\kappa_t), g^A(\kappa_t)\right)}. \quad (5.2)$$

Second, we know from Proposition 3 that the equilibrium task intensity, κ_t, increases in K_t/L. More precisely, one readily verifies that

$$\frac{d\ln\kappa_t}{d\ln\left(K_t/L\right)} = \frac{1}{1+\epsilon_\kappa^A\left(\kappa_t\right)+\epsilon_\kappa^B\left(\kappa_t\right)}, \quad (5.3)$$

where

$$\epsilon_\kappa^A\left(\kappa_t\right) \equiv \frac{\kappa_t g_\kappa^A\left(\kappa_t\right)}{1+g^A\left(\kappa_t\right)} > 0 \quad \text{and} \quad \epsilon_\kappa^B\left(\kappa_t\right) \equiv \frac{-\kappa_t g_\kappa^B\left(\kappa_t\right)}{1+g^B\left(\kappa_t\right)} > 0$$

denote the respective elasticity of productivity growth with respect to κ_t.

Then, the relevant elasticity of substitution is defined as

$$\sigma(\kappa_t) \equiv -\left[\frac{d\ln MRTS_{K,L}\left(\kappa_t\right)}{d\ln\kappa_t}\frac{d\ln\kappa_t}{d\ln\left(K_t/L\right)}\right]^{-1}. \quad (5.4)$$

We refer to $\sigma(\kappa_t)$ as the *equilibrium elasticity of substitution*.

Proposition 11 *(Equilibrium Elasticity of Substitution)*

The equilibrium elasticity of substitution is equal to

$$\sigma(\kappa_t) = \frac{\left[f'(\kappa_t) - i(g^B(\kappa_t))\right]\left[f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\right]}{-\kappa_t f''(\kappa_t)\left[f(\kappa_t) - \kappa_t i(g^B(\kappa_t)) - i(g^A(\kappa_t))\right]}\left(1 + \epsilon_\kappa^A(\kappa_t) + \epsilon_\kappa^B(\kappa_t)\right). \quad (5.5)$$

The equilibrium elasticity of substitution looks surprisingly simple. The main reason for this is that in equilibrium

$$\frac{\partial MRTS_{K,L}\left(\kappa_t, q_t^B, q_t^A\right)}{\partial q_t^B} = \frac{\partial MRTS_{K,L}\left(\kappa_t, q_t^B, q_t^A\right)}{\partial q_t^A} = 0. \quad (5.6)$$

In the parlance of Acemoglu (see, Acemoglu (2007), Definition 5 and 6), the technology represented by q_t^B and q_t^A does not give rise to a weak equilibrium bias since it does not induce a relative bias. Accordingly, the first factor in $\sigma(\kappa_t)$ reflects only the effects of changing κ_t on $MRTS_{K,L}\left(\kappa_t, q_t^B, q_t^A\right)$ of (5.1). Intuitively, it differs from the standard expression of the elasticity of substitution in three respects.[21] In the numerator the rental rate and the wage per efficiency unit are now $f' - i(g^B)$ and $f - \kappa f' - i(g^A)$, respectively, and take investment outlays into account. In the denominator, f is replaced by the expression for the equilibrium net output per unit of efficient labor, $f - \kappa i(g^B) - i(g^A)$.

The second factor picks up the strength with which changes in the capital-labor ratio impinge on the equilibrium task intensity κ_t. The following example reveals that $\sigma(\kappa_t)$ may change in a non-monotonous way to changes in κ_t. Moreover, depending on κ_t, $\sigma(\kappa_t)$ may exceed or fall short of unity.

Example 4 *Let F exhibit a constant elasticity of substitution equal to .95, i. e.,*

$$F(M, N) = \Gamma \cdot \left[\alpha M^{\frac{-1}{19}} + (1-\alpha) N^{\frac{-1}{19}}\right]^{19}$$

Moreover, let $\alpha = 1/2$ and $\Gamma = 1$ and $\Gamma' = 2$. Then, the equilibrium elasticity of substitution is continuous on $\mathbb{R}_+$ with

$$\sigma(\kappa) > 0.95 \quad \textit{for all } \kappa \in (0, \infty) \textit{ and } \quad \lim_{\kappa \to 0} \sigma(\kappa) = \lim_{\kappa \to \infty} \sigma(\kappa) = .95 < 1. \quad (5.7)$$

Moreover, as shown in Figure 5.1 for intermediate values of κ it holds that $\sigma(\kappa) > 1$.

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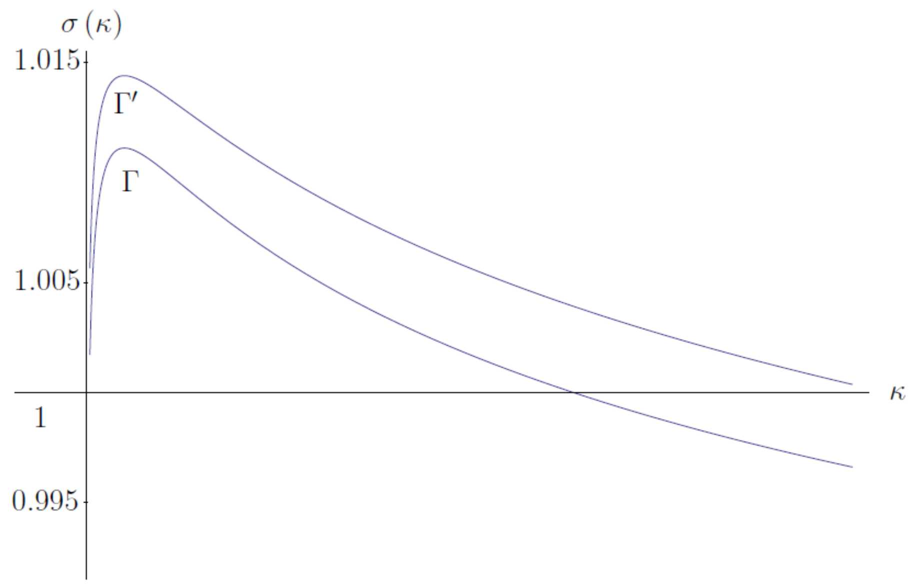
[21] Recall that the elasticity of substitution of the (gross) production function $F(B_t K_t, A_t L)$ at a given technology $\left(q_t^B, q_t^A\right)$ gives rise to the standard expression $f'(\kappa_t)\left[f(\kappa_t) - \kappa_t f'(\kappa_t)\right] / \left[-\kappa_t f''(\kappa_t) f(\kappa_t)\right]$.

Figure 5.1: Example 4: The equilibrium elasticity of substitution may be smaller or greater than unity as κ varies.



5.1.2 Functional Income Distribution and Relative Factor Endowments

The focus of this subsection is on the role of relative factor endowments for the functional income distribution. To derive this distribution, we interpret net output as the economy's gross domestic product, GDP. In equilibrium, GDP is equal to the sum of (gross) capital income, $R_t K_t$, and wage income, $w_t L$. To see this, recall that the final-good sector earns zero profits. With (3.9) this implies

$$F(M_t, N_t) - M_t i\left(q_t^B\right) - N_t i\left(q_t^A\right) = M_t \frac{R_t}{b_t} + N_t \frac{w_t}{a_t}.$$

Clearly, the left-hand side is net output. Moreover, with full employment of capital and labor we have $M_t / b_t = K_t$ and $N_t / a_t = L_t$ which turns the right-hand side into $R_t K_t + w_t L_t$. Absent of indirect taxes or subsidies, we obtain net national income, NNI_t

as $NNI_t = GDP_t - \delta^K K_t$, hence,[22]

$$NNI_t = K_t \left(R_t - \delta^K \right) + w_t L_t. \tag{5.8}$$

Thus, in equilibrium, aggregate income in absolute terms is equal to the sum of wage income of workers and net capital income of capital owners. In what follows we assume that net capital income is strictly positive, i. e., $R_t > \delta^K$. In the presence of physical capital depreciation it is important to distinguish between *factor shares*, i. e., the expenditure for production factors that accrues at the level of the firm, and the *functional income distribution*, i. e., the income that capital owners and workers actually receive.[23]

Therefore, we study the determinants of *factor shares* through the lens of labor's share, i. e., $LS_t = w_t L / GDP_t$. Upon dividing the numerator and the denominator by $A_t L$ the latter becomes $LS_t = (w_t / A_t) / v_t$, or

$$LS_t = \frac{f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right)}{f(\kappa_t) - \kappa_t i\left(g^B(\kappa_t)\right) - i\left(g^A(\kappa_t)\right)}, \tag{5.9}$$

The determinants of the *functional income distribution* are analyzed through the lens of the relative income of capital owners

$$s_t = \frac{\left(R_t - \delta^K\right) K_t}{w_t L_t}. \tag{5.10}$$

Using (5.1) the latter may be expressed as

$$s_t = \frac{\kappa_t \left[f'(\kappa_t) - i\left(g^B(\kappa_t)\right) - \frac{\delta^K}{B_t} \right]}{f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right)}. \tag{5.11}$$

Since in steady state we have $\kappa_t = \kappa^*$ and $B_t = B^*$, the steady-state factor shares as well as the functional income distribution are stable, and the steady state of Proposition 5 is shown to be consistent with Kaldor's facts.

An interesting question is then how the relative endowment with the production factors affects the distribution of these income streams.

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[22] Observe that NNI_t may also be written as $NNI_t = F(M_t, N_t) - M_t i\left(q_t^B\right) - N_t i\left(q_t^A\right) - \delta^K K_t$. This reveals two possible interpretations of our setup that turn out to be equivalent for the analysis of the functional income distribution. The first interpretation views $M_t i\left(q_t^B\right)$ and $N_t i\left(q_t^A\right)$ as machines (capital) that fully depreciate after use whereas K_t is capital that depreciates slower at rate δ^K. Then, Y_t is the total value added at t, i. e., GDP_t, and subtracting all forms of depreciation delivers net national income. The second interpretation views $M_t i\left(q_t^B\right)$ and $N_t i\left(q_t^A\right)$ as intermediates. Then, GDP_t is equal to net output, and net national income is net output minus capital allowances.

[23] See, e. g., Bertola, Foellmi, and Zweimüller (2006), Chapter 4, for a discussion of these concepts.

Proposition 12 *(Functional Income Distribution)*

Consider two economies that differ only with respect to their capital-labor ratio such that $(K_t/L)' > (K_t/L)$*. Then, in steady state the following holds.*

1. *The labor share in GDP satisfies*

$$(LS^*)' \gtreqless LS^* \quad \text{if and only if} \quad \sigma(\kappa^*) \lesseqgtr 1. \tag{5.12}$$

2. *The relative income of capital owners satisfies*

$$(s^*)' \gtreqless s^* \quad \text{if and only if} \quad \sigma(\kappa^*) \gtreqless \sigma_c$$

where

$$\sigma_c = \frac{\frac{f' - i(g^B)}{f - \kappa^* i(g^B) - i(g^A)}}{\frac{f' - i(g^B) - (\delta^K/B^*)}{f - \kappa i(g^B) - i(g^A) - (\kappa^* \delta^K/B^*)}} > 1, \tag{5.13}$$

and f*,* g^B *and* g^A *are evaluated at* κ^**.*

Statement 1 is in line with the well-known intuition that a higher capital-labor ratio increases the share of labor in GDP if and only if the elasticity of substitution between capital and labor is smaller than one. However, here what matters is the equilibrium elasticity of substitution. Constant returns to scale in the net output function then implies that the opposite is true for capital's share in GDP.

Statement 2 emphasizes the distinction between *factor shares* and the *functional income distribution* when depreciation of the capital stock is allowed for. For capital owners what matters is their net factor income, $\left(R_t - \delta^K\right) K_t$. Moreover, whether the latter increases in response to a higher capital-labor ratio depends on a critical value of the elasticity of substitution that exceeds unity for $\delta^K > 0$. In fact,

$$s_c = \frac{\frac{R_t K_t}{GDP_t}}{\frac{(R_t - \delta^K) K_t}{NNI_t}} \geq 1 \quad \text{with strict inequality whenever } \delta^K > 0.$$

Clearly, the fraction in the denominator necessarily falls as δ^K increases above zero. Hence, the share of gross capital income in GDP is greater than the share of net capital income in NNI.[24]

[24]Notice that this property of the functional income distribution is not specific to our framework with endogenous technical change.

In what follows we focus our analysis on the steady-state functional income distribution and emphasize the potential role of capital depreciation at rate $\delta^K > 0$. We show next that unlike the economy's steady-state growth rate, its functional income distribution depends on the discount factor.

Proposition 13 *(Functional Income Distribution and the Discount Factor)*

Consider two economies that differ only with respect to their discount factor such that $\beta' > \beta$. Then, in steady state, the relative income of their capital owners satisfies

$$(s^*)' < s^*. \tag{5.14}$$

Hence, the more patient economy is predicted to have a lower capital share in the long run. The intuition for (and the proof) of this finding are closely linked to Proposition 6 stating that $\beta' > \beta$ implies $(B^*)' < B^*$. Then, it is immediate from (5.11) that $(s^*)' < s^*$. Hence, contrary to what Acemoglu (2003b) claims the presence of endogenous capital-augmenting technical change does not cut off the link between the steady-state functional income distribution and the household side of the economy.

The intuition for this finding is straightforward. From (5.11) it is obvious that depreciation rate per efficiency unit of capital, δ^K / B_t matters for the relative income of capital owners. Then, Proposition 13 just says that in the more patient economy this rate will be lower since the steady-state efficiency units of capital are greater.[25]

5.2 Fiscal Policy

5.2.1 A Linear Tax on Capital

Suppose the government levies a tax on capital such that the net after-tax rate of return per unit of capital at t is $(1-\tau)\left(R_t - \delta^K\right)$, $\tau \in [0,1)$.Then, the Euler condition (3.3) becomes $C_{t+1}/C_t = \beta\left((1-\tau)\left(R_{t+1} - \delta^K\right) + 1\right)$, i. e., from the household's point of view,

[25] An alternative interpretation highlights that technological knowledge is *disembodied*. Indeed, write the steady-state net capital income as $R^* K_t - \delta^K K_t = B^*\left(f'(\kappa^*) - i\left(\delta^B / \left(1-\delta^B\right)\right)\right) K_t - \delta^K K_t$. Disembodied technological knowledge, B^*, is a determinant of the marginal value product of capital, i. e., of the rental rate of capital, R^*. However, B^* survives if capital decays. If capital depreciation was proportionate to the amount of efficient capital, e. g., if $\tilde{\delta}^K(B_t K_t)$, then the steady-state relative share of capital would only depend on κ^* and no longer on β. However, such a specification runs counter to the idea of disembodied technological knowledge that has a life (and a death) of its own.

the relative price of consumption tomorrow increases with the tax. Due to the lump-sum transfer of the government's tax receipts the household's flow budget constraint (3.2) remains unchanged.

How does this policy affect the economy's growth rate and its functional income distribution in steady state? First, observe that the tax does not affect the steady-state growth rate of the economy. According to Proposition 5, the latter is equal to the growth rate of labor-augmenting technological knowledge and hinges on $\left(q^A\right)^* = g^A\left(\kappa^*\right)$. As κ^* is determined by (4.17) and reflects only the production side of the economy, the tax on capital leaves the economy's steady-state growth rate unchanged.

However, there are steady-state level effects that also matter for the functional income distribution.

Next, consider the relative income of capital owners. In the presence of a tax on capital it is the net after-tax rate of return on capital that matters for capital owners so that (5.10) becomes[26]

$$s_t = \frac{(1-\tau)\left(R_t - \delta^K\right)K_t}{w_t L}. \tag{5.15}$$

One readily verifies that in steady state we have

$$s_\tau^* = \frac{(1-\tau)\kappa^*}{f\left(\kappa^*\right) - \kappa^* f'\left(\kappa^*\right) - i\left(g^A\left(\kappa^*\right)\right)}\left(f'\left(\kappa^*\right) - i\left(\frac{\delta^B}{1-\delta^B}\right) - \frac{\delta^K}{B_\tau^*}\right) \tag{5.16}$$

which generalizes (5.11) to $\tau > 0$. Hence, there are two channels through which the tax affects the steady-state functional income distribution. First, there is the direct effect on the net after-tax rate of return on capital which is negative. Second, there is an indirect effect of opposite sign through B_τ^*. It captures the necessary adjustment of the depreciation rate per unit of efficient capital. Indeed, the same steps that lead to (4.16) deliver now

$$B_\tau^* = \frac{(1+g^*) - \beta\left(1-(1-\tau)\delta^K\right)}{\beta(1-\tau)\left(f'\left(\kappa^*\right) - i\left(\frac{\delta^B}{1-\delta^B}\right)\right)}, \tag{5.17}$$

with $B_\tau^* = B^*$ for $\tau = 0$. Observe that B_τ^* increases in τ. To see why, consider the steady-state Euler condition $1 + g^* = \beta\left((1-\tau)\left(R_\tau^* - \delta^K\right) + 1\right)$. It requires that any pair (τ, R_τ^*) is such that the household's desired consumption growth rate is g^*. Accordingly, changes in τ and R_τ^* must satisfy

$$\frac{dR_\tau^*}{d\tau} = \frac{R_\tau^* - \delta^K}{1-\tau} > 0. \tag{5.18}$$

[26]Observe that the redistribution of the government's tax receipts gives rise to a third kind of household income which is neither capital nor labor income. In what follows we neglect this aspect. Mutatis mutandis, the same remark applies to the two policies of subsidizing innovation investments studied below.

Hence, the rental rate of capital increases in τ since the net rate of return on capital, $R_\tau^* - \delta^K$, is strictly positive. Intuitively, the tax reduces the net rate of return on capital so that R^* must increase. As $R_\tau^* = B_\tau^* \left(f'\left(\kappa^*\right) - i\left(\delta^B / \left(1-\delta^B\right)\right)\right)$ we have sign $\left[dB_\tau^*/d\tau\right] =$ sign $\left[dR_\tau^*/d\tau\right] > 0$. However, the direct effect can be shown to dominate. Hence, the relative income of capital owners unequivocally falls in τ.

The following proposition summarizes the above results.

Proposition 14 *(Linear Tax on Capital, Steady-State Growth and Functional Income Distribution)*

Consider two economies that differ only with respect to their linear tax on capital such that $1 > \tau' > \tau \geq 0$*. Then, in steady state the following holds:*

$$\left(g_\tau^*\right)' = g_\tau^*, \tag{5.19}$$

$$\left(R_\tau^*\right)' > R_\tau^*, \tag{5.20}$$

$$\left(s_\tau^*\right)' < s_\tau^*. \tag{5.21}$$

5.2.2 A Subsidy to Capital-Augmenting Innovation Investments

Suppose the government pays a subsidy $\sigma^B i\left(q_t^B(m)\right)$ for all innovation investments that raise the productivity of capital at t, where $\sigma^B \in [0,1)$ is the subsidy rate. Such a subsidy reduces the (minimized) cost per task performed by capital and, therefore, renders innovation investments more attractive. This shows up in the conditions for profit-maximization that now involve

$$f'\left(\kappa_t\right) = \left(1-\sigma^B\right) c\left(q_t^B\right). \tag{5.22}$$

The latter generalizes (3.15) to the case where $\sigma^B > 0$ and leads to changes in Proposition 1. In fact, we now have[27]

$$q_t^B = g^B(\kappa_t, \sigma^B), \quad \text{with} \quad g_\kappa^B(\kappa_t, \sigma^B) < 0 \quad \text{and} \quad g_{\sigma^B}^B(\kappa_t, \sigma^B) > 0, \tag{5.23}$$

$$R_t = R\left(\kappa_t, B_{t-1}, \sigma^B\right) \quad \text{with} \quad R_{\sigma^B}\left(\kappa_t, B_{t-1}, \sigma^B\right) > 0. \tag{5.24}$$

How does the policy of subsidizing capital-augmenting innovation investments affect the economy's growth rate and its functional income distribution in steady state? First, observe that the subsidy increases the steady-state growth rate of the economy. Again, the latter is equal to the growth rate of labor-augmenting technological knowledge and hinges on $\left(q^A\right)^* = g^A\left(\kappa^*\right)$. However, here κ^* increases in σ^B. Indeed, in light of (5.23) condition (4.17) must be replaced by

$$g^B(\kappa^*_{\sigma^B}, \sigma^B) = \frac{\delta^B}{1-\delta^B}, \tag{5.25}$$

where $\kappa^*_{\sigma^B}$ is the steady-state equilibrium task intensity consistent with a subsidy rate $\sigma^B > 0$. Implicit differentiation reveals that the steady-state equilibrium task intensity, $\kappa^*_{\sigma^B}$, increases in σ^B, i. e.,

$$\frac{d\kappa^*_{\sigma^B}}{d\sigma^B} = -\frac{g_\sigma^B\left(\kappa^*_{\sigma^B}, \sigma^B\right)}{g_\kappa^B\left(\kappa^*_{\sigma^B}, \sigma^B\right)} > 0. \tag{5.26}$$

The intuition for this is straightforward. In steady state q^B must be equal to $\delta^B / \left(1-\delta^B\right)$. A subsidy rate $\sigma^B > 0$ reduces the cost of each task performed by capital at any given level of the equilibrium task intensity and makes it worthwhile to acquire a higher value of q^B. To offset this effect $\kappa^*_{\sigma^B}$ must increase.

Through this channel a higher subsidy rate also induces faster steady state growth. To see this formally, observe that the steady-state growth rate of the economy is now

$$g^*_{\sigma^B} = \left(1-\delta^A\right)\left(1+g^A\left(\kappa^*_{\sigma^B}\right)\right) - 1. \tag{5.27}$$

Then, with (5.26) we have

$$\frac{dg^*_{\sigma^B}}{d\sigma^B} = \left(1-\delta^A\right) g_\kappa^A\left(\kappa^*_{\sigma^B}\right)\frac{d\kappa^*_{\sigma^B}}{d\sigma^B} > 0. \tag{5.28}$$

[27] Total differentiation of (5.22) delivers (5.23). Moreover, observe that due to the subsidy the rental rate of capital consistent with profit-maximizing behavior becomes $R_t = B_{t-1}\left(1-\delta^B\right)\left(1+g^B(\kappa_t,\sigma^B)\right)\left[f'\left(\kappa_t\right) - \left(1-\sigma^B\right) i\left(g^B(\kappa_t,\sigma^B)\right)\right]$. Then, one readily verifies that $R_{\sigma^B}\left(\kappa_t, B_{t-1}, \sigma^B\right) > 0$.

Next, consider the relative income of capital owners (5.11). In the presence of σ^B we obtain in a steady state

$$s^*_{\sigma^B} = \frac{\kappa^*_{\sigma^B}\left(f'\left(\kappa^*_{\sigma^B}\right) - \left(1-\sigma^B\right) i\left(\frac{\delta^B}{1-\delta^B}\right) - \frac{\delta^K}{B^*_{\sigma^B}}\right)}{f\left(\kappa^*_{\sigma^B}\right) - \kappa^*_{\sigma^B} f'\left(\kappa^*_{\sigma^B}\right) - i\left(g^A\left(\kappa^*_{\sigma^B}\right)\right)}. \tag{5.29}$$

A change in σ^B affects the relative share of capital in an intricate way by changing both the income that accrues to capital owners and the wage income. Let us start with the effect on capital income, i. e., on the numerator of $s^*_{\sigma^B}$. First, there is a direct effect that increases net capital income. Intuitively, the subsidy allows for a higher rental rate of capital. Second, there is an indirect effect through an adjustment in $B^*_{\sigma^B}$ which is also strictly positive. Intuitively, the depreciation rate of efficient capital declines. Finally, there is an indirect effect through $\kappa^*_{\sigma^B}$ which is indeterminate. As a result the effect of changing σ^B on the numerator of (5.29) remains undeterminate. The effect of a greater σ^B on the steady-state wage income in efficiency units, i. e., on the denominator of (5.29), is strictly positive. The positive effect of a higher marginal product of tasks performed by labor dominates the negative effect arising from greater investment outlays. However, the total effect of increasing σ^B on $s^*_{\sigma^B}$ remains undecided in general.

The following proposition summarizes the main results on long-run effects of a subsidy to capital-augmenting innovation investments.

Proposition 15 *(Subsidy to Capital-Augmenting Innovation Investments, Steady-State Growth and Functional Income Distribution)*

Consider two economies that differ only with respect to the subsidy rate to capital-augmenting innovation investments such that $1 > \left(\sigma^B\right)' > \sigma^B \geq 0$. Then, in steady state the following holds:

$$\left(g^*_{\sigma^B}\right)' \quad > \quad g^*_{\sigma^B}, \tag{5.30}$$

$$\left(R^*_{\sigma^B}\right)' \quad > \quad R^*_{\sigma^B} \quad \text{and} \quad \left(\left(\frac{w_t}{A_t}\right)^*\right)' > \left(\frac{w_t}{A_t}\right)^*, \tag{5.31}$$

$$\left(s^*_{\sigma^B}\right)' \quad \gtreqless \quad s^*_{\sigma^B}. \tag{5.32}$$

5.2.3 A Subsidy to Labor-Augmenting Innovation Investments

Suppose the government pays a subsidy $\sigma^A i\left(q^A_t(n)\right)$ for all innovation investments that raise the productivity of labor at t, where $\sigma^A \in [0, 1)$ denotes the subsidy rate. Such a subsidy reduces the cost per task performed by labor and, therefore, renders innovation investments more attractive. This shows up in the conditions for profit-maximization that now involve

$$f\left(\kappa_t\right)-\kappa_t f'\left(\kappa_t\right)=\left(1-\sigma^A\right)c\left(q_t^A\right). \tag{5.33}$$

The latter generalizes (3.16) to the case where $\sigma^A>0$. The concomitant changes to Proposition 1 are as follows. In fact, we now have[28]

$$q_t^A = g^A(\kappa_t,\sigma^A),\quad \text{with}\quad g_\kappa^A(\kappa_t,\sigma^A)>0 \quad\text{and}\quad g_{\sigma^A}^A(\kappa_t,\sigma^A)>0, \tag{5.34}$$

$$w_t = w\left(\kappa_t,A_{t-1},\sigma^A\right)\quad\text{with}\quad w_{\sigma^A}^A\left(\kappa_t,A_{t-1},\sigma^A\right)>0. \tag{5.35}$$

How does the policy of subsidizing labor-augmenting innovation investments affect the steady-state growth rate of the economy and its functional income distribution? First, observe that the subsidy leaves the equilibrium task intensity κ^* unchanged. Nevertheless, it increases the steady-state growth rate of the economy through its direct effect on $q_t^A=g^A(\kappa_t,\sigma^A)$. Notice that the steady state requires κ^* to be determined by (4.17). But the subsidy has made innovation investments in tasks performed by labor more attractive at any level of equilibrium task intensity. In view of Proposition 5, the steady-state growth rate of the economy may now be written as

$$g_{\sigma^A}^*=\left(1-\delta^A\right)\left(1+g^A\left(\kappa^*,\sigma^A\right)\right)-1. \tag{5.36}$$

Then, using (5.34) we have indeed that

$$\frac{dg_{\sigma^A}^*}{d\sigma^A}=\left(1-\delta^A\right)g_\sigma^A\left(\kappa^*,\sigma^A\right)>0. \tag{5.37}$$

Next, consider the relative income of capital owners (5.11). In the presence of σ^A we obtain in steady state

$$s_{\sigma^A}^*=\frac{\kappa^*\left(f'\left(\kappa^*\right)-i\left(\frac{\delta^B}{1-\delta^B}\right)-\frac{\delta^K}{B_{\sigma^A}^*}\right)}{f\left(\kappa^*\right)-\kappa^*f'\left(\kappa^*\right)-\left(1-\sigma^A\right)i\left(g^A\left(\kappa^*,\sigma^A\right)\right)}. \tag{5.38}$$

A subsidy to labor-augmenting innovation investments affects the relative share of capital by changing both the income that accrues to capital owners and the wage income.

[28] Total differentiation of (5.33) delivers (5.34). Moreover, observe that due to the subsidy the real wage consistent with profit-maximizing behavior becomes $w_t = A_{t-1}\left(1-\delta^A\right)\times\left(1+g^A(\kappa_t,\sigma^A)\right)\left[f\left(\kappa_t\right)-\kappa_t f'\left(\kappa_t\right)-\left(1-\sigma^A\right)i\left(g^A(\kappa_t,\sigma^A)\right)\right]$. Then, we have $w_{\sigma^A}^A\left(\kappa_t,A_{t-1},\sigma^A\right)>0$.

Capital owners gain from the subsidy as it increases the net rate of return to capital. The intuition for this finding comes from the Euler condition in steady state,

$$1 + g^*_{\sigma^A} = \beta \left(R^*_{\sigma^A} + 1 - \delta^K \right). \tag{5.39}$$

As $g^*_{\sigma^A}$ increases in response to a higher subsidy rate, $R^*_{\sigma^A}$ must also rise. More precisely, as the steady-state rental rate of capital is $R^*_{\sigma^A} = B^*_{\sigma^A} \left[f'(\kappa^*) - i\left(\delta^B / \left(1 - \delta^B\right)\right)\right]$ it is immediate that $dR^*_{\sigma^A}/d\sigma^A = dB^*_{\sigma^A}/d\sigma^A > 0$. Since the steady-state value of the equilibrium task intensity remains unchanged neither the marginal product of tasks performed by capital nor the investment outlays for tasks performed by capital are affected. Hence, the subsidy only affects the steady-state level of capital-augmenting technological knowledge.

At the same time the subsidy affects the denominator of (5.38), i. e., the steady-state wage income in efficiency units. Two effects must be considered. First, there is a direct effect that unequivocally increases the wage in efficiency units. Second, there is an offsetting indirect effect since a greater subsidy induces higher investment outlays per task. In general this tension cannot be signed. However, with a little more structure it is possible to show that the denominator falls in response to a greater σ^A. For this to hold it is sufficient for the investment requirement function $i\left(q^A\right)$ to be a power function as assumed in Example 1. Hence, suppose that, $i\left(q^A\right) = \gamma \left(q^A\right)^\nu$ with $\nu > 1$, $\gamma > 0$. Then, a greater subsidy σ^A will reduce the wage rate per efficiency unit. In this case, a higher subsidy rate unequivocally benefits the owners of capital.

The following proposition summarizes the main results about the long-run effects of a subsidy to labor-augmenting innovation investments.

Proposition 16 *(Subsidy to Labor-Augmenting Innovation Investments, Steady-State Growth and Functional Income Distribution)*

Consider two economies that differ only with respect to the subsidy rate for labor-augmenting innovation investments such that $1 > \left(\sigma^A\right)' > \sigma^A \geq 0$. Moreover, let the innovation investment function be of the form $i\left(q^A\right) = \gamma \left(q^A\right)^\nu, \nu > 1, \gamma > 0$. Then, in steady state the following holds:

$$\left(g^*_{\sigma^A}\right)' > g^*_{\sigma^A}, \tag{5.40}$$

$$\left(R^*_{\sigma^A}\right)' > R^*_{\sigma^A} \quad \text{and} \quad \left(\left(\frac{w_t}{A_t}\right)^*\right)' < \left(\frac{w_t}{A_t}\right)^*, \tag{5.41}$$

$$\left(s^*_{\sigma^A}\right)' > s^*_{\sigma^A}. \tag{5.42}$$

It is worth mentioning that the effect of a higher subsidy, σ^A, on $\left(1-\sigma^A\right) i\left(q_t^A(n)\right)$ is identical to the effect of a decline in the parameter γ^A in the specification of Example 1 where a productivity growth rate of q_t^A requires investment outlays of $\gamma^A \left(q_t^A\right)^2/2$. In the latter case, we may interpret γ^A as the price in units of the final good that firms have to pay for an investment good that generates productivity growth at rate q_t^A. A decline in γ^A is then tantamount to a decline in the relative price of investment goods. Moreover, Proposition 16 predicts a lower price of investment goods to induce a decline in the labor share. Recently, Karabarbounis and Neiman (2014b) use this reasoning to explain the global decline of the labor share observed since the early 1980s. In line with, e. g., Greenwood, Hercowitz, and Huffman (1988), these authors attribute the decline in the relative price of investment goods to investment-specific technical change.

6 Normative Implications: Optimal Growth

The purpose of this section is to elicit the welfare properties of the dynamic competitive equilibrium. We derive two main results. First, we show that the equilibrium is not Pareto optimal. Second, we establish that the Pareto-efficient steady state may be implemented with an appropriate policy of investment subsidies.

6.1 The Planner's Problem

To derive the Pareto-efficient allocation, we assess allocations with regard to their effects on the overall utility of the representative household of (3.1). Moreover, we focus on allocations with the same structural properties as the decentralized equilibrium allocation. In particular, we restrict attention to symmetric configurations that involve $q_t^B(m) = q_t^B$ and $q_t^A(n) = q_t^A$.[29] To save space, we directly take capital and labor as fully employed.

Then, given $L > 0$, initial values of the capital stock, $K_0 > 0$, and of technological knowledge, $A_{-1} > 0$ and $B_{-1} > 0$, the planner solves

$$\max_{\left\{q_t^B, q_t^A, C_t\right\}_{t=0}^{\infty} \in \mathbb{R}_+^3} \quad \sum_{t=0}^{\infty} \beta^t \ln C_t, \tag{6.1}$$

[29]This excludes, e. g., asymmetric patterns where the planner chooses, say, $q_t^B(m) > 0$ for a small subset of tasks $m \in [0, \bar{m}_t]$, $\bar{m}_t < M_t$, and does not undertake innovation investments in all other tasks performed by capital. This strategy reduces current outlays for innovation investments and, at the same time, allows to start period $t+1$ with a high level of $B_t\left(1-\delta^B\right)$ as knowledge accumulates according to (3.26). Of course, such a strategy also has a downside since the productivity of capital at t in all tasks $m \in [\bar{m}_t, M_t]$ is $B_{t-1}\left(1-\delta^B\right)$. At all events, in a decentralized economy such patterns cannot arise in equilibrium and would even be very difficult to implement by a planning authority.

subject to the resource constraint

$$C_t + K_{t+1} = F\left(B_t K_t, A_t L\right) - B_t K_t i(q_t^B) - A_t L i(q_t^A) + \left(1-\delta^K\right) K_t, \tag{6.2}$$

the evolution of the two stocks of technological knowledge of (3.27) and a set of appropriate non-negativity constraints.

Besides these constraints and three transversality constraints,[30] the planner's problem satisfies the following first-order conditions for K_{t+1}, q_t^B, and q_t^A:

$$K_{t+1}: \quad \frac{C_{t+1}}{C_t} = \beta\left[B_t(1-\delta^B)(1+q_{t+1}^B)\left(f'\left(\kappa_{t+1}\right) - i\left(q_{t+1}^B\right)\right) + \left(1-\delta^K\right)\right], \tag{6.3}$$

$$\begin{aligned} q_t^B: \quad 0 &= \frac{K_t}{C_t}\left[f'\left(\kappa_t\right) - i\left(q_t^B\right) - \left(1+q_t^B\right) i'\left(q_t^B\right)\right], \\ &+ \beta \cdot \frac{K_{t+1}}{C_{t+1}}\left(1-\delta^B\right)\left(1+q_{t+1}^B\right)^2 i'\left(q_{t+1}^B\right) \end{aligned} \tag{6.4}$$

$$\begin{aligned} q_t^A: \quad 0 &= \frac{K_t}{C_t}\left[f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right) - i\left(q_t^A\right) - \left(1+q_t^A\right) i'\left(q_t^A\right)\right] \\ &+ \beta \cdot \frac{K_{t+1}}{C_{t+1}}\left(1-\delta^A\right)\left(1+q_{t+1}^A\right)^2 i'\left(q_{t+1}^A\right). \end{aligned} \tag{6.5}$$

Condition (6.3) is the Euler condition of the planner's problem. The comparison with the Euler condition (4.9) of the competitive equilibrium reveals that the inter-temporal equilibrium allocation of capital is the efficient one. This, however, is not the case for the equilibrium choices of q_t^B and q_t^A. To see this compare (6.4) and (6.5) to their respective equilibrium counterparts (3.15) and (3.16). The decentralized equilibrium has productivity growth rates such that the minimum costs per task are equal to the respective value product of the marginal task. If these conditions hold for t and $t+1$ then the first lines of (6.4) and (6.5) vanish whereas the second remain positive.[31] In other words, evaluated at the equilibrium allocation the (marginal) value of q_t^B and q_t^A is strictly positive for the planner. The presence of β suggests that the additional advantage is of an inter-temporal

[30]The transversality constraints are $\lim_{t\to\infty} \beta^t \mu_t^K K_{t+1} = 0$, $\lim_{t\to\infty} \beta^t \mu_t^A A_t = 0$, and $\lim_{t\to\infty} \beta_t^t \mu_t^B B_t = 0$, respectively, where μ_t^K, μ_t^A, μ_t^B are the Lagranage multipliers associated with the resource constraint and the appropriate technological constraint, respectively.

[31]Equations (8.1) and (8.2) in the proof of proposition 1 make this fully obvious.

nature. To see this more clearly, consider the following variational argument applied to the first-order condition (6.4) that describes the social planner's choice of q_t^B. Mutatis mutandis, an analogous argument applies to (6.5).

Suppose the economy evolves along an optimal path $\left\{A_t, B_t, K_{t+1}, C_t, q_t^B, q_t^A\right\}_{t=0}^{\infty}$. Now, the planner considers an increase in q_t^B at some period $t \geq 0$ in conjunction with a decrease in q_{t+1}^B such that the sequence $\left\{A_{\tau-1}, B_\tau, K_\tau, q_{\tau+1}^B, q_{\tau-1}^A\right\}_{\tau=t+1}^{\infty}$ remains unchanged. To study the effects of such variation, consider the planner's net output at t

$$V_t = F\left(B_t K_t, A_t L\right) - A_t L i\left(q_t^A\right) - B_t K_t i\left(q_t^B\right) \tag{6.6}$$

in conjunction with the evolution of the two stocks of technological knowledge of (3.27). Then, a small increase in q_t^B implies

$$dV_t = B_{t-1}\left(1-\delta^B\right) K_t \left[f'\left(\kappa_t\right) - i\left(q_t^B\right) - \left(1+q_t^B\right) i'\left(q_t^B\right)\right] dq_t^B, \tag{6.7}$$

where the first term in the bracketed expression represents the increase in net output due to a greater productivity of capital. The second term captures the additional investment outlays that arise since a greater productivity of capital increases the number of tasks performed by capital under full employment. The third term represents the additional investment outlays that arise since the investment outlays of each performed task increase.

Since B_{t+1} is unaffected by the variation in q_t^B and q_{t+1}^B, the changes in these variables must satisfy $dq_{t+1}^B = -\left(1+q_{t+1}^B\right) / \left(1+q_t^B\right) dq_t^B$. Then, the effect of dq_t^B on net output in $t+1$ is

$$\begin{aligned} dV_{t+1} &= -B_{t+1} K_{t+1} i'\left(q_{t+1}^B\right) dq_{t+1}^B \\ &= B_{t+1} K_{t+1} i'\left(q_{t+1}^B\right) \left(\frac{1+q_{t+1}^B}{1+q_t^B}\right) dq_t^B \\ &= B_{t-1}\left(1-\delta^B\right)^2 K_{t+1} \left(1+q_{t+1}^B\right)^2 i'\left(q_{t+1}^B\right) dq_t^B. \end{aligned} \tag{6.8}$$

Hence, net output in $t+1$ increases since $dq_t^B > 0$ and the concomitant decline in q_{t+1}^B reduces the investment outlays for all $M_{t+1} = B_{t+1} K_{t+1}$ tasks performed by capital. To link these findings to the first-order condition (6.4) observe that the latter may be written as

$$0 = \frac{dV_t}{C_t} + \beta \frac{dV_{t+1}}{C_{t+1}}. \tag{6.9}$$

Hence, along the optimal path the sum of the contemporaneous and the inter-temporal effect of a variation in dq_t^B vanishes when compared in "utils" of period t. In other words, the respective second terms in conditions (6.4) and (6.5) represent the inter-temporal advantage of greater values for q_t^B and q_t^A that are not taken into account in the dynamic competitive equilibrium.

6.2 Steady State Analysis

The economy of the planner involves the net output function (6.6) with constant returns to scale in capital and labor and the resource constraint (6.2). Therefore, the generalized steady state growth theorem of Irmen (2013b) applies, i. e., a steady state has $B_t = B^{**}$ and the growth rate of the economy given by the growth rate of labor-augmenting technological knowledge, $g^{**} = g_A^{**}$. Moreover, $g_V^{**} = g_Y^{**} = g_C^{**} = g_K^{**} = g^{**}$. The evolution of technological knowledge (3.27) is consistent with this pattern. The task of this section is to establish the existence of such a steady state and to compare it to the steady state of the dynamic competitive economy.

To support $B_t = B^{**}$, the evolution of the stock of capital-augmenting technological knowledge of (3.27) requires

$$\left(q^B\right)^{**} = \frac{\delta^B}{1-\delta^B} = \left(q^B\right)^{*}. \tag{6.10}$$

Hence, the steady state of the planner's problem involves the same q^B as the competitive equilibrium. This is so, even though the planner internalizes the inter-temporal knowledge spill-over associated with innovation investments that increase the productivity of capital.

Constant consumption growth in (6.3) and (6.10) imply $\kappa_t = \kappa^{**}$. Then, in steady state conditions (6.4) and (6.5) boil down to

$$c\left(\frac{\delta^B}{1-\delta^B}\right) = f'\left(\kappa^{**}\right) + \beta \cdot \frac{i'\left(\frac{\delta^B}{1-\delta^B}\right)}{1-\delta^B}, \tag{6.11}$$

$$c\left(\left(q^A\right)^{**}\right) = f\left(\kappa^{**}\right) - \kappa^{**} f'\left(\kappa^{**}\right) + \beta\left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right)^2 i'\left(\left(q^A\right)^{**}\right) \tag{6.12}$$

Hence, the planner's steady-state choice of q^B and q^A is such that the minimized costs per task are equal to the sum of the contemporaneous marginal product of the respective task and the inter-temporal advantage arising from the knowledge spill-over. The comparison with (3.15) and (3.16) of the competitive equilibrium shows that the inter-temporal effect is neglected by the competitive production sector. The reason for this is straightforward. While innovation investments give rise to new technological knowledge that increases the productivity of the factors of production, this advantage is confined to the period in which the innovation investment is made. For all subsequent periods the newly created technological knowledge becomes publicly available and can be used by any firm free of charge. Hence, the investment behavior of firms does not reflect the future.[32]

[32] In other words, newly created technological knowledge is partially excludable, i. e., for one period, and

Condition (6.11) determines $\kappa^{**} > 0$. The comparison with (3.15) evaluated at the steady state reveals immediately that $\kappa^{**} > \kappa^{*}$. This reflects the presence of the inter-temporal effect that forces the marginal product of tasks performed by capital must fall. Condition (6.12) determines $\left(q^A\right)^{**} > \left(q^A\right)^{*}$. There are two reinforcing reasons for this inequality. First, $\kappa^{**} > \kappa^{*}$ means that the marginal product of tasks performed by labor increases. Therefore, the cost-minimizing level of q^A will be higher. Second, the inter-temporal advantage is itself increasing in q^A. In steady state, the Euler condition, (6.3), pins down B^{**} such that consumption grows at the same rate as the economy. The following proposition summarizes our results.

Proposition 17 *(Planner's Steady-State Allocation)*

1. *Suppose Assumptions (1) and (2) hold. Then, the planner's problem has a unique steady state involving* $\left(\kappa^{**}, \left(q^B\right)^{**}, \left(q^A\right)^{**}\right) \in \mathbb{R}^3_{++}$, *and*

$$c^{**} = f\left(\kappa^{**}\right) - \kappa^{**} i\left(\frac{\delta^B}{1-\delta^B}\right) - i\left(\left(q^A\right)^{**}\right) - \frac{\kappa^{**}}{B^{**}}\left(g^{**} + \delta^K\right), \quad (6.13)$$

$$B^{**} = \frac{\left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right) - \beta\left(1-\delta^K\right)}{\beta\left(f'\left(\kappa^{**}\right) - i\left(\frac{\delta^B}{1-\delta^B}\right)\right)} > B^{*}, \quad (6.14)$$

2. *The welfare-maximizing steady-state growth rate of the economy is*

$$g^{**} \equiv \frac{A_{t+1}}{A_t} - 1 = \left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right) - 1 > g^{*}.$$

Moreover, along the planner's steady-state path, it holds that

a) $$\frac{Y_{t+1}}{Y_t} = \frac{V_{t+1}}{V_t} = \frac{K_{t+1}}{K_t} = \frac{C_{t+1}}{C_t} = \frac{M_{t+1}}{M_t} = \frac{N_{t+1}}{N_t} = 1 + g^{**},$$

b) $$k^{*} = \frac{1}{B^{**}}, \quad \frac{l_{t+1}}{l_t} = \frac{1}{1+g^{**}}$$

6.3 Pareto-Improving Fiscal Policy

The discrepancy between the steady-state growth rates of the dynamic competitive equilibrium and the planner's solution suggests the possibility of introducing Pareto-improving

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non-rival. One may contrast this with a dynamic competitive economy where knowledge is perfectly excludable, i. e., excludable forever. Then, the knowledge accumulation equation of (3.26) become firm specific and are taken into account when firms maximize profits. In this scenario, the dynamic competitive equilibrium chooses the efficient steady-state levels of q^B and q^A as characterized by (6.11) and (6.12). However, a deviation from perfect excludability implies the same qualitative results as derived above.

policy measures. The following proposition establishes that an appropriately chosen investment subsidy accompanied by a lump-sum tax to balance the government's budget may close the gap between the allocation chosen by the planner and the one obtained under laissez-faire.

Proposition 18 *Suppose the government subsidizes innovation investments that increase the productivity of capital at rate $\sigma^B \in (0,1)$ and those that increase the productivity of labor at rate $\sigma^A \in (0,1)$. Then, the pair $\left((\sigma^B)^{**}, (\sigma^A)^{**}\right)$ implements the planner's allocation, where*

$$\left(\sigma^B\right)^{**} = \frac{\beta i'}{i' + (1-\delta^B)\, i} \tag{6.15}$$

and i is evaluated at $q^B = \delta^B / \left(1-\delta^B\right)$, and

$$\left(\sigma^A\right)^{**} = \frac{\beta\left(1-\delta^A\right)\left(1+\left(q^A\right)^{**}\right) i'\left(\left(q^A\right)^{**}\right)}{c\left(\left(q^A\right)^{**}\right)}. \tag{6.16}$$

Notice that the two subsidies differ. The reason for this is that they fulfill different purposes. The subsidy to innovation investments that increase the productivity of capital induces a higher steady-state value of the equilibrium task intensity. Since q^B is pinned down in the steady state by the requirement to offset depreciation of the stock of capital-augmenting knowledge, the subsidy rate σ^B has to be exactly such that the inter-temporal spill-over yields κ^{**}. Since the equilibrium task intensity has already been pushed up to κ^{**} the subsidy to innovation investments that increase the productivity of labor makes firms to internalize the inter-temporal effect of q^A .

7 Concluding Remarks

The idea that the rate and direction of technical change result from costly and purposeful activities, motivated by economic forces, is an old one and even predates the analytic representation of exogenous technical change. This paper drafts an original competitive growth model to derive new insights both for the positive and the normative implications of endogenous capital- and labor-augmenting technical change.

The reader may recall that our results are obtained for a logarithmic per-period utility function of the representative household. This begs the question of whether our qualitative findings are different under a more general utility function allowing for a constant inter-temporal elasticity of substitution (CIES) different from unity. In fact, little will change. To see this, remember that neither the steady-state efficient capital intensity nor the steady-state growth rate hinge on household preferences. Both are entirely determined by the production sector of the economy. The CIES will only affect steady-state levels as well as the transition while leaving the local stability property of the steady state intact.

For an CIES exceeding unity an additional restriction on permissible parameter constellations is called for. This condition assures that the steady-state growth rate is not too large so that the household's problem remains well-defined. The presence of a CIES also introduces an additional parameter for comparative static exercises. However, since the household becomes less willing to accept deviations from a uniform consumption profile the smaller the CIES, the comparative-static effect of a decline in the CIES delivers the same sign as a decline in the discount factor, β.

In the main text we also abstract from population growth. Let us now consider the main consequences of a constant population growth rate, g_L, where $\beta(1+g_L) < 1$. Again, since the steady-state growth rate is determined within the production sector it will not depend on g_L or the size of the population. Moreover, from (4.6) evaluated at the steady state it becomes apparent that in steady state the equilibrium net output per capita does not depend on g_L or the size of the population either. Hence, as suggested in Footnote 3 there are neither "strong" nor "weak" scale effects. Clearly, there will be level effects along the transition associated with g_L that, however, do not affect the local stability property of the steady state.

It is worth noting that the respective roles of the CIES and of g_L are quite different in the model variant of Section 4.4 where only labor-augmenting technical change is feasible. For instance, changing either parameter will change the steady-state growth rate. This suggests that the analysis of the present paper may be interpreted as a robustness check for the qualitative results of the more "standard setting" with labor-augmenting technical change only. Our analysis highlights several new circumstances where the presence of capital-augmenting technical change leads to predictions that substantially differ from the standard setting.

Our analysis also contributes the discussion about why technical change is eventually only labor-augmenting. As in the models of von Weizsäcker (1962), Drandakis and Phelps (1966), Samuelson (1966), and Acemoglu (2003b) we deduce labor-augmenting technical change instead of assuming it, i. e., in the neighborhood of the steady state it results from the equilibrium incentives faced by innovating firms. However, contrary to these four contributions, the forces that push the economy towards its steady state with labor-augmenting technical change only are independent of the elasticity of substitution between capital and labor.[33]

[33]Drandakis and Phelps (1966) show that their dynamical system exhibits a saddle path if the elasticity of

Our paper leaves some important issues unresolved. They include the plausibility of a time-invariant equilibrium innovation possibility frontier (EIPF). Should this frontier move over time? Empirical studies such as Caselli and Coleman (2006) suggest the existence of country-specific frontiers which may actually change over time. One way to think about this is in terms of investment-specific technical change that lowers the relative price of the resources used as innovation investments. This route may also open the door to a new mechanism that links the empirically observed decline of labor shares to the relative price of investment goods (see, e. g., Karabarbounis and Neiman (2014b)).

Another desirable feature would be a more flexible role for tasks. So far, we restrict attention to time-invariant factor-specific tasks. However, in practice the boundary between tasks performed by labor and those performed by capital shifts over time. Technical change may tend to transfer tasks from one factor of production to another. Moreover, history shows that technical change may make certain tasks redundant altogether and eliminate them from the production process. We leave these challenging questions for future research.

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substitution exceeds unity. However, with their assumption of an exogenous savings behavior there is no mechanism leading the economy onto the saddle path.

8 Appendix: Proofs

8.1 Proof of Proposition 1

We consider each claim of Proposition 1 in turn. Without loss of generality, we suppress the time argument.

1. Recall that $c\left(q^B\right)$ and $c\left(q^A\right)$ denote the minimum cost per task performed by capital and labor, respectively, i. e.,

$$\begin{aligned} c\left(q^B\right) &= \min_{q^B \geq 0} \left[i\left(q^B\right) + \frac{R}{B_{-1}\,(1-\delta^B)\,(1+q^B)} \right] \\ &= i\left(q^B\right) + \left(1+q^B\right) i'\left(q^B\right), \end{aligned} \tag{8.1}$$

$$\begin{aligned} c\left(q^A\right) &= \min_{q^A \geq 0} \left[i\left(q^A\right) + \frac{w}{A_{-1}\,(1-\delta^A)\,(1+q^A)} \right] \\ &= i\left(q^A\right) + \left(1+q^A\right) i'\left(q^A\right), \end{aligned} \tag{8.2}$$

where use is made of (3.11) and (3.12).

The properties of $f(\kappa)$ and $i(q^j)$ ensure the existence of some function $g^j : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$, $j = A, B$. Implicit differentiation of (3.15) and (3.16) reveals that $g^A_\kappa(\kappa_t) > 0 > g^B_\kappa(\kappa_t)$ as claimed in (3.17) and (3.18).

2. Solving (3.11) and (3.12) for the respective factor price R and w and using Claim 1 delivers

$$\begin{aligned} R &= B_{-1}\left(1-\delta^B\right)\left(1+g^B(\kappa)\right)^2 i'\left(g^B(\kappa)\right) \equiv R\left(\kappa, B_{-1}\right), \\ w &= A_{-1}\left(1-\delta^A\right)\left(1+g^A(\kappa)\right)^2 i'\left(g^A(\kappa)\right) \equiv w(\kappa, A_{-1}), \end{aligned}$$

where $R : \mathbb{R}^2_{++} \rightarrow \mathbb{R}_{++}$ and $w : \mathbb{R}^2_{++} \rightarrow \mathbb{R}_{++}$. The partial derivatives indicated in (3.19) and (3.20) follow immediately from Claim 1 and the properties of the function i. ■

8.2 Proof of Proposition 2

We consider each claim of Proposition 2 in turn. Without loss of generality, we suppress the time argument.

1. Consider equations (3.17) and (3.18). Since g^B is strictly increasing on its domain it is invertible. Let $G^B : \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}$ denote the inverse of g^B. Then, from (3.17), $\kappa = G^B\left(q^B\right)$. Hence, with (3.18), we may write

$$q^A = g^A\left(G^B\left(q^B\right)\right) \equiv g\left(q^B\right).$$

The slope of the function $g(q^B)$ is given by

$$g'\left(q^B\right) \equiv \frac{dq^A}{dq^B} = \frac{dg^A(\kappa)}{d\kappa}\frac{dG^B\left(q^B\right)}{dq^B} = \frac{g^A_\kappa(\kappa)}{g^B_\kappa(\kappa)} < 0. \tag{8.3}$$

2. Consider equations (3.19) and (3.20). From equation (3.20), the function $w(\kappa, A_{-1})$ is strictly increasing in κ on its domain. Hence, given A_{-1}, this function is invertible in κ. Let $W : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ denote this inverse. Then, $\kappa = W(w : A_{-1})$. Hence, with (3.19), we may write

$$R = R(W(w : A_{-1}), B_{-1}) \equiv h(w, A_{-1}, B_{-1}),$$

where $h : \mathbb{R}^3_{++} \to \mathbb{R}_{++}$. The partial derivative of $h(w, A_{-1}, B_{-1})$ with respect to w is given by

$$h_w(w, A_{-1}, B_{-1}) \equiv \frac{dR}{dw} = \frac{dR(\kappa, B_{-1})}{d\kappa}\frac{dW(w : A_{-1})}{dw} = \frac{R_\kappa(\kappa, B_{-1})}{w_\kappa(\kappa, A_{-1})} < 0. \tag{8.4}$$

■

8.3 Proof of Proposition 3

Without loss of generality, we suppress the time argument.

Consider equation (4.7). Given $\left(B_{-1}\left(1-\delta^B\right)K/A_{-1}(1-\delta^A)L\right) > 0$, its right-hand side defines a continuous function $RHS\left(\kappa, B_{-1}\left(1-\delta^B\right)K/A_{-1}\left(1-\delta^A\right)L\right) > 0$ for all $\kappa > 0$. Moreover, following Proposition 1, the properties of $g^A(\kappa)$ and $g^B(\kappa)$ imply that $RHS\left(\kappa, B_{-1}\left(1-\delta^B\right)K/A_{-1}(1-\delta^A)L\right)$ is continuous and strictly decreasing in $\kappa > 0$. Hence, $\lim_{\kappa\to 0} RHS\left(\kappa, B_{-1}\left(1-\delta^B\right)K/A_{-1}(1-\delta^A)\right) > 0$. By continuity, there is a unique $\kappa > 0$ that satisfies $\kappa = RHS\left(\kappa, B_{-1}\left(1-\delta^B\right)K/A_{-1}(1-\delta^A)\right)$. Implicit differentiation delivers that this value increases whenever $B_{t-1}\left(1-\delta^B\right)K_t/\left[A_{t-1}\left(1-\delta^A\right)L\right]$ increases. ■

8.4 Proof of Proposition 4

Proposition 4 claims that for given $L > 0$ and initial values $(A_{-1}, B_{-1}, K_0) > 0$ the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$. To prove this we pursue the following steps. First, we derive the three-dimensional system of Proposition 4. Second, we establish that κ_t and B_t describe the state of the economy at t. Finally, we show that for any admissible $(\kappa_t, B_t, c_t) > 0$ there is a corresponding unique $\{\kappa_{t+1}, c_{t+1}, B_{t+1}\}$ for all t.

1. To obtain the three-dimensional system we proceed as follows. To derive (4.8) use equations 8.1, 8.2 and Claim 1 of Proposition (1) in (3.15) and (3.16) to substitute for R_t and w_t in the household's budget constraint. We may then write the latter as

$$K_{t+1} = B_t\Big(f'(\kappa_t) - i(g^B(\kappa_t))\Big)K_t + A_tL\Big(f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\Big) - C_t + (1-\delta^K)K_t.$$

By definition we have $K_t = (A_tL/B_t)\kappa_t$ and $C_t = A_tLc_t$. Together with equation (4.6) above equation may be written in terms of efficiency units as

$$\frac{A_{t+1}B_t}{A_tB_{t+1}}\kappa_{t+1} \quad = B_t\left(v\left(\kappa_t\right) - c_t\right) + (1-\delta^K)\kappa_t.$$

Equation 4.9 is obtained upon employing equations (3.17) and (3.15) in (3.3).

Now, the system describing the evolution of the economy is four-dimensional and given by

$$\frac{A_{t+1}B_t}{A_t B_{t+1}}\kappa_{t+1} = B_t\left(v(\kappa_t) - c_t\right) + (1-\delta^K)\kappa_t, \tag{8.5}$$

$$\frac{A_{t+1}}{A_t}\frac{c_{t+1}}{c_t} = \beta\left(B_{t+1}\left[f'(\kappa_{t+1}) - i(g^B(\kappa_{t+1}))\right] + (1-\delta^K)\right), \tag{8.6}$$

$$A_t = A_{t-1}\left(1-\delta^A\right)(1+g^A(\kappa_t)), \tag{8.7}$$

$$B_t = B_{t-1}\left(1-\delta^B\right)(1+g^B(\kappa_t)). \tag{8.8}$$

Observe that this system of four first-order, non-linear difference equations may be reduced to a system of three equations. Forwarding (8.7) and (8.8) and substituting in (8.5) and (8.6) we obtain the three-dimensional system of Proposition 4.

2. Notice that given $(A_{-1}, B_{-1}, K_0, L) > 0$ the state of the economy in period t is fully characterized by two variables, the two state variables κ_t and B_t. In period $t = 0$ κ_0 is determined by (4.12) in accordance with Proposition 3. Having κ_0 pins down the value of the other state variable B_0 in accordance with (4.10).

3. To show that for any initial values (A_{-1}, B_{-1}, K_0, L) the transitional dynmaics is given by a unique sequence $\{\kappa_t, c_t, B_t\}_{t=0}^{\infty}$ we introduce

$$\Omega^\kappa(\kappa_{t+1}) \equiv \frac{\left(1+g^A(\kappa_{t+1})\right)}{\left(1+g^B(\kappa_{t+1})\right)}\kappa_{t+1}. \tag{8.9}$$

Then, equation (4.8) may be rewritten as

$$\Omega^\kappa(\kappa_{t+1}) = \frac{(1-\delta^B)}{(1-\delta^A)}\left[B_t\Big(v(\kappa_t) - c_t\Big) + (1-\delta^K)\kappa_t\right]. \tag{8.10}$$

For any triple $(\kappa_t, B_t, c_t) \in \mathbb{R}^3_{++}$ such that the right-hand side of (8.10) is strictly positive, there will be a unique value of $\kappa_{t+1} > 0$ satisfying equation (8.10) if $\Omega^\kappa(\kappa_{t+1})$ is strictly positive, continuous and monotone in $\kappa_{t+1} > 0$ and may take any value in $\mathbb{R}_{++}$.

Observe first that $\Omega^\kappa(\kappa_{t+1}) > 0$ indeed holds for all $\kappa_{t+1} > 0$ and is a consequence of the properties of the functions g^A and g^B, as established in Proposition 1. It remains to be shown that $\lim_{\kappa\to 0}\Omega^\kappa(\kappa_{t+1}) = 0$ and $\lim_{\kappa\to\infty}\Omega^\kappa(\kappa_{t+1}) = \infty$. To show this, consider the right-hand side of (8.9). Recall from Proposition 1 that $g^B(\kappa)$ is decreasing on $\mathbb{R}_{++}$ and bounded below by zero. Hence, $\lim_{\kappa\to\infty} g^B(\kappa)$ is finite, while $\lim_{\kappa\to 0} g^B(\kappa)$ is either finite or infinite. Moreover, Proposition 1 implies that $\lim_{\kappa\to 0} g^A(\kappa)$ is finite and bounded below by zero while $\lim_{\kappa\to\infty} g^A(\kappa)$ is finite or infinite since g^A is increasing on $\mathbb{R}_{++}$. Consequently, as κ tends to zero we have $\lim_{\kappa\to 0}\Omega^\kappa(\kappa_{t+1}) = 0$ and as κ tends to infinity we have $\lim_{\kappa\to\infty}\Omega^\kappa(\kappa_{t+1}) = \infty$.

It follows that $\Omega^\kappa(\kappa_{t+1})$ is increasing in $\kappa_{t+1} > 0$, approaches zero as $\kappa \to 0$ and approaches infinity as $\kappa \to \infty$. Therefore, there is a unique $\kappa_{t+1} > 0$ that satisfies eq. (4.8) for given $(\kappa_t, B_t, c_t) \in \mathbb{R}^3_{++}$.

Given a unique $\kappa_{t+1} > 0$ equation (4.9) delivers a unique $c_{t+1} > 0$ and equation 4.10 a unique $B_{t+1} > 0$. ■

8.5 Proof of Proposition 5

We consider both claims of Proposition 5 in turn.

1. Equations (4.15) - (4.17) follow immediately from the corresponding equations (4.8) - (4.10) of the dynamical system for the reasons discussed in the main text. Obviously, in steady state the transversality condition (4.11) is also satisfied. As explained in the main text, Assumption 1 and Assumption 2 guarantee a strictly positive solution to (4.17) and (4.16), respectively. It remains to be shown that $c^* > 0$ or $B^* v^* (\kappa^*) > \kappa^* \left(g^* + \delta^K\right)$. Using (4.16) the latter inequality may be written as

$$\left[(1+g^*) - \beta\left(1-\delta^K\right)\right] v^*(\kappa^*) > \beta\left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)\kappa^*\left(g^* + \delta^K\right) \tag{8.11}$$

A sufficient condition for this to hold is obtained for $\beta = 1$ on both sides of the inequality. This gives

$$v^*(\kappa^*) > \left(f'(\kappa^*) - i\left(g^B(\kappa^*)\right)\right)\kappa^*. \tag{8.12}$$

Finally, observe that due to constant returns to scale of F it holds in equilibrium that $f(\kappa) = c\left(q^A\right) + \kappa c\left(q^B\right)$. Using the latter for the steady state delivers $v^*(\kappa^*) = \left(1 + g^A(\kappa)^*\right) i'\left(g^A(\kappa)^*\right) + \kappa^*\left(1 + g^B(\kappa)^*\right) i'\left(g^B(\kappa)^*\right)$. Then, with the understanding that both g^A and g^B are evaluated at κ^*, inequality (8.12) becomes

$$\begin{aligned}
\left(1+g^A\right) i'\left(g^A\right) + \kappa^*\left(1+g^B\right) i'\left(g^B\right) &> \left(f'(\kappa^*) - i\left(g^B\right)\right)\kappa^*, \\
\left(1+g^A\right) i'\left(g^A\right) &> \left[f'(\kappa^*) - i\left(g^B\right) - \left(1+g^B\right) i'\left(g^B\right)\right]\kappa^*, \\
\left(1+g^A\right) i'\left(g^A\right) &> 0,
\end{aligned}$$

as $f'(\kappa^*) - c\left(\left(q^B\right)^*\right) = 0$.

2. The expression for the steady-state growth rate, g^*, follows from (3.27) and Proposition 1. The derivation of the remaining findings is explained in the main text. ■

8.6 Proof of Proposition 6

1. From (4.17), κ^* is independent of β. Therefore, g^* does not depend on β either. From (4.16) it is immediate that a higher β requires a lower B^*. According to the expression for R^* in Claim 2 of Proposition 5 the rental rate of capital must also fall. From (4.15) the same is true for c^*.

2. Implicit differentiation of (4.17) delivers

$$\frac{d\kappa^*}{d\delta^B} = \frac{1}{(1-\delta^B)^2 g_\kappa^B(\kappa^*)} < 0 \tag{8.13}$$

as $g_\kappa^B < 0$. Hence, $(\kappa^*)' < \kappa^*$. The concomitant effect on the steady-state growth rate, g^*, is

$$\frac{dg^*}{d\delta^B} = \frac{dg^*}{d\kappa}\frac{d\kappa^*}{d\delta^B} < 0.$$

The sign follows since $dg^*/d\kappa = \left(\partial g^*/\partial q^A\right)\left(dg^A(\kappa^*)/d\kappa\right) > 0$. Hence, $(g^*)' < g^*$.
To obtain the effect of δ^B on B^* consider (4.16). Then,

$$\begin{aligned}
\frac{dB^*}{d\delta^B} &= \frac{dB^*}{d\kappa}\frac{d\kappa^*}{d\delta^B} \\
&= \frac{1}{\beta\left(f' - i\left(g^B\right)\right)}\left[\frac{dg^*}{d\kappa} - \frac{\left(g^* + \delta^K\right)\left(f'' - i'\left(g^B\right) g_\kappa^B\right)}{f' - i\left(g^B\right)}\right]\frac{d\kappa^*}{d\delta^B} < 0,
\end{aligned} \tag{8.14}$$

where f and g^B are evaluated at κ^*. To verify the sign of this expression note from (8.1) that

$$g_\kappa^B(\kappa) \equiv \frac{dq^B}{d\kappa} = \frac{f''(\kappa)}{2i'\left(q^B\right) + \left(1+q^B\right) i''\left(q^B\right)}. \tag{8.15}$$

Hence, $f''(\kappa^*) - i'\left(g^B(\kappa^*)\right) g^B_\kappa(\kappa^*) < 0$ and $(B^*)' < B^*$.

Using (4.16) the steady-state rental rate of capital may be written as $R^* = (1 + g^*)/\beta - \left(1 - \delta^K\right))$. Following $dg^*/d\delta^B < 0$ it is immediate that $(R^*)' < R^*$.

Finally, consider the effect of δ^B on c^* of (4.15). It is given by

$$\begin{aligned} \frac{dc^*}{d\delta^B} &= \frac{dc^*}{d\kappa}\frac{d\kappa*}{d\delta^B} \\ &= \left[v'(\kappa^*) - \frac{\left(g^* + \delta^K + \kappa^* \frac{dg^*}{d\kappa}\right) B^* - \frac{dB^*}{d\kappa}\left(g^* + \delta^K\right)\kappa^*}{(B^*)^2}\right]\frac{d\kappa*}{d\delta^B} \end{aligned} \tag{8.16}$$

and is indeterminate in general.

3. From (4.17), κ^* is independent of δ^A. Therefore, $(\kappa')^* = \kappa^*$. Since $g^* = \left(1 - \delta^A\right)\left(1 + g^A\left(\kappa^*\right)\right) - 1$ it is immediate that $\left(\delta^A\right)' > \delta^A \Rightarrow (g^*)' < g^*$.

   From (4.16) it is immediate that $\text{sign}[dB^*/d\delta^A] = \text{sign}[dg^*/d\delta^A] < 0$. Hence, $(B^*)' < B^*$. Invoking $R^* = (1 + g^*)/\beta - \left(1 - \delta^K\right))$ and (4.15) one also finds that $\text{sign}[dR^*/d\delta^A] = -\text{sign}[dc^*/d\delta^A] = \text{sign}[dg^*/d\delta^A] < 0$. Hence, $(R^*)' < R^*$ and $(c^*)' > c^*$. ■

8.7 Proof of Proposition 7

We characterize the local stability of the dynamical system of Proposition 4 in the proximity of its steady-state equilibrium, (κ^*, c^*, B^*).[34]

Consider first the system of autonomous, nonlinear, first-order difference equations (4.8), (4.9) and (4.10), and notice that equations (4.8) and (4.9) define continuously differentiable functions, $\Phi^i : \mathbb{R}^3_{++} \rightarrow \mathbb{R}_{++}$, where $i = \kappa, c$, such that

$$\kappa_{t+1} = \Phi^\kappa(\kappa_t, c_t, B_t), \quad c_{t+1} = \Phi^c(\kappa_t, c_t, B_t).$$

Next, forward equation (4.10) one period and substitute $\kappa_{t+1} \equiv \Phi^\kappa(\kappa_t, c_t, B_t)$ to obtain

$$B_{t+1} = B_t(1-\delta)(1 + g^B(\Phi^\kappa(\kappa_t, c_t, B_t))) \equiv \Phi^B(\kappa_t, c_t, B_t),$$

where Φ^B is also a continuously differentiable function.

Thus, with this notation, the dynamic system may be rewritten as

$$\begin{aligned} \kappa_{t+1} &= \Phi^\kappa(\kappa_t, c_t, B_t) \\ c_{t+1} &= \Phi^c(\kappa_t, c_t, B_t) \\ B_{t+1} &= \Phi^B(\kappa_t, c_t, B_t) \end{aligned}$$

The above nonlinear dynamical system is approximated locally around its steady-state equilibrium, (κ^*, c^*, B^*) by the following linear system:

$$\begin{bmatrix} \kappa_{t+1} \\ c_{t+1} \\ B_{t+1} \end{bmatrix} = J \begin{bmatrix} \kappa_t \\ c_t \\ B_t \end{bmatrix} + X,$$

[34] See Tabaković (2014) for a general discussion of the local stability analysis for three-dimensional discrete dynamical systems.

where

$$J \equiv \begin{bmatrix} \Phi^{\kappa}_{\kappa}(\kappa^*, c^*, B^*) & \Phi^{\kappa}_{c}(\kappa^*, c^*, B^*) & \Phi^{\kappa}_{B}(\kappa^*, c^*, B^*) \\ \Phi^{c}_{\kappa}(\kappa^*, c^*, B^*) & \Phi^{c}_{c}(\kappa^*, c^*, B^*) & \Phi^{c}_{B}(\kappa^*, c^*, B^*) \\ \Phi^{B}_{\kappa}(\kappa^*, c^*, B^*) & \Phi^{B}_{c}(\kappa^*, c^*, B^*) & \Phi^{B}_{B}(\kappa^*, c^*, B^*) \end{bmatrix} \tag{8.17}$$

is the Jacobian matrix of the dynamical system evaluated at (κ^*, c^*, B^*), and

$$X \equiv \begin{bmatrix} \kappa^* - \Phi^{\kappa}_{\kappa}(\kappa^*, c^*, B^*)\kappa^* - \Phi^{\kappa}_{c}(\kappa^*, c^*, B^*)c^* - \Phi^{\kappa}_{B}(\kappa^*, c^*, B^*)B^* \\ c^* - \Phi^{c}_{\kappa}(\kappa^*, c^*, B^*)\kappa^* - \Phi^{c}_{c}(\kappa^*, c^*, B^*)c^* - \Phi^{c}_{B}(\kappa^*, c^*, B^*)B^* \\ B^* - \Phi^{B}_{\kappa}(\kappa^*, c^*, B^*)\kappa^* - \Phi^{B}_{c}(\kappa^*, c^*, B^*)c^* - \Phi^{B}_{B}(\kappa^*, c^*, B^*)B^* \end{bmatrix}$$

is a constant column vector.

To obtain the elements of J take the total differential of

$$\frac{(1-\delta^A)\left(1+g^A(\kappa_{t+1})\right)}{(1-\delta^B)\left(1+g^B(\kappa_{t+1})\right)}\kappa_{t+1} \equiv \Omega(\kappa_{t+1}) = B_t\left(v(\kappa_t) - c_t\right) + (1-\delta^K)\kappa_t,$$

$$c_{t+1} = \beta\frac{\left[B_{t+1}\left(f'(\kappa_{t+1}) - i\left(g^B(\kappa_{t+1})\right)\right) + (1-\delta^K)\right]c_t}{(1-\delta^A)\left(1+g^A(\kappa_{t+1})\right)},$$

$$B_{t+1} = B_t(1-\delta^B)\Big(1+g^B(\kappa_{t+1})\Big),$$

which gives:

$$\Phi^{\kappa}_{\kappa} = \frac{B^* v'(\kappa^*) + 1 - \delta^K}{(1+g^*)\left(1+\epsilon^A_{\kappa}+\epsilon^B_{\kappa}\right)} > 0,$$

$$\Phi^{\kappa}_{c} = -\frac{B^*}{(1+g^*)\left(1+\epsilon^A_{\kappa}+\epsilon^B_{\kappa}\right)} < 0,$$

$$\Phi^{\kappa}_{B} = \frac{v(\kappa^*) - c^*}{(1+g^*)\left(1+\epsilon^A_{\kappa}+\epsilon^B_{\kappa}\right)} > 0,$$

$$\Phi^{c}_{\kappa} = \frac{\beta B^*\left[\left(f''(\kappa^*) - i' g^B_{\kappa}\right) + \left(f'(\kappa^*) - i(g^B(\kappa^*))\right)(1-\delta^B)g^B_{\kappa}\right] - (1-\delta^A)g^A_{\kappa}}{1+g^*}c^*\Phi^{\kappa}_{\kappa} < 0,$$

$$\Phi^{c}_{c} = 1 + \frac{\Phi^{c}_{\kappa}\Phi^{\kappa}_{c}}{\Phi^{\kappa}_{\kappa}} > 1,$$

$$\Phi^{c}_{B} = \frac{\beta\left(f'(\kappa^*) - i(g^B(\kappa^*))\right)c^*}{(1+g^*)} + \frac{\Phi^{c}_{\kappa}\Phi^{\kappa}_{B}}{\Phi^{\kappa}_{\kappa}} \quad \text{(sign indeterminate)},$$

$$\Phi^{B}_{\kappa} = B^*(1-\delta^B)g^B_{\kappa}\Phi^{\kappa}_{\kappa} < 0,$$

$$\Phi^{B}_{c} = B^*(1-\delta^B)g^B_{\kappa}\Phi^{\kappa}_{c} > 0,$$

$$\Phi^{B}_{B} = 1 + B^*(1-\delta^B)g^B_{\kappa}\Phi^{\kappa}_{B} = 1 + \frac{\Phi^{B}_{\kappa}\Phi^{\kappa}_{B}}{\Phi^{\kappa}_{\kappa}} \in (0,1).$$

The local stability properties of our three-dimensional system are fully determined by the eigenvalues λ_1, λ_2 and λ_3 of the Jacobian matrix. To find the eigenvalues obtain the solution to

$$\det(J - \lambda I) = 0,$$

which gives rise to the following characteristic polynomial:

$$c(\lambda) \equiv \lambda^3 - tr(J)\lambda^2 + \sum M_2(J)\lambda - \det(J), \tag{8.18}$$

where $tr(J)$ denotes the trace, $\sum M_2(J)$ the sum of principal minors of order two and $\det(J)$ the determinant of the Jacobian matrix. One can show that

$$tr(J) = \quad \lambda_1 + \lambda_2 + \lambda_3 \quad = \Phi^\kappa_\kappa + \Phi^c_c + \Phi^B_B > 0 \tag{8.19}$$

$$\sum M_2(J) = \quad \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \quad = 2\Phi^\kappa_\kappa + \Phi^c_c\Phi^B_B - \Phi^c_B\Phi^B_c > 0 \tag{8.20}$$

$$det(J) = \quad \lambda_1\lambda_2\lambda_3 \quad = \Phi^\kappa_\kappa > 0 \tag{8.21}$$

By Descartes' rule of signs we know that if the terms of a polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is either equal to the number of sign differences between consecutive nonzero coefficients, or is less than it by an even number. Moreover, the number of negative roots is at most equal to the number of continuations in the signs of the coefficients. Inspection of equation (8.18) reveals that it has

(α) either three real positive roots,

(β) or one real positive root and one pair of complex conjugate roots.

Next, evaluate the characteristic polynomial at $\lambda = 1$ to obtain $c(1) = -\Phi^B_c \frac{c^*}{B^*} < 0$, implying that one eigenvalue is of magnitude greater than 1, say $\lambda_1 > 1$. If the remaining two eigenvalues are real,

($\alpha 1$) either both have magnitude greater than one,

($\alpha 2$) or both have magnitude smaller than one.

Otherwise, the remaining two eigenvalues are complex and

($\beta 1$) either have modulus greater than one, i. e., $\mid x \pm \omega i \mid > 1$,

($\beta 2$) or have modulus smaller than one, i. e., $\mid x \pm \omega i \mid < 1$.

To determine the magnitude of the remaining two eigenvalues use equations $(8.19) - (8.21)$ to obtain

$$\mathcal{C}(\lambda_p\lambda_q) = (\lambda_p\lambda_q)^3 - \sum M_2(J)(\lambda_p\lambda_q)^2 + tr(J)det(J)(\lambda_p\lambda_q) - det(J)^2 = 0, \tag{8.22}$$

where $(\lambda_p\lambda_q)$ represents any of the three product pairs of the eigenvalues of $c(\lambda)$. Moreover, we have that, by construction, the three roots of $\mathcal{C}(\lambda_p\lambda_q)$ are the three product pairs of the eigenvalues of $c(\lambda)$ and therefore all roots of $\mathcal{C}(\lambda_p\lambda_q)$ are greater than zero.

(α) Consider the case in which all eigenvalues are real. Without loss of generality, let $\lambda_1 > 1$. We can determine the magnitude of the remaining eigenvalues by evaluating $\mathcal{C}(\lambda_p\lambda_q)$ at $\lambda_p\lambda_q = 1$. Some algebra delivers

$$\mathcal{C}(1) = \underbrace{B^*\left(1 - \delta^B\right) g^B_\kappa}_{<0} \underbrace{\left[\frac{\beta\left(f' - i\right)}{1 + g^*} c^* \Phi^\kappa_c\right]}_{<0} + \underbrace{\Phi^\kappa_B}_{>0} \underbrace{\left[\Phi^B_\kappa - B^*\left(1 - \delta^B\right) g^B_\kappa\right]}_{>0} > 0,$$

which implies that either all three roots of $\mathcal{C}$ are smaller than one or that one is smaller than one and two are greater than one. Since the three roots of $\mathcal{C}(\lambda_p\lambda_q)$ are given by the three product pairs of the eigenvalues of $c(\lambda)$ it follows that only alternative $(\alpha 2)$ is compatible with $c(1) < 0$ and $\mathcal{C}(1) > 0$. Therefore, we may conclude that if all eigenvalues are real and positive, the system is asymptotically locally stable in the state space.

(β) Consider now the case of one real eigenvalue and a pair of complex eigenvalues. Without loss of generality let $\lambda_1 > 1$ be the real eigenvalue and let λ_2, λ_3 be the complex conjugate pair. First notice that in this case only one root of $\mathcal{C}(\lambda_p\lambda_q)$ is real, namely, the product of the two complex conjugate eigenvalues of $c(\lambda)$. Then the fact that $\mathcal{C}(1) > 0$ implies that the only real root of $\mathcal{C}$ must be smaller than one which is only possible if $\mid \lambda_{2,3} \mid = \mid x \pm \omega i \mid < 1$. Therefore, we may conclude that if the system features one real eigenvalue and a pair of complex conjugate eigenvalues, it will be asymptotically locally stable in the state space. ■

8.8 Proof of Proposition 8

Proposition 4 claims that for given $L > 0$ and initial values $(A_{-1}, K_0) > 0$ the transitional dynamics of the dynamic competitive equilibrium is given by a unique sequence $\{\kappa_t, c_t\}_{t=0}^{\infty}$. To prove this we first derive the two-dimensional system of Proposition 4. Then, we show that for any admissible $(\kappa_t, c_t) > 0$ there is a corresponding unique value $\{\kappa_{t+1}, c_{t+1}\}$ for all t.

1. To obtain the two-dimensional system we proceed as follows. To derive (4.24) use equations 8.1 and 3.18 in (3.16) to substitute for w_t in the household's budget constraint. We may then write the latter as

$$K_{t+1} = f'(\kappa_t)K_t + A_t L\Big(f(\kappa_t) - \kappa_t f'(\kappa_t) - i(g^A(\kappa_t))\Big) - C_t + (1-\delta^K)K_t.$$

By definition we have $K_t = (A_t L)\,\kappa_t$ and $C_t = A_t L c_t$ so that above equation may be written in terms of efficiency units as

$$\frac{A_{t+1}}{A_t}\kappa_{t+1} = f\,(\kappa_t) - i\left(g^A(\kappa_t)\right) - c_t + (1-\delta^K)\kappa_t.$$

Equation 4.25 is obtained upon employing (3.15) in (3.3).

Now, the system describing the evolution of the economy is three-dimensional and given by

$$\frac{A_{t+1}}{A_t}\kappa_{t+1} = v(\kappa_t) - c_t + (1-\delta^K)\kappa_t, \tag{8.23}$$

$$\frac{A_{t+1}}{A_t}\frac{c_{t+1}}{c_t} = \beta\left(f'(\kappa_{t+1}) + \left(1-\delta^K\right)\right), \tag{8.24}$$

$$A_t = A_{t-1}\left(1-\delta^A\right)(1+g^A(\kappa_t)), \tag{8.25}$$

where $v(\kappa_t) = f(\kappa_t) - i\left(g^A(\kappa_t)\right)$. Observe that this system of three first-order, non-linear difference equations may be reduced to a system of two equations. Forwarding (8.25) and substituting in (8.23) and (8.24) one obtains the two-dimensional system of Proposition 8.

Given $L > 0$ and initial values $\left(A_{-1}, K_0\right) > 0$, eq. (4.27) admits a unique solution $\kappa_0 > 0$. To show uniqueness of the equilibrium sequence $\{c_t, \kappa_t\}_{t=0}^{\infty}$ consider first equation (4.24) and define

$$\Psi(\kappa_{t+1}) \equiv (1+\lambda)(1-\delta)\Big(1+g^A(\kappa_{t+1})\Big)\kappa_{t+1} \tag{8.26}$$

For any $(\kappa_t, c_t) \in \mathbb{R}^2_{++}$ such that the right-hand side of (4.24) is strictly positive, there will be a unique value of $\kappa_{t+1} > 0$ satisfying equation (4.24) if $\Psi(\kappa_{t+1})$ is strictly positive, continuous and monotone in $\kappa_{t+1} > 0$ and may take any value in $\mathbb{R}_{++}$.

Observe that g^A has the same properties as established in Proposition 1 for the economy with capital-augmenting technical change, so that indeed $\Psi(\kappa_{t+1}) > 0$ for all $\kappa_{t+1} > 0$.

It remains to be shown that $\lim_{\kappa\to 0} \Psi^{\kappa}(\kappa_{t+1}) = 0$ and $\lim_{\kappa\to\infty} \Psi^{\kappa}(\kappa_{t+1}) = \infty$. Since g^A is bounded below by zero and may be finite or infinite for $\kappa \to \infty$ we have that $\lim_{\kappa\to 0} \Psi^{\kappa}(\kappa_{t+1}) = 0$ and $\lim_{\kappa\to\infty} \Psi^{\kappa}(\kappa_{t+1}) = \infty$.

It follows that the left hand side of (4.24) is increasing in $\kappa_{t+1} > 0$, approaches zero as $\kappa \to 0$ and approaches infinity as $\kappa \to \infty$. Therefore, there is a unique $\kappa_{t+1} > 0$ that satisfies eq. (4.24) for any $(\kappa_t, c_t) \in \mathbb{R}_{++}$ such that the right-hand side of (4.24) is strictly positive.

Given this value of κ_{t+1} and $c_t > 0$, (4.25) delivers a unique $c_{t+1} > 0$. ■

8.9 Proof of Proposition 9

We consider both claims of Proposition 9 in turn.

1. Equations (4.29) and (4.30) follow immediately from the corresponding equations (4.24)-(4.25) of the dynamical system for reasons discussed in the main text. Obviously, in the steady state the transversality condition (4.26) is also satisfied. Assumption 3 ensures a strictly positive solution to (**??**). It remains to be shown that $c^* > 0$, i. e., $v(\kappa^*) > \kappa^*(g^* + \delta^K)$. A sufficient condition for this to hold may be obtained for $\beta = 1$. Using $\beta = 1$ in (4.30) the latter inequality may be expressed as

$$v(\kappa^*) > \kappa^* f'(\kappa^*). \tag{8.27}$$

Since F has constant returns to scale it holds in equilibrium that $f(\kappa) = \kappa f'(\kappa) + c(q^A)$. Evaluating the latter for the steady state yields $v(\kappa^*) = \kappa^* f'(\kappa^*) + (1 + (g^A)^*)\, i'(g^A)$, where g^A is evaluated at κ^*. Then, inequality (8.27) becomes

$$\begin{aligned} \kappa^* f'(\kappa^*) + \left(1 + \left(g^A\right)^*\right) i'\left(\left(g^A\right)^*\right) &> \kappa^* f'(\kappa^*), \\ \left(1 + \left(g^A\right)^*\right) i'\left(\left(g^A\right)^*\right) &> 0. \end{aligned}$$

2. The expression for the steady-state growth rate follows from (3.27) and Proposition 1. The explanation of the other findings is contained in the main text. ■

8.10 Proof of Proposition 10

1. Implicit differentiation of (4.30) reveals that $d\kappa^*/d\beta > 0$, hence $(\kappa^*)' > \kappa^*$. Moreover, $dg^*/d\beta = \left(dg^*/dq^A\right)\left(dg^A(\kappa^*)/d\kappa\right)\left(d\kappa^*/d\beta\right) > 0$ as all three derivatives are strictly positive. Hence, $(g^*)' > g^*$. Diminishing returns to capital and $d\kappa^*/d\beta > 0$ deliver $(R^*)' < R^*$. Since $v'(\kappa^*)$ cannot be signed in general, the effect of the discount factor on c^* is indeterminate in general.
2. Implicit differentiation of the Euler equation (4.30) delivers $d\kappa^*/d\delta^A > 0$, hence $(\kappa^*)' > \kappa^*$. In conjunction with diminishing returns to capital, we have $(R^*)' < R^*$. The effect of δ^A on g^* is immediate from

   $$\frac{\partial g^*}{\partial \delta^A} = -\left(1 + g^A(\kappa^*)\right) < 0.$$

   Again, the effect on c^* through equation (4.29) remains indeterminate in general. ■

8.11 Proof of Proposition 11

To prove Proposition 11 apply the definition of the elasticity of substitution given in (5.4) to

$$MRTS_{K,L}(\kappa_t) = \frac{B_{t-1}\left(1-\delta^B\right)\left(1+g^B(\kappa_t)\right)\left[f'(\kappa_t) - i\left(g^B(\kappa_t)\right)\right]}{A_{t-1}\left(1-\delta^A\right)\left(1+g^A(\kappa_t)\right)\left[f(\kappa_t) - \kappa_t f'(\kappa_t) - i\left(g^A(\kappa_t)\right)\right]}.$$

■

8.12 Proof of Proposition 12

Statements 1 and 2 of Proposition 12 are obtained by taking respectively the partial derivative of (5.9) and of (5.11) with respect to K_t/L. ■

8.13 Proof of Proposition 13

To be found in the main text. ■

8.14 Proof of Proposition 14

The proof of equations (5.19) and (5.20) are to be found in the main text. As to equation (5.21) consider the effect of τ on s_τ^*,

$$\frac{ds_\tau^*}{d\tau} = \frac{-\kappa^* B_\tau^*\left(f' - i(g^B)\right)\left(B_\tau^*\left(f' - i(g^B)\right) - \delta^K\right) + \kappa^*\delta^K\left(B_\tau^*\left(f' - i(g^B)\right) - \delta^K\right)}{\left(B_\tau^*\right)^2\left(f' - i(g^B)\right)\left(f - \kappa^* f' - i(g^A)\right)},$$

where it is understood that f, g^B, g^A are evaluated at κ^*. Then, it is straightforward to see that $ds_\tau^*/d\tau < 0$ since $R_\tau^* = B_\tau^*\left(f' - i(g^B)\right) > \delta^K$. ■

8.15 Proof of Proposition 15

The proof of equation (5.30) may be found in the main text.

To prove (5.31) consider first the Euler equation in steady state which is

$$1 + g_{\sigma^B}^* = \beta\left(R_{\sigma^B}^* + 1 - \delta^K\right),$$

Taking the derivative with respect to σ^B on both sides it is clear that $dR_{\sigma^B}^*/d\sigma^B > 0$ must hold to match $dg_{\sigma^B}^*/d\sigma^B > 0$. To prove that the steady-state wage rate in efficiency units is also increasing in σ^B consider

$$\begin{aligned}\left(\frac{w_t}{A_t}\right)^* &= f\left(\kappa_{\sigma^B}^*\right) - \kappa_{\sigma^B}^* f'\left(\kappa_{\sigma^B}^*\right) - i\left(g^A\left(\kappa_{\sigma^B}^*\right)\right), \\ &= \left(1 + g^A\left(\kappa_{\sigma^B}^*\right)\right) i'\left(\kappa_{\sigma^B}^*\right).\end{aligned}$$

Implicit differentiation reveals the effect of an increase in σ^B on $(w_t/A_t)^*$ is given by

$$\left(i'\left(g^A(\kappa_{\sigma^B}^*)\right) + \left(1 + g^A(\kappa_{\sigma^B}^*)\right) i''\left(g^A(\kappa_{\sigma^B}^*)\right)\right)\frac{d\kappa_{\sigma^B}^*}{d\sigma^B} g_\kappa^A\left(\kappa_{\sigma^B}^*\right) > 0.$$

As to equation (5.32) we note that all effects of σ^B on $s_{\sigma^B}^*$ are identified and signed in the main text. The algebra necessary to determine the overall effect is straightforward but quite involved. Moreover, it does not reveal clear-cut results. We therefore abstain from presenting it here. ■

8.16 Proof of Proposition 16

The proof of (5.40) as well as the one of $\left(R^*_{\sigma^A}\right)' > R^*_{\sigma^A}$ as claimed in (5.41) can be found in the main text.

The proof of the second claim in (5.41) needs the assumed functional form of the innovation investment function $i\left(q^A\right) = \gamma\left(q^A\right)^\nu$ with $\gamma > 0$ and $\nu > 1$. To see this, consider

$$\left(\frac{w_t}{A_t}\right)^* = f\left(\kappa^*\right) - \kappa^* f'\left(\kappa^*\right) - \left(1-\sigma^A\right) i\left(g^A\left(\kappa^*, \sigma^A\right)\right).$$

Implicit differentiation reveals that

$$\frac{d\left(\frac{w_t}{A_t}\right)^*}{d\sigma^A} = i\left(g^A\left(\kappa^*, \sigma^A\right)\right) - \left(1-\sigma^A\right)\left(g^A_{\sigma^A}\left(\kappa^*, \sigma^A\right) i'\left(g^A\left(\kappa^*, \sigma^A\right)\right)\right).$$

Hence, the wage rate in efficiency units is increasing, remains constant, or, decreasing in σ^A if and only if

$$i\left(g^A\right) \gtreqless \left(1-\sigma^A\right) g^A_{\sigma^A} i'\left(g^A\right). \tag{8.28}$$

The first-order condition (5.33) delivers

$$g^A_{\sigma^A} = \frac{dq^A}{d\sigma^A} = \frac{\left(1+q^A\right) i'\left(q^A\right) + i\left(q^A\right)}{\left(1-\sigma^A\right)\left(\left(1+q^A\right) i''\left(q^A\right) + 2i'\left(q^A\right)\right)}.$$

Evaluating the latter in steady state and substituting into (8.28) yields

$$\frac{d\left(\frac{w_t}{A_t}\right)^*}{d\sigma^A} \gtreqless 0 \quad \text{if and only if} \quad \frac{1}{1+g^A} \gtreqless \frac{i'}{i} - \frac{i''}{i'},$$

where it is understood that g^A is evaluated at $\left(\kappa^*, \sigma^A\right)$ and i is evaluated at g^A. Now, observe that the left-hand side of the above inequality is smaller than one. It is then straightforward to show that the right-hand side is always greater than the left-hand side provided $i\left(q^A\right) = \gamma\left(q^A\right)^\nu$, with $\nu > 1, \gamma > 0$.

Turning to (5.42) observe that the effect of changing σ^B on $s^*_{\sigma^B}$ of equation (5.29) is indeterminate in general. However, if $i\left(q^A\right) = \gamma\left(q^A\right)^\nu$, with $\nu > 1, \gamma > 0$, then some tedious but straightforward algebra using (5.42) reveals that $\left(s^*_{\sigma^A}\right)' > s^*_{\sigma^A}$. ■

8.17 Proof of Proposition 17

We consider each claim of Proposition 17 in turn.

1. Equations (6.13) and (6.14) follow immediately from equations (6.2) and (6.3) for reasons given in the main text. Moreover, the tranversality conditions are satisfied. Assumption 1 and Assumption 2 guarantee a strictly positive solution to (6.11) and (6.14). Showing that $c^{**} > 0$ is analogous to showing that $c^* > 0$ in the competitive equilibrium.

2. The expression for the steady-state growth rate, g^{**}, follows from (3.27) and Proposition 1. The remaining results follow from the discussion in the main text. ■

8.18 Proof of Proposition 18

With $\sigma^B \in (0,1)$ the first-order condition (3.15) is replaced by (5.22). In steady state, the latter gives rise to a function $\kappa\left(\sigma^B\right)$ with $\kappa'\left(\sigma^B\right) > 0$ that satisfies

$$f'\left(\kappa\left(\sigma^B\right)\right) = \left(1-\sigma^B\right) c\left(\frac{\delta^B}{1-\delta^B}\right). \tag{8.29}$$

Hence, the desired value for σ^B is such that $\kappa\left(\sigma^B\right) = \kappa^{**}$. Using (6.4) gives $\left(\sigma^B\right)^{**}$ of (6.15).

With $\sigma^A \in (0,1)$ the first-order condition (3.16) becomes

$$f\left(\kappa_t\right) - \kappa_t f'\left(\kappa_t\right) = \left(1-\sigma^A\right) c\left(q_t^A\right). \tag{8.30}$$

Using (6.5) at $\kappa_t = \kappa^{**}$ and $q_t^A = \left(q^A\right)^{**}$, the latter determines $\left(\sigma^A\right)^{**}$ as stated in (6.16). ■

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