Endogenous ambiguity in cheap talk

Christian Kellner and Mark Thordal-Le Quement
University of Bonn

Abstract

We provide a rationale for ambiguous communication. We do so by considering a cheap talk game in which a (possibly ambiguity averse) sender (S) able to randomize according to unknown probabilities faces an ambiguity averse receiver (R). We show that under fairly general circumstances, there exist equilibria featuring Ellsbergian communication strategies that allow both S and R to obtain a higher ex ante payoff than any non Ellsbergian equilibrium. Ambiguity allows to shift R's response to information towards S's favorite action. R also benefits because ambiguous equilibria involve a larger amount of information transmission.

Keywords: cheap talk, ambiguity.

JEL classification: D81, D83.

1 Introduction

Ambiguous language abounds in everyday interaction. We think of ambiguous language as different from vague language. *A vague* statement is one that generates (and is intended to generate) a commonly agreed and imprecise interpretation. An *ambiguous* statement instead suggests a variety of more precise meanings and hints that if one were equipped with the right key, one could successfully decode the more informative statement nested within it.

This paper rationalizes ambiguous language by showing that under fairly general circumstances, a rational sender (S) facing an ambiguity averse receiver (R) (and possibly herself ambiguity averse, which plays no role in our analysis) can benefit from the use of ambiguous communication strategies. This holds true in the sense that the maximal equilibrium expected payoff obtainable by S is achieved in equilibria in which S makes use of an ambiguous communication strategy. Ambiguity allows S to mitigate conflict by shifting R's response to information in a direction more congruent with S's preferences. We furthermore show that R may also benefit from equilibrium ambiguity. While R's ability to optimally respond to given information is negatively affected by the added ambiguity, ambiguous equilibria feature more information transmission (more partitions or more equally spaced ones).

Communication is ambiguous here in the sense that S randomizes over messages by conditioning on a draw from an Ellsberg urn containing red and black balls. A so-called Ellsbergian partitional communication strategy simply adds an element of Ellsbergian randomization to the classical partitional communication strategy. Consider the following simple 2-partitions Ellsbergian communication strategy. Let there be four messages $m_1^A, m_1^B, m_2^A, m_2^B$. Let there be three thresholds $t \in (0,1)$, $c_1 \in (0,t]$ and $c_2 \in (t,1]$. In other words, t splits $[0,1]$ into two partitions $[0,t]$ and $(t,1]$ while c_1 and c_2 split each of these partitions into two subpartitions. We may now describe S's strategy:

- If $\omega \in [0,t]$ and S draws a red ball, she sends m_1^A if $\omega \leq c_1$ and m_1^B if $\omega > c_1$.
- If $\omega \in [0,t]$ and S draws a black ball, she sends m_1^B if $\omega \leq c_1$ and m_1^A if $\omega > c_1$.
- If $\omega \in (t,1]$ and S draws a red ball, she sends m_2^A if $\omega \leq c_2$ and m_2^B if $\omega > c_2$.
- If $\omega \in (t,1]$ and S draws a black ball, she sends m_2^B if $\omega \leq c_2$ and m_2^A if $\omega > c_2$.

In other words, in an equilibrium featuring the above communication strategy, the subscript of the message received by R allows her to learn with certainty whether $\omega \leq t$ or $\omega > t$. Conditional on receiving a message endowed with a subscript $i = 1, 2$, it however remains unclear whether $\omega \leq c_i$ or $\omega > c_i$. Our equilibrium construction thus features two types of uncertainty. Classical probabilistic uncertainty relates to the standard partitions denoted by message subscripts. Knightian uncertainty relates to the subpartitions denoted by message superscripts. This relates to our definition of ambiguous statements: The missing key here is the share ρ of red balls contained in the Ellsberg urn used by S to randomize over messages. If this share were known, R could indeed infer a more precise meaning from the messages that she receives in equilibrium.

The Ellsbergian randomization featured in equilibria also relates to the concept of Necker messages that we have introduced for didactic purposes in a previous paper (Kellner and Le Quement (2013)). A Necker message is one that, if picked by S, determines a distribution p over possible observed messages, just as a Necker cube drawing determines a distribution over three dimensional cube interpretations by receivers. The added element here would be that the distribution p is unknown in the knightean sense. The above described Ellsbergian partitional communication strategy could be implemented by the use of four Necker messages, each of them being described by a vector of probabilities $(p_1^A, p_1^B, p_2^A, p_2^B)$ over $m_1^A, m_1^B, m_2^A, m_2^B$. Let messages be defined as follows.

If $\omega \in [0, c_1)$, S sends $m_I = (\rho, 1-\rho, 0, 0)$.
If $\omega \in [c_1, t)$, S sends $m_{II} = (1-\rho, \rho, 0, 0)$.
If $\omega \in [t, c_2)$, S sends $m_{III} = (0, 0, \rho, 1-\rho)$.
If $\omega \in [c_2, 1)$, S sends $m_{IV} = (0, 0, 1-\rho, \rho)$.
Assume that ρ is unknown.

Technically speaking, our analysis makes use of concepts initially introduced in Riedel and Sass (2014) and Bose and Renou (2014). Riedel and Sass (2014) introduce the general idea of Ellsberg strategies, i.e. randomizing conditional on draws from an Ellsberg urn. Bose and Renou (2014) explore the idea of belief manipulation through communication devices that generate ambiguous beliefs by generating signals according to an unknown distribution. Bose and Renou (2014) analyze a mechanism design problem and do not, as far as known to us, examine implications for the classical cheap-talk game.

Our contribution lies at the intersection of the literatures on respectively cheap talk communication and ambiguity. The first was initiated by the seminal model of Crawford and Sobel (1982) (CS in what follows). Our analysis bears many connections to Board, Blume and Kawamura (2007) as well as Blume and Board (2013, 2014). The endogeneous randomization over messages featured in our model is reminiscent of the exogenous randomization studied in the first of these papers. In the latter model, an emitted message may be randomly swapped with another during the transmission process. This helps mediate conflict and allows more information communication to be incentive compatible, to the extent that exogenous randomization may be welfare beneficial. Note however that if S had access to non-noisy messages, she would strictly favour these over noisy messages. In Blume and Board (2014), a sender instead voluntarily adds vagueness to her messages because this allows her to mitigate conflict with receivers. The type of vagueness inserted into messages is one that corresponds to our definition of ambiguous communication: It increases the likelihood that an identical message is interpreted differently by different receivers. A key aspect of the above mentioned papers is that they all operate within a non-knightian expected utility framework. Our main contribution thus resides in the introduction of knightian uncertainty.

Our paper also relates to a rich literature of ambiguity. We model rational behavior in the presence of ambiguity based on the max-min model (Gilboa (1987), Gilboa and Schmeidler (1989)). A consensus has yet to emerge on the right modelling of updating of ambiguity averse preferences. We refer to Siniscalchi (2011) as well as Hanany and Klibanoff (2007, 2009) for a discussion of this issue. Recently, ambiguity has been brought to strategic settings by a number of authors. Azrieli and Teper (2011), Bade (2010) as well as Riedel and Sass (2011) define general equilibrium concepts under ambiguity. A large array of papers study more specific applications to finance, tournaments or contract theory. More directly related contributions include a number of studies of mechanism design under ambiguity (Bose and Renou (2014), Di Tillio et al. (2011)). Kellner and Le Quement (2014) examines cheap talk communication in a simple binary model featuring an ambiguous state distribution. Equilibrium communication exhibits randomization by both S and R. We argue that these two forms of randomization correspond to two different modes of ambiguous communication. Increased bias may be advantageous to S in the presence of restrictions on the message space and increased ambiguity may require the use of more messages in S-optimal equilibria.

2 Model

There are two players, a sender S and a receiver R. The state of the world ω is drawn from a commonly know distribution endowed with the continuously differentiable cdf F and density f on the support $[0,1]$. S can choose a message $m \in M$, where M is a rich message space. R can pick an action a belonging to the set of real numbers. The timing of the game is as follows. S learns the value of ω and sends a message m. R takes an action a after having observed m. None of the two players has any commitment ability. The game ends and payoffs are realized. Given an action and a state, the utility functions of respectively S and R are given by $U^S(a,\omega)$ and $U^R(a,\omega)$. We have

$$U^S(a,\omega) = G(\omega + \beta(\omega) - a),$$

where $\beta(\omega) > 0$, $\forall \omega \in [0,1]$ and $G(x)$ is a concave and single peaked function of x. If We also have

$$U^R(a,\omega) = G(a - \omega).$$

The utility functions satisfy the following further conditions, letting subscripts denote partial derivatives. For $j = R, S$, $U_1^j = 0$ for some a, $U_{11}^j < 0$, $U_{12}^j > 0$. We furthermore assume that both S and R are ambiguity averse and react to ambiguous information by following the max-min decision rule proposed by (Gilboa (1987), Gilboa and Schmeidler (1989)).

In the classical non knightian framework, a communication strategy is simply a mapping from $[0,1] \to \Delta^M$, where Δ^M is the set of distributions over the message space M. A central class of communication strategies is given by the set of partitional strategies. A partitional communication strategy is described by a vector of thresholds $t_0 = 0 < t_1 < . < t_N = 1$ s.t. all sender types belonging to the same interval $(t_i, t_{i+1}]$ send the same message m_i. We now introduce the notion of Ellsbergian partitional communication strategies. We first introduce the notion of Ellsberg randomization.

Definition 1 ***Ellsberg randomization***

Let S have access to an Ellsberg urn containing exclusively red and black balls. The share ρ of red balls is unknown and belongs to $[0,1]$. Consider an interval $(\underline{\omega}, \overline{\omega}] \subseteq [0,1]$ and let

$c \in (\underline{\omega}, \overline{\omega}]$. Let there be two messages m and m'. The Ellsberg randomization $\varphi(\underline{\omega}, \overline{\omega}, c, m, m')$ consists in sending messages according to the following process. If S picks a red ball, she sends m if $\omega \in (\underline{\omega}, c)$ and m' if $\omega \in [c, \overline{\omega}]$. If S instead picks a black ball, she sends m' if $\omega \in (\underline{\omega}, c)$ and m if $\omega \in [c, \overline{\omega}]$.

An Ellsberg randomization is thus a randomization with unknown probabilities over two classical partitional messaging strategies on the interval $(\underline{\omega}, \overline{\omega}]$, each strategy being the mirror image of the other.

Definition 2 ***Ellsbergian partitional communication strategy***

Let there be two profiles of thresholds $t_0 = 0 < t_1 < . < t_{N-1} < t_N = 1$ and $0 \leq c_{0,1} < c_{1,2} < . < c_{N-1,N} \leq 1$ s.t for every $i \in \{1, ., N-1\}$, $c_{i,i+1} \in (t_i, t_{i+1}]$. If $\omega \in (t_i, t_{i+1}]$, for $i \in \{0, ., N-1\}$, S uses the simple Ellsberg randomization $\varphi\left(t_i, t_{i+1}, c_{i,i+1}, m_i^A, m_i^B\right)$. An Ellsbergian partitional communication strategy is thus summarized by $\{t_r\}_{r=1}^{N-1}$ and $\{c_{r,r+1}\}_{r=1}^{N-1}$.

An Ellsbergian partitional communication strategy simply adds an element of randomization to the classical non-Ellsbergian partitional communication strategy. It specifies a simple two steps procedure. S first determines in which partition $(t_i, t_{i+1}]$ the state ω is located. She then applies the Ellsberg randomization $\varphi\left(t_i, t_{i+1}, c_{i,i+1}, m_i^A, m_i^B\right)$. We say that the profile $\{c_{r,r+1}\}_{r=1}^{N-1}$ is compatible with the profile $\{t_r\}_{r=1}^{N-1}$ if for every $i \in \{1, ., N-1\}$, $c_{i,i+1} \in (t_i, t_{i+1}]$. In an Ellsbergian partitional equilibrium, we still refer to any given $(t_i, t_{i+1}]$ as a partition and to N as the number of partitions featured in the considered equilibrium.

We examine Perfect Bayesian equilibria of our game, which implies that each player chooses the ex post optimal action given available information and the known equilibrium communication strategy of the opponent. Players respond to ambiguity by engaging in prior by prior Bayesian updating and subsequently deciding according to the max-min rule.

3 The non-Ellsbergian case

We start by observing that our utility functions can in a specific sense be interpreted as a special case of those assumed in CS. The insight is important because it allows us to directly invoke comparative static results obtained in CS, as we shall do in Proposition 1. Consider two hypothetical senders with utility function $G(\omega + \beta(\omega) - a)$ and $G(\omega + \beta'(\omega) - a)$ s.t. for any ω, $\beta'(\omega) > \beta(\omega)$. One sender is thus strictly more biased than the other for any ω. We show below that the two utility functions can be generated from a common underlying function $U^S(a, \omega, b)$ satisfying the assumptions made in the comparative statics section of CS. I.e. $U^S(a, \omega, b)$ is s.t. b is a scalar parameter measuring interest misalignment, $U^S_{13} \geq 0$ everywhere and $U^S(a, \omega, 0) = U^R(a, \omega)$.

Remark 1 *Consider $G(\omega + \beta(\omega) - a)$ and $G(\omega + \beta'(\omega) - a)$ s.t. for any ω, $\beta'(\omega) > \beta(\omega)$. One can find a function $U^S(a, \omega, b)$ s.t:*
a) $U^S(a, \omega, 1) = G(\omega + \beta(\omega) - a)$.
b) $U^S(a, \omega, 2) = G(\omega + \beta'(\omega) - a)$.
c) $U^S(a, \omega, 0) = U^R(a, \omega)$.
d) $U^S_{13}(a, \omega, b)$ is strictly positive everywhere.

Proof: The proof is constructive. Simply define

$$U^S(a, \omega, 0) = \begin{cases} G(a - \omega - (\beta(\omega) + (b-1)\left[\beta'(\omega) - \beta(\omega)\right])) \text{ if } b \geq 1 \\ G\left(a - \omega - \left(\beta(\omega) b^{\frac{\beta'(\omega) - \beta(\omega)}{\beta(\omega)}}\right)\right) \text{ if } b \leq 1. \end{cases}$$

This function clearly satisfies a), b) and c). As to d), note that $U^S_1(a, \omega, b) = G'(.)$ and that

$$U^S_{13}(a, \omega, b) = \begin{cases} -G''(.)\left[\beta'(\omega) - \beta(\omega)\right] \text{ if } b \geq 1 \\ -G''(.)\left[\beta'(\omega) - \beta(\omega)\right] b^{\frac{\beta'(\omega) - \beta(\omega)}{\beta(\omega)}} \text{ if } b \leq 1. \end{cases}$$

Given that G'' is negative everywhere and $\beta'(\omega) - \beta(\omega) > 0$, it follows that $U^S_{13} > 0$ everywhere. Note also that $U^S_1(a, \omega, b)$ is indeed continuously differentiable in b given that $\lim_{b \to 1-} U^S_{13}(a, \omega, b) = \lim_{b \to 1+} U^S_{13}(a, \omega, b) = 1$.■

We know from CS that in the absence of Ellsbergian strategies, any equilibrium of the game is equivalent to a partitional equilibrium. Denote by $a^*_{ne}(t_{i-1}, t_i)$ the optimal action of R conditional on $\omega \in (t_{i-1}, t_i]$. The profile of thresholds $\{t_r\}_{r=1}^{N-1}$ constitutes an equilibrium if and only if for any $i \in \{1, ., N-1\}$

$$U^S(a^*_{ne}(t_{i-1}, t_i), t_i) = U^S(a^*_{ne}(t_i, t_{i+1}), t_i). \tag{1}$$

CS states the following monotonicity condition:

Condition M For a given value of b, if t and $\widetilde{t}$ are two solutions of (1) with $t_0 = \widetilde{t}_0$ and $\widetilde{t}_1 > t_1$, then $\widetilde{t}_i > t_i$, for any $i \geq 2$.

There are two sets of known sufficient condition for condition M. One appears in CS and is given as follows:

$$U^S_2(a, \omega) + U^S_1(a, \omega) \text{ is nondecreasing in } a \tag{2}$$

and

$$\int_0^d U^R_{11}(a, \omega) f(\omega) d\omega + U^R_1(a, d) f(d) \text{ is non decreasing in } d. \tag{3}$$

A second set of sufficient conditions appears in Szalay (2012) and is given as follows:

$$U^R_1(a, \omega) + U^R_2(a, \omega) \text{ is nondecreasing in } a \tag{4}$$

and

$$f(\omega) \text{ is log concave.}$$

In what follows, for two bias function $\beta(\omega)$ and $\beta'(\omega)$ s.t. $\beta'(\omega) \geq \beta(\omega)$ for any ω, with strict inequality for some ω, we simply write $\beta' > \beta$. For any given N, let $\Gamma(N)$ denote the set of bias functions for which there exists an N-partitions equilibrium.

Proposition 1 *Assume that condition M is satisfied and assume that S is restricted to using non-Ellsbergian strategies.*

1. For any given bias function β, there is a finite $\overline{N}(\beta)$ s.t for any integer $N \leq \overline{N}(\beta)$, there exists a unique equilibrium featuring N partitions. Call it $E(\beta, N)$ and denote the corresponding *profile of thresholds by* $\{t_r(\beta,N)\}_{r=1}^{N-1}$. *Given* β, *there exists no equilibrium featuring strictly more than* $\overline{N}(\beta)$ *partitions.*

2. $\overline{N}(\beta) \geq \overline{N}(\beta')$ *if* $\beta' > \beta$.

3. Denote by $\pi^j(\beta, E(\beta,N))$ *the expected payoff of* $j = S, R$ *in* $E(\beta,N)$.

a) $\pi^R(\beta, E(\beta,N)) > \pi^R(\beta', E(\beta',N))$ *for* $\beta < \beta' \in \Gamma(N)$.

b) $\pi^j(\beta, E(\beta,N-1)) < \pi^j(\beta, E(\beta,N))$ *for* $\beta \in \Gamma(N)$, *for* $j \in \{S,R\}$.

Point 3.a), for the case of R, corresponds to Theorem 4 in CS. Point 3.b) corresponds to Theorems 3 and 5 in CS.■

In w

We state the following result which does not appear in CS. In what follows, for any given function β and $\varepsilon > 0$, let $\beta + \varepsilon$ denote the function β' s.t. $\beta'(\omega) = \beta(\omega) + \varepsilon$, for any ω.

Lemma 1 *Assume that Condition M is satisfied.* $\pi^S(\beta, E(\beta,N)) > \pi^S(\beta+\varepsilon, E(\beta+\varepsilon,N))$ *for* β *s.t.* $\beta, \beta+\varepsilon \in \Gamma(N)$ *s.t.*

Proof: see in Appendix A.

4 Main analysis

4.1 Preliminary results

We assume that the following joint condition on preferences and utilities is satisfied.

Assumption 1 *Let* $0 \leq \underline{\omega} < \overline{\omega} \leq 1$ *and* $c \in [\underline{\omega}, \overline{\omega}]$.

$$E\left[U^R(a^*_{ne}(\underline{\omega},c),\omega)\,|\omega \in [\underline{\omega},c]\right] > E\left[U^R(a^*_{ne}(\underline{\omega},c),\omega)\,|\omega \in (c,\overline{\omega}]\right], \tag{5}$$

and

$$E\left[U^R(a^*_{ne}(c,\overline{\omega}),\omega)\,|\omega \in (c,\overline{\omega}]\right] > E\left[U^R(a^*_{ne}(c,\overline{\omega}),\omega)\,||\omega \in [\underline{\omega},c]\right].$$

We know that Assumption 1 is satisfied in the case of the Uniform Quadratic model. The condition is compatible with the stated sufficient conditions for Condition M. Evaluating exactly how constraining it is is left for future work. We shall assume from now on that Assumption 1 and Condition M are satisfied.

Denote by $a_e^*(t_i, t_{i+1}, c_{i,i+1})$ R's (identical) best response to messages m_i^A or m_i^B, if these are known to be generated by the Ellsbergian randomization $\varphi\left(t_i, t_{i+1}, c_{i,i+1}, m_i^A, m_i^B\right)$.

Lemma 2 *a) $a_e^*(t_i, t_{i+1}, c_{i,i+1})$ is unique. It satisfies*

$$\int_{t_i}^{c_{i,i+1}} U^R(a_e^*(t_i, t_{i+1}, c_{i,i+1}), \omega) f(\omega)\, d\omega = \int_{c_{i,i+1}}^{t_{i+1}} U^R(a_e^*(t_i, t_{i+1}, c_{i,i+1}), \omega) f(\omega)\, d\omega.$$

b) $a_e^*(t_i, t_{i+1}, c_{i,i+1})$ is a continuous and strictly increasing function of $c_{i,i+1}$ and

$$a_e^*(t_i, t_{i+1}, t_i) < a_{ne}^*(t_i, t_{i+1}) < a_e^*(t_i, t_{i+1}, t_{i+1}).$$

Proof: see Appendix B.

Figure 1 below illustrates the above Lemma and shows how R's best response shifts to the right as $c_{i,i+1}$ moves towards t_{i+1}. We let $t_i = 0$ and $t_{i+1} = .75$ in what follows, and consider the cases of $c_{i,i+1} = .2$ and $c_{i,i+1} = .6$. Continuous curves correspond to $E\left[U^R(a,\omega)\,|\omega \in [0,.2]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [.2,.75]\right]$. Dashed curves correspond to $E\left[U^R(a,\omega)\,|\omega \in [0,.6]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [.6,.75]\right]$.

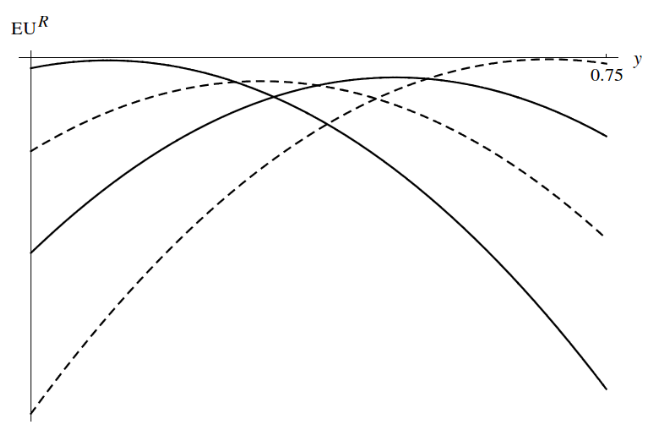

Figure 1.

The best response $a_e^*(t_i, t_{i+1}, c_{i,i+1})$ is simply the action that equalizes the expected payoff of R under both priors $(t_i, c_{i,i+1}]$ and $(c_{i,i+1}, t_{i+1}]$. S can thus bias R's best response to a given partition upwards or downwards by using Ellsberg randomization on a given interval. The intuition is as follows: for a given interval $(t_i, t_{i+1}]$, by shifting the Ellsberg randomization parameter $c_{i,i+1}$ upwards, S shifts upwards the action that equalizes the expected payoff of R under both priors $(t_i, c_{i,i+1}]$ and $(c_{i,i+1}, t_{i+1}]$.

Lemma 3 *An equilibrium featuring the Ellsbergian partitional communication strategy*

$$\left(\{t_r(\beta, N)\}_{r=1}^{N-1}, \{c_{r,r+1}\}_{r=1}^{N-1}\right)$$

exists iff for any $i \in \{1, ., N-1\}$,

$$U^S(a_e^*(t_{i-1}, t_i, c_{i-1,i}), t_i) = U^S(a_e^*(t_i, t_{i+1}, c_{i,i+1}), t_i). \tag{6}$$

Proof: Recall that for any $i \in \{1, ., N-1\}$, m_i^A and m_i^B trigger an identical best response, so that S is indifferent between sending the two messages m_i^A and m_i^B for any $\omega \in (t_i, t_{i+1}]$. It follows that all we need to consider are deviations across messages carrying a different subscript.■

4.2 Main result

Lemma 4 *Let* $N \geq 2$ *and let* β *s.t. the non-Ellsbergian equilibrium* $E(\beta, N)$ *exists with* $t_1(\beta, N) > 0$ *and* $t_{N-1}(\beta, N) < 1$. *There is an* $\bar{\varepsilon} > 0$ *s.t. for any* $\varepsilon \leq \bar{\varepsilon}$, *there exists an equilibrium* $\widetilde{E}(\beta, \beta - \varepsilon, N)$ *featuring an Ellsbergian partitional communication strategy* $\{t_i\}_{i=1}^{N-1}$ *and* $\{c_{i,i+1}\}_{i=1}^{N-1}$ *that satisfies the following:*

a) $t_i = t_i(\beta - \varepsilon, N)$ *and* $a_e^*(t_{i-1}, t_i, c_{i-1,i}) = a_{ne}^*(t_{i-1}, t_i) + \varepsilon$, *for any* $i \in \{1, ., N-1\}$.

b) The expected payoff of S in $\widetilde{E}(\beta, \beta - \varepsilon, N)$ *is equal to* $\pi^S(\beta - \varepsilon, E(\beta - \varepsilon, N))$.

c) The expected payoff of R in $\widetilde{E}(\beta, \beta - \varepsilon, N)$ *is strictly larger than* $\pi^R(E(\beta, N))$.

Proof: See in Appendix C.

We may now state our main result.

Proposition 2 *For any β s.t. there exists an influential equilibrium featuring a non Ellsbergian communication strategy, there exists an equilibrium $\widetilde{E}$ featuring an Ellsberg strategy that improves on the maximal equilibrium payoff of both S and R.*

Proof**:** We know from Proposition 1 that given β, S and R both prefer, among all equilibria, the equilibrium featuring the most steps (which we denote by $E(\beta, N(\beta))$). We also know from Proposition 1 and Lemma 1 that the equilibrium payoff of S and R, given a fixed number of partitions $N \geq 2$, is decreasing in β. Lemma 3 shows that given β, there exists an equilibrium featuring an Ellsbergian strategy ensuring S the payoff $\pi^S(\beta - \varepsilon, E(\beta - \varepsilon, N(\beta)))$, for some $\varepsilon > 0$ and ensuring R a payoff strictly larger than $\pi^R(\beta, E(\beta, N(\beta)))$. ■

Proposition 3 *Let β be s.t. there exists no influential equilibrium if no Ellsbergian strategy is allowed. There exists an equilibrium featuring an Ellsberg strategy in which S obtains an expected payoff strictly larger than the payoff $\pi^S(\beta, E(\beta, 1))$ obtained in the non-Ellsbergian babbling equilibrium.*

Proof: There is an $\bar{\varepsilon} > 0$, s.t. for any $\varepsilon \leq \bar{\varepsilon}$, there exists a $c \in [0,1]$ s.t.

$$a_e^*(0,1,c) = a_{ne}^*(0,1) + \varepsilon.$$

Now, given that we assume $\beta(\omega) > 0$ for any $\omega \in [0,1]$, it follows that there is some $\bar{\delta} > 0$ s.t. for any $\delta \leq \bar{\delta}$,

$$\int_0^1 U^S\left(a_{ne}^*(0,1) + \delta, \omega\right) f(\omega) d\omega > \int_0^1 U^S\left(a_{ne}^*(0,1), \omega\right) f(\omega) d\omega.$$

■

5 The Uniform Quadratic example

We now suppose that ω is uniformly distributed on $[0,1]$ and that utility functions are

$$U^S(\omega, a, b) = -(a - (\omega + b))^2,$$

$$U^R(\omega, a) = -(a-\omega)^2.$$

We know from CS that for any $N \geq 2$, there is a finite $b(N) > 0$ s.t. for $b \leq b(N)$, there exists a unique N-partitions equilibrium. Given $b \leq b(N)$, the unique N-partitions equilibrium is defined by the following arbitrage condition for every $i \in \{1, ., N\}$:

$$-\left(\frac{t_i(N,b)+t_{i+1}(N,b)}{2} - t_i(N,b) - b\right)^2 = -\left(\frac{t_{i-1}(N,b)+t_i(N,b)}{2} - t_i(N,b) - b\right)^2. \tag{7}$$

We first characterize R's best response to messages m_i^A or m_i^B, if these are known to be generated by the Ellsbergian randomization $\varphi\left(t_i, t_{i+1}, c_{i,i+1}, m_i^A, m_i^B\right)$.

Remark 2 .

$$a_e^*(t_i, t_{i+1}, c) = \frac{t_i + t_{i+1} + c}{3}.$$

Proof: **Step 1** Let us simply calculate the expected payoff of a given pure action a given that ω is uniformly distributed on some interval $[c,d] \subseteq [0,1]$.

$$E\left[U^R(a,\omega)\,|\omega \in [c,d]\right] = -\int_c^d (a-\omega)^2 \frac{1}{d-c} d\omega = -a^2 + ac + ad - \frac{1}{3}c^2 - \frac{1}{3}cd - \frac{1}{3}d^2.$$

Now, consider parameters $e < c < d$ s.t. $0 < e$ and $d < 1$. Suppose that R receives a message that leaves her (knighteanly) uncertain as to whether $\omega \in [e,c]$ or $\omega \in (c,d]$. For a given a, R compares $E\left[U^R(a,\omega)\,|\omega \in [e,c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in (c,d\right]$ and computes the following difference:

$$E\left[U^R(a,\omega)\,|\omega \in (c,d\right] - E\left[U^R(a,\omega)\,|\omega \in [e,c]\right] = -\frac{1}{3}(d-e)(c-3a+d+e). \tag{8}$$

If the above expression is positive, then the worse prior for R given a, is $[e,c)$ (i.e. the low interval). Otherwise it is $(c,d]$ (i.e. the high interval). We know that $d-e>0$. Now, the sign of (8) is determined by the sign of $(c+d+e-3a)$. If $a > \frac{e+c+d}{3}$, the expression is positive so that the worse interval for R is the low interval $[e,c]$. On the other hand, if $a < \frac{e+c+d}{3}$, the expression is negative and the worse interval for R is the high interval $(c,d]$. Let us recapitulate. If $a > \frac{e+c+d}{3}$, the loss of R under the worse prior is given by $E\left[U^R(a,\omega)\,|\omega \in [e,c)\right]$. If $a < \frac{e+c+d}{3}$, the loss of R under the worse prior is given by $E\left[U^R(a,\omega)\,|\omega \in [c,d]\right]$.

Step 2 Assume that $\frac{e+c+d}{3} - a < 0$. Now, note that

$$\frac{\partial E\left[U^R(a,\omega)\,|\omega \in [e,c)\right]}{\partial a} = e + c - 2a, \tag{9}$$

which has the same sign as $\frac{e+c}{2} - a$. Given $\frac{e+c+d}{3} - a < 0$, it follows that $\frac{e+c}{2} - a < 0$ because $\frac{e+c+d}{3} > \frac{e+c}{2}$. So the derivative (9) is negative, implying that R should pick the smallest action in the considered set, i.e. $\frac{e+c+d}{3}$. Assume now that $\frac{e+c+d}{3} - a > 0$. Now, note that

$$\frac{\partial E\left[U^R(a,\omega)\,|\omega \in [c,d]\right]}{\partial a} = c + d - 2a, \tag{10}$$

which has the same sign as $\frac{c+d}{2} - a$. Given $\frac{e+c+d}{3} - a > 0$, it follows that $\frac{c+d}{2} - a > 0$ because $\frac{e+c+d}{3} < \frac{c+d}{2}$. So the derivative (10) is positive, implying that R should pick the highest action in the considered set, i.e. $\frac{e+c+d}{3}$. To conclude, it follows that the optimal action of the agent (i.e. the max-min action) conditional on m_i^A or m_i^B is given by $\frac{e+c+d}{3}$, yielding the same payoff

$$E\left[U^R(\frac{e+c+d}{3},\omega)\,|\omega \in [e,c)\right] = E\left[U^R(\frac{e+c+d}{3},\omega)\,|\omega \in [c,d]\right]$$

conditional on any of the two priors under consideration.■

In what follows, we examine how the availability of Ellsberg strategies affects the expected payoff achievable in equilibrium. We consider two types of outcomes that are achievable. First, starting from a given non-Ellsbergian partitional equilibrium and keeping the number of equilibrium partitions fixed, one can achieve a more attractive allocation of partitions and thereby increase the expected payoffs of S and R. This replicates the argument appearing in Lemma 3 and we repeat its main steps for the Uniform Quadratic example for the sake of concreteness. The second type of outcome achievable through Ellsberg strategies, which we did not consider in the main analysis, is an increase in the number of equilibrium partitions. We give a description of the procedure by which this can be accomplished and identify a simple case in which this can be advantageous to S.

Finetuning partitions through Ellsberg strategies We know from Lemma 3 that for a given $N \geq 2$, there is an $\bar{\varepsilon}$ s.t. for any $\varepsilon < \bar{\varepsilon}$, a sender with $b \leq b(N)$ can achieve the payoff $\pi^S(E\,(b-\varepsilon, N))$ by using an Ellsberg strategy. We explicitly construct the strategy involved for the Uniform Quadratic case in what follows. Fix $\varepsilon > 0$. Let the Ellsbergian strategy be given by $\{t_i(N, b-\varepsilon)\}_{i=1}^{N-1}$ and a profile $\{c_{i,i+1}\}_{i=1}^{N-1}$ satisfying, for every $i \in \{1, ., N\}$,

$$c_{i,i+1} = \frac{t_i(N, b-\varepsilon) + t_{i+1}(N, b-\varepsilon)}{2} + 3\varepsilon.$$

In order for our strategy to be well defined, we need to ensure that $c_{i,i+1}$ belongs to $(t_i(N, b-\varepsilon), t_{i+1}(N, b-\varepsilon))$ for every $i \in \{1, ., N\}$. Given that $t_{i+1}(N, b) - t_{i+1}(N, b)$ is increasing in i, we simply need prove that there is some $\bar{\varepsilon}$ s.t. for any $\varepsilon \leq \bar{\varepsilon}$,

$$t_1(N, b-\varepsilon) > \frac{t_1(N, b-\varepsilon)}{2} + 3\varepsilon \Leftrightarrow \frac{t_1(N, b-\varepsilon)}{6} > \varepsilon.$$

Now, if an N-partitions equilibrium exists given b, then $t_1(N, b) > 0$, so there is some $\varepsilon > 0$ s.t. $\frac{t_1(N,b)}{6} > \varepsilon$. Furthermore, $t_1(N, b-\varepsilon) > t_1(N, b)$, for $\varepsilon > 0$. We now examine the question of welfare in the constructed equilibrium. Note the following equality:

$$\begin{aligned} & \sum_0^{N-1} \int_{t_i(N,b-\varepsilon)}^{t_{i+1}(N,b-\varepsilon)} \left(\frac{t_i(N, b-\varepsilon) + t_{i+1}(N, b-\varepsilon) + c_{i,i+1}}{3} - (\omega + b) \right)^2 d\omega \\ = & \sum_0^{N-1} \int_{t_i(N,b-\varepsilon)}^{t_{i+1}(N,b-\varepsilon)} \left(\frac{t_i(N, b-\varepsilon) + t_{i+1}(N, b-\varepsilon)}{2} - \omega - (b-\varepsilon) \right)^2 d\omega. \end{aligned}$$

It follows that the expected payoff of S is indeed $\pi^S(b-\varepsilon, N(b))$.

We now show that R can also benefit from the use of Ellsberg strategies in the Uniform Quadratic case.

Remark 3 *For a given $N \geq 2$, if $b \leq b(N)$, there exists an Ellsberg equilibrium in which R achieves an expected payoff strictly larger than $\pi^R(b, N(b))$. It follows that for any b s.t. that there exists an influential equilibrium in the absence of Ellsberg strategies, there exists an Ellsberg equilibrium that yields a higher payoff than R's preferred non-Ellsbergian equilibrium.*

Proof: We simply give the main argument behind the proof. For given $N \geq 2$ and $b \leq b(N)$, consider the equilibrium constructed above, allowing S to obtain the payoff

$\pi^S(E(b-\varepsilon,N))$. In the latter equilibrium, the loss of R is given by:

$$\sum_{i=1}^{N}\int_{t_{i-1}(b-\varepsilon,N)}^{t_i(b-\varepsilon,N)}\left(\omega-\left(\frac{t_{i-1}(b-\varepsilon,N)+t_i(b-\varepsilon,N)}{2}+\varepsilon\right)\right)^2 d\omega.$$

Noting that:

$$\int_a^b\left(\omega-\frac{a+b}{2}-\varepsilon\right)^2 d\omega=-\frac{1}{12}(a-b)^3-(a-b)\,\varepsilon^2,$$

it follows that the loss of R in the constructed Ellsbergian partitional equilibrium is given by:

$$\begin{aligned}&-\frac{1}{12}\sum_{i=1}^{N}\left(\frac{1}{N}+2(b-\varepsilon)(2i-N-1)\right)^3-\sum_{i=1}^{N}\left(\frac{1}{N}+2(b-\varepsilon)(2i-N-1)\right)\varepsilon^2\\ =\;&-\frac{1}{12N^2}\left(4N^4b^2-8N^4b\varepsilon+4N^4\varepsilon^2-4N^2b^2+8N^2b\varepsilon+8N^2\varepsilon^2+1\right).\end{aligned}$$

Now, note that:

$$\frac{\partial\left(-\frac{1}{12N^2}\left(4N^4b^2-8N^4b\varepsilon+4N^4\varepsilon^2-4N^2b^2+8N^2b\varepsilon+8N^2\varepsilon^2+1\right)\right)}{\partial\varepsilon} \tag{11}$$

$$=\frac{2}{3}b\left(N^2-1\right)-\frac{2}{3}(2+N^2)\varepsilon. \tag{12}$$

Expression (11) implies that for ε sufficiently small, R favours the constructed equilibrium over $E(b,N)$.■

Adding more partitions through Ellsberg strategies We now examine the possibility of increasing the maximum number of equilibrium partitions by making use of an Ellsberg strategy. We first show a simple construction, similar to that used previously to finetune partitions for any given $N\geq 2$ and $b\leq b(N)$, that is guaranteed to also allow an improvement in S's expected payoff.

Remark 4 *Let $N\geq 5$. There is $\overline{\varepsilon}$ s.t. for any $\varepsilon<\overline{\varepsilon}$, if $b=b(N)+\varepsilon$, there exists an Ellsbergian N-partitions equilibrium ensuring S the expected payoff $\pi^S(E(b(N)-\varepsilon',N))$ for some $\varepsilon'>0$.*

Proof: **Step 1** Let $N \geq 5$ and let $b = b(N) + \varepsilon$. It is easily seen that given $\varepsilon > 0$ and $X > 0$ s.t. $b(N) - \frac{\varepsilon}{X} > 0$:

$$t_1(N, b(N) - \frac{\varepsilon}{X}) = \frac{1}{N} + 2\left(\frac{1}{\frac{1}{2}(2N-1)^2 - \frac{1}{2}} - \frac{\varepsilon}{X}\right)(1-N) = 2(N-1)\frac{\varepsilon}{X}.$$

Now, consider the profile of thresholds $t_1(N, b(N) - \frac{\varepsilon}{X}), .., t_{N-1}(N, b(N) - \frac{\varepsilon}{X})$. Construct an Ellsberg partitional strategy by setting for every i,

$$c_{i,i+1} = \frac{t_i(N, b(N) - \frac{\varepsilon}{X}) + t_{i+1}(N, b(N) - \frac{\varepsilon}{X})}{2} + 3(\varepsilon + \frac{\varepsilon}{X}).$$

Note that for given N and $b \leq b(N)$, $t_{i+1}(N, b) - t_i(N, b)$ is increasing in i, which implies that

$$t_{i+1}(N, b) - \frac{t_i(N, b) + t_{i+1}(N, b)}{2}.$$

is increasing in i. In order to check that our strategy constitutes an equilibrium in the sense that $c_{i,i+1} \in (t_i(N, b(N) - \frac{\varepsilon}{X}), t_{i+1}(N, b(N) - \frac{\varepsilon}{X})]$, for any i, we thus simply need to check that $c_{0,1} < t_1(N, b(N) - \frac{\varepsilon}{X})$. Now,

$$\begin{aligned} c_{0,1} &\leq t_1(N, b(N) - \frac{\varepsilon}{X}) \Leftrightarrow \\ (N-1)\frac{\varepsilon}{X} + 3(\varepsilon + \frac{\varepsilon}{X}) &\leq 2(N-1)\frac{\varepsilon}{X} \Leftrightarrow \\ 1 + \frac{1}{X} &< \frac{N-1}{3}. \end{aligned}$$

Clearly, for $N \geq 5$, this equality is satisfied for X large enough. We now examine the question of welfare. If the above constructed N-partitions equilibrium exists, the expected loss of S is given by:

$$\begin{aligned} &\sum_1^N \int_{t_i(N,b(N)-\frac{\varepsilon}{X})}^{t_{i+1}(N,b(N)-\frac{\varepsilon}{X})} \left(\frac{t_i(N, b(N) - \frac{\varepsilon}{X}) + t_{i+1}(N, b(N) - \frac{\varepsilon}{X}) + c_{i,i+1}}{3} - (\omega + b)\right)^2 d\omega \\ = &\sum_1^N \int_{t_i(N,b(N)-\frac{\varepsilon}{X})}^{t_{i+1}(N,b(N)-\frac{\varepsilon}{X})} \left(\frac{t_i(N, b(N) - \frac{\varepsilon}{X}) + t_{i+1}(N, b(N) - \frac{\varepsilon}{X})}{2} - \left(\omega + b(N) - \frac{\varepsilon}{X}\right)\right)^2 d\omega. \end{aligned}$$

It follows that in the obtained equilibrium, S is obtaining payoff $E(b(N) - \frac{\varepsilon}{X}, N)$.■

We now explore a different way of constructing equilibria with an increased number of partitions, for a given b. The advantage of this second method is that it also works for the case of $N \in \{2,3,4\}$, as opposed to the previous construction. We however have no simple characterization regarding the impact on S's expected payoff, though Remark 5 offers a preliminary positive insight.

Remark 5 *Let $N \geq 2$. There is $\bar{\varepsilon}$ s.t. for any $\varepsilon < \bar{\varepsilon}$, if $b = b(N) + \varepsilon$, there exists an Ellsbergian N-partitions equilibrium.*

Proof: We construct the following Ellsbergian partitional communication strategy. Consider an arbitrary profile of thresholds $0 < t_1 < . < t_N = 1$ and set, for every $i \in \{1,.,N\}$, $c_{i,i+1} = t_{i+1}$. For such a simple Ellsbergian N-partitions strategy to be incentive compatible for S, it must be that for every $i \in \{1,.,N-1\}$:

$$-\left(\frac{t_i + 2t_{i+1}}{3} - t_i - b\right)^2 = -\left(\frac{t_{i-1} + 2t_i}{3} - t_i - b\right)^2 \Leftrightarrow \tag{13}$$

$$t_{i+1} = \frac{3}{2}t_i - \frac{1}{2}t_{i-1} + 3b. \tag{14}$$

Solving the above linear difference equation, we obtain a solution parameterized by t_1 that is given as follows:

$$t_i\left(b, t_1\right) = 2^{1-i}\left[6b - 3b2^{i+1} + 3\left(2^i\right)bi - t_1 + 2^i t_1\right].$$

We may now look for the maximal value of b compatible with the existence of a simple Ellsbergian N-partitions equilibrium of the form assumed above. Call this value $b_e(N)$. To find this value, we simply assume $t_1 = 0$ and find that value of b that solves:

$$t_N(b,0) = 1 \Leftrightarrow 2^{1-i}\left[6b - 3b2^{i+1} + 3\left(2^i\right)bi\right] = 1.$$

We obtain

$$b_e(N) = \frac{1}{2^{1-N}\left[6 - 32^{N+1} + 3\left(2^N\right)N\right]}.$$

Now, it is easily shown that $b_e(N) > b\left(N\right), \forall N \geq 2$. Note the following: $b_e(2) = \frac{1}{3} > b(2) = \frac{1}{4}$; $b_e(3) = \frac{2}{15} > b(3) = \frac{1}{12}$, etc.■

The next Remark shows that increasing the number of equilibrium partitions through an Ellsberg strategy of the above type can be advantageous for S.

Remark 6 *Let $b \in (\frac{1}{4}, \frac{1}{3}]$ so that only the babbling equilibrium $E(b,1)$ exists in the absence of Ellsberg strategies. There exists a 2-partitions Ellsbergian equilibrium that ensures S a greater expected payoff than $\pi^S\left(E(b,1)\right)$.*

Proof: Let $b \in (\frac{1}{4}, \frac{1}{3}]$. We know that without an Ellsberg strategy, no influential communication is possible. It follows that S's expected payoff is simply given by $\pi^S(E(b,1)) = -b^2 - \frac{1}{12}$. We also know that there exists a Ellsbergian 2-partitions equilibrium for $b \in (\frac{1}{4}, \frac{1}{3}]$ given that $b_e(2) = \frac{1}{3}$. In this equilibrium, the threshold t_1 solves:

$$\frac{t_1+2}{3} - t_1 - b = t_1 + b - \frac{2t_1}{3}.$$

The solution is given by $t_1 = \frac{2}{3} - 2b$. The expected payoff of S in this equilibrium is given by:

$$\begin{aligned} & -\int_0^{\frac{2}{3}-2b}\left(\left(\frac{2(\frac{2}{3}-2b)}{3}\right)-(\omega+b)\right)^2 d\omega - \int_{\frac{2}{3}-2b}^{1}\left(\left(\frac{\frac{2}{3}-2b+2}{3}\right)-(\omega+b)\right)^2 d\omega \\ = \; & \frac{8}{3}b^3 - \frac{25}{9}b^2 + \frac{11}{27}b - \frac{1}{27}. \end{aligned}$$

It is easily shown that this latter expression is strictly larger than $\pi^S(E(b,1)) = -b^2 - \frac{1}{12}$ for any $b \in (\frac{1}{4}, \frac{1}{3}]$.■

The babbling case For $b \in (\frac{1}{3}, 1]$, we are not able to show that an Ellsberg communication strategy can ensure the existence of an equilibrium featuring influential communication. However, as stated in Proposition 3, there still exists an Ellsbergian equilibrium in which S improves his expected payoff w.r.t. the classical babbling equilibrium. Consider the Ellsbergian communication strategy given by $t_0 = 0, t_1 = 1$ and $c_{0,1} = 1$. The best response ot R to m_1^A or m_1^B is given by $\frac{2}{3}$. Now, note that:

$$-\int_0^1\left(\frac{2}{3}-(\omega+b)\right)^2 d\omega = -b^2 + \frac{1}{3}b - \frac{1}{9} > -\int_0^1\left(\frac{1}{2}-(\omega+b)\right)^2 d\omega = -b^2 - \frac{1}{12}.$$

For any $b \geq \frac{1}{3}$, the above described equilibrium thus improves on $\pi^S\left(E(b,1)\right)$.

The highest of the two curves (in blue) appearing in Figure 2 shows, for every given value of b, the expected payoff obtained by S in the Ellsbergian partitional equilibrium featuring the most partitions, considering only the subclass of Ellsbergian partitional equilibria specified by $c_{i,i+1} = t_{i+1}$ for every i. The lower curve represents S's payoff in the non Ellsbergian equilibrium featuring the most partitions, which is S's preferred equilibrium for a given b. As we we, Ellsbergian equilibria allow to improve S's expected payoff for any b. Figure 3 is an equivalent for R's expected payoff. Consider the curve that is highest for any b small enough (in blue). It shows, for every given value of b, the expected payoff obtained by R in the Ellsbergian partitional equilibrium featuring the most partitions, considering only the subclass of Ellsbergian partitional equilibria specified by $c_{i,i+1} = t_{i+1}$ for every i. The other curve represents R's payoff in the non Ellsbergian equilibrium featuring the most partitions, which is R's preferred equilibrium for a given b. As we see, Ellsbergian equilibria allow to improve R's expected payoff for any b s.t. there exists at least a two partitions Ellsbergian equilibrium within the considered classof partitional Ellsbergian equilibria (i.e. $b \leq b_e(2)$).

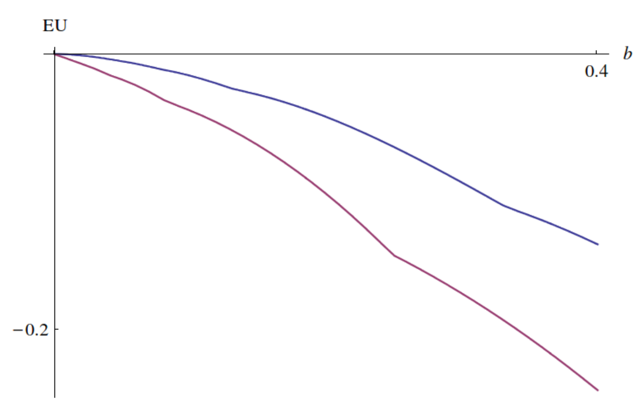

Figure 2.

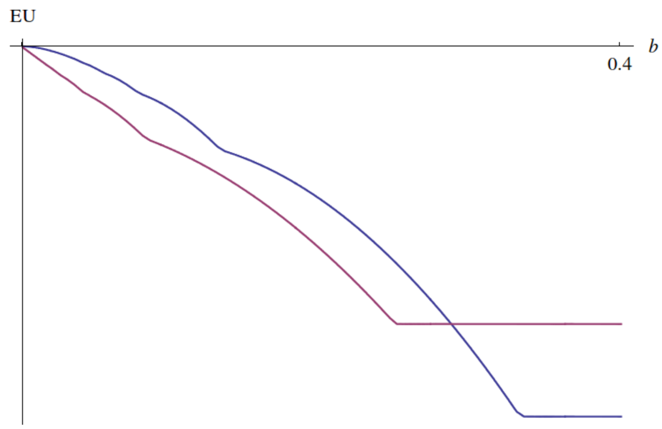

Figure 3.

6 Conclusion

This paper rationalizes ambiguous language by showing that under fairly general circumstances, a rational sender facing an ambiguity averse receiver can benefit from the use of ambiguous communication strategies. Future research ought to examine how our insights generalize. Do our basic results carry over to settings that do not satisfy our assumptions (Condition M, Assumption 1)? Is the class of simple Ellsberg randomizations examined sufficient or could one further increase expected payoffs by allowing for more complex ambiguous strategies? Do there generally exist Pareto improving ambiguous equilibria? We hope to address these questions in the future.

7 Appendix A

In what follows, we abuse notation and denote the utility function of S by $U^S(a,\omega,\beta)$, thus explicitly referring to the bias function β. Note that β is not a scalar parameter as in the original CS setup. We have:

$$\pi^S(\beta, E(\beta,N)) = \sum_{i=0}^{N-1} \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} U^S(a_e^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta) f(\omega)\, d\omega.$$

Let us define

$$\frac{d\pi^S(\beta, E(\beta,N))}{d\beta} = \lim_{\varepsilon \to 0} \frac{\pi^S(\beta+\varepsilon, E(\beta+\varepsilon,N)) - \pi^S(\beta, E(\beta,N))}{\varepsilon}.$$

This corresponds to the marginal effect on the payoff of S of a change in her bias function from $\beta(\omega)$ to $\beta'(\omega) = \beta(\omega) + \varepsilon$, for any ω. So let us examine:

$$\frac{d\pi^S(\beta, E(\beta,N))}{d\beta} \tag{15}$$

$$= \sum_{i=0}^{N-1} \frac{d\left(\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta) f(\omega)\, d\omega\right)}{d\beta} \tag{16}$$

$$= \sum_{i=0}^{N-1} \begin{pmatrix} \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{dU^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{d\beta} f(\omega)\, d\omega \\ +U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), t_{i+1}(\beta,N), \beta) f(t_{i+1}(\beta,N)) \frac{\partial t_{i+1}(\beta,N)}{\partial \beta} \\ -U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), t_i(\beta,N), \beta) f(t_i(\beta,N)) \frac{\partial t_i(\beta,N)}{\partial \beta} \end{pmatrix}$$

$$= \sum_{i=0}^{N-1} \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{dU^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{d\beta} f(\omega)\, d\omega \tag{17}$$

$$+ \sum_{i=1}^{N-1} \left(\begin{bmatrix} -U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), t_i(\beta,N), \beta) \\ +U^S(a_{ne}^*(t_{i-1}(\beta,N), t_i(\beta,N)), t_i(\beta,N), \beta) \end{bmatrix} f(t_i(\beta,N)) \frac{\partial t_i(\beta,N)}{\partial \beta} \right) \tag{18}$$

$$+U^S(a_{ne}^*(t_0(\beta,N), t_1(\beta,N)), t_0(\beta,N), \beta) f(t_0(\beta,N)) \frac{\partial t_0(\beta,N)}{\partial \beta} \tag{19}$$

$$-U^S(a_{ne}^*(t_{N-1}(\beta,N), t_N(\beta,N)), t_N(\beta,N), \beta) f(t_N(\beta,N)) \frac{\partial t_N(\beta,N)}{\partial \beta}. \tag{20}$$

Note first that the second line of the above expression is equal to 0 given that for every $i \in \{1, ., N-1\}$,

$$U^S(a_{ne}^*(t_{i-1}(\beta,N), t_i(\beta,N)), t_i(\beta,N), \beta) - U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), t_i(\beta,N), \beta) = 0.$$

Note furthermore that by definition

$$\frac{\partial t_0(\beta,N)}{\partial \beta} = \frac{\partial t_N(\beta,N)}{\partial \beta} = 0,$$

given that $t_0(\beta, N) = 0$ and $t_N(\beta, N) = 1$. We now show that for every $i \in \{0, ., N-1\}$,

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{dU^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{d\beta} f(\omega)\, d\omega < 0.$$

Note that for every $i \in \{0, ., N-1\}$,

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{dU^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{d\beta} f(\omega)\, d\omega$$

$$= \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial\beta} f(\omega)\, d\omega + \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial a} f(\omega)\, d\omega \left(\begin{array}{c} \frac{\partial a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N))}{\partial t_i} \frac{\partial t_i(\beta,N)}{\partial \beta} + \\ \frac{\partial a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N))}{\partial t_{i+1}} \frac{\partial t_{i+1}(\beta,N)}{\partial \beta} \end{array} \right).$$

Note first that

$$\frac{\partial U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial\beta} = -\frac{\partial U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial a}.$$

The above is true because we have assumed that $U^S(a, \omega, \beta(\omega)) = G(a - \omega + \beta(\omega))$ for some concave and single peaked function G. Note now that

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^S(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial a} f(\omega)\, d\omega > 0. \tag{21}$$

Indeed, $a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N))$ by definition satisfies:

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^R(a_{ne}^*(t_i(\beta,N), t_{i+1}(\beta,N)), \omega, \beta)}{\partial a} f(\omega)\, d\omega = 0. \tag{22}$$

Furthermore, we have assumed that $U^S(a, \omega, 0) = U^R(a, \omega)$ and $U_{13}^S > 0$. It follows that (22) implies (21). Intuitively, R's favorite action conditional on $\omega \in (t_i(\beta,N), t_{i+1}(\beta,N)]$ is smaller than S's favoured action, thus implying that the derivtative of S's expected payoff function w.r.t. a at the chosen a^*_{ne} must be strictly positive. Finally, note that

$$\begin{aligned}
\frac{\partial t_i(\beta, N)}{\partial \beta} &< 0, \\
\frac{\partial t_{i+1}(\beta, N)}{\partial \beta} &< 0, \\
\frac{\partial a^*_{ne}(t_i(\beta, N), t_{i+1}(\beta, N))}{\partial t_i} &> 0, \\
\frac{\partial a^*_{ne}(t_i(\beta, N), t_{i+1}(\beta, N))}{\partial t_{i+1}} &> 0.
\end{aligned}$$

■

8 Appendix B

8.1 Three claims

We first state three claims that hold independently of Assumption 1.

Claim 1 ***Concavity and single peakedness***

Note that $E\left[U^R(a, \omega)\,|\omega \in [\underline{\omega}, \overline{\omega}]\right]$ *is a concave and single peaked function with a unique maximizer* $a^*_{ne}(\underline{\omega}, \overline{\omega})$.

Proof of Claim 1: Omitted.■

Claim 2 ***Shift in maximum***

For any $\underline{\omega}, \overline{\omega}, \Delta_1, \Delta_2$ *s.t.*$\Delta_1 \geq 0$ *and* $\Delta_2 \geq 0$ *(with strict inequality for at least one of the two),* $0 \leq \underline{\omega} < \overline{\omega} \leq 1$ *and* $0 \leq \underline{\omega} + \Delta_1 < \overline{\omega} + \Delta_2 \leq 1$,

$$a^*_{ne}(\underline{\omega}, \overline{\omega}) < a^*_{ne}(\underline{\omega} + \Delta_1, \overline{\omega} + \Delta_2).$$

Proof of Claim 2: We prove the statement for $\Delta_1 = 0$ and $\Delta_2 > 0$. The proof for remaining cases is similar. By definition, it is true that

$$\int_{\underline{\omega}}^{\overline{\omega}} U_1^R(a^*_{ne}(\underline{\omega}, \overline{\omega}), \omega) f(\omega)\, d\omega = 0 \tag{23}$$

Now, given the concavity of $\int_{\underline{\omega}}^{\overline{\omega}+\Delta_2} U^R(a,\omega) f(\omega)\, d\omega$, we simply need to prove that

$$\int_{\underline{\omega}}^{\overline{\omega}+\Delta_2} U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) f(\omega)\, d\omega > 0.$$

Note that

$$\begin{aligned} & \int_{\underline{\omega}}^{\overline{\omega}+\Delta_2} U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) f(\omega)\, d\omega \\ = & \int_{\underline{\omega}}^{\overline{\omega}} U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) f(\omega)\, d\omega + \int_{\overline{\omega}}^{\overline{\omega}+\Delta_2} U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) f(\omega)\, d\omega > 0. \end{aligned} \quad (24)$$

The first integral in (24) is equal to 0, so we simply need to prove that the second integral is strictly positive. Now, given the assumption that $U_{12}^R > 0$, note that (23) trivially implies that

$$U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) > 0, \text{ for any } \omega \geq \overline{\omega} \text{ (with stricty inequality for } \omega \geq \overline{\omega}),$$

which in turn implies that

$$\int_{\overline{\omega}}^{\overline{\omega}+\Delta_2} U_1^R(a_{ne}^*(\underline{\omega},\overline{\omega}),\omega) f(\omega)\, d\omega > 0.$$

■

Claim 3 ***Single crossing condition***

If a^* is s.t.

$$E\left[U^R(a^*,\omega)\,|\omega \in [\underline{\omega}+\Delta_1, \overline{\omega}+\Delta_2]\right] = E\left[U^R(a^*,\omega)\,|\omega \in [\underline{\omega},\overline{\omega}]\right],$$

then for any $a > a^*$

$$E\left[U^R(a^*,\omega)\,|\omega \in [\underline{\omega}+\Delta_1, \overline{\omega}+\Delta_2]\right] > E\left[U^R(a^*,\omega)\,|\omega \in [\underline{\omega},\overline{\omega}]\right].$$

Proof of Claim 3: Given $U_{12} > 0$, it must be true that

$$\frac{\partial E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}+\Delta_1, \overline{\omega}+\Delta_2]\right]}{\partial a} > \frac{\partial E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega},\overline{\omega}]\right]}{\partial a}, \forall a > a^*,$$

which implies that for for any $a > a^*$,

$$E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}+\Delta_1, \overline{\omega}+\Delta_2]\right] > E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega},\overline{\omega}]\right].$$

■

8.2 Proof of Lemma 2

Step 1 Consider a mixed action $\widetilde{a}$ of R given by a distribution $\widetilde{g}$ over $[0,1]$. Denote by $\overline{a}(\widetilde{a})$ the action satisfying $\overline{a}(\widetilde{a}) = \int_0^1 a\widetilde{g}(a)da$. Note that the payoff function U^R is concave. It follows by Jensen's inequality that the expected payoff of $\widetilde{a}$ is weakly smaller than that of $\overline{a}(\widetilde{a})$ for any prior distribution F of the state ω, i.e.

$$\int_0^1 \left(\int_0^1 U^R(a,\omega)\widetilde{g}(a)da \right) f(\omega)d\omega \leq \int_0^1 U^R(\overline{a}(\widetilde{a}), \omega) f(\omega)d\omega.$$

R is a max-min decision maker, i.e. chooses the (possible mixed action) a^* (given by the distribution g^*) that maximizes

$$\min_f \int_0^1 \left(\int_0^1 U^R(a,\omega)g^*(a)da \right) f(\omega)d\omega.$$

Suppose the optimal max-min action assigns positive probability to multiple pure actions, i.e.that g^* is not degenerate. We know that the action $\overline{a}(a^*) = \int_0^1 ag^*(a)da$ does weakly better for any prior distribution f of the state. It follows that two cases are possible. Either a^* and $\overline{a}(a^*)$ are both solutions to the max-min problem or $\overline{a}(a^*)$ is while a^* is not. Consequently, we may without loss of generality focus on pure actions in searching for the max-min solution.

Step 2 Point a) follows immediately from Claims 1-3 and Assumption 1 given above. Given $0 \leq \underline{\omega} \leq \overline{\omega} \leq 1$, note that $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega},\overline{\omega}]\right]$ is a concave and single peaked function with a unique maximizer $a^*_{ne}(\underline{\omega},\overline{\omega})$. The functions $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega},c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in (c,\overline{\omega}]\right]$ cross once at some value $\widetilde{a}$ on the interval

$$(a^*_{ne}(\underline{\omega},c), a^*_{ne}(c,\overline{\omega})).$$

It follows immediately that $\widetilde{a}$ is the unique max-min solution.

Step 3 This proves Point b). We simply state and prove the following Lemma.

Lemma 5 ***Comparative statics result***

Assume that Assumption 1 holds. Let $0 \leq \underline{\omega} < \overline{\omega} \leq 1$. Let $a^*(c)$ be the unique value a s.t.

$$E\left[U^R(a^*(c),\omega)\,|\omega \in [\underline{\omega},c]\right] = E\left[U^R(a^*(c),\omega)\,|\omega \in (c,\overline{\omega}]\right]. \tag{25}$$

It follows that $a^*(c)$ is strictly increasing in c on $[\underline{\omega},\overline{\omega}]$.

Proof of above Lemma:

Step 3.1 Let $\underline{\omega} \leq c < c' \leq \overline{\omega}$. Consider the three functions $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]$, $E\left[U^R(a,\omega)\,|\omega \in (c, c']\right]$ and $E\left[U^R(a,\omega)\,|\omega \in (c', \overline{\omega}]\right]$. We know that $a^*_{ne}(\underline{\omega}, c) < a^*_{ne}(c, c') < a^*_{ne}(c', \overline{\omega})$. Furthermore, using Assumption 1, the unique crossing point a_1 of

$$E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] \text{ and } E\left[U^R(a,\omega)\,|\omega \in (c, c']\right]$$

belongs to $(a^*_{ne}(\underline{\omega}, c), a^*_{ne}(c, c'))$. Similarly, by Assumption 1, the unique crossing point a_3 of

$$E\left[U^R(a,\omega)\,|\omega \in (c, c']\right] \text{ and } E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$$

belongs to $(a^*_{ne}(c, c'), a^*_{ne}(c', \overline{\omega}))$. It also follows that the unique crossing point a_2 of

$$E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] \text{ and } E\left[U^R(a,\omega)\,|\omega \in (c', \overline{\omega}]\right]$$

belongs to (a_1, a_3). We thus have $a_1 < a_2 < a_3$

Step 3.2 Now, let us first compare $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$. Note that for every a, there is some $\alpha \in (0,1)$ s.t.

$$E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right] = \alpha E\left[U^R(a,\omega)\,|\omega \in (c, c']\right] + (1-\alpha) E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right].$$

We know from step 1 that for any $a < a_1$,

$$\max\left\{E\left[U^R(a,\omega)\,|\omega \in (c, c']\right], E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]\right\} < E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right].$$

It follows that for $a < a_1$, $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] > E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$. We also know from step 1 that for any $a \geq a_2$,

$$\begin{aligned} E\left[U^R(a,\omega)\,|\omega \in (c, c']\right] &> E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] \cap \\ E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right] &\geq E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]. \end{aligned}$$

It follows that for $a \geq a_2$, $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] < E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$. We may conclude that $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$ cross somewhere on $[a_1, a_2)$.

Step 3.3 Let us now compare $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c']\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$. Note that for every a, there is some $\widetilde{\alpha} \in (0,1)$ s.t.

$$E\left[U^R(a,\omega)\,\middle|\omega \in [\underline{\omega}, c']\right] = \widetilde{\alpha} E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] + (1-\widetilde{\alpha}) E\left[U^R(a,\omega)\,\middle|\omega \in (c, c']\right].$$

We know from step 1 that for any $a \leq a_2$,

$$\begin{aligned} E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right] &\geq E\left[U^R(a,\omega)\,\middle|\omega \in [c', \overline{\omega}]\right] \cap \\ E\left[U^R(a,\omega)\,\middle|\omega \in (c, c']\right] &> E\left[U^R(a,\omega)\,\middle|\omega \in [c', \overline{\omega}]\right]. \end{aligned}$$

It follows that for $a \leq a_2$, $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c']\right] > E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$. We also know from step 1 that for any $a > a_3$,

$$\max\left\{E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right], E\left[U^R(a,\omega)\,\middle|\omega \in (c, c']\right]\right\} < E\left[U^R(a,\omega)\,\middle|\omega \in [c', \overline{\omega}]\right].$$

It follows that for $a > a_3$, $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c']\right] < E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$. We may conclude that $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c']\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$ cross somewhere on $(a_2, a_3]$.

Step 3.4 Having now proved that $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$ cross somewhere on $[a_1, a_2)$ while $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c']\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c', \overline{\omega}]\right]$ cross somewhere on $(a_2, a_3]$, it follows that $a^*(c) < a^*(c')$. The continuity of $a^*(c)$ in c follows from the continuity of $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, c]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [c, \overline{\omega}]\right]$ in c.

Step 3.5 To see that $a^*_{ne}(t_i, t_{i+1}) < a^*_e(t_i, t_{i+1}, t_{i+1})$, note that it follows immediately from Claims 1-3 and Assumption 1 that the value of a ensuring equality of $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, \overline{\omega}]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [\overline{\omega}, \overline{\omega}]\right]$ is strictly larger than $a^*_{ne}(\underline{\omega}, \overline{\omega})$. If similarly follows that the value of a ensuring equality of $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, \underline{\omega}]\right]$ and $E\left[U^R(a,\omega)\,|\omega \in [\underline{\omega}, \overline{\omega}]\right]$ is strictly smaller than $a^*_{ne}(\underline{\omega}, \overline{\omega})$.■

9 Appendix C

Step 1 This step proves Point a). In what follows, as in Appendix A, we abuse notation and denote the utility function of S by $U^S(a, \omega, \beta)$, thus explicitly referring to the bias function β. Note that β is not a scalar parameter as in the original CS setup.First, note that given $t_1(\beta, N), ., t_{N-1}(\beta, N)$, there is some maximal $\bar{\varepsilon}$ s.t. for any $\varepsilon \leq \bar{\varepsilon}$, one can pick a profile $\{c_{i,i+1}\}_{i=1}^{N-1}$ s.t. for every $i \in \{0, ., N-1\}$:

$$a_e^*(t_i(\beta, N), t_{i+1}(\beta, N), c_{i,i+1}) = a_{ne}^*(t_i(\beta, N), t_{i+1}(\beta, N)) + \varepsilon.$$

By continuity of a_e^* and t_i, for any given N, there exists a $\overline{\delta}$ s.t. for every $\delta \leq \overline{\delta}$ and every $\varepsilon \leq \frac{\bar{\varepsilon}}{2}$, there is some profile $\{c_{i,i+1}\}_{i=1}^{N-1}$ compatible with $\{t_i(\beta - \delta, N)\}_{i=1}^{N-1}$ s.t. for every $i \in \{0, ., N-1\}$:

$$a_e^*(t_i(\beta - \delta, N), t_{i+1}(\beta - \delta, N), c_{i,i+1}) = a_{ne}^*(t_i(\beta - \delta, N), t_{i+1}(\beta - \delta, N)) + \varepsilon$$

Choosing $\delta = \varepsilon \leq \min\left\{\overline{\delta}, \frac{\bar{\varepsilon}}{2}\right\}$, there thus exists some profile $\{c_{i,i+1}\}_{i=1}^{N-1}$ s.t. for every $i \in \{0, ., N-1\}$:

$$a_e^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N), c_{i,i+1}) = a_{ne}^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N)) + \varepsilon.$$

Clearly, having picked such a profile $\{c_{i,i+1}\}_{i=1}^{N-1}$, note that for every $i \in \{0, ., N-1\}$ and ω :

$$\begin{aligned} & U^S\left(a_e^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N), c_{i,i+1}), \omega, \beta\right) \\ = \; & U^S\left(a_{ne}^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N)), \omega, \beta - \varepsilon\right). \end{aligned}$$

It follows that if for every $i \in \{1, ., N-1\}$

$$\begin{aligned} & U^S\left(a_{ne}^*(t_{i-1}(\beta - \varepsilon, N), t_i(\beta - \varepsilon, N)), t_i(\beta - \varepsilon, N), \beta - \varepsilon\right) \\ = \; & U^S\left(a_{ne}^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N)), t_i(\beta - \varepsilon, N), \beta - \varepsilon\right), \end{aligned}$$

as implied by the existence of the non-Ellsbergian partitional equilibrium $E(\beta - \varepsilon, N)$ for a sender bias given by $\beta - \varepsilon$, then it must be true that for every $i \in \{1, ., N-1\}$,

$$\begin{aligned} & U^S\left(a_e^*(t_{i-1}(\beta - \varepsilon, N), t_i(\beta - \varepsilon, N), c_{i-1,i}), t_i(\beta - \varepsilon, N), \beta\right) \\ = \; & U^S\left(a_e^*(t_i(\beta - \varepsilon, N), t_{i+1}(\beta - \varepsilon, N), c_{i,i+1}), t_i(\beta - \varepsilon, N), \beta\right), \end{aligned}$$

which implies that the equilibrium $\widetilde{E}(\beta, \beta - \varepsilon, N)$ exists.

Step 2 This proves Point b). The expected payoff of S in $\widetilde{E}(\beta, \beta-\varepsilon, N)$ is given by:

$$\begin{aligned} & \sum_{i=0}^{N-1} \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} U^S\left(a_e^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N), c_{i,i+1}), \omega, \beta\right) f(\omega) d\omega \\ = & \sum_{i=0}^{N-1} \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} U^S\left(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)), \omega, \beta-\varepsilon\right) f(\omega) d\omega \\ = & \pi^S\left(\beta-\varepsilon, E\left(\beta-\varepsilon, N\right)\right) > \pi^S\left(\beta, E\left(\beta, N\right)\right). \end{aligned}$$

Step 3 Note first that

$$\pi^R\left(\widetilde{E}\left(\beta,\beta-\varepsilon,N\right)\right) = \sum_{i=0}^{N-1} \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} U^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, \omega) f\left(\omega\right) d\omega.$$

We have:

$$\frac{\partial \pi^R\left(\widetilde{E}\left(\beta,\beta-\varepsilon,N\right)\right)}{\partial \varepsilon} = \left(\sum_{i=0}^{N-1} \frac{\partial\left(\int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} U^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, \omega) f\left(\omega\right) d\omega\right)}{\partial \varepsilon}\right).$$

By Leibniz rule, the above can be rewritten as

$$\sum_{i=1}^{N-1} \left(\begin{array}{c} \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} \frac{dU^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N))+\varepsilon,\omega) f(\omega)}{d\varepsilon} d\omega \\ +U^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, t_{i+1}(\beta-\varepsilon,N)) f\left(t_{i+1}(\beta-\varepsilon,N)\right) \frac{\partial t_{i+1}(\beta-\varepsilon,N)}{\partial \varepsilon} \\ -U^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, t_i(\beta-\varepsilon,N)) f\left(t_i(\beta-\varepsilon,N)\right) \frac{\partial t_i(\beta-\varepsilon,N)}{\partial \varepsilon} \end{array} \right).$$

Note that:

$$\begin{aligned} & \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} \frac{dU^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, \omega) f\left(\omega\right)}{d\varepsilon} d\omega \\ = & \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} L(i,\beta,\varepsilon,N) f\left(\omega\right) d\omega, \end{aligned}$$

where

$$L(i,\beta,\varepsilon,N) = \left(\frac{\partial U^R(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon, \omega)}{\partial a}\right)\left(\frac{\partial\left(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon\right)}{\partial \varepsilon}\right)$$

Note furthermore that:

$$\begin{aligned} & \frac{\partial\left(a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N)) + \varepsilon\right)}{\partial \varepsilon} \\ = & \frac{\partial a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N))}{\partial t_i} \frac{\partial t_i(\beta-\varepsilon,N)}{\partial \varepsilon} + \\ & \frac{\partial a_{ne}^*(t_i(\beta-\varepsilon,N), t_{i+1}(\beta-\varepsilon,N))}{\partial t_{i+1}} \frac{\partial t_{i+1}(\beta-\varepsilon,N)}{\partial \varepsilon} + 1. \end{aligned}$$

It follows that we may write:n

$$\frac{\partial \pi^R\left(\widetilde{E}\left(\beta,\beta-\varepsilon,N\right)\right)}{\partial \varepsilon}$$

$$= \sum_{i=0}^{N-1}\left(\begin{array}{c} \int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} P(i,\beta,\varepsilon,N) f\left(\omega\right) d\omega \\ +U^R(a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))+\varepsilon,t_{i+1}(\beta-\varepsilon,N)) f\left(t_{i+1}(\beta-\varepsilon,N)\right) \frac{\partial t_{i+1}(\beta-\varepsilon,N)}{\partial \varepsilon} \\ -U^R(a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))+\varepsilon,t_i(\beta-\varepsilon,N)) f\left(t_i(\beta-\varepsilon,N)\right) \frac{\partial t_i(\beta-\varepsilon,N)}{\partial \varepsilon} \end{array}\right)$$
$$+\sum_{i=0}^{N-1}\int_{t_i(\beta-\varepsilon,N)}^{t_{i+1}(\beta-\varepsilon,N)} \frac{\partial U^R(a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))+\varepsilon,\omega)}{\partial a} f\left(\omega\right) d\omega,$$

where

$$P(i,\beta,\varepsilon,N) = \frac{\partial U^R(a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))+\varepsilon,\omega)}{\partial a} \times \left(\begin{array}{c} \frac{\partial a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))}{\partial t_i}\frac{\partial t_i(\beta-\varepsilon,N)}{\partial \varepsilon} \\ +\frac{\partial a_{ne}^*(t_i(\beta-\varepsilon,N),t_{i+1}(\beta-\varepsilon,N))}{\partial t_{i+1}}\frac{\partial t_{i+1}(\beta-\varepsilon,N)}{\partial \varepsilon} \end{array}\right).$$

It follows that

$$\left.\frac{\partial \pi^R\left(\widetilde{E}\left(\beta,\beta-\varepsilon,N\right)\right)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

$$= \sum_{i=0}^{N-1}\left(\begin{array}{c} \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} Q(i,\beta,N) f\left(\omega\right) d\omega \\ -U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_{i+1}(\beta,N)) f\left(t_{i+1}(\beta,N)\right) \frac{\partial t_{i+1}(\beta,N)}{\partial \beta} \\ +U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_i(\beta,N)) f\left(t_i(\beta,N)\right) \frac{\partial t_i(\beta,N)}{\partial \beta} \end{array}\right)$$
$$+\sum_{i=0}^{N-1}\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),\omega)}{\partial a} f\left(\omega\right) d\omega,$$

where

$$Q(i,\beta,N) = \frac{\partial U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),\omega)}{\partial a} \times \left(-\frac{\partial a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N))}{\partial t_i}\frac{\partial t_i(\beta,N)}{\partial \beta} - \frac{\partial a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N))}{\partial t_{i+1}}\frac{\partial t_{i+1}(\beta,N)}{\partial \beta}\right)$$

Given the FOC characterizing $a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N))$, it follows that for every $i \in \{0,.,N-1\}$,

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} \frac{\partial U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),\omega)}{\partial a} f\left(\omega\right) d\omega = 0$$

and that

$$\int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} Q(i,\beta,N) f(\omega)\, d\omega = 0$$

It follows that

$$\sum_{i=0}^{N-1}\left(\begin{array}{c} \int_{t_i(\beta,N)}^{t_{i+1}(\beta,N)} Q(i,\beta,N) f(\omega)\, d\omega f(\omega)\, d\omega \\ -U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_{i+1}(\beta,N)) f(t_{i+1}(\beta,N)) \frac{\partial t_{i+1}(\beta,N)}{\partial\beta} \\ +U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_i(\beta,N)) f(t_i(\beta,N)) \frac{\partial t_i(\beta,N)}{\partial\beta} \end{array}\right)$$

$$= \sum_{i=1}^{N-1}\left[\begin{array}{c} -U^R(a_{ne}^*(t_{i-1}(\beta,N),t_i(\beta,N)),t_i(\beta,N)) \\ +U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_i(\beta,N)) \end{array}\right] f(t_i(\beta,N)) \frac{\partial t_i(\beta,N)}{\partial\beta}.$$

Given Condition M, $\frac{\partial t_i(\beta,N)}{\partial\beta} < 0$ for every $i \in \{1,.,N-1\}$. Furthermore, for every $i \in \{1,.,N-1\}$,

$$\begin{array}{rcl} U^R(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_i(\beta,N)) - U^R(a_{ne}^*(t_{i-1}(\beta,N),t_i(\beta,N)),t_i(\beta,N)) & < \\ U^S(a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N)),t_i(\beta,N),\beta) - U^S(a_{ne}^*(t_{i-1}(\beta,N),t_i(\beta,N)),t_i(\beta,N),\beta) & = & 0 \end{array}$$

The first inequality (appearing on the first line) holds true because $a_{ne}^*(t_{i-1}(\beta,N),t_i(\beta,N)) < a_{ne}^*(t_i(\beta,N),t_{i+1}(\beta,N))$, $U^S(a,\omega,0) = U^R(a,\omega)$ and $U_{13}^S > 0$ everywhere. The second inequality (appearing on the second line) holds true by definition because $\{t_i(\beta,N)\}_{i=1}^N$ is a non-Ellsbergian equilibrium partitional communication strategy.■

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