Dynamic OCE Choice: Time Consistency and the Separation of Time and Risk

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Abstract

No existing dynamic preference model can simultaneously satisfy time consistency, the full separation of time and risk preferences, and temporal resolution of risk indifference. In the context of consumption-saving and consumption-portfolio optimization problems, we derive necessary and sufficient conditions such that all three of these properties are satisfied by the dynamic ordinal certainty equivalent (DOCE) preference structure axiomatized in Selden and Stux (1978). These conditions ensure that DOCE resolute, naive and sophisticated consumption and asset demands are (i) identical and (ii) the same as the demands generated by Kreps and Porteus (1978) (KP) preferences. When the conditions are violated, the elasticity of intertemporal substitution can play a key role in determining whether axiomatic differences between the DOCE and KP preference models imply significantly different demand behavior.

KEYWORDS. Kreps-Porteus-Selden preferences, time consistency, separation of time and risk, temporal resolution indifference, consumption-portfolio problem

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1 Introduction

For economic models where consumers solve dynamic optimization problems under risk, assumptions on preferences play a key role in the resulting solutions and their comparative statics. At the level of preferences, the following three properties are often mentioned as being desirable: (i) time consistency (TC), (ii) the ability to fully separate time and risk preferences (SEP) and (iii) the ability to accommodate temporal resolution of risk indifference (TRI). Currently, no single dynamic preference model can simultaneously accommodate all three properties. This paper makes three contributions. First, it provides special conditions under which the three properties can be satisfied. Second, these results are shown to have important implications for the use of the preference models of SS (Selden and Stux 1978) and KP (Kreps and Porteus 1978) in dynamic consumption-saving and consumption-portfolio applications. Third when the conditions do not hold, we show that optimal consumption, saving and asset demand behavior can still be surprisingly similar for the two preference models so long as the consumer's time preferences exhibit sufficient aversion to intertemporal substitution.

Although KP (1978) motivate the introduction of their recursive preference structure on the basis of being able to accommodate a preference for early or late resolution of uncertainty, Epstein, Farhi and Strzalecki (2014) among others argue that while early resolution can be beneficial in decision making, it may not be desirable to require this property at the pure preference level. In fact in the EZ (Epstein and Zin 1989) homothetic version of KP utility, two parameters govern the seemingly distinct time preferences, risk preferences and a preference for the resolution of uncertainty. However, when setting the parameters at different values to achieve SEP, an analyst loses her ability to control temporal resolution preferences. This limitation has been recognized from the start in EZ and in part motivates us to explore in this paper the prospect of using DOCE (dynamic ordinal certainty equivalent) preferences. As a natural generalization of Selden (1978), DOCE preferences are based on independent risk and time preference building blocks which, respectively, are used to replace risky consumption in each period by certainty equivalent consumption and evaluate the resulting vector of certain and certainty equivalent consumption. Thus, DOCE preferences exhibit SEP. By assumption, they also exhibit TRI. In contrast to KP preferences, the attitude toward the resolution of risk is independent of the form of time and risk preferences as well as their interrelationship. However as suggested by Johnsen and Donaldson (1985), DOCE preferences in general violate time consistency. It should be noted that although the KP and DOCE utilities in general differ, they become ordinally equivalent in a two period setting where the first period is certain. The common representation is typically referred to as the KPS (Kreps-Porteus-Selden) utility.

Given that in general, neither KP nor DOCE preferences can simultaneously satisfy TC, SEP and TRI, are there any special circumstances under which either model can satisfy the three properties? We show that in a consumption-portfolio setting if the distribution of asset returns is independent over time, then the consumer's demands will be time consistent if and only if her underlying building block representations of time and risk preferences both exhibit homotheticity. In this case, the time and risk preference utilities, respectively take the CES (constant elasticity of substitution) and CRRA (constant relative risk aversion) forms. As a result, DOCE preferences which are defined over the subset of dynamic consumption trees where consumption along branches exhibits a special proportionality will satisfy TC as well as SEP and TRI. While the restriction that asset returns be independent over time is clearly a special case, the stronger assumption that asset returns are i.i.d. (identically and independently distributed), has been made in a number of important papers such as Levhari and Srinivasan (1969), Samuelson (1969), Weil (1993), Campbell and Cochrane (1999) and Barro (2009). Moreover, the assumption that the representations of time and risk preferences are homothetic has also been widely used for instance in the EZ special case of KP preferences. The intuition for our result is that the combination of independent returns and homotheticity permits the transformation of the choice over a multi-date-event-branch consumption tree into the choice over an equivalent single branch tree analogous to what the consumer confronts in a pure certainty time consistent setting.

It would clearly be desirable to weaken the restriction that time and risk preferences must be homothetic. In fact, it is possible to extend our result to the full class of HARA (hyperbolic absolute risk aversion)[1] time and risk preferences in a consumption-portfolio setting if one adds to independent asset returns the assumption that one of the available assets is risk free. The quasihomothetic members of the HARA class include the translated origin CRRA utility used for instance in the external habit model of Campbell and Cochrane (1999) and the familiar CARA (constant absolute risk aversion) form. Both of these can be viewed as being homothetic to translated origins (see Pollak 1971). The risk free asset assumption in our result is crucial in dealing with the translations. If either the homotheticity or HARA conditions is satisfied, DOCE preferences will exhibit TC, SEP and TRI on a restricted domain corresponding to the specific choice

[1] See Gollier (2001) for a characterization of HARA preferences and their properties.

problem.[2] Moreover as discussed in Section 3 below, satisfaction of the DOCE conditions for TC allows one to greatly simplify the complex T-period dynamic consumption-portfolio problem by decomposing it into $T-1$ single period portfolio optimizations and solve an elementary time consistent consumption-saving problem based on certainty time preferences and a budget constraint involving equivalent portfolio rates of return.

Given these results, it is natural to wonder how the time consistent DOCE and KP demands relate to one another assuming the corresponding dynamic preferences are based on the same time and risk preference building blocks and asset returns are independent over time. Since the utilities are not ordinally equivalent and both sets of demands are time consistent, one would expect that the demands would differ due to the corresponding preferences differing in terms of the properties SEP and TRI. However the DOCE and KP demand functions are identical.[3] As a result, under the assumptions outlined above a number of key consumption saving and asset demand properties present in two period KPS applications[4] extend to the dynamic setting. For instance, the classic two period portfolio result that the ratio of risk free to risky asset demands depends only on risk preferences and not time preferences extends to the HARA versions of DOCE and KP preferences. Giovannini and Weil (1989) prove that if asset returns are i.i.d., then the EZ special case of KP preferences result in the same consumption and asset demand behavior as generated by single period EU (expected utility) preferences or the comparable two period KPS (Kreps-Porteus-Selden) preferences.[5] In addition to extending their result to DOCE preferences, we also extend it to the KP case based on the full class of HARA utilities. We weaken the restriction on asset returns from being identical over time and show that the conditions are necessary as well as sufficient. Finally it should be noted that when the conditions for the convergence of the KP and DOCE demands are satisfied, the complex KP consumption-portfolio optimization can be significantly simplified by employing the same intuitive certainty consumption-saving problem referred to in the prior paragraph.

[2] As discussed in Subsection 4.3, on this restricted domain early resolution consumption trees do not exist and KP preferences will not exhibit a preference for early versus late resolution trees as typically defined.

[3] We observe in Subsection 4.3 that DOCE and KP demands continue to be identical when both preference models are based on the same time inconsistent quasi-hyperbolic discounted homothetic time preference utility. In this case, for each model resolute, naive and sophisticated choice diverge. However, respectively the DOCE and KP resolute, naive and sophisticated demands are the same.

[4] See, for example, Selden (1979), Barsky (1989) and Kimball and Weil (2009).

[5] A related argument is made by Kocherlakota (1990).

Given that there is significant empirical evidence suggesting that asset returns can deviate from being independent over time, how does this affect the relationship between DOCE and KP demands? In this case, DOCE preferences become time inconsistent and it becomes necessary to consider the standard (Strotz-Pollak) resolute, naive and sophisticated solution techniques for consumption-saving and consumption-portfolio problems. In order to analyze the case where asset returns are not independent over time, we generally assume a three period setting and the DOCE and KP preferences share the same CES time and CRRA risk preference building blocks. We focus on differences in demand for the resolute, naive and sophisticated DOCE and KP cases, often based on numerical simulations. Our analysis suggests that two quite different sets of conclusions can be obtained depending on the value of the EIS (elasticity of intertemporal substitution). First, when the EIS is in the range of roughly 0.20 to 0.40, as estimated in a number of certainty empirical studies, we find that the KP and DOCE resolute, naive and sophisticated period 1 consumption and asset demands exhibit the same qualitative properties and can be surprisingly close in absolute value. This suggests that axiomatic differences in the two models may not be critical. Second, if the EIS is considerably larger in the range of 1.5 to 2.0, as suggested by calibrations of some finance and macro models, then the DOCE and KP demands can differ quite substantially.[6] In general DOCE resolute and KP demands continue to be similar, but DOCE sophisticated and KP demands can diverge significantly in terms of qualitative behavior. This difference is surprising since in a three period setting, the two models follow backward induction and the period 2 conditional saving decisions are the same. In particular as the EIS approaches infinity, the models differ in whether (i) period 1 consumption increases or decreases and (ii) the risk free to risky asset demand ratio dramatically increases or remains more or less unchanged. Both of these differences can be explained by the impact of a strong preference for intertemporal substitution. This effect is clear even in the very simple two period certainty consumption-saving problem. A substitution oriented consumer when confronting returns larger than unity tends to substantially reduce current consumption and greatly increase saving to maximize period 2 consumption. Our results suggest the critical importance of developing experimental laboratory tests to determine whether in risky consumption-saving and consumption-portfolio problems, consumer behavior can best explained by EIS values less than 0.40 or greater than 1.5.

The rest of the paper is organized as follows. In the next section, we introduce

[6] For a review of the literature on the size of the EIS, see for instance Attanasio and Weber (2010), Havranek (2015) and Thimme (2017) and the references cited in these papers.

notation, definitions and the optimization problems. Section 3 illustrates (i) the intuitive appeal of DOCE utility and (ii) the significant simplification of complex dynamic consumption-portfolio problems when our conditions for time consistent DOCE demands hold. In Section 4, we provide our main theorems for DOCE preferences to be time consistent and provide results relating DOCE and KP demands. Section 5 provides comparisons of consumption and asset demands for the DOCE resolute, naive, sophisticated and KP cases when asset returns deviate from being independent over time. Section 6 contains concluding comments. Proofs are given in Appendix A and supporting materials are provided in Supplemental Appendix B.

2 Preliminaries

2.1 Notation and Definitions

Assume time is indexed by $t = 1, \ldots, T$. Exogenous shocks s_t realize in a finite set S. A history of shocks up to some date t is denoted by $s^t = (s_1, s_2, \ldots, s_t)$ and called a date event. Since each chance node in a tree can be reached only through one historical path, we also use s^t to denote a chance node. The notation $s^{t+1} \succ s^t$ refers to the node s^{t+1} following node s^t. Let $\mathcal{S}$ denote the set of all nodes, s^t, of the tree. We consider an agent's choices over T periods, $t = 1, \ldots, T$. For simplicity, we often focus on the $T = 3$ case where we use a different notation and denote nodes at $t = 2$ by (21), (22),... and at $t = 3$ by (31), (32),....

We next briefly describe the DOCE utility axiomatized in Selden and Stux (1978). (Their paper, although unpublished, is available on the website of Larry Selden, Columbia University Graduate Business School.) Assume a T period setting, where consumption in period $t = 1$ is certain and risky in periods $t = 2, ..., T$. In period t, the consumer's certainty time preferences over degenerate consumption streams $(c_t, ..., c_T)$ $(t \in \{1, ..., T\})$ are represented by the following additively separable utility

$$U_t(c_t, ..., c_T) = u(c_t) + \sum_{i=t+1}^{T} \beta^{i-t} u(c_i), \tag{1}$$

where $0 < \beta < 1$ is the standard discount function. The consumer's risk preferences in each period $t \in \{2, ..., T\}$ are identical and represented by the EU function

$$\sum_{s^t} \pi(s^t) V(c(s^t)),$$

where $\pi(s^t)$ is the probability of the date-event (node) s^t and V is the NM (von

Neumann-Morgenstern) index. DOCE preferences are assumed to be independent of when risk is resolved. This preference axiom, referred to as TRI, is one important difference from KP preferences described below. The stationary time preference u and NM index V will be assumed to satisfy $u' > 0, u'' < 0, V' > 0$ and $V'' < 0$ unless stated otherwise. In what follows, we use preferences over current and future consumption conditional on the current date-event node being s^τ.

The period t certainty equivalent evaluated at node s^τ is defined by

$$(\widehat{c}_t|s^\tau) = V^{-1}\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau)V(c(s^t))\right),$$

where $\pi(s^t|s^\tau)$ is the probability of date-event s^t conditional on being at node s^τ. Thus, for a given s^τ, the DOCE representation is given by

$$\mathcal{U}(\mathbf{c}|s^\tau) = u(c(s^\tau)) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\widehat{c}_t|s^\tau).$$

In period 1, the utility is given by

$$\mathcal{U}(\mathbf{c}) = u(c_1) + \sum_{t=2}^{T} \beta^{t-1} u(\widehat{c}_t|s_1).$$

For the DOCE preference model, (i) risk preferences are constant over time, (ii) there is a complete separation of time and risk preferences corresponding to u and V and (iii) EU preferences are a special case where u and V are affinely equivalent. EU preferences exhibit the TRI axiom.

Kreps and Porteus (1978) derived the recursive representation

$$\mathcal{U}(\mathbf{c}|s^\tau) = U\left(c(s^\tau), \sum_{s^{\tau+1} \succ s^\tau} \pi(s^{\tau+1}|s^\tau)\mathcal{U}(\mathbf{c}|s^{\tau+1})\right),$$

where U is continuous and strictly increasing.[7] Note that if U is linear in the second argument, the KP representation converges to the EU special case. The EZ representation is a special case of the KP utility,[8] where

$$U(c_t, x) = -\frac{\left(c_t^{-\delta_1} + \beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \text{and} \quad V_T(x) = -\frac{x^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1).$$

If $\delta_1 = \delta_2 = \delta$, the EZ representation converges to the EU function

$$\mathcal{U}(\mathbf{c}|s^\tau) = -\frac{(c(s^\tau))^{-\delta}}{\delta} - E\left[\sum_{t=\tau+1}^{T} \beta^{t-\tau} \frac{(c_t(s^\tau))^{-\delta}}{\delta}\right].$$

[7] Unlike the DOCE and EZ cases, the KP preference building blocks are U and $\mathcal{U}$. An EU index V can be induced from the KP utility for the final time period T.

[8] Weil (1990) derives an alternative specialization of the KP preference model.

Both the KP and EZ recursive preference structures can accommodate a preference for early or late resolution of risk. However as was mentioned in the prior section, this temporal resolution preference cannot be varied independently from time and risk preferences.

The following special pairs of utilities will be used extensively in our analysis[9]

$$u(c_t) = -\frac{1}{\delta_1}(c_t - b)^{-\delta_1} \quad \text{and} \quad V(c_t) = -\frac{1}{\delta_2}(c_t - b)^{-\delta_2} \ \ (b \gtreqless 0, \delta_1, \delta_2 > -1), \tag{2}$$

$$u(c_t) = -\frac{\exp(-\kappa_1 c_t)}{\kappa_1} \quad \text{and} \quad V(c_t) = -\frac{\exp(-\kappa_2 c_t)}{\kappa_2} \ \ (\kappa_1, \kappa_2 > 0) \tag{3}$$

and

$$u(c_t) = \frac{1}{\delta_1}(b - c_t)^{-\delta_1} \quad \text{and} \quad V(c_t) = \frac{1}{\delta_2}(b - c_t)^{-\delta_2} \ \ (\delta_1, \delta_2 < -1), \tag{4}$$

where for (2) $c_t > \max(0, b)$ and for (4) $b > c_t > 0$. For the NM indices in (2)-(4), respectively, the risk preferences exhibit decreasing, constant and increasing absolute risk aversion. This collection of NM indices is typically referred to as the HARA class. The corresponding certainty utilities are frequently referred to as the Modified Bergson family.[10] One important special case of (2) is the following CES time and CRRA risk preference utilities used in the EZ special case of KP preferences

$$u(c_t) = -\frac{1}{\delta_1}c_t^{-\delta_1} \quad \text{and} \quad V(c_t) = -\frac{1}{\delta_2}c_t^{-\delta_2} \ \ (\delta_1, \delta_2 > -1), \tag{5}$$

where the EIS (elasticity of intertemporal substitution) and Arrow-Pratt relative risk aversion measures are given by, respectively,

$$EIS = \frac{1}{1 + \delta_1} \qquad \text{and} \qquad -c_t\frac{V''(c_t)}{V'(c_t)} = 1 + \delta_2. \tag{6}$$

For the popular DARA (decreasing absolute risk aversion) case (2), it is standard to interpret $b > 0$ as a certain subsistence requirement.[11]

2.2 Optimization Problems

In this subsection, we formally define the consumption-saving and consumption-portfolio problems and describe the three solution techniques of resolute, naive

[9] To avoid corner solutions for the consumption-portfolio problems considered below, we do not include the $\delta_2 = -1$ case.

[10] See Pollak (1971) for a description of the Modified Bergson class.

[11] For the DARA case we can have $b < 0$, but then the subsistence interpretation does not make sense (see Pollak 1970, p. 748). For the IARA (increasing absolute risk aversion) case (4), b can be interpreted as a bliss point.

and sophisticated choice that are typically considered when preferences fail to be time consistent.[12]

At the beginning of each period $t = 1, \ldots, T-1$ there are J assets available for trade with returns $\mathbf{R}(s^{t+1}) = (R_j(s^{t+1}))_{j=1}^{J} \geq 0$ being realized at node s^{t+1}. We assume that asset returns preclude arbitrage in that there exist $\rho(s^t) > 0$ for all s^t such that

$$\sum_{s^{t+1} \succ s^t} \rho(s^{t+1}) R_j(s^{t+1}) = \rho(s^t) \ \ \forall s^t, j. \tag{7}$$

Suppose the special case of complete asset markets holds where the number of assets is the same as the number of states, or more formally at each s^t, $t < T$, the matrix $(R(s^{t+1}))_{\{s^{t+1} \succ s^t\}}$ has rank S. Then $\rho(s^{t+1})$ in eqn. (7) can be interpreted as the contingent claim price for $c(s^{t+1})$.

A much weaker assumption which plays a prominent role in our analysis is that there exists a one period risk free asset at each date-event.

Assumption [RF] For each s^t, $t = 1, \ldots, T-1$, there exists an $\boldsymbol{\omega}(s^t)$, where $\boldsymbol{\omega} = \omega_1, ..., \omega_J$, such that

$$\sum_j \omega_j \left(s^t\right) R_j \left(s^{t+1}\right) = 1 \ \ \forall s^{t+1} \succ s^t.$$

Note that this assumption is automatically satisfied when markets are complete.

In the next section, we focus on the case where the probability distribution of returns is independent over time. Formally, this is stated as follows.

Assumption [IR] Assume that for each s^t, $\mathbf{R}(s^t) = \overline{\mathbf{R}}_t(s_t)$ for some function $\overline{\mathbf{R}}_t(.)$ that might vary with time t, but only depends on the shock, s_t and does not depend on history. Furthermore, the probabilities satisfy $\pi(s^t|s^{t-1}) = \overline{\pi}_t(s_t)$ for some function $\overline{\pi}_t(s_t)$.

An individual is assumed to choose consumption and assets in periods $t = 1, \ldots, T-1$ so as to maximize utility. We assume throughout that the individual has rational expectations in that she knows future asset returns contingent on the nodes.

In period $t \in \{1, ..., T-1\}$, at the node s^t, denote the demand for asset $j \in \{1, ..., J\}$ by $n_j(s^t)$ and the vector of asset holdings by $\mathbf{n} = (\mathbf{n}(s^1), \ldots \{\mathbf{n}(s^t)\}, \ldots \{\mathbf{n}(s^{T-1})\})$, where $\mathbf{n}(s^t) = (n_1(s^t), ..., n_j(s^t))$. Define the stream of time $t = 1, ..., T$ consumption and contingent claim quantities $\mathbf{c} = (c(s^1), ..., \{c(s^t)\}, ..., \{c(s^T)\})$.

[12] Phelps and Pollak (1968) and Peleg and Yaari (1973) argue that one should think of the time inconsistent choice problem as being equivalent to a game between divergent individuals – myself today and my selves in future periods. Caplin and Leahy (2006) argue that the sophisticated and game theoretic approaches result in equivalent solutions.

Let I and $I(s^t)$ denote, respectively, initial income and the income received from investment in period $t-1$ at the beginning of period $t>1$ at the node s^t and $I\left(s^t\right)=I$ when $t=1$.

The period 1 consumption-portfolio problem is defined as follows

$$\max_{\mathbf{c},\mathbf{n}} \mathcal{U}\left(\mathbf{c}\right) \quad S.T. \tag{8}$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t=1, \tag{9}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2<t<T, \tag{10}$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t=T. \tag{11}$$

A special case results when there is a single asset, $J=1$. In this case, the problem can be rewritten as

$$\max_{\mathbf{c}} \mathcal{U}\left(\mathbf{c}\right) \quad S.T. \tag{12}$$

$$I\left(s_1\right) = I, \tag{13}$$

$$I(s^{t+1}) = R(s^{t+1})\left(I(s^t) - c(s^t)\right) \quad \left(t=1,...,T-1, s^{t+1} \succ s^t\right), \tag{14}$$

$$c\left(s^T\right) = I(s^T). \tag{15}$$

For the consumption-saving and consumption-portfolio problems, it is assumed that in any period t the consumer can only purchase assets with maturity of one time period.

To simplify notation, we use $(\mathbf{c}^{\circ},\mathbf{n}^{\circ})$, $(\mathbf{c}^{*},\mathbf{n}^{*})$ and $(\mathbf{c}^{**},\mathbf{n}^{**})$ to denote resolute, naive and sophisticated demands, respectively. To facilitate the comparison with KP preferences below, we will use $\left(\mathbf{c}^{KP},\mathbf{n}^{KP}\right)$ to denote the optimal demands corresponding to KP preferences. Consistent with the certainty analysis of Strotz (1956) and Pollak (1968), DOCE demands are said to be time consistent if and only if $(\mathbf{c}^{\circ},\mathbf{n}^{\circ}) = (\mathbf{c}^{*},\mathbf{n}^{*}) = (\mathbf{c}^{**},\mathbf{n}^{**})$, for all prices. Formally, we have the following definitions.[13]

Definition 1 *The consumption-portfolio problem (8)-(11) is said to be solved via resolute choice if and only if the agent makes all choices at $t=1$ and these choices are not revised over time as new choices become optimal. Given returns and initial income, we define resolute choice as*

$$(c^{\circ}(s^t),\mathbf{n}^{\circ}(s^t))_{s^t\in\mathcal{S}}\left((\mathbf{R}(s^t))_{s^t\in\mathcal{S}}, I\right) = \arg\max_{c(s^t),\mathbf{n}(s^t)} U(\mathbf{c}|s^1) \quad S.T.$$

[13]For a more basic discussion in a certainty setting, see Selden and Wei (2016, p. 1916).

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = 1,$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad 2 < t < T,$$

and

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T.$$

Definition 2 *The consumption-portfolio problem (8)-(11) is said to be solved via naive choice if and only if the agent reoptimizes and revises her choices every period based on her current period preferences. Naive choice is defined sequentially for* $\tau = 1, 2, ..., T$ *as*

$$(c^*(s^\tau), \mathbf{n}^*(s^\tau))\,(I\,(s^\tau)) = (c^\circ(s^\tau), \mathbf{n}^\circ(s^\tau))\,\big((\mathbf{R}(s^t))_{s^t \in \mathcal{S}}, I\big)$$

where

$$(c^\circ(s^t), \mathbf{n}^\circ(s^t))_{s^t \succeq s^\tau}\,\big((\mathbf{R}(s^t))_{s^t \in \mathcal{S}}, I\big) = \underset{c(s^t),\mathbf{n}(s^t)}{\arg\max}\; U(\mathbf{c}|s^\tau) \quad S.T.$$

$$c(s^t) = I - \sum_j n_j(s^t), \quad t = \tau,$$

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t) - \sum_j n_j(s^t), \quad \tau < t < T,$$

and

$$c(s^t) = \mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t), \quad t = T.$$

Definition 3 *The consumption-portfolio problem (8)-(11) is said to be solved via sophisticated choice if and only if the agent takes into account her future period preferences when making her choices in earlier periods. The sophisticated choice can be defined recursively for* $\tau = T, T-1, \ldots$ *as*[14]

$$(c^{**}(s^\tau), \mathbf{n}^{**}(s^\tau))\,(I\,(s^\tau)) = \underset{c(s^\tau),\mathbf{n}(s^\tau)}{\arg\max}\; u\,(c(s^\tau)) + \sum_{t=\tau+1}^{T} \beta^{t-\tau} u(\widehat{c}_t|s^\tau) \quad S.T.$$

$$c(s^t) = I(s^t) - \sum_j n_j(s^t), \quad t = \tau,$$

and

$$(\widehat{c}_t|s^\tau) = V^{-1}\left(\sum_{s^t \succ s^\tau} \pi(s^t|s^\tau) V(c^{**}(s^t)(\mathbf{n}(s^{t-1}) \cdot \mathbf{R}(s^t)))\right).$$

[14] It should be noted that in general a unique sophisticated choice may not exist in the recursive solution process. However for the utility functions we consider in this paper, a unique solution always exists since (quasi)homotheticity ensures concavity of the corresponding utility functions. Also, note that we have written $U(\mathbf{c}|s^\tau)$ as a separable form in order to highlight the role of $(\widehat{c}_t|s^\tau)$.

One can also define time consistency at the preference as opposed to the demand level. Denote the continuation of a consumption tree starting from node s^t by $\mathbf{c}(s \succeq s^t)$ which includes consumption at s^t. Then following Epstein and Zin (1989), TC can be defined as follows.

Definition 4 *The consumer's preferences satisfy TC if and only if at time t with some payoff history s^t,*

$$\mathbf{c}\left(s \succeq s^{t+1}\right) \succeq \mathbf{c}'\left(s \succeq s^{t+1}\right) \quad \left(\forall s^{t+1} \succ s^t\right) \Rightarrow \mathbf{c}\left(s \succeq s^t\right) \succeq \mathbf{c}'\left(s \succeq s^t\right),$$

where $c(s^t) = c'(s^t)$.

In the certainty case, Blackorby, et al. (1973) prove that time consistency holds if and only if each period $t+1$ utility can be embedded into the period t utility for all $t \in \{1, ..., T-1\}$ utilities. Johnsen and Donaldson (1985) extend this notion to the risky case, where time consistency holds if and only if the future utility function in each state can be embedded into the utility function of prior periods. Following Blackorby et al. (1973), in Section 4 we link the demand and preference definitions of time consistency in our setting.

3 Intuition

In this section, we first illustrate the very simple and intuitive calculation of DOCE utility. Second, we argue that if the necessary and sufficient conditions derived in the next section for DOCE preferences to be time consistent are satisfied, the consumer's dynamic consumption-portfolio problem can be reformulated as a sequence of single period portfolio optimizations and a certainty consumption-saving optimization where the former optimizations are based on risk preferences and the latter is based on time preferences. Moreover, the relatively complex KP process for solving the dynamic consumption-portfolio problem can also be reformulated using this same highly intuitive simplification when the conditions for KP and DOCE demands to converge are satisfied. Moreover, the relatively complex KP process for solving the dynamic consumption-portfolio problem can also be reformulated using this same highly intuitive simplification when the conditions for KP and DOCE demands to converge are satisfied.

Consider the three period consumption tree in Figure 1. The assumed TRI property implies that the tree on the left hand side of the figure can be decomposed into the two single period trees on the right hand side. The upper tree corresponds to the distribution of period 2 consumption. The bottom tree can be thought of

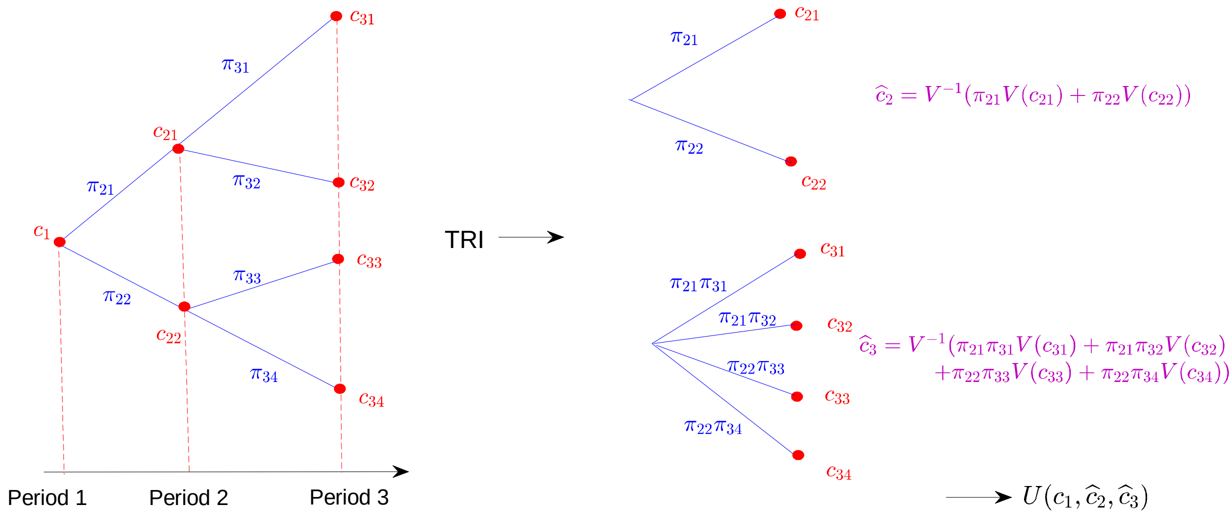


Figure 1:

as a single stage compound lottery paying off period 3 consumption. It further follows from TRI that the two subtrees can be evaluated independently using the assumed risk preferences. As shown in the figure, it is straightforward to compute the corresponding certainty equivalents $\widehat{c}_2$ and $\widehat{c}_3$. Then the utility of the three period tree on the left hand side of Figure 1 can be viewed as being equivalent to the utility of the degenerate tree $(c_1, \widehat{c}_2, \widehat{c}_3)$ computed by $U(c_1, \widehat{c}_2, \widehat{c}_3)$.

Next consider the three period consumption-portfolio problem (8) - (11) corresponding to Figure 2, where time and risk preferences are respectively represented by the CES and CRRA utilities in (5). The consumer can be viewed as solving single period portfolio optimizations at the period 1 node and at the two period 2 nodes. It follows from Proposition 5 below that if Assumptions [IR] and [RF] hold and DOCE preferences are homothetic following (5), these three portfolio optimizations are independent of time preferences and only depend on risk preferences. Denote the (gross) returns on a period 1 risky asset in the two states by R_{21} and R_{22} and the (gross) return on the risk free asset R_{f2}. With slight abuse of our general notation, period 1 risky and risk free asset holdings are respectively denoted by n_1 and n_{f1}. Thus in period 1, the consumer can be viewed as maximizing the following conditional on her level of period 1 saving $(I - c_1)$

$$\max_{n_1, n_{f1}} \pi_{21}V(R_{21}n_1 + R_{f2}n_{f1}) + \pi_{22}V(R_{22}n_1 + R_{f2}n_{f1}) \quad S.T.\ I - c_1 = n_1 + n_{f1}. \tag{16}$$

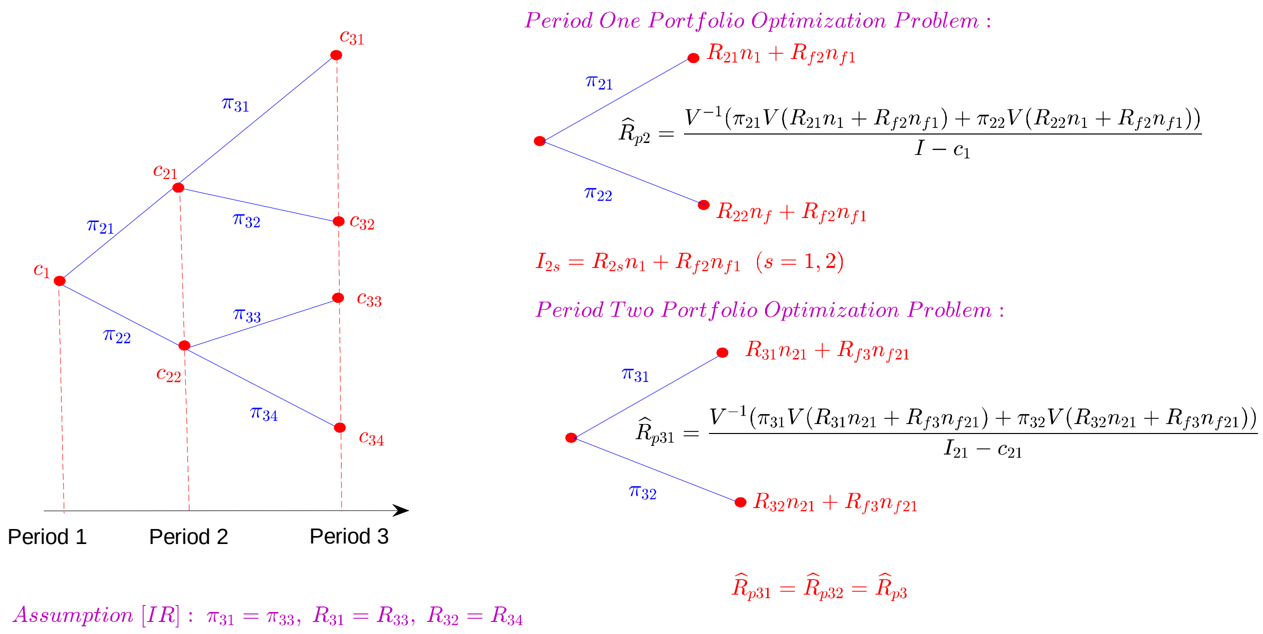


Figure 2:

Conditional on her optimal period 1 asset demands, period 2 income for the two branches is given by

$$I_{21} = R_{21}n_1 + R_{f2}n_{f1} \quad \text{and} \quad I_{22} = R_{22}n_1 + R_{f2}n_{f1}.$$

Then in period 2, conditional on the upper node being realized and income equalling I_{21}, the consumer faces the following portfolio problem

$$\max_{n_{21}, n_{f21}} \pi_{31}V(R_{31}n_{21} + R_{f3}n_{f21}) + \pi_{32}V(R_{32}n_{21} + R_{f3}n_{f21}) \quad S.T.\ I_{21} - c_{21} = n_{21} + n_{f21}. \tag{17}$$

Now if Assumption [IR] holds, it follows that

$$\pi_{31} = \pi_{33}, \quad \pi_{32} = \pi_{34} \quad \text{and} \quad R_{31} = R_{33},\ R_{32} = R_{34}$$

and the period 2 portfolio optimization confronted on the lower branch in Figure 2 is identical to the one the upper branch (17) except for investable income being $I_{22} - c_{22}$. Then given the homothetic form of the CRRA NM index V, certainty equivalent consumption in the upper and lower branches can be expressed as

$$\widehat{c}_{31} = (I_{21} - c_{21})\widehat{R}_{p31} \quad \text{and} \quad \widehat{c}_{32} = (I_{22} - c_{22})\widehat{R}_{p32},$$

where $\widehat{R}_{p31}$ and $\widehat{R}_{p32}$ denote respectively the portfolio certainty equivalent returns on the upper and lower branches. Assumption [IR] implies that $\widehat{R}_{p31} = \widehat{R}_{p32} = \widehat{R}_{p3}$.

It follows from Theorem 1 below that if Assumption [IR] holds and the consumer's time and risk preferences take the homothetic form (5), resolute, naive and sophisticated consumption and asset demands are the same. The consumer's DOCE preferences exhibit time consistency on a restricted domain corresponding to the consumption-portfolio problem. And the consumption-saving problem in Figure 2, can equivalently be expressed as the following very simple certainty problem

$$\max_{c_1,\widehat{c}_2,\widehat{c}_3} \left(c_1^{-\delta_1} + \beta\widehat{c}_2^{-\delta_1} + \beta^2\widehat{c}_3^{-\delta_1}\right)^{-\frac{1}{\delta_1}} \quad S.T.\ I = c_1 + \frac{\widehat{c}_2}{\widehat{R}_{p2}} + \frac{\widehat{c}_3}{\widehat{R}_{p2}\widehat{R}_{p3}},$$

where

$$\widehat{c}_2 = V^{-1}\left(\pi_{21}V(c_{21}) + \pi_{22}V(c_{22})\right) \quad \text{and} \quad \widehat{c}_3 = V^{-1}\left(\pi_{21}V(\widehat{c}_{31}) + \pi_{22}V(\widehat{c}_{32})\right).$$

Solving this problem yields

$$c_1 = \frac{I}{1 + \beta^{\frac{1}{1+\delta_1}}\widehat{R}_{p2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}}\widehat{R}_{p2}^{-\frac{\delta_1}{1+\delta_1}}\widehat{R}_{p3}^{-\frac{\delta_1}{1+\delta_1}}}. \tag{18}$$

Given the conditional asset demands resulting from solving the single period portfolio problems, optimal unconditional asset demands can be computed utilizing period 1 consumption (18).[15]

It should be noted that this same transformation of the dynamic consumption-portfolio problem into a sequence of single period portfolio problems and a certainty consumption-saving problem is possible even if the consumer's representation of time preferences takes the popular changing taste quasi-hyperbolic discounted form. (See the discussion at the end of Subsection 4.3.)

4 Time Consistent DOCE Demand

In this section, we discuss time consistency when DOCE preferences are homothetic and quasihomothetic. When DOCE demands are time consistent, we show that (i) they can also be rationalized by time consistent KP preferences based on the same assumed building blocks utilities (u, V) and (ii) key properties relating to saving and asset demand behavior derived for two period KPS preferences also hold for T-period DOCE and KP preferences.

[15] A similar simplification of the dynamic consumption-portfolio problem can be done for the DOCE time and risk preference utilities (2), (3) and (4).

4.1 Homothetic Preferences

In this subsection, first necessary and sufficient conditions are given such that DOCE preferences generate time consistent consumption and asset demands. Second, intuition is provided for this surprising result.

4.1.1 Main Result

It is easy to see that DOCE preferences will be homothetic if and only if the building block time and risk preference representations take the CES time and CRRA risk preference forms in (5). Then, we have the following result.

Theorem 1 *Suppose Assumption [IR] holds and the consumer solves the consumption-portfolio problem (8)-(11). Then DOCE demands are time consistent if and only if one of the following holds*

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1 > -1, \delta_2 > -1, \delta_1, \delta_2 \neq 0),$$

$$u(c) = \ln c \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_2 > -1, \delta_2 \neq 0),$$

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \ln c \quad (\delta_1 > -1, \delta_1 \neq 0),$$

or

$$u(c) = \ln c \quad \text{and} \quad V(c) = \ln c. \tag{19}$$

It should be noted that eqn. (19) in Theorem 1 corresponds to a time consistent EU special case of DOCE preferences.

At first glance, the theorem seems very surprising: How can DOCE preferences be generally time inconsistent, but still generate time consistent demands when asset returns are independent over time? While our detailed proof of Theorem 1 gives a formal answer to this puzzle, it is useful to understand the issue in the simplest possible case. We argue next that while DOCE preferences are generally time inconsistent, one can find a well defined restricted domain on which the preferences (and corresponding demands) are time consistent.

4.1.2 Time Consistent Preferences over Restricted Domains

Without loss of generality, assume the three time period tree structure in Figure 3, and denote the nodes by the following sequence of numbers $1, 21, 22, 31$ and 32 corresponding naturally to the subscripts for consumption at each node.

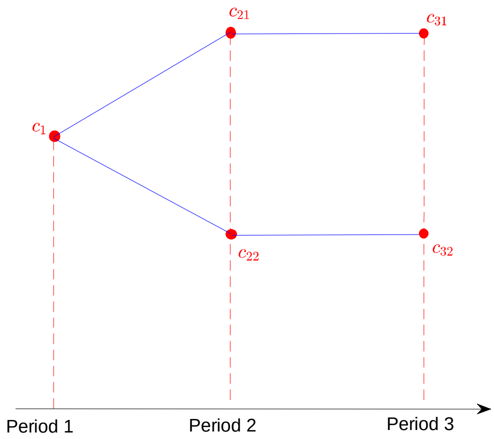


Figure 3:

Given the fixed tree structure in Figure 3 and a set of probabilities, a given consumption tree is fully characterized by the consumption vector

$$\mathbf{c} = (c_1, c_{21}, c_{22}, c_{31}, c_{32}) \in \mathbb{R}^5_+. \tag{20}$$

The vectors $(c_{21}, c_{31}), (c_{22}, c_{32}) \in \mathbb{R}^2_+$, respectively, characterize in a natural way the upper and lower subtrees. Preferences over the full tree, upper subtree and lower subtree consumption vectors are denoted respectively by $\succeq_1$, $\succeq_{21}$ and $\succeq_{22}$. Without loss of generality, it will turn out to be useful to view $\succeq_{21}$ and $\succeq_{22}$ as preferences over $\mathbb{R}^5_+$ with the requirement that $\forall \mathbf{c}, \mathbf{c}' \in \mathbb{R}^5_+$ and $\mathbf{d}, \mathbf{d}' \in \mathbb{R}^2_+$, if $\mathbf{c} \succeq_{21} \mathbf{c}'$ then

$$\mathbf{d} \succeq_{21} \mathbf{d}' \text{ whenever } (c_{21}, c_{31}) = (d_{21}, d_{31}),\ (c'_{21}, c'_{31}) = (d'_{21}, d'_{31})$$

and if $\mathbf{c} \succeq_{22} \mathbf{c}'$ then

$$\mathbf{d} \succeq_{22} \mathbf{d}' \text{ whenever } (c_{22}, c_{32}) = (d_{22}, d_{32}),\ (c'_{22}, c'_{32}) = (d'_{22}, d'_{32}).$$

In the current setting, the TC Definition 4 specializes to the following. The preference relations $\succeq_1$, $\succeq_{21}$ and $\succeq_{22}$ are said to satisfy TC over a given domain $\mathcal{I} \subset \mathbb{R}^5_+$ if and only if whenever for all $\mathbf{c}, \mathbf{c}' \in \mathcal{I}$ with $c_1 = c'_1$

$$\mathbf{c} \succeq_{2s} \mathbf{c}' \text{ for } s = 1, 2 \Rightarrow \mathbf{c} \succeq_1 \mathbf{c}',$$

with $\mathbf{c} \succ_1 \mathbf{c}'$ if at least one of $\succeq_{21}, \succeq_{22}$ holds strictly.

As noted earlier for $\mathbb{R}^5_+$, DOCE preferences do not satisfy time consistency. As illustrated in Example 2 below and Example B.1 in Supplemental Appendix B.4, there can be significant differences between sophisticated and resolute choice when Assumption [IR] does not hold. However, it turns out that one can restrict the domain of preference so that they become time consistent over the restricted domain. For example, it is easy to see that for any $\overline{\mathbf{c}} \in \mathbb{R}^5_+$ DOCE preferences are time consistent over $\{\mathbf{c} \in \mathbb{R}^5_+ : \mathbf{c} = \alpha\overline{\mathbf{c}}, \alpha > 0\}$. It turns out to be more interesting and relevant to consider the set

$$\mathcal{I}_0 = \left\{\mathbf{c} \in \mathbb{R}^5_+ : \mathbf{c} = (c_1, c_{21}, c_{22}, \alpha c_{21}, \alpha c_{22}), \alpha \in \mathbb{R}_+\right\}. \tag{21}$$

With homothetic preferences, optimal intertemporal choices will lie in this set if asset returns are independent over time. Assuming homotheticity, the period 1 utility function for DOCE preferences can be written as

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= u(c_1) + \beta u \circ V^{-1}\left(\sum_s \pi_s V(c_{2s})\right) + \beta^2 u \circ V^{-1}\left(\sum_s \pi_s V(\alpha c_{2s})\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_s \pi_s V(c_{2s})\right) + \\
&\quad \beta^2\left(u \circ V^{-1}\left(\sum_s \pi_s V(c_{2s})\right) u \circ V^{-1}\left(V(\alpha)\right)\right) \\
&= u(c_1) + \beta u \circ V^{-1}\left(\sum_s \pi_s V(c_{2s})\right)(1 + \beta u(\alpha)) \qquad (22)
\end{aligned}$$

and depending on which state $s = 1, 2$ is realized,

$$\mathcal{U}(\mathbf{c}_2|s) = u(c_{2s})(1 + \beta u(\alpha)).$$

But then it is easy to see that for any c_{21}, c_{22}, α and $c'_{21}, c'_{22}, \alpha'$, since preferences are homothetic

$$u(c_{2s})(1 + \beta u(\alpha)) \geq u(c'_{2s})(1 + \beta u(\alpha')) \text{ for } s = 1, 2$$

if

$$\beta u \circ V^{-1}\left(\sum_s \pi_s V(c_{2s})\right)(1 + \beta u(\alpha)) \geq \beta u \circ V^{-1}\left(\sum_s \pi_s V(c'_{2s})\right)(1 + \beta u(\alpha')).$$

Thus, we have the following proposition.

Proposition 1 *Assume the set of consumption trees corresponding to Figure 3. Homothetic DOCE preferences are TC in terms of Definition 4 over the domain* $\mathcal{I}_0$.

Connecting Proposition 1 back to Theorem 1, one can observe that $\mathcal{I}_0$ includes the optimal demands derived from Theorem 1. Once it is established that optimal demands lie in $\mathcal{I}_0$, this implies that homothetic preferences are TC. It should be noted that although we assume [IR] for asset returns, this does not imply that the consumption distribution is also independent over time. For the two state case, since second period residual income is different for the upper and lower branches, the consumption distribution will be different for the two branches as reflected in the construction of $\mathcal{I}_0$.

4.2 HARA Preferences

We next derive necessary and sufficient conditions for demands in the consumption-portfolio problem to be time consistent.

Theorem 2 *Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8) - (11). Then DOCE demands are time consistent if and only if one of the following holds*

(i)

$$\begin{aligned} u(c) &= -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \\ (\delta_1, \delta_2 &> -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0,b)), \end{aligned}$$

$$\begin{aligned} u(c) &= \ln(c-b) \quad \textit{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \\ (\delta_2 &> -1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0,b)), \end{aligned}$$

$$\begin{aligned} u(c) &= -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \ln(c-b) \\ (\delta_1 &> -1, \delta_1 \neq 0, b \in \mathbb{R}, c > \max(0,b)) \end{aligned}$$

or

$$\begin{aligned} u(c) &= \ln(c-b) \quad \textit{and} \quad V(c) = \ln(c-b) \\ (b &\in \mathbb{R}, c > b); \ \textit{or} \end{aligned}$$

(ii)

$$u(c) = -\frac{\exp(-\kappa_1 c_1)}{\kappa_1} \quad \textit{and} \quad V(c) = \frac{\exp(-\kappa_2 c_2)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0); \ \textit{or}$$

(iii)

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0).$$

Remark 1 *It should be noted that (i) for both Theorems 1 and 2, the additive time preference U, eqn. (1), can have an arbitrary period* 1 *utility $u_1(c_1)$ which satisfies $u_1' > 0$ and $u_1'' < 0$ and (ii) for Theorem 2, the cases covered for risk preferences include the full HARA class.*

What is the intuition in Theorem 2 for why time independent returns and quasihomothetic preferences result in time consistent demands and what role is played by the assumed presence of a risk free asset? First note that following Theorem 1, we have time consistency for the special case of Theorem 2(i) where $b = 0$ and preferences take the homothetic form corresponding to the CES time and CRRA risk utilities (5). To see the intuition for the quasihomothetic consumption-portfolio case, consider the more general case where $b \neq 0$. Then note that the risk free asset is used to fund subsistence consumption and a portfolio of assets funds saving and supernumerary consumption. Thus the presence of the risk free asset essentially translates the quasihomothetic case into the homothetic case. For the consumption-saving problem (12) - (15) since there is no risk free asset, we have time consistent demands only for homothetic preferences.

4.3 Another Time Consistent Rationalization

We have shown that when appropriate restrictions are imposed on asset markets and DOCE time and risk preferences, demands are time consistent. Suppose that KP preferences are constructed from the same time and risk preference building block utilities (5) as in the time consistent DOCE case and one assumes that asset returns satisfy [IR]. Quite surprisingly, we next show that the two preference relations which are not ordinally equivalent over the full choice space, nevertheless result in the same demands.

Proposition 2 *Suppose Assumption [IR] holds and the consumer solves the consumption-portfolio problem (8)-(11). For DOCE preferences, further assume that*

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, \delta_1, \delta_2 \neq 0).$$

Then the optimal demands can also be rationalized by KP preferences, where

$$U(c_t, x) = -\frac{\left(c_t^{-\delta_1} + \beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \textit{and} \quad V_T(x) = -\frac{x^{-\delta_2}}{\delta_2}.$$

Proposition 2 shows that when DOCE and KP preferences are based on the same homothetic and time and risk preference building blocks and Assumption [IR] holds, they generate the same solution to the consumption-portfolio problem. The proof shows that the DOCE and KP utilities generate identically the same indirect utility over the proportionality contingent claim proportionality factor α_t (introduced in the definition of $\mathcal{I}_0$, (21)). But this doesn't address the basic question of how the DOCE and KP utilities converge. The following example seeks to provide an intuitive answer to this question.

Example 1 *Assume the tree structure in Figure 3, the consumer solves the consumption-portfolio problem (8)-(11). Asset returns satisfy Assumption [IR]. The DOCE and KP preference structures take the forms in Proposition 2. Then for DOCE sophisticated choice, it can be verified that*[16]

$$\frac{c_{22}}{c_{21}} = \frac{c_{32}}{c_{31}} = \left(\frac{\pi_2(R_{f2}-R_{22})}{\pi_1(R_{21}-R_{f2})}\right)^{\frac{1}{1+\delta_2}}. \tag{23}$$

The key to converting DOCE certainty equivalent terms such as $\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{-\frac{1}{\delta_2}}$ *into a "certainty" expression proportional to* c_{21} *is to use the relation between* c_{22} *and* c_{21} *given in (23). Therefore, the period* 1 *DOCE utility function can be written as*

$$\begin{aligned}\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1} + \beta\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} + \beta^2\left(\pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\ &= \left(c_1^{-\delta_1} + \beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)\right)^{-\frac{1}{\delta_1}}\end{aligned} \tag{24}$$

and the budget constraint can be written as

$$I = c_1 + \left(\frac{R_{f2}-R_{22}}{\left(R_{21}-R_{22}\right)R_{f2}} + \frac{R_{21}-R_{f2}}{\left(R_{21}-R_{22}\right)R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}\right)\left(c_{21} + \frac{c_{31}}{R_{f3}}\right). \tag{25}$$

For KP preferences, it can be verified that

$$\frac{\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)^{-\frac{1}{\delta_1}}}{\left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1}\right)^{-\frac{1}{\delta_1}}} = \frac{\pi_2(R_{f2}-R_{22})}{\pi_1(R_{21}-R_{f2})}, \tag{26}$$

[16] For the tree structure in Figure 3, since $\pi_{31|21} = \pi_{32|22} = 1$, the probability structure can be fully captured by (π_{21}, π_{22}). To simplify the notation, we use (π_1, π_2) throughout. The same argument applies to the analogous examples below.

where the numerator and denominator are respectively the utility value of the upper and lower branches. Using (26), the KP period 1 *utility function becomes*

$$\begin{aligned}\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1}+\beta\left(\pi_1\left(c_{21}^{-\delta_1}+\beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}+\pi_2\left(c_{22}^{-\delta_1}+\beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\ &= \left(c_1^{-\delta_1}+\beta\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}\left(c_{21}^{-\delta_1}+\beta c_{31}^{-\delta_1}\right)\right)^{-\frac{1}{\delta_1}},\end{aligned}$$

which is the same as the utility in eqn. (24) and the budget constraint is also the same as eqn. (25). The expression (26) is key to converting the KP formulation into a certainty optimization involving only c_1, c_{21} *and* c_{31}. *Therefore if Assumption [IR] holds and DOCE and KP share the same homothetic building blocks* (u, V), *one can always transform the DOCE and KP consumption-portfolio problem into a same certainty, time consistent optimization along a single branch. Once the single-branch solution has been derived, all of the optimal asset holdings and other contingent claim consumption values can be simply computed. Finally, the coefficient in front of the period two and three consumption in the utility function 24, satisfies*

$$\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} \gtreqqless 1 \Leftrightarrow \delta_1 \gtreqqless 0.$$

Thus if $\delta_1 > (<)\, 0$, *the risk effects involving probabilities, asset payoffs and the risky preference parameter* δ_2, *can be viewed as a negative (postive) discount function. (See Supplemental Appendix B.1 for supporting calculations).*

We next show that the DOCE and KP demands are the same for the Theorem 2 case of HARA preferences, assuming independent returns over time and the presence of a risk free asset.

Proposition 3 *Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11). For DOCE preferences,*

(i) *if we assume that*

$$\begin{aligned} u(c) &= -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \\ (\delta_1, \delta_2 &> -1, \delta_1, \delta_2 \neq 0, b \in \mathbb{R}, c > \max(0, b)), \end{aligned}$$

then the optimal demands can also be rationalized by KP preferences, where

$$U\left(c_t, x\right) = -\frac{\left(\left(c_t-b\right)^{-\delta_1}+\beta\left(-\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \textit{and} \quad V_T\left(x\right) = -\frac{(x-b)^{-\delta_2}}{\delta_2};$$

(ii) *if we assume that*

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \textit{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

then the optimal demands can also be rationalized by KP preferences, where

$$U(c_t, x) = -\frac{\left(\exp(-\kappa_1 c_t) + \beta(-\kappa_2 x)^{\frac{\kappa_1}{\kappa_2}}\right)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2} \quad \textit{and} \quad V_T(x) = -\frac{\exp(-\kappa_2 x)}{\kappa_2};$$

(iii) *if we assume that*

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 < -1, b > c > 0),$$

then the optimal demands can also be rationalized by KP preferences, where

$$U(c_t, x) = \frac{\left((b-c_t)^{-\delta_1} + \beta(\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad \textit{and} \quad V_T(x) = \frac{(b-x)^{-\delta_2}}{\delta_2}.$$

The intuition for Propositions 2 and 3 is that since Assumption [IR] holds, effectively we do not receive any new information with the passage of the time. Thus the preference for early or late resolution for KP preferences cannot be distinguished from temporal resolution indifference for DOCE preferences. In fact, [IR] rules out the canonical early resolution consumption tree corresponding to the case in Figure 3 where $c_{21} = c_{22}$.[17] Moreover, as proved in Proposition 1, over the domain $\mathcal{I}_0$, DOCE and KP preferences are indistinguishable in terms of time consistency. It is clear that property SEP holds for KP preferences. Moreover assuming homothetic preferences and Assumption [IR], it can be seen from the last part of the proof of Proposition 2 that the KP utility takes the same form as the DOCE utility over the domain $\mathcal{I}_0$. A similar argument can be made for quasihomothetic preferences.

So far, we have assumed that the consumer's time and risk preferences corresponding to a given (U, V)-pair do not change over time. Time preferences in

[17] Suppose a consumer prefers this early resolution tree to a second late resolution consumption tree which has the same c_1 and c_2, but risk is resolved at the end of period 2 rather than at the end of period 1 in Figure 3. Then following Kreps and Porteus (1978), she is said to have a preference for early resolution. Given that $c_{21} = c_{22}$, the assumption that asset returns are independent over time implies that no matter how much is saved in period 1, period 2 income will be the same on the upper and lower branches. Since preferences are also the same on the upper and lower branches, optimal c_2 and c_3 will also be the same on the two branches. Thus the restricted domain will necessarily exclude early resolution consumption trees with different c_3-values.

future periods are represented by the continuation of the current U and each of the NM indices in periods $2, ..., T$ are equivalent up to a positive affine transformation. The time inconsistency inherent in DOCE preferences is attributable to asset returns failing to be independent over time. But what if time preferences change over time? Suppose in each period t, U_t takes the quasi-hyperbolic discounted utility form first introduced by Phelps and Pollak (1968)[18]

$$U_t(c_t, ..., c_T) = u(c_t) + \gamma \sum_{i=t+1}^{T} \beta^{i-t} u_t(c_i). \tag{27}$$

When $\gamma = 1$, the above utility converges to the exponential discounted utility (1). When $\gamma \neq 1$, the period 2 continuation of (27), corresponding to U_2, cannot be nested in the period 1 utility U_1. This implies that time preferences exhibit changing tastes and in applications such as the certainty consumption-saving problem, resolute, naive and sophisticated demands will diverge.[19] The three solution techniques also yield different demands in the consumption-portfolio problem with risky asset returns for both the DOCE and KP cases. Quite surprisingly, at least for us, the equivalence of the DOCE and KP demands established in Proposition 3 extends to the case of quasi-hyperbolic time preferences. That is, respectively the resolute, naive and sophisticated DOCE and KP demand functions are the same. (See Proposition B.2 in Supplemental Appendix B.2.)

4.4 Extension of Two Period KPS Demand Properties

In this subsection, we show that two key demand properties which hold for two period KPS preferences extend to the DOCE setting if the conditions in Theorems 1 and 2 are satisfied. The first relates to precautionary saving which in recent years has received considerable attention in finance and macroeconomics. Gollier (2001, chapter 19) analyzes in a two period setting the properties of excess saving $\theta = s_1^{risky} - s_1^{certain}$, where s_1^{risky} and $s_1^{certain}$ denote, respectively, optimal period 1 saving when the return on the investment asset is risky and certain. The certain return equals the mean of the risky return. Selden and Wei (2018) prove that for KPS preferences corresponding to the CES and CRRA utilities in eqn. (5), the existence of a positive θ depends on a comparison of the EIS and unity and is independent of the risk aversion parameter δ_2.

[18] In order to be consistent with the use of β as the discount function in the rest of this paper, we have interchanged the normal roles of β and γ typically used in the quasi-hyperbolic discounting literature.

[19] The economic implications of the quasi-hyperbolic discounted form have been studied extensively (e.g., Laibson 1997 and Diamond and Koszegi 2003)

Proposition 4 *Suppose Assumption [IR] holds and the consumer solves the consumption-saving problem (12)-(15). Further assume*

$$u(c) = -\frac{c^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{c^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1).$$

Then for KP and DOCE preferences, using (6) we have

$$\theta \gtreqless 0 \Leftrightarrow EIS \lesseqgtr 1,$$

which is the same as the two period case. (The proof of this proposition is provided in Supplemental Appendix B.3.)

One attractive feature of the complete separation of time and risk preferences implicit in the KPS utility corresponding to (5) is that in the classic consumption-portfolio problem, optimal asset ratios are determined by risk preferences and are independent of time preferences. This result extends to the dynamic setting if the conditions in Theorem 2 are satisfied.

Proposition 5 *Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11). In each period $t \in \{1, ..., T-1\}$, given the node s^t, denote the return on the risk free asset on the branch starting from node s^t by $R_f(s^t)$,*[20] *the demands for risky and risk free assets by $n_j(s^t)$ and $n_f(s^t)$, respectively. If we further assume*

(i)

$$u(c) = -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, c > \max(0, b)),$$

then in each period $t \in \{1, ..., T-1\}$,[21]

$$\frac{n_f(s^t) - \frac{b}{R_f(s^t)}}{n_j(s^t)} = \eta_j(s^t)$$

are the same for KP and DOCE preferences and independent of δ_1 and β;

(ii)

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

[20] We use the notation $R_f(s^t)$ instead of $R_f(s^{t+1})$ since the risk free rate only depends on the starting node s^t and is the same for each $s^{t+1} \succ s^t$.

[21] If $\delta_i = 0$, one can use $u(c) = \ln(c)$ instead of power utility as in Theorem 2. This statement also applies to subsequent results unless indicated otherwise.

then in each period $t \in \{1, ..., T-1\}$,

$$n_j\left(s^t\right) = \eta_j\left(s^t\right)$$

are the same for KP and DOCE preferences and independent of κ_1 *and* β*; or*

(iii)

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, b > c > 0),$$

then in each period $t \in \{1, ..., T-1\}$,

$$\frac{\frac{b}{R_f(s^t)} - n_f\left(s^t\right)}{n_j\left(s^t\right)} = \eta_j\left(s^t\right)$$

are the same for KP and DOCE preferences and independent of δ_1 *and* β.

Remark 2 *There is a direct connection between Proposition 5 and a widely referenced result in Giovannini and Weil (1989, section 2.5). They prove that corresponding to the EZ special case of KP preferences associated with eqn. (5), the portfolio optimization is identical to that of a single period EU optimization and hence is independent of the consumer's time preference parameters* δ_1 *and* β*. Proposition 5 extends this result to a more general set of KP preferences and establishes the connection to DOCE preferences. Also, Proposition 5 when combined with Theorem 2 can be viewed as providing necessary as well as sufficient conditions for asset ratios to be independent of time preferences since DOCE preferences are time consistent only under the indicated conditions.*

(In Supplemental Appendix B.2, Proposition 5 is shown to hold even when the DOCE and KP time preference utilities take the quasi-hyperbolic form (27).)

5 Departures from Independent Asset Returns

In the prior section when Assumption [IR] holds, the KP and DOCE models were shown to generate the same time consistent demands and satisfy SEP. In this section, we assume the DOCE and KP preferences have the same homothetic time and risk preference building blocks (u, V) corresponding to (5) and relax Assumption [IR]. As a result resolute, naive and sophisticated DOCE and KP demands all diverge. As a short digression, the first subsection demonstrates in a certainty setting, when CES time preferences exhibit time inconsistency, the sophisticated consumption-saving solution can diverge dramatically from the resolute and naive solutions when the consumer has a strong preference for intertemporal substitution. In Subsection 5.2, a similar phenomena is shown to arise for the DOCE solutions to consumption-portfolio problem. Conversely when the consumer exhibits an aversion to intertemporal substitution, period 1 consumption (and saving) and portfolio composition (as reflected by the ratio n_{f1}/n_1)[22] can exhibit very similar behavior for the KP and DOCE resolute, naive and sophisticated cases.[23]

5.1 Strong Preference for Intertemporal Substitution: Divergent Sophisticated Saving Behavior

Once one relaxes Assumption [IR], resolute, naive and sophisticated choice diverge for DOCE preferences. The question naturally arises whether one solution technique can be said to be preferred. Strotz (1956, p. 173) argues for the superiority of sophisticated choice based on his well-known argument that in the last time period, an individual optimizes subject to contraints and in each earlier period she chooses "the best plan among those she will actually follow". Caplin and Leahy (2006) build on this logic and suggest that properly structured sophisticated choice is preferable to or in some cases equivalent to the popular alternative of a game-theoretic solution (e.g., Phelps and Pollak 1968 and Peleg and Yaari 1973). In this subsection, we present a simple example to illustrate that when consumers exhibit a strong preference for intertemporal substitution, the argument for sophisticated choice may be less compelling since it can result in counterintuitive saving behavior.

Assume a three period certainty consumption-saving problem, where the consumer can invest in periods 1 and 2 in a risk free asset with a (gross) return R_f in each period. The period 1 budget constraint is given by

$$c_1 + \frac{c_2}{R_f} + \frac{c_3}{R_f^2} = I.$$

Assume quasi-hyperbolic discounted time preferences (e.g., Laibson 1997) corresponding to the following period 1 and 2 utilities, respectively,

$$U^{(1)}(c_1, c_2, c_3) = -\frac{c_1^{-\delta_1}}{\delta_1} - \gamma\beta\frac{c_2^{-\delta_1}}{\delta_1} - \gamma\beta^2\frac{c_3^{-\delta_1}}{\delta_1}$$

[22] Since there is one node in period 1, we simplify the notation by denoting the risky and risk free asset holdings respectively by n_1 and n_{f1} instead of $n(s^1)$ and $n_f(s^1)$.

[23] Paralleling Subsection 5.2, which focuses on the consumption-portfolio problem, Supplemental Appendix B.4 provides a similar analysis for the consumption-saving problem.

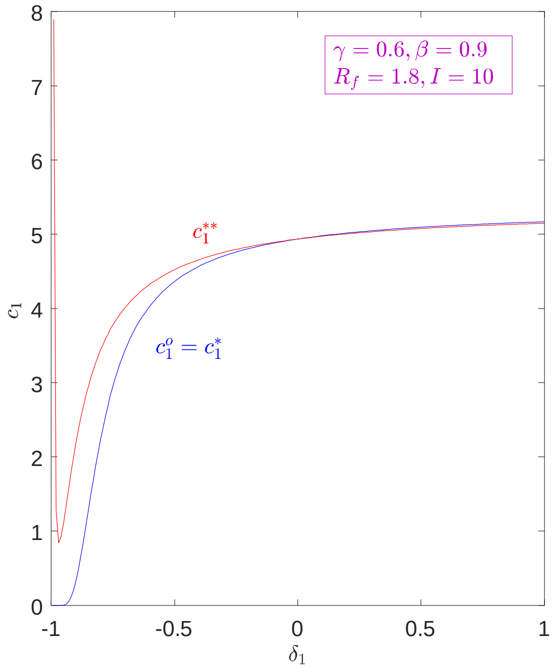


Figure 4:

and

$$U^{(2)}(c_2, c_3) = -\frac{c_2^{-\delta_1}}{\delta_1} - \gamma\beta\frac{c_3^{-\delta_1}}{\delta_1},$$

where $\delta_1 > -1$ and $\delta_1 \neq 0$ to rule out the log additive special case. Solving for $c_1^\circ = c_1^*$ and c_1^{**} and letting $\delta_1 \to -1$, one obtains

$$c_1^\circ = c_1^* \longrightarrow 0$$

if and only if

$$\gamma\beta R_f > 1 \quad \text{or} \quad \gamma\beta^2 R_f^2 > 1.$$

Based on numerical simulations, c_1^* and c_1^{**} can behave quite differently based on variations in the time preference parameter δ_1. For example, if[24]

$$\gamma = 0.6, \beta = 0.9, R_f = 1.8,$$

we have

$$\lim_{\delta_1 \to -1} c_1^* = 0 \text{ and } \lim_{\delta_1 \to -1} c_1^{**} = I$$

as shown in Figure 4. To see the intuition for this result, notice that

[24]The γ and β values assumed here are standard in the existing literature (see, for example, Laibson 1997, p. 456).

$$\gamma\beta R_f = 0.918 \quad \text{and} \quad \gamma\beta^2 R_f^2 = 1.405. \tag{28}$$

Since $\gamma\beta^2 R_f^2 = 1.405 > 1$, for resolute (naive) choice, the return in the last period after taking into account the discount seems quite attractive. Assume the consumer is completely substitute oriented where $\delta_1 \to -1$ and the EIS, (6), goes to infinity. Then, the resolute (naive) consumer will not consume in periods 1 and 2 and will invest all of her initial income for period 3 consumption. The same substitute oriented consumer when following sophisticated choice behaves very differently. Again assuming $\delta_1 \to -1$, the sophisticated consumer in period 2 views the return paying off in the third period $\gamma\beta R_f = 0.918 < 1$ as not good enough to merit saving. Hence she is inclined to consume everything in period 2. However, when viewed from period 1 perspective, the period 2 return which is the same $\gamma\beta R_f = 0.918 < 1$ and is not attractive. Thus it will seem best for the sophisticated consumer to do no saving and consume everything in period 1. Therefore, we have $c_1^{**} = I$ when $\delta_1 \to -1$. This backward folding response to strong intertemporal substitution orientation of not saving seem less intuitive than the resolute (naive) view of taking advantage of two years of very risk free returns and only consuming in period 3.

5.2 Disentangling the Effects of Time and Risk on Asset Demand

Despite the difficulties raised in the prior subsection, we next show that the relaxation of Assumption [IR] can result in KP and resolute, naive and sophisticated DOCE optimal demands being quite similar in a consumption-portfolio problem.[25] We focus on asset demands and in particular the portfolio composition.

Assume the CES time and CRRA risk preference utilities in (5) and the tree structure in Figure 3. In period 1, the consumer can buy short term risk free and risky assets, which pay off in period 2. For simplicity, consider the two state case. The short term risk free asset has the return R_{f2}. The short term risky asset has the return R_{2s} with the probability π_s $(s = 1, 2)$. In period 2, depending on which state is realized, there exists a risk free asset with return R_{f31} or R_{f32}. With slight abuse of our general notation, period 1 asset holdings will be denoted by n_1 and n_{f1}. Then period 2 income for the two branches is given by

$$I_{2s} = R_{2s} n_1 + R_{f2} n_{f1} \;\; (s = 1, 2).$$

[25] One may wonder whether the trees in Figure **??** can ever occur as the result of an optimization process such as in the consumption-portfolio problem. We show in Supplemental Appendix B.5 that tree τ_2 can be the result of an optimization and that τ_1 is a feasible tree consisent with the budget constraints that is not optimal.

For DOCE resolute choice, the period 1 utility function is

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta \left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} + \beta^2 \left(\pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}}$$

and the budget constraints are

$$c_{31} = R_{f31} \left(R_{21} n_1 + R_{f2} n_{f1} - c_{21} \right), \quad c_{32} = R_{f32} \left(R_{22} n_1 + R_{f2} n_{f1} - c_{22} \right)$$

and

$$I = c_1 + n_1 + n_{f1}.$$

Straightforward, although tedious calculations result in

$$\frac{n_{f1}^{\circ}}{n_1^{\circ}} = \frac{\left(\begin{array}{c} \frac{\kappa R_{21}}{R_{f32}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} + R_{21} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}} \\ - \left(\frac{\kappa R_{22}}{R_{f31}} + R_{22} \right) \end{array} \right)}{R_{f2} \left(\frac{\kappa}{R_{f31}} + 1 - \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \frac{\kappa}{R_{f32}} - \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}} \right)},$$

where κ is a function depending on β and δ_1. (Supporting calculations for this resolute case including the expression for κ are given in Supplemental Appendix B.6.1.) One key observation is that κ depends on the time preference parameters δ_1 and β and hence so does the optimal asset ratio $n_{f1}^{\circ}/n_1^{\circ}$. This differs from the analogous two period KPS solution as well as the case where Assumptions [IR] holds as shown in Proposition 5. (It should be noted that it is not possible to derive a closed form expression for the optimal asset ratio for DOCE sophisticated choice. However, we demonstrate via simulations in Example 2 below that when Assumption [IR] does not hold, the sophisticated n_{f1}^{**}/n_1^{**} in general depends on the consumer's time preference parameters.)

For the comparable KP model, the period 1 utility function is

$$\begin{aligned} \mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1} + \beta \left(\pi_1 \left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{\frac{\delta_2}{\delta_1}} + \pi_2 \left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{\frac{\delta_2}{\delta_1}} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}} \\ &= \left(c_1^{-\delta_1} + \beta \left(\begin{array}{c} \pi_1 \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{21} n_1 + R_{f2} n_{f1} \right)^{-\delta_2} + \\ \pi_2 \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{22} n_1 + R_{f2} n_{f1} \right)^{-\delta_2} \end{array} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}} \end{aligned} \quad (29)$$

Solving for asset demands, one obtains

$$\frac{n_{f1}^{KP}}{n_1^{KP}} = \frac{R_{21} k_2^{\frac{1}{1+\delta_2}} - R_{22}}{\left(1 - k_2^{\frac{1}{1+\delta_2}} \right) R_{f2}},$$

where

$$k_2 = \frac{\pi_2 \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} (R_{f2} - R_{22})}{\pi_1 \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} (R_{21} - R_{f2})}.$$

As in the DOCE resolute case, the asset ratio n_{f1}^{KP}/n_1^{KP} depends on δ_1 and β. It should be noted that when $R_{f31} = R_{f32}$, we have

$$k_2 = \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})}$$

and the conditional portfolio problem converges to the two period case with the asset ratio being independent of δ_1 and β. Moreover, for this case, if

$$\pi_1 R_{21} + \pi_2 R_{22} > R_{f2},$$

we have $k_2 < 1$, implying that $n_1^{KP} > 0$. However, if $R_{f31} \neq R_{f32}$ then it is possible for $k_2 > 1$ and $n_1^{KP} < 0$. The general condition for determining the sign of n_1^{KP} is given by the following expression

$$n_1^{KP} \gtreqless 0 \Leftrightarrow \frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})} \gtreqless \frac{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}}. \tag{30}$$

Note that when $n_1^{KP} < 0$, we require that period 2 income

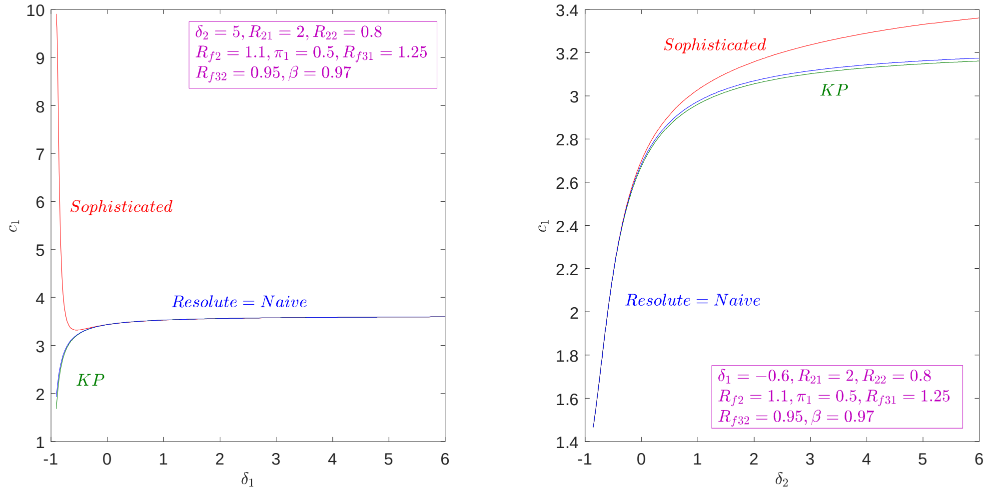
$$I_{2s} = R_{2s} n_1 + R_{f2} n_{f1} > 0 \ (s = 1, 2)$$

in order for c_{21}, c_{22}, c_{31} and c_{32} to be positive and for the consumer's utility function to be well-defined. The increase in the risk free asset holdings financed by the shorting of the risky asset should not be viewed as reflecting increased intraperiod risk aversion. Rather as discussed in Supplemental Appendix B.6.2, it should be viewed as dynamic hedging of intertemporal risk.

The following example illustrates the differences in the behavior of c_1 and n_{f1}/n_1 in response to variations in the time and risk preference parameters δ_1 and δ_2 for the KP and DOCE resolute, naive and sophisticated cases.

Example 2 *Assume a consumption-portfolio setting consistent with Figure 3. The time and risk preference building blocks are given by (5). Assume the following parameter values*

$R_{21} = 2, R_{22} = 0.8, R_{f2} = 1.1, R_{f31} = 1.25, R_{f32} = 0.95, \pi_1 = 0.5, \beta = 0.97, I = 10.$

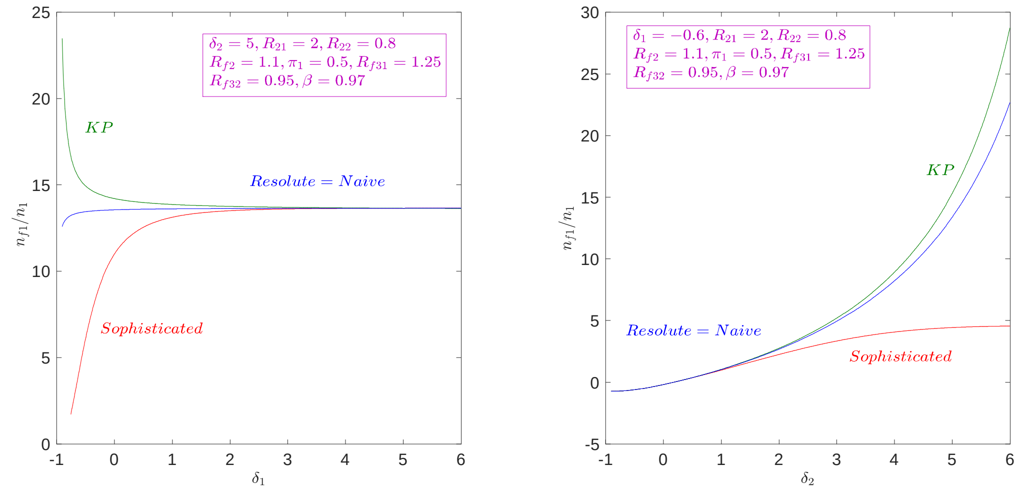


(a) (b)

Figure 5:

The results from numerical simulations of optimal c_1 *as functions of* δ_1 *and* δ_2 *are plotted in Figures 5(a) and (b). Paralleling Example B.1, period one DOCE resolute and sophisticated and KP consumption values are generally quite close in value except when* δ_1 *is close to* -1*. The pattern of DOCE sophisticated and resolute (naive) period* 1 *consumption diverging is similar to that of the certainty quasi-hyperbolic discounting case discussed in the prior subsection. Simulations of the optimal asset ratio* n_{f1}/n_1 *as functions of* δ_1 *and* δ_2 *are given in Figures 6(a) and (b). Based on the definitions of resolute and naive choice,* $n^*_{f1}/n^*_1 = n^\circ_{f1}/n^\circ_1$*. The KP and DOCE resolute and sophisticated asset ratios* n_{f1}/n_1 *converge for the EU special case where* $\delta_1 = \delta_2$*. In contrast to Proposition 5 where Assumption [IR] holds, in Figure 6(a),* n_{f1}/n_1 *varies with* δ_1*. In Figure 6(b) when* $\delta_2 = 5$ *and* $\delta_1 = -0.6$*, the KP and DOCE sophisticated asset ratios equal* 15.28 *and* 4.43*, respectively.*[26] *Given that the two models share the same time and risk preference building blocks, how can this divergence be explained? Because for this example asset returns are positively correlated over time, the upper and lower branches in Figure 3 are associated respectively with (good, good) and (bad, bad) asset payoffs. But the two models react quite differently to this intertemporal risk. Consider first the KP optimization, which is based on evaluating a lottery of utility values.*

[26] If one assumes for the [IR] case that $R_{f3} = \pi_1 R_{f31} + \pi_2 R_{f32} = 1.10$, then the constant $n_{f1}/n_1 = 4.70$ which is the same as the two period case.



(a) (b)

Figure 6:

Referring to eqn. (29), the utility levels in the two states are respectively given by

$$U_{21} = \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\frac{(1+\delta_1)}{\delta_1}} (R_{21}n_1 + R_{f2}n_{f1})$$

and

$$U_{22} = \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\frac{(1+\delta_1)}{\delta_1}} (R_{22}n_1 + R_{f2}n_{f1}) .$$

Independent of whether $\delta_1 < 0$ *or* $\delta_1 > 0$,

$$\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\frac{(1+\delta_1)}{\delta_1}} > \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\frac{(1+\delta_1)}{\delta_1}} . \tag{31}$$

This inequality implies that if $R_{21} > R_{22}$ *and* $n_1 > 0$, *the consumer faces more risk when considering the certainty equivalent of utility values*

$$\left(\pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2}\right)^{-\frac{1}{\delta_2}}$$

than considering the certainty equivalent of consumption values

$$\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{-\frac{1}{\delta_2}} ,$$

where

$$c_{21} = R_{21}n_1 + R_{f2}n_{f1} \quad \text{and} \quad c_{22} = R_{22}n_1 + R_{f2}n_{f1}.$$

The spread in period 2 *consumption values* c_{21} *and* c_{22} *is increased by the period* 3 *return factors in (31) and the consumer faces more risk than in the two period case. As a result she attempts to compensate for this greater risk by increasing her* n_{f1}/n_1 *ratio beyond that of the two period or [IR] case where there is no intertemporal risk. In contrast, why does the DOCE sophisticated consumer seemingly perceive the risk as less worrisome as evidenced by her much smaller* n_{f1}/n_1 *ratio? This difference in perception occurs even though the KP and DOCE optimization processes both proceed via backward induction and consider the conditional optimal saving decisions based on* I_{21} *and* I_{22}*. For the sophisticated DOCE consumer, the fact that* $\delta_1 < 0$ *(*$EIS > 1$*) has two distinct, important consequences for her asset allocation. First as in Example B.1, for the certainty consumption allocation along the upper and lower branches, the substitution effect dominates the income effect. Depending on whether* $R_{f3s} > (<) 1$ $(s = 1, 2)$ *the consumer substitutes* c_3 *for* c_2 *(*c_2 *for* c_3*). It follows that* $c_{31} > c_{21}$ *on the upper branch and* $c_{22} > c_{32}$ *on the lower branch. This results in the period* 2 *and* 3 *consumption spreads,* $c_{22} - c_{21}$ *and* $c_{31} - c_{32}$*, being respectively smaller and larger than the spread of* $I_{21} - I_{22}$*. It follows that* c_{31} *and* c_{32} *are respectively the best and worst of the four contingent claim consumption values.*[27] *The assumption that* $\delta_2 = 5$ *suggests a relatively risk averse consumer which in turn implies that the certainty equivalents* $\widehat{c}_2$ *and* $\widehat{c}_3$ *are near their respective lowest contingent claim outcomes. The second important implication of* $\delta_1 < 0$ *relates to the evaluation of* $\widehat{c}_2$ *and* $\widehat{c}_3$*. Since* $\widehat{c}_3 < \widehat{c}_2$ *and the DOCE consumer is substitute oriented, she will overvalue period* 2 *versus period* 3 *and the asset allocation decision will largely be determined by the period* 2 *spread. But as argued above, the period* 2 *consumption spread is smaller than the* (I_{21}, I_{22}) *distribution. Hence the DOCE sophisticated choice consumer perceives her period* 2 *risk as not being increased by the positive correlation of asset returns and hence she does not increase the* n_{f1}/n_1 *ratio as the KP consumer does. As indicated by Figure 6(a), if* $\delta_1 > 0$ *(*$EIS < 1$*) the above argument does not apply and the asset ratios for the four models are close. (For a discussion of cases where* n_1 *and the asset ratio can be negative, see Supplemental Appendix B.6.2. This appendix also contains supporting calculations for Example 2.)*

The results of this example suggest that when asset returns are not independent over time, if one follows much of the certainty empirical literature in assuming that the EIS is in the range of 0 and 0.4 (or using eqn. (6) $\delta_1 > 1.5$), then the DOCE and KP preference models generate qualitatively quite similar consumption and asset demand behavior. Alternatively, if one accepts the long-run risk and some

[27] The same is true for the KP consumer.

macro EIS calibrations of 1.5 to 2.0 (or equivalently $-0.5 < \delta_1 < -0.33$), then the demands differ significantly and differences in the respective preferences and in particular their underlying properties of TC, SEP and TRI become critical.

6 Concluding Comments

In this paper, we provide conditions such that DOCE preferences exhibit TC, SEP and TRI on a restricted domain of consumption trees corresponding to consumption-saving and consumption-portfolio problems. Under these same conditions, optimal consumption and asset demands for KP preferences are the same as the common DOCE resolute, naive and sophisticated demands. When the key Assumption [IR] is relaxed, the demands for the KP and different DOCE solution techniques can be close but also can diverge significantly.

Two extensions of our work would seem interesting. The first relates to the critical role played by the value of the EIS measure for the case of KP and DOCE preferences based on the CRRA and translated CRRA preference models (5) and (2). Significant differences in both optimal consumption and asset demands can arise when the $EIS > 1$ (or $\delta_1 < 0$). Given that existing empirical research is inconclusive on whether the EIS measure is larger or smaller than unity, it would seem desirable to investigate this question particularly in the context of the simple dynamic structure in Example 2. Although a number of challenges exist in applying parametric or non-parametric tests to this setting, it would nevertheless seem to be an important area for future research.

The second extension relates to comparing equilibrium asset returns based on the KP and three DOCE dynamic solution techniques when Assumption [IR] does not hold. For instance for the CES and CRRA utilities (5) as one varies δ_1 and δ_2 as in Figure 6, how do the divergent behaviors of n_{f1}^{KP}/n_1^{KP} and n_{f1}^{**}/n_1^{**} get reflected in terms of the equilibrium risky and risky free returns and the equity risk premium?[28]

Appendix

[28]One important caveat relates to possible inconsistencies that may arise between micro-demand and equilibrium return properties as noted by Selden and Wei (2018).

A Proofs

A.1 Proof of Theorem 1

To prove Theorems 1, it is useful to first state the following lemma that follows from generalizing the argument in Subsection 4.1.2.

Lemma 1 *Suppose preferences are homothetic and for each* $\tau = 1, \ldots, T-1$ *there exist* $\alpha_t^\tau(s)$, $t = \tau+1, \ldots, T$, $s = 1 \ldots S$ *such that at each* τ *naive choice satisfies*

$$c(s^t) = \alpha_t^\tau(s_t) c\left(s^{t-1}\right) \quad \forall t = \tau+1, \ldots, T, \; s_t = 1 \ldots, S.$$

Then choices are time consistent and naive and resolute choice are identical.[29]

Proof. Generalizing eqn. (22) we obtain

$$\begin{aligned}
\mathcal{U}(\alpha|s^\tau) &= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1}) V(\alpha_\tau(s_{\tau+1}) c(s^\tau))\right) + \ldots + \\
& \quad \beta^T u \circ V^{-1}\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1}) \ldots \sum_{s_T} \bar{\pi}_T(s_T) V(\alpha_\tau(s_{\tau+1}) \cdot \ldots \cdot \alpha_{T-1}(s_T) c(s^\tau))\right) \\
&= u(c(s^\tau)) + \beta u \circ V^{-1}\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1}) V(\alpha_\tau(s_{\tau+1}) c(s^\tau))\right) K_{\tau+1},
\end{aligned}$$

where $K_{\tau+1}$ is recursively defined as

$$K_T = 1 + \beta u \circ V^{-1}\left(\sum_{s_T} \bar{\pi}_T(s_T) V(\alpha_{T-1}(s_T))\right)$$

and

$$K_t = 1 + \beta u \circ V^{-1}\left(\sum_{s_t} \bar{\pi}_t(s_t) V(\alpha_{t-1}(s_t))\right) K_{t+1}$$

for $t = \tau+1, \ldots T-1$. By the same argument as in the proof of Proposition 1, it is now clear that if $\mathcal{U}(\alpha|s^\tau) > \mathcal{U}(\tilde{\alpha}|s^\tau)$ and $\tilde{\alpha}_t(s) = \alpha_t(s)$ for all $t \leq \tau$ and all s then $\mathcal{U}(\alpha|s^{\tau-1}) > \mathcal{U}(\tilde{\alpha}|s^{\tau-1})$. Therefore if α is preferred to $\tilde{\alpha}$ at τ following naive choice so must it be preferred following naive choice at $\tau - 1$ and, by induction, preferred by resolute choice. This completes the proof. ■

We are now in a position to prove Theorem 1.

Proof of Theorem 1 The following first order conditions are necessary and

[29] If resolute choice and naive choice coincide, it can easily be shown that sophisticated choice is the same.

sufficient for naive choice at s^τ under Assumption [IR]

$$u'(c(\overline{s}^t))\left(V\circ u^{-1}\right)'\left(\sum_{s^t\succ s^\tau}\pi(s^t)u(c(s^t))\right)=$$
$$\beta(V\circ u^{-1})'\left(\sum_{s^{t+1}\succ s^\tau}\pi(s^{t+1})u\left(c(s^{t+1})\right)\right)\sum_{s^{t+1}\succ \overline{s}^t}R(s_{t+1})\pi(s^{t+1})u'\left(c\left(s^{t+1}\right)\right),$$

for all $\overline{s}^t$, $t<T$. Since $u(.)$ and $V(.)$ are assumed to be homothetic, it is clear that any budget-feasible solution must satisfy

$$c(s^t)=\alpha_{t-1}(s_t)c\left(s^{t-1}\right)\quad \forall t=1,\ldots,T,\ s_t=1\ldots,S$$

and sufficiency follows directly from Lemma 1.

To prove necessity consider a simple version of the model with three periods, $t=1,2,3$, and an event tree as depicted in Figure 3. Suppose markets are complete. To satisfy Assumption [IR] suppose that the prices of the contingent claims are identical and denoted by $p(2)$.

The first order conditions for optimal naive choice at $t=2$ are

$$p(2)u'(c_{2s})=\beta u'(c_{3s}),\quad (s=1,2)$$

and, at $t=1$, planning for $t=2$, are

$$p(2)V'(c_{2s})(u\circ V^{-1})'\left(\sum_s\pi_sV(c_{2s})\right)=\beta(u\circ V^{-1})'\left(\sum_s\pi_sV(c_{3s})\right)V'(c_{3s}).$$

The first equation implies

$$c_{3s}=u'^{-1}\left(\frac{p(2)}{\beta}u'(c_{2s})\right)$$

and substituting this into the second equation we obtain

$$\begin{aligned}&p(1)V'(c_{2s})(u\circ V^{-1})'\left(\sum_s\pi_sV(c_{2s})\right)=\\&\beta(u\circ V^{-1})'\left(\sum_s\pi_sV\left(u'^{-1}\left(\frac{p(1)}{\beta}u'(c_{2s})\right)\right)\right)\\&V'\left(u'^{-1}\left(\frac{p(2)}{\beta}u'(c_{2s})\right)\right).\end{aligned}\tag{A.1}$$

Denote the price $p(2)$ simply by p. Then we consider variations in $p(2)=p$ as well as first period prices $p(1)$ that keep second period consumption fixed. Taking the derivative with respect to p on both sides and then setting $p = \beta$ one obtains

$$1 = \frac{(u \circ V^{-1})'' \left(\sum_s \pi_s V(c_{2s})\right) \sum_s \pi_s \left(V'(c_{2s}) u'^{-1\prime} \circ u'(c_{2s}) u'(c_{2s})\right)}{(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s})\right)} + \frac{V''(u'^{-1\prime} u'(c_{2s})) u'(c_{2s})}{V'(c_{2s})}.$$

Taking the derivatives with respect to c_{2s}, $s = 1, 2$, we obtain[30]

$$\frac{d}{dc} \frac{f'^{-1\prime}(g(c)) g(c)}{f(c)} = 0,$$

where $f(c) = V'(c)$ and $g(c) = u'(c)$.

Since

$$g^{-1\prime}(g(c)) g'(c) = 1,$$

we obtain

$$\frac{d}{dc} \frac{f'(c) g(c)}{g'(c) f(c)} = 0.$$

Consider the following ordinary differential equation

$$\frac{d}{dc} \left(\frac{f'(c)\, g(c)}{f(c)\, g'(c)} \right) = 0.$$

We have

$$\frac{f'(c)\, g(c)}{f(c)\, g'(c)} = K_1,$$

where K_1 is a constant. Therefore,

$$\frac{f'(c)}{f(c)} = (\ln f(c))' = K_1 \frac{g'(c)}{g(c)} = K_1 (\ln g(c))',$$

implying that

$$\ln f(c) = K_1 \ln g(c) + K_2,$$

where K_2 is a constant. Thus we have

$$f(c) = K_3 (g(c))^{K_1},$$

where K_3 is a constant.

Assuming $K > 0$, we can write $V'(c) = u'^K$ and $V'^{-1}(x) = u'^{-1}(x^{\frac{1}{K}})$. Substituting this into (A.1) we obtain

$$p(u \circ V^{-1})' \left(\sum_s \pi_s V(c_{2s}) \right) = \beta (u \circ V^{-1})' \left(\sum_s \pi_s V \left(u'^{-1} \left(\frac{p}{\beta} u'(c_{2s}) \right) \right) \right) \left(\frac{p}{\beta} \right)^{\frac{1}{K}}.$$

[30] This is possible since we can vary the prices of both Arrow securities at $t = 1$ independently.

Since $u \circ V^{-1}(x) = x^{\nu}$ for some ν it follows that the above can only hold if $u\left((u')^{-1}(x)\right)$ is homothetic. In this case we can write

$$u\left((u')^{-1}(x)\right) = ax^{\delta}.$$

Then we have

$$(u')^{-1}(x) = u^{-1}\left(ax^{\delta}\right).$$

Assuming

$$(u')^{-1}(x) = y,$$

then

$$u^{-1}\left(ax^{\delta}\right) = y \Leftrightarrow x = \left(\frac{u(y)}{a}\right)^{\frac{1}{\delta}}.$$

Therefore, we have

$$u'(x) = a\left(u(x)\right)^{\delta}.$$

Thus if $\delta \neq 1$, we have

$$\frac{d\left(u(x)\right)^{1-\delta}}{dx} = a(1-\delta) \Rightarrow u(x) = \left(a(1-\delta)x + c\right)^{-\frac{1}{1-\delta}}.$$

This corresponds to the DARA or IARA case of the HARA class. If $\delta = 1$,

$$\frac{d \ln u(x)}{dx} = a(1-\delta) \Rightarrow u(x) = \exp\left(a(1-\delta)x + c\right).$$

A simple numerical example shows that DARA and IARA utilities within the HARA class do not produce time consistent demand unless the condition of Theorem 2 holds. This completes the proof.

A.2 Proof of Theorem 2

For translated CRRA preferences (2), notice that the demand is identical to demand with homothetic preferences and tradable endowments.

For negative exponential preferences, the result follows from the fact (see Pollak 1971) that

$$\lim_{d \to -\infty} \left(1 + \frac{-\beta}{d}x\right)^{d} = -\exp(\beta x).$$

Thus, the result follows directly from the proof of Theorem 1.

A.3 Proof of Proposition 2

It follows from Theorem 1 that DOCE demands are time consistent. Next we prove that DOCE and KP preferences generate the same demands. First, observe that for both KP and DOCE preferences, homogeneity of the utility function, together with Assumption [IR] implies that the optimal solution must satisfy

$$c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}),$$

where $\alpha_{t-1}(s_t)$ are constants that depend on the shock s_t and the previous time period $t-1$. This implies that we can write the choice problems alternatively by having agents choose over α_t as well as initial consumption (subject to budget constraints). The key insight is that DOCE and KP preferences generate identical indirect utility functions over α_t. To see this, observe that DOCE utility can be written as follows

$$\begin{aligned}
&\mathcal{U}^{DOCE}(\mathbf{c}, \alpha | s^\tau) \\
= \; & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \frac{c(s^\tau)^{-\delta_1}}{\delta_1}\beta\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})\alpha_\tau(s_{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} - \\
& \frac{c(s^\tau)^{-\delta_1}}{\delta_1}\beta^2\left(\begin{array}{c}\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})\alpha_\tau(s_{\tau+1})^{-\delta_2} \\ \left(\sum_{s_{\tau+2}} \bar{\pi}_{\tau+2}(s_{\tau+2})\alpha_{\tau+1}(s_{\tau+2})^{-\delta_2}\right)\end{array}\right)^{\frac{\delta_1}{\delta_2}} - \dots - \\
& \frac{c(s^\tau)^{-\delta_1}}{\delta_1}\beta^{T-\tau}\left(\begin{array}{c}\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})\alpha_\tau(s_{\tau+1})^{-\delta_2} \\ \left(\dots\left(\sum_{s_T} \bar{\pi}_T(s_T)\alpha_{T-1}(s_T)^{-\delta_2}\right)\right)\end{array}\right)^{\frac{\delta_1}{\delta_2}}.
\end{aligned}$$

KP utility can be written as

$$\begin{aligned}
&\mathcal{U}^{KP}(\mathbf{c}, \alpha | s^\tau) \\
= \; & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \frac{\beta\left(\sum_{s^{\tau+1}\succ s^\tau} \pi(s^{\tau+1}|s^\tau)U(\mathbf{c}|s^{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}}{\delta_1} \\
= \; & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \beta\frac{c(s^\tau)^{-\delta_1}}{\delta_1} \\
& \left(\begin{array}{c}\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})\alpha_\tau(s_{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} \\ \left(1+\beta\left(\sum_{s_{\tau+2}} \bar{\pi}_{\tau+2}(s_{\tau+2})\alpha_{\tau+1}(s_{\tau+2})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}(1+\dots)\right)\end{array}\right),
\end{aligned}$$

which, when multiplied out, is identical to DOCE utility. Thus KP and DOCE preferences generate the same demands.

A.4 Proof of Proposition 3

It follows from Theorem 2 that DOCE demands are time consistent. It follows from the proof of Theorem 2 and Proposition 2 that the DOCE and KP preferences generate the same demands assuming the same time and risk preferences.

A.5 Proof of Proposition 5

It follows from Proposition 3 that DOCE demands are time consistent and the same as those for KP preferences. Therefore, it is enough to consider DOCE resolute choice. Consider case (i) with $b = 0$. As in the proof of Proposition 2, homogeneity of the utility function, together with Assumption [IR] implies that the optimal solution must satisfy

$$c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}),$$

where $\alpha_{t-1}(s_t)$ are constants that depend on the shock s_t and the previous time period $t-1$. Note that

$$\begin{aligned} & \mathcal{U}^{DOCE}\left(\mathbf{c}, \alpha | s^{\tau}\right) \\ = \quad & -\frac{c(s^{\tau})^{-\delta_1}}{\delta_1} - \frac{1}{\delta_1}\beta\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} - \\ & \frac{1}{\delta_1}\beta^2\left(\begin{array}{c} \sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \\ \left(\sum_{s_{\tau+2}} \bar{\pi}_{\tau+2}(s_{\tau+2})\alpha_{\tau+1}(s_{\tau+2})^{-\delta_2}\right) \end{array}\right)^{\frac{\delta_1}{\delta_2}} - \dots - \\ & \frac{1}{\delta_1}\beta^{T-\tau}\left(\begin{array}{c} \sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \\ \left(\dots\left(\sum_{s_T} \bar{\pi}_T(s_T)\alpha_{T-1}(s_T)^{-\delta_2}\right)\right) \end{array}\right)^{\frac{\delta_1}{\delta_2}}. \end{aligned}$$

Following resolute choice, the optimal asset ratios $n_f\left(s^{\tau}\right)/n_j\left(s^{\tau}\right)$ are determined by maximizing the EU

$$\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2},$$

which is independent of time preference parameters δ_1 and β. Since Assumption [IR] holds, in period τ, each s^{τ} gives the same result and hence

$$\frac{n_f\left(s^{\tau}\right)}{n_j\left(s^{\tau}\right)} = \eta_j\left(s^{\tau}\right).$$

For other cases, the argument is the same as the proof of Theorem 2.

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B For Online Publication: Supplemental Appendix

B.1 Supporting Calculations for Example 1

First consider DOCE sophisticated choice. In period two, if the upper branch occurs, then the utility function is

$$\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)^{-\frac{1}{\delta_1}}$$

and the budget contraint is

$$c_{31} = R_{f3}\left(R_{21}n_1 + R_{f2}n_{f1} - c_{21}\right).$$

Therefore, the first order condition is

$$\frac{c_{21}^{-\delta_1-1}}{\beta c_{31}^{-\delta_1-1}} = R_{f3},$$

implying that

$$c_{21} = \frac{R_{21}n_1 + R_{f2}n_{f1}}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}} \quad \text{and} \quad c_{31} = \left(\beta R_{f3}\right)^{\frac{1}{1+\delta_1}} c_{21}.$$

Similarly, we have

$$c_{22} = \frac{R_{22}n_1 + R_{f2}n_{f1}}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}} \quad \text{and} \quad c_{32} = \left(\beta R_{f3}\right)^{\frac{1}{1+\delta_1}} c_{22},$$

implying that

$$\frac{c_{21}}{c_{22}} = \frac{c_{31}}{c_{32}} = \frac{R_{21}n_1 + R_{f2}n_{f1}}{R_{22}n_1 + R_{f2}n_{f1}}.$$

It follows from the first order condition that

$$\frac{R_{21}n_1 + R_{f2}n_{f1}}{R_{22}n_1 + R_{f2}n_{f1}} = \left(\frac{\pi_2\left(R_{f2} - R_{22}\right)}{\pi_1\left(R_{21} - R_{f2}\right)}\right)^{-\frac{1}{1+\delta_2}}.$$

Therefore,

$$\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} = \left(\pi_1 + \pi_2\left(\frac{\pi_2(R_{f2} - R_{22})}{\pi_1(R_{21} - R_{f2})}\right)^{\frac{-\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1}$$

and

$$\left(\pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} = \left(\pi_1 + \pi_2\left(\frac{\pi_2(R_{f2} - R_{22})}{\pi_1(R_{21} - R_{f2})}\right)^{\frac{-\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1},$$

implying that

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1} + \beta\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} + \beta^2\left(\pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\
&= \left(\begin{array}{c}
c_1^{-\delta_1} + \beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1} \\
+\beta^2\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1}
\end{array}\right)^{-\frac{1}{\delta_1}}.
\end{aligned}$$

Next consider the KP preferences. Since the period 2 optimization problem is the same as DOCE sophisticated choice, we have

$$\frac{c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}}{c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1}} = \frac{\left(R_{21}n_1 + R_{f2}n_{f1}\right)^{-\delta_1}}{\left(R_{22}n_1 + R_{f2}n_{f1}\right)^{-\delta_1}}$$

and thus

$$\begin{aligned}
\mathcal{U}(\mathbf{c}) &= \left(c_1^{-\delta_1} + \beta\left(\pi_1\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} + \pi_2\left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}} \\
&= \left(c_1^{-\delta_1} + \beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)\right)^{-\frac{1}{\delta_1}}.
\end{aligned}$$

Since $\pi_1 R_{21} + \pi_2 R_{22} > R_{f2}$,

$$\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)} < 1.$$

If $-1 < \delta_2 < 0$,

$$\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}} < 1,$$

implying that

$$\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{1}{\delta_2}} > 1$$

and if $\delta_2 > 0$

$$\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}} > 1,$$

implying that

$$\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{1}{\delta_2}} > 1.$$

Therefore, we have

$$\left(\pi_1 + \pi_2 \left(\frac{\pi_2 (R_{f2} - R_{22})}{\pi_1 (R_{21} - R_{f2})}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} \gtreqless 1 \Leftrightarrow \delta_1 \gtreqless 0.$$

B.2 Quasi-hyperbolic Time Preferences

In this appendix we show first that the equivalence of the DOCE and KP demands established in Proposition 3 extends to the case of quasi-hyperbolic time preferences (27) and second that the asset allocation is independent of time preference parameters.

Proposition B.1 *Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11), where U_t takes the quasi-hyperbolic form (27). The consumer employs the resolute, naive and sophisticated solution techniques (Definitions 1 - 3) for both the DOCE and KP cases. For DOCE preferences,*

(i) *if we assume that*

$$\begin{aligned} u(c) &= -\frac{(c-b)^{-\delta_1}}{\delta_1} \quad \text{and} \quad V(c) = -\frac{(c-b)^{-\delta_2}}{\delta_2} \\ (\delta_1, \delta_2 &> -1, \delta_1, \delta_2 \neq 0, b \gtreqless 0, c > \max(0, b)), \end{aligned}$$

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

$$U_1(c_1, x) = -\frac{\left((c_1 - b)^{-\delta_1} + \gamma\beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2}$$

and

$$U_t(c_t, x) = -\frac{\left((c_t - b)^{-\delta_1} + \beta(-\delta_2 x)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2} \quad (t \geq 2) \quad \text{and} \quad V_T(x) = -\frac{(x-b)^{-\delta_2}}{\delta_2};$$

(ii) *if we assume that*

$$u(c) = -\frac{\exp(-\kappa_1 c)}{\kappa_1} \quad \text{and} \quad V(c) = -\frac{\exp(-\kappa_2 c)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

$$U_1(c_1, x) = -\frac{\left(\exp(-\kappa_1 c_1) + \gamma\beta(-\kappa_2 x)^{\frac{\kappa_1}{\kappa_2}}\right)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2}$$

and

$$U_t\left(c_t,x\right)=-\frac{\left(\exp\left(-\kappa_1 c_t\right)+\beta\left(-\kappa_2 x\right)^{\frac{\kappa_1}{\kappa_2}}\right)^{\frac{\kappa_2}{\kappa_1}}}{\kappa_2}\quad (t\geq 2)\quad \textit{and}\quad V_T\left(x\right)=-\frac{\exp\left(-\kappa_2 x\right)}{\kappa_2};$$

(iii) *if we assume that*

$$u(c)=\frac{(b-c)^{-\delta_1}}{\delta_1}\quad \textit{and}\quad V(c)=\frac{(b-c)^{-\delta_2}}{\delta_2}\quad (\delta_1,\delta_2<-1,b>c>0).$$

then the optimal resolute, naive and sophisticated demands can also be rationalized by KP preferences, where

$$U_1\left(c_1,x\right)=\frac{\left(\left(b-c_1\right)^{-\delta_1}+\gamma\beta\left(\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2}$$

and

$$U_t\left(c_t,x\right)=\frac{\left(\left(b-c_t\right)^{-\delta_1}+\beta\left(\delta_2 x\right)^{\frac{\delta_1}{\delta_2}}\right)^{\frac{\delta_2}{\delta_1}}}{\delta_2}\quad (t\geq 2)\quad \textit{and}\quad V_T\left(x\right)=\frac{(b-x)^{-\delta_2}}{\delta_2}.$$

Proof. The proof directly follows that of Propositions 2 and 3 where exponential discounting is replaced by quasi-hyperbolic discounting. ■

We next show that despite the presence of time inconsistent quasi-hyperbolic time preferences corresponding to (27), the common optimal asset allocation for DOCE and KP preferences is independent of the time preference parameters $(\delta_1,\kappa_1,\gamma,\beta)$ if Assumptions [IR] and [RF] hold. Thus, Proposition 5 extends to the case of quasi-hyperbolic time preferences.[31]

Proposition B.2 *Suppose Assumptions [IR] and [RF] hold and the consumer solves the consumption-portfolio problem (8)-(11), where U_t takes the quasi-hyperbolic form (27). In each period $t\in\{1,...,T-1\}$, given the node s^t, denote the return on the risk free asset on the branch starting from node s^t by $R_f\left(s^t\right)$ and the demands for risky and risk free assets by $n_j\left(s^t\right)$ and $n_f\left(s^t\right)$, respectively. If we further assume*

(i)

$$\begin{aligned} u(c) &= -\frac{(c-b)^{-\delta_1}}{\delta_1}\quad \textit{and}\quad V(c)=-\frac{(c-b)^{-\delta_2}}{\delta_2} \\ (\delta_1,\delta_2 &> -1, b \gtreqless 0, c>\max(0,b)), \end{aligned} \tag{B.1}$$

[31] It should be noted that versions of this result are shown by Palacios-Huertay and Pérez-Kakabadsez (2017) for EU preferences and by Love and Phelan (2015) for the EZ special case of KP preferences. Both papers assume different settings, only investigate sophisticated choice and never consider DOCE preferences.

then in each period $t \in \{1, ..., T-1\}$,

$$\frac{n_f\left(s^t\right) - \frac{b}{R_f(s^t)}}{n_j\left(s^t\right)} = \eta_j\left(s^t\right)$$

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of δ_1, γ *and* β;

(ii)

$$u(c) = -\frac{\exp\left(-\kappa_1 c\right)}{\kappa_1} \quad \textit{and} \quad V(c) = -\frac{\exp\left(-\kappa_2 c\right)}{\kappa_2} \quad (\kappa_1, \kappa_2 > 0),$$

then in each period $t \in \{1, ..., T-1\}$

$$n_j\left(s^t\right) = \eta_j\left(s^t\right)$$

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of κ_1, γ *and* β; *or*

(iii)

$$u(c) = \frac{(b-c)^{-\delta_1}}{\delta_1} \quad \textit{and} \quad V(c) = \frac{(b-c)^{-\delta_2}}{\delta_2} \quad (\delta_1, \delta_2 > -1, b > c > 0),$$

then in each period t

$$\frac{\frac{b}{R_f(s^t)} - n_f\left(s^t\right)}{n_j\left(s^t\right)} = \eta_j\left(s^t\right)$$

are the same for KP and DOCE resolute, naive and sophisticated choice and are independent of δ_1, γ *and* β.

Proof. Consider case (i) with $b = 0$. For both KP and DOCE, homogeneity of the utility function, together with Assumption [IR] implies that resolute choice or sophisticated choice must satisfy

$$c(s^t) = \alpha_{t-1}(s_t)c(s^{t-1}),$$

where $\alpha_{t-1}(s_t)$ are constants that depend on the shock s_t and the previous time period $t-1$. Note that

$$\begin{aligned}
& \mathcal{U}^{DOCE}\left(\mathbf{c}, \alpha | s^\tau\right) \\
= \; & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1} - \frac{\gamma\beta}{\delta_1}\left(\sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} - \\
& \frac{\gamma\beta^2}{\delta_1}\left(\begin{array}{c} \sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \\ \left(\sum_{s_{\tau+2}} \bar{\pi}_{\tau+2}(s_{\tau+2})\alpha_{\tau+1}(s_{\tau+2})^{-\delta_2}\right) \end{array}\right)^{\frac{\delta_1}{\delta_2}} - ... - \\
& \frac{\gamma\beta^{T-\tau}}{\delta_1}\left(\begin{array}{c} \sum_{s_{\tau+1}} \bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2} \\ \left(\ldots\left(\sum_{s_T} \bar{\pi}_T(s_T)\alpha_{T-1}(s_T)^{-\delta_2}\right)\right) \end{array}\right)^{\frac{\delta_1}{\delta_2}}
\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{U}^{KP}(\mathbf{c},\alpha|s^\tau)\\
=\ & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1}-\frac{\gamma\beta\left(\sum_{s^{\tau+1}\succ s^\tau}\pi(s^{\tau+1}|s^\tau)U(\mathbf{c}|s^{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}}{\delta_{1\tau}}\\
=\ & -\frac{c(s^\tau)^{-\delta_1}}{\delta_1}-\frac{\gamma\beta}{\delta_1}\left(\begin{array}{c}\left(\sum_{s_{\tau+1}}\bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\\ \left(1+\beta\left(\sum_{s_{\tau+2}}\bar{\pi}_{\tau+2}(s_{\tau+2})\alpha_{\tau+1}(s_{\tau+2})^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}(1+\ldots)\right)\end{array}\right).
\end{aligned}
$$

Therefore, the optimal asset ratios $n_f(s^\tau)/n_j(s^\tau)$ are determined by maximizing the EU

$$\sum_{s_{\tau+1}}\bar{\pi}_{\tau+1}(s_{\tau+1})c(s_{\tau+1})^{-\delta_2},$$

which is independent of time preference parameters δ_1, β and γ. Since Assumption [IR] holds, in period τ, each s^τ gives the same result and hence

$$\frac{n_f(s^\tau)}{n_j(s^\tau)}=\eta_j(s^\tau).$$

For other cases, the argument is the same as the proof of Theorem 2. ■

B.3 Proof of Proposition 4

As proved in Proposition 2, KP and DOCE preferences generate the same demands. Therefore, it is enough to consider DOCE sophisticated choice. Without loss of generality, consider the three period case with identical returns in each period. It can be verified that

$$c_1^{DOCE}=\frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}}\widehat{R}^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1}\widehat{R}^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}},$$

where

$$\widehat{R}=\left(E\widetilde{R}^{-\delta_2}\right)^{-\frac{1}{\delta_2}}.$$

For the certainty case, it can be verified that

$$c_1^{certain}=\frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}}R_f^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1}R_f^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}}.$$

Since

$$\theta=s_1^{DOCE}-s_1^{certain}=c_1^{certain}-c_1^{DOCE},$$

where $s_1 = I - c_1$, we have

$$\theta \gtreqless 0 \Leftrightarrow \left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1} \widehat{R}^{-\delta_1} \gtreqless \left(1+\beta^{\frac{1}{1+\delta_1}} R_f^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1} R_f^{-\delta_1}.$$

Since

$$\widehat{R} < E\widetilde{R} = R_f,$$

we have

$$\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1} \widehat{R}^{-\delta_1} \gtreqless \left(1+\beta^{\frac{1}{1+\delta_1}} R_f^{-\frac{\delta_1}{1+\delta_1}}\right)^{1+\delta_1} R_f^{-\delta_1} \Leftrightarrow \delta_1 \gtreqless 0,$$

implying that

$$\theta \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0.$$

This is consistent with the result in the consumption-saving setting for the two period case in Selden and Wei (2018). Next consider the four period case with identical returns in each period. In the consumption-saving setting, for the DOCE preferences, it can be verified that

$$c_1^{DOCE} = \frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}^{-\frac{\delta_1}{1+\delta_1}}\left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\delta_1+1}\right)^{\delta_1+1} \widehat{R}^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}}.$$

For the certainty case, it can be verified that

$$c_1^{certain} = \frac{I}{1+\left(\beta\left(1+\beta^{\frac{1}{1+\delta_1}} R_f^{-\frac{\delta_1}{1+\delta_1}}\left(1+\beta^{\frac{1}{1+\delta_1}} R_f^{-\frac{\delta_1}{1+\delta_1}}\right)^{\delta_1+1}\right)^{\delta_1+1} R_f^{-\delta_1}\right)^{\frac{1}{1+\delta_1}}}.$$

Therefore, we still have

$$\theta \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0.$$

Based on induction, the above result can be extended to the T period case given Assumption [IR].

B.4 Disentangling the Effects of Time and Risk on Saving

Consider the special case portrayed in Figure 7. In period 1, there is a risk free asset with return R_{f2}. In period 2 if the upper state is realized (with probability π_1), there exists a risk free asset with return R_{f31}. If the lower state is realized (with probability π_2), the risk free return is R_{f32}. Assume the CES time and CRRA risk preference utilities in (5). Employing these same utility building blocks, we derive optimal DOCE sophisticated and KP demands based on backward induction.

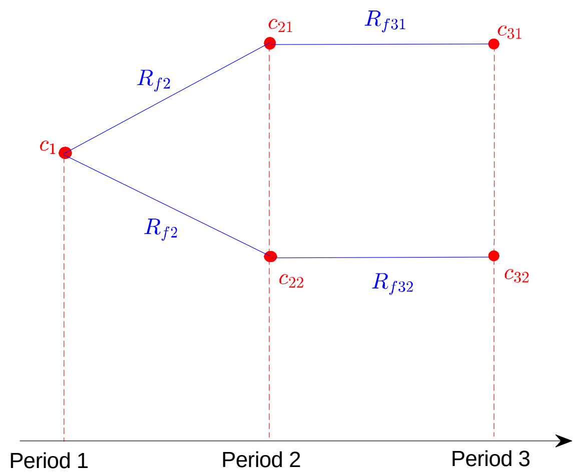


Figure 7:

Figure 7 facilitates a particularly clear comparison of the two sets of demands. It should be noted that for the consumption-saving setup in Figure 7, there is neither exogenous income nor capital risk. However, the KP and DOCE solution processes create risky consumption in periods 2 and 3 when viewed from the perspective of period 1. The saving in period 2 conditional on being on the up or down branch is certain. The income realized from period 1 saving is $(I - c_1)R_{f2}$, which is the same at both period 2 nodes. However, since $R_{f31} \neq R_{f32}$ optimal c_{21} and c_{22} will in general differ as will c_{31} and c_{32}.

First we investigate the special case of risk neutral risk preferences defined by $\delta_2 = -1$. We show that risk neutral DOCE resolute choice results in boundary solutions. We also derive closed form analytic expressions for optimal period 1 consumption for the cases of DOCE sophisticated and KP preferences and discuss the differences.

Clearly Assumption [IR] is violated for the case in Figure 7 since the risk free returns for the upper and lower branches differ. Given the assumed form of preferences, if Assumption [IR] holds and $R_{f31} = R_{f32} = R_{f3}$ it follows from Theorem 1 that the DOCE demands will be time consistent and resolute, naive and sophisticated choice will agree. It follows from Proposition 2 that the KP and DOCE demands will be the same. To provide intuition for the impact of relaxing

Assumption [IR], it is first useful to consider the following certainty problem

$$\max_{c_1,c_2,c_3} \left(c_1^{-\delta_1} + \beta c_2^{-\delta_1} + \beta^2 c_3^{-\delta_1}\right)^{-\frac{1}{\delta_1}} \quad S.T. \;\; I = c_1 + \frac{c_2}{R_{f2}} + \frac{c_3}{R_{f2}R_{f3}}. \tag{B.2}$$

From the first order conditions, optimal first period consumption is given by

$$c_1 = \frac{I}{1 + B_1^{\frac{1}{1+\delta_1}}},$$

where

$$B_1 = \beta R_{f2}^{-\delta_1} \left[\left(\frac{1}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{\beta^{\frac{1}{1+\delta_1}} R_{f3}^{\frac{1}{1+\delta_1}}}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \right] \tag{B.3}$$

or equivalently,

$$B_1 = \beta R_{f2}^{-\delta_1} \left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}} \right)^{1+\delta_1}. \tag{B.4}$$

To provide some intuition on how to interpret the two terms inside the bracket on the right hand side of eqn. (B.3), note first that based on the constraint in (B.2) one can introduce the following convention for defining prices for consumption in each period as follows:

$$p_1 = 1, \quad p_2 = \frac{1}{R_{f2}} \quad \text{and} \quad p_3 = \frac{1}{R_{f2}R_{f3}}.$$

It can be verified that

$$p_1 c_1 = \frac{I}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}},$$

$$p_2 c_2 = \frac{\beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} I}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}}$$

and

$$p_3 c_3 = \frac{\beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}} I}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}}.$$

Computing the relative marginal propensities to consume (MPCs), we obtain the following

$$\frac{\partial \left(p_2 c_2\right)/\partial I}{\partial \left(p_1 c_1\right)/\partial I} = \beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} \quad \text{and} \quad \frac{\partial \left(p_3 c_3\right)/\partial I}{\partial \left(p_1 c_1\right)/\partial I} = \beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}$$

and the expression (B.4) can be rewritten in terms of the relative MPCs as

$$B_1^{\frac{1}{1+\delta_1}} = \frac{\partial (p_2 c_2) / \partial I}{\partial (p_1 c_1) / \partial I} + \frac{\partial (p_3 c_3) / \partial I}{\partial (p_1 c_1) / \partial I}.$$

To derive the DOCE sophisticated demands when $R_{f31} \neq R_{f32}$, we follow the solution process in Definition 3. Assuming the upper state is realized, the sophisticated DOCE consumer solves the optimization problem

$$\max_{c_{21}} \left(c_{21}^{-\delta_1} + \beta \left(R_{f31} \left(R_{f2} \left(I - c_1 \right) - c_{21} \right) \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}}.$$

The first order condition is

$$c_{21}^{-\delta_1 - 1} = \frac{\beta R_{f31}}{\left(R_{f31} \left(R_{f2} \left(I - c_1 \right) - c_{21} \right) \right)^{1+\delta_1}}.$$

It follows that

$$c_{21}^{-\delta_1 - 1} = \frac{\beta \left(R_{f2} \left(I - c_1 \right) - c_{21} \right)^{-\delta_1 - 1}}{R_{f31}^{\delta_1}},$$

implying that

$$c_{21} = \frac{R_{f2} \left(I - c_1 \right)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}}.$$

For the lower branch, similarly, one can obtain

$$c_{22} = \frac{R_{f2} \left(I - c_1 \right)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}}.$$

Hence

$$c_{31} = \frac{R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f31}^{\frac{1}{1+\delta_1}} \left(I - c_1 \right)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} \quad \text{and} \quad c_{32} = \frac{R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}} \left(I - c_1 \right)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}}.$$

Therefore, the period 1 utility function becomes

$$\begin{aligned} & \left(c_1^{-\delta_1} + \beta \left(\pi_1 c_{21} + \pi_2 c_{22} \right)^{-\delta_1} + \beta^2 \left(\pi_1 c_{31} + \pi_2 c_{32} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \\ = & \left(\begin{array}{c} c_1^{-\delta_1} + \beta \left(\frac{\pi_1 R_{f2}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} + \frac{\pi_2 R_{f2}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \left(I - c_1 \right)^{-\delta_1} + \\ \beta^2 \left(\frac{\pi_1 R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f31}^{\frac{1}{1+\delta_1}}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} + \frac{\pi_2 R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \left(I - c_1 \right)^{-\delta_1} \end{array} \right)^{-\frac{1}{\delta_1}} \end{aligned}$$

and the first order condition is

$$0 = c_1^{-\delta_1 - 1} - \beta \left(\frac{\pi_1 R_{f2}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} + \frac{\pi_2 R_{f2}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} (I - c_1)^{-1-\delta_1} -$$
$$\beta^2 \left(\frac{\pi_1 R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f31}^{\frac{1}{1+\delta_1}}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} + \frac{\pi_2 R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}}}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} (I - c_1)^{-1-\delta_1}.$$

The optimal DOCE sophisticated solution is given by

$$c_1^{**} = \frac{I}{1 + B_1^{\frac{1}{1+\delta_1}}}, \tag{B.5}$$

where

$$B_1 = \beta R_{f2}^{-\delta_1} \left(\frac{\pi_1}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} + \frac{\pi_2}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} \right)^{\frac{\delta_1}{\delta_2}} +$$
$$\beta^2 R_{f2}^{-\delta_1} \left(\frac{\pi_1 \beta^{-\frac{\delta_2}{1+\delta_1}} R_{f31}^{-\frac{\delta_2}{1+\delta_1}}}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} + \frac{\pi_2 \beta^{-\frac{\delta_2}{1+\delta_1}} R_{f32}^{-\frac{\delta_2}{1+\delta_1}}}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} \right)^{\frac{\delta_1}{\delta_2}} . \tag{B.6}$$

It is clear that the risk confronting the consumer in period 1 is which pair of period 2 and 3 conditional demands will hold, which in turn depends on whether R_{f31} or R_{f32} is realized. This explains why terms such as $1 + \beta^{\frac{1}{1+\delta_1}} R_{f3s}^{-\frac{\delta_1}{1+\delta_1}}$ $(s = 1, 2)$ appear in the solution (B.5)-(B.6).

Define the certainty equivalent returns ${}_1\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}}$ and ${}_2\widehat{R}_{f3}^{\frac{1}{1+\delta_1}}$ as follows

$$\frac{1}{1 + {}_1\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}} = \left(\frac{\pi_1}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} + \frac{\pi_2}{\left(1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} \right)^{-\frac{1}{\delta_2}} \tag{B.7}$$

and

$$\frac{{}_2\widehat{R}_{f3}^{\frac{1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}}{1 + {}_2\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}} = \left(\pi_1 \left(\frac{\beta^{\frac{1}{1+\delta_1}} R_{f31}^{\frac{1}{1+\delta_1}}}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_2} + \pi_2 \left(\frac{\beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}}}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_2} \right)^{-\frac{1}{\delta_2}}.$$

Then B_1 in eqn. (B.6) can be rewritten as

$$B_1 = \beta R_{f2}^{-\delta_1} \left[\left(\frac{1}{1 + {}_1\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{{}_2\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}}{1 + {}_2\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \beta^{\frac{1}{1+\delta_1}}} \right)^{-\delta_1} \right], \quad \text{(B.8)}$$

which is exactly analogous to the certainty eqn. (B.3). Clearly, the certainty equivalent returns ${}_1\widehat{R}_{f3}$ and ${}_2\widehat{R}_{f3}$ depend not just on the risk preference parameter δ_2 but also on the time preference parameter δ_1. Thus although at the DOCE preference level time and risk preferences can be specified independently, at the sophisticated demand level when Assumption [IR] does not hold time and risk preferences are clearly intertwined in the sense that the certainty equivalents depend on δ_1 and β as well as δ_2.[32]

Next we solve for optimal period 1 consumption assuming KP preferences based on the same CES and CRRA utilities in (5). The period 2 optimization problems are the same as for the DOCE case. The period 1 utility is given by

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta \left(\pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}}, \quad \text{(B.9)}$$

where

$$U_{21} = \left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \quad \text{and} \quad U_{22} = \left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{-\frac{1}{\delta_1}}.$$

Since the conditional demands c_{21}, c_{22}, c_{31} and c_{32} are the same as those for DOCE preferences, we have

$$U_{21} = \left(\left(\frac{R_{f2}(I - c_1)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f31}^{\frac{1}{1+\delta_1}} (I - c_1)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}}$$

and

$$U_{22} = \left(\left(\frac{R_{f2}(I - c_1)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{R_{f2} \beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}} (I - c_1)}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}}.$$

[32] Since ${}_1\widehat{R}_{f3}$ and ${}_2\widehat{R}_{f3}$ are in general different, eqn. (B.8) cannot be simplified to a form analogous to eqn. (B.4).

Therefore, the period 1 utility function is

$$\left(c_1^{-\delta_1} + \beta \left(\begin{array}{c} \pi_1 \left(\left(\frac{R_{f2}(I-c_1)}{1+\beta^{\frac{1}{1+\delta_1}} R_{f21}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{R_{f2}\beta^{\frac{1}{1+\delta_1}} ER_{f31}^{\frac{1}{1+\delta_1}} (I-c_1)}{1+\beta^{\frac{1}{1+\delta_1}} ER_{f31}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \\ +\pi_2 \left(\left(\frac{R_{f2}(I-c_1)}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} + \beta \left(\frac{R_{f2}\beta^{\frac{1}{1+\delta_1}} R_{f32}^{\frac{1}{1+\delta_1}} (I-c_1)}{1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \end{array} \right)^{-\delta_1} \right)^{-\frac{1}{\delta_1}} .$$

The first order condition is

$$c_1^{-\delta_1-1} = \beta \left(\begin{array}{c} \pi_1 \left(\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}} \right)^{1+\delta_1} R_{f2}^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \\ +\pi_2 \left(\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}} \right)^{1+\delta_1} R_{f2}^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \end{array} \right)^{-\delta_1} (I-c_1)^{-1-\delta_1} .$$

Solving for optimal demands recursively yields

$$c_1 = \frac{I}{1+B_2^{\frac{1}{1+\delta_1}}},$$

where

$$B_2 = \beta R_{f2}^{-\delta_1} \left(\pi_1 \left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} + \pi_2 \left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \right)^{\frac{\delta_1}{\delta_2}} . \tag{B.10}$$

Comparing (B.10) with the corresponding expression for the certainty case (B.4), if we define the certainty equivalent $\widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}}$ based on the following

$$1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} = \left(\begin{array}{c} \pi_1 \left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \\ +\pi_2 \left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}} \right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \end{array} \right)^{\frac{\delta_1}{(1+\delta_1)\delta_2}} , \tag{B.11}$$

then B_2 in eqn. (B.10) can be rewritten as

$$B_2 = \beta R_{f2}^{-\delta_1} \left(1+\beta^{\frac{1}{1+\delta_1}} \widehat{R}_{f3}^{-\frac{\delta_1}{1+\delta_1}} \right)^{1+\delta_1} ,$$

which is exactly analogous to eqn. (B.4). It should be noted that in eqn. (B.11) δ_1 enters into the intertemporal risk terms in each state

$$1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}} \quad \text{and} \quad 1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}$$

as in the sophisticated DOCE expression (B.7). However, for the KP case, δ_1 also enters into the outer exponent $\frac{(1+\delta_1)\delta_2}{\delta_1}$. This is an important difference from the DOCE sophisticated demands.

Next we show how to apply Karush–Kuhn–Tucker (KKT) conditions to solve for the resolute choice in the risk neutral case.

Proposition B.3 *Consider the consumption-saving problem (12) - (15) with the tree structure in Figure 7 and* $\delta_2 = -1$

$$\mathcal{U}(c_1, c_{21}, c_{22}) = \left(\begin{array}{c} c_1^{-\delta_1} + \beta\left(\pi_1 c_{21} + \pi_2 c_{22}\right)^{-\delta_1} + \\ \beta^2 \left(\begin{array}{c} \pi_1 R_{f31}\left(R_{f2}\left(I - c_1\right) - c_{21}\right) \\ +\pi_2 R_{f32}\left(R_{f2}\left(I - c_1\right) - c_{22}\right) \end{array} \right)^{-\delta_1} \end{array} \right)^{-\frac{1}{\delta_1}}.$$

We have the following resolute choice:

(i) *if*

$$\pi_1 \geq \frac{1}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}},$$

then

$$c_1^\circ = \frac{\beta^{-\frac{2}{1+\delta_1}} \left(R_{f2} R_{f31}\right)^{\frac{\delta_1}{1+\delta_1}} I}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}} + \beta^{-\frac{2}{1+\delta_1}} \left(R_{f2} R_{f31}\right)^{\frac{\delta_1}{1+\delta_1}}};$$

(ii) *if*

$$\frac{1}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}} > \pi_1 > \frac{1}{1 + R_{f31} \beta^{-\frac{1}{1+\delta_1}} R_{f32}^{-\frac{1}{1+\delta_1}}},$$

then

$$c_1^\circ = \frac{\left(R_{f2}(\beta - 1)\left(\pi_2 R_{f2}\right)^{-1-\delta_1} + \beta R_{f2} R_{f31}\left(\pi_1 R_{f2} R_{f31}\right)^{-1-\delta_1}\right)^{-\frac{1}{1+\delta_1}} I}{1 + \left(R_{f2}(\beta - 1)\left(\pi_2 R_{f2}\right)^{-1-\delta_1} + \beta R_{f2} R_{f31}\left(\pi_1 R_{f2} R_{f31}\right)^{-1-\delta_1}\right)^{-\frac{1}{1+\delta_1}}};$$

(iii) *if*

$$\pi_1 \leq \frac{1}{1 + R_{f31} \beta^{-\frac{1}{1+\delta_1}} R_{f32}^{-\frac{1}{1+\delta_1}}},$$

then

$$c_1^\circ = \frac{\left(\left(\beta - 1\right) R_{f2} \pi_2 (R_{f2} - \gamma) + \beta R_{f2} R_{f31}\left(\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma\right)\right)^{-\frac{1}{1+\delta_1}} I}{1 + \left(\left(\beta - 1\right) R_{f2} \pi_2 (R_{f2} - \gamma) + \beta R_{f2} R_{f31}\left(\pi_1 R_{f2} R_{f31} + \pi_2 R_{f32} \gamma\right)\right)^{-\frac{1}{1+\delta_1}}}.$$

Proof. An agent solves

$$\max u(c_1) + \beta u\left(\sum_{s=1}^{2} \pi_s c_{2s}\right) + \beta^2 u\left(\sum_{s=1}^{2} \pi_s c_{3s}\right) \quad S.T.$$

$$c_1 = I - n_1, \quad c_{2s} = n_1 R_{f2} - n_{2s}, c_{3s} = n_{2s} R_{f3s}, s = 1, 2,$$

$$c_1 \geq 0, c_{2s} \geq 0, c_{3s} \geq 0, s = 1, 2.$$

Assuming that $u(.)$ satisfies an Inada condition we can drop the first constraint and obtain

$$n_1 R_{f2} - n_{2s} \geq 0, \quad n_{2s} \geq 0, s = 1, 2.$$

Denote by μ_{1s} $(s = 1, 2)$ the Lagrange multiplier associated with the inequality constraint $n_1 R_{f2} - n_{2s}$ and μ_{2s} $(s = 1, 2)$ the Lagrange multiplier associated with the inequality constraint n_{2s}. The KKT conditions are

$$-u'(c_1) + \beta R_{f2} u'\left(\sum_s \pi_s c_{2s}\right) + \sum_s R_{f2} \mu_{1s} = 0, \tag{B.12}$$

$$-u'\left(\sum_s \pi_s c_{2s}\right) + \beta R_{f3s} u'\left(\sum_s \pi_s c_{3s}\right) - \mu_{1s} + \mu_{2s} = 0 \quad (s = 1, 2), \tag{B.13}$$

$$\mu_{ts} \geq 0 \ \ (t = 1, 2, \ \ s = 1, 2),$$

and

$$\mu_{1s} (n_1 R_{f2} - n_{2s}) = 0 \quad \text{and } \mu_{2s} n_{2s} = 0 \ \ (s = 1, 2).$$

Without loss of generality, assume $R_{f31} > R_{f32}$. It is easy to see that the equations

$$-u'\left(\sum_s \pi_s c_{2s}\right) + \beta R_{f3s} u'\left(\sum_s \pi_s c_{3s}\right) - \mu_{1s} + \mu_{2s} = 0 \quad (s = 1, 2)$$

can only have a solution if $\mu_{22} > 0$ or $\mu_{11} > 0$. Complementary slackness implies that $c_{22} = 0$ or $c_{21} = 0$. As we will show, there are three cases.

1. Under the condition
$$\pi_1 \geq \frac{1}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}}, \tag{B.14}$$
we have $\mu_{22} > 0, \mu_{11} = 0$, i.e. $c_{22} = n_1 R_{f2}, c_{32} = 0$.

2. Under the condition
$$\frac{1}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}} > \pi_1 > \frac{1}{1 + R_{f31} \beta^{-\frac{1}{1+\delta_1}} R_{f32}^{-\frac{1}{1+\delta_1}}}, \tag{B.15}$$
we have $\mu_{22} > 0, \mu_{11} > 0$, and $c_{21} = n_1 R_{f2}, c_{31} = 0$, $c_{22} = 0$ and $c_{32} = n_1 R_{f2} R_{f32}$.

3. Under the condition

$$\pi_1 \leq \frac{1}{1 + R_{f31}\beta^{-\frac{1}{1+\delta_1}} R_{f32}^{-\frac{1}{1+\delta_1}}},$$

we have $\mu_{22} = 0, \mu_{11} > 0$, and $c_{22} = 0$, $c_{32} = n_1 R_{f2} R_{f32}$.

We will solve the three cases, one by one.

Case 1: First solve for a solution for $\mu_{22} > 0, \mu_{11} = 0$. We have $n_{22} = 0$ and from the two first order conditions that hold with equality we obtain two linear equations

$$I - n_1 = (\beta R_{f2})^{-\frac{1}{1+\delta_1}} \left(\pi_1 \left(n_1 R_{f2} - n_{21}\right) + \pi_2 n_1 R_{f2}\right)$$

and

$$n_1 R_{f2} - \pi_1 n_{21} = (R_{f31}\beta)^{-\frac{1}{1+\delta_1}} n_{21} \pi_1 R_{f31}.$$

This implies

$$n_{21} = \frac{n_1 R_{f2}}{\pi_1 \left(1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}\right)} \quad \text{and} \quad s_1 = \frac{\left(1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}}\right) I}{1 + \beta^{-\frac{1}{1+\delta_1}} R_{f31}^{\frac{\delta_1}{1+\delta_1}} + \beta^{-\frac{2}{1+\delta_1}} \left(R_{f2} R_{f31}\right)^{\frac{\delta_1}{1+\delta_1}}}.$$

Since we require $c_{21} \geq 0$, which is equivalent to $n_1 R_{f2} - n_{21} \geq 0$, condition (B.14) is obtained.

Case 2: In the next step we solve for $\mu_{22} > 0, \mu_{11} > 0$ – in this case we have

$$n_{21} = R_{f2} n_1 \quad \text{and} \quad n_{22} = 0.$$

This is the correct solution if inequality (B.14) does not hold and if

$$u'(\pi_2 R_{f2} n_1) > \beta R_{f32} u'(\pi_1 R_{f2} R_{f31} n_1)$$

or

$$\pi_2 < \pi_1 (\beta R_{f32})^{-\frac{1}{1+\delta_1}} \pi_1 R_{f31}.$$

Using $\pi_2 = 1 - \pi_1$ this gives eqn. (B.15). In this case we obtain from the KKT condition (B.13) at (B.15) that

$$\mu_{11} = -u'(\pi_2 n_1 R_{f2}) + \beta R_{f31} u'(\pi_1 R_{f2} R_{f31} n_1).$$

Plugging this into the KKT condition (B.12) we obtain the following non-linear equation:

$$-u'(I - n_1) + \beta R_{f2} u'(\pi_2 n_1 R_{f2}) + R_{f2} \left(-u'(\pi_2 n_1 R_{f2}) + \beta R_{f31} u'(\pi_1 R_{f2} R_{f31} n_1)\right) = 0.$$

This is equivalent to

$$(I-n_1)^{-\delta_1-1}=R_{f2}(\beta-1)n_1^{-1-\delta_1}\left(\pi_2R_{f2}\right)^{-1-\delta_1}+n_1^{-1-\delta_1}\beta R_{f2}R_{f31}\left(\pi_1R_{f2}R_{f31}\right)^{-1-\delta_1},$$

which implies

$$I-n_1=n_1\left(R_{f2}(\beta-1)\left(\pi_2R_{f2}\right)^{-1-\delta_1}+\beta R_{f2}R_{f31}\left(\pi_1R_{f2}R_{f31}\right)^{-1-\delta_1}\right)^{-\frac{1}{1+\delta_1}}$$

and

$$n_1=\frac{I}{1+\left(R_{f2}(\beta-1)\left(\pi_2R_{f2}\right)^{-1-\delta_1}+\beta R_{f2}R_{f31}\left(\pi_1R_{f2}R_{f31}\right)^{-1-\delta_1}\right)^{-\frac{1}{1+\delta_1}}}.$$

Case 3: Finally, the third and last case is $\mu_{11}>0$ and $\mu_{22}=0$. The optimal n_{22} is determined by

$$u'(\pi_2(n_1R_{f2}-n_{22}))=\beta R_{f32}u'(\pi_1n_1R_{f2}R_{f31}+\pi_2n_{22}R_{f32}),$$

or substituting for u' yields

$$\pi_2(n_1R_{f2}-n_{22})=(\beta R_{f32})^{-\frac{1}{1+\delta_1}}(\pi_1n_1R_{f2}R_{f31}+\pi_2n_{22}R_{f32}),$$

which implies

$$n_{22}=\gamma n_1,\quad \gamma=\frac{\pi_2R_{f2}-(\beta R_{f32})^{-\frac{1}{1+\delta_1}}\pi_1R_{f2}R_{f31}}{\pi_2+(\beta R_{f32})^{-\frac{1}{1+\delta_1}}\pi_2R_{f32}}.$$

Similarly as in Case 2 the optimal choice is determined by

$$\begin{aligned}0 \;=\; & -u'(I-n_1)+\beta R_{f2}u'(\pi_2(n_1R_{f2}-n_{22}))\\ & +R_{f2}(-u'(\pi_2(n_1R_{f2}-n_{22}))\\ & +\beta R_{f31}u'(\pi_1R_{f2}R_{f31}n_1+\pi_2R_{f32}n_{22})).\end{aligned}$$

Plugging in n_{22} we obtain

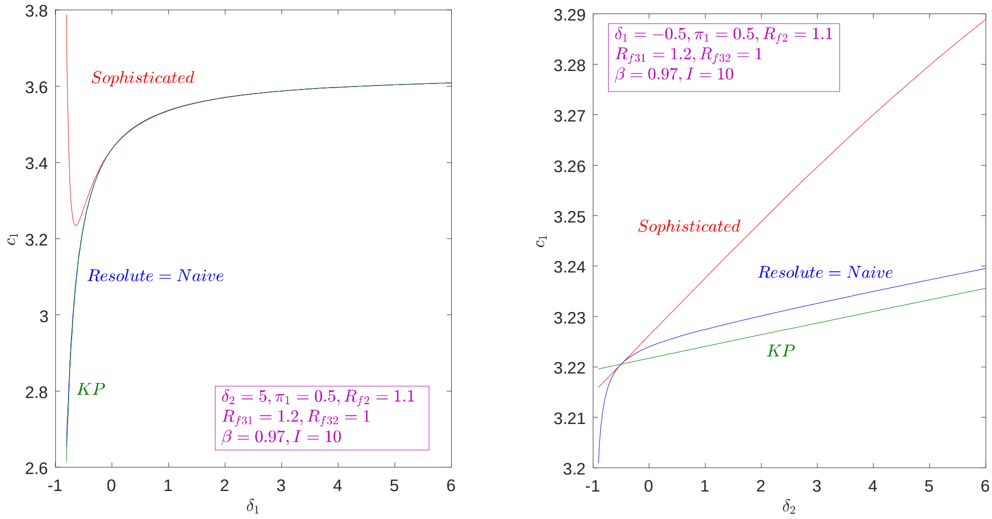
$$I-n_1=n_1\left((\beta-1)\,R_{f2}\pi_2(R_{f2}-\gamma)+\beta R_{f2}R_{f31}\left(\pi_1R_{f2}R_{f31}+\pi_2R_{f32}\gamma\right)\right)^{-\frac{1}{1+\delta_1}}$$

and hence

$$n_1=\frac{I}{1+\left((\beta-1)\,R_{f2}\pi_2(R_{f2}-\gamma)+\beta R_{f2}R_{f31}\left(\pi_1R_{f2}R_{f31}+\pi_2R_{f32}\gamma\right)\right)^{-\frac{1}{1+\delta_1}}}.$$

■

We next illustrate important differences in optimal period 1 consumption particularly for the sophisticated DOCE and KP cases when Assumption [IR] does not hold and provide intuition for why c_1^{KP} can be significantly lower than c_1^{**}.



(a) (b)

Figure 8:

Example B.1 *Assume the consumption-saving setting associated with Figure 7 and that the time and risk preference utilities take the form in (5). Based on the following parameter values*

$$R_{f2} = 1.1, R_{f31} = 1.2, R_{f32} = 1, \pi_1 = 0.5, \beta = 0.97, I = 10,$$

we performed numerical simulations of optimal c_1 as functions of δ_1 and δ_2 which are summarized respectively in Figures 8(a) and (b). Based on the definitions of resolute and naive choice (Definitions 1 and 2), $c_1^* = c_1^\circ$ always holds. In both Figures 8(a) and (b), c_1°, c_1^{**} and c_1^{KP} are generally close in value as δ_1 and δ_2 are varied (with respectively $\delta_2 = 5$ and $\delta_1 = -0.5$ being fixed).[33] *The three curves intersect at the EU special cases in Figures 8(a) and (b) where respectively $\delta_1 = \delta_2 = 5$ and $\delta_1 = \delta_2 = -0.5$.*[34] *In addition to c_1^{**} and c_1^{KP} differing in their monotonicity with respect to δ_1 in Figure 8(a), they also diverge significantly in value as δ_1 goes to -1. To provide intuition for why these differences arise, first consider sophisticated choice. Following the recursive solution process, the consumer faces a two period certainty consumption-saving problem. As $\delta_1 \to -1$, on the upper branch in Figure 7 where $R_{f31} > 1$, the consumer substitutes c_{31} for c_{21} and saves all of her period 2 income $(I - c_1)R_{f2}$ resulting in c_{21} going to zero. Similarly, on the lower branch since $R_{f32} = 1$ and $\beta < 1$, the consumer*

[33] It should be noted that the scale of the vertical axes in Figures 8(a) and (b) are different.
[34] The three curves also intersect at $\delta_1 = 0$ in Figure 8(a).

consumes all of her income in period 2 resulting in c_{32} going to zero. Assuming $\delta_2 \geq 0$ as well as $\delta_1 \to -1$, $\widehat{c}_2$, $\widehat{c}_3 \to 0$ and the highly substitute oriented consumer maximizes her three period certainty utility by setting $c_1^{**} = I$. For the KP case, note that the period 1 utility is given by

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta \left(\pi_1 U_{21}^{-\delta_2} + \pi_2 U_{22}^{-\delta_2} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}},$$

where

$$U_{21} = \left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1} \right)^{-\frac{1}{\delta_1}} \quad \text{and} \quad U_{22} = \left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1} \right)^{-\frac{1}{\delta_1}}.$$

Applying the same argument for KP utility as was done for sophisticated choice, we have $c_{21} = 0$ and $c_{32} = 0$ when $\delta_1 \to -1$. Substituting into the above formula for the KP utility yields

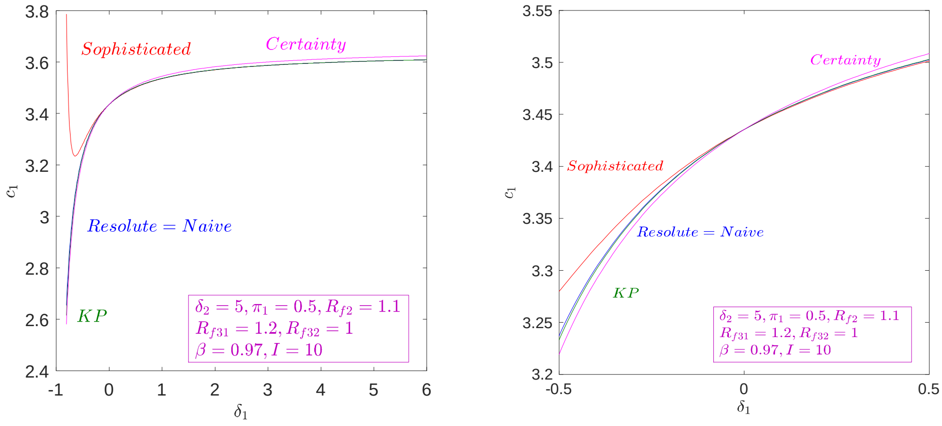
$$\begin{aligned}
& \left(\pi_1 \left(c_{21} + \beta c_{31} \right)^{-\delta_2} + \pi_2 \left(c_{22} + \beta c_{32} \right)^{-\delta_2} \right)^{-\frac{1}{\delta_2}} \\
= & \left(\pi_1 \left(\beta c_{31} \right)^{-\delta_2} + \pi_2 c_{22}^{-\delta_2} \right)^{-\frac{1}{\delta_2}} \\
= & \left(\pi_1 \left(\beta R_{f2} \left(I - c_1 \right) \right)^{-\delta_2} + \pi_2 \left(R_{f2} \left(I - c_1 \right) \right)^{-\delta_2} \right)^{-\frac{1}{\delta_2}} \\
= & \left(\pi_1 \beta^{-\delta_2} + \pi_2 \right)^{-\frac{1}{\delta_2}} R_{f2} \left(I - c_1 \right).
\end{aligned}$$

Since $\left(\pi_1 \beta^{-\delta_2} + \pi_2 \right)^{-\frac{1}{\delta_2}} R_{f2} > 1$, it follows from maximizing the KP period one utility function that $c_1 \to 0$ when $\delta_1 \to -1$. For resolute choice, where unlike the KP case one does not use backward induction, as $\delta_1 \to -1$ the boundary solution $c_1 \to 0$ is realized.

The question of how excess saving θ differs for the DOCE resolute and sophisticated and KP models is of interest since, as shown in Example B.1, they can exhibit very similar behavior in terms of optimal first period consumption. They also share the same time preference utility and hence the same $c_1^{certain}$. The following shows that the conclusions of Proposition 4 relating to the sign of excess saving can still hold when Assumption [IR] is violated.

Example B.2 *Assume the same setting as in Example B.1. To define an appropriate $c_1^{certain}$ for comparison purposes, assume that the certainty period 3 risk free return $R_{f3} = \pi_1 R_{f31} + \pi_2 R_{f32} = R_{f2}$. For the certainty case, it can be easily verified that*

$$c_1 = \frac{I}{1 + \beta^{\frac{1}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} + \beta^{\frac{2}{1+\delta_1}} R_{f2}^{-\frac{\delta_1}{1+\delta_1}} R_{f3}^{-\frac{\delta_1}{1+\delta_1}}}.$$



(a) (b)

Figure 9:

For DOCE resolute choice, the resulting first order conditions are

$$
\begin{aligned}
c_1^{-\delta_1-1} &= \beta^2 \begin{pmatrix} \pi_1 \left(R_{f31} \left(R_{f2} \left(I - c_1\right) - c_{21}\right)\right)^{-\delta_2} \\ +\pi_2 \left(R_{f32} \left(R_{f2} \left(I - c_1\right) - c_{22}\right)\right)^{-\delta_2} \end{pmatrix}^{\frac{\delta_1}{\delta_2}-1} \times \\
& \begin{pmatrix} \pi_1 R_{f31} R_{f2} \left(R_{f31} \left(R_{f2} \left(I - c_1\right) - c_{21}\right)\right)^{-\delta_2-1} \\ +\pi_2 R_{f32} R_{f2} \left(R_{f32} \left(R_{f2} \left(I - c_1\right) - c_{22}\right)\right)^{-\delta_2-1} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
& \left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}-1} c_{21}^{-\delta_2-1} \\
= \; & \beta R_{f31} \begin{pmatrix} \pi_1 \left(R_{f31} \left(R_{f2} \left(I - c_1\right) - c_{21}\right)\right)^{-\delta_2} \\ +\pi_2 \left(R_{f32} \left(R_{f2} \left(I - c_1\right) - c_{22}\right)\right)^{-\delta_2} \end{pmatrix}^{\frac{\delta_1}{\delta_2}-1} \\
& \times \left(R_{f31} \left(R_{f2} \left(I - c_1\right) - c_{21}\right)\right)^{-\delta_2-1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}-1} c_{22}^{-\delta_2-1} \\
= \; & \beta R_{f32} \begin{pmatrix} \pi_1 \left(R_{f31} \left(R_{f2} \left(I - c_1\right) - c_{21}\right)\right)^{-\delta_2} \\ +\pi_2 \left(R_{f32} \left(R_{f2} \left(I - c_1\right) - c_{22}\right)\right)^{-\delta_2} \end{pmatrix}^{\frac{\delta_1}{\delta_2}-1} \\
& \times \left(R_{f32} \left(R_{f2} \left(I - c_1\right) - c_{22}\right)\right)^{-\delta_2-1}.
\end{aligned}
$$

Figure 9(a) shows that the value of the certainty c_1 is surprisingly close to the corresponding values for the DOCE sophisticated, resolute (=naive) and KP cases

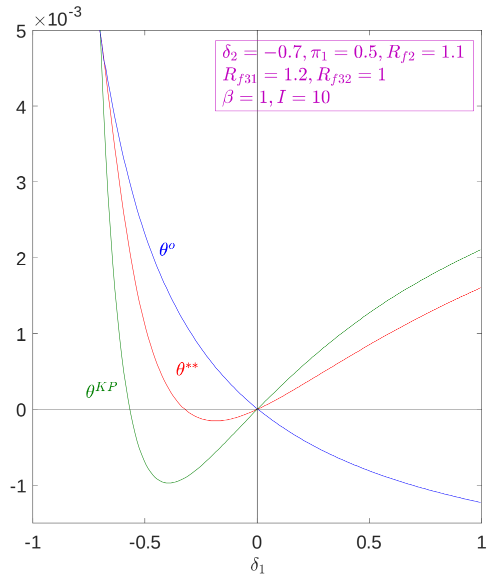


Figure 10:

as long as δ_1 *is not too close to* -1.[35] *Figure 9(b) confirms that* $c_1^{certain}$ *is larger (smaller) than resolute, sophisticated and KP period* 1 *consumption if* $\delta_1 > (<)0$. *This in turn implies that* $\theta > (<)0$ *if* $\delta_1 > (<)0$ *for the DOCE and KP models. However unlike Proposition 4, in the current case the conclusion on the sign of* θ *can depend on the value of* δ_2. *When* $\delta_2 = 5$ *all of the curves in Figure 9(b) intersect at* $\delta_1 = 0$ *and using (6), we have*

$$\theta \gtreqless 0 \Leftrightarrow \delta_1 \gtreqless 0 \Leftrightarrow EIS \lesseqgtr 1$$

for the DOCE and KP models. However, as in Figure 10, when δ_2 *takes on low values such as* -0.7 *positive excess saving can occur for the DOCE and KP models when* $\delta_1 < 0$.

B.5 Supporting Calculations for Subsection ??

For the positive correlation tree in Figure **??**(b), we need the residue income

$$I_{21} = 30 \text{ and } I_{22} = 10.$$

Assuming

$$R_{21} = 1.5, R_{22} = 0.5 \text{ and } R_f = 1, \tag{B.16}$$

[35]Not surprisingly, the distance between the certainty c_1 curve and the other curves in Figure 9(a) increases if we consider more dispersed distributions of risk free returns such as $R_{f31} = 1.5$ and $R_{f32} = 0.7$.

then these residue incomes can be realized by holding $n = 20$ and $n_f = 0$. Note that for the returns in eqn. (B.16),[36]

$$E\widetilde{R} > R_f \text{ iff } \pi_1 > 0.5.$$

For the negative correlation tree Figure **??**(a), we have

$$I_{21} = I_{22} = 20,$$

which can be realized by holding $n = 0$ and $n_f = 20$. Therefore, the asset holdings for both trees in Figure **??** can be realized with the the budget constraint

$$n + n_f = 20.$$

Then if $\beta = 1$ and $R_{f31} = R_{f32} = 1$, we have the optimal demands

$$c_{21} = 15, c_{31} = 15, c_{22} = 5 \text{ and } c_{32} = 5$$

when $n = 20$ and $n_f = 0$. Moreover, when $n = 0$ and $n_f = 20$,

$$c_{21} = 5, c_{31} = 15, c_{22} = 15, c_{32} = 5$$

are also feasible. Thus the trees in Figure **??** can be replicated by optimization.

B.6 Supporting Materials for Section 5.2

B.6.1 Optimal Asset Ratio: DOCE Resolute Case

To solve for the asset ratio for the DOCE resolute case, first note that from the constraints

$$c_{31} = R_{f31}\left(R_{21}n_1 + R_{f2}n_{f1} - c_{21}\right) \quad \text{and} \quad c_{32} = R_{f32}\left(R_{22}n_1 + R_{f2}n_{f1} - c_{22}\right).$$

It follows that

$$n_1 = \frac{\frac{c_{31}}{R_{f31}} + c_{21} - \frac{c_{32}}{R_{f32}} - c_{22}}{R_{21} - R_{22}} \quad \text{and} \quad n_{f1} = \frac{R_{21}\left(\frac{c_{32}}{R_{f32}} + c_{22}\right) - R_{22}\left(\frac{c_{31}}{R_{f31}} + c_{21}\right)}{\left(R_{21} - R_{22}\right)R_{f2}}.$$

[36] Strictly speaking, we cannot exactly replicate the trees in Figure **??** for $\pi_1 = 0.5$ with optimization but can replicate them for any $\pi_1 > 0.5$.

Therefore, the period 1 budget constraint is

$$
\begin{aligned}
I &= c_1 + n_1 + n_{f1} \\
&= c_1 + \frac{\frac{c_{31}}{R_{f31}} + c_{21} - \frac{c_{32}}{R_{f32}} - c_{22}}{R_{21} - R_{22}} + \\
&\quad \frac{R_{21}\left(\frac{c_{32}}{R_{f32}} + c_{22}\right) - R_{22}\left(\frac{c_{31}}{R_{f31}} + c_{21}\right)}{\left(R_{21} - R_{22}\right) R_{f2}} \\
&= c_1 + \frac{R_{f2} - R_{22}}{\left(R_{21} - R_{22}\right) R_{f2}} c_{21} + \frac{R_{21} - R_{f2}}{\left(R_{21} - R_{22}\right) R_{f2}} c_{22} \\
&\quad + \frac{R_{f2} - R_{22}}{\left(R_{21} - R_{22}\right) R_{f31} R_{f2}} c_{31} + \frac{R_{21} - R_{f2}}{\left(R_{21} - R_{22}\right) R_{f32} R_{f2}} c_{32}.
\end{aligned}
$$

Thus the first order conditions for DOCE resolute choice are

$$
\frac{\pi_1 c_{21}^{-1-\delta_2}}{\pi_2 c_{22}^{-1-\delta_2}} = \frac{R_{f2} - R_{22}}{R_{21} - R_{f2}} \Leftrightarrow c_{22} = \left(\frac{\pi_2\left(R_{f2} - R_{22}\right)}{\pi_1\left(R_{21} - R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}} c_{21}
$$

and

$$
\frac{\pi_1 c_{31}^{-1-\delta_2}}{\pi_2 c_{32}^{-1-\delta_2}} = \frac{\left(R_{f2} - R_{22}\right) R_{f32}}{\left(R_{21} - R_{f2}\right) R_{f31}} \Leftrightarrow c_{32} = \left(\frac{\pi_2\left(R_{f2} - R_{22}\right) R_{f32}}{\pi_1\left(R_{21} - R_{f2}\right) R_{f31}}\right)^{\frac{1}{1+\delta_2}} c_{31}.
$$

Therefore, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch (c_1, c_{21}, c_{31}), which is

$$
\left(\begin{array}{c}
c_1^{-\delta_1} + \beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1} \\
+\beta^2\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1}
\end{array}\right)^{-\frac{1}{\delta_1}} \tag{B.17}
$$

and the budget constraint can be rewritten as

$$
\begin{aligned}
I &= c_1 + \left(\frac{R_{f2} - R_{22}}{\left(R_{21} - R_{22}\right) R_{f2}} + \frac{R_{21} - R_{f2}}{\left(R_{21} - R_{22}\right) R_{f2}}\left(\frac{\pi_2\left(R_{f2} - R_{22}\right)}{\pi_1\left(R_{21} - R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}\right) c_{21} + \\
&\quad \left(\begin{array}{c}
\frac{R_{f2}-R_{22}}{\left(R_{21}-R_{22}\right)R_{f31}R_{f2}} + \\
\frac{R_{21}-R_{f2}}{\left(R_{21}-R_{22}\right)R_{f32}R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}
\end{array}\right) c_{31}.
\end{aligned} \tag{B.18}
$$

The first order condition is

$$\frac{\beta\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}c_{21}^{-1-\delta_1}}{\beta^2\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}c_{31}^{-1-\delta_1}}$$

$$= \frac{\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f2}}+\frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}}{\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f31}R_{f2}}+\frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f32}R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}},$$

or equivalently,

$$c_{31}=\kappa c_{21},$$

where

$$\kappa=\left(\frac{\beta\left(\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f2}}+\frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}\right)\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}}{\left(\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f31}R_{f2}}+\frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f32}R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}\right)\left(\pi_1+\pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}}}\right)^{\frac{1}{1+\delta_1}}. \tag{B.19}$$

Therefore, we have

$$\begin{aligned}\frac{n_{f1}^{\circ}}{n_1^{\circ}} &= \frac{R_{21}\left(\frac{c_{32}}{R_{f32}}+c_{22}\right)-R_{22}\left(\frac{c_{31}}{R_{f31}}+c_{21}\right)}{R_f\left(\frac{c_{31}}{R_{f31}}+c_{21}-\frac{c_{32}}{R_{f32}}-c_{22}\right)} \\ &= \frac{\left(\frac{\kappa R_{21}}{R_{f32}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}+R_{21}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}-\left(\frac{\kappa R_{22}}{R_{f31}}+R_{22}\right)\right)}{R_{f2}\left(\frac{\kappa}{R_{f31}}+1-\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)R_{f32}}{\pi_1\left(R_{21}-R_{f2}\right)R_{f31}}\right)^{\frac{1}{1+\delta_2}}\frac{\kappa}{R_{f32}}-\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}\right)}.\end{aligned}$$

B.6.2 Supporting Calculations and Intuition for Example 2

For DOCE sophisticated choice, the period 1 utility function is

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta\left(\pi_1 c_{21}^{-\delta_2} + \pi_2 c_{22}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}} + \beta^2\left(\pi_1 c_{31}^{-\delta_2} + \pi_2 c_{32}^{-\delta_2}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}$$

$$= \left(\begin{array}{c} c_1^{-\delta_1} + \beta\left(\pi_1 \frac{\left(R_{21}n_1 + R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} + \pi_2 \frac{\left(R_{22}n_1 + R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} + \\ \beta^2 \left(\begin{array}{c} \pi_1 \frac{\beta^{-\frac{\delta_2}{1+\delta_1}} R_{f31}^{-\frac{\delta_2}{1+\delta_1}} \left(R_{21}n_1 + R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} + \\ \pi_2 \frac{\beta^{-\frac{\delta_2}{1+\delta_1}} R_{f32}^{-\frac{\delta_2}{1+\delta_1}} \left(R_{22}n_1 + R_{f2}n_{f1}\right)^{-\delta_2}}{\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{-\delta_2}} \end{array} \right)^{\frac{\delta_1}{\delta_2}} \end{array} \right)^{-\frac{1}{\delta_1}} \tag{B.20}$$

The DOCE sophisticated case simulations in Example 2 can follow the first order conditions based on the above equation. For KP preferences, the period 1 utility function is

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta\left(\pi_1\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} + \pi_2\left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}$$

$$= \left(c_1^{-\delta_1} + \beta \left(\begin{array}{c} \pi_1\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{21}n_1 + R_{f2}n_{f1}\right)^{-\delta_2} + \\ \pi_2\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{22}n_1 + R_{f2}n_{f1}\right)^{-\delta_2} \end{array} \right)^{\frac{\delta_1}{\delta_2}} \right)^{-\frac{1}{\delta_1}}.$$

Defining

$$k_2 = \frac{\pi_2\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{f2} - R_{22}\right)}{\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}} R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} \left(R_{21} - R_{f2}\right)},$$

we have the following conditional demands

$$n_{f1} = \frac{\left(R_{21}k_2^{\frac{1}{1+\delta_2}} - R_{22}\right)\left(I - c_1\right)}{R_{f2} - R_{22} + k_2^{\frac{1}{1+\delta_2}}\left(R_{21} - R_{f2}\right)} \quad \text{and} \quad n_1 = \frac{\left(1 - k_2^{\frac{1}{1+\delta_2}}\right) R_{f2}\left(I - c_1\right)}{R_{f2} - R_{22} + k_2^{\frac{1}{1+\delta_2}}\left(R_{21} - R_{f2}\right)}.$$

The period 1 utility function can be rewritten as

$$\mathcal{U}(\mathbf{c}) = \left(c_1^{-\delta_1} + \beta\left(\pi_1\left(c_{21}^{-\delta_1} + \beta c_{31}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}} + \pi_2\left(c_{22}^{-\delta_1} + \beta c_{32}^{-\delta_1}\right)^{\frac{\delta_2}{\delta_1}}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}$$

$$= \left(c_1^{-\delta_1} + \beta\left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(\frac{(R_{21}-R_{22})R_{f2}(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2} + \\ \pi_2\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\left(\frac{(R_{21}-R_{22})R_{f2}k_2^{\frac{1}{1+\delta_2}}(I-c_1)}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2}\end{array}\right)^{\frac{\delta_1}{\delta_2}}\right)^{-\frac{1}{\delta_1}}.$$

The first order condition is

$$c_1^{-\delta_1-1} = \beta\left(\begin{array}{c}\left(\frac{(R_{21}-R_{22})R_{f2}}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2} \\ \left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} + \\ \pi_2 k_2^{-\frac{\delta_2}{1+\delta_2}}\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\end{array}\right)\end{array}\right)^{\frac{\delta_1}{\delta_2}}(I-c_1)^{-1-\delta_1},$$

implying that

$$c_1^{KP} = \frac{I}{1+\beta^{\frac{1}{1+\delta_1}}\left(\begin{array}{c}\left(\frac{(R_{21}-R_{22})R_{f2}}{R_{f2}-R_{22}+k_2^{\frac{1}{1+\delta_2}}\left(R_{21}-R_{f2}\right)}\right)^{-\delta_2} \\ \left(\begin{array}{c}\pi_1\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f31}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}} + \\ \pi_2 k_2^{-\frac{\delta_2}{1+\delta_2}}\left(1+\beta^{\frac{1}{1+\delta_1}}R_{f32}^{-\frac{\delta_1}{1+\delta_1}}\right)^{\frac{(1+\delta_1)\delta_2}{\delta_1}}\end{array}\right)\end{array}\right)^{\frac{\delta_1}{(1+\delta_1)\delta_2}}}.$$

For resolute choice, the period 1 DOCE utility function can be transformed into a certainty utility of the single branch (c_1, c_{21}, c_{31}) as in eqn. (B.17) and the budget constraint can be rewritten as eqn. (B.18). The first order conditions are

$$c_1^{-1-\delta_1} = \frac{\beta\left(\pi_1 + \pi_2\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{-\frac{\delta_2}{1+\delta_2}}\right)^{\frac{\delta_1}{\delta_2}} c_{21}^{-\delta_1-1}}{\frac{R_{f2}-R_{22}}{(R_{21}-R_{22})R_{f2}} + \frac{R_{21}-R_{f2}}{(R_{21}-R_{22})R_{f2}}\left(\frac{\pi_2\left(R_{f2}-R_{22}\right)}{\pi_1\left(R_{21}-R_{f2}\right)}\right)^{\frac{1}{1+\delta_2}}}$$

and

$$c_1^{-1-\delta_1} = \frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}} c_{31}^{-\delta_1 - 1}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}}.$$

Therefore

$$c_{21} = \left(\frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} c_1$$

and

$$c_{31} = \left(\frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} c_1.$$

It follows that

$$c_1^\circ = \frac{I}{\left(\begin{array}{c} 1 + \left(\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}} \right) \\ \times \left(\frac{\beta \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right)}{\pi_1 \left(R_{21} - R_{f2} \right)} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} + \\ \left(\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}} \right) \\ \times \left(\frac{\beta^2 \left(\pi_1 + \pi_2 \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{-\frac{\delta_2}{1+\delta_2}} \right)^{\frac{\delta_1}{\delta_2}}}{\frac{R_{f2} - R_{22}}{(R_{21} - R_{22}) R_{f31} R_{f2}} + \frac{R_{21} - R_{f2}}{(R_{21} - R_{22}) R_{f32} R_{f2}} \left(\frac{\pi_2 \left(R_{f2} - R_{22} \right) R_{f32}}{\pi_1 \left(R_{21} - R_{f2} \right) R_{f31}} \right)^{\frac{1}{1+\delta_2}}} \right)^{\frac{1}{1+\delta_1}} \end{array} \right)}.$$

The considerable variation in the n_{f1}/n_1 ratio in Figure 6(b) suggests a significant difference in risk attitudes. As argued in Subsection 5.2, there are two

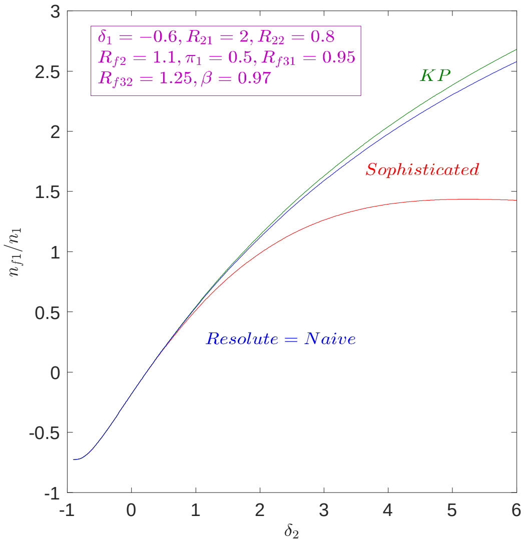


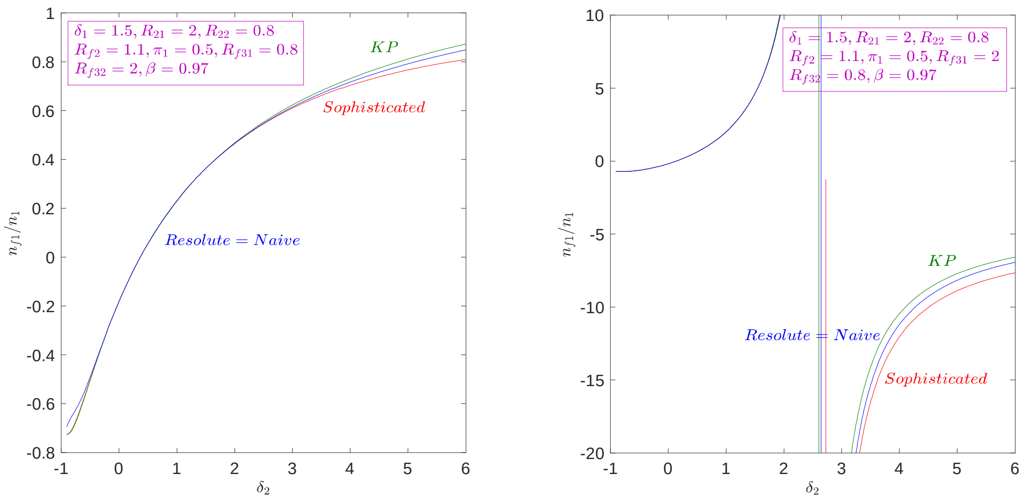
Figure 11:

dimensions of risk – the intraperiod portfolio risk in period 2 and the interperiod risk corresponding to the correlation pattern of the period 2 risky and period 3 risk free asset returns. The latter phenomenon can very clearly be observed if we switch the pattern of returns for R_{f31} and R_{f32} in Figure 6(b) to that shown in Figure 11. In the latter case, the period 2 risk can be viewed as being partially hedged by the period 3 risk as the intertemporal correlation has gone from positive to negative. As a result, continuing to assume that $\delta_1 = -0.6$ and $\delta_2 = 5$, the n_{f1}/n_1 ratio for all four models drops substantially from the case in Figure 6(b), where the asset return intertemporal correlation is positive. Moreover, it is not surprising that the asset ratio for each of the models is the same when $R_{f31} = R_{f32}$ since there is no intertemporal risk. Also, the common ratio is intermediate between the positive correlation case of Figure 6(b) and the negative correlation case of Figure 11.

The behavioral response to interperiod risk in the form of explicit intertemporal hedging can take a different form if we consider the more extreme negative and positive correlation cases, respectively,

$$(R_{f31}, R_{f32}) = (0.8, 2.0) \quad \text{and} \quad (R_{f31}, R_{f32}) = (2.0, 0.8).$$

See Figure 12(a) and (b) where $\delta_1 = 1.5$ and $\delta_2 = 5$, and the corresponding Table 1. Note that in a two period setting, if $E\widetilde{R} > R_f$ a risk averse investor will always hold the risky asset in positive quantity. However, that is not the case in Figure



(a) (b)

Figure 12:

12(b). The KP and DOCE resolute asset ratios at first increase with δ_2 as the consumer increases n_{f1} and reduces n_1.[37] As the risky asset demand approaches 0, the graphs become discontinuous. As δ_2 increases past 2.8, the consumer begins to short the risky asset (eqn. (30) provides the necessary and sufficient condition for $n_1^{KP} < 0$). One can view the reduction in the positive quantity of n_1 as an attempt to decrease the period 2 portfolio intraperiod risk whereas the shift to shorting the risky asset can be thought of a move to decrease the intertemporal risk via dynamic hedging. To see this more explicitly, refer to the row corresponding to n_1 and column "$R_{f31} = 2$ and $R_{f32} = 0.8$" in Table 1. For both the KP and sophisticated DOCE consumer, $n_1 < 0$. The motivation for this short-selling of the risky asset is to bolster the lower income $I_{22} = n_1 R_{22} + n_{f1} R_{f2}$ in the down state and thereby realize a better (c_{22}, c_{32})-pair. By shorting n and increasing the purchase of n_{f1}, the consumer reduces her income I_{21} because $R_{21} > R_{f2}$ and increases I_{22} since $R_{f2} > R_{22}$. Therefore, the period 2 income is higher when the bad state is realized and the consumer reduces her "bad, bad" outcome of a low (R_{22}, R_{f32}) outcome by sacrificing gains in the "good, good" case (R_{21}, R_{f31}) associated with I_{21}. It should be emphasized that this phenomenon only occurs

[37] In Figure 12(b), we do not show the sophisticated choice curve for $\delta_2 < 2$ due to instability in the numerical simulation. Also for $\delta_2 < 2$, the curve increasing with δ_2 corresponds to both the DOCE resolute and KP cases as the n_{f1}/n_1 ratios are too close to distinguish.

in the positive correlation case where $(R_{f31}, R_{f32}) = (2.0, 0.8)$.[38] For the negative correlation case, since the risk is already partially hedged by the asset returns automatically, the consumer will hold more risky assets to increase her expected returns. This is consistent with both the KP and DOCE sophisticated values of n_1 being positive and larger than the risk free asset holdings in the negative correlation column "$R_{f31} = 0.8$ and $R_{f32} = 2.0$" in Table 1.

To discuss the asset ratio behavior analytically, assume $\delta_1 > -1$ and $\delta_2 \geq 0$. Define[39]

$$\hat{c}_3^* = \max_{n_1+n_{f1}=I} V^{-1}\left(\pi_1 V((n_1R_{21} + n_{f1}R_{f2})R_{f31}) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2})R_{f32})\right),$$

$$(n^{3*}, n_f^{3*}) = \arg\max_{n_1+n_{f1}=I} V^{-1}\left(\pi_1 V((n_1R_{21} + n_{f1}R_{f2})R_{f31}) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2})R_{f32})\right), \tag{B.21}$$

$$\hat{c}_2^* = \max_{n_1+n_{f1}=I} V^{-1}\left(\pi_1 V((n_1R_{21} + n_{f1}R_{f2})) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2}))\right),$$

$$(n^{2*}, n_f^{2*}) = \arg\max_{n_1+n_{f1}=I} V^{-1}\left(\pi_1 V((n_1R_{21} + n_{f1}R_{f2})) + \pi_2 V((n_1R_{22} + n_{f1}R_{f2}))\right).$$

It can be verified that

$$R_{f31}\beta > 1, \quad R_{f32}\beta < 1, \hat{c}_2^* < \beta\hat{c}_3^* \text{ and } I < \beta^2\hat{c}_3^*.$$

It is easy to compute the optimal first period portfolios in the limit. It is clear that as $\delta_1 \to -1$ we obtain for resolute choice that

$$n_{f1}^\circ \to n_f^{3*},\ n_1^\circ \to n^{3*}.$$

Note that in the period 3 optimization problem since $R_{f31} > R_{f32}$, we effectively have larger dispersion between the up and down states and hence risk than the two period case, which is the reason why we have a larger ratio of n_{f1}/n_1 for the resolute choice than the two period case. For KP preferences, we obtain

$$(n^{KP*}, n_f^{KP*}) = \arg\max_{n+n_f=I} \beta\pi_1 V((n_1R_{21}+n_{f1}R_{f2})R_{f31}) + \pi_2 V((n_1R_{22}+n_{f1}R_{f2})). \tag{B.22}$$

Comparing the optimization problems (B.21) and (B.22), since $\beta < 1$ and $R_{f32} < 1$, it follows that in eqn. (B.22) the consumer is essentially underweighting or discounting the good state $n_1R_{21}+n_{f1}R_{f2}$ and overweighting the bad state $n_1R_{22}+$

[38] In Figure 12(a) and (b), the asset ratio is also negative when δ_2 is close to -1. However in this case, we have $n_{f1} < 0$ instead of $n_1 < 0$ and the negative ratio does not correspond to dynamic hedging. In contrast, it simply suggests that when the investor is almost risk neutral, she pursues the higher return through shorting the risk free asset and buying more risky asset.

[39] Note that in the rest of this appendix, $*$ denotes optimization instead of naive choice.

		$R_{f31} = 0.8$ $R_{f32} = 2$	$R_{f31} = 2$ $R_{f32} = 0.8$
c_1	KP	3.772	3.51
	$Sophisticated$	3.753	3.523
$\frac{n_{f1}}{n_1}$	KP	0.811	−7.725
	$Sophisticated$	0.766	−8.892
s_1	KP	6.228	6.49
	$Sophisticated$	6.247	6.477
n_{f1}	KP	2.789	7.455
	$Sophisticated$	2.71	7.298
n_1	KP	3.439	−0.965
	$Sophisticated$	3.537	−0.821
I_{21}	KP	9.946	6.271
	$Sophisticated$	10.055	6.376
I_{22}	KP	5.819	7.429
	$Sophisticated$	5.811	7.361
c_{21}	KP	4.671	3.797
	$Sophisticated$	4.722	3.86
c_{22}	KP	3.553	3.489
	$Sophisticated$	3.518	3.457
c_{31}	KP	4.220	4.949
	$Sophisticated$	4.266	5.032
c_{32}	KP	4.592	3.152
	$Sophisticated$	4.586	3.123
$\delta_1 = 1.5, \delta_2 = 5, R_{21} = 2, R_{22} = 0.8$ $R_{f2} = 1.1, \pi_1 = 0.5, \beta = 0.97, I = 10$			

Table 1:

$n_{f1}R_{f2}$. Because the agent is risk averse, it is easy to see that for δ_1 sufficiently close to -1 we have

$$\frac{n_{f1}^{\circ}}{n_1^{\circ}} < \frac{n_{f1}^{KP}}{n_1^{KP}},$$

and that the difference in these ratios can be large, depending on returns. Clearly for the case of sophisticated choice we have $n^{**} = n_f^{**} = 0$. Unfortunately, this says little about the limit n_1^{**}/n_{f1}^{**} as $\delta_1 \to -1$.

It is useful to understand to what degree the ratio between risky and risk-free asset holdings in period 1 is a useful measure of risk-aversion. Clearly, for recursive utility (KP or EU) we can write the period 1 maximization problem as

$$\max_{n,n_f} \; u(I - n_1 - n_{f1}) + \beta \sum_s \pi_s W_s(n_1 R_{2s} + n_{f1} R_{f2}).$$

For the case of independent returns the value functions W_s do not depend on s and it makes sense to talk about risk-aversion. Otherwise it generally makes no sense to map portfolio-choice in the first period to any kind of risk-aversion. However, for the case of homothetic utility, holding everything else fixed, it is easy to see that we can write

$$W_s(n_1 R_{2s} + n_{f1} R_{f2}) = \Gamma_s(\delta_1) V(n_1 R_{2s} + n_{f1} R_{f2})$$

for some Γ_s that depends on all parameters but in particular on δ_1. A change of δ_1 there does not change the risk-aversion but it changes the probabilities. For our example above

$$W_s(I) = \max_c \; (c^{-\delta_1} + \beta((I-c)R_{f3s})^{-\delta_1})^{-\frac{1}{\delta_1}}.$$

By Pratt's theorem it is decreasing in δ_1. It is useful to consider the two cases $\delta_1 = 0$ and $\delta_1 = -1$. For the former we obtain

$$W_s(I) = \frac{1}{2} I \sqrt{\beta R_{f3s}}.$$

For $\delta_1 = -1$ we obtain

$$W_s(I) = \max \; (I, \beta R_{f3s} I).$$

This gives a simple condition under which the preference for the risk-free asset increases as δ_1 decreases from $\delta_1 = 0$ to $\delta_1 = -1$. For $\delta_1 = 0$ the probabilities for each state s are twisted by a factor $\sqrt{\beta R_{f3s}}$. For $\delta_1 = -1$ they are twisted by βR_{f31} in state 1 and by 1 in state 2. For many parameter values this makes state 2 relatively more likely and therefore the risky asset a better asset.