Sequential Vote Buying

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Abstract

The paper studies a vote-buying model in which leader of a committee sequentially approaches committee members in an attempt to purchase enough votes for her policy proposal to pass. Upon approaching a member, the leader offers the member a transfer in return for the member's vote. We study equilibrium sequences in which the leader approaches the members, the transfers the leader offers and the leader's ability to purchase sufficient number of votes for her policy to pass. We show that the equilibrium choice of a member to approach reflects how strongly the member is opposed to the policy and how strongly the rest of the committee is opposed to the policy; the opposition of the un-approached rest of the committee determines how aggressively the leader can bargain with the member. The equilibrium choice of a member to approach thus resolves a trade-off between approaching a weakly opposed member with whom the leader cannot bargain aggressively and approaching a strongly opposed member with whom the leader can bargain aggressively.

1 Introduction

2 Model

A leader wants to pass a new policy. She sequentially approaches the members of a committee. With each approached member, she tries to reach a bilateral agreement, offering a transfer in return for the member's support.

Formally, the game is played by the leader and a set of committee members $N = \{1, \dots, n\}$, where $n \geq 2$. Passing the policy requires $q \in \{1, \dots, n\}$ votes from the committee members (other than the leader) and provides, relative to maintaining the status quo, the leader with payoff $y > 0$ and each committee member $i \in N$ with payoff $-x_i$. We assume that $x_i > 0$ for each $i \in N$ and index the committee members such that $x_i \leq x_{i+1}$, so a member with a higher index is more strongly opposed to the policy.[1]

The leader approaches the committee members in consecutive periods. We assume that the players' actions are publicly observable. Hence, a public history at the beginning of a period includes what members have been approached previously, the transfers offered to them and the members' support decisions.

Suppose that at the beginning of a period, the set of un-approached members is U and the number of approached members who have accepted the offers is n_a. The leader can choose to *approach* a member in U, or *initiate a vote* or *stop*. If the leader decides to approach a member $i \in U$, then she offers him a non-negative transfer in exchange for his vote. Member i either accepts the offer, thus giving the leader control of his vote, or rejects the offer, and the game proceeds to the next period. If the leader decides to initiate a vote, the policy passes if $n_a \geq q$ and the status quo is maintained if $n_a < q$, and the game ends. If the leader decides to stop, the policy does not pass and the game ends. We consider two variants of this game which differ in terms of the nature of the leader's offer.

[1] Assuming that $x_i > 0$ is without loss of generality since any member with $x_i \leq 0$ prefers the new policy to the status quo and can thus be ignored in the analysis.

(1) Transfer promises: the leader's offer is a promise to make a transfer if and when a vote on the policy is held in exchange for the member's vote.

(2) Up-front payments: the leader's offer is an up-front payment in exchange for the member's vote.

In the transfer-promise game, the leader does not make a payment if she chooses not to put the policy to a vote in the end, and therefore the transfer promises are *non-sunk cost.* In the up-front-payment game, the transfer is made irrespective of whether or not a vote on the policy is held. In this sense, the payment is *sunk cost* to the leader.

The game ends either when the leader stops or a vote is held on the policy. To describes the players' payoffs at the terminal nodes, let N_a be the set of members who accepted the leader's offers and for each $i \in N_a$, let τ_i be the period in which the offer t_i was accepted. If a vote is held, let τ denote the period. We assume that the players have a common discount factor $\delta \in (0,1)$.

First consider the transfer-promise game. If no vote is held by the end of the game, each player receives a payoff of 0. If a vote is held and the policy passes, the leader receives a payoff of $\delta^{\tau-1}(y - \sum_{i \in N_a} t_i)$, and member i receives a payoff of $\delta^{\tau-1}(t_i - x_i)$ if $i \in N_a$ and $\delta^{\tau-1}(-x_i)$ if $i \notin N_a$. If a vote is held and the policy does not pass, the leader receives a payoff of $\delta^{\tau-1}(-\sum_{i \in N_a} t_i)$, and member i receives a payoff of $\delta^{\tau-1}(t_i)$ if $i \in N_a$ and 0 if $i \notin N_a$.

In the up-front-payments game, if no vote is held or is held but the policy does not pass, the leader receives a payoff of $-\sum_{i \in N_a} \delta^{\tau_i - 1} t_i$, and member i receives a payoff of $\delta^{\tau_i - 1} t_i$ if $i \in N_a$ and 0 if $i \notin N_a$. If a vote is held and the policy passes, the leader receives a payoff of $\delta^{\tau-1} y - \sum_{i \in N_a} \delta^{\tau_i - 1} t_i$, and member i receives a payoff of $-\delta^{\tau-1} x_i + \delta^{\tau_i - 1} t_i$ if $i \in N_a$ and $-\delta^{\tau-1} x_i$ if $i \notin N_a$.

In the transfer-promise game, the transfers are paid when the policy passes and hence the payoffs from the policy and from the transfers are discounted by the same factor. In contrast, in the up-front-payment game, the transfers are paid immediately upon the acceptance of the offers while the passage of the policy happens only at the end of the game and hence the payoffs from the policy and from the transfers are discounted by a different factor.

Defining strategies, maps from histories into available actions, is standard. The solution concept we use is subgame perfect equilibrium, and for the rest of the paper, we simply use the term equilibrium to refer to subgame perfect equilibrium.

Before proceeding, we discuss several features of our model. First, assuming that transfers in the sunk cost variant of the game are paid only once the leader calls a vote or stops is without loss of generality. (We have changed this, but a similar discussion would be useful.) To see this, the transfer compensating a pivotal member in period τ for supporting the leader in period τ' when the vote is called solves $\delta^{\tau'-\tau}(-x_i + t) = 0$ providing the leader with ex-ante payoff $\delta^{\tau'-1}(y - t) = \delta^{\tau'-1}(y - x_i)$. If transfers were paid immediately, the transfer to a pivotal member would solve $-\delta^{\tau'-\tau}x_i + t = 0$ providing the leader with $\delta^{\tau'-1}y - \delta^{\tau-1}t = \delta^{\tau'-1}y - \delta^{\tau-1}\delta^{\tau'-\tau}x_i$. Second, the specification of the game assumes that offers and support decisions are public and that the leader has all the bargaining power, cannot re-approach members and cannot extend offers simultaneously to multiple members. Any of these extensions, while certainly interesting, would significantly complicate the analysis and obscure the main strategic forces we study below. Third, the game allows the leader to choose an arbitrary sequence in which she approaches the members, including an arbitrary history dependence of this sequence. The underlying principles that shape the leader's decision how to approach the members are the main focus of our analysis.

We introduce the notion of states to facilitate the analysis. Recall that a history at the beginning of a period records the set of members who have been previously approached, the transfers offered to them, and the members' acceptance decision. For the model with transfer promises, a state is (S, r, t), where $S \subseteq N$, $S \neq \varnothing$, $r \in \{1, \ldots, q\}$ and $t \geq 0$, corresponds to a set of histories such that the set of members who have been previously approached is $N \setminus S$, the number of members who promised to support the leader is $q - r$ and the sum of transfers to these members is t. That is, state (S, r, t) corresponds to histories in which the set of un-approached agents is S, the leader still needs support from r members in order for the policy to pass and the leader has already promised to pay a total of t to the members who have accepted the offers. For the model with up front payments, we do not include the transfers that are already accepted and paid in a state since they do not affect the equilibria in the subgames. Hence, we denote a state by (S, r) in the up-front-payment model. Let $\mathcal{S} = \{(S, r, t) \in 2^N \times \mathbb{Z} \times \mathbb{R}_+ | S \neq \varnothing \wedge 1 \leq r \leq |S|\}$ be the set of all states (dropping the $\mathbb{R}_+$ dimension for the sunk cost model).[2] Note that given any state in $\mathcal{S}$ and any two histories inducing that state, the subgames following the two histories are identical and hence the two subgames have the same set of equilibria. Let $\Gamma(S, r, t)$ denote a subgame starting with state (S, r, t) in the transfer-promise model and $\Gamma(S, r)$ denote a subgame starting with state (S, r) in the up-front-payment model. Then the entire game is $\Gamma(N, q, 0)$ in the transfer-promise game and $\Gamma(N, q)$ in the up-front-payment game.

3 Transfer promises

We begin by studying the model in which the leader offers a transfer promise in exchange for a member's vote. We first establish conditions under which the policy passes in equilibrium and then characterize the optimal sequence in which the leader approaches the members and how much transfer promises she offers them in equilibrium.

3.1 When does the policy pass?

Consider a subgame $\Gamma(S, r, t)$. Proposition 1 below says that whether the policy passes in equilibrium depends on how the leader's gain (net of the transfer promises already accepted) compares with the sum of the losses of the r members in S who are least opposed to the policy. Applying this result to the whole game immediately implies that whether the policy passes in equilibrium depends on whether the leader's gain from the policy is strictly higher

[2] There are states not in $\mathcal{S}$ that could arise in the extensive form: for example, there are subgames in which $r > |S|$, but since they are not interesting to analyze, we exclude them from $\mathcal{S}$.

than the sum of the losses of the q members who are least opposed to the policy.

Given $S \subseteq N$ and $0 \leq r \leq |S|$, let $S^r \subseteq S$ denote the set of the r agents in S who have the lowest losses from the policy. Let $S^r = \varnothing$ if $r = 0$.

Proposition 1. *Suppose the leader offers transfer promises. Consider a subgame $\Gamma(S, r, t)$ where $r \leq |S|$. In any equilibrium, (a) if $y - t > \sum_{j \in S^r} x_j$, the policy passes, and (b) if $y - t < \sum_{j \in S^r} x_j$, the policy does not pass.*

To gain some intuition for part (a), note that a member i would accept an offer greater than his loss.[3] Hence, when the leader needs r votes, if her gain from the policy (net of the transfer promises that have already been offered and accepted) is larger than the sum of r members' losses, she can guarantee a strictly positive payoff by approaching these r members at the end (so that each understands that he is pivotal) and making each an offer that just compensates for the loss. Since the leader's payoff is only 0 when the policy does not pass, she is better off if she buys these members' votes and therefore the policy passes in any equilibrium.

For part (b), it is straightforward to see that if the cardinality of S equals r (corresponding to unanimity), then the leader needs the vote of every member in S, who will accept an offer only if it at least compensates for the member's loss. Hence, if the leader's gain from the policy (net of the transfer promises already accepted) is lower than the sum of the members' losses, then she can only receive a negative payoff by getting the policy passed, whereas she receives a payoff of 0 if the policy does not pass. Given that the leader is better off if the policy does not pass and she can choose to stop, the policy does not pass in any equilibrium. If cardinality of S is equal to $r + 1$, then the member approached in the first period will accept an offer only if it at least compensates for his loss since the member foresees that without his support, the policy will not pass in the continuation game. Since every member can reason like this and thus demands an offer that makes him at least even, again the leader

[3] If we restrict attention to undominated strategies, then clearly a member i would accept an offer greater than his loss, but our result still holds even if weakly dominated strategies are allowed.

cannot buy enough votes without offering transfer promises that exceed her gain from the policy. Induction shows that the policy does not pass in any equilibrium.

3.2 Optimal sequence of vote buying with transfer promises

We have established in Proposition 1 that if $y > \sum_{i=1}^{q} x_i$, then the policy passes in any equilibrium. In what sequence should the leader approach the members and what transfer promises does she offer them in equilibrium? These are the questions we address in this subsection. The answer is immediate under unanimity ($q = n$): the leader offers each member i a transfer promise equal to x_i and the sequence of approaching does not matter. In what follows, we consider $q < n$. It is useful to introduce the notion of "(in)dispensability."

Definition 1. *In the transfer-promise game, consider state (S, r, t) where $r < |S|$. (a) We say that member $i \in S$ is indispensable in (S, r, t) if $\sum_{j \in S^r} x_j < y - t < \sum_{j \in S^r_{-i}} x_j$. (b) We say that member $i \in S$ is dispensable in (S, r, t) if $y - t > \sum_{j \in S^r_{-i}} x_j$.*

Intuitively, a member is indispensable in a state if the policy does not pass in equilibrium without the leader securing the member's vote in that state whereas a member is dispensable in a state if the policy still passes in equilibrium even without the leader securing the member's vote in that state. A member's strategic position is stronger when he is indispensable than when he is dispensable. When a member is indispensable, he accepts the leader's offer only if it at least compensates for his loss since by rejecting the offer, the policy will fail to pass. When a member is dispensable, however, he anticipates that the policy still passes even if he rejects the offer. Since his rejection only delays the passage of the policy, he is willing to accept an offer that just compensates him for a sooner passage ($(1 - \delta)x_i$ to member i). It follows that, when δ is sufficiently high, a dispensable member is willing to accept an offer close to 0, an offer lower than any offer that the leader needs to make in order to secure an indispensable member's vote.

As Proposition 2 below shows, what members' votes the leader buys depends on whether the qth member is dispensable at the beginning of the game. If he is, then it is optimal for the leader to approach the members who have the lowest q losses in descending order (that is, first approach the member with the highest loss among the set, then the second highest, and so on). Along this sequence, every member i being approached is dispensable and thus accepts an offer $(1-\delta)x_i$. Since they also have the lowest losses among all members, the leader cannot improve her payoff by approaching others.

If the qth member is indispensable at the beginning of the game, then any member with a lower loss is also indispensable at the beginning of the game. Hence, if the leader approaches any of them, she has to make an offer equal to the loss. However, since $y > \sum_{i=1}^{q} x_i$, any member in $\{q+1, q+2, ..., n\}$ is dispensable at the beginning of the game, implying that if the leader approaches member $q+1$, she only needs to offer $(1-\delta)x_{q+1}$. Moreover, after securing this member's vote with a transfer promise close to 0, any member in $\{1, 2, ..., q\}$ becomes dispensable at the beginning of the continuation game, and therefore it is optimal for for the leader to approach the members who have the lowest $q-1$ losses in descending order. The following proposition formalizes the results.

Proposition 2. *Suppose the leader offers transfer promises. Suppose $n > q$ and $y > \sum_{i=1}^{q} x_i$. There exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, the following results hold.*
(a) In any equilibrium, q members are approached and each accepts the leader's offer.
(b) If $y > \sum_{i=1}^{q-1} x_i + x_{q+1}$, then (i) there exists an equilibrium in which the leader approaches $i \in \{1, ..., q\}$ in descending order; (ii) in any equilibrium, the leader approaches members $i \in \{1, ..., q\}$ and offers $t_i = (1-\delta)x_i$.
(c) If $y < \sum_{i=1}^{q-1} x_i + x_{q+1}$, then (i) there exists an equilibrium in which the leader approaches $i \in \{1, ..., q-1, q+1\}$ in descending order; (ii) in any equilibrium, the leader approaches members $i \in \{1, ..., q-1, q+1\}$ and offers $t_i = (1-\delta)x_i$.

4 Up-front payments

We now turn to the model in which the leader offers an up-front payment in exchange for a member's vote. Note that under unanimity, since every member i has the right to veto, he accepts an offer if and only if it compensates for his loss x_i, appropriately discounted. Specifically, in any state (S, r) such that $|S| = r$, if member i is approached in equilibrium, he accepts the offer t_i if and only if $t_i \geq \delta^r x_i$. (Since the policy passes after the leader buys the votes of the remaining r members but the payment is up front, the member is willing to accept any offer greater than $\delta^r x_i$.) It follows that the leader's payoff is $\delta^r(y - \sum_{i \in S} x_i)$ by getting the policy pass, and therefore the policy passes in equilibrium if and only if this payoff is positive, that is, $y \geq \sum_{i \in S} x_i$. Another special case is when $r = 1$. Since the leader needs only one vote for the policy to pass, once the leader buys one member's vote, she does not approach any more members and initiates voting immediately. Whether the offer is an up-front payment or a transfer promise does not matter for the incentives, implying that the condition for the policy to pass in equilibrium is $y \geq x_1$. In contrast, when the leader needs more than one vote for the policy to pass, whether the offer she makes is up-front payment or transfer promises has important implications, which we illustrate by the following example.

Example 1. *Suppose $n = 3$ and $q = 2$. We first show that when $y > x_1 + x_2$, then the policy passes in equilibrium with offers close to 0 when the players are patient, similar to what happens in transfer-promise game. We then show that if $x_2 < y < x_1 + x_2$, then the policy still passes in equilibrium in the up-front-payment game, even though it does not in the transfer-promise game, but in this case, not all offers are close to 0. We finally show that if $y < x_2$, then the policy does not pass even in the up-front-payment game.*

First consider $y > x_1 + x_2$. If member 3 is approached first, then he is willing to accept any offer greater than $x_3\delta^2(1 - \delta)$ because even if he rejects the offer, the policy will still pass in the continuation game and therefore his rejection only delays the passing of the policy by one period. Hence, he accepts any offer t such that $t - \delta^2 x_3 > \delta^3 x_3$. After member 3's vote is bought, member 1 is willing to accept any offer greater than $x_1\delta(1 - \delta)$ because his rejection *only delays the passing of the policy by one period. Note that both offers are close to* 0 *for patient players – we refer to them as "exploitation" offers.*[4]

Now consider $x_2 < y < x_1 + x_2$*. If member* i *is approached first, he is willing to accept an offer if and only if* $t_i \geq \delta^2 x_i$*. To see this, note that if member* i *rejects the offer, then the policy will fail to pass since the leader would need to buy each remaining member's vote, which is too costly given that* $y < x_1 + x_2$*. But after securing member* 1*'s vote by offering him* $t_1 = \delta^2 x_1$ *(we call this an "temptation" offer), now the leader can buy member* 3*'s vote by making him an exploitation offer* $(1 - \delta)x_3$*. Since the leader can buy enough votes at a cost lower than* y*, the policy passes in equilibrium.*

Finally consider $y < x_2$*. For the same reason as discussed above, the leader has to make a temptation offer to the member approached first. Since* $y < x_2$*, it is too costly for the leader to tempt member* 2 *or* 3*. Furthermore, even if the leader buys member* 1*'s vote first, whoever the leader approaches next would still accept only a temptation offer, which would be too costly. Hence, the policy does not pass in equilibrium even in the up-front-payment game.*

The next proposition says that given a state (S, r), if the players are sufficiently patient, the policy passes if and only if y is above a threshold $W(S, r)$, defined recursively as follows. For any state (S, r), denote by $maxS$ the member in S with the highest loss and let $S' = S \setminus \{maxS\}$. Let $W(S, r) = \sum_{i \in S} x_i$ when $r = |S|$ and

$$W(S, r) = \min_{T \in 2^S} \max \left\{ \sum_{j \in T} x_j, W((S \setminus T)', r - |T|) \right\} \tag{1}$$

when $r < |S|$.

Proposition 3. *Suppose the leader offers up-front payments. Consider a subgame* $\Gamma(S, r)$*. For generic* y*, there exists* $\bar{\delta} < 1$ *such that for* $\delta > \bar{\delta}$ *in any equilibrium, the policy passes if* $y > W(S, r)$ *and the policy does not pass if* $y < W(S, r)$*.*

[4] This is the optimal sequence if $y < x_1 + x_3$; if $y > x_1 + x_3$, then it is optimal to approach member 2 first, followed by member 1.

To understand why $W(S,r)$ is the threshold that determines whether the policy passes in equilibrium, it is useful to classify the members in terms of their bargaining position given a state.

Definition 2. *A member $i \in S$ in state (S,r) is*

1. *dispensable if $y > W(S \setminus \{i\}, r)$,*

2. *indispensable if $y \in (W(S \setminus \{i\}, r-1), W(S \setminus \{i\}, r))$.*

The definition of dispensability and indispensability here parallel those in the model of transfer promises. As implied by Proposition 3, a member is indispensable in a state if the policy does not pass in equilibrium without the leader securing the member's vote in that state whereas a member is dispensable in a state if the policy still passes in equilibrium even without the leader securing the member's vote in that state. As before, when a member is indispensable, he has a strong bargaining position and thus accepts the leader's offer if and only if it at least compensates for his loss (with the appropriate discounting), but when a member is dispensable, he has a weak bargaining position and is therefore willing to accept an offer that just compensates him for a sooner passage of the policy. We referred to these two distinct kinds of offers as temptation and exploitation offers, as formalized in the following definition.

Definition 3. *Fix a profile of strategies and consider the resulting sequence of approached members. Suppose member i is offered t_i in state (S,r). We say that member i is tempted if $t_i = \delta^r x_i$ and that member i is exploited if $t_i = \delta^r x_i(1-\delta)$. We say a profile is in a temptation phase when the approached member is tempted and is in an exploitation phase when the approached member is exploited.*

Lemma 1. *Suppose the leader offers up-front payments. For generic y, there exists $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$ in any equilibrium, if member i is approached in a state in which he is indispensable, then he is tempted; and if member i is approached in a state in which he is dispensable, then he is exploited.*

Lemma 2. *Given a state* (S, r)*, if there exists a member* $i \in S$ *who is dispensable, then (i) member* $maxS$ *is dispensable in* (S, r)*; (ii) any member in* $S \setminus \{i\}$ *is dispensable in state* $(S \setminus \{i\}, r-1)$*, which implies that starting in state* (S, r)*, there exists a sequence of* r *members along which each member is dispensable.*

Since the payment made to a dispensable member goes to 0 as the discount factor goes to 1 by Lemma 1, once an exploitation phase starts in equilibrium, it remains in that phase until the leader buys all the votes she needs. Moreover, since the payment made to an indispensable member equals his loss (in the limit as the discount factor goes to 1), the total payment that the leader makes is the sum of the losses of the members approached in the temptation phase. Hence, if there exists a set of members such that the following two conditions hold: (i) the sum of their losses is below y, and (ii) after the leader buys their votes, at least one remaining member is dispensable, then the policy will pass in equilibrium. Conditions (i) and (ii) are reflected in the definition of $W(S, r)$.

To see which members are tempted in equilibrium, consider the following problem for any state (S, r):

$$\min_{T \in 2^S} \sum_{j \in T} x_j \text{ s.t. } y > W((S \setminus T)', r - |T|). \tag{2}$$

The constraint ensures that after the leader buys the votes of members in the set T, there exists one member in the remaining set who is dispensable. As discussed above, the leader's payments to members who are dispensable are close to 0 and therefore she is only concerned about her payments to member in T, which equals the sum of their losses. Hence, the least costly way for her to buy enough votes to get the policy passed involves tempting members in set T. It also follows that the payment that the leader makes in equilibrium is the value of the problem, that is,

$$\Pi(S, r, y) = \min_{T \in 2^S} \sum_{j \in T} x_j \text{ s.t. } y > W((S \setminus T)', r - |T|).$$

We summarizes these characterizations of the equilibrium in Proposition 4 below.

Proposition 4. *Suppose the leader offers up-front payments and* $y > W(N, q)$. *For generic* y, *there exists* $\bar{\bar{\delta}} < 1$ *such that for* $\delta > \bar{\bar{\delta}}$ *following results hold.*

(a) In any equilibrium, q *members are approached and each accepts the leader's offer.*

(b) Any equilibrium consists of two phases (with one possibly empty), a temptation phase followed by an exploitation phase.

(c) In any equilibrium, the set of members included in the temptation phase, T, *solves the optimization problem (2). For any order of the members in* T, *there exists an equilibrium in which the members in the temptation phase are approached in that order. There exists an equilibrium in which the first member in the exploitation phase is the member with the lowest loss among the members who are dispensable in* $(N \setminus T, q - |T|)$ *and the members subsequently approached in the exploitation phase are those with the lowest losses in* $N \setminus T$.

(d) The leader's equilibrium payoff is constant across equilibria and tends to $y - \Pi(N, q, y)$ *as* $\delta \to 1$.

5 Conclusion

A Proofs