Competitive information discovery

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Abstract

In classical models of markets, the state of nature is revealed regardless of the actions agents take. Firms for example cannot cause information to be revealed by deciding to experiment with new technologies. If instead agents can uncover information they will determine which states can be distinguished and thus which goods are traded. Perfectly competitive equilibria can then be inefficient. One source of inefficiency is self-confirmation: price expectations can lead agents not to discover the information that would invalidate those expectations. Inefficiency can also occur if agents fail to be risk-averse or share a common prior, assumptions normally unnecessary for the first welfare theorem. Restoring optimality requires these assumptions and a *competitive price rule* must hold: the prices anticipated when agents contemplate an information discovery must be proportional to the probabilities of the events that could be revealed. As an application, these conditions imply that firms will efficiently experiment with uncertain technologies without the prospect of earning a monopoly reward.

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1 Introduction

In classical models of competitive markets, the revelation of information descends without human intervention. The archetype is the weather which is learned regardless of how agents act. But in at least as many cases of economic interest agents must take an action to reveal the state: a firm must experiment with a new technology to find out if it is productive, a consumer must try out a good to gauge its utility. The predominant analysis of firms' information discoveries follows Schumpeter's noncompetitive theory: firms try out new technologies to win a measure of monopoly power and would not conduct experiments without a chance of that reward.[1]

This paper builds a theory of competitive markets in which agents decide whether to discover information. There are two main messages.

First, the conditions under which competitive equilibria deliver efficient outcomes deviate markedly from the standard lessons of the welfare theorems. Even if no informational externalities are present and agents are not excluded from the information discoveries that others make, equilibria can be Pareto inefficient. In a characteristic example, it can be efficient for a firm to invest in testing an innovation but the output price that agents anticipate if the test is conducted and the experiment is successful will not be high enough to justify the investment. The firm therefore does not experiment with the new technology, leaving agents ignorant of the state. This example flips the Schumpeterian story: instead of an innovator using its informational monopoly to manipulate prices in its favor, firms face perfect competition and expected prices move to an innovator's disadvantage.

We can neutralize this problem by requiring agents to hold 'competitive' price expectations that do not discourage efficient investment. But equilibria still need not be Pareto efficient. For the first welfare theorem to hold, we must also assume that agents share a common prior and are risk-averse. These assumptions are alien to the contemporary understanding of the efficiency of markets. One of the insights of Arrow (1951) and Debreu (1951) was that the first welfare theorem is nearly assumption-free: the weak Pareto efficiency of equilibria requires no assumptions and strong efficiency requires only that preferences are

[1] See Schumpeter (1942). More recent work in this vein includes Aghion and Howitt (1998), Grossman and Helpman (1991), and Romer (1990).

transitive and locally nonsatiated. But when opportunities to discover information are present, the absence of risk aversion and have common priors will imply that an agent that decides not to discover information can harm other agents by denying them access to utility-increasing gambles. Inefficiency therefore results. The Arrow-Debreu model can make do with weaker assumptions only because it assumes that information is revealed automatically: if some agent has a coin in his pocket that only he can decide to toss then the first welfare theorem will fail when agents seek risk or their priors for heads and tails differ.

The second message runs counter to Schumpeter's pessimism that firms will efficiently sample new technologies only when they can sometimes win a monopoly pay-off. With competitive price expectations, and assuming common priors and risk aversion, firms will experiment efficiently, even when those experiments would seem doomed to lose money. Suppose the output of a firm is stochastic and that the firm can undertake an investment that will make the distribution of its output riskier. Since the firm's technology can be shelved in favor of alternative technologies when it has a poor productivity realization, such increases in risk will expand economy-wide expected output and can deliver a Pareto improvement. Efficiency can therefore require that firms adopt riskier technologies despite the fact that it is the firms that run the alternative technologies that reap part of the benefit (Mandler (2017)). The main result of this paper shows that firms in competitive equilibrium can select the efficient technologies even in cases like this.

The competitive price expectations mentioned above require that the prices agents anticipate for a good k when an information discovery is contemplated equal the prices for k that obtain under ignorance multiplied by the probabilities of the events that could be discovered. This *competitive price rule* extends price-taking to settings where the set of tradable goods is determined endogenously: it ensures that as agents consider information discoveries the price they anticipate paying for an increment in the probability of receiving a good remains constant.

Equilibria that violate the competitive price rule and lead to inefficiency can be 'self-confirming': agents can hold price expectations that lead them not to make the information discoveries that would disconfirm those expectations, similarly to Fudenberg and Levine (1993) though the analysis here applies to markets rather than games. Ruling out self confirmation is delicate. If we simply impose a subgame perfection requirement that agents accurately perceive the equilibrium that results as they change their actions they would no longer behave as price-takers: they would adjust their demands to optimize their effect on prices, a noncompetitive behavior that would lead to inefficiency.

The competitive price rule solves this problem. First, the price expectations it mandates will match the equilibrium prices that would rule if we could force the state to be revealed and there were no supply-and-demand consequences of the information learned. When information does have a demand impact, this match will no longer be exact: knowledge of the state will change demand and the prices that clear markets. But if the value of information is small – a concept I will make precise in section 4 – the price rule lays out price expectations that approximate what would occur in equilibrium when revelation is forced. So, comparably to standard models of competitive markets, equilibrium prices under the price rule will provide good predictions of the consequences of small perturbations of the environment. In addition to providing an equilibrium rationale for the price rule, the small-information assumption will pave the way for a characterization of the price rule.

The main impediment to efficiency that information discovery introduces stems from the market power that information discoverers potentially wield: since goods are distinguished by state, an agent that causes the state to be revealed can affect which goods are traded and thus indirectly the prices of goods. This leverage resembles the monopoly power that Schumpeterian firms gain when they discover an innovation. But information discovery need not always be accompanied by the acquisition of market power. Unlike Schumpeterian analyses, the competitive price rule disentangles these two features of innovation. Although innovative firms can still influence prices through their discoveries, the price rule identifies a neutral ground where this influence does not lead to inefficiency.

The existence of equilibrium will be a side issue in this paper. Although discrete costs for information discoveries will introduce nonconvexities that can interfere with existence, those difficulties have classical fixes. And when firms are the discoverers, the convexity and continuity assumptions that guarantee existence can still be imposed (see section 6).

Boldrin and Levine (2002, 2017a, 2017b) pursue a compatible agenda where innovative goods are competitively produced under constant returns and optimality obtains. Boldrin and Levine do not allow for uncertainty, however, a prominent feature of technological development; one goal of this paper is to fill this gap. My aim though is not to argue that efficiency is the norm. For competitive equilibria to be efficient, one must assume that information externalities are absent. There are cases where that assumption makes sense (consumers figuring out their own tastes) and other Schumpeterian cases where it does not (expensive technological research that free riders can copy). As usual with first welfare theorems, the point is not to claim that the world is in fact efficient but to provide a classification of when and how efficiency and inefficiency obtain.

2 Equilibrium with information discovery

Consumers, firms, states, and goods.

The sets of consumers $\mathcal{I}$, firms $\mathcal{J}$, and states Ω are all finite with I, J, and S elements respectively. There are L_1 goods in the first period and L_2 goods at each state in the second period. The total number of goods is therefore $L = L_1 + SL_2$.

A consumption for $i \in \mathcal{I}$ and a production for $j \in \mathcal{J}$ are given by

$$x^i = \left(x_1^i(1), ..., x_{L_1}^i(1), \left(x_1^i(\omega), ..., x_{L_2}^i(\omega)\right)_{\omega\in\Omega}\right) \in \mathbb{R}_+^L,$$
$$y^j = \left(y_1^j(1), ..., y_{L_1}^j(1), \left(y_1^j(\omega), ..., y_{L_2}^j(\omega)\right)_{\omega\in\Omega}\right) \in \mathbb{R}^L.$$

Define also

$$x^i(1) = \left(x_1^i(1), ..., x_{L_1}^i(1)\right), \; x^i(\omega) = \left(x_1^i(\omega), ..., x_{L_2}^i(\omega)\right), \; x = \left(x^i\right)_{i\in\mathcal{I}},$$
$$y^j(1) = \left(y_1^j(1), ..., y_{L_1}^j(1)\right), \; y^j(\omega) = \left(y_1^j(\omega), ..., y_{L_2}^j(\omega)\right), \; y = \left(y^j\right)_{j\in\mathcal{J}},$$

and let x^{-i} and y^{-j} denote $(x^{i'})_{i'\in\mathcal{I}\setminus\{i\}}$ and $(y^{j'})_{j'\in\mathcal{J}\setminus\{j\}}$.

The probabilities of states and events in Ω are given by $\pi(\cdot)$ and we fix, for any event $E \subset \Omega$, conditional probabilities $\pi(\cdot|E)$ that satisfy Bayes rule when applicable.

Each consumer i at each state ω has a concave and locally nonsatiated utility u_ω^i : $\mathbb{R}_+^{L_1+L_2} \to \mathbb{R}$, henceforth called a *vNM utility*, which defines an expected utility function

$U^i : \mathbb{R}^L_+ \to \mathbb{R}$ by $U^i(x^i) = \sum_{\omega\in\Omega} \pi(\omega) u^i_\omega(x^i(1), x^i(\omega))$.

Notice that we have imposed risk aversion and a common prior. Both assumptions will be indispensable for the first welfare theorem to hold.

Consumer i's endowments are given by $e^i = \left(e^i(1), (e^i(\omega))_{\omega\in\Omega}\right) \in \mathbb{R}^L_+$.

Each firm $j \in \mathcal{J}$ has a production set $Y^j \subset \mathbb{R}^L$. Consumer i's ownership share of j is $\theta^{ij} \geq 0$, where $\sum_{i\in\mathcal{I}} \theta^{ij} = 1$. For each state ω, let the *single-state production set* Y^j_ω be the projection of Y^j onto its first L_1 coordinates and its L_2 ω-coordinates (the first-period goods and the second-period goods that appear at ω).[2] Although weaker conditions that I state in a footnote would do, I assume that Y^j and the Y^j_ω are related by the rule that firm j's production set at state ω equals Y^j_ω regardless of what j chooses at other states: the firm chooses $y^j(1)$ in the first period, nature then selects the state ω, and the firm then chooses any vector in $\left\{y^j(\omega) \in \mathbb{R}^{L_2} : (y^j(1), y^j(\omega)) \in Y^j_\omega\right\}$ in the second period. Firm j's production set therefore equals $Y^j = \left\{\left(y^j(1), (y^j(\omega))_{\omega\in\Omega}\right) \in \mathbb{R}^L : (y^j(1), y^j(\omega)) \in Y^j_\omega \text{ for all } \omega \in \Omega\right\}$.[3]

Information.

The actions agents take in the first period can uncover information on which second-period actions can then be conditioned. For each consumer i, the choice x^i informs all agents at the end of date 1 of a cell of the partition $\mathcal{P}^i(x^i)$ of Ω, where, since it is i's first-period consumption that reveals the information, $\mathcal{P}^i(x^i) = \mathcal{P}^i(x^{i\prime})$ if $x^i(1) = x^{i\prime}(1)$. Similarly, for each firm j, the production $y^j \in Y^j$ informs agents of a cell of the partition $\mathcal{P}^j(y^j)$, where $\mathcal{P}^j(y^j) = \mathcal{P}^j(y^{j\prime})$ if $y^j(1) = y^{j\prime}(1)$. Since some goods can be useless and have a 0 price, the model can let agents costlessly select a partition from an arbitrary menu of partitions without affecting their useful consumption or production (see section 5).

The following examples consider the information that one agent can discover; typically

[2] That is,

$$Y^j_\omega = \left\{(y^j(1), y^j(\omega)) \in \mathbb{R}^{L_1+L_2} : (y^j(1), \left(y^j(\omega')\right)_{\omega'\in\Omega}) \in Y^j \text{ for some } \left(y^j(\omega')\right)_{\omega'\in\Omega\setminus\{\omega\}} \in \mathbb{R}^{L_2(S-1)}\right\}.$$

[3] For the sake of generality, in the paper's proofs I will not rule out all inter-dependencies of production across states. The important requirement is that if the Y^j_ω coincide at some set of states then the same productions can be chosen simultaneously at these states, fixing what j does elsewhere. Formally, the assumption in the proofs is that if $Y^j_{\omega'} = Y^j_{\omega''}$ and $y^j \in Y^j$ then the $\widehat{y}^j$ defined by $\widehat{y}^j(\sigma) = y^j(\sigma)$ for $\sigma = 1$ and $\sigma \in \Omega\setminus\{\omega''\}$ and $\widehat{y}^j(\omega'') = y^j(\omega')$ is an element of Y^j: j can choose at ω'' whatever it chooses at ω', all else remaining fixed.

many agents will make discoveries simultaneously.

Example 1 Suppose consumer i can uncover information by buying $c \in \mathbb{R}^{L_1}_{+}$ or more in the first period: for some partition $\widehat{\mathcal{P}}$ of Ω, $\mathcal{P}^i(x^i) = \widehat{\mathcal{P}}$ if $x^i(1) \geq c$ and $\mathcal{P}^i(x^i) = \{\Omega\}$ otherwise. The partition $\widehat{\mathcal{P}}$ could indicate whether i likes good 1 with, for example, the cells of $\widehat{\mathcal{P}}$ given by

$$P_I = \{\omega \in \Omega : u^i_\omega \text{ is increasing in } x^i_1(\omega)\},$$
$$P_D = \{\omega \in \Omega : u^i_\omega \text{ is decreasing in } x^i_1(\omega)\},$$
$$P_N = \Omega \backslash (P_I \cup P_D).$$

The cost c might consist of a minimum-size sampling of the good in the first period: $c_1 > 0$ and $c_k = 0$ for $2 \leq k \leq L_1$. ■

Example 1 exhibits a nonconvexity where information is discovered only when a consumer makes a threshold payment. Information discoveries by firms could work the same way, but it is just as plausible for the harnessing of inputs to reveal information directly without any violation of standard convexity assumptions. In the following example, mentioned in the Introduction, a firm can build a capital good in the first period to produce output in the second period. In the course of constructing the capital, the firm finds out if its experimental technology 'works', whether it has high or low productivity.

Example 2 There are two states ω_H and ω_L indicating high and low productivity and one good at date 1 and each state. A firm j can use a linear activity with coefficients $(-2, 3, 1)$ for the first-period, ω_H, and ω_L goods respectively and therefore has the production set

$$Y^j = \{(y(1), y(\omega_H), y(\omega_L)) : (y(1), y(\omega_H), y(\omega_L)) \leq \lambda(-2, 3, 1) \text{ for some } \lambda \geq 0\}.$$

The state is revealed if and only if the firm chooses $y^j(1) < 0$: $\mathcal{P}^j(y^j) = \{\{\omega_H\}, \{\omega_L\}\}$ if $y^j(1) < 0$ and $\mathcal{P}^j(y^j) = \{\Omega\}$ otherwise. Notably, the production set Y^j is convex. ■

The next example gives firms the option to abandon a variant of the uncertain technology of Example 2 in favor of an alternative technology. By exercising this option when the low-productivity state obtains, firms can use the risky technology to raise the expected output of the economy as a whole. Seemingly however agents should refuse to undertake such risky investments: it is the owners of the alternative technology that receive the payout in the low-productivity states.

Example 3 The dates and states are as in Example 2. An endowment of capital at date 1 can build either 'risky' or 'safe' capital that two corresponding linear activities at date 2 can use as inputs. The input and output coefficients of these activities and of a pure-labor activity are:

	risky activity	safe activity	pure-labor activity
risky capital input	-1	0	0
safe capital input	0	-1	0
labor input	-1	-1	-1
ω_H-output	3	2	$\frac{3}{2}$
ω_L-output	1	2	$\frac{3}{2}$

Letting the probabilities be $\pi(\omega_H) = \pi(\omega_L) = \frac{1}{2}$, the potential outputs of a risky investment are a mean-preserving spread of the outputs of a safe investment. If the endowment of date-2 labor can fully employ all of the date-2 capital that could be created then maximization of expected output requires that only risky capital is created: since the labor that uses risky capital when ω_H obtains can resort to the pure-labor activity when ω_L obtains, one unit of risky capital generates an expected output of $\left(\frac{1}{2} \times 3\right) + \left(\frac{1}{2} \times \frac{3}{2}\right) = 2\frac{1}{4}$, while one unit of safe capital generates an output of 2.

The definitions of the production sets, which are again convex, proceed as in Example 2 and we assume as before that agents face the partition $\{\{\omega_H\}, \{\omega_L\}\}$ if and only if some firm builds risky capital. Otherwise agents face $\{\Omega\}$. ■

As indicated already, agents will face a common information partition, one cell of which is revealed at the end of the first period. Common information is not essential and our results do not depend on it: we could employ the Radner (1968) competitive equilibria where have agents have differential information. We assume symmetric information for three reasons. The first is the Radner (1979) rational expectations argument that equilibrium prices will reveal agents' private information. Second, symmetric information ensures that both sides to any trade of state-contingent goods can confirm its execution. The third reason is paramount: equilibria that fail to reveal agents' private information would introduce an independent source of inefficiency. Asymmetric information would therefore obscure the conclusion that information discovery per se can lead to inefficiency.

Formally, when agents take the actions $(x, y) \in \mathbb{R}^{IL}_{+} \times \prod_{j \in \mathcal{J}} Y^j$ they face the coarsest common refinement of the partitions in $\{\mathcal{P}^i(x^i) : i \in \mathcal{I}\}$ and $\{\mathcal{P}^j(y^j) : j \in \mathcal{J}\}$ which we denote by $\mathcal{P}_{x,y}$. This framework allows agents to learn information regardless of what actions they take: if the partition $\mathcal{P}$ represents this information then the $\mathcal{P}_{x,y}$ would be finer than $\mathcal{P}$ for every (x, y). As in the Arrow-Debreu model, all agents might learn the weather at date 2.[4]

Markets and equilibrium.

Let $p = \left(p_1(1), ..., p_{L_1}(1), (p_1(\omega), ..., p_{L_2}(\omega))_{\omega \in \Omega}\right) \in \mathbb{R}^L_+$ denote a price vector. Prices for second-period goods have operational meaning only in relation to the partition $\mathcal{P}_{x,y}$ that agents face: the price of good k in the event $P \in \mathcal{P}_{x,y}$ will be $\sum_{\omega \in P} p_k(\omega)$. While $p_k(\omega)$ need not indicate the price of k at ω (since $\{\omega\}$ might not form a cell of a partition that agents face), the $p_k(\omega)$ nevertheless play an important role: they determine the prices agents expect as they alter their actions and thereby change $\mathcal{P}_{x,y}$.

Given prices p, productions y, and consumptions x^{-i} for all consumers besides i, the budget set for consumer i is

$$B^i(p, x^{-i}, y) = \left\{ x^i \in \mathbb{R}^L_+ : x^i \text{ is } \mathcal{P}_{x^i, x^{-i}, y}\text{-measurable and } p \cdot x^i \leq p \cdot e^i + \sum_j \theta^{ij} p \cdot y^j \right\}.$$

Given the consumptions x and productions y^{-j} for all firms besides j, the action set for firm

[4] To ensure that a consumer i always has the option of selling any portion of his endowment to the market, e^i should be known to i in the second period whatever actions the agents take. For example, $e^i(\omega)$ could be a constant function of ω. To permit random endowments, for $i \in \mathcal{I}$ and $j \in \mathcal{J}$ let $\bigwedge \mathcal{P}^i$ and $\bigwedge \mathcal{P}^j$ be finest common coarsening of the partitions in $\left\{\mathcal{P}^i(x^i) : x^i \geq 0\right\}$ and $\left\{\mathcal{P}^j(y^j) : y^j \in Y^j\right\}$ respectively: $\bigwedge \mathcal{P}^l$ represents the information agent l will necessarily uncover regardless of the action l takes. We assume that, for each consumer i, the random variable e^i is measurable with respect to the coarsest common refinement of the partitions in $\left\{\bigwedge \mathcal{P}^l : l \in \mathcal{I} \cup \mathcal{J}\right\}$. For example, if there are good and bad weather cells in each $\bigwedge \mathcal{P}^i$ and $\bigwedge \mathcal{P}^j$ then our assumption would allow agents' endowments to be influenced by the weather. If the assumption were violated and second-period endowments were publicly observed in the second period then, for some set of first-period actions the agents might take, the agents would be failing to learn all that could be inferred from their observations. The assumption also ensures that if the market for delivery of a good at a state in $P \in \mathcal{P}_{x,y}$ clears then so will the markets for the same good at other states in P.

j is given by

$$A^j(x, y^{-j}) = \left\{ y^j \in Y^j : y^j \text{ is } \mathcal{P}_{x,y^j,y^{-j}}\text{-measurable} \right\}.$$

These measurability requirements ensure that agents cannot take actions that vary with respect to information they are not privy to. But since the measurability requirements adjust as agents vary their x^i's or y^j's, agents do take account of how their information and hence the prices they face would change with their actions. Our measurability requirements and the fact that, even as p stays fixed, different information partitions will give agents the opportunity to buy different goods follow Radner (1968). But unlike Radner our individuals select information partitions via their actions. Price-taking therefore has no clear-cut meaning: since actions can change the set of purchasable goods, agents enjoy a measure of power over the prices they anticipate paying.

Definition 1 *An **equilibrium** is a (p, x, y) such that, for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$,*

- $x^i \in B^i(p, x^{-i}, y)$ *and* $U^i(x^i) \geq U^i(x^{i\prime})$ *for each* $x^{i\prime} \in B^i(p, x^{-i}, y)$,
- $y^j \in A^j(x, y^{-j})$ *and* $p \cdot y^j \geq p \cdot y^{j\prime}$ *for each* $y^{j\prime} \in A^j(x, y^{-j})$,
- $\sum_{i \in \mathcal{I}} x^i \leq \sum_{j \in \mathcal{J}} y^j + \sum_{i \in \mathcal{I}} e^i$, *with* $p_k(\sigma) = 0$ *if strict inequality obtains for good* k *and period* $\sigma = 1$ *or state* $\sigma = \omega$.

3 Inefficiency and its cure

We begin with two examples of inefficiency. The first returns to the firm in Example 2 that tested a convex technology in the course of building capital equipment. Although it is socially efficient for the firm to perform this test, if the firm expects a low price for output when the test reveals that productivity is high then its expected return on capital investment will be negative and hence the firm will not conduct the test. Outside of information discovery being an active decision, the example is entirely orthodox.

Example 2 Continued There is one firm, an arbitrary number of consumers, states and dates as previously specified, and probabilities $\pi(\omega_H) = \frac{2}{3}$ and $\pi(\omega_L) = \frac{1}{3}$. Endowments satisfy $\sum_{i\in\mathcal{I}} e^i(1) > 0$ and $e^i(\omega_L) = e^i(\omega_H) > 0$ for each i, which ensures that agents cannot deduce the state from the endowment profile. Since there is one good at every date and state, we omit the subscripts denoting goods.

As described in the original example, the firm can use a linear activity with coefficients $(-2, 3, 1)$ for the first-period, ω_H, and ω_L goods respectively and the state is revealed if and only if the firm chooses $y(1) < 0$. Each consumer i is risk-neutral and does not discount the future: $U^i(x^i(1), x^i(\omega_H), x^i(\omega_L)) = x^i(1) + \frac{2}{3}x^i(\omega_H) + \frac{1}{3}x^i(\omega_L)$.

With the prices $p(1) = 3$, $p(\omega_H) = 1$, $p(\omega_L) = 2$, the firm will choose $y = (0, 0, 0)$ rather than produce: an investment of two units of the first-period good would cost 6 and earn a return of 5. When the state is not revealed, the consumers are happy to consume their endowments since one unit of first-period consumption and one unit of second-period consumption (the latter delivered at both states) each costs 3.

This equilibrium is inefficient: a sacrifice of 2 units of the first-period consumption yields $\left(\frac{2}{3} \times 3\right) + \left(\frac{1}{3} \times 1\right) = 2\frac{1}{3}$ units of expected second-period consumption. ■

The equilibrium in Example 2 Continued is self-confirming: agents' price expectations lead the firm not to discover the information that could disconfirm those expectations. The price expectations consistent with a no-discovery equilibrium can vary widely. If the state is unknown, the price that consumers pay for second-period consumption is the sum $p(\omega_H) + p(\omega_L)$ and as long as $p(1) = p(\omega_H) + p(\omega_L)$ the decomposition of that sum into state-by-state prices is irrelevant: any decomposition that leads the firm to refrain from information discovery will be consistent with equilibrium. I examine self-confirmation in more detail in section 4. Inefficient equilibria can be built for Example 3 as well but that is less surprising given that the risky technology is abandoned at ω_L.

In the next Example, a consumer rather than a firm can discover information. Since a consumer can make a separate consumption decision at each state (unlike a firm whose input decision might fix its output across states) the prices that lead to inefficiency have to be chosen more carefully in some cases. The convexity that Example 2 enjoys is less plausible with a consumer as the discoverer and I therefore use a variant of Example 1: the consumer reveals information by paying a discrete cost. Since information is valuable only to the discoverer, no externality is thereby introduced.

Example 4 There are two consumers a and b, no firms, two states ω_H and ω_L, one good at each date and state, and probabilities $\pi(\omega_H) = \frac{1}{2}$ and $\pi(\omega_L) = \frac{1}{2}$. Both agents are risk-neutral. Consumer a has vNM utilities $2x^a(1) + 3x^a(\omega_H)$ at ω_H and $2x^a(1) + x^a(\omega_L)$ at ω_L and b has the vNM utility $x^b(1) + x^b(\omega)$ at each ω. Thus ω_H and ω_L indicate whether a's marginal utility of consumption is high or low. We omit subscripts denoting goods.

The only agent that can discover information is consumer a who can reveal the state by applying $c \geq 0$ of the first-period good to an information-discovery technology.[5] Let endowments satisfy $e^a(1) > c$ and $e^i(\omega_L) = e^i(\omega_H) > 0$ for each i.

With the prices $p(1) = 4$, $p(\omega_H) = 3$, $p(\omega_L) = 1$, consumer a has no incentive to discover the state. Whether or not the state is known, a dollar buys $\frac{1}{4}$ units of $x^a(1)$ and hence a utility gain of $\frac{1}{2} = 2 \times \frac{1}{4}$. If the state is not known a dollar buys a bundle of $\frac{1}{4}$ units each of $x^a(\omega_H)$ and $x^a(\omega_L)$ and thus a utility gain of $\frac{1}{2} = \left(\frac{1}{2} \times 3 \times \frac{1}{4}\right) + \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right)$, while if the state is known a dollar buys $\frac{1}{3}$ units of $x^a(\omega_H)$ for a utility gain of $\frac{1}{2} = \left(\frac{1}{2} \times 3 \times \frac{1}{3}\right)$ or 1 unit of $x^a(\omega_L)$ for a utility gain of $\frac{1}{2} = \left(\frac{1}{2} \times 1 \times 1\right)$. Thus, putting aside the discovery cost, consumer a experiences neither a gain nor a loss if the state is revealed. Consequently, if $c > 0$ the only equilibrium with the above prices is for consumer a not to discover.

For c sufficiently small, this equilibrium is inefficient: if the state were known, both a and b would be strictly better off if b transferred a unit of second-period consumption to a at state ω_H and received a unit from a at ω_L. ■

When the discovery cost c in Example 4 is 0, the price expectations that will lead consumer a not to discover the state must equate his marginal utility of income across states; otherwise a would discover the state in order to concentrate consumption in the state with the higher marginal utility. If $c > 0$ the price expectations consistent with nondiscovery enjoy greater latitude.

[5] When $c > 0$, this discovery can be modeled as the purchase of c units of an additional first-period good produced by a firm using c units of the original first-period good as its input. So $\mathcal{P}^a(x^a) = \{\{\omega_L\}, \{\omega_H\}\}$ if $x^a_2(1) \geq c$ and $\mathcal{P}^a(x^a) = \{\Omega\}$ otherwise. We can leave out further mention of these details by having the firm buy and sell at the same price, thus leaving no profits to distribute to its owners. When $c = 0$, discovery can be modeled as the purchase of a free good: see section 5.

3.1 The competitive price rule

When agents can discover information, competition need not lead to Pareto efficiency. As the above Examples show, there are price expectations that can lead to inefficiency even when externalities are absent, that is, no agent that can discover information valuable to others. We propose a rule for price expectations that will block this source of inefficiency.

Definition 2 *An equilibrium (p, x, y) satisfies the* ***competitive price rule*** *if, for each $P \in \mathcal{P}_{x,y}$, $\omega \in P$, and good k,*

$$p_k(\omega) = \pi(\omega|P) \sum_{\omega' \in P} p_k(\omega').$$

To understand the sense in which this rule is competitive, suppose at some equilibrium (p, x, y) that consumer i, by deviating to $x^{i\prime}$, reveals some event E contained in an event that is observable at (p, x, y). That is, suppose $E \subset P \in \mathcal{P}_{x,y}$ and $E \in \mathcal{P}_{x^{i\prime},x^{-i},,y}$. The competitive price rule leads to a price expectation for good k at E given by

$$\sum_{\omega \in E} p_k(\omega) = \sum_{\omega \in E} \pi(\omega|P) \sum_{\omega' \in P} p_k(\omega') = \pi(E|P) \sum_{\omega' \in P} p_k(\omega').$$

Thus the expected price of k at E is a *linear* function of the conditional probability of E given P: when an agent, by adjusting his or her discoveries, can change which event in P will be revealed the effect on expected prices will be proportional to the change in the probability. So, although agents can change the set of purchasable goods and therefore cannot be traditional price-takers, under the competitive price rule they face a constant expected price for good k per increment of likelihood. When E is an arbitrary event in Ω the expected price of k at E according to the price rule will equal

$$\sum_{\omega \in E} p_k(\omega) = \sum_{P \in \mathcal{P}_{x,y}} \sum_{\omega \in E} \pi(\omega|P) \sum_{\omega' \in P} p_k(\omega') = \sum_{P \in \mathcal{P}_{x,y}} \pi(E|P) \sum_{\omega' \in P} p_k(\omega').$$

The interpretation is similar. At any $P \in \mathcal{P}_{x,y}$ agents face a constant expected price for good k per increment of likelihood, and hence the effect on expected price of changing $\pi(E|P)$ by the same amount at every $P \in \mathcal{P}_{x,y}$ is again linear.

Consider a weaker version of the price rule that has been satisfied in all our market examples: require the expected price of good k at a subevent E of $P \in \mathcal{P}_{x,y}$ plus the expected price of k at the complementary event $P \backslash E$ to equal the price of k at P. Using obvious notation, $p_k(E) + p_k(P \backslash E) = p_k(P)$. This condition on expected prices will hold in equilibrium if agents can costlessly undertake a discovery that refines P into the subevents E and $P \backslash E$: if it were violated then either buyers or sellers of k would gain by making the discovery. See Theorem 4 in section 5 for a result in this vein. The competitive price rule imposes this weak requirement and then goes further: the prices of k at subevents must be proportional to their conditional likelihoods.

As in any model of price-taking, agents whose expectations conform to the competitive price rule do not behave strategically by plotting out the equilibrium consequences of their information discoveries. I will consider the relationship between the competitive price rule and the equilibria that obtain when information is uncovered in section 4.

The following Example illustrates the price rule and makes clear that it can be applied even when informational externalities make efficiency impossible.

Example 5 There are three firms, one consumer, and $\Omega = \{\omega_g, \omega_b\}$, with labor and output present at each date and state. The probabilities are $\pi(\omega_g) = \pi(\omega_b) = \frac{1}{2}$. Each firm at each period and state can use labor to produce output with the linear production function $f(l)$ subject to the capacity constraint $f(l) = \min[l, 1]$ for $l \geq 0$. In the first period, any of the firms can use a unit of output to test a production function h equal to $h(l) = \min[2l, 2]$ in state ω_g and $h(l) = 0$ in state ω_b.[6] If some firm tests the technology then agents face the partition $\{\{\omega_g\}, \{\omega_b\}\}$ while if no firm tests they face $\{\Omega\}$. In the latter case, the measurability restrictions on the firms imply that they cannot productively use h.

The consumer is risk-neutral and elastically supplies labor at a price equal to half a unit of output: letting the subscripts o and l indicate output and labor/leisure and omitting the

[6] Formally, this discovery can be modeled as the production by some firm j of an additional first-period good d with a technology that produces one unit of d only if at least one unit of first-period output is applied. So $\mathcal{P}^j(y^j) = \{\{\omega_g\}, \{\omega_b\}\}$ if $y^j_d(1) \geq 1$ and $\mathcal{P}^j(y^j) = \{\Omega\}$ otherwise. Implicitly good d has price 0 below.

superscript, the consumer's utility is

$$U(x) = x_o(1) + \frac{1}{2}x_l(1) + \sum_{\omega \in \{\omega_g, \omega_b\}} \frac{1}{2}\left(x_o(\omega) + \frac{1}{2}x_l(\omega)\right).$$

The consumer has the same endowment of labor $e_l > 3$ at each date and state.

In the only equilibrium that satisfies the competitive price rule (up to a normalization), no firm tests the uncertain technology, prices are $p_o(1) = 1$, $p_l(1) = \frac{1}{2}$, $p_o(\omega_g) = p_o(\omega_b) = \frac{1}{2}$, $p_l(\omega_g) = p_l(\omega_b) = \frac{1}{4}$, each firm j chooses $y^j(1) = y^j(\omega_g) = y^j(\omega_b) = (1, -1)$ (output has the first index). At each date and state, consumption is 3 plus any output endowment. Since $p_o(\omega_g) = p_o(\omega_b) = \frac{1}{2}(p_o(\omega_g) + p_o(\omega_b))$ and $p_l(\omega_g) = p_l(\omega_b) = \frac{1}{2}(p_l(\omega_g) + p_l(\omega_b))$, the price rule is satisfied. If a firm tests the technology it gains $p_o(\omega_g) \times 1 = \frac{1}{2}$ in revenue but loses the cost of 1. So the decision not to test is profit-maximizing. Due to the externality, that decision is socially inefficient: expected output would increase by $\frac{1}{2} \times 3$ while the output cost of testing is 1. ■

3.2 The first welfare theorem

To achieve efficiency, information about an agent's utility function or production set must be conditionally independent of the information that the remaining agents can discover, given one of the events that could be revealed in equilibrium. Otherwise, as in Example 5, there would be an externality: one agent's discovery would be valuable to another agent. Conditional independence, unlike unconditional independence, lets agents receive common information, e.g., they all read the same weather report or learn some fact that some agent always uncovers. But given the weather report conditional independence requires that one agent j's information discovery, for example about his own tastes, must not reveal any further information about another agent's tastes.

Let $\mathcal{Q}^{-l}$ for $l \in \mathcal{I} \cup \mathcal{J}$ equal the coarsest common refinement of the partitions in

$$\left\{\mathcal{P}^i(x^i) : i \in \mathcal{I}\backslash\{l\} \text{ and } x^i \geq 0\right\} \text{ and } \left\{\mathcal{P}^j(y^j) : j \in \mathcal{J}\backslash\{l\} \text{ and } y^j \in Y^j\right\}.$$

The $\mathcal{Q}^{-l}$ partition represents the information that all agents besides l can reveal.

For $i \in \mathcal{I}$, let $\mathcal{V}^i$ be the partition of Ω that demarcates the utilities consumer i might have: $W \in \mathcal{V}^i$ if and only if there exists a vNM utility v such that $W = \{\omega \in \Omega : u^i_\omega = v\}$. Similarly, for $j \in \mathcal{J}$, let $\mathcal{Y}^j$ be the partition of Ω that demarcates the single-state production sets firm j might have: $W \in \mathcal{Y}^j$ if and only there exists a $Y \subset \mathbb{R}^{L_1+L_2}$ such that $W = \{\omega \in \Omega : Y^j_\omega = Y\}$.

Definition 3 ***No externalities*** *is satisfied at an equilibrium* (p, x, y) *if the realization of an agent's utility function or production set is conditionally independent of any information other agents can discover, given the information all agents do discover in equilibrium: for all* $P \in \mathcal{P}_{x,y}$, $l \in \mathcal{I} \cup \mathcal{J}$, $W \in \mathcal{V}^l$ *for* $l \in \mathcal{I}$, $W \in \mathcal{Y}^l$ *for* $l \in \mathcal{J}$, *and* $Q \in \mathcal{Q}^{-l}$,

$$\pi\left(W \cap Q|P\right) = \pi(W|P)\pi(Q|P).$$[7]

Under no externalities, what all other agents besides l can learn above and beyond $P \in \mathcal{P}_{x,y}$ provides no information about l's utility if l is a consumer or about l's production possibilities if l is a firm. But the event P itself can reveal information about agents' tastes, e.g., that rain has made some agents like umbrellas.

Examples 2 Continued and 4 satisfy no externalities, in each case because the only agent with a state-dependent utility function or production set is also the only agent that can discover information. Example 3 when completed with non-discovering consumers will also satisfy no externalities.

That efficiency requires an absence of information externalities is no surprise; the notable fact will be that ruling them out is not enough.

An allocation $x \in \mathbb{R}^{IL}_+$ is *feasible* if x^i is $\mathcal{P}_{x,y}$ measurable for each $i \in \mathcal{I}$ and there exists a $y \in \mathbb{R}^{JL}$ such that $y^j \in A^j(x, y^{-j})$ for each $j \in \mathcal{J}$ and $\sum_{i\in\mathcal{I}} x^i \leq \sum_{j\in\mathcal{J}} y^j + \sum_{i\in\mathcal{I}} e^i$. An equilibrium (p, x, y) is *Pareto efficient* if there does not exist a feasible allocation x' such $U^i(x^{i\prime}) \geq U(x^i)$ for all $i \in \mathcal{I}$ and with strict inequality for some i.

[7] When footnote 3 is applied, we assume that Definition 3 additionally requires that $W \in \mathcal{Y}^l$ and $W \cap Q \cap P \neq \varnothing$ imply $W' \cap Q \cap P \neq \varnothing$ for any $W' \in \mathcal{Y}^l$ such that $W' \cap P \neq \varnothing$. This requirement extends conditional independence to 0-probability events: even if $\pi(W|P) = 0$ or $\pi(Q|P) = 0$, for $W \in \mathcal{Y}^l$ and $Q \in \mathcal{Q}^{-l}$ that intersect P, a switch of the single-state production set from W to W' still defines a non-null event in P (though with probability 0). Since this additional assumption applies only to firms, not consumers, it is irrelevant in an exchange economy.

Theorem 1 *If an equilibrium satisfies no externalities and the competitive price rule then it is Pareto efficient.*

Thus for Example 2 Continued, the completion of Example 3, and Example 4, equilibria that satisfy the competitive price rule are efficient: the market can induce agents to undertake the optimal information discoveries. In Example 2 Continued, for instance, the firm can earn enough when the technology it is testing has high productivity to make up for its losses when productivity is low.

Efficient information discovery is more counterintuitive when there is a choice between riskier and safer technologies: since risky technologies with poor realizations can be abandoned in favor of alternative methods, as in Example 3, it can be efficient to channel resources to and uncover information about the technologies with the riskiest distribution of productivities (see Mandler (2017)). Theorem 1 says that those efficient decisions will occur in any equilibrium that satisfies the competitive price rule even though the firms that operate the alternative methods collect part of the gain. Experimentation with risky technologies does not require a compensating reward of market power as Schumpeterians argue; competitive markets can induce firms to undertake the efficient experiments.

The proof constructs an artificial economy where agents have to satisfy only those measurability requirements implied by their own information discoveries but are free to let their actions vary with respect to what others can discover – in effect an Arrow-Debreu economy where the state is revealed at date 2 and agents have unusual consumption and production sets. I show that an equilibrium of the true economy is an equilibrium of the artificial economy. If there were a utility gain to a consumer i in the artificial economy from letting x^i vary with respect to information that could be uncovered by other agents then by no externalities and the concavity of i's vNM utilities an 'average' of that action that would also be a gain for i in the true economy, and the competitive price rule implies that the averaged action would be affordable. Similarly, if an increase in profits were possible for a firm j in the artificial economy by letting y^j vary with respect to information that could be uncovered by other agents then due to no externalities j could, in the true economy, choose whichever action from among these new possibilities yields the greatest profits, and the competitive price rule implies that this choice would also lead to an increase in profits in the true economy. The presence of an advantageous deviation in the artificial economy for either a consumer or a firm thus implies that the true economy could not have been in equilibrium. Since the standard first welfare theorem implies that the equilibrium allocation x is Pareto efficient in the artificial economy and since the set of feasible allocations of the artificial economy contains the feasible allocations of the true economy, x must be Pareto efficient in the true economy as well.[8] Proofs are in Appendix B.

3.3 Why are common priors and risk aversion required?

The assumptions needed for the first welfare theorem are famously weak. The Pareto efficiency of competitive equilibria requires only that preferences be transitive and locally nonsatiated. Weak Pareto efficiency – the absence of an allocation that gives a strict improvement to every consumer – requires no assumptions at all.

Several assumptions in contrast underlie Theorem 1. No externalities and the competitive price rule address points that do not arise in the standard general equilibrium model, namely actions that reveal information and the expected prices for goods that are not marketed in equilibrium. The need for no externalities is plain: if a firm can discover the productivity of its technology or if a consumer can discover his tastes only when another agent pays the costs then inefficiency can result. The competitive price rule is part of the definition of equilibrium rather than an assumption on primitives and will be the focus of sections 4 and 5.

Theorem 1 implicitly requires the concavity of the vNM utilities and that agents share a common probability π. These conditions are unnecessary for the standard first welfare theorem and their presence is surprising at first glance.

A consumer with a nonconcave vNM utility can prefer a lottery of consumption bundles over receiving the lottery's expected value with certainty, which can lead potentially to a mutually advantageous trade (bet) with a risk-neutral agent. But the feasibility of that trade can depend on a third party undertaking an information discovery: without the discovery

[8] An equilibrium of the true economy will not generally be an equilibrium of a different artificial economy that omits all measurability requirements on agents: a consumer i that is not measurability restricted could want to take advantage of information about his own utility but not be willing to pay to reveal that information.

the two agents may not be able to buy and sell state-contingent goods and achieve the Pareto improvement. Example 7 in Appendix A illustrates the inefficiency.

Similarly, if two consumers hold different prior probabilities then each can gain from a bet between the two. But they may not be able to make such a bet if the lottery that allows them to do so is conducted only when some other agent undertakes an information discovery. See Example 8 in Appendix A.

These examples challenge the accepted view that the efficiency of competitive equilibria rests on weak assumptions. That weakness requires an implicit assumption that information is revealed automatically: every coin must be tossed.

Since consumers can have state-dependent utilities, any model where consumers have diverse probabilities is behaviorally identical to a model with common priors and utilities that are rescaled separately by state. This modification would however convert a model that satisfies no externalities – such as Example 8 – to one where it would be violated. If agent 1 tosses a coin that reveals no information about 2's utility but, in order to align 2's probabilities with 1's, we raise 2's marginal utilities at 'heads' then the coin will reveal utility information about 2. While the inefficiency that can accompany diverse probabilities can therefore be diagnosed as a type of externality, the story becomes convoluted: one agent's discovery might provide information to others only with respect to a hypothetical probability distribution that none of the agents holds.

Outside of no externalities, Theorem 1 imposes no extra assumptions on firms: the convexity of the Y^j is not needed and, as in the standard general equilibrium model, probabilities do not enter into firm decision-making.

4 Self-confirming equilibria

One rationale for the competitive price rule is that violations can lead to equilibria that are self-confirming: the hypothetical prices that agents assign to events that they do not learn in equilibrium would not be borne out if agents could trade all state-contingent goods (they are released from their measurability requirements). These misguided price expectations can in turn validate the equilibrium decision not to discover information. We first return to our earlier examples of inefficiency to underscore the mismatch between the prices agents assign to events and the prices that would rule if markets for all goods were open.

Examples 2 Continued and 4 (revisited) In the equilibrium in Example 2 Continued, the price ratio $\frac{p(\omega_H)}{p(\omega_L)} = \frac{1}{2}$ that agents assign to the goods that appear at the high and low states does not coincide with the ratio given by the competitive price rule, namely $\frac{\pi(\omega_H)}{\pi(\omega_L)} = \frac{2}{1}$. If we could force the state to be revealed – say by requiring the firm to use ε units of the first-period good as an input – the market-determined price ratio for the high and low state goods will equal the ratio of the consumer's marginal utilities, $\frac{2}{1}$, in accord with the price rule. Although the prices that agents assign to states do not match the prices that would rule if revelation of the state were forced, the fact that the state is not discovered in equilibrium allows these beliefs to be sustained.

In Example 4, suppose consumer 1 is forced to pay the ε revelation cost that will reveal the state. If consumer a is small relative to b (specifically if $2e^a(1) + e^a(\omega_L) \leq e^b(\omega_H)$) then the ratio of equilibrium prices $\frac{p(\omega_H)}{p(\omega_L)}$ will be determined by consumer b's ratio of high-state to low-state marginal utility, $\frac{1}{1}$, which is also the ratio given by the competitive price rule.[9] In contrast, agents assign a price ratio of $\frac{3}{1}$ to the high-state and low-state goods in the equilibrium in Example 4 where the state is not revealed. ■

The revisited examples are unusual in that the market prices that obtain if revelation of the state is forced will *exactly* equal the price expectations prescribed by the competitive price rule. This precision is the exception since in most cases knowledge of the state will change demand and affect prices. In Example 4, for instance, the variation by state of a's demands will change equilibrium prices when revelation of the state is forced and a fails to be small relative to b.

We cannot eliminate the combination of self-confirmation and inefficiency simply by re-

[9] To find the equilibrium when the state is revealed, set $p(1) = 1$. Since for both a and b the ratio of the marginal utility of first-period consumption to state-invariant second-period consumption equals 1, $p(\omega_H) + p(\omega_L) = p(1) = 1$ must hold in equilibrium. If $p(\omega_H) \leq \frac{1}{2}$ then a consumes only $x^a(\omega_H)$ and thus has an excess demand $z^a(\omega_H)$ determined by $p(\omega_H)z^a(\omega_H) = e^a(1) + p(\omega_L)e^a(\omega_L)$. If $p(\omega_H) < \frac{1}{2}$ then $x^b(\omega_L) = 0$ but $x^a(\omega_L) + x^b(\omega_L) = 0$ cannot occur in equilibrium. If $p(\omega_H) > \frac{1}{2}$ then $x^b(\omega_H) = 0$ and $z^a(\omega_H) \leq \frac{1}{p(\omega_H)}e^a(1) + \frac{p(\omega_L)}{p(\omega_H)}e^a(\omega_L) < 2e^a(1) + e^a(\omega_L)$. The assumption that $2e^a(1) + e^a(\omega_L) \leq e^b(\omega_H)$ thus implies $z^a(\omega_H) < e^b(\omega_H)$ and hence $z^a(\omega_H) + z^b(\omega_H) < 0$, which also cannot occur in equilibrium. Hence $p(\omega_H) = \frac{1}{2}$.

quiring, in the spirit of subgame perfection, that agents' price expectations must coincide with the market equilibrium prices that would obtain as they adjust their information discoveries and demands. That requirement would give agents a strategic, noncompetitive power over prices and would allow inefficiency to return.

The customary response to the manipulation problem in general equilibrium theory is to argue that for 'small' agents deviations from equilibrium actions will have a small impact on prices and hence the incentive to deviate is also small. So even though price taking cannot hold literally, it can hold approximately. I follow a similar path by showing that the competitive price rule will be a good predictor of the prices that obtain when revelation of the state is forced if and only if the demand-and-supply effects of revelation are small. The assumption that the demand effect of information is small will serve an additional purpose as well, a full characterization of the price rule.

I first show that if the true state does not convey valuable information beyond what is learned in equilibrium then the predictions of the competitive price rule hold exactly if and only if there is no difference between an equilibrium as previously defined and the equilibrium that would occur if revelation of the state were forced. Then I show that, when the content of further information and the costs of discovery are small, the price rule holds approximately if and only if the difference between an equilibrium and a forced-revelation equilibrium is small.

To concentrate on essentials, we consider exchange economies ($J = 0$) and therefore omit all y's from our notation. Recall that $\mathcal{V}^i$ is the partition of Ω that defines the utilities consumer i might have: $V \in \mathcal{V}^i$ if and only if there exists a vNM utility v such that $V = \{\omega \in \Omega : u^i_\omega = v\}$.

Definition 4 *Information discoveries are* ***conclusive*** *if, for all $x \geq 0$, $P \in \mathcal{P}_x$, $i \in \mathcal{I}$, and $V \in \mathcal{V}^i$, $\pi(V|P)$ equals 0 or 1.*

When information discoveries are conclusive, a $P \in \mathcal{P}_x$ for a potential equilibrium (p, x) fully reveals the utilities agents have. When information discoveries are not conclusive, revelation of the exact $\omega \in P$ provides some consumer i with utility information and changes i's demand, which will typically invalidate the price predictions of the competitive price rule.

Forcing the revelation of the state translates formally into releasing agents from their measurability requirements. Any result seeking a correspondence between equilibrium prices and the competitive price rule has to drop those requirements: if delivery of a good k at ω is always bundled with delivery at ω', no market mechanism could patrol the relative price of k at these states.

Definition 5 *A **forced-revelation equilibrium** satisfies all of the requirements of an equilibrium except that, for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$, the measurability conditions in B^i and A^j are omitted.*[10]

When, for each $i \in \mathcal{I}$ and $\omega \in \Omega$, u^i_ω is differentiable and strictly increasing in each good, the economy is *differentiable*. An equilibrium (p, x) (in the sense of Definition 1) is *interior* if $x^i \gg 0$ for each $i \in \mathcal{I}$.

Theorem 2 *For a differentiable exchange economy in which information discoveries are conclusive, any interior equilibrium is a forced-revelation equilibrium if and only if the competitive price rule holds.*

So, when information discoveries are conclusive, a violation of the competitive price rule implies that an equilibrium cannot persist undisturbed if agents gain the right to trade for delivery at any state: the equilibrium would have to jump in response. The proof in this direction ('only if') argues that if a forced-revelation equilibrium does not satisfy the price rule then there must be a mismatch between the ratio of probabilities for two subevents of some $P \in \mathcal{P}_x$ and the ratio of the prices of goods delivered at those subevents; a shift of consumption to the underpriced subevent would then increase a consumer i's utility at P (i has only one utility function at P due to conclusiveness), violating the assumption of equilibrium. For 'if', a standard equilibrium that satisfies the price rule must also be a forced-revelation equilibrium: if a utility improvement were available to some i when we drop i's measurability requirements then an 'averaged' version of that improvement would also be an improvement and would satisfy the measurability requirements, a contradiction similar to

[10] We will later use forced-revelation equilibria when firms are present.

the one used in the proof of Theorem 1. The equivalence of equilibria and forced-revelation equilibria (given conclusiveness) thus provides a characterization of the price rule.

We extend Theorem 2 to information discoveries that are less than conclusive by considering when equilibria that nearly satisfy the competitive price rule will approximate forced-revelation equilibria. Both the costs of information discovery as well as information content will need to be small. No agent will pay for an information discovery if the revelation of the state is forced: so if costs were large then the demand effect of this information 'gift' would lead to lead to a sizable discrepancy between equilibrium demand and forced-revelation demand.

An information discovery undertaken by a consumer i will now incur a distinct expenditure of first-period goods, $c^i(\mathcal{P}^i) \in \mathbb{R}_+^{L_1} \times \{0\}^{SL_2}$, for a discovery $\mathcal{P}^i$ (a partition of Ω). The discoveries available to i are given by a menu of partitions $\mathcal{M}^i$. Given the partitions $\mathcal{P}^{-i}$ and letting $\mathcal{R}_{\mathcal{P}^i,\mathcal{P}^{-i}}$ denote the coarsest common refinement of $\mathcal{P}^i$ and the $\mathcal{P}^{-i}$, consumer i's budget set will now be

$$B^i(p, \mathcal{P}^{-i}) = \left\{ \left(x^i, \mathcal{P}^i\right) \in \mathbb{R}_+^L \times \mathcal{M}^i : x^i \text{ is } \mathcal{R}_{\mathcal{P}^i,\mathcal{P}^{-i}}\text{-measurable and } p \cdot x^i + p \cdot c^i(\mathcal{P}^i) \leq p \cdot e^i \right\}.$$

Accordingly an equilibrium is now a (p, x) such that there exists $\left(\mathcal{P}^1, ..., \mathcal{P}^I\right) \in \prod_{i \in \mathcal{I}} \mathcal{M}^i$ where, for every consumer i, $(x^i, \mathcal{P}^i) \in B^i(p, \mathcal{P}^{-i})$ and $U^i(x^i) \geq U^i(x^{i\prime})$ for each $(x^{i\prime}, \mathcal{P}^{i\prime}) \in B^i(p, \mathcal{P}^{-i})$ and where the market-clearing inequality in Definition 1 is replaced by $\sum_{i \in \mathcal{I}}(x^i + c^i(\mathcal{P}^i)) \leq \sum_{i \in \mathcal{I}} e^i$.

If, for each $i \in \mathcal{I}$, u^i is differentiably strictly concave and increasing then the economy is *smooth*. Fixing u^i, e^i, and $\mathcal{M}^i$ for each agent $i \in \mathcal{I}$, a *model* $\mathcal{E}$ consists of a cost $c^i(\mathcal{P})$ for each $i \in \mathcal{I}$ and $\mathcal{P} \in \mathcal{M}^i$ and a probability π. For a sequence of models $\mathcal{E}_n$, the *costs of information converge to* 0 if $c_n^i(\mathcal{P}) \to 0$ for each $i \in \mathcal{I}$ and $\mathcal{P} \in \mathcal{M}^i$ and the *inconclusiveness of information converges to* 0 if there is a π, called the *probability identified* by the sequence $\mathcal{E}_n$, that satisfies Definition 4 and $\pi_n \to \pi$.

If $\mathcal{E}_n$ is a sequence for a smooth economy such that the inconclusiveness of information converges to 0 then each consumer i has well-defined forced-revelation demands $x^i(p)$ when facing $p \gg 0$ and the probability identified by $\mathcal{E}_n$.[11] A (p, x) is *regular* if $D \sum_{i \in \mathcal{I}} x^i(p)$ has rank $L - 1$ and $(p, x) \gg 0$. The rank condition is generically satisfied in equilibrium.[12]

Given $\mathcal{E}_n$, the sequence (p_n, x_n) *satisfies the competitive price rule in the limit* if

$$p_{k,n}(\omega) - \pi_n(\omega | P_n(\omega)) \sum_{\omega' \in P_n(\omega)} p_{k,n}(\omega') \to 0$$

for each state ω and good k, where $P_n(\omega)$ indicates the cell of $\mathcal{P}_{x_n}$ that contains ω.

Theorem 3 *If for a sequence of smooth exchange economies the costs and inconclusiveness of information converge to* 0 *and the equilibria* (p_n, x_n) *converge to a regular point then there exist forced-revelation equilibria* (p_n^*, x_n^*) *such that the distance between* (p_n^*, x_n^*) *and* (p_n, x_n) *converges to* 0 *if and only if* (p_n, x_n) *satisfies the competitive price rule in the limit.*

So if the costs and inconclusiveness of information are small the equilibria that satisfy the competitive price rule would not be disturbed by much if markets for all state-contingent goods were to open. The competitive-price-rule equilibria thus do not display the self-confirmation pattern where the hypothetical prices of goods at events that will not be observed lie far from the values that would obtain if markets for goods at those events were open.

5 The price rule as a positive feature of equilibrium

Theorems 2 and 3 raise the question of whether the competitive price rule will be a necessary property of (standard, not forced-revelation) equilibria when information discoveries are conclusive and costless. A violation of the price rule means that the prices that agents assign to goods at some unobserved event E will be disproportionately low relative to the probability of E, which gives agents an incentive to discover information that can distinguish E from its complement. As we will now see, this argument is correct but it does not imply the price rule, which typically requires the coordinated discoveries of many agents.

[11] That is, $x^i(p)$ is the solution to $\max \sum_{\omega \in \Omega} \pi(\omega) u^i_\omega(x^i_\omega)$ s.t. $p \cdot x^i \leq p \cdot e^i$, $x^i \geq 0$.

[12] See, e.g., Mas-Colell (1985), chapter 8.

The positive result that a partial price rule must hold in equilibrium requires only a weaker form of conclusiveness. Since the information discoveries we consider might not reveal the exact state, we do not need to assume that agents know their utilities once they learn $P \in \mathcal{P}_{x,y}$. It is enough that any further information beyond P is independent of the information agents receive about their utilities from P.

The information discovery (partition) $\mathcal{D}$ is *conditionally conclusive for consumer i at the equilibrium* (p, x, y) if i's vNM utility and the information i can discover are conditionally independent given i's information in equilibrium: for all $D \in \mathcal{D}$, $P \in \mathcal{P}_{x,y}$, and $V \in \mathcal{V}^i$,

$$\pi(D \cap V|P) = \pi(D|P)\pi(V|P).$$

The information discovery $\mathcal{D}$ is *costless* for consumer i at equilibrium (p, x, y) if there is a $x^{i\prime} \geq 0$ such that (i) $\mathcal{D} = \mathcal{P}_{x^{i\prime},x^{-i},y}$, (ii) $\mathcal{D}$ refines $\mathcal{P}_{x,y}$, (iii) $p \cdot x^{i\prime} = p \cdot x^i$, and (iv) $U^i(x^{i\prime}) = U^i(x^i)$. So $\mathcal{D}$ is costless to a consumer if paying for the discovery does not lead to a utility loss and the consumer gains information relative to what he knows in equilibrium. As mentioned earlier, the general model of section 2 can accommodate costless information discoveries if we introduce goods that provide no utility with a price of 0. Define consumer i's *utility for k to be differentiable at* $(\overline{p}, \overline{x}, \overline{y})$ if, for each $\omega \in \Omega$, u^i_ω is differentiable and strictly increasing with respect to $x^i_k(\omega)$ and in addition $\overline{x}^i_k(\omega) > 0$.

Theorem 4 *Assume consumer's i utility for k is differentiable at the equilibrium (p, x, y). If $\mathcal{D}$ is conditionally conclusive and costless for consumer i at (p, x, y) then the competitive price rule obtains with respect to $\mathcal{D}$ and good k: for each $P \in \mathcal{P}_{x,y}$ and $D \in \mathcal{D}$ with $D \subset P$,*

$$\sum_{\omega \in D} p_k(\omega) = \pi(D|P) \sum_{\omega \in P} p_k(\omega).$$

The reasoning behind Theorem 4 is similar to the proof of Theorem 2: a violation of the price rule at some event D would induce i to reveal D and buy a little bit more of good k either at D or its complement in P at a disproportionately low price.

Theorem 4 is as far as we can go and even its strong assumptions do not imply the competitive price rule, as the following Example shows. To avoid the inefficiency that accompanies information discovery, we have to impose the price rule: it will not impose itself.

Example 6 Set $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with probabilities given by:

$\pi(\omega_1) = .1$	$\pi(\omega_2) = .4$
$\pi(\omega_3) = .4$	$\pi(\omega_4) = .1$

There are two consumers and no firms. Consumer 1 can costlessly discover the rows above, the cells $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$, while 2 can costlessly discover the columns, $\{\omega_1, \omega_3\}$ and $\{\omega_2, \omega_4\}$.

Assume there is one first-period good (in addition to a useless good that triggers information discoveries) and one second-period good per state. If each consumer i has the vNM utility $u^i_\omega(x^i(1), x^i(\omega)) = x^i(1) + x^i(\omega)$ at state ω then it is an equilibrium action for each agent to consume his endowment at the prices $p(1) = 1$ and $p(\omega) = .25$ for all $\omega \in \Omega$ and to not uncover the information he could discover: the sum of the prices at any event D an agent can discover, $\sum_{\omega \in D} p(\omega)$, equals $\frac{1}{2}$ as does $\pi(D) \sum_{\omega \in \Omega} p(\omega)$. The conclusion of Theorem 4 is therefore satisfied. The competitive price rule in contrast requires that $(p(\omega))_{\omega \in \Omega}$ be proportional to $(\pi(\omega))_{\omega \in \Omega}$. Notice that if at least one of the two agents does undertake discovery then the competitive price rule must hold. ■

The limited enforcement of the price rule given in Theorem 4 applies only to the discoveries of consumers, not firms. Firms do not hold probability judgements and do not maximize an expected value of any random variable: they therefore are not in a position to exploit expected prices that are not proportional to probabilities. See Example 2 Continued.

6 Conclusion

When agents can discover information, competitive equilibria can be inefficient. Ruling out information externalities will not by itself solve the problem; a competitive price rule that the prices assigned to goods at undiscovered events must be proportional to the probabilities of those events is needed and agents must be risk-averse and share a common prior. The latter two assumptions would normally be irrelevant for efficiency. From the glass-is-half-full perspective, the price rule extends the concept of price-taking to settings where agents can discover information: the rule holds constant the price of increases in the likelihood that a good is received. When the value of information is small, the price rule also avoids the self-confirmation phenomenon where prices would jump from their equilibrium values if markets for goods at unobserved events were to open.

Though several examples have illustrated the existence of equilibrium, this paper has focused on efficiency. To scrutinize the Schumpeterian view that optimal experimentation with new technologies leads inevitably to market power, it is the compatibility of efficiency and competition that is relevant. Moreover the existence issues that do arise can be addressed by known tools. For instance, one natural case of information acquisition occurs when agents must pay a discrete cost to make a discovery (Example 1). This nonconvexity can block existence since it introduces a discontinuity of demand: agents can respond to a small price change by discretely deciding to start or stop paying the discovery cost. The Starr (1969) approach to existence can tackle this problem: a continuum of agents can bridge the discontinuities by letting a fraction of agents (a continuous variable) pay the discrete discovery cost.

The existence of equilibria is easier to establish when information discoveries are made by firms. As Example 2 indicated, discovery can occur as a byproduct of trying out a technology in the first period or building capital equipment. The conditions that guarantee existence of an Arrow-Debreu equilibrium – convexity, continuity, and positive endowments – will then imply that a forced-revelation equilibrium exists.[13] As long as second-period production requires first-period inputs, one can then build a Definition 1 equilibrium from a forced-revelation equilibrium. We close with a brief sketch.

Recall that $\mathcal{Y}^j$ is the partition that indicates firm j's single-state production sets.

Definition 6 *If there exists a partition $\mathcal{P}$ of Ω and a $Y_1^j \subset \mathbb{R}^{L_1}$ for each $j \in \mathcal{J}$ such that*

1. *for all $j \in \mathcal{J}$, $y^j(1) \in Y_1^j$ implies $\mathcal{P}^j(y^j)$ is the coarsest common refinement of $\mathcal{Y}^j$ and $\mathcal{P}$ while $y^j(1) \notin Y_1^j$ implies $\mathcal{P}^j(y^j) = \mathcal{P}$ and $y^j(\omega) = 0$ for all $\omega \in \Omega$,*

[13] A forced-revelation equilibrium amounts to an Arrow-Debreu equilibrium in which the state is revealed independently of agent actions at the end of the first period.

2. *for all $i \in \mathcal{I}$ and $x^i \geq 0$, $\mathcal{P}^i(x^i) = \mathcal{P}$,*

3. *the coarsest common refinement of $\mathcal{P}$ and the $\mathcal{Y}^j$, $j \in \mathcal{J}$, equals the partition of singletons $\{\{\omega\} : \omega \in \Omega\}$,*

then ***first-period production is fully revealing****.*

The partition $\mathcal{P}$ above indicates the information revealed by nature independently of agent actions. Condition 1 says that each firm j can discover its second-period production possibilities and go on to produce in the second period if and only if it tries out production in the first period by taking an action in Y_1^j. Conditions 2 and 3 say that the information revealed by firms and nature form the whole of the economy's uncertainty.

Proposition 1 *If a forced-revelation equilibrium exists and production sets are convex, no externalities is satisfied, and first-period production is fully revealing then an equilibrium exists.*

The proof, which we omit, builds an equilibrium from a forced-revelation equilibrium by having each agent, at each cell P of the partition $\mathcal{P}_{x,y}$ that arises in a forced-revelation equilibrium (p, x, y), instead take the average of the actions the agent takes at P. As in the proof of Theorem 1, the new actions are feasible given the convexity assumption and deliver the same utility or profits.[14] It may be that some information is not revealed in equilibrium since some firm j may decide not to take an action in Y_1^j. But the assumption that first-period production is fully revealing ensures that this information loss does not hamper firm j: by Definition 6-1, when j fails to take an action in Y_1^j the missing information is useless.

A Appendix: risk aversion and common priors

Example 7 Let there be two consumers a and b and two states in an exchange economy with one consumption good at each date and state and probabilities given by $\pi(\omega_1) = \pi(\omega_2) = \frac{1}{2}$. Endowments for both consumers are constant across dates and states, $e^a = e^b = (1, 1, 1)$. Consumer a is risk-loving with vNM utility $x^a(1) + 2(x^a(\omega))^2$ at each state ω while

[14] In effect, this is a full-revelation equilibrium that is constant across sunspots. See Cass and Polemarchakis (1990) for a similar argument.

consumer b is risk-neutral with vNM utility $x^b(1) + 2x^b(\omega)$ at each ω. (We omit subscripts on consumptions and prices.) Since the vNM utilities do not vary by state, no externalities is vacuously satisfied.

Consumer b can reveal the state by spending $\varepsilon \geq 0$ of the first-period good on an information-discovery technology.[15] In the following equilibrium, b does not reveal the state – with a strict disincentive if $\varepsilon > 0$ – but it would be socially efficient to do so. The prices are $p = (p(1), p(\omega_1), p(\omega_2)) = (1, 1, 1)$, which satisfies the competitive price rule, and equilibrium consumption is

$$\begin{aligned}(x^a(1), x^a(\omega_1), x^a(\omega_2)) &= (0, \tfrac{3}{2}, \tfrac{3}{2}),\\ (x^b(1), x^b(\omega_1), x^b(\omega_2)) &= (2, \tfrac{1}{2}, \tfrac{1}{2}).\end{aligned}$$

But if $\varepsilon \geq 0$ and $\delta > 0$ are sufficiently small, the allocation

$$\begin{aligned}(x^a(1), x^a(\omega_1), x^a(\omega_2)) &= (1 - \varepsilon - \delta, 2, 0),\\ (x^b(1), x^b(\omega_1), x^b(\omega_2)) &= (1 + \delta, 0, 2),\end{aligned}$$

strictly Pareto dominates equilibrium consumption. ■

Example 8 Suppose there are three consumers a, b, and c, and two states in an exchange economy with one good at each date and state, with probabilities now varying by individual and given by $\pi^a(\omega_1) = \frac{1}{2}$, $\pi^b(\omega_1) = \frac{1}{4}$, $\pi^c(\omega_1) = \frac{3}{4}$. Each consumer i is endowed with one unit of the good at each date and state, $e^i = (1, 1, 1)$.

Consumer a by devoting $\varepsilon \geq 0$ of the first-period good to a production technology can reveal the state. We build an equilibrium where the consumer does not reveal the state but where it is socially efficient to do so. The price vector will be $p = (p(1), p(\omega_1), p(\omega_2)) = (1, \frac{1}{2}, \frac{1}{2})$ which satisfies the price rule when $\pi(\omega_1) = \frac{1}{2}$. (We again omit the subscripts on prices and consumptions.)

For consumers b and c, set the utility functions so that at p: if the state is revealed, the agent consumes $\frac{3}{2}$ at the state to which the agent assigns the probability $\frac{3}{4}$ and consumes $\frac{1}{2}$ at the state to which the agent assigns probability $\frac{1}{4}$ and if the state is not revealed, the agent consumes 1 unit of each good. That is, $(x^b(1), x^b(\omega_1), x^b(\omega_2)) = (1, \frac{1}{2}, \frac{3}{2})$ and $(x^c_1, x^c(\omega_1), x^c(\omega_2)) = (1, \frac{3}{2}, \frac{1}{2})$ if the state is revealed and $(x^b_1, x^b(\omega_1), x^b(\omega_2)) = (x^c_1, x^c(\omega_1), x^c(\omega_2)) = (1, 1, 1)$ if the state is not revealed. To satisfy these features, let $\widehat{u}$ be differentiable and strictly concave such that $\frac{1}{4}\widehat{u}'(\frac{1}{2}) = \frac{1}{2}\widehat{u}'(1) = \frac{3}{4}\widehat{u}'(\frac{3}{2})$ and assume that each consumer i (including $i = a$) has the expected utility

$$U^i(x^i) = \widehat{u}(x^i(1)) + \pi^i(\omega_1)\widehat{u}(x^i(\omega_1)) + \pi^i(\omega_2)\widehat{u}(x^i(\omega_2)).$$

As the vNM utilities do not vary by state, no externalities is satisfied regardless of which distribution π^i is used in Definition 3.

If there were no discovery cost, agent a would consume $(1, 1, 1)$ at p and thus enjoy the same expected utility level (2) whether or not the state is revealed. Thus a has no

[15] As in Example 4, discovery must be modeled formally as the purchase of an additional first-period good produced by a firm that uses the original first-period good as its input. See footnote 5.

incentive to reveal the state and has a strict disincentive if $\varepsilon > 0$. But when b and c both consume $(1,1,1)$, the ratio of b's marginal utilities for consumption at ω_1 and ω_2, $\frac{\pi^b(\omega_1)}{\pi^b(\omega_2)} = \frac{1}{3}$, does not equal the same ratio for c, $\frac{\pi^c(\omega_1)}{\pi^c(\omega_2)} = \frac{3}{1}$. Consequently if the state were revealed a reallocation between b and c of their consumption at ω_1 and ω_2 can increase both agents' expected utility. Hence, when ε is sufficiently small, consumer a could be paid to reveal the state while preserving the utility increase for b and c.

Observe that the price rule is satisfied both with respect to π^a, the belief of the consumer who makes the decision to reveal the state, and the average beliefs of the consumers, $\frac{1}{3}\left(\pi^a + \pi^b + \pi^c\right)$. ■

B Appendix: proofs

Throughout Appendix B, we use the notation $x^i_\omega = (x^i(1), x^i(\omega))$ and $y^j_\omega = (y^j(1), y^j(\omega))$.

Proof of Theorem 1. Let $(\overline{p}, \overline{x}, \overline{y})$ be an equilibrium that satisfies the price rule. We show that $(\overline{p}, \overline{x}, \overline{y})$ is a competitive equilibrium of the following model $\widehat{\mathcal{E}}$. Let each $i \in \mathcal{I}$ have the consumption set

$$\widehat{X}^i = \left\{x^i \in \mathbb{R}^L_+ : x^i \text{ is measurable w.r.t. the coarsest common refinement of } \mathcal{P}^i(x^i) \text{ and } \mathcal{Q}^{-i}\right\},$$

and, given $p \in \mathbb{R}^L_+$ and $y \in \mathbb{R}^{LJ}_+$, the budget set $\widehat{B}^i(p,y) = \{x^i \in \widehat{X}^i : p \cdot x^i \leq p \cdot e^i + \sum_j \theta^{ij} p \cdot y^j\}$. Let each $j \in \mathcal{J}$ have the production set

$$\widehat{Y}^j = \left\{y^j \in Y^j : y^j \text{ is measurable w.r.t. the coarsest common refinement of } \mathcal{P}^j(y^j) \text{ and } \mathcal{Q}^{-j}\right\}.$$

Each $i \in \mathcal{I}$ must choose a $x^i \in \widehat{B}^i(p,y)$ but can violate the further measurability requirements in $B^i(p, x^{-i}, y)$ and each $j \in \mathcal{J}$ must choose a $y^j \in \widehat{Y}^j$ but can violate the further measurability requirements in $A^j(x, y^{-j})$. Otherwise the definition of equilibrium remains unchanged. Given that preferences are utility-representable and hence transitive and our local nonsatiation assumption, the first welfare theorem applies to $\widehat{\mathcal{E}}$ and $\overline{x}$ is therefore Pareto efficient among allocations in

$$\widehat{F} = \left\{x \in \prod_{i\in\mathcal{I}} \widehat{X}^i : \text{ there exists } y \in \prod_{j\in\mathcal{J}} \widehat{Y}^j \text{ such that } \sum_{i\in\mathcal{I}} \widehat{x}^i \leq \sum_{j\in\mathcal{J}} y^j + \sum_{i\in\mathcal{I}} e^i\right\}.$$

Since $\widehat{F}$ contains the set of feasible allocations for the original model, Pareto efficiency in $\widehat{F}$ implies that $\overline{x}$ is Pareto efficient in the original model.

To conclude that $(\overline{p}, \overline{x}, \overline{y})$ is a competitive equilibrium of $\widehat{\mathcal{E}}$, suppose to the contrary that there is either (1) a $i \in \mathcal{I}$ and $\widehat{x}^i \in \widehat{X}^i$ such that $\overline{p} \cdot \widehat{x}^i \leq \overline{p} \cdot e^i + \sum_{j\in\mathcal{J}} \theta^{ij} \overline{p} \cdot \overline{y}^j$ and $U^i\left(\widehat{x}^i\right) > U^i\left(\overline{x}^i\right)$ (an affordable but possibly non-$\mathcal{P}_{\overline{x},\overline{y}}$-measurable $\widehat{x}^i$ that increases i's utility relative to $\overline{x}^i$), or (2) a $j \in \mathcal{J}$ and $\widehat{y}^j \in \widehat{Y}^j$ such that $\overline{p} \cdot \widehat{y}^j > \overline{p} \cdot \overline{y}^j$ (a feasible but possibly non-$\mathcal{P}_{\overline{x},\overline{y}}$-measurable $\widehat{y}^j$ that increases j's profits relative to $\overline{y}^j$).

For (1), define the $\mathcal{P}_{\widehat{x}^i, \overline{x}^{-i}, \overline{y}}$-measurable $\widetilde{x}^i$ by setting $\widetilde{x}^i(1) = \widehat{x}^i(1)$ and, for each $P \in \mathcal{P}_{\widehat{x}^i, \overline{x}^{-i}, \overline{y}}$ and $\omega \in P$, $\widetilde{x}^i(\omega) = \sum_{\omega' \in P} \pi(\omega'|P) \widehat{x}^i(\omega')$. Since $\widetilde{x}^i$ is $\mathcal{P}_{\widehat{x}^i, \overline{x}^{-i}, \overline{y}}$-measurable, $\widetilde{x}^i \in \widehat{X}^i$.

We show that $U^i(\widetilde{x}^i) \geq U^i(\widehat{x}^i)$ and $\widetilde{x}^i \in B^i(p, \overline{x}^{-i}, \overline{y})$, thus contradicting the fact that $(\overline{p}, \overline{x}, \overline{y})$ is a competitive equilibrium.

Fix $P \in \mathcal{P}_{\widehat{x}^i, \overline{x}^{-i}, \overline{y}}$ and $V^* \in \mathcal{V}^i$ such that $\pi(V^* \cap P) > 0$. For any $Q \in \mathcal{Q}^{-i}$, the assumption that $\widehat{x}^i \in \widehat{X}^i$ implies there is a $\widehat{x}^i_Q \in \mathbb{R}^{L_1+L_2}$ such that $\widehat{x}^i_\omega = \widehat{x}^i_Q$ for all $\omega \in Q \cap P$. Hence, for any $\omega \in P$,

$$\widetilde{x}^i(\omega) = \sum\nolimits_{V \in \mathcal{V}^i} \pi(V|P) \sum\nolimits_{Q \in \mathcal{Q}^{-i}} \pi(Q|V \cap P)\widehat{x}^i_Q.$$

For all $V \in \mathcal{V}^i$ with $\pi(V \cap P) > 0$ and all $Q \in \mathcal{Q}^{-i}$, no externalities implies $\pi(Q|V \cap P) = \pi(Q|V^* \cap P)$ and thus $\sum_{V \in \mathcal{V}^i} \pi(V|P) \sum_{Q \in \mathcal{Q}^{-i}} \pi(Q|V \cap P)\widehat{x}^i_Q = \sum_{Q \in \mathcal{Q}^{-i}} \pi(Q|V^* \cap P)\widehat{x}^i_Q$. For $x^i \in \mathbb{R}^L_+$, let $v^i(x^i)$ denote the random variable equal to $u^i_\omega(x^i_\omega)$ at $\omega \in \Omega$ and let $u^i_{V^*}$ denote the vNM utility u^i_ω where ω is any state in V^*. Then

$$E[v^i(\widetilde{x}^i)|V^* \cap P] = u^i_{V^*}\left(\sum\nolimits_{Q \in \mathcal{Q}^{-i}} \pi(Q|V^* \cap P)\widehat{x}^i_Q\right).$$

Since

$$E[v^i(\widehat{x}^i)|V^* \cap P] = \sum\nolimits_{Q \in \mathcal{Q}^{-i}} \pi(Q|V^* \cap P) u^i_{V^*}(\widehat{x}^i_Q),$$

the concavity of the u^i_ω and Jensen's inequality imply $E[v^i(\widetilde{x}^i)|V^* \cap P] \geq E[v^i(\widehat{x}^i)|V^* \cap P]$. Consequently $U^i(\widetilde{x}^i) \geq U^i(\widehat{x}^i)$.

To confirm that $\widetilde{x}^i \in B^i(p, \overline{x}^{-i}, \overline{y})$, fix $P \in \mathcal{P}_{\widehat{x}^i, \overline{x}^{-i}, \overline{y}}$ and a good k. The competitive price rule then implies the third equality below:

$$\sum_{\omega \in P} \overline{p}_k(\omega)\widetilde{x}^i_k(\omega) = \sum_{\omega \in P} \overline{p}_k(\omega) \sum_{\omega' \in P} \pi(\omega'|P)\widehat{x}^i_k(\omega') = \sum_{\omega' \in P} \pi(\omega'|P) \sum_{\omega \in P} \overline{p}_k(\omega)\widehat{x}^i_k(\omega') = \sum_{\omega' \in P} \overline{p}_k(\omega')\widehat{x}^i_k(\omega').$$

Hence $p \cdot \widetilde{x}^i = p \cdot \widehat{x}^i$. Since $\overline{p} \cdot \widehat{x}^i \leq \overline{p} \cdot e^i + \sum_{j \in \mathcal{J}} \theta^{ij} \overline{p} \cdot \overline{y}^j$, $\widetilde{x}^i \in B^i(p, \overline{x}^{-i}, \overline{y})$. Combined with $U^i(\widetilde{x}^i) \geq U^i(\widehat{x}^i) > U^i(\overline{x}^i)$, this contradicts $\overline{x}^i$ being an equilibrium choice for i.

For (2), the assumption that $\widehat{y}^j \in \widehat{Y}^j$ implies that for each $P \in \mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$ and $Q \in \mathcal{Q}^{-j}$ there is a $\widehat{y}^j_{P \cap Q} \in \mathbb{R}^{L_2}$ such that $\widehat{y}^j(\omega) = \widehat{y}^j_{P \cap Q}$ for $\omega \in P \cap Q$. For each $P \in \mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$, set Q_P to be an element of $\arg\max_Q \sum_{\omega \in P} \overline{p}(\omega) \cdot \widehat{y}^j_{P \cap Q}$ s.t. $Q \cap P \neq \varnothing$, where $\overline{p}(\omega) = \left(\overline{p}_1(\omega), ..., \overline{p}_{L_2}(\omega)\right)$, and define $\widetilde{y}^j \in \mathbb{R}^L$ by $\widetilde{y}^j(1) = \widehat{y}^j(1)$ and $\widetilde{y}^j(\omega) = \widehat{y}^j_{P(\omega) \cap Q_P}$ for each $\omega \in P$, where $P(\omega)$ denotes the $P \in \mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$ such that $\omega \in P$.

We first show that $\widetilde{y}^j \in A^j(x, y^{-j})$. Since $\widetilde{y}^j$ is $\mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$-measurable, we need to show that $\widetilde{y}^j \in Y^j$. To that end, enumerate the cells in the coarsest common refinement of $\mathcal{Y}^j$ and $\mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$ as $\mathcal{R} = \{R_1, ..., R_n\}$ and let $P(R_m) \in \mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$ satisfy $P(R_m) \supset R_m$. Beginning with $\widehat{y}^j \in Y^j$, change $\widehat{y}^j(\omega)$ to $\widetilde{y}^j(\omega)$ for ω in the sequence of cells $R_1, ..., R_n$ until arriving at $\widetilde{y}^j$. Formally, for $m \in \{0, ..., n\}$, define $y^j(m) \in \mathbb{R}^L$ by

$$y^j_\omega(m) = \begin{cases} \widetilde{y}^j_\omega & \text{if } \omega \in R_l \text{ and } 1 \leq l \leq m, \\ \widehat{y}^j_\omega & \text{otherwise.} \end{cases}$$

To argue by induction that $y^j(n) = \widetilde{y}^j \in Y^j$, note that $y^j(0) \in Y^j$ and suppose $y^j(m-1) \in Y^j$ for some $m \in \{1, ..., n\}$. Recall that in footnote 3 we impose the assumption (weaker than the assumption in the text) that if $Y^j_{\omega'} = Y^j_{\omega''}$ and $y^j \in Y^j$ then $y^{j\prime} \in Y^j$ for the $y^{j\prime}$ defined by $y^{j\prime}_{\omega} = y^j_{\omega}$ for $\omega \in \Omega \backslash \{\omega''\}$ and $y^{j\prime}_{\omega''} = y^j_{\omega'}$. See also footnote 7 for the amendment to no externalities that holds when footnote 3 applies. Since (i) $Y^j_{\omega'} = Y^j_{\omega''}$ for all $\omega', \omega'' \in R_m$, (ii) $y^j(m-1) \in Y^j$, and (iii) $Q_{P(R_m)} \cap P(R_m) \neq \varnothing$ and no externalities imply that $Q_{P(R_m)} \cap R_m \neq \varnothing$ and hence there is a $\omega \in R_m$ such that $y^j_{\omega}(m-1) = (\widehat{y}^j(1), \widehat{y}^j_{P(R_m)\cap Q_{P(R_m)}})$, we conclude that $y^j(m) \in Y^j$.

To finish, we show that $\overline{p} \cdot \widetilde{y}^j \geq \overline{p} \cdot \widehat{y}^j$ and therefore $\overline{p} \cdot \widetilde{y}^j > \overline{p} \cdot \overline{y}^j$ which contradicts $\overline{y}^j$ being an equilibrium choice for j. Fix $P \in \mathcal{P}_{\overline{x}, \widehat{y}^j, \overline{y}^{-j}}$. For $y^j \in \mathbb{R}^L$, define $\Pi(y^j) = \sum_{\omega \in P} \overline{p}(\omega) \cdot y^j(\omega)$ (the profits earned by y^j at P) and let $Q(\omega)$ denotes the $Q \in \mathcal{Q}^{-j}$ such that $\omega \in Q$. Then $\Pi(\widetilde{y}^j) = \sum_{\omega \in P} \overline{p}(\omega) \cdot \widehat{y}^j_{P \cap Q_P}$ while the competitive price rule implies

$$\Pi(\widehat{y}^j) = \sum_{\omega \in P} \overline{p}(\omega) \cdot \widehat{y}^j_{P \cap Q(\omega)} = \sum_{\omega \in P} \left(\pi(\omega|P) \sum_{\omega' \in P} \overline{p}(\omega') \right) \cdot \widehat{y}^j_{P \cap Q(\omega)} = \sum_{\omega \in P} \pi(\omega|P) \left(\sum_{\omega' \in P} \overline{p}(\omega') \cdot \widehat{y}^j_{P \cap Q(\omega)} \right).$$

Since the definition of Q_P implies $\sum_{\omega \in P} \overline{p}(\omega) \cdot \widehat{y}^j_{P \cap Q_P} \geq \sum_{\omega \in P} \overline{p}(\omega) \cdot \widehat{y}^j_{P \cap Q(\omega)}$, we have $\Pi(\widetilde{y}^j) \geq \Pi(\widehat{y}^j)$ and hence $\overline{p} \cdot \widetilde{y}^j \geq \overline{p} \cdot \widehat{y}^j > \overline{p} \cdot \overline{y}^j$. ■

Proof of Theorem 2. Suppose (p, x) is a forced-revelation equilibrium and the competitive price rule fails: there exist $P \in \mathcal{P}_x$, $\omega' \in P$, and a good k such that $p_k(\omega') \neq \pi(\omega'|P) \sum_{\omega \in P} p_k(\omega)$.

Observation 1: for any $\widehat{P} \subset \Omega$, $\pi(\widehat{P}) = 0$ iff $\sum_{\omega \in \widehat{P}} p_{k'}(\omega) = 0$ for each good k'. Proof: if $\pi(\widehat{P}) = 0$ then $x^i \gg 0$ implies $p_{k'}(\omega) = 0$ for each $\omega \in \widehat{P}$, while if $\sum_{\omega \in \widehat{P}} p_{k'}(\omega) = 0$ then the increasingness of the utilities implies $\pi(\widehat{P}) = 0$.

Observation 2: $\pi(P) > 0$. Proof: if $\pi(P) = 0$ then, by Observation 1, $\sum_{\omega \in P} p_k(\omega) = 0$ and $p_k(\omega') = 0$ which imply $p_k(\omega') = \pi(\omega'|P) \sum_{\omega \in P} p_k(\omega)$, a contradiction.

To conclude that there is a $P' \subset P$ such that $\sum_{\omega \in P'} p_k(\omega) < \pi(P'|P) \sum_{\omega \in P} p_k(\omega)$, note that $\{\omega'\}$ can serve as P' if $p_k(\omega') < \pi(\omega'|P) \sum_{\omega \in P} p_k(\omega)$. If $p_k(\omega') > \pi(\omega'|P) \sum_{\omega \in P} p_k(\omega)$ then $\sum_{\omega \in P \backslash \{\omega'\}} p_k(\omega) < \pi(P \backslash \{\omega'\}|P) \sum_{\omega \in P} p_k(\omega)$ and so $P \backslash \{\omega'\}$ can serve as P'.

Next we show that $\sum_{\omega \in P \backslash P'} p_k(\omega) > 0$. If instead $\sum_{\omega \in P \backslash P'} p_k(\omega) = 0$ then, by Observation 1, $\pi(P'|P) = 1$ and hence $\sum_{\omega \in P'} p_k(\omega) = \pi(P'|P) \sum_{\omega \in P} p_k(\omega)$, a contradiction. This fact permits the following definitions.

For $\varepsilon > 0$ and $\omega \in \Omega$, define $\widetilde{x}^i(\varepsilon, \omega) \in \mathbb{R}^{L_2}$ by setting, for each good k',

$$\widetilde{x}^i_{k'}(\varepsilon, \omega) = \begin{cases} x^i_k(\omega) + \varepsilon & \text{if } k' = k \text{ and } \omega \in P', \\ x^i_k(\omega) - \frac{\sum_{\widehat{\omega} \in P'} p_k(\widehat{\omega})}{\sum_{\widehat{\omega} \in P \backslash P'} p_k(\widehat{\omega})} \varepsilon & \text{if } k' = k \text{ and } \omega \in P \backslash P', \\ x^i_{k'}(\omega) & \text{otherwise,} \end{cases}$$

and also $\widetilde{x}^i_{\omega}(\varepsilon) = (x^i(1), \widetilde{x}^i(\varepsilon, \omega))$ and $\widetilde{x}^i(\varepsilon) = \left(x^i(1), (\widetilde{x}^i(\varepsilon, \omega))_{\omega \in \Omega}\right)$.

Since $\sum_{\omega \in P'} p_k(\omega) < \pi(P'|P) \sum_{\omega \in P} p_k(\omega)$ implies $\sum_{\omega \in P \backslash P'} p_k(\omega) > \pi(P \backslash P'|P) \sum_{\omega \in P} p_k(\omega)$ and therefore $\pi(P') > \pi(P \backslash P') \frac{\sum_{\omega \in P'} p_k(\omega)}{\sum_{\omega \in P \backslash P'} p_k(\omega)}$, we have

$$E\left[\widetilde{x}^i_k(\varepsilon, \cdot)\right] - E\left[x^i_k(\cdot)\right] = \left(\pi(P') - \pi(P \backslash P') \frac{\sum_{\omega \in P'} p_k(\omega)}{\sum_{\omega \in P \backslash P'} p_k(\omega)} \right) \varepsilon > 0.$$

Given our differentiability assumption, Arrow (1965) implies that, for any vNM utility u and all $\varepsilon > 0$ sufficiently small, $\sum_{\omega \in P} \pi(\omega|P) u(\widetilde{x}^i_\omega(\varepsilon)) > \sum_{\omega \in P} \pi(\omega|P) u(x^i_\omega)$.

For $x^i \in \mathbb{R}^L_+$, let $v^i(x^i)$ denote the random variable equal to $u^i_\omega(x^i_\omega)$ at ω; for $V \in \mathcal{V}^i$, let u^i_V denote the vNM utility u^i_ω where ω is any state in V. Since $\pi(V|P)$ equals either 0 or 1 for each $V \in \mathcal{V}^i$ there is one $V \in \mathcal{V}^i$ such that $\pi(V|P) = 1$, which we label V^*. Hence $E[v^i(\widetilde{x}^i(\varepsilon))|P] = \sum_{\omega \in P} \pi(\omega|P) u^i_{V^*}(\widetilde{x}^i_\omega(\varepsilon))$. Since $\sum_{\omega \in P} \pi(\omega|P) u^i_{V^*}(\widetilde{x}^i_\omega(\varepsilon)) > \sum_{\omega \in P} \pi(\omega|P) u^i_{V^*}(x^i_\omega)$, we conclude that

$$E[v^i(\widetilde{x}^i(\varepsilon))|P] > u^i_{V^*}\left(x^i_P\right) = E[v^i(x^i)|P].$$

Since, by Observation 2, $\pi(P) > 0$, $U^i\left(\widetilde{x}^i(\varepsilon)\right) > U^i(x^i)$ for all $\varepsilon > 0$ sufficiently small, and since

$$\sum_{\omega \in P} p_k(\omega) \widetilde{x}^i_k(\varepsilon, \omega) = \sum_{\omega \in P'} p_k(\omega) \left(x^i_k(\omega) + \varepsilon\right) + \sum_{\omega \in P \setminus P'} p_k(\omega) \left(x^i_k(\omega) - \frac{\sum_{\omega \in P'} p_k(\omega)}{\sum_{\omega \in P \setminus P'} p_k(\omega)} \varepsilon \right)$$
$$= \sum_{\omega \in P} p_k(\omega) x^i_k(\omega),$$

$\widetilde{x}^i(\varepsilon) \in B^i(p, x^{-i})$.

Since $\widetilde{x}^i(\varepsilon)$ for any ε sufficiently small is therefore a utility-increasing deviation, (p, x) could not be an equilibrium.

Conversely suppose the competitive price rule holds at the equilibrium (p, x) but (p, x) is not a forced-revelation equilibrium: there is a $i \in \mathcal{I}$ and $\widehat{x}^i \geq 0$ such that $U^i(\widehat{x}^i) > U^i(x^i)$ and $p \cdot \widehat{x}^i \leq p \cdot e^i$.

Fix $P \in \mathcal{P}_x$ such that $\pi(P) > 0$, and let V^* be the sole element of $\mathcal{V}^i$ such that $\pi(V^* \cap P) > 0$ and let $u^i_{V^*}$ denote the vNM utility u^i_ω where ω is any state in V^*. Following the proof of Theorem 1, define the $\mathcal{P}_x$-measurable $\widetilde{x}^i$ by setting $\widetilde{x}^i(1) = \widehat{x}^i(1)$ and, for each $P \in \mathcal{P}_x$ and $\omega \in P$, $\widetilde{x}^i(\omega) = \sum_{\omega' \in P} \pi(\omega'|P) \widehat{x}^i(\omega')$. Let $v^i(x^i)$ denote the random variable defined by $v^i(x^i)(\omega) = u^i_\omega(x^i_\omega)$. Since $E[v^i(\widetilde{x}^i)|P] = u^i_{V^*}\left(\sum_{\omega' \in P} \pi(\omega'|P) \widehat{x}^i(\omega')\right)$ and $E[v^i(\widehat{x}^i)|P] = \sum_{\omega' \in P} \pi(\omega'|P) u^i_{V^*}(\widehat{x}^i(\omega'))$, the concavity of the u^i_ω and Jensen's inequality imply $E[v^i(\widetilde{x}^i)|P] \geq E[v^i(\widehat{x}^i)|P]$ and consequently $U^i\left(\widetilde{x}^i\right) \geq U^i\left(\widehat{x}^i\right)$. Mention budget constraint. As in the proof of Theorem 1, $\sum_{\omega \in P} p_k(\omega) \widetilde{x}^i_k(\omega) \leq \sum_{\omega \in P} p_k(\omega) \widehat{x}^i_k(\omega)$ for each good k and hence $p \cdot \widetilde{x}^i \leq p \cdot e^i$. Thus (p, x) could not be an equilibrium. ■

Proof of Theorem 3. Let $(\overline{p}, \overline{x})$ denote the regular point to which the equilibria (p_n, x_n) converge, let $\overline{\pi}$ denote the probabilities to which π_n converges, and let $\left(\mathcal{P}^1_n, ..., \mathcal{P}^I_n\right) \in \prod_{j \in \mathcal{I}} \mathcal{M}^j$ be the partitions chosen at equilibrium (p_n, x_n). Given the finiteness of Ω, there must be a $\left(\mathcal{P}^1, ..., \mathcal{P}^I\right)$ and a subsequence of positive integers $\langle n' \rangle$ such that $\left(\mathcal{P}^1, ..., \mathcal{P}^I\right) = \left(\mathcal{P}^1_n, ..., \mathcal{P}^I_n\right)$ for all n' in the subsequence. Let $\mathcal{R}$ denote the coarsest common refinement of the $\mathcal{P}^j$.

Note that since $(\overline{p}, \overline{x}) \gg 0$ our smoothness assumption implies $\overline{\pi} \gg 0$.

If. Suppose (p_n, x_n) satisfies the competitive price rule in the limit. We begin by showing that $(\overline{p}, \overline{x})$ is a forced-revelation equilibrium when probabilities equal $\overline{\pi}$. Suppose to the contrary that there is a $i \in \mathcal{I}$ and $\widehat{x}^i \geq 0$ such that $\overline{p} \cdot \widehat{x}^i \leq \overline{p} \cdot e^i$ and $\sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\widehat{x}^i_\omega) > \sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\overline{x}^i_\omega)$. Define the $\mathcal{R}$-measurable $\widetilde{x}^i$ by setting $\widetilde{x}^i(1) =$

$\widehat{x}^i(1)$ and, for each $P \in \mathcal{R}$ and $\omega \in P$, $\widetilde{x}^i(\omega) = \sum_{\omega' \in P} \overline{\pi}(\omega'|P)\widehat{x}^i(\omega')$. Since $p_{k,n'}(\omega) - \pi_{n'}(\omega|P)\sum_{\omega' \in P} p_{k,n'}(\omega') \to 0$ as $n' \to \infty$ for each good k and $\overline{\pi} \gg 0$, $P \in \mathcal{R}$, and $\omega \in P$, we have $\overline{p}_k(\omega) = \overline{\pi}(\omega|P)\sum_{\omega' \in P} \overline{p}_k(\omega')$. As in the proofs of Theorem 1 and 2, $\overline{p} \cdot \widetilde{x}^i = \overline{p} \cdot \widehat{x}^i$ and

$$\sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\widetilde{x}^i_\omega) \geq \sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\widehat{x}^i_\omega) > \sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\overline{x}^i_\omega).$$

Next define, for each n' in the subsequence, the $\mathcal{P}$-measurable $\widetilde{x}^i_{n'}$ by setting $\widetilde{x}^i_{n'}(1) = \widehat{x}^i(1) - c^i_{n'}(\mathcal{P}^i)$ and $\widetilde{x}^i_{n'}(\omega) = \widetilde{x}^i(\omega)$ for each $\omega \in \Omega$. Since $\pi_{n'} \to \overline{\pi}$, $c^i_{n'}(\mathcal{P}^i) \to 0$, $(p_n, x_n) \to (\overline{p}, \overline{x})$, and the u^i_ω are continuous,

$$\begin{aligned}
\sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(\widetilde{x}^i_{n'}(1), \widetilde{x}^i_{n'}(\omega)) &\to \sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\widetilde{x}^i(1), \widetilde{x}^i(\omega)), \\
\sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(x^i_{n'}(1), x^i_{n'}(\omega)) &\to \sum_{\omega \in \Omega} \overline{\pi}(\omega) u^i_\omega(\overline{x}^i(1), \overline{x}^i_\omega), \\
p_{n'} \cdot \widetilde{x}^i_{n'} &\to \overline{p} \cdot \widetilde{x}^i.
\end{aligned}$$

Thus $p_{n'} \cdot \widetilde{x}^i_{n'} \to \overline{p} \cdot \widehat{x}^i$ and hence $p_{n'} \cdot \widetilde{x}^i_{n'} + p_{n'} \cdot c^i_{n'}(\mathcal{P}^i) - p_{n'} \cdot e^i \to 0$ and there exists a $\varepsilon > 0$ such that, for all n' sufficiently large,

$$\sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(\widetilde{x}^i_{n'}(1), \widetilde{x}^i_{n'}(\omega)) \geq \sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(x^i_{n'}(1), x^i_{n'}(\omega)) + \varepsilon.$$

There is consequently a $\delta > 0$ such that

$$\sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(\widetilde{x}^i_{n'}(1) - (\delta, ..., \delta), \widetilde{x}^i_{n'}(\omega)) > \sum_{\omega \in \Omega} \pi_{n'}(\omega) u^i_\omega(x^i_{n'}(1), x^i_{n'}(\omega))$$

and $p_{n'} \cdot (\widetilde{x}^i_{n'}(1) - (\delta, ..., \delta), \widetilde{x}^i_{n'}(\omega)) \leq p_{n'} \cdot e^i$ for all n' sufficiently large, which contradicts the assumption that each (p_n, x_n) is an equilibrium. Thus $(\overline{p}, \overline{x})$ is a forced-revelation equilibrium when probabilities are $\overline{\pi}$.

Let $x^i(p, \pi)$ denote agent i's forced-revelation demand as a function of p and π. Given that u^i is differentiably strictly concave and weakly increasing, u^i is strictly increasing which combined with concavity implies $DU^i(x^i) \gg 0$ for any $x^i \geq 0$. Using this fact and the differentiably strict concavity of u^i, a standard application of the implicit function theorem to i's optimization problem implies that $x^i(\cdot)$ is continuously differentiable. Since $(\overline{p}, \overline{x})$ is a forced-revelation equilibrium, $p = \overline{p}$ is a solution of $x^i(p, \overline{\pi}) = \sum_{i \in \mathcal{I}} e^i$. Since $(\overline{p}, \overline{x})$ is regular, the implicit function theorem implies that if $\pi_n \to \overline{\pi}$ then for all n sufficiently large $x^i(p, \pi_n) = \sum_{i \in \mathcal{I}} e^i$ has a solution $p = p^*_n$ such that $p^*_n \to \overline{p}$, and so $\left(p^*_n, (x^i(p^*_n, \pi_n))_{i \in \mathcal{I}}\right)$ provides the desired sequence of forced-revelation equilibria.

Only if. Suppose there is a forced-revelation equilibrium (p^*_n, x^*_n) for each $\mathcal{E}_n$ such that $(p^*_n, x^*_n) - (p_n, x_n) \to 0$ and that, for some good k and state ω',

$$p_{k,n}(\omega') - \pi_n(\omega'|P_n(\omega')) \sum_{\omega \in P_n(\omega)} p_{k,n}(\omega)$$

fails to converge to 0. Taking a further subsequence of $\langle n' \rangle$ if necessary, there must be a $P \in \mathcal{R}$ and $a \neq 0$ such that $P_{n'}(\omega') = P$ for all n' and $p_{k,n'}(\omega') - \pi_{n'}(\omega'|P)) \sum_{\omega \in P} p_{k,n'}(\omega) \to a$. Therefore $\overline{p}_k(\omega') \neq \overline{\pi}(\omega'|P) \sum_{\omega \in P} \overline{p}_k(\omega)$.

We now follow the proof of Theorem 2 and its notation except that $\overline{x}^i_{k'}(\omega)$ replaces $x^i_{k'}(\omega)$ in the definition of $\widetilde{x}^i(\varepsilon, \omega)$ and E_n (resp. U^i_n) and $E_{\overline{\pi}}$ (resp. $U^i_{\overline{\pi}}$) indicate expectations (resp. expected utilities) calculated using π_n and $\overline{\pi}$ respectively. Since there exists a $P' \subset P$ such that $\sum_{\omega \in P'} \overline{p}_k(\omega) < \overline{\pi}(P'|P) \sum_{\omega \in P} \overline{p}_k(\omega)$, $E_{\overline{\pi}}\left[\widetilde{x}^i_k(\varepsilon, \cdot)\right] - E_{\overline{\pi}}\left[\overline{x}^i_k(\cdot)\right] > 0$. Hence for any vNM utility u and all $\varepsilon > 0$ sufficiently small, $\sum_{\omega \in P} \overline{\pi}(\omega|P) u(\widetilde{x}^i_\omega(\varepsilon)) > \sum_{\omega \in P} \overline{\pi}(\omega|P) u(\overline{x}^i_\omega)$. Since the inconclusiveness of information converges to 0, there is a $V^* \in \mathcal{V}^i$ such that $\overline{\pi}(V^*|P) = 1$. Hence $E_{\overline{\pi}}[v^i(\widetilde{x}^i(\varepsilon))|P] = \sum_{\omega \in P} \overline{\pi}(\omega|P) u^i_{V^*}(\widetilde{x}^i_\omega(\varepsilon))$ and therefore

$$E_{\overline{\pi}}[v^i(\widetilde{x}^i(\varepsilon))|P] > u^i_{V^*}\left(x^i_P\right) = E_{\overline{\pi}}[v^i(x^i)|P].$$

Since $E_n[v^i(\widetilde{x}^i(\varepsilon))|P] \to E_{\overline{\pi}}[v^i(\widetilde{x}^i(\varepsilon))|P]$ and $E_n[v^i(\overline{x}^i)|P] \to E_{\overline{\pi}}[v^i(\overline{x}^i)|P]$, we conclude that $E_n[v^i(\widetilde{x}^i(\varepsilon))|P] > E_{\overline{\pi}}[v^i(\overline{x}^i)|P]$ for all large n. Hence $U^i_n(\widetilde{x}^i(\varepsilon)) > U^i_n(\overline{x}^i)$ for large n. Since $\widetilde{x}^i(\varepsilon) \in B^i(p_n, x^{-i}_n)$, (p_n, x_n) could not be an equilibrium for large n. ■

Proof of Theorem 4. Suppose that the partition $\mathcal{D}$ is conditionally conclusive and costless for consumer i at an equilibrium (p, x, y), $P \in \mathcal{P}_{x,y}$, $D \in \mathcal{D}$ with $D \subset P$, and there is a good k such that $\sum_{\omega \in D} p_k(\omega) \neq \pi(D|P) \sum_{\omega \in P} p_k(\omega)$ and $x^i_k(\omega) > 0$ for $\omega \in P$.

Following the proof of Theorem 2, $\pi(D) > 0$ and there is no loss in generality in assuming $\sum_{\omega \in D} p_k(\omega) < \pi(D|P) \sum_{\omega \in P} p_k(\omega)$. With $D = P'$, let $\widetilde{x}^i(\varepsilon, \omega)$, $\widetilde{x}^i_\omega(\varepsilon)$, $\widetilde{x}^i(\varepsilon)$, the random variable $v^i(x^i)$, and the vNM utility u^i_V assume their earlier definitions. Then

$$E\left[\widetilde{x}^i_k(\varepsilon, \cdot)\right] - E\left[x^i_k(\cdot)\right] = \left(\pi(D) - \pi(P \backslash D) \frac{\sum_{\omega \in D} p_k(\omega)}{\sum_{\omega \in P \backslash D} p_k(\omega)}\right) \varepsilon > 0$$

and hence, for any vNM utility u and all $\varepsilon > 0$ sufficiently small, $\sum_{\omega \in P} \pi(\omega|P) u(\widetilde{x}^i_\omega(\varepsilon)) > \sum_{\omega \in P} \pi(\omega|P) u(x^i_\omega)$. Letting $\widetilde{x}^i_D(\varepsilon)$ (resp. $\widetilde{x}^i_{P \backslash D}$) denote $\widetilde{x}^i_\omega(\varepsilon)$ for $\omega \in D$ (resp. $\omega \in P \backslash D$), we therefore have $\pi(D|P) u\left(\widetilde{x}^i_D(\varepsilon)\right) + \pi(P \backslash D|P) u\left(\widetilde{x}^i_{P \backslash D}(\varepsilon)\right) > u\left(x^i_P\right)$ for small $\varepsilon > 0$. Using this fact for the inequality, the fact that $\mathcal{D}$ refines $\mathcal{P}$ for the first equality, and the conditional conclusiveness of $\mathcal{D}$ for the third equality, we have, for all $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
E[v^i(\widetilde{x}^i(\varepsilon))|P] &= \pi(D|P) \sum_{V \in \mathcal{V}^i} \pi(V|D) u^i_V\left(\widetilde{x}^i_D(\varepsilon)\right) + \pi(P \backslash D|P) \sum_{V \in \mathcal{V}^i} \pi(V|P \backslash D) u^i_V\left(\widetilde{x}^i_{P \backslash D}(\varepsilon)\right) \\
&= \sum_{V \in \mathcal{V}^i} \left(\pi(V|D)\pi(D|P) u^i_V\left(\widetilde{x}^i_D(\varepsilon)\right) + \pi(V|P \backslash D)\pi(P \backslash D|P) u^i_V\left(\widetilde{x}^i_{P \backslash D}(\varepsilon)\right)\right) \\
&= \sum_{V \in \mathcal{V}^i} \pi(V|P) \left(\pi(D|P) u^i_V\left(\widetilde{x}^i_D(\varepsilon)\right) + \pi(P \backslash D|P) u^i_V\left(\widetilde{x}^i_{P \backslash D}(\varepsilon)\right)\right) \\
&> \sum_{V \in \mathcal{V}^i} \pi(V|P) u^i_V\left(x^i_P\right) = E[v^i(x^i)|P].
\end{aligned}$$

Since $\pi(D) > 0$ and hence $\pi(P) > 0$, $U^i\left(\widetilde{x}^i(\varepsilon)\right) > U^i(x^i)$ for all $\varepsilon > 0$ sufficiently small, and since

$$\sum_{\omega \in P} p_k(\omega)\widetilde{x}_k^i(\varepsilon,\omega) = \sum_{\omega \in D} p_k(\omega)\left(x_k^i(\omega)+\varepsilon\right) + \sum_{\omega \in P\setminus D} p_k(\omega)\left(x_k^i(\omega) - \frac{\sum_{\omega\in D} p_k(\omega)}{\sum_{\omega \in P\setminus D} p_k(\omega)}\varepsilon\right) = \sum_{\omega\in P} p_k(\omega)x_k^i(\omega),$$

and $\mathcal{D}$ is costless, $\widetilde{x}^i(\varepsilon) \in B^i(p, x^{-i}, y)$.

Since $\widetilde{x}^i(\varepsilon)$ for any ε sufficiently small is therefore a utility-increasing deviation, (p, x, y) could not be an equilibrium. ∎

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