Redistributive Income Taxation with Directed Technical Change*

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This paper studies the implications of (endogenously) directed technical change for the design of non-linear labor income taxes in a Mirrleesian economy augmented to include endogenous technology development and adoption choices by firms. First, I identify conditions under which any progressive tax reform induces equalizing technical change, that is, technical change that compresses the pre-tax wage distribution. The key intuition is that progressive tax reforms tend to reduce labor supply of more skilled relative to less skilled workers, while the increased relative supply of less skilled workers induces firms to develop and use technologies that are more complementary to less skilled workers. Second, I provide conditions under which the endogenous response of technology raises the welfare gains from progressive tax reforms. Third, I show that the endogenous technical change effects tend to make the optimal tax scheme more progressive, raising marginal tax rates at the right tail of the income distribution and lowering them (potentially below zero) at the left tail. Finally, I calibrate the model based on the empirical literature on directed technical change. While the directed technical change effects of tax reforms on wage inequality turn out to be small, the impact of directed technical change on optimal taxes is considerable. For an optimistic calibration of directed technical change effects, optimal marginal tax rates increase monotonically with income instead of being U-shaped (as in most of the previous literature) and marginal tax rates on incomes below the median are reduced substantially.

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1. Introduction

Technical change is widely considered an important determinant of changes in the wage structure of an economy and hence of first-order importance for the design of redistributive tax

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schemes. Existing work analyzes how redistributive taxes respond optimally to exogenous changes in production technology that affect the wage distribution (e.g. Ales, Kurnaz and Sleet, 2015). But technologies are developed and adopted by firms pursuing economic objectives (cf. Acemoglu, 1998, 2007), so they should respond to perturbations of the economy such as tax reforms. In particular, previous work on directed technical change has theoretically proposed and empirically substantiated that the supply of skills in an economy is an important determinant of the extent to which technology favors skilled workers and thereby raises wage inequality (e.g Acemoglu, 1998; Morrow and Trefler, 2017; Carneiro, Liu and Salvanes, 2019). At the same time a large literature on redistributive taxation shows that (non-linear) labor income taxes distort the supply of labor at different levels of skill in a quantitatively significant way (cf. Saez, Slemrod and Giertz, 2012). Changes in labor income taxes should thus be expected to induce changes in technology, which in turn affect pre-tax wage inequality. Taking into account these technology responses in the analysis of tax policy seems an important task for taxation theory.

This paper therefore analyzes the design of non-linear labor income taxes when technology is determined endogenously through the profit-maximizing decisions of firms. For that purpose, I develop a general but tractable model of the economy that features both endogenous labor supply of a continuum of differentially skilled workers and endogenous technology development and adoption choices of firms.

In the model, technical change is driven by technology firms' decisions in which type of technology to invest. Some types are more complementary to high-skilled workers, some are more complementary to low-skilled workers. Technology firms' investment decisions depend on final good firms' demand for intermediate goods that embody the different types of technologies. This intermediate good demand in turn crucially depends on the structure of labor supply firms face on the labor market. If there is a relatively large supply of low-skilled workers, firms demand technologies that are relatively complementary to the low-skilled; if the supply of high-skilled workers is relatively large, firms demand more skill-biased technologies.

Income taxes interact with directed technical change via the structure of labor supply. For example, raising marginal tax rates for high incomes and reducing them for low incomes discourages labor supply of high-skilled and encourages labor supply of low-skilled workers. This shifts firms' demand towards less skill-biased technologies, to which technology firms respond by shifting investment towards such technologies. Intuitively, progressive tax reforms should therefore induce technical change in favor of less skilled workers.

I examine this intuition formally and investigate its implications for the welfare effects of tax reforms and the design of optimal taxes. To this end, I first show that the model's equilibrium has a parsimonious reduced form, which makes the tax analysis tractable. Importantly, this reduced form is well studied by the theory of directed technical change (e.g. Acemoglu, 2007; Loebbing, 2018). Moreover, Acemoglu (2007) shows that it applies to a large set of directed technical change models studied in the literature. This makes my tax analysis, which is based exclusively on the reduced form, generic within the theory of directed technical change.

Turning to the analysis of income taxes, I first study the effects of tax reforms on the direction of technical change. In line with the intuition developed above, I find that, under certain conditions, progressive tax reforms induce technical change that compresses the wage distribution.

In the next step I analyze how the welfare effects of tax reforms are affected by the presence of directed technical change.[1] Somewhat surprisingly given the preceding result, I find that directed technical change does not unambiguously raise the welfare gains from a given progressive reform. This is because the positive redistributive effect of the induced technical change may be counteracted by an adverse effect on tax revenue if the induced technical change shifts wage income from workers with high to workers with low marginal tax rates. This resembles the finding of Sachs, Tsyvinski and Werquin (2019) that accounting for substitution effects between workers – as first analyzed by Stiglitz (1987) in optimal taxation – does not necessarily lower the welfare gain from progressive reforms because of the associated revenue effects.

Yet, I show that once one considers the scope for welfare improvements by means of progressive reforms instead of the welfare effects of a given reform, the results align with the preceding findings and the intuition given above. In particular, I find that the set of tax schedules that can be improved in terms of welfare by means of progressive reforms increases when taking into account directed technical change. In this sense, accounting for directed technical change makes progressive tax reforms more attractive.[2]

Next I characterize optimal tax rates when accounting for directed technical change. I find that directed technical change renders the optimal tax more progressive in the sense that it raises the optimal marginal tax rates in the upper tail and lowers them in the lower tail of the income distribution. The optimal marginal tax rates in the lower tail may even become negative. Intuitively, the optimal tax capitalizes on the redistributive effect of the technical change induced by a more progressive tax system.

The results are based on a comparison between the true optimal tax rates and those perceived as optimal by an exogenous technology planner who believes, mistakenly within the model, that technology is fixed at its level observed under some arbitrary initial tax. Conceptually, the exogenous technology planner's perspective is a strict generalization over the self-confirming policy equilibrium by Rothschild and Scheuer (2013). The two coincide when the initial tax is set to its self-confirming policy equilibrium value. Comparing the true optimal tax with the preferred tax of the exogenous technology planner thus includes a comparison with the self-confirming policy equilibrium.

Finally, I quantify the previous results using estimates of directed technical change effects from the empirical literature. I first consider a hypothetical tax reform that reverses the regressive reforms of the US income tax system during the last 50 years and restores tax progressivity to its 1970 level. I find that, even with an optimistic calibration of directed technical change effects, the impact of this hypothetical reform on wage inequality is small compared to the actual change in US wage inequality observed since 1970. These limited effects are due to the

[1] I use a general Bergson-Samuelson welfare function and impose a mild condition ensuring that the welfare function values equity across workers.

[2] The logic behind this result transfers to the analysis of substitution effects between workers. It can thereby bridge the gap between the seemingly disparate findings for the impact of substitution effects on the welfare assessment of tax reforms and on the shape of optimal taxes in Sachs et al. (2019).

empirically small effects of tax reforms on labor supply. The result suggests that the regressive reforms since 1970 in conjunction with directed technical change are not a likely driver of the observed increase in US wage inequality.

The impact of directed technical change on optimal taxes, however, is quantitatively significant. With an optimistic calibration of directed technical change effects, the U-shape of optimal marginal tax rates familiar from the existing literature (Diamond, 1998) vanishes. Instead, optimal marginal tax rates rise almost monotonically with income. The ensuing adjustment of optimal marginal tax rates is particularly pronounced for incomes below the current US median income: for workers who currently earn about half of the median, optimal marginal tax rates decrease by between 5 and 15 percentage points.

The structure of the paper is as follows. Section 3 presents the model and introduces notation. Section 4 states important results from the theory of directed technical change, which provide the basis for the analysis in the present paper. Section 5 contains the analysis of tax reforms. Section 5.1 considers the effects of tax reforms on directed technical change and Section 5.2 considers the impact of directed technical change on the welfare effects of tax reforms. Section 6 analyzes optimal taxes, Section 7 quantifies the results from the preceding sections, and Section 8 concludes.

2. Related Literature

The paper connects the literatures on the optimal design of non-linear labor income taxes and on (endogenously) directed technical change. It is the first to incorporate endogenous technology responses into an analysis of non-linear labor income taxation and the first to rigorously explore normative implications of the theory of directed technical change.

Starting from the literature on optimal taxation, the paper extends Sachs et al. (2019), who analyze the implications of (within-technology) substitution effects between workers for the design of non-linear labor income taxes. I use and extend their techniques for the analysis of non-linear taxes in general equilibrium. Especially my result on the scope for welfare improvements through progressive tax reforms can easily be transferred to their analysis and provides a qualification to their results regarding the impact of substitution effects on the welfare assessment of progressive reforms. Most importantly, however, I extend their analysis to incorporate directed technical change effects, which are qualitatively very different from the within-technology substitution effects studied by Sachs et al. (2019).

Compared to Ales et al. (2015), who analyze how exogenous technical change affects the optimal tax schedule, I treat technical change as endogenous such that it responds to changes in the tax system.

The paper is also related to recent studies of the taxation of robots (Guerreiro, Rebelo and Teles, 2018; Thuemmel, 2018). In contrast to these studies, I show that technology can be affected indirectly through the income tax system without resorting to direct taxes on specific technologies, which might be challenging in practice due to informational and administrative constraints.

Starting from the theory of directed technical change, I build on the seminal ideas of Acemoglu

(1998) and Kiley (1999) and explore their normative implications, in particular for the design of redistributive labor income taxes. In doing so, I use the theoretical advances by Acemoglu (2007) and Loebbing (2018) as a building block in my analysis. Specifically, their results lend structure to the relationship between labor supply and production technology, which I exploit to analyze the relationship between taxes and technology.

I use empirical work on directed technical change to quantify my results in Section 7. In particular, I use estimates from Lewis (2011), Morrow and Trefler (2017), and Carneiro et al. (2019) to calibrate the strength of directed technical change effects in my model. The empirical literature on directed technical change is discussed in more detail in Section 7.

3. Setup

The model is a general equilibrium model with endogenous production technology embodied in intermediate inputs. The intermediate inputs are supplied under monopolistic competition as for example in Romer (1990). The monopolistically competitive suppliers can improve the quality of their intermediate goods by investing R&D resources. Crucially, the model features multiple types of technology-embodying intermediate inputs, which differ in their complementarity relationships with different types of labor. Hence, the wage distribution is affected differentially by improvements in the quality of different types of intermediates. Changes in the distribution of R&D resources over intermediate good types induced by exogenous shocks (such as tax reforms) constitute endogenous technical change.

The tax analysis will build on a parsimonious subset of the model's equilibrium conditions. While the model itself imposes several specific assumptions, the conditions used for the tax analysis are much more general: they can be obtained from a variety of different models of endogenously directed technical change, as shown in Acemoglu (2007).[3]

3.1. Model

The model features heterogeneous workers, perfectly competitive final good firms, monopolistically competitive technology firms, and a government that levies taxes.

Workers There is a continuum of workers with different types $\theta \in \Theta = [\underline{\theta}, \overline{\theta}] \subset \mathbb{R}$. Types are distributed according to the density function $h : \theta \mapsto h_\theta$, with cumulative distribution function H.

Workers' utilities depend on consumption c_θ and labor supply l_θ according to

$$u_\theta = c_\theta - v(l_\theta) \ ,$$

where v represents disutility from labor. Linearity in consumption implies that there are no income effects on labor supply.

[3] See Acemoglu (2007) and Loebbing (2018) for complementary lists of models that all give rise to the relevant subset of equilibrium conditions derived below.

Workers' pre-tax incomes are $y_\theta = w_\theta l_\theta$ and income taxes are given by the tax function $T : y_\theta \mapsto T(y_\theta)$. The retention function corresponding to tax T is denoted R_T. Hence, workers' budget constraints are

$$c_\theta = R_T(w_\theta l_\theta) + S \,, \tag{1}$$

where S is a lump-sum transfer used to neutralize the government's budget constraint.
Workers choose their labor supply to maximize utility, taking wages as given. The first-order condition is[4]

$$v'(l_\theta) = R_T'(w_\theta l_\theta) w_\theta \,. \tag{2}$$

Firms There is a continuum of mass one of identical final good firms indexed by i. They produce a final consumption good (the numéraire) according to the C^2 production function $G(L_i, Q_i)$. The first input $L_i = \{L_{i,\theta}\}_{\theta\in\Theta}$ collects the amounts of all different types of labor used by firm i. The second input $Q_i = \left\{Q_{i,j}\right\}_{j\in\{1,2,\dots,J\}}$ collects the variables $Q_{i,j}$, each of which is an aggregate of a continuum of technology-embodying intermediate goods:[5]

$$Q_{i,j} = \int_0^1 \phi_{j,k} q_{i,j,k}^\alpha \, dk \,.$$

The variables $q_{i,j,k}$ denote the amount of intermediate good (j,k) used by firm i, while the parameter $\alpha \in (0,1)$ governs the substitutability of intermediates with the same j-index. The variables $\phi_{j,k}$ give the quality of the corresponding intermediate goods and they represent the endogenous part of production technology in the model. Their determination is described in detail below.
With this structure of final good production, we can write the output of firm i as $\tilde{G}(L_i, \phi, q_i)$ where $\phi = \left\{\phi_{j,k}\right\}_{(j,k)\in\{1,2,\dots,J\}\times[0,1]}$ and $q_i = \left\{q_{i,j,k}\right\}_{(j,k)\in\{1,2,\dots,J\}\times[0,1]}$ collect qualities and quantities of all different intermediates. I assume that the function $\tilde{G}$ is linear homogeneous and concave in the rival inputs (L, q), satisfying the standard microeconomic replication argument (e.g. Romer, 1994). Since in addition all final good firms are price takers, the final good sector admits a representative firm, so I drop the index i in what follows.
Final good firms' profit maximization leads to the following demand for labor:

$$w_\theta = D_{L_\theta} \tilde{G}(L, \phi, q) \,.$$

The operator D_{L_θ} denotes Gateaux differentiation with respect to L in direction of the Dirac measure at θ, which I define rigorously in Section 3.2. Labor market clearing requires that the aggregate labor demand L_θ equals the sum of individual workers' labor supply,

$$L_\theta = l_\theta h_\theta \quad \text{for all}\, \theta \,.$$

[4]Assumptions that guarantee the existence of the derivatives used in the following are imposed in Section 3.3.
[5]The case with a continuum of different intermediate good types j, $j \in [0, J]$, can be treated analogously.

Demand for intermediate good $q_{j,k}$ is given by

$$p_{j,k} = \alpha \phi_{j,k} q_{j,k}^{\alpha-1} \frac{\partial G(L,Q)}{\partial Q_j} \,, \tag{3}$$

where $p_{j,k}$ is the price of the intermediate good.

The technology-embodying intermediate goods are produced under monopolistic competition by technology firms. Each good (j,k) is produced by a single technology firm, which I label by the index (j,k) of its output. Technology firm (j,k) produces its output at constant marginal cost η_j from final good and receives an ad valorem sales subsidy of ξ (see the description of the government for details). It sets the post-subsidy price $p_{j,k}$ to maximize profits

$$\left((1+\xi)p_{j,k} - \eta_j\right) q_{j,k}$$

subject to the demand from final good firms (equation (3)). Since the demand from final good firms is iso elastic, the profit-maximizing price is given by a constant markup over marginal cost net of the subsidy:

$$p_{j,k} = \frac{\eta_j}{(1+\xi)\alpha} \,. \tag{4}$$

Technology firms can invest R&D resources to improve the quality of their output. In particular, a quality level of $\phi_{j,k}$ costs $C_j(\phi_{j,k})$ units of R&D resources, where the cost function C_j is smooth, convex, and strictly increasing for every j. Firm (j,k)'s profits as a function of its quality level $\phi_{j,k}$ are

$$\pi_{j,k}(\phi_{j,k}) = \max_q \left\{ \alpha \phi_{j,k} \frac{\partial G(L,Q)}{\partial Q_j} q^{\alpha} - \eta_j q - p^r C_j(\phi_{j,k}) \right\} \,,$$

where p^r denotes the (competitive) market price of R&D resources. Via an envelope argument, the first-order condition for the choice of quality is given by

$$\alpha \frac{\partial G(L,Q)}{\partial Q_j} q_{j,k}^{\alpha} = p^r \frac{dC_j(\phi_{j,k})}{d\phi_{j,k}} \,,$$

where $q_{j,k}$ is assumed to take its profit-maximizing value implied by equation (4). One can verify that the optimal $q_{j,k}$ grows at the rate $1/(1-\alpha)$ in $\phi_{j,k}$, such that the left-hand side of equation (3.1) grows at rate $\alpha/(1-\alpha)$ in $\phi_{j,k}$. I assume henceforth that $dC_j/d\phi_{j,k}$ grows at a rate greater than $\alpha/(1-\alpha)$ in $\phi_{j,k}$, which ensures that the first-order condition identifies the unique profit maximum. Since profits are symmetric across all firms (j,k) with the same j-index, uniqueness of the profit maximum implies that the choices of all firms with index j are the same and we can drop the k-index henceforth.

The supply of R&D resources is exogenous and given by $\overline{C}$. Their price adjusts to guarantee market clearing,

$$\sum_{j=1}^{J} C_j(\phi_j) = \overline{C} \,.$$

The assumption of a fixed amount of R&D resources allows to focus on the effects of labor income taxes on the direction instead of the speed of technical change. For an analysis of capital and labor income taxes when the speed, but not the direction, of technical change is endogenous, see Jagadeesan (2019).[6]

Government The government levies different types of taxes/subsidies. First, it subsidizes the sale of technology-embodying intermediate goods by the ad valorem subsidy ξ to counteract the inefficiency created by the market power of technology firms. I assume that $\xi = (1-\alpha)/\alpha$, such that post-subsidy prices of intermediate goods equal marginal costs, $p_j = \eta_j$.[7] This implies that absent any other taxes, the equilibrium allocation will be efficient. Hence, income taxes are used for redistributive purposes only and do not contain any Pigouvian elements, which would deviate the focus away from the central points of the paper. Importantly, the government is restricted to impose a uniform subsidy across all technology types j. This precludes policies aimed at changing the relative utilization of different technologies to reduce pre-tax wage inequality, as analyzed, for example, in Thuemmel (2018) and Guerreiro et al. (2018).[8]

Second, the government taxes the profits of technology firms and of the owners of R&D resources. As is standard in the literature on labor income taxation, I assume that these taxes are confiscatory to avoid a role for the distribution of firm ownership without a meaningful theory of wealth formation in the model.[9] Alternatively, I could assume that firm ownership and the ownership of R&D resources are uniformly distributed across workers without changing any of the results.

Third, the government taxes income according to the tax function T. Reforms of T and its optimal shape are the central objects of the paper.

Taken together, taxes and subsidies generate the following government revenue,

$$S(y) = \int_{\Theta} T(y_\theta) h_\theta \, d\theta + p^r \overline{C} + \sum_{j=1}^{J} \pi_j - \sum_{j=1}^{J} \xi p_j q_j ,$$

which is redistributed lump-sum across workers.

Equilibrium An equilibrium of the model, given a tax function T, is a collection of quantities and prices such that all firms maximize profits, workers maximize utility, and all markets clear.

[6] Endogenizing the total amount of R&D investment in the present framework would lead to increasing returns in aggregate production. This would add a constant to all wage elasticities and hence slightly alter the expressions for the effects of tax reforms in Section 5. The main insights regarding the effects of tax reforms on relative wages, however, would remain unchanged. The same holds for the optimal tax analysis. Only if the government's ability to tax the profits associated with R&D investment were restricted and the distribution of these profits were not uniform, results would change substantially. In that case, the government would distort R&D investment downwards for redistributive reasons. To counteract the ensuing inefficiency, marginal tax rates on labor income would be optimally reduced (Jagadeesan, 2019).

[7] This level of subsidies would also be chosen as part of the optimal tax policy if it were included in the optimal tax analysis of Section 6.

[8] See Section 2 on the relationship of my approach to Thuemmel (2018) and Guerreiro et al. (2018).

[9] Note that confiscatory profit taxes are part of the optimal tax policy whenever ownership shares of firms increase and marginal welfare weights decrease in workers' income levels at the optimum.

Despite the detailed micro structure of the model, the equilibrium variables of interest for the tax analysis can be characterized by a parsimonious set of equations. I call this set of equations a reduced form, as many endogenous variables that are related to the specific micro structure of the model are eliminated from it using appropriate equilibrium conditions. To derive the reduced form, note first that aggregate production at labor input l and a given set of quality levels ϕ can be written as (because intermediate goods prices equal marginal costs):

$$F(l,\phi) := \max_{q} \left\{ \tilde{G}(\{h_\theta l_\theta\}_{\theta\in\Theta},\phi,q) - \sum_{j=1}^{J} \eta_j q_j \right\} . \tag{5}$$

Note that I used labor market clearing (equation (3.1)) to replace the aggregate labor input L by the individual labor input l to save on notation in the following. Via an envelope argument, the labor demand equation (3.1) then implies that in equilibrium, wages are given by

$$w_\theta(l,\phi) = \frac{1}{h_\theta} D_{l_\theta} F(l,\phi) , \tag{6}$$

where the adjustment factor $1/h_\theta$ is necessitated by the switch from aggregate to individual labor inputs in the aggregate production function.

The condition for profit-maximizing quality choices of technology firms (equation (3.1)) coincides with the first-order condition for a maximum of aggregate production with respect to quality ϕ (simply called technology, henceforth) when ϕ is restricted to the set of feasible technologies $\Phi = \left\{\phi \in \mathbb{R}^J_+ \mid \sum_{j=1}^J C_j(\phi_j) \le \overline{C}\right\}$. Thus,

$$\phi^*(l) := \operatorname*{argmax}_{\phi\in\Phi} F(l,\phi) \tag{7}$$

is an equilibrium technology. In the following I focus on equilibria in which technology satisfies equation (7). Existence of other equilibria can be ruled out by imposing assumptions that guarantee strict quasiconcavity of F in ϕ under the constraint $\phi \in \Phi$.[10]

Finally, we can simplify the expression for the government's budget surplus. To this end, note that marginal cost pricing of intermediate goods implies that technology firms' profits are equal to the total amount of subsidies minus the cost for R&D resources:

$$\sum_{j=1}^{J} \pi_j = \sum_{j=1}^{J} \left((1+\xi)p_j - \eta_j\right) q_j - p^r\overline{C} = \sum_{j=1}^{J} \xi p_j q_j - p^r \overline{C} .$$

It follows that the revenue from corporate taxes and the expenses on technology good subsi-

[10]In particular, if the constrained function

$$\tilde{F}(l,\phi_{-J}) := F(l,\phi_{-J},\tilde{\phi}_J(\phi_{-J})), \text{ where } \phi_{-J} = \{\phi_j\}_{j\in\{1,2,\dots,J-1\}} \text{ and } \tilde{\phi}_J(\phi_{-J}) = C_J^{-1}\left(\overline{C} - \sum_{j=1}^{J-1} C_j(\phi_j)\right),$$

is strictly quasiconcave in ϕ_{-J}, the first-order conditions for a maximum of $\tilde{F}$ in ϕ_{-J} are necessary and sufficient and there is a unique value $\phi^*_{-J}(l)$ that satisfies them. Equivalently, there is a unique value $\phi^*(l)$ satisfying the first-order conditions of the program (7), which are identical to the equilibrium condition (3.1), and this unique value indeed solves the program.

dies offset each other exactly in equation (3.1), such that government revenue reduces to

$$S(y) = \int_\Theta T(y_\theta) h_\theta \, d\theta \, . \tag{8}$$

We can now define a reduced form equilibrium for a given income tax T as a collection of labor inputs l, a technology ϕ, government revenue S, consumption levels c, and wages w, such that workers' first-order conditions (2), their budget constraints (1), the wage equation (6), the technology condition (7), and the equation for government revenue (8) are satisfied. These reduced form equations provide the starting point for the tax analysis in the next sections.

3.2. Notation

The tax analysis uses functional derivatives and various elasticities. To simplify the exposition I define a specific notation for several frequently used expressions.

Derivatives For derivatives in finite-dimensional spaces I use standard notation. For perturbations of the tax function T and labor input l I will frequently use the following functional derivatives.

Let $x : (T, z) \mapsto x(T, z)$ be a function of the tax T and, potentially, further variables z. Then,

$$D_\tau x(T, z) := \left. \frac{dx(T + \mu\tau, z)}{d\mu} \right|_{\mu=0}$$

denotes the directional derivative of x with respect to T in direction of the tax reform τ. Similarly, let $x : (l, z) \mapsto x(l, z)$ be a function of labor input l and, potentially, further variables z. I formalize the derivative of x with respect to labor supply of a given type θ, l_θ, as[11]

$$D_{l_\theta} x(l, z) := \lim_{\Delta \to 0} \frac{1}{\Delta} \left. \frac{dx(l + \mu \tilde{l}_{\Delta,\theta}, z)}{d\mu} \right|_{\mu=0} ,$$

where $\tilde{l}_{\Delta,\theta} : \tilde{\theta} \mapsto \tilde{l}_{\Delta,\theta,\tilde{\theta}}$ is a real-valued function on the type space given by

$$\tilde{l}_{\Delta,\theta,\tilde{\theta}} = \begin{cases} 0 & \text{for } \tilde{\theta} < \theta - \Delta \\ \frac{\tilde{\theta} - \theta + \Delta}{\Delta} & \text{for } \tilde{\theta} \in [\theta - \Delta, \theta] \\ \frac{\theta - \tilde{\theta} + \Delta}{\Delta} & \text{for } \tilde{\theta} \in [\theta, \theta + \Delta] \\ 0 & \text{for } \tilde{\theta} > \theta + \Delta \end{cases}$$

for all interior types $\theta \in (\underline{\theta}, \overline{\theta})$; by

$$\tilde{l}_{,\Delta,\overline{\theta},\tilde{\theta}} = \begin{cases} 0 & \text{for } \tilde{\theta} < \overline{\theta} - \Delta \\ \frac{2(\tilde{\theta} - \overline{\theta} + \Delta)}{\Delta} & \text{for } \tilde{\theta} \in [\overline{\theta} - \Delta, \overline{\theta}] \end{cases}$$

[11] The derivative of a function with respect to aggregate labor supply L_θ is defined analogously.

for the highest type $\overline{\theta}$; and by

$$\tilde{l}_{\Delta,\underline{\theta},\tilde{\theta}} = \begin{cases} \frac{2(\underline{\theta}-\tilde{\theta}+\Delta)}{\Delta} & \text{for } \tilde{\theta} \in [\underline{\theta}, \underline{\theta}+\Delta] \\ 0 & \text{for } \tilde{\theta} > \underline{\theta}+\Delta \end{cases}$$

for the lowest type $\underline{\theta}$. Intuitively, the derivative is obtained by perturbing the labor supply function in a continuous way in a neighborhood of type θ and letting this neighborhood converge to θ. Appendix A.1 demonstrates that the thus defined derivative works in a natural way by showing in detail that

$$D_{L_\theta} \int_{\underline{\theta}}^{\overline{\theta}} w_{\tilde{\theta}} L_{\tilde{\theta}} \, d\tilde{\theta} = w_\theta \quad \forall \theta \; .$$

This also proves the labor demand equation (3.1).

The tax analysis below often distinguishes between the direct effect of changes in T or l on an outcome x and the indirect effect mediated through the response of technology ϕ^*. In particular, suppose $x : (T, \phi) \mapsto x(T, \phi)$ depends (directly) on taxes T and technology ϕ. The direct effect of a tax reform in direction τ, holding technology fixed, is then given by $D_\tau x(T, \phi)$ as defined above. For the indirect effect of the tax reform via technology (the induced technical change effect, henceforth) I introduce the following notation:

$$D_{\phi,\tau} x(T, \phi^*(T)) := \left. \frac{dx(T, \phi^*(T + \mu\tau))}{d\mu} \right|_{\mu=0} \; .$$

Here, $\phi^*(T)$ denotes the equilibrium technology at tax function T. The total effect of the reform on x is then obtained as the sum of the direct and the induced technical change effect. Writing $x^*(T) := x(T, \phi^*(T))$, we get

$$D_\tau x^*(T) = D_\tau x(T, \phi^*(T)) + D_{\phi,\tau} x(T, \phi^*(T)) \; .$$

Analogously, if $x : (l, \phi) \mapsto x(l, \phi)$ is a function of labor input l and technology ϕ, the induced technical change effect of a labor input change in direction l_θ is

$$D_{\phi,l_\theta} = \lim_{\Delta \to 0} \frac{1}{\Delta} \left. \frac{dx(l, \phi^*(l + \mu\tilde{l}_{\Delta,\theta})}{d\mu} \right|_{\mu=0} \; ,$$

where $\phi^*(l)$ is given by equation (7).

Wage Elasticities The response of wages to labor input changes plays a central role in the tax analysis. Consider wages as given by (6), that is, for each type θ the wage w_θ is a function of labor inputs l and technology ϕ.

The first set of wage elasticities is concerned with the direct effect of labor inputs on wages, holding technology constant. I call these elasticities the within-technology substitution elasticities (sometimes just substitution elasticities, for brevity), as they describe the changes in marginal productivities induced by factor substitution within a given technology. The own-wage substitution elasticity, that is, the elasticity of w_θ with respect to l_θ, is defined as

$$\gamma_{\theta,\theta} := \frac{l_\theta}{w_\theta} \lim_{\Delta\to 0} \left. \frac{dw_\theta(l+\mu \tilde{l}_{\Delta,\theta}, \phi)}{d\mu} \right|_{\mu=0} .$$

Alternatively, we could write the wage w_θ as a function of ϕ, l, and type θ's labor input l_θ separately, as typically a type's labor input affects its own wage in a way distinct from the labor input function l (see for example the CES case in Section 3.4). Then, the own-wage substitution elasticity is simply

$$\gamma_{\theta,\theta} = \frac{l_\theta}{w_\theta} \frac{\partial w_\theta(l_\theta, l, \phi)}{\partial l_\theta} .$$

The cross-wage substitution elasticity, that is, the elasticity of w_θ with respect to a different type's labor input $l_{\tilde{\theta}}$ (with $\tilde{\theta} \neq \theta$), is given by

$$\gamma_{\theta,\tilde{\theta}} := \frac{l_{\tilde{\theta}}}{w_\theta} D_{l_{\tilde{\theta}}} w_\theta(l, \phi) ,$$

with the derivative $D_{l_{\tilde{\theta}}}$ as defined above.

The second set of wage elasticities captures the induced technical change effects of changes in labor inputs on wages. These elasticities are called the technical change elasticities in the following. The own-wage technical change elasticity is defined as

$$\rho_{\theta,\theta} := \frac{l_\theta}{w_\theta} \lim_{\Delta\to 0} \left. \frac{dw_\theta(l, \phi^*(l+\mu \tilde{l}_{\Delta,\theta}))}{d\mu} \right|_{\mu=0} .$$

Again, the CES case in Section 3.4 clarifies why this is a natural definition of the own-wage technical change elasticity and how it can be expressed in terms of conventional partial derivatives.

The cross-wage technical change elasticity measures how wage w_θ is affected by a change in another type's labor supply $l_{\tilde{\theta}}$ via induced technical change. Formally, it is given by

$$\rho_{\theta,\tilde{\theta}} := \frac{l_{\tilde{\theta}}}{w_\theta} D_{\phi, l_{\tilde{\theta}}} w_\theta(l, \phi^*(l)) ,$$

where the derivative $D_{\phi, l_{\tilde{\theta}}}$ has been defined above.

Rate of Progressivity The rate of progressivity of a tax schedule T is defined as minus the elasticity of the marginal retention rate R'_T with respect to income,

$$P_T(y) := -\frac{R''_T(y) y}{R'_T(y)} .$$

It measures the progression of marginal tax rates as income increases. If the income tax is linear such that marginal tax rates are constant, $P_T(y)$ is zero. If the income tax is progressive (regressive) in the sense that marginal tax rates increase (decrease) with income, the rate of progressivity is positive (negative).

Labor Supply Elasticities As is usual in the literature, I also define some standard concepts of labor supply elasticities to express the effects of tax reforms compactly. The first is the hypothetical elasticity of labor supply with respect to the marginal retention rate that would obtain if the retention function were linear:

$$e_\theta(l) := \frac{v'(l_\theta)}{v''(l_\theta)l_\theta} .$$

Consider now the labor supply of an arbitrary worker type θ, given by workers' first-order condition (2), as a function of T and w_θ. The true elasticity of labor supply with respect to the marginal retention rate must take into account potential non linearities of the retention function, which cause the worker's marginal retention rate to change as labor supply changes. This elasticity is given by

$$\epsilon^R_\theta(T, l, w) := \frac{R'_T(w_\theta l_\theta)}{l_\theta} D_{\tilde{\tau}} l_\theta(T, w_\theta) ,$$

where the auxiliary tax reform $\tilde{\tau}$ is chosen such that, as the scaling factor μ of the reform goes to zero, it raises the marginal retention rate by one infinitesimal unit:

$$\forall y : \; \tilde{\tau}(y) = -y , \quad \text{and thus:} \quad (y - (T(y) + \mu\tilde{\tau}(y)))' = 1 - T'(y) + \mu .$$

Inserting this into workers' first-order condition and differentiating with respect to μ (at $\mu = 0$) then gives exactly the local response of individual labor supply to a one unit increase in the marginal retention rate. This leads to the following expression for the elasticity of labor supply with respect to the marginal retention rate (see Appendix A.2 for details):

$$\epsilon^R_\theta(T, l, w) = \frac{e_\theta(l)}{1 + e_\theta(l) P_T(w_\theta l_\theta)} . \tag{9}$$

For a locally linear tax function, that is, for $P_T(w_\theta l_\theta) = 0$, the elasticity coincides with the hypothetical elasticity e_θ defined above.

Finally, define the elasticity of labor supply with respect to the wage as

$$\epsilon^w_\theta(T, l, w) := \frac{w_\theta}{l_\theta} \frac{\partial l_\theta(T, w_\theta)}{\partial w_\theta} .$$

It is a standard result that this elasticity can be written as (see Appendix A.2 for details)

$$\epsilon^w_\theta(T, l, w) = \frac{(1 - P_T(w_\theta l_\theta)) e_\theta(l)}{1 + e_\theta(l) P_T(w_\theta l_\theta)} . \tag{10}$$

3.3. Global Assumptions

Using the notation for derivatives and elasticities just defined, I now impose a few assumptions that are maintained throughout the paper. They mainly justify the use of derivatives to characterize equilibrium. Further assumptions, which are only relevant for certain parts of the analysis, are imposed at the start of the sections in which they are needed.

I impose the assumptions directly on the parameters of the reduced form equations. They can of course be mapped into assumptions on the fundamentals of the specific model presented above. I decide to impose them on the reduced form because this seems more transparent and, as argued before, the reduced form is far more general than the model itself.

Assumption 1. *The parameters of the reduced form equations (2), (1), (6), (7), and (8) satisfy the following.*

1. *The derivative $D_{l_\theta} F$ exists and is strictly positive for all θ.*
2. *The wage elasticities $\gamma_{\theta,\tilde{\theta}}$ and $\rho_{\theta,\tilde{\theta}}$ exist for all $\theta, \tilde{\theta}$.*
3. *The density h is C^1 and strictly positive for all θ.*
4. *The maximizer $\operatorname{argmax}_{\phi \in \Phi} F(l, \phi)$ exists and is unique for all l.*
5. *The disutility of labor v is C^2 with $v' > 0$ and $v'' > 0$ everywhere.*
6. *Whenever an exogenous tax T is considered, it is C^2 and satisfies $T' < 1$ everywhere.*

The first two points say that the aggregate production function F is twice differentiable under the differentiation operator D_{l_θ}. Together with the third point (especially $h > 0$ everywhere), they ensure that wages and wage elasticities are always well defined. The fourth point implies that the equilibrium technology $\phi^*(l)$ is unique for any equilibrium labor input l. This allows me to write results in terms of equilibrium values instead of sets of equilibrium values. The latter would complicate the notation without adding interesting substance. Finally, the differentiability assumptions on v and T ensure that the labor supply elasticity e_θ and the rate of progressivity of the tax schedule are well defined.

3.4. Special Cases

Parts of the results of the tax analysis require further structural assumptions on production functions, utility functions, or the tax function. I introduce these special cases here and refer to them in the tax analysis whenever needed.

CES Production An important special case of the model is obtained when the aggregate production function F features a constant elasticity of substitution (CES) between worker types while the research cost functions are iso elastic. I refer to this configuration of the model as the CES case.

The CES case is obtained via the following assumptions on the fundamentals of the model presented above.

$$\tilde{G}(L, \tilde{\phi}, q) = \left[\int_{\underline{\theta}}^{\overline{\theta}} \left(\tilde{\kappa}_\theta L_\theta^{1-\alpha} \int_0^1 \tilde{\phi}_{\theta,k} q_{\theta,k}^\alpha \, dk \right)^{\frac{\tilde{\sigma}-1}{\tilde{\sigma}}} d\theta \right]^{\frac{\tilde{\sigma}}{\tilde{\sigma}-1}}$$
$$C_\theta(\tilde{\phi}_{\theta,k}) = \tilde{\phi}_{\theta,k}^{\tilde{\delta}} .$$

The function $\tilde{\kappa}$ is an exogenous component of technology and assumed to be continuously differentiable; $\tilde{\sigma} > 0$ measures the elasticity of substitution between differentially skilled workers in the production of an individual final good firm; and $\tilde{\delta}$ determines the convexity of the research cost function. The endogenous component of technology is $\tilde{\phi}$.[12]

Appendix A.3 shows that under these assumptions aggregate production indeed takes the CES form. In particular, the aggregate production function F becomes

$$F(l,\phi) = \left[\int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta \phi_\theta l_\theta h_\theta)^{\frac{\sigma-1}{\sigma}} \, d\theta \right]^{\frac{\sigma}{\sigma-1}} , \tag{11}$$

where κ and σ are functions of the parameters $\tilde{\kappa}$, η (the unit cost of intermediate goods), α, and $\tilde{\sigma}$. Appendix A.3 provides the precise relationships between fundamentals and the parameters of F. These relationships show that continuous differentiability of $\tilde{\kappa}$ translates into continuous differentiability of κ, while the restriction $\tilde{\sigma} > 0$ implies $\sigma > \alpha$. The new technology variable ϕ is a bijection of the original technology $\tilde{\phi}$, $\phi_\theta := \tilde{\phi}_\theta^{1/(1-\alpha)}$ for all θ. These substitutions are done purely for notational convenience. The new technology ϕ satisfies the reduced form equation (7) with the set of feasible technologies Φ given by (see Appendix A.3)

$$\Phi = \left\{ \phi : \theta \mapsto \phi_\theta \in \mathbb{R}_+ \mid \int_{\underline{\theta}}^{\overline{\theta}} \phi_\theta^\delta \, d\theta \leq \overline{C} \right\} , \tag{12}$$

where $\delta := (1-\alpha)\tilde{\delta}$.

Moreover, following the reduced form equation (6), wages in the CES case are (see Appendix A.3)

$$w_\theta(l,\phi) = (\kappa_\theta \phi_\theta)^{\frac{\sigma-1}{\sigma}} (l_\theta h_\theta)^{-\frac{1}{\sigma}} F(l,\phi)^{\frac{1}{\sigma}} . \tag{13}$$

Finally and crucially, the wage elasticities defined in Section 3.2 take a particularly simple form in the CES case (see again Appendix A.3 for the derivations). The own-wage substitution elasticity is given by

$$\gamma_{\theta,\theta} = -\frac{1}{\sigma} =: \gamma^{CES} \tag{14}$$

and the cross-wage substitution elasticity becomes

$$\gamma_{\theta,\tilde{\theta}} = \frac{1}{\sigma} \frac{l_{\tilde{\theta}} w_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l,\phi)} . \tag{15}$$

The own-wage technical change elasticity is

$$\rho_{\theta,\theta} = \frac{(\sigma-1)^2}{(\delta-1)\sigma^2 + \sigma} =: \rho^{CES} , \tag{16}$$

[12] Note that here the set of technology types is equated with the set of worker types, such that technology and research costs are now indexed by θ. This reflects the assumption that for every worker type θ there exists a type of technology, embodied in the intermediate goods $q_{\theta,k}$, that raises the efficiency of labor of type θ in the production process. Moreover, the set of technology types is a continuum here, in contrast to the finite set $\{1, 2, ..., J\}$ in the general model above. As mentioned in footnote 5, the case with a continuum of technology types can be treated analogously to the finite case presented above and is therefore omitted from the presentation of the general model.

and the cross-wage technical change elasticity is given by

$$\rho_{\theta,\tilde{\theta}} = -\frac{(\sigma-1)^2}{(\delta-1)\sigma^2+\sigma}\frac{l_{\tilde{\theta}}w_{\tilde{\theta}}h_{\tilde{\theta}}}{F(l,\phi)} \,. \tag{17}$$

Iso-Elastic Disutility of Labor When the disutility of labor is iso elastic, workers' utility functions take the form

$$u_\theta = c_\theta - \frac{e}{e+1} l_\theta^{\frac{e+1}{e}} \,.$$

In this case, the hypothetical labor supply elasticity $e_\theta(l)$ is constant across θ and l:

$$e_\theta(l) = e \quad \text{for all } \theta, l. \,.$$

Constant-Rate-of-Progressivity Taxes A constant-rate-of-progressivity (CRP) tax function takes the form (e.g. Feldstein, 1969; Heathcote, Storesletten and Violante, 2017)

$$T(y) = y - \lambda y^{1-P} \,.$$

For any CRP tax schedule T the rate of progressivity P_T is constant across income levels:

$$P_T(y) = P \quad \text{for all } y \,.$$

This special case, when combined with iso-elastic disutility of labor, ensures that the labor supply elasticities ϵ_θ^R and ϵ_θ^w are constant in θ.

The combination of iso-elastic disutility of labor and CRP tax schedules plays an important role in simplifying and clarifying the results of the tax analysis by suppressing heterogeneity in labor supply responses to changes in marginal tax rates.

4. Directed Technical Change

Income tax reforms affect technical change via differential changes in labor supply across worker types. An important building block of the tax analysis below is therefore the relationship between the structure of labor supply and technical change. This relationship is studied by the theory of directed technical change.

To review the central results (for the purpose of the tax analysis below) of this theory, take labor supply l as exogenous for the moment and consider wages and equilibrium technology as determined by the reduced form equations (6) and (7) given labor inputs (copied here for the reader's convenience):

$$w_\theta(l,\phi) = \frac{1}{h_\theta} D_{l_\theta} F(l,\phi)$$
$$\phi^*(l) := \operatorname*{argmax}_{\phi\in\Phi} F(l,\phi) \,.$$

4.1. Weak Relative Bias

Starting from the previous two equations, a first major results of directed technical change theory identifies conditions under which any increase in the relative supply of skill in the economy induces skill-biased technical change.

In particular, let dl be a change in the labor input l. We say that dl is an increase in the relative supply of skill if it raises the supply of more skilled relative to the supply of less skilled workers, that is, dl_θ / l_θ increases in θ. Similarly, a technology ϕ is skill-biased relative to another technology $\tilde{\phi}$ if more skilled workers earn higher wages relative to less skilled workers under ϕ than under $\tilde{\phi}$, that is, $w_\theta(l, \phi)/w_{\tilde{\theta}}(l, \phi) \geq w_\theta(l, \tilde{\phi})/w_{\tilde{\theta}}(l, \tilde{\phi})$ for all $\theta \geq \tilde{\theta}$ and all l. In this case we write $\phi \succeq \tilde{\phi}$.

For any increase in relative skill supply to induce skill-biased technical change, I have shown in previous work that the following non-parametric restrictions on the aggregate production function F are sufficient and close to necessary (Loebbing, 2018).

Assumption 2. *The aggregate production function F is homogeneous in l and quasisupermodular in ϕ.*

Homogeneity of F in labor is in fact already implied by the structure of the model presented in Section 3.1, so it imposes no further restrictions (see Appendix A.4). By quasisupermodularity I mean the following here. For any l and any two technologies ϕ and $\tilde{\phi}$, if $F(l, \underline{\phi}) \leq F(l, \phi)$ for all $\underline{\phi} \preceq \phi, \tilde{\phi}$, then there must exist a $\overline{\phi} \succeq \phi, \tilde{\phi}$ such that $F(l, \overline{\phi}) \geq F(l, \tilde{\phi})$.[13]

For two technologies that can are ordered according to their skill-bias, the condition is not restrictive. So, to understand the restrictions involved, suppose that neither $\phi \preceq \tilde{\phi}$ nor $\phi \succeq \tilde{\phi}$. For illustration, consider a setting with three different types and let ϕ induce a higher skill premium at the top (i.e., between the high and the middle skill) while $\tilde{\phi}$ induces a higher skill premium at the bottom (i.e., between the middle and the low skill). Moreover, suppose that moving from any technology $\underline{\phi}$ that induces lower skill premia than both ϕ and $\tilde{\phi}$ to ϕ increases output. Note that such a technical change raises the skill premium at the top. Quasisupermodularity then says that there must also be a technical change starting from $\tilde{\phi}$ that raises the skill premium at the top and increases output. In short, if we can raise output by an increase in the skill premium at the top when starting from low skill premia everywhere, then we must also be able to raise output by an increase in the skill premium at the top when the skill premium at the bottom is already elevated.

More generally, this implies that technical changes that increase inequality on different segments of the wage distribution must not be substitutes: technical change that raises inequality on some segment must not reduce the profitability of technical change raising inequality on another segment. The CES case introduced in Section 3.4 is exactly the case where technical

[13]Note that this slightly deviates from the original definition of quasisupermodularity given by Milgrom and Shannon (1994). For their definition, we would first have to assume that the set $(\Phi, \succeq)$ has a lattice structure, that is, for any two technologies ϕ and $\tilde{\phi}$ there exist supremum and infimum in Φ. Then, quasisupermodularity would be defined using infimum and supremum instead of arbitrary technologies below and above ϕ and $\tilde{\phi}$. In particular, for any l and any $\phi, \tilde{\phi}$, if $F(l, \underline{\phi}) \leq F(l, \phi)$, then $F(l, \overline{\phi}) \geq F(l, \tilde{\phi})$, where $\underline{\phi}$ and $\overline{\phi}$ denote infimum and supremum of ϕ and $\tilde{\phi}$. My definition is slightly less restrictive (and sufficiently restrictive for the present purpose), but more importantly it does not require to introduce the lattice structure of Φ, which would unnecessarily complicate the exposition.

changes affecting different parts of the wage distribution are independent of each other: the profitability of increasing productivity of some worker type θ at the cost of reducing productivity for some other type $\tilde{\theta}$ only depends on productivity and labor input levels of these two types, but not on the productivity distribution over other types of workers.[14]

In the absence of precise empirical evidence about the complementarity relationships between different forms of technical change, the agnostic view (neither strict complementarity nor strict substitutability) of the CES function seems a reasonable benchmark. Note also that, while I am not aware of empirical tests of quasisupermodularity in the aggregate production process, the implications of quasisupermodularity presented in Lemma 1 below receive support in the empirical literature. I discuss this literature in Section 7, when quantifying the results of the tax analysis.[15]

Taking Assumption 2 now as given, the following result applies.

Lemma 1. *Take any labor input l and let dl be a change in the labor input such that the relative change dl_θ/l_θ increases in θ. Then, the technical change induced by dl raises more skilled workers' wages relative to less skilled workers' wages, that is,*

$$\frac{1}{w_\theta} D_{\phi,dl} w_\theta(l, \phi^*(l)) \geq \frac{1}{w_{\tilde{\theta}}} D_{\phi,dl} w_{\tilde{\theta}}(l, \phi^*(l)) \tag{18}$$

for all $\theta \geq \tilde{\theta}$, where

$$D_{\phi,dl} w_\theta(l, \phi^*(l)) := \left.\frac{dw_\theta(l, \phi^*(l + \mu dl))}{d\mu}\right|_{\mu=0} .$$

In words, any increase in relative skill supply induces skill-biased technical change.

Proof. See Appendix B. □

The result of Lemma 1 is called weak relative bias of technology in the literature on directed technical change (Acemoglu, 2007). The intuition behind this result relies on the complementarity relationships between labor and technology. An increase in the relative supply of high-skilled over low-skilled workers raises the profitability of technologies that are (relatively) complementary to high-skilled workers, as these worker types are now more abundant. But since complementarity is a symmetric relation, technologies that are relatively complementary to high-skilled workers also raise their relative productivity, which increases skill premia. Hence, an increase in relative skill supply induces skill-biased technical change.

The relative wage effects of the induced technical change in equation (18) can alternatively be expressed using the induced technical change wage elasticities defined in Section 3.2. Using these elasticities, we obtain

$$\frac{1}{w_\theta} D_{\phi,dl} w_\theta(l, \phi^*(l)) = \rho_{\theta,\theta} \frac{dl_\theta}{l_\theta} + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\theta'} \frac{dl_{\theta'}}{l_{\theta'}} \, d\theta' .$$

[14] Hence, the CES production function satisfies quasisupermodularity in ϕ. I discuss directed technical change results for the CES case in more detail below.

[15] Finally, it is worth to note that quasisupermodularity is indispensable for the weak bias result presented in Lemma 1. In this sense, it is the minimal restriction required to make progress in the analysis of the implications of directed technical change for the design of non-linear labor income taxes.

This expression offers an easy way to verify the statement of Lemma 1 for the CES case. In particular, consider the difference

$$\rho_{\theta,\theta} \frac{dl_\theta}{l_\theta} + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\theta'} \frac{dl_{\theta'}}{l_{\theta'}} \, d\theta' - \rho_{\tilde{\theta},\tilde{\theta}} \frac{dl_{\tilde{\theta}}}{l_{\tilde{\theta}}} - \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\tilde{\theta},\theta'} \frac{dl_{\theta'}}{l_{\theta'}} \, d\theta'$$

for some types $\theta \geq \tilde{\theta}$. Using equations (16) and (17), which specify the induced technical change elasticities in the CES case, this difference becomes

$$\rho^{CES} \left(\frac{dl_\theta}{l_\theta} - \frac{dl_{\tilde{\theta}}}{l_{\tilde{\theta}}} \right) ,$$

which is clearly positive whenever dl_θ / l_θ increases in θ. Hence, if dl is an increase in relative skill supply, the induced technical change raises more skilled workers' wages relative to the wages of less skilled workers, which is the statement of Lemma 1.

4.2. Strong Relative Bias

The weak bias results only concern the effects of labor supply changes on wages mediated by induced technical change. These effects are important to identify precisely what is added by accounting for directed technical change. Yet in general, wages are also affected by factor substitution within a given technology, that is, by the within-technology substitution effects described in Section 3.2. These work typically against the induced technical change effects, reducing skill premia when relative skill supply rises and increasing them otherwise. An important question is then whether the induced technical change effects are strong enough to outweigh the effects coming from within-technology factor substitution, or vice versa.

The strong bias results of directed technical change theory provide an answer to this question. The theory says that there is strong relative bias of technology if the induced technical change effects dominate the within-technology substitution effects, such that the total effect of an increase in relative skill supply is to raise skill premia. In Loebbing (2018), I show that strong relative bias can occur if the production function fails to be quasisconcave. Quasiconvexity in aggregate production can in turn be microfounded by models with imperfect competition, Arrow (1962) type external effects between firms, or endogenous firm entry and inframarginal rents (Acemoglu, 2007; Loebbing, 2018). The model presented in Section 3.1 features imperfect competition on technology markets. So strong bias is a possibility.

For the CES case, there is an exact parametric condition for strong bias. In particular, the total effect of a labor supply change dl on the relative wage between types $\theta \geq \tilde{\theta}$ is

$$\begin{aligned} (\rho_{\theta,\theta} + \gamma_{\theta,\theta}) \frac{dl_\theta}{l_\theta} + \int_{\underline{\theta}}^{\overline{\theta}} (\rho_{\theta,\theta'} + \gamma_{\theta,\theta'}) \frac{dl_{\theta'}}{l_{\theta'}} \, d\theta' - (\rho_{\tilde{\theta},\tilde{\theta}} + \gamma_{\tilde{\theta},\tilde{\theta}}) \frac{dl_{\tilde{\theta}}}{l_{\tilde{\theta}}} - \int_{\underline{\theta}}^{\overline{\theta}} (\rho_{\tilde{\theta},\theta'} + \gamma_{\tilde{\theta},\theta'}) \frac{dl_{\theta'}}{l_{\theta'}} \, d\theta' \\ = (\rho^{CES} + \gamma^{CES}) \left(\frac{dl_\theta}{l_\theta} - \frac{dl_{\tilde{\theta}}}{l_{\tilde{\theta}}} \right) . \end{aligned}$$

Considering again an increase in the relative input of more skilled workers, dl such that

$dl_\theta / l_\theta \geq dl_{\tilde{\theta}} / l_{\tilde{\theta}}$, the effect on the relative wage is positive if and only if

$$\rho^{CES} + \gamma^{CES} \geq 0 . \tag{19}$$

Hence, if condition (19) is satisfied, induced technical change effects dominate substitution effects and the total effect of an increase in relative skill supply on skill premia is positive.

5. Tax Reforms

Starting from a given tax T, a tax reform is represented by the change from T to $T + \mu\tau$, where $\mu \in \mathbb{R}_+$ and $\tau : y \mapsto \tau(y) \in \mathbb{R}$ is a C^2 real-valued function. In this notation, μ is the scaling factor of the tax reform while τ indicates its direction: If $\tau(y)$ is positive (negative) at some income level y, the reform raises (lowers) the tax burden for workers who earn y.
The curvature of τ, relative to the curvature of T, governs the progressivity of the reform. More precisely, I call a reform progressive if the post-reform tax schedule has a higher rate of progressivity than the pre-reform schedule everywhere.

Definition 1. Starting from tax T the tax reform (τ, μ) is progressive if and only if

$$P_{\tilde{T}}(y) \geq P_T(y) \quad \forall y ,$$

where $\tilde{T} := T + \mu\tau$ denotes the post-reform tax function.

This definition is equivalent to the following characterizations of progressivity.

Lemma 2. *Take any tax function T. The following statements are equivalent.*

1. *The reform (τ, μ) is progressive according to Definition 1.*

2. *The post-reform tax $\tilde{T} = T + \mu\tau$ can be obtained by taxing post-tax income under the initial tax in a progressive way, that is, by means of a tax function with increasing marginal tax rates:*

$$R_{\tilde{T}} = r \circ R_T$$

   for some concave function r.

3. *The reform (τ, μ) satisfies*

$$\frac{\tau'(y)}{1 - T'(y)} \geq \frac{\tau'(\tilde{y})}{1 - T'(\tilde{y})} \quad \forall y \geq \tilde{y} .$$

Proof. See Appendix C. □

The first equivalence provides an intuitive interpretation of progressivity: a reform is progressive if and only if it can be obtained by augmenting the initial tax by an additional tax on post-tax income that features increasing marginal tax rates. In this sense, a progressive reform is obtained by taxing the initial post-tax income in a progressive way.

The second equivalence shows that a progressive reform raises the marginal tax rate relative to the initial marginal retention rate by more for higher incomes. Alternatively, the marginal retention rate for higher incomes is reduced relative to that for lower incomes. This equivalence will turn out useful in the analysis below.

In the following I focus on the local effects of a reform in the direction of τ, that is, the effects on economic outcomes of changing T to $T + \mu\tau$ as $\mu \to 0$. Note that this does not lead to confusion with the definition of progressivity, because, as indicated by the second equivalence in Lemma 2, Definition 1 only depends on the direction τ of a reform but not on the scaling factor μ. Moreover, I assume without loss of generality that worker types are ordered according to their wages under the initial tax schedule, that is, $w_\theta \leq w_{\tilde{\theta}}$ if $\theta \leq \tilde{\theta}$ under the initial tax.

To describe the effects of tax reforms on economic outcomes formally, I write equilibrium variables as a function of the tax, that is, the equilibrium value of a variable x (e.g. wages or labor inputs) under tax T is denoted by $x(T)$.[16]

5.1. Direction of Induced Technical Change

A key step in the analysis of the effects of tax reforms is to characterize the responses of labor inputs to a given reform. Let

$$\hat{l}_{\theta,\tau}(T) := \frac{1}{l_\theta} D_\tau l_\theta(T)$$

denote the relative change in the labor input of type θ in response to reform τ. The relative labor input changes must satisfy the following fixed point equation (see Appendix C.1 for details).[17]

$$\hat{l}_{\theta,\tau}(T) = -\epsilon_\theta^R \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \epsilon_\theta^w (\gamma_{\theta,\theta} + \rho_{\theta,\theta}) \hat{l}_{\theta,\tau}(T) + \epsilon_\theta^w \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\theta,\tilde{\theta}} + \rho_{\theta,\tilde{\theta}}) \hat{l}_{\tilde{\theta},\tau} \, d\tilde{\theta} \,. \tag{20}$$

In equilibrium, labor inputs respond directly to the tax reform (the first term in equation (20)) but they also cause wages to adjust. These wage adjustments in turn feed back to labor inputs, which is captured by the second and third terms in equation (20). Accounting for these feedback effects gives rise to the fixed point character of equation (20).

I solve for the fixed point of equation (20) by an iteration procedure. Within the iteration steps I disentangle the feedback effects purely transmitted via induced technical change from those transmitted via within-technology factor substitution. Thereby, I obtain a decomposition of the total labor input response into a substitution and an induced technical change component. The slope of the induced technical change component over the type space can then be signed for the case of a progressive tax reform, using the structure of the induced technical change effects predicted by the theory of directed technical change.[18]

[16] Note that in some cases this involves an abuse of notation. I write for example $w_\theta(l, \phi)$ in equation (6) to denote wages as a function of labor inputs and technology; now I use $w_\theta(T, \phi^*(T))$ to denote wages as a function of the tax. The latter is meant as a short cut for $w_\theta(l(T), \phi^*(l(T)))$, where $l(T)$ denotes labor inputs under tax T.

[17] All elasticities in this section are evaluated at the equilibrium under the initial tax T. I do not write this dependence explicitly to save on notation.

[18] The expression for labor input responses in Lemma 3 differs from that provided by Sachs et al. (2019) even when ignoring induced technical change effects and the ensuing decomposition of the total effect (i.e., when

Lemma 3. *Fix an initial tax T and suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2). Moreover, suppose that under T,*

$$\sup_{\theta\in\Theta}\left[(\epsilon^w_\theta\rho_{\theta,\theta})^2\right]+\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}(\epsilon^w_\theta\rho_{\theta,\tilde{\theta}})^2\,d\tilde{\theta}\,d\theta+2\sqrt{\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\left(\epsilon^w_\theta\rho_{\theta,\theta}\epsilon^w_\theta\rho_{\theta,\tilde{\theta}}\right)^2\,d\tilde{\theta}\,d\theta}<1 \tag{21}$$

$$\sup_{\theta\in\Theta}\left[(\epsilon^w_\theta\zeta_{\theta,\theta})^2\right]+\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}(\epsilon^w_\theta\zeta_{\theta,\tilde{\theta}})^2\,d\tilde{\theta}\,d\theta+2\sqrt{\int_{\underline{\theta}}^{\bar{\theta}}\int_{\underline{\theta}}^{\bar{\theta}}\left(\epsilon^w_\theta\zeta_{\theta,\theta}\epsilon^w_\theta\zeta_{\theta,\tilde{\theta}}\right)^2\,d\tilde{\theta}\,d\theta}<1\,, \tag{22}$$

where $\zeta_{\theta,\tilde{\theta}}:=\gamma_{\theta,\tilde{\theta}}+\rho_{\theta,\tilde{\theta}}$.[19]
Then, the effect of tax reform τ on the labor input of type θ can be written as

$$\hat{l}_{\theta,\tau}(T)=\sum_{n=0}^{\infty}\hat{l}^{(n)}_{\theta,\tau}(T) \tag{23}$$

for all $\theta\in\Theta$, where

$$\hat{l}^{(0)}_{\theta,\tau}(T)=-\epsilon^R_\theta\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))}$$
$$\hat{l}^{(n)}_{\theta,\tau}(T)=\epsilon^w_\theta\zeta_{\theta,\theta}\hat{l}^{(n-1)}_{\theta,\tau}(T)+\epsilon^w_\theta\int_{\underline{\theta}}^{\bar{\theta}}\zeta_{\theta,\tilde{\theta}}\hat{l}^{(n-1)}_{\tilde{\theta},\tau}(T)\,d\tilde{\theta}\quad\forall\,n>0\,.$$

The total effect on labor inputs can be decomposed as follows,

$$\hat{l}_{\theta,\tau}(T)=-\epsilon^R_\theta\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))}+\underbrace{\sum_{n=1}^{\infty}\tilde{TE}^{(n)}_{\theta,\tau}(T)}_{=:\tilde{TE}_{\theta,\tau}(T)}+\underbrace{\sum_{n=1}^{\infty}\tilde{SE}^{(n)}_{\theta,\tau}(T)}_{=:\tilde{SE}_{\theta,\tau}(T)}\,, \tag{24}$$

where (omitting the argument T)

$$\tilde{TE}^{(1)}_{\theta,\tau}=\epsilon^w_\theta\rho_{\theta,\theta}(-\epsilon^R_\theta)\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))}+\epsilon^w_\theta\int_{\underline{\theta}}^{\bar{\theta}}\rho_{\theta,\tilde{\theta}}(-\epsilon^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\,d\tilde{\theta}$$
$$\tilde{TE}^{(n)}_{\theta,\tau}=\epsilon^w_\theta\rho_{\theta,\theta}\tilde{TE}^{(n-1)}_{\theta,\tau}+\epsilon^w_\theta\int_{\underline{\theta}}^{\bar{\theta}}\rho_{\theta,\tilde{\theta}}\tilde{TE}^{(n-1)}_{\tilde{\theta},\tau}\,d\tilde{\theta}\quad\forall\,n>1$$
$$\tilde{SE}^{(1)}_{\theta,\tau}=\epsilon^w_\theta\gamma_{\theta,\theta}(-\epsilon^R_\theta)\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))}+\epsilon^w_\theta\int_{\underline{\theta}}^{\bar{\theta}}\gamma_{\theta,\tilde{\theta}}(-\epsilon^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\,d\tilde{\theta}$$
$$\tilde{SE}^{(n)}_{\theta,\tau}=\epsilon^w_\theta\gamma_{\theta,\theta}(\tilde{TE}^{(n-1)}_{\theta,\tau}+\tilde{SE}^{(n-1)}_{\theta,\tau})+\rho_{\theta,\theta}\tilde{SE}^{(n-1)}_{\theta,\tau}$$
$$+\epsilon^w_\theta\int_{\underline{\theta}}^{\bar{\theta}}\left[\gamma_{\theta,\theta}(\tilde{TE}^{(n-1)}_{\theta,\tau}+\tilde{SE}^{(n-1)}_{\theta,\tau})+\rho_{\theta,\theta}\tilde{SE}^{(n-1)}_{\theta,\tau}\right]\,d\tilde{\theta}\quad\forall\,n>1\,.$$

setting $\rho_{\theta,\tilde{\theta}}=0$ for all $\theta,\tilde{\theta}$). I discuss the relationship between Lemma 3 and the results of Sachs et al. (2019) in Appendix C.4. In short, my approach has the advantage that, after decomposing the total effect, it allows me to derive analytical insights into the structure of the induced technical change component.

[19] Conditions (21) and (22) ensure that the series in equations (23) and (24) converge. They are sufficient but generally not necessary for convergence. If the conditions are not satisfied, the equilibrium may be unstable in the sense that an increase in some types' labor inputs may trigger a wage adjustment that is more than sufficient to justify the initial increase in labor inputs. I check that the conditions are satisfied in the quantitative analysis.

If ϵ_θ^w is constant in θ (e.g. because the disutility of labor is iso elastic and T is CRP), then $\tilde{TE}_{\theta,\tau}$ is decreasing in θ if $\epsilon_\theta^R \tau'(y_\theta(T))/(1 - T'(y_\theta(T)))$ increases in θ.
Hence, if ϵ_θ^R is also constant in θ, $\tilde{TE}_{\theta,\tau}$ decreases in θ for any progressive reform τ.

Proof. See Appendix C.1. □

Equation (23) expresses the labor input change induced by reform τ as the sum over successive rounds of general equilibrium adjustments, capturing feedback loops from labor supply to wages and back to labor supply. The first summand $\hat{l}_{\theta,\tau}^{(0)}(T)$ is the direct effect of the reform on labor supply, holding wages constant. The direct adjustment of labor supply in turn changes wages, which then feeds back into labor supply. This first-round feedback effect is captured by $\hat{l}_{\theta,\tau}^{(1)}(T)$. The labor supply change $\hat{l}_{\theta,\tau}^{(1)}(T)$ then induces another adjustment of wages, which again affects labor supply, and so on.[20]
Equation (24) decomposes the total labor input change into three components. The first is the direct effect of reform τ, holding wages constant. The second term isolates the part of the general equilibrium feedback in which the effect of labor supply on wages is purely transmitted via induced technical change. The third term collects the remaining parts of the feedback, containing within-technology substitution effects from labor supply on wages.
With constant labor supply elasticities across workers, the induced technical change component $\tilde{TE}_{\theta,\tau}(T)$ is decreasing in θ for any progressive tax reform; that is, the induced technical change component reduces the labor supply of more relative to less skilled workers. This follows from the weak bias result of directed technical change theory. Intuitively, with constant labor supply elasticities, the direct effect of a progressive tax reform on labor supply is to reduce relative skill supply. By weak bias, this induces technical change reducing skill premia (equalizing technical change, henceforth). Again under constant labor supply elasticities, such equalizing technical change feeds back into a further reduction in relative skill supply, which in turn induces further equalizing technical change. Summing over the thus induced rounds of reductions in relative skill supply eventually gives rise to the term $\tilde{TE}_{\theta,\tau}(T)$, which must then also reduce relative skill supply (i.e., decrease in θ).
We can now use Lemma 3 to study the effects of tax reforms on technical change. More precisely, consider the relative changes in wages that are caused by the technical change induced by a reform τ. Using the derivative $D_{\phi,\tau}$ introduced in Section 3.2, these relative wage changes are given by

$$\frac{1}{w_\theta} D_{\phi,\tau} w_\theta(T, \phi^*(T)) \ .$$

They can be expressed in terms of induced technical change elasticities and labor input responses as follows.

$$\frac{1}{w_\theta} D_{\phi,\tau} w_\theta(T, \phi^*(T)) = \rho_{\theta,\theta} \hat{l}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \hat{l}_{\tilde{\theta},\tau}(T)\, d\tilde{\theta} \ . \tag{25}$$

[20] Mathematically, the series representation in equation (23) is the von Neumann series expansion of the solution to the fixed point equation (20). In particular, the fixed point equation can be written abstractly as $(I - X)\hat{l}_\tau = Z$, where I denotes the identity function, X is a linear operator on the space of real-valued functions on Θ, and Z is the direct effect of τ on labor supply. Inverting $I - X$ yields $\hat{l}_\tau = (I - X)^{-1} Z$. By von Neumann series expansion, this is equivalent to $\hat{l}_\tau = \sum_{n=0}^{\infty} X^n Z$.

Inserting expression (24) from Lemma 3 into equation (25) yields an expression for the induced technical change effects of reform τ, consisting of three terms with intuitive interpretations. The slope of two of these terms can be signed using the structure of induced technical change effects imposed by weak bias.

Proposition 1. *Fix an initial tax T, suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ_θ^R and ϵ_θ^w are well defined (see Appendix A.2), and let conditions (21) and (22) be satisfied.*
Then, the relative effect of the technical change induced by tax reform τ on wages can be written as

$$\begin{aligned}\frac{1}{w_\theta} D_{\phi,\tau} w_\theta(T, \phi^*(T)) = & \underbrace{\rho_{\theta,\theta}(-\epsilon_\theta^R) \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}(-\epsilon_{\tilde{\theta}}^R) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta}}_{:= DE_{\theta,\tau}(T)} \\ & + \underbrace{\rho_{\theta,\theta} \tilde{TE}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \tilde{TE}_{\tilde{\theta},\tau}(T) \, d\tilde{\theta}}_{=: TE_{\theta,\tau}(T)} \\ & + \underbrace{\rho_{\theta,\theta} \tilde{SE}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \tilde{SE}_{\tilde{\theta},\tau}(T) \, d\tilde{\theta}}_{=: SE_{\theta,\tau}(T)}, \end{aligned} \tag{26}$$

for all $\theta \in \Theta$, where $\tilde{TE}_{\theta,\tau}(T)$ and $\tilde{SE}_{\theta,\tau}(T)$ are defined in Lemma 3.
If ϵ_θ^w is constant in θ (e.g. because the disutility of labor is iso elastic and T is CRP), then $DE_{\theta,\tau}(T)$ and $TE_{\theta,\tau}(T)$ are decreasing in θ if $\epsilon_\theta^R \tau'(y_\theta(T))/(1 - T'(y_\theta(T)))$ increases in θ.
Hence, if ϵ_θ^R is also constant in θ, $DE_{\theta,\tau}(T)$ and $TE_{\theta,\tau}(T)$ decrease in θ for any progressive reform τ.

Proof. See Appendix C.2. □

The terms in equation (26) derive directly from Lemma 3. The first line of equation (26) is the technical change effect on wages induced by the direct component of the labor supply response to the tax reform τ. It decreases in θ for any progressive reform because, by Lemma 3, the direct effect of a progressive reform is to reduce relative skill supply; and by weak bias, a reduction in relative skill supply induces equalizing technical change.
The term $TE_{\theta,\tau}(T)$ captures the technical change effect induced by the $\tilde{TE}_{\theta,\tau}(T)$ component of the labor supply response to τ. Recall from Lemma 3 that this component decreases in θ for any progressive reform. Hence, by weak bias, it induces equalizing technical change. The term $TE_{\theta,\tau}(T)$ must therefore decrease in θ. Intuitively, it captures the successive rounds of general equilibrium feedback from induced technical change to labor supply and back to technical change. The direct response of labor supply to a progressive reform τ induces equalizing technical change (see above). This equalizing technical change further reduces relative skill supply, which then again induces equalizing technical change, and so on. We thus obtain a sum of equalizing technical changes, which must be equalizing itself (i.e., decreasing in θ).
Finally, the slope of the term $SE_{\theta,\tau}(T)$ cannot be signed without further restrictions. The reason is that this term includes within-technology substitution effects. To sign within-technology substitution effects, however, we have imposed too little structure on the aggregate production function F so far.

Since we cannot sign the slope of the term $SE_{\theta,\tau}(T)$, we can also not sign the total induced technical change effect of a progressive tax reform on relative wages. To do so, additional restrictions are needed. The most radical approach is to restrict the aggregate production function F to be linear in l for any technology ϕ. Then, there are no within-technology substitution effects. The only remaining are the direct effect and the feedback effects via induced technical change, both of which compress the wage distribution.

Corollary 1. *Fix an initial tax T, suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ_θ^R and ϵ_θ^w are well defined (see Appendix A.2), and let condition (21) be satisfied. In addition, suppose that aggregate production F is linear in l such that $\gamma_{\theta,\tilde{\theta}} = 0$ for all $\theta, \tilde{\theta}$. Then, $SE_{\theta,\tau}(T) = 0$.*
Moreover, if ϵ_θ^w and ϵ_θ^R are constant across types (e.g. because the disutility of labor is iso-elastic and T is CRP), any progressive tax reform induces technical change that reduces all skill premia.

Proof. If $\gamma_{\theta,\tilde{\theta}} = 0$ for all $\theta, \tilde{\theta}$, conditions (21) and (22) are identical, so the conditions of Proposition 1 are satisfied. The definition of $SE_{\theta,\tau}$ then immediately implies $SE_{\theta,\tau} = 0$. The last part of Corollary 1 also follows directly from Proposition 1. □

A similarly clear pattern emerges when we assume that aggregate production takes the CES form introduced in Section 3.4. In that case the wage elasticities $\gamma_{\theta,\tilde{\theta}}$ and $\rho_{\theta,\tilde{\theta}}$ are independent of θ. If in addition the labor supply elasticity ϵ_θ^w is the same for all types, the general equilibrium feedback from wages to labor supply is the same for all workers (see Lemma 6 in Appendix C.1). Hence, the effect of reform τ on relative labor supply is fully determined by its direct effect holding wages fixed. As a consequence, the direction of the technical change induced by τ is also fully determined by the direct effect of τ on labor supply. Given constant labor supply elasticities and a progressive tax reform, this direct effect is to reduce relative skill supply, which in turn directs technical change to be equalizing.

Corollary 2. *Fix an initial tax T and suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ_θ^R and ϵ_θ^w are well defined (see Appendix A.2). Moreover, assume that F and Φ are CES as introduced in Section 3.4 and the elasticity ϵ_θ^w is constant in θ, that is, $\epsilon_\theta^w = \epsilon^w$ for all $\theta \in \Theta$. Then the relative wage effect of the technical change induced by tax reform τ satisfies*

$$\frac{1}{w_\theta} D_{\phi,\tau} w_\theta(T, \phi^*(T)) = \rho^{CES}(-\bar{\epsilon}_\theta^R) \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} - \rho^{CES} \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T), \phi^*(T))} (-\bar{\epsilon}_{\tilde{\theta}}^R) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta} \quad (27)$$

for all $\theta \in \Theta$, where

$$\bar{\epsilon}_\theta^R := \frac{\epsilon_\theta^R}{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w} \, .$$

Hence, any reform τ with $\bar{\epsilon}_\theta^R \tau'(y_\theta(T))/(1 - T'(y_\theta(T)))$ increasing in θ induces technical change that reduces all skill premia.
If in addition ϵ_θ^R is constant across types (e.g. because the disutility of labor is iso-elastic and T is CRP), any progressive tax reform induces technical change that reduces all skill premia.

Proof. See Appendix C.2.[21] □

By Corollary 2, we finally have a set of conditions under which the intuition developed in the introduction is true and indeed any progressive tax reform induces equalizing technical change. This provides the starting point for the analysis of welfare implications of tax reforms and optimal taxes in the next sections.

5.2. Welfare Implications

Given that under certain conditions progressive tax reforms induce equalizing technical change it is natural to suspect that taking into account the induced technical change effect of tax reforms should raise the expected welfare gains from progressive reforms. This conjecture is examined in the following.

Welfare is measured by a Bergson-Samuelson welfare function $V : \{u_\theta\}_{\theta\in\Theta} \mapsto V\left(\{u_\theta\}_{\theta\in\Theta}\right)$ that is strictly increasing in all arguments. The marginal welfare weight of an individual worker of type θ is obtained as

$$g_\theta\left(\{u_\theta\}_{\theta\in\Theta}\right) = \frac{1}{h_\theta} D_{u_\theta} V\left(\{u_\theta\}_{\theta\in\Theta}\right) ,$$

where the derivative D_{u_θ} is defined analogously to the definition of D_{l_θ} in Section 3.2.

I assume that V is constructed in a way that the average welfare weight is normalized to one everywhere and impose that g is continuous in θ whenever u is continuous. In addition and more substantially, I assume that the welfare function values equity across workers in the following sense.

Assumption 3. *For any utility profile $\{u_\theta\}_{\theta\in\Theta}$ such that u_θ increases in θ, the marginal welfare weights $g_\theta\left(\{u_\theta\}_{\theta\in\Theta}\right)$ decrease in θ.*

The assumption ensures that redistributing consumption from workers with high utility to workers with low utility improves welfare.[22]

To analyze the welfare effects of a tax reform τ, write welfare as a function of the tax system:

$$W(T) := V\left(\{u_\theta(c_\theta(T), l_\theta(T))\}_{\theta\in\Theta}\right) .$$

Given the welfare function $W(T)$, the welfare effect of a tax reform can now be decomposed as follows.

Proposition 2. *Fix an initial tax T and suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2). Then,*

[21] Note at this point that Corollary 2 does not include Corollary 1. If F is linear in l it necessarily takes the CES form. But the CES case as introduced in Section 3.4 additionally restricts research cost functions to be iso elastic such that the constraint $\phi \in \Phi$ can be represented by a CES function (see equation (12)). This restriction is not required for Corollary 1.

[22] Note that, since preferences are identical across workers, the interpersonal comparison of utilities inherent in Assumption 3 is not problematic.

the welfare effect of a tax reform τ can be written as

$$
\begin{aligned}
D_\tau W(T) = & \underbrace{\int_{\underline{\theta}}^{\overline{\theta}} (1-g_\theta)\tau(y_\theta(T))h_\theta \, d\theta}_{:=ME_\tau(T)} + \underbrace{\int_{\underline{\theta}}^{\overline{\theta}} T'(y_\theta(T))y_\theta(T)(-\epsilon_\theta^R)\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))}h_\theta \, d\theta}_{:=BE_\tau(T)} \\
& + \underbrace{\int_{\underline{\theta}}^{\overline{\theta}} \left[g_\theta(1-T'(y_\theta(T))) + T'(y_\theta(T))(1+\epsilon_\theta^w)\right] y_\theta(T)\frac{1}{w_\theta(T)}D_{\phi,\tau}w_\theta(T,\phi^*(T))h_\theta \, d\theta}_{:=TE_\tau^W(T)} \\
& + \underbrace{\int_{\underline{\theta}}^{\overline{\theta}} \left[g_\theta(1-T'(y_\theta(T))) + T'(y_\theta(T))(1+\epsilon_\theta^w)\right] y_\theta(T)\frac{1}{w_\theta(T)}D_\tau w_\theta(T,\phi^*(T))h_\theta \, d\theta}_{:=SE_\tau^W(T)} \ .
\end{aligned}
$$

Proof. See Appendix C.5. □

Proposition 2 shows that a tax reform has four distinct effects on welfare. The mechanical effect $ME_\tau(T)$ captures the effect from changing taxes and redistributing revenue in the absence of any behavioral responses. The behavioral effect $BE_\tau(T)$ captures the effect of the direct response of labor supply, holding wages constant. Both effects are well known in the literature.

The third term $TE_\tau^W(T)$ represents the welfare implications of the technical change induced by the tax reform. The first part,

$$
\int_{\underline{\theta}}^{\overline{\theta}} g_\theta(1-T'(y_\theta(T)))y_\theta(T)\frac{1}{w_\theta(T)}D_{\phi,\tau}w_\theta(T,\phi^(T))h_\theta \, d\theta \ ,
$$

captures the direct effect of the technology-induced wage changes on workers' utility: from the change in pre-tax income, only the share $1-T'(y_\theta(T))$ translates directly into a change of utility as the remaining share is taxed away. The second part,

$$
\int_{\underline{\theta}}^{\overline{\theta}} T'(y_\theta(T))(1+\epsilon_\theta^w)y_\theta(T)\frac{1}{w_\theta(T)}D_{\phi,\tau}w_\theta(T,\phi^(T))h_\theta \, d\theta \ ,
$$

is the welfare effect of the lump-sum redistribution of the revenue gain or loss induced by the wage adjustments to technical change. Here, the pre-tax income change is scaled by $1+\epsilon_\theta^w$, as the wage change induces a labor supply adjustment of ϵ_θ^w.[23]

Importantly, even if the induced technical change reduces the skill premium (e.g., because τ is progressive and the conditions of Corollary 2 are satisfied), we cannot sign the induced technical change effects on welfare unambiguously. This is because, when for example starting from a progressive tax T, the reduction in high-skilled workers' wages passes through to the government budget to a larger extent than the simultaneous rise in low-skilled workers' wages, as marginal tax rates are higher for the high-skilled. Hence, directed technical change may reduce tax revenue following a progressive reform, which affects welfare negatively via reduced lump-sum transfers. This negative welfare effect potentially outweighs the positive

[23]The labor supply adjustment does not enter the first part of $TE_\tau^W(T)$ because it does not affect workers' utility by the envelope theorem.

effect coming from the reduction in pre-tax wage inequality through the induced technical change.[24]

The final term in Proposition 2, $SE^W_\tau(T)$ captures the welfare effect of the within-technology substitution effects on wages caused by the tax reform. Its structure is analogous to that of $TE^W_\tau(T)$. Given that even the induced technical change component $TE^W_\tau(T)$ has an ambiguous effect on welfare, it is not surprising that also the substitution component $SE^W_\tau(T)$ can generally not be signed.

Importantly, however, Proposition 2 can be combined with equations (26) and (51) for the relative wage effects of tax reforms from Propositions 1 and 6 (in Appendix C.3). This yields a formula for the welfare effects of tax reforms in terms of empirically observable quantities and welfare weights. I use this combination of expressions to quantify the welfare effects of tax reforms and the contribution of directed technical change in Section 7.

The implications of Proposition 2 may be somewhat unexpected in light of the previous section's result. After all, if a progressive reform induces equalizing technical change and the welfare function values equity, directed technical change effects should make progressive reforms in some way more attractive. To see precisely in which way this is indeed true, we must slightly adjust the question posed by Proposition 2.

Concretely, instead of asking how directed technical change alters the welfare effects of a given progressive tax reform, we now study how accounting for directed technical change affects the set of initial taxes under which welfare can be improved by some progressive reform. In particular, let

$$\mathcal{T} := \{T \mid T \text{ is CRP, } \exists \tau \text{ progressive s.t. } D_\tau W(T) > 0\}$$

denote the set of CRP tax schedules that can be improved in a welfare sense by a progressive tax reform. The restriction to CRP taxes is imposed to invoke Corollary 2. Specifically, combining the CRP restriction with iso-elastic disutility of labor and the CES production structure from Section 3.4 ensures, according to Corollary 2, that any progressive tax reform induces equalizing technical change.

As a benchmark for comparison that does not include directed technical change effects, let

$$D^{ex}_\tau W(T) := D_\tau W(T)|_{\rho_{\theta,\tilde{\theta}}=0\ \forall\, \theta,\tilde{\theta}} \tag{28}$$

denote the welfare effect of a reform τ when counterfactually setting all technical change elasticities to zero (or, put differently, when holding technology fixed). Then, we can define

$$\mathcal{T}^{ex} := \{T \mid T \text{ is CRP, } \exists \tau \text{ progressive s.t. } D^{ex}_\tau W(T) > 0\}$$

as the set of CRP schedules that one would perceive to be improvable by progressive reforms if one were to ignore directed technical change.

Comparing the two thus defined sets we find that accounting for directed technical change expands the set of tax schedules under which welfare can be improved by a progressive reform.

[24]This is similar to the observation by Sachs et al. (2019) that within-technology substitution effects may increase the revenue gains from progressive tax reforms if the initial tax schedule is already progressive.

Proposition 3. *Suppose F and Φ are CES as introduced in Section 3.4 and the disutility of labor is iso elastic. Then,*

$$\mathcal{T}^{ex} \subseteq \mathcal{T} ,$$

that is, the set of initial tax schedules that can be improved by a progressive reform becomes larger when accounting for directed technical change effects.

Proof. See Appendix C.5. □

This result proposes a way in which directed technical change effects make progressive reforms more attractive. Specifically, accounting for directed technical change increases the scope for welfare improvements by progressive tax reforms. It aligns neatly with Corollary 2, whereby progressive reforms induce equalizing technical change, and with the corresponding intuition developed in the introduction.

The idea behind Proposition 3 relies on the mechanism design approach to income taxation. It goes as follows. Consider a progressive tax reform that a tax planner who neglects directed technical change effects (the exogenous technology planner, henceforth) expects to raise welfare. For any such reform, a planner who correctly anticipates directed technical change effects (the endogenous technology planner, henceforth) can find another progressive reform that exactly replicates the labor allocation expected by the exogenous technology planner following his reform. But since progressive tax reforms induce equalizing technical change, the endogenous technology planner anticipates a more equal wage distribution after her reform than the exogenous technology planner expects to find after his reform. Via incentive compatibility constraints, a more equal wage distribution allows to distribute consumption more equally as well. Hence, while the two planners expect the same labor allocation to materialize, the endogenous technology planner anticipates a more equal consumption distribution than the exogenous technology planner. Since this reasoning holds for any progressive reform of the exogenous technology planner, the endogenous technology planner can find a welfare-improving progressive reform whenever the exogenous technology planner can find one. Hence, the endogenous technology planner perceives the scope for welfare improvements through progressive tax reforms to be greater.

6. Optimal Taxes

The results from the tax reform analysis suggest that directed technical change effects should render the optimal tax scheme more progressive. I examine this conjecture in the following.

For the characterization of optimal tax rates it is convenient to denote the average welfare weight across all types above a given type θ as

$$\tilde{g}_\theta := \frac{1}{1 - H_\theta} \int_{\theta}^{\overline{\theta}} g_{\tilde{\theta}} h_{\tilde{\theta}} \, d\tilde{\theta} .$$

Moreover, for a function $x : (\theta, z) \mapsto x_\theta(z)$ (e.g., wages or labor inputs) that depends on θ and potentially further variables z, I denote the derivative of x with respect to θ by

$$x'_\theta(z) := \frac{dx_\theta(z)}{d\theta}$$

and the corresponding semi-elasticity by

$$\hat{x}_\theta(z) := \frac{x'_\theta(z)}{x_\theta(z)} \, .$$

Finally, without loss of generality let worker types be ordered according to their wages under the optimum tax schedule, that is, under the optimal tax $w_\theta \leq w_{\tilde{\theta}}$ if $\theta \leq \tilde{\theta}$.

To derive optimal tax rates, I follow the mechanism design approach to optimal taxation as it provides the fastest route ot the central results presented below.[25] For that, write welfare as a function of consumption and labor allocations instead of utility levels:

$$\tilde{W}(c,l) := V\left(\{u_\theta(c_\theta, l_\theta)\}_{\theta\in\Theta}\right) \, .$$

The goal is to find the consumption-labor allocation that maximizes welfare $W(c,l)$ subject to the aggregate resource constraint and to incentive compatibility constraints across worker types. The optimal tax schedule is then obtained as the tax that implements the welfare-maximizing allocation.

The aggregate resource constraint is given by

$$\int_{\underline{\theta}}^{\overline{\theta}} c_\theta h_\theta \, d\theta = F(l, \phi^*(l)) \, . \tag{29}$$

Incentive compatibility requires

$$u_\theta = \max_{\tilde{\theta}\in\Theta} \left\{ c_{\tilde{\theta}} - v\left(\frac{w_{\tilde{\theta}} l_{\tilde{\theta}}}{w_\theta}\right) \right\} \quad \forall\, \theta \, .$$

I restrict attention to instances of the model where the labor input under the optimal tax is C^1 almost everywhere and continuous in θ. Moreover, I assume that this property of labor inputs extends to wages as follows.

Assumption 4. *Aggregate production F satisfies the following. If l is C^1 almost everywhere and continuous in θ, then $D_{l_\theta}F(l,\phi)$ is C^1 almost everywhere and continuous in θ for all $\phi \in \Phi$.*

Under this restriction and with the wage function w_θ increasing in θ at the optimum, the incentive compatibility constraint is equivalent to the following conditions:

$$c'_\theta = v'(l_\theta)(w'_\theta l_\theta + w_\theta l'_\theta)\frac{1}{w_\theta} \quad \text{for almost every } \theta, \tag{30}$$

$$y'_\theta \geq 0 \quad \text{for almost every } \theta \, . \tag{31}$$

[25] The alternative approach would be to use the formulas for the welfare effects of tax reforms from Section 5.2 and impose that these effects are zero for all reforms at the optimum. The two approaches yield the same results.

As is usual in the literature, I drop the monotonicity requirement (31) and study the relaxed problem of maximizing welfare subject to (29) and (30).[26]

From the incentive compatibility and resource constraints, consumption levels can be derived as a function of labor inputs. I substitute this function into the welfare function $\tilde{W}$ and take first-order conditions with respect to labor inputs. Using workers' first-order condition to reintroduce marginal tax rates into the problem then yields the following expression for optimal marginal tax rates.

Proposition 4. *Suppose the labor input l under the optimal tax is C^1 in θ almost everywhere. Then, at almost every type θ, optimal marginal tax rates satisfy the following conditions.*

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = PE^*_\theta + TE^*_\theta + SE^*_\theta \, ,$$

where

$$PE^*_\theta = \left(1+\frac{1}{e_\theta}\right)\frac{1-N_{w_\theta}}{n_{w_\theta} w_\theta}(1-\tilde{g}_\theta)$$

$$TE^*_\theta = \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} \frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}} \left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0} d\tilde{\theta}$$

$$SE^*_\theta = \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} \frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}} \left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} d\tilde{\theta} \, ,$$

all variables are evaluated at equilibrium under the optimal tax T, and N and m denote the cumulative distribution and the density function of wages at the optimum.

Moreover, if l'_θ exists on some left (right) neighborhood of $\bar{\theta}$ ($\underline{\theta}$) and $\limsup_{\theta\to\bar{\theta}} l'_\theta < \infty$ ($\liminf_{\theta\to\underline{\theta}} l'_\theta > -\infty$) under the optimal tax,[27] then

$$TE^*_{\underline{\theta}} \leq 0 \quad \textit{and} \quad TE^*_{\bar{\theta}} \geq 0 \, .$$

Proof. See Appendix D.1. □

Proposition 4 provides an expression that decomposes the optimal tax rates into three terms. The first term PE^*_θ is the standard expression that would be obtained in a setting with exogenous wages. It is zero at the bottom and the top income level, reflecting the well-known result that the optimal marginal tax rate is zero for the highest and the lowest income earner when wages are exogenous.

The second term, TE^*_θ, captures the impact of directed technical change effects on the optimal tax. It is negative at the bottom and positive at the top income. The intuition behind these signs is closely related to the intuition for the positive influence of directed technical change effects on the scope for welfare improvements through progressive tax reforms described in

[26] In all quantitative simulations of optimal taxes, I verify that the monotonicity condition (31) holds at the optimum.

[27] This assumption guarantees that the distribution of labor inputs is well behaved at the top and at the bottom, in the sense that its density is continuous and strictly positive on some neighborhood of the top or the bottom type, respectively. This in turn allows to evaluate the effects of the labor input perturbations $\tilde{l}_{\Delta,\bar{\theta}}$ and $\tilde{l}_{\Delta,\underline{\theta}}$ on relative labor inputs and invoke the directed technical change results of Section 4.

Section 5.2. By reducing marginal tax rates at the bottom and increasing them at the top, the optimal tax schedule stimulates the relative labor supply of less skilled workers, thus inducing firms to operate technologies with a higher relative productivity for low-skilled workers. This raises low-skilled workers' wages relative to those of high-skilled workers. In the mechanism design problem, the ensuing compression in the pre-tax wage distribution slackens high-skilled workers' incentive compatibility constraints and widens the scope for redistribution.

The third term, SE^*_θ, comes from within-technology substitution effects. Without further restriction, there is too little structure imposed on the aggregate production function F to sign the impact of within-technology substitution effects on optimal marginal tax rates at any point of the income distribution.

For the CES case, Lemma 9 in Appendix D.2 provides specific versions of the terms SE^*_θ and TE^*_θ. There, TE^*_θ is strictly positive at the top and strictly negative at the bottom. The substitution term has the opposite signs. It is strictly negative at the top and strictly positive at the bottom. This recovers the results from Stiglitz (1987), who shows, in a model with only two types of workers, that within-technology substitution effects raise optimal marginal tax rates at the bottom and lower them at the top.

Appendix D.2 also shows that the combined impact of directed technical change and within-technology substitution effects depends on condition (19) for strong bias. If there is strong bias, the technical change term TE^*_θ dominates the substitution term SE^*_θ, leading to a positive marginal tax rate at the top and a negative marginal tax at the bottom. Inversely, if there is no strong bias, the optimal marginal tax rate is positive at the bottom and negative at the top, following the logic of Stiglitz (1987).

It is well known in the literature that results for optimal marginal tax rates at the highest income level are very local. They do not extend approximately to incomes even slightly below the top. To derive analytical insights into the structure of optimal tax rates over more relevant ranges of the income distribution, it is useful to consider the CES case in more detail.

Using the CES versions of TE^*_θ and SE^*_θ from Appendix D.2, the conditions for optimal marginal tax rates simplify as follows.

Proposition 5. *Suppose the conditions of Proposition 4 are satisfied and F and Φ take the CES form introduced in Section 3.4. Then, at almost every type θ, optimal marginal tax rates satisfy the following conditions.*

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta}(1-\tilde{g}_\theta) + \gamma^{CES}(1-g_\theta) + \rho^{CES}(1-g_\theta)\,, \tag{32}$$

where all variables are evaluated at equilibrium under the optimal tax T, the function $\beta : \theta \mapsto \beta_\theta$ is given by

$$\beta_\theta := \kappa_\theta^{1+\gamma^{CES}+\rho^{CES}} h_\theta^{\gamma^{CES}+\rho^{CES}} \quad \forall\,\theta\,,$$

while B and b are the cumulative distribution and the density function of β.

Proof. See Appendix D.2. □

Proposition 5 provides a Diamond (1998) style expression for optimal marginal tax rates, taking into account the endogeneity of wages through directed technical change and within-technology substitution effects.

Remarkably, when welfare weights are exogenous and the disutility of labor is iso-elastic (such that e_θ is exogenous), equation (32) offers a closed-form solution for optimal marginal tax rates. Hence, once the model is calibrated appropriately, optimal marginal tax rates can be simulated directly from equation (32). I do so in the quantitative assessment of optimal taxes with directed technical change in Section 7.

From an analytical point of view, the fact that equation (32) has closed form (under certain conditions) means that it can be used to precisely identify the role of directed technical change in shaping the optimal tax schedule. For that, I use again the concepts of endogenous and exogenous technology planners introduced in Section 5.2. The endogenous technology planner fully understands how the economy works and computes optimal taxes according to equation (32). The exogenous technology planner knows about all fundamentals of the model but mistakenly believes that technology is exogenous. In particular, he observes the economy under a given tax $\overline{T}$ and believes that technology remains fixed at its current state $\phi^*(\overline{T})$, independently of the tax schedule.[28] Lemma 10 in Appendix D.3 shows that the tax $T^{ex}_{\overline{T}}$ perceived as optimal by the exogenous technology planner satisfies the following condition:

$$\frac{T^{ex\prime}_{\overline{T}}(y_\theta)}{1-T^{ex\prime}_{\overline{T}}(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-\overline{B}_{\overline{\beta}_\theta}}{\overline{b}_{\overline{\beta}_\theta}\overline{\beta}_\theta}(1-\tilde{g}_\theta) + \gamma^{CES}(1-g_\theta)\ , \tag{33}$$

where the function $\overline{\beta}:\theta\mapsto\overline{\beta}_\theta$ is given by

$$\overline{\beta}_\theta := \kappa_\theta^{1+\gamma^{CES}} h_\theta^{\gamma^{CES}}\left(\phi^*(\overline{T})\right)^{1+\gamma^{CES}} \quad \forall\,\theta\ ,$$

while $\overline{B}$ and $\overline{b}$ are the cumulative distribution and the density function of $\overline{\beta}$.[29]

Comparing the exogenous technology planner's tax rates (33) with those of the endogenous technology planner in equation (32), there are two differences. First, the endogenous technology planner takes into account the directed technical change adjustment

$$\rho^{CES}(1-g_\theta)\ .$$

This term is increasing in θ (as welfare weights are decreasing in θ at the optimum) and in this sense necessitates a progressive adjustment of the tax schedule. The intuition for this adjustment is the same as for the top and bottom tax rate adjustments in the general case:

[28] Formally, the exogenous technology planner bases his computation of optimal taxes on the reduced form equations (2), (1), (6), and (8), but replaces the equilibrium technology equation (7) by the "wrong" equation

$$\phi^*(l) = \phi^*(l(\overline{T})) = \operatorname*{argmax}_{\phi\in\Phi} F(l(\overline{T}),\phi) \quad \forall l\ .$$

[29] Note that in the present environment the concept of an exogenous technology planner strictly generalizes the self-confirming policy equilibrium of Rothschild and Scheuer (2013). Specifically, when setting $\overline{T}$ to the tax in the self-confirming policy equilibrium, the exogenous technology planner's preferred tax $T^{ex}_{\overline{T}}$ is exactly the self-confirming policy equilibrium tax. Hence, comparing the exogenous and endogenous technology planners' preferred taxes for arbitrary initial taxes $\overline{T}$ strictly includes the comparison between the true optimal tax and the self-confirming policy equilibrium tax.

lowering marginal tax rates at the bottom and raising them at the top induces technical change that compresses the wage distribution and hence improves equity.

The second difference is that the endogenous technology planner uses the hazard ratio of β whereas the exogenous technology planner uses that of $\overline{\beta}$. The function β can be interpreted as the degree of exogenous inequality in the model: if labor supply was identical across all workers, wages would be proportional to β. The function $\overline{\beta}$ instead is the exogenous technology planner's wrong inference about the degree of exogenous inequality. The exogenous technology planner believes that, if all labor types' supply was identical, wages would be proportional to $\overline{\beta}$ instead of β.

It can be shown that (see Appendix D.4)

$$\frac{1 - B_{\beta_\theta}}{b_{\beta_\theta} \beta_\theta} < \frac{1 - \overline{B}_{\overline{\beta}\theta}}{\overline{b}_{\overline{\beta}_\theta} \overline{\beta}_\theta} \quad \forall \theta \tag{34}$$

if the tax schedule $\overline{T}$, under which the exogenous technology planner observes the economy, has a rate of progressivity smaller than one. I assume this to be the case henceforth.[30] The second adjustment therefore reduces marginal tax rates everywhere.

Intuitively, the exogenous technology planner overestimates the degree of exogenous inequality in the economy, because he mistakenly believes that the skill bias of the equilibrium technology under $\overline{T}$ is exogenous. The endogenous technology planner in contrast understands that this skill bias is the endogenous adaptation of technology to the fact that more skilled workers supply more labor under $\overline{T}$. Since more exogenous inequality calls for higher marginal tax rates to increase the redistributive lump-sum payment to all workers, the exogenous technology planner chooses elevated marginal tax rates everywhere.[31]

To summarize, both directed technical change adjustments reduce marginal tax rates in the lower part of the income distribution, whereas they work in opposite directions in the upper part. I next consider the lower and upper tail of the income distribution and show that the directed technical change adjustments can be signed unambiguously in both tails.

6.1. Optimal Taxes in the Lower Tail

For low income levels both directed technical change adjustments reduce marginal tax rates, so we directly arrive at the following conclusion.

Corollary 3. *Suppose the conditions of Proposition 4 are satisfied, aggregate production (F, Φ) is CES, welfare weights are exogenous, continuous and strictly decreasing in θ, and the tax $\overline{T}$ satisfies $P_{\overline{T}}(y) < 1$ for all y. Let T denote the optimal tax schedule as characterized by Proposition 5 and $T^{ex}_{\overline{T}}$ the tax schedule perceived as optimal by the exogenous technology planner as given by equation* (33).

[30] If $P_{\overline{T}}(y) > 1$, the labor supply elasticity with respect to the wage, ϵ^w_θ, becomes negative, which produces some counterintuitive results. Assuming that $P_{\overline{T}}(y) < 1$ everywhere is reasonable, as empirical tax schedules typically have rates of progressivity far below one.

[31] An intuition for the positive impact of exogenous inequality on marginal tax rates is that a higher degree of exogenous inequality implies that the pre-tax income distribution will respond less strongly to rising tax rates, such that redistribution can be achieved at a lower efficiency loss. See Proposition 7 in Appendix D.2 and the subsequent discussion for details.

Then, we have

$$T'(y_\theta(T)) < T^{ex\prime}_{\overline{T}}(y_\theta(T^{ex}_{\overline{T}})) \quad \forall \theta \leq \tilde{\theta}_1$$

for some $\tilde{\theta}_1 > \underline{\theta}$.
Moreover, if $\gamma^{CES} + \rho^{CES} > 0$,

$$T'(y_\theta(T)) < 0 \quad \forall \theta < \tilde{\theta}_2$$

for some $\tilde{\theta}_2 > \underline{\theta}$.

Proof. See Appendix D.4. □

Corollary 3 establishes that accounting for directed technical change unambiguously lowers marginal tax rates for a non-degenerate interval of types at the lower end of the income distribution. If there is strong bias, that is, if $\gamma^{CES} + \rho^{CES} > 0$, directed technical change effects even call for negative marginal tax rates in the lower tail of the income distribution. The idea behind this result is that negative marginal tax rates at the bottom stimulate the labor supply of low-skilled workers. This in turn induces firms to operate technologies with a higher complementarity to low-skilled workers, which raises their wages. In the case of strong bias, this induced technical change effect is strong enough to raise low-skilled workers' wages above their initial level despite them being in greater supply. This provides utility gains for the low-skilled and improves equity.

6.2. Optimal Taxes in the Upper Tail

For high incomes the directed technical change adjustments work in opposite directions. Yet, when assuming realistically that the upper tail of the income distribution resembles the tail of a Pareto distribution, the ambiguity resolves.
In particular, suppose that we observe the economy at a tax $\overline{T}$ with a constant marginal top tax rate. Suppose in addition that under this tax the inverse hazard rate of the income distribution is proportional to y at high incomes, and hence $(1 - M_{y_\theta})/(m_{y_\theta} y_\theta)$ converges to a constant when θ approaches $\overline{\theta}$ (as is the case for a Pareto distribution). One can then trace this property back to properties of the distribution of the exogenous inequality measure β that gives rise to the observed income distribution. Specifically, it turns out that $(1 - B_{\beta_\theta})/(b_{\beta_\theta} \beta_\theta)$ must also converge to a constant for high types. Moreover, this constant can be recovered from the tail parameter of the observed income distribution. Using the optimal marginal tax rates from Proposition 5 then yields a parametric expression for the optimal marginal tax rate in the upper tail of the income distribution.

Corollary 4. *Let the conditions of Proposition 4 be satisfied and F and* Φ *take the CES form introduced in Section 3.4. Suppose at a tax* $\overline{T}$, *with* $\overline{T}'(y) = \tau^{top}$ *for all* $y \geq \tilde{y}$ *and some threshold* $\tilde{y}$, *the income distribution satisfies*

$$\lim_{\theta \to \overline{\theta}} \frac{1 - M_{y_\theta}}{m_{y_\theta} y_\theta} = \frac{1}{a}$$

for some $a > 1$. *Moreover, let the disutility of labor be iso elastic with* $e_\theta = e$ *for all* θ, *and welfare weights satisfy*

$$\lim_{\theta \to \overline{\theta}} g_\theta = g^{top}$$

at the optimal tax.
Then, the optimal tax T satisfies

$$\lim_{\theta \to \overline{\theta}} \frac{T'(y_\theta)}{1 - T'(y_\theta)} = \frac{1 - g^{top}}{ae} + \frac{a-1}{a} \gamma^{CES} (1 - g^{top}) + \frac{a-1}{a} \rho^{CES} (1 - g^{top}) \,. \tag{35}$$

Proof. See Appendix D.4. □

Complementing Corollary 4, Corollary 6 in Appendix D.4 shows that under the same conditions the exogenous technology planner's preferred tax satisfies

$$\lim_{\theta \to \overline{\theta}} \frac{T^{ex\prime}_{\overline{T}}(y_\theta)}{1 - T^{ex\prime}_{\overline{T}}(y_\theta)} = \frac{1 - g^{top}}{ae} + \frac{a-1}{a} \gamma^{CES} (1 - g^{top}) \,.$$

This expression is smaller than the endogenous technology planner's preferred rate (35). Hence, taking into account directed technical change when designing optimal taxes leads to unambiguously higher marginal tax rates in the upper (Pareto) tail of the income distribution.

7. Quantitative Analysis

To assess the quantitative relevance of directed technical change effects, I calibrate the CES version of the model to estimates from the empirical literature on directed technical change. I use the calibration to simulate the effects of tax reforms and to compute optimal taxes for the endogenous and the exogenous technology planner.

7.1. Calibration

The calibration proceeds as follows. First, I set the wage elasticity parameters γ^{CES} and ρ^{CES} (equivalently, σ and δ), the labor supply elasticity e (assuming iso-elastic disutility of labor), and the initial tax function $\overline{T}$ (approximating the US income tax system in 2005) on the basis of existing empirical estimates. In the second step, I infer the exogenous technology parameter κ from the US income distribution in 2005.

Within-Technology Substitution Effects The within-technology substitution elasticity γ^{CES} and the induced technical change elasticity ρ^{CES} govern the response of relative wages to changes in relative labor inputs. Induced technical change effects are likely to arise with considerable delay, implying that to measure ρ^{CES}, one has to track relative wages over a long period of time after an exogenous change in labor inputs occurred. Within-technology substitution, in contrast, does not require firms to change their production technology, so its effects are likely to occur over a much shorter period of time. The timing of the effects therefore provides an opportunity to identify γ^{CES} and ρ^{CES} separately.
The empirical literature that aims to identify an elasticity of substitution between differentially skilled worker groups without explicit reference to directed technical change has typically focused on comparably short time periods of about one year or slightly more. I take these estimates to set γ^{CES}.

Besides the timing of the effects, an important property in which many empirical studies differ is the definition of the skill groups between which an elasticity of substitution is estimated. Many studies focus on college graduates versus those without a college degree. Others, however, consider high school graduates versus high school dropouts. Dustmann, Frattini and Preston (2013) stand out in that they estimate substitution elasticities between workers located at 20 different points in the wage distribution. They test for heterogeneity in these elasticities but find no evidence for it. In light of this result, the CES assumption, which imposes a single elasticity of substitution between any two disjoint groups of workers, seems an acceptable simplification. It implies that all estimates, irrespective of the definition of skill groups, are equally relevant for the calibration of γ^{CES}.

Acemoglu (2002) summarizes the consensus of the literature at that time as σ being somewhere between 1.4 and 2, which implies that γ^{CES} falls between -0.5 and -0.7. The results of Carneiro et al. (2019) imply a short-run elasticity, measured within two years after the skill supply shock, of -0.5 (for a detailed description of Carneiro et al. 2019 see below). This value falls within the consensus range observed by Acemoglu (2002). Moreover, Carneiro et al. (2019) is the only study that estimates wage responses at different points in time. Thereby, it provides estimates of γ^{CES} and ρ^{CES} obtained consistently within a single framework. For these reasons I set $\gamma^{CES} = -0.5$, the estimate implied by Carneiro et al. (2019).

Directed Technical Change Effects A few studies measure the response of wages to skill supply shocks over substantially longer periods of time (about 10 years or more). Most of them explicitly reference directed technical change and provide evidence for technology adjustments being an important driver of the long-run wage responses. Since this applies only to a handful of papers, I give a brief overview over each of them in Appendix E.1.

Table 1 shows the results of these papers. The short-run estimates are -0.55 and -0.53, which (further) motivates my choice of γ^{CES}. Estimates over a period of about 10 years are consistently close to zero, ranging from -0.1 to 0. Finally, the estimate from Carneiro et al. (2019) for an adjustment period of 17 years shows an effect of 0.5. These long-run effects are total effects, in the sense that they include both within-technology and between-technology (directed technical change) substitution. Hence, they map into the sum of γ^{CES} and ρ^{CES}.

Based on Table 1 I will consider two cases. The first case, derived from the 10 year estimates, sets $\gamma^{CES} + \rho^{CES}$ to -0.1, which, given the set value for γ^{CES}, implies $\rho^{CES} = 0.4$. In this case, within-technology substitution and directed technical change effects are of similar magnitudes and almost cancel each other (given that they work in opposite directions). Hence, accounting for directed technical change puts the analysis back close to the case with exogenous wages studied extensively in the literature on optimal taxation. In the second case, based on the 17 year estimate of Carneiro et al. (2019), I set $\gamma^{CES} + \rho^{CES}$ to 0.5, such that $\rho^{CES} = 1$. In this case, directed technical change dominates within-technology substitution, that is, there is strong bias (see Section 4). I therefore call this the strong bias case.

The conservative case is supported by all four studies in Table 1. Moreover, there are at least two further papers that, for different reasons, do not provide estimates that could be used to infer γ^{CES} and ρ^{CES}, but nevertheless support the view of the conservative case that the

Study	Skill Groups	Time Horizon	Geographical Level	Cross-wage Effect
Carneiro et al. (2019)	College vs. non-college	2 years	Norwegian municipalities	−0.55
Carneiro et al. (2019)	College vs. non-college	11 years	Norwegian municipalities	0
Carneiro et al. (2019)	College vs. non-college	17 years	Norwegian municipalities	0.5
Lewis (2011)	High school vs. high-school dropout	10 years	US metro areas	−0.14
Dustmann and Glitz (2015)	Postsecondary vocational degree or apprenticeship versus no postsecondary education	10 years	German local labor markets (aggregates of German counties)	−0.09
Morrow and Trefler (2017)	Some tertiary versus no tertiary education	Short (see description in Appendix E.1)	38 countries	−0.53
Morrow and Trefler (2017)	Some tertiary versus no tertiary education	Long (see description in Appendix E.1)	38 countries	−0.11

Table 1. The table shows estimates of the effect of relative skill supply changes on relative wages from a set of empirical studies. A brief outline of each study with an explanation of how the numbers in the last column are derived from the respective study's results is provided in Appendix E.1.

long-run wage effects of skill supply shocks are close to zero. First, Blundell, Green and Jin (2018) document that a large and sudden increase in the share of individuals holding a college degree in the 1990s in the UK left the wage premium associated with college education basically unchanged. They provide empirical results suggesting that firms responded to the hike in the relative supply of college graduates by adopting production forms that granted higher degrees of autonomy and responsibility to their workers, which likely benefited highly qualified workers' productivity. They argue that these endogenous technology adjustments offset the negative within-technology substitution effect on the college premium. Second, Clemens, Lewis and Postel (2018) study the effect of the exclusion of half a million Mexican farm workers (braceros) from the US in 1965 on domestic farm workers' wages and find no evidence for differential wage changes following the event in states heavily exposed to the bracero exclusion relative to less exposed states. They provide striking evidence for rapid adoption of labor-replacing technologies on farms in heavily exposed states after the exclusion.

The strong bias case is supported directly only by Carneiro et al. (2019). Nevertheless, I believe that the case for strong bias is stronger than it might appear from this. The studies in Table 1 analyze the differential evolution of wages between often quite narrowly defined geographical areas, which were hit by plausibly exogenous skill supply shocks differentially. By construction, such estimates miss all directed technical change effects that manifest themselves on a higher geographical level. Since the relevant markets for innovative technologies are plausibly much larger than most of the geographical units listed in Table 1, the estimates are likely to capture mostly the effects of investments into adopting already existing technologies, rather than the effects of re-directed innovative activity. The model developed in Section 3.1 implies that endogenous adoption and innovation work in the same direction, cumulating in the total directed technical change effect that is represented by ρ^{CES}. Hence, the estimates of Table 1 likely miss part of ρ^{CES} and therefore underestimate it.[32][33]

Finally, it is important to note at this point that there are at least two empirical studies that are in more or less open contradiction to the predictions of directed technical change theory. First, Blum (2010) finds that in a panel of countries, increases in the relative supply of skilled workers reduce their relative wages by more in the long-run than in the short-run. Second, Ciccone and Peri (2005) report long-run estimates for the elasticity of substitution between college graduates and non-college workers of about 1.5, which maps into a total wage elasticity $\gamma^{CES} + \rho^{CES}$ of -0.7. With $\gamma^{CES} = -0.5$, this implies a negative ρ^{CES}, inconsistent with theory. These results should serve as a word of caution regarding the simulation results below. Yet, I do not respect them directly in the simulations. After all, calibrating a model to empirical results that contradict the qualitative predictions of the model makes no sense.

Labor Supply Elasticity I assume an iso-elastic disutility of labor as introduced in Section 3.4. This necessitates calibration of the hypothetical labor supply elasticity along the linearized budget set represented by the parameter e. I choose e such that the elasticity of taxable income with respect to changes in the marginal retention rate implied by the model matches empirical estimates of this elasticity. Starting from e, the elasticity of taxable income has to account for potential non-linearities in the tax scheme and for the equilibrium response of wages to changes in the aggregate labor supply of workers of a given type. The first adjustment is accommodated by the elasticity ϵ^R, as explained in Section 3.2. The second adjustment leads to the following expression for the model-implied elasticity of taxable income (see Appendix C.4 for a more detailed explanation of this elasticity):

$$\frac{\epsilon_\theta^R}{1-(\gamma^{CES}+\rho^{CES})\epsilon_\theta^w} = \frac{e}{1+eP_{\overline{T}}(y_\theta)-(\gamma^{CES}+\rho^{CES})(1-P_{\overline{T}}(y_\theta))e}\,.$$

The wage response to a change in a type's labor supply is again likely to differ between the short and the long run. The above expression incorporates the long-run response, as evidenced by its use of $\gamma^{CES} + \rho^{CES}$. Most reliable estimates of the elasticity of taxable income, however, measure income responses over rather short periods of time (Saez et al., 2012).

[32] Another piece of evidence in favor of strong bias is provided by Fadinger and Mayr (2014). They show that in a cross section of countries, relative skill supply measures are negatively correlated with relative unemployment rates of more versus less skilled workers and with relative emigration rates of skilled workers. In a directed technical change model with frictional labor markets and endogenous migration, they show that both correlations can be interpreted as signs of strong relative bias of technology.

[33] A potential source of upwards bias in directed technical change effects obtained by comparing small geographical units are Rybczynski effects: a rise in relative skill supply in one region increases the region's exports of skill-intensive goods, which raises skilled workers' wages. This wage increase may be mistakenly attributed to directed technical change. All the studies listed in Table 1, however, provide different forms of evidence suggesting that adjustments in the output mix of their observation units are not driving their results. See the respective papers for details.

Hence, a more appropriate theoretical counterpart of these estimates is given by the expression

$$\tilde{\varepsilon}_\theta^R := \frac{e}{1 + eP_{\overline{T}}(y_\theta) - \gamma^{CES}(1 - P_{\overline{T}}(y_\theta))e},$$

which only includes the wage effects of within-technology factor substitution. I use this expression to calibrate e given estimates of $\tilde{\varepsilon}_\theta^R$. In doing so, I set $P_{\overline{T}}$ to 0, as empirical tax systems are piece-wise linear.

Following Saez et al. (2012), who partially review the extensive literature on the elasticity of taxable income, I choose a value of 0.25 for $\tilde{\varepsilon}_\theta^R$. This maps into a value for e of 0.27. In the simulations of tax reforms effects on wages, I will also experiment with a larger labor supply elasticity located more towards the upper bound of empirical estimates. For that, I use the taxable income elasticity of 0.57 from Gruber and Saez (2002), which Diamond and Saez (2011) call a conservative upper bound. The implied value for e is 0.64.

Initial Tax System I set the initial tax system, denoted by $\overline{T}$, as an approximation to the US income tax in 2005. I follow Heathcote et al. (2017), who show that a constant-rate-of-progressivity schedule as introduced in Section 3.4 provides a good approximation. Heathcote et al. (2017) estimate the parameters of such a tax function on 2000 to 2005 income and tax data for the US and obtain values of $p = 0.181$ and $\lambda = 5.568$. I use these values in all simulations.

Exogenous Technology With the parameters γ^{CES}, ρ^{CES}, e, and $\overline{T}$ calibrated, the exogenous technology parameter κ is identified by the income distribution under the initial tax system $\overline{T}$. I approximate the income distribution by smoothly combining a lognormal distribution for incomes below \$200k and a Pareto distribution with tail parameter 1.5 above \$200k (Diamond and Saez, 2011). Moreover, I assume that the type distribution h is standard uniform on $[\underline{\theta}, \overline{\theta}] = [0, 1]$. In the CES case, this assumption is insubstantial, because the cross-wage elasticity between any two distinct types of workers is independent of the types' locations in the type space. Given an estimate of the income distribution, it is straightforward to compute the function κ from workers' first-order condition (2) and the wage equation (13). The procedure is described in more detail in Appendix E.2.

Welfare Function Finally, I use a welfare function of the type

$$V(\{u_\theta\}_{\theta\in\Theta}) = \int_{\underline{\theta}}^{\overline{\theta}} \frac{1}{1-r} u_\theta^{1-r} h_\theta \, d\theta \, ,$$

where the relative inequality aversion parameter r allows to vary the strength of the preference for equity inherent in the welfare function in a flexible way (Atkinson, 1970).

7.2. Simulation

Given the calibrated CES version of the model, I simulate the effect of tax reforms on wages and welfare and compute optimal taxes on the basis of the analytical results of Sections 5 and 6.

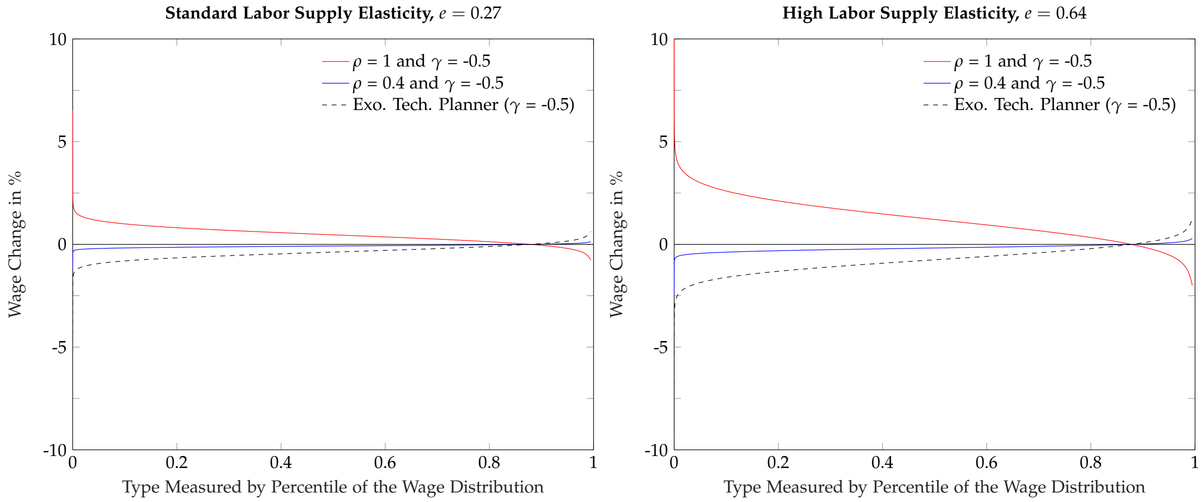

Figure 1. The figure displays the total wage changes in log points induced by the progressive tax reform described in the text. In the left panel, the labor supply elasticity takes its standard value of $e = 0.27$. The right panel shows wage changes for a high value of $e = 0.64$. Wage changes are given for workers at each percentile in the (pre- and post-reform) wage distribution. In each panel, the three plots show the wage changes predicted in the strong bias case ($\rho = 1$), in the conservative case ($\rho = 0.4$), and by the exogenous technology planner ($\rho = 0$).

Tax Reforms To assess the results on the effects of tax reforms quantitatively, I study a hypothetical tax reform that reverses the cumulative impact on tax progressivity of US income tax reforms from 1970 to 2005. As documented by Piketty and Saez (2007), the US income tax system underwent a series of regressive reforms in this period. Heathcote et al. (2017) estimate the decline in tax progressivity between 1970 and 2005 to be 0.034 when measured by the progressivity parameter of a constant-rate-of-progressivity tax schedule. Taking as a starting value the progressivity estimate of $p = 0.181$, I hence ask what are the effects of raising the progressivity of a constant-rate-of-progressivity tax from 0.181 (its 2005 US value) to 0.215 (its 1970 US value).[34]

In the case of strong bias ($\rho^{CES} + \gamma^{CES} > 0$), such a progressive reform will reduce wage inequality. A particular focus of the analysis will thus be whether taking back the regressive tax reforms of the past decades would result in a meaningful reduction in US wage inequality when accepting the strong bias calibration described above. Reversedly, the results are informative about whether regressive tax reforms could have played a role, via directed technical change effects, in the rise of US wage inequality observed over the period under consideration.

I compute the wage effects of the described reform using the expressions provided by Corollaries 2 and 5. For the exogenous technology planner, who ignores directed technical change, the effects are given by equation (53) in Corollary 5. For the two cases with directed technical change effects, total wage effects are given by the sum of (the right-hand-sides of) equations (27) and (53).

The results are displayed in the left panel of Figure 1. In the conservative case, the reform has almost no effect on wages. This was expected, because directed technical change and

[34] I choose the post-reform value for the parameter λ (the second parameter of a constant-rate-of-progressivity tax function) such that, in the conservative case described above ($\rho^{CES} = 0.4$), the reform leaves tax revenue unchanged.

within-technology substitution effects cancel each other approximately in this case. The exogenous technology planner predicts moderate wage decreases for low-skilled workers and even smaller gains for the high-skilled. Even in the strong bias case, wage effects are modest. Wages for low-skilled workers rise by less than 2% while wages for high-skilled workers fall by less than 1%. Towards the very ends of the wage distribution, the effects become somewhat stronger, but only few workers are affected by these more substantial changes.

To put the results into perspective, I compute the predicted change in the ratio between the 90th and the 10th percentile of the wage distribution. In the conservative case, this amounts to -0.2%, whereas in the strong bias case the ratio changes by about -1% percent. These changes can be compared to the actually observed changes in the US wage distribution between 1970 and 2005. The data presented in Acemoglu and Autor (2011) suggest that the 90-10-percentile ratio of wages rose by about 30% percent between 1970 and 2005. Compared to this increase in inequality, the effects of the tax reforms that took place between 1970 and 2005 on the US wage distribution seem rather small even in the strong bias case.

A likely explanation for this somewhat sobering result is that the labor supply responses to tax reforms found in the empirical literature, reflected in the elasticity parameter e, are typically small. Since the entire directed technical change effect of tax reforms is transmitted via labor supply responses, the directed technical change effects tend to be small as well.

To assess how sensitive the results are to the choice of the labor supply elasticity, the right panel of Figure 1 plots wage changes when e is set to 0.64. As argued in the previous section, this value falls into the high range of estimates from the empirical literature on taxable income elasticities. Expectedly, the results do not change remarkably for the conservative case. For the strong bias case, the predicted wage growth for low-wage earners approximately doubles from below 2 to almost 4%. Again computing the effect on the 90-10-percentile ratio yields a value of -2.7%. Still, this is orders of magnitude below the total changes observed in the US wage distribution over the past 50 years.

Finally, Figure 2 computes the percentage change in welfare induced by the tax reform for different values of the relative inequality aversion parameter r (based on Proposition 2). As observed analytically, the influence of directed technical change effects on the welfare assessment of a given progressive reform is ambiguous. The figure shows that for lower degrees of inequality aversion, directed technical change raises the welfare gains from the progressive reform. Here, the reduction in wage inequality induced by directed technical change outweighs the loss in tax revenue, which translates into a reduction in the lump-sum payment to all workers. For high values of inequality aversion, for which the welfare function approaches a Rawlsian objective, the negative revenue effect from directed technical change becomes dominant, reducing the welfare gains from the reform. Interestingly, for very high levels of inequality aversion, accounting for directed technical change even switches the sign of the welfare effect. When ignoring directed technical change, the reform appears to raise welfare; taking into account directed technical change effects makes the reform undesirable.

Notwithstanding these results, Proposition 3 implies that even for such very high values of inequality aversion, there must be a different progressive reform that raises welfare when accounting for directed technical change.

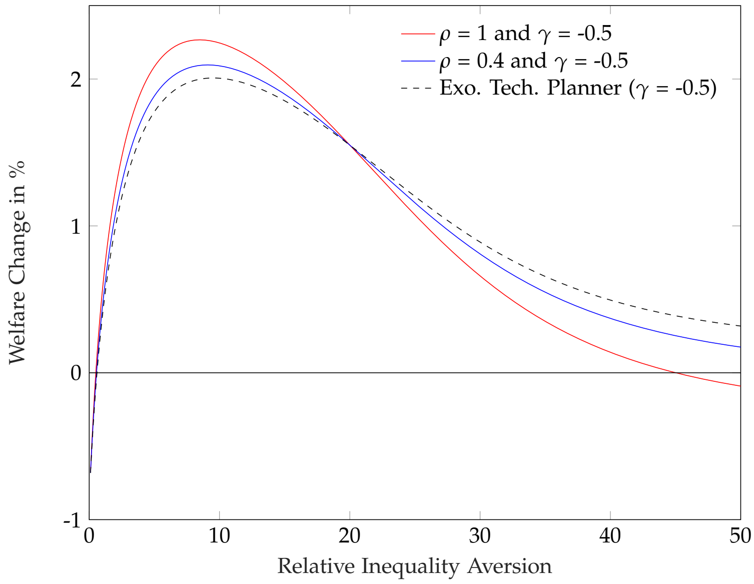

Figure 2. The figure displays welfare changes in log points induced by the progressive tax reform described in the text. Welfare changes are given for different values of the relative inequality aversion parameter r. The three plots show the welfare changes predicted in the strong bias case ($\rho = 1$), the conservative case ($\rho = 0.4$), and by the exogenous technology planner ($\rho = 0$).

Optimal Taxes Turing to optimal taxes, I consider two cases. First, I set the relative inequality aversion parameter to 1, which corresponds to a welfare function that is logarithmic in individual utilities. Second, I set the parameter to 50, approximating a Rawlsian welfare function. I then compute optimal taxes for the endogenous and the exogenous technology planners from Proposition 5 (endogenous technology planner) and Lemma 10 (exogenous technology planner). For the endogenous technology planner, I again distinguish between the conservative and the strong bias case regarding the strength of directed technical change effects.

The resulting optimal marginal tax rates are displayed in Figure 3. As predicted by theory, directed technical change effects reduce optimal marginal tax rates in the lower part of the income distribution and increase them in the upper part.

For a low equity concern (the left panel), the point where directed technical change effects reverse their sign is close to the US average income, between \$60k and \$70k. The effect on optimal top tax rates is small. While the exogenous technology planner prefers an asymptotic top tax rate of marginally below 70%, the preferred rate in the conservative directed technical change case is marginally above 70%. In the strong bias case, it amounts to 72.5%. More significant are the changes in marginal tax rates between \$200k and \$400k and below the median income (indicated by the vertical line in the left panel). For the range between \$200k and \$400k, the strong bias case points towards an upwards adjustment of marginal tax rates of more than 5 percentage points relative to the exogenous technology planner. For below median incomes, the difference is even stronger. Optimal marginal tax rates in the strong bias case decline quickly below the median income, reaching a low of slightly above 40% for low incomes. The exogenous technology planner's tax rates and, to a lesser extent, taxes in the conservative case increase when moving below the median income, reaching about 65% for

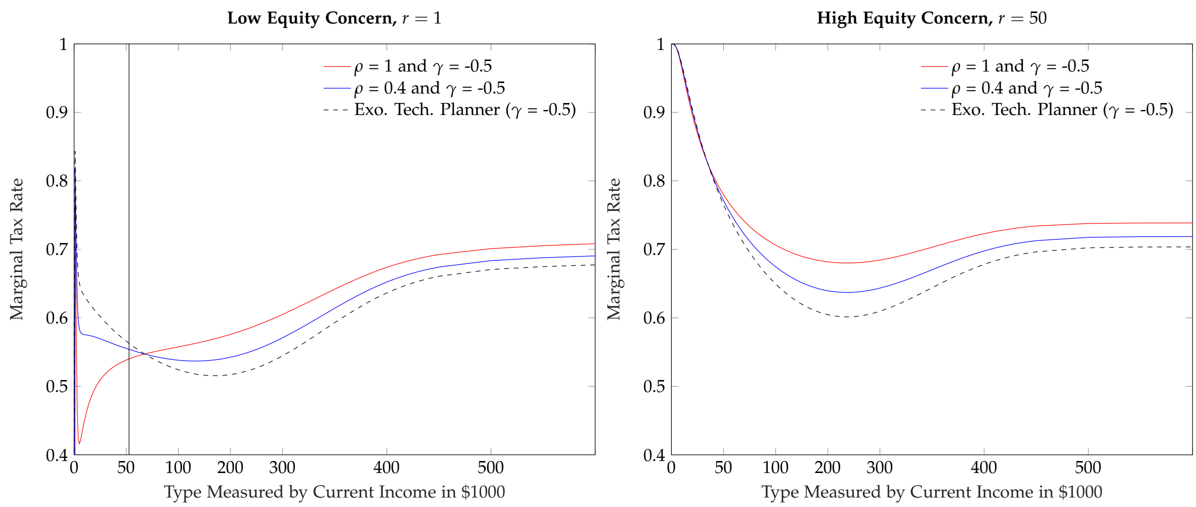

Figure 3. The figure displays optimal marginal tax rates for worker types, where worker types are identified by their income under the current tax system. In the left panel, the relative inequality aversion parameter is $r = 1$. In the right panel, $r = 50$. In each panel, the three plots show optimal marginal taxes as computed by the endogenous technology planner in the case of strong bias ($\rho = 1$), in the conservative case ($\rho = 0.4$), and as computed by the exogenous technology planner. The vertical line in the left panel indicates the US median income in 2005 of about $52k.

low incomes in the case of the exogenous technology planner.

The striking difference in the evolution of marginal tax rates below the median is driven by the desire to stimulate labor supply of below-median income earners in the strong bias case. With strong bias, stimulating these workers' labor supply induces technological developments in favor of low-income workers that outweigh the substitution effects at fixed technology. Hence, decreasing marginal tax rates for below-median earners provides a way to complement the redistribution of income via the tax system by redistribution of pre-tax incomes.

Remarkably, optimal marginal tax rates in the strong bias case do no longer follow the familiar U-shape (Diamond, 1998). Ignoring their volatile behavior at the very bottom of the income distribution, marginal taxes increase monotonically in income.

Towards the very bottom of the income distribution, marginal tax rates approach 100% for the exogenous technology planner and in the conservative case. In the strong bias case, they first shoot up to almost 70% and decline then to a value below zero (compare Corollary 3). This somewhat erratic behavior is driven by extreme values of the inverse hazard ratio of the exogenous technology parameter κ, which in turn are inherited from extreme behavior of the inverse hazard ratio of the income distribution at the bottom of the income distribution under current taxes. This is likely due to the fact that a lognormal distribution does not approximate the very low end of the empirical income distribution very well. The quantitative results for tax rates at the very bottom should therefore not be overrated.

Turning to the right panel of the figure, the striking result for marginal tax rates below the median income in the strong bias case disappears. With a close-to-Rawlsian welfare function, most incomes below the median are not relevant from a social perspective, except for their contribution to tax revenue. Hence, the incentive to redistribute pre-tax income from high earners to those between the very bottom and the median disappears. Marginal taxes on income earners between \$100k and \$400k are, however, still much higher when accounting for directed technical change. The negative marginal tax rates at the bottom of the income distribution predicted by Corollary 3 for the strong bias case are concentrated that close to zero that they are not discernible in the figure. So, when welfare weights are shifted from low income earners to very low income earners, the U-shape of optimal marginal tax rates is restored even with strong bias. The only substantial remaining difference to the exogenous technology case is the upwards adjustment of marginal tax rates on high incomes.

8. Conclusion

I investigate the implications of directed technical change for the design of non-linear labor income taxes. First, I develop a model with directed technical change and endogenous labor supply, in which the structure of labor supply determines the direction of technical change. Tax reforms affect the direction of technical change by altering the structure of labor supply.
I derive conditions under which any progressive income tax reform induces technical change that compresses the wage distribution. Relatedly, using a welfare measure that values equity across workers, I show that accounting for directed technical change unambiguously increases the set of tax schedules that can be improved in terms of welfare by means of progressive tax reforms. Finally, when directed technical change is taken into account – as opposed to treating technology as exogenous – optimal marginal tax rates are higher in the upper tail and lower in the lower tail of the income distribution. Optimal marginal tax rates in the lower tail may even become negative.
I quantify the results based on estimates from the empirical literature on directed technical change. Even with relatively strong directed technical change effects, the impact of progressive tax reforms on wage inequality is modest. The reason are empirically small responses of labor supply to marginal tax rates. The impact of directed technical change effects on optimal marginal tax rates, however, is substantial. With estimates for the strength of directed technical change effects in the upper range of the empirical results, optimal marginal tax rates are no longer U-shaped but increase almost monotonically in income. Optimal marginal tax rates for workers who earn about half of the 2005 US median income are reduced by more than 10 percentage points relative to the benchmark where technology is treated as exogenous.

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A. Proofs and Additional Results for the Setup

Here I provide proofs, derivations, and some additional results omitted from the main text for Section 3. I start by deriving the labor demand equation (3.1) in detail to demonstrate that the functional derivative D_{l_θ} works as expected.

A.1. Derivation of the labor demand equation (3.1)

Final good firm profits are given by

$$\tilde{G}(L, \phi, q) - \int_{\underline{\theta}}^{\overline{\theta}} w_\theta L_\theta \, d\theta - \sum_{j=1}^{J} \int_0^1 p_{j,k} q_{j,k} \, dk.$$

Taking the derivative D_{L_θ} as defined in Section 3.2 and equating it with zero yields:

$$D_{L_\theta} \tilde{G}(L, \phi, q) = D_{L_\theta} \int_{\underline{\theta}}^{\overline{\theta}} w_{\tilde{\theta}} L_{\tilde{\theta}} \, d\tilde{\theta}$$

The remaining task is to show that

$$D_{L_\theta} \int_{\underline{\theta}}^{\overline{\theta}} w_{\tilde{\theta}} L_{\tilde{\theta}} \, d\tilde{\theta} = w_\theta.$$

I derive this equality for interior types $\theta \in (\underline{\theta}, \overline{\theta})$ in detail to demonstrate the working of the functional derivative D_{L_θ}. The derivations for the highest and lowest types $\overline{\theta}$ and $\underline{\theta}$ proceed analogously and are therefore omitted.
By definition:

$$D_{L_\theta} \int_{\underline{\theta}}^{\overline{\theta}} w_{\tilde{\theta}} L_{\tilde{\theta}} \, d\tilde{\theta} = \lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\underline{\theta}}^{\overline{\theta}} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \tilde{L}_{\Delta,\theta,\tilde{\theta}} \right) \Big|_{\mu=0} \, d\tilde{\theta} \, .$$

Moreover, by definition of $\tilde{L}_{\Delta,\theta}$:

$$\int_{\underline{\theta}}^{\overline{\theta}} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \tilde{L}_{\Delta,\theta,\tilde{\theta}} \right) \Big|_{\mu=0} \, d\tilde{\theta} = \int_{\theta-\Delta}^{\theta} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \frac{\tilde{\theta} - \theta + \Delta}{\Delta} \right) \Big|_{\mu=0} \, d\tilde{\theta}$$
$$+ \int_{\theta}^{\theta+\Delta} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \frac{\theta - \tilde{\theta} + \Delta}{\Delta} \right) \Big|_{\mu=0} \, d\tilde{\theta} \, .$$

Hence:

$$\int_{\underline{\theta}}^{\overline{\theta}} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \tilde{L}_{\Delta,\theta,\tilde{\theta}} \right) \Big|_{\mu=0} \, d\tilde{\theta} = \int_{\theta-\Delta}^{\theta} w_{\tilde{\theta}} \frac{\tilde{\theta} - \theta + \Delta}{\Delta} \, d\tilde{\theta} + \int_{\theta}^{\theta+\Delta} w_{\tilde{\theta}} \frac{\theta - \tilde{\theta} + \Delta}{\Delta} \, d\tilde{\theta} \, .$$

Then, by L'Hôspital's rule:

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\underline{\theta}}^{\overline{\theta}} \frac{d}{d\mu} w_{\tilde{\theta}} \left(L_{\tilde{\theta}} + \mu \tilde{L}_{\Delta,\theta,\tilde{\theta}} \right) \Big|_{\mu=0} \, d\tilde{\theta} = \lim_{\Delta \to 0} \frac{1}{2\Delta} \int_{\theta-\Delta}^{\theta} w_{\tilde{\theta}} \, d\tilde{\theta} + \lim_{\Delta \to 0} \frac{1}{2\Delta} \int_{\theta}^{\theta+\Delta} w_{\tilde{\theta}} \, d\tilde{\theta} \, .$$

Applying L'Hôspital's rule again, we obtain:

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}}\frac{d}{d\mu}w_{\tilde{\theta}}\left(L_{\tilde{\theta}}+\mu\tilde{L}_{\Delta,\theta,\tilde{\theta}}\right)\Big|_{\mu=0}d\tilde{\theta}=\lim_{\Delta\to 0}\frac{w_{\theta-\Delta}}{2}+\lim_{\Delta\to 0}\frac{w_{\theta+\Delta}}{2}=w_{\theta}\ ,$$

where the last equality requires continuity of w in θ, which I assume is given in equilibrium.

A.2. Labor Supply Elasticities

Next I derive expressions (9) and (10) for the labor supply elasticities with respect to the marginal retention rate and the wage. The starting point is workers' first-order condition (2):

$$v'(l_{\theta}(T,w_{\theta}))=R'_{T}(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}.$$

Taking the derivative $D_{\tilde{\tau}}$ on both sides of the equation yields:

$$v''(l_{\theta}(T,w_{\theta}))D_{\tilde{\tau}}l_{\theta}(T,w_{\theta})=w_{\theta}\ \frac{d}{d\mu}\left(1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))+\mu\right)\Big|_{\mu=0}-T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}^{2}D_{\tilde{\tau}}l_{\theta}(T,w_{\theta})$$

and hence:

$$D_{\tilde{\tau}}l_{\theta}(T,w_{\theta})=\frac{w_{\theta}}{v''(l_{\theta}(T,w_{\theta}))+T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}^{2}}.$$

By definition of ϵ_{θ}^{R} we obtain

$$\epsilon_{\theta}^{R}=\frac{\frac{w_{\theta}(1-T'(w_{\theta}l_{\theta}(T,w_{\theta})))}{v''(l_{\theta}(T,w_{\theta}))l_{\theta}(T,w_{\theta})}}{1+\frac{T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}l_{\theta}(T,w_{\theta})}{1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))}\frac{(1-T'(w_{\theta}l_{\theta}(T,w_{\theta})))w_{\theta}}{v''(l_{\theta}(T,w_{\theta}))l_{\theta}(T,w_{\theta})}}.$$

Again using the first-order condition to replace $(1-T'(wl))w$ by $v'(l)$, we obtain equation (9). For equation (10) differentiate the first-order condition with respect to w_{θ} on both sides,

$$\begin{aligned}v''(l_{\theta}(T,w_{\theta}))\frac{\partial l_{\theta}(T,w_{\theta})}{\partial w_{\theta}}&=1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))\\&\quad-T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}^{2}\frac{\partial l_{\theta}(T,w_{\theta})}{\partial w_{\theta}}-T''(w_{\theta},l_{\theta}(T,w_{\theta}))w_{\theta}l_{\theta}(T,w_{\theta}),\end{aligned}$$

and rearrange it to obtain

$$\frac{\partial l_{\theta}(T,w_{\theta})}{\partial w_{\theta}}=\frac{1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))-T''(w_{\theta}l_{\theta}(T,w_{\theta}))}{v''(l_{\theta}(T,w_{\theta}))+T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}^{2}}.$$

Then, use the definition of ϵ_{θ}^{w} to get

$$\epsilon_{\theta}^{w}=\frac{\left(1-\frac{T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}l_{\theta}(T,w_{\theta})}{1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))}\right)\frac{(1-T'(w_{\theta}l_{\theta}(T,w_{\theta})))w_{\theta}}{v''(l_{\theta}(T,w_{\theta}))l_{\theta}(T,w_{\theta})}}{1+\frac{T''(w_{\theta}l_{\theta}(T,w_{\theta}))w_{\theta}l_{\theta}(T,w_{\theta})}{1-T'(w_{\theta}l_{\theta}(T,w_{\theta}))}\frac{(1-T'(w_{\theta}l_{\theta}(T,w_{\theta})))w_{\theta}}{v''(l_{\theta}(T,w_{\theta}))l_{\theta}(T,w_{\theta})}}.$$

Replacing $(1-T'(wl))w$ by $v'(l)$ yields equation (10).

Note at this point that the second-order condition of workers' utility maximization requires

$$v''(l_\theta) + T'(w_\theta l_\theta) w_\theta^2 \geq 0 \; .$$

If this is satisfied strictly, we can use workers' first-order condition to rearrange the inequality as follows.

$$\frac{T''(w_\theta l_\theta) w_\theta l_\theta}{1 - T'(w_\theta l_\theta)} \frac{v'(l_\theta)}{v''(l_\theta) l_\theta} = P_T(w_\theta l_\theta) e_\theta(l_\theta) > -1 \; .$$

This ensures that the denominators of the expressions for ϵ_θ^w and ϵ_θ^R are strictly positive and the expressions themselves are well defined.

A.3. CES Production Function

Here I derive the expressions for the aggregate production function, the set of feasible technologies, wages, and wage elasticities for the special case of a CES production function for final goods introduced in Section 3.4.

Aggregate Production To derive the aggregate production function $F(l, \phi)$ as given by equation (11), start from its definition:

$$F(l, \phi) = \max_{\{q_\theta\}_{\theta \in \Theta}} \left\{ \tilde{G}(\{h_\theta l_\theta\}_{\theta \in \Theta}, \tilde{\phi}, \{q_\theta\}_{\theta \in \Theta}) - \int_{\underline{\theta}}^{\overline{\theta}} \eta_\theta q_\theta \, d\theta \right\} \; .$$

The first-order conditions for the maximization with respect to q are:

$$\tilde{G}^{\frac{1}{\tilde{\sigma}}} \left(\tilde{\kappa}_\theta \tilde{\phi}_\theta h_\theta^{1-\alpha} l_\theta^{1-\alpha} \right)^{\frac{\tilde{\sigma}-1}{\tilde{\sigma}}} \alpha q_\theta^{\frac{\alpha\tilde{\sigma} - \alpha - \tilde{\sigma}}{\tilde{\sigma}}} = \eta_\theta \quad \forall \theta \; ,$$

which can be rearranged to yield an explicit expression for the maximizer:

$$q_\theta = \left(\frac{\alpha}{\eta_\theta} \right)^{\frac{\tilde{\sigma}}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}} \left(\tilde{\kappa}_\theta \tilde{\phi}_\theta h_\theta^{1-\alpha} l_\theta^{1-\alpha} \right)^{\frac{\tilde{\sigma}-1}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}} \tilde{G}^{\frac{1}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}} \quad \forall \theta \; . \tag{36}$$

Denoting this maximizer by q^* and inserting it into $\tilde{G}$ yields

$$\tilde{G}(\{h_\theta l_\theta\}_{\theta \in \Theta}, \tilde{\phi}, q^*) = \left[\int_{\underline{\theta}}^{\overline{\theta}} \left(\frac{\alpha}{\eta_\theta} \right)^{\frac{\alpha(\tilde{\sigma}-1)}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}} \left(\tilde{\kappa}_\theta \tilde{\phi}_\theta h_\theta^{1-\alpha} l_\theta^{1-\alpha} \right)^{\frac{\tilde{\sigma}-1}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}} \tilde{G}^{\frac{\alpha(\tilde{\sigma}-1)}{(\alpha + \tilde{\sigma} - \alpha\tilde{\sigma})\tilde{\sigma}}} \, d\theta \right]^{\frac{\tilde{\sigma}}{\tilde{\sigma}-1}} ,$$

which can be solved for $\tilde{G}$:

$$\tilde{G}(\{h_\theta l_\theta\}_{\theta \in \Theta}, \tilde{\phi}, q^*) = \alpha^{\frac{\alpha}{1-\alpha}} \left[\int_{\underline{\theta}}^{\overline{\theta}} \left(\eta_\theta^{-\frac{\alpha}{1-\alpha}} \tilde{\kappa}_\theta^{\frac{1}{1-\alpha}} \tilde{\phi}_\theta^{\frac{1}{1-\alpha}} h_\theta l_\theta \right)^{\frac{(1-\alpha)\tilde{\sigma}}{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}} \frac{\tilde{\sigma}-1}{\tilde{\sigma}}} d\theta \right]^{\frac{\alpha + \tilde{\sigma} - \alpha\tilde{\sigma}}{(1-\alpha)\tilde{\sigma}} \frac{\tilde{\sigma}}{\tilde{\sigma}-1}} . \tag{37}$$

This provides an expression for gross aggregate production. Using the maximizer q^* from equation (36) again, the part of gross output that goes into the production of intermediate goods becomes

$$\int_{\underline{\theta}}^{\overline{\theta}} \eta_\theta q_\theta^* \, d\theta = \int_{\underline{\theta}}^{\overline{\theta}} \eta_\theta^{\frac{\alpha-\alpha\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \alpha^{\frac{\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \left(\tilde{\kappa}_\theta \tilde{\phi}_\theta h_\theta^{1-\alpha} l_\theta^{1-\alpha}\right)^{\frac{\tilde{\sigma}-1}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \tilde{G}^{\frac{1}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \, d\theta$$

$$= \alpha^{\frac{\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \tilde{G}^{\frac{1}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\eta_\theta^{-\frac{\alpha}{1-\alpha}} \tilde{\kappa}_\theta^{\frac{1}{1-\alpha}} \tilde{\phi}_\theta^{\frac{1}{1-\alpha}} h_\theta l_\theta\right)^{\frac{(1-\alpha)\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}} \frac{\tilde{\sigma}-1}{\tilde{\sigma}}} \, d\theta \tag{38}$$

$$= \alpha \tilde{G} \, . \tag{39}$$

Combining equations (37) and (38), we obtain net aggregate production F as follows:

$$F(l, \phi) = (1-\alpha)\tilde{G}(\{h_\theta l_\theta\}_{\theta\in\Theta}, \tilde{\phi}, q^*)$$

$$= \left[\int_{\underline{\theta}}^{\overline{\theta}} \left((1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \eta_\theta^{-\frac{\alpha}{1-\alpha}} \tilde{\kappa}_\theta^{\frac{1}{1-\alpha}} \tilde{\phi}_\theta^{\frac{1}{1-\alpha}} h_\theta l_\theta\right)^{\frac{(1-\alpha)\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}} \frac{\tilde{\sigma}-1}{\tilde{\sigma}}} \, d\theta\right]^{\frac{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}}{(1-\alpha)\tilde{\sigma}} \frac{\tilde{\sigma}}{\tilde{\sigma}-1}} .$$

Defining

$$\frac{\sigma-1}{\sigma} := \frac{(1-\alpha)\tilde{\sigma}}{\alpha+\tilde{\sigma}-\alpha\tilde{\sigma}} \frac{\tilde{\sigma}-1}{\tilde{\sigma}}$$

$$\kappa_\theta := (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \eta_\theta^{-\frac{\alpha}{1-\alpha}} \tilde{\kappa}_\theta^{\frac{1}{1-\alpha}} \quad \forall\theta$$

$$\phi_\theta := \tilde{\phi}_\theta^{\frac{1}{1-\alpha}} \quad \forall\theta \, ,$$

net aggregate production becomes

$$F(l, \phi) = \left[\int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta \phi_\theta l_\theta h_\theta)^{\frac{\sigma-1}{\sigma}} \, d\theta\right]^{\frac{\sigma}{\sigma-1}} ,$$

which is equation (11) from the main text.

Set of Feasible Technologies From the R&D resource constraint and the R&D cost function, the set of feasible technologies $\tilde{\phi}$ follows as

$$\left\{\tilde{\phi} : \theta \mapsto \tilde{\phi}_\theta \in \mathbb{R}_+ \mid \int_{\underline{\theta}}^{\overline{\theta}} \tilde{\phi}^{\tilde{\delta}} \, d\theta \leq \overline{C}\right\} .$$

Using the substitution

$$\phi_\theta := \tilde{\phi}_\theta^{\frac{1}{1-\alpha}} \quad \forall\theta \, ,$$

the set of feasible ϕ becomes

$$\Phi = \left\{\phi : \theta \mapsto \phi_\theta \in \mathbb{R}_+ \mid \int_{\underline{\theta}}^{\overline{\theta}} \phi_\theta^{\delta} \, d\theta \leq \overline{C}\right\} ,$$

where $\delta := (1-\alpha)\tilde{\delta}$, as given in the main text.

Wages I derive expression (13) for interior types $\theta \in (\underline{\theta}, \overline{\theta})$. For the boundary types $\underline{\theta}$ and $\overline{\theta}$ the derivations proceed analogously and yield the same result.
Consider first the derivative

$$D_{l_\theta} F(l,\phi) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left. \frac{dF(l+\mu \tilde{l}_{\Delta,\theta},\phi)}{d\mu} \right|_{\mu=0} .$$

Using the definition of $\tilde{l}_{\Delta,\theta}$ this derivative becomes

$$\left. \frac{dF(l+\mu \tilde{l}_{\Delta,\theta},\phi)}{d\mu} \right|_{\mu=0}$$
$$= F(l,\phi)^{\frac{1}{\sigma}} \left[\int_{\theta-\Delta}^{\theta} (\kappa_{\tilde{\theta}}\phi_{\tilde{\theta}}h_{\tilde{\theta}})^{\frac{\sigma-1}{\sigma}} \, l_{\tilde{\theta}}^{-\frac{1}{\sigma}} \frac{\tilde{\theta}-\theta+\Delta}{\Delta} \, d\tilde{\theta} + \int_{\theta}^{\theta+\Delta} (\kappa_{\tilde{\theta}}\phi_{\tilde{\theta}}h_{\tilde{\theta}})^{\frac{\sigma-1}{\sigma}} \, l_{\tilde{\theta}}^{-\frac{1}{\sigma}} \frac{\theta-\tilde{\theta}+\Delta}{\Delta} \, d\tilde{\theta} \right] .$$

Taking limits and applying L'Hôspital's rule yields:

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\theta-\Delta}^{\theta} (\kappa_{\tilde{\theta}}\phi_{\tilde{\theta}}h_{\tilde{\theta}})^{\frac{\sigma-1}{\sigma}} \, l_{\tilde{\theta}}^{-\frac{1}{\sigma}} \frac{\tilde{\theta}-\theta+\Delta}{\Delta} \, d\tilde{\theta} = \frac{1}{2} (\kappa_{\theta}\phi_{\theta}h_{\theta})^{\frac{\sigma-1}{\sigma}} \, l_{\theta}^{-\frac{1}{\sigma}}$$

and

$$\lim_{\Delta \to 0} \frac{1}{\Delta} \int_{\theta}^{\theta+\Delta} (\kappa_{\tilde{\theta}}\phi_{\tilde{\theta}}h_{\tilde{\theta}})^{\frac{\sigma-1}{\sigma}} \, l_{\tilde{\theta}}^{-\frac{1}{\sigma}} \frac{\theta-\tilde{\theta}+\Delta}{\Delta} \, d\tilde{\theta} = \frac{1}{2} (\kappa_{\theta}\phi_{\theta}h_{\theta})^{\frac{\sigma-1}{\sigma}} \, l_{\theta}^{-\frac{1}{\sigma}} ,$$

where I used continuity of κ, h, ϕ, and l in θ. The former two are continuous by assumption; ϕ is continuous in equilibrium if l is continuous, as evident from equation (41) below; and continuity of l is presumed in all equilibria under consideration.
Finally, combine the two previous expressions to obtain

$$D_{l_\theta} F(l,\phi) = (\kappa_{\theta}\phi_{\theta}h_{\theta})^{\frac{\sigma-1}{\sigma}} \, l_{\theta}^{-\frac{1}{\sigma}} F(l,\phi)^{\frac{1}{\sigma}}$$

and therewith

$$w_\theta(l,\phi) = (\kappa_\theta \phi_\theta)^{\frac{\sigma-1}{\sigma}} \, (l_\theta h_\theta)^{-\frac{1}{\sigma}} \, F(l,\phi)^{\frac{1}{\sigma}} .$$

Wage Elasticities Again I focus on the derivations for interior types. Given expression (13), the own-wage substitution elasticity is simply the elasticity of w_θ with respect to l_θ:

$$\gamma_{\theta,\theta} = -\frac{1}{\sigma} .$$

The cross-wage substitution elasticity $\gamma_{\theta,\tilde{\theta}}$ is

$$\begin{aligned} \gamma_{\theta,\tilde{\theta}} &= \frac{l_{\tilde{\theta}}}{w_\theta} D_{l_{\tilde{\theta}}} \, (\kappa_\theta \phi_\theta)^{\frac{\sigma-1}{\sigma}} \, (l_\theta h_\theta)^{-\frac{1}{\sigma}} \, F(l,\phi)^{\frac{1}{\sigma}} \\ &= \frac{l_{\tilde{\theta}}}{w_\theta} \, (\kappa_\theta \phi_\theta)^{\frac{\sigma-1}{\sigma}} \, (l_\theta h_\theta)^{-\frac{1}{\sigma}} \, \frac{1}{\sigma} F(l,\phi)^{\frac{1}{\sigma}-1} w_{\tilde{\theta}} h_{\tilde{\theta}} \\ &= \frac{1}{\sigma} \frac{w_{\tilde{\theta}}(l,\phi) h_{\tilde{\theta}} l_{\tilde{\theta}}}{F(l,\phi)} . \end{aligned}$$

For the technical change elasticities, consider first the determination of equilibrium technology described by equation (7). First-order conditions for the maximization problem in equation (7) are

$$\delta \phi_\theta^{*\delta-1} \lambda = (\kappa_\theta h_\theta l_\theta)^{\frac{\sigma-1}{\sigma}} \phi_\theta^{*-\frac{1}{\sigma}} F(l,\phi)^{\frac{1}{\delta}} \quad \forall \theta \,, \tag{40}$$

where λ is the Lagrange multiplier for the R&D resource constraint. The conditions equate the marginal R&D cost of raising ϕ_θ, converted into units of final good via λ, with the marginal gain in production. The latter is given by the derivative $D_{\phi_\theta} F$, which is computed analogously to $D_{l_\theta} F$ above.

Solving the first-order conditions for ϕ_θ yields

$$\phi_\theta^* = (\delta\lambda)^{\frac{-\sigma}{(\delta-1)\sigma+1}} (\kappa_\theta h_\theta l_\theta)^{\frac{\sigma-1}{(\delta-1)\sigma+1}} F(l,\phi)^{\frac{1}{(\delta-1)\sigma+1}} \quad \forall \theta \,. \tag{41}$$

Then, we can use the R&D resource constraint

$$\int_{\underline{\theta}}^{\overline{\theta}} \phi_\theta^{*\delta} \, d\theta = \overline{C}$$

to solve for the Lagrange multiplier:

$$\lambda = \frac{1}{\delta} \overline{C}^{-\frac{(\delta-1)\sigma+1}{\delta\sigma}} \left[\int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta h_\theta l_\theta)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}} \, d\theta \right]^{\frac{(\delta-1)\sigma+1}{\sigma\delta}} F(l,\phi)^{\frac{1}{\sigma}} \,.$$

Plugging this into equation (41), we obtain the following expression for the equilibrium technology ϕ^*:

$$\phi_\theta^* = \overline{C}^{\frac{1}{\delta}} (\kappa_\theta h_\theta l_\theta)^{\frac{\sigma-1}{(\delta-1)\sigma+1}} \left[\int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta h_\theta l_\theta)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}} \, d\theta \right]^{-\frac{1}{\delta}} \,. \tag{42}$$

We can now use equation (42) to derive the technical change elasticities. The own-wage technical change elasticity is simply derived from equations (13) and (42) as

$$\begin{aligned} \rho_{\theta,\theta} &= \frac{\phi_\theta^*}{w_\theta} \frac{\partial w_\theta}{\partial \phi_\theta^*} \frac{l_\theta}{\phi_\theta^*} \frac{\partial \phi_\theta^*}{\partial l_\theta} \\ &= \frac{\sigma-1}{\sigma} \frac{\sigma-1}{(\delta-1)\sigma+1} \,. \end{aligned}$$

For the cross-wage technical change elasticity, start from its definition:

$$\begin{aligned} \rho_{\theta,\tilde{\theta}} &= \frac{l_{\tilde{\theta}}}{w_\theta} D_{\phi,l_{\tilde{\theta}}} (\kappa_\theta \phi_\theta^*(l))^{\frac{\sigma-1}{\sigma}} (l_\theta h_\theta)^{-\frac{1}{\sigma}} F(l,\phi^*(l))^{\frac{1}{\sigma}} \\ &= \frac{\sigma-1}{\sigma} \frac{l_{\tilde{\theta}}}{\phi_\theta^*} D_{\phi,l_{\tilde{\theta}}} \phi_\theta^*(l) + \frac{l_{\tilde{\theta}}}{w_\theta} (\kappa_\theta \phi_\theta^*(l))^{\frac{\sigma-1}{\sigma}} (l_\theta h_\theta)^{-\frac{1}{\sigma}} D_{\phi,l_{\tilde{\theta}}} F(l,\phi^*(l))^{\frac{1}{\sigma}} \,. \end{aligned}$$

The second term of the sum in the second row is zero by the envelope theorem. So, we obtain

$$\rho_{\theta,\tilde{\theta}} = \frac{\sigma-1}{\sigma} \frac{l_{\tilde{\theta}}}{\phi_\theta^*} D_{\phi,l_{\tilde{\theta}}} \phi_\theta^*(l) \,.$$

Analogously to the computation of the derivative $D_{l_\theta} F(l,\phi)$ above, we can compute $D_{\phi,l_{\tilde{\theta}}} \phi_\theta^*(l)$

using equation (42):

$$D_{\phi,l_{\tilde\theta}}\phi^*_\theta(l) = -\frac{\sigma-1}{(\delta-1)\sigma+1}\overline{C}^{-\frac{1}{\delta}}\left(\kappa_\theta h_\theta l_\theta\right)^{\frac{\sigma-1}{(\delta-1)\sigma+1}}\left[\int_{\underline{\theta}}^{\overline{\theta}}\left(\kappa_\theta h_\theta l_\theta\right)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}}\,d\theta\right]^{-\frac{1}{\delta}}$$
$$\times\left[\int_{\underline{\theta}}^{\overline{\theta}}\left(\kappa_\theta h_\theta l_\theta\right)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}}\,d\theta\right]^{-1}\left(\kappa_{\tilde\theta} h_{\tilde\theta} l_{\tilde\theta}\right)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}} l_{\tilde\theta}^{-1}$$
$$= -\frac{\sigma-1}{(\delta-1)\sigma+1}\frac{\phi^*_\theta(l)}{l_{\tilde\theta}}\frac{\phi^*_{\tilde\theta}(l)}{\overline{C}}\,.$$

Thereby,

$$\rho_{\theta,\tilde\theta} = -\frac{\sigma-1}{\sigma}\frac{\sigma-1}{(\delta-1)\sigma+1}\frac{\phi^*_{\tilde\theta}(l)}{\overline{C}}\,. \tag{43}$$

To derive the expression from the main text, note that we can rewrite the first-order condition (40) as

$$\delta\phi_\theta^{*\delta-1}\lambda = w_\theta h_\theta l_\theta \phi_\theta^{*-1}\,,$$

which implies

$$\phi_\theta^{*\delta} = \frac{1}{\lambda\delta}w_\theta h_\theta l_\theta\,. \tag{44}$$

We now integrate this over θ and use Euler's homogeneous function theorem to obtain

$$\overline{C} = \frac{1}{\lambda\delta}F\,.$$

Using this to eliminate λ in equation (44), we obtain

$$\phi_\theta^{*\delta} = \overline{C}\frac{w_\theta h_\theta l_\theta}{F}\,.$$

Finally, combining this with equation (43) yields

$$\rho_{\theta,\tilde\theta} = -\frac{\sigma-1}{\sigma}\frac{\sigma-1}{(\delta-1)\sigma+1}\frac{w_{\tilde\theta}(l,\phi)h_{\tilde\theta}l_{\tilde\theta}}{F(l,\phi)}\,,$$

which is the expression given in the main text.

A.4. Linear Homogeneity of Aggregate Production

In the main text I assume that final good firms' production function $\tilde{G}$ is linear homogeneous in the rival inputs l and q. Here I show that the aggregate production function F and its equilibrium version F^* (defined below) inherit this property.

Lemma 4. *The aggregate production function F defined in* (5) *is linear homogeneous in l.*

Proof. Aggregate production $F(l,\phi)$ for some labor input l and some technology ϕ is defined as

$$\max_q\left\{\tilde{G}(\{h_\theta l_\theta\}_{\theta\in\Theta},\phi,q) - \sum_{j=1}^{J}\eta_j q_j\right\}\,.$$

Let $q^*(l, \phi)$ denote a solution to this maximization problem.
Consider now the labor input λl for some $\lambda > 0$ and the intermediate input $\lambda q^*(l, \phi)$. Since $\tilde{G}$ is linear homogeneous in l and q, the first-order conditions of the maximization problem are satisfied at λl, $\lambda q^*(l, \phi)$, and ϕ. Since $\tilde{G}$ is concave in l and q, first-order conditions are sufficient for a maximum, and $\lambda q^*(l, \phi)$ is a maximizer of $\tilde{G}$ at λl and ϕ. So, using linear homogeneity of $\tilde{G}$ again,

$$\begin{aligned} F(\lambda l, \phi) &= \tilde{G}(\{\lambda h_\theta l_\theta\}_{\theta \in \Theta}, \phi, \lambda q^*(l, \phi)) - \sum_{j=1}^{J} \eta_j \lambda q_j^*(l, \phi) \\ &= \lambda \tilde{G}(\{h_\theta l_\theta\}_{\theta \in \Theta}, \phi, q^*(l, \phi)) - \sum_{j=1}^{J} \eta_j q_j^*(l, \phi) \\ &= F(l, \phi) . \end{aligned}$$

□

Consider next the equilibrium aggregate production function

$$F^*(l) := F(l, \phi^*(l)) . \tag{45}$$

Lemma 5. *The equilibrium aggregate production function F^* defined in (45) is linear homogeneous in l.*

Proof. By the reduced form equation for equilibrium technology $\phi^*(l)$, the equilibrium aggregate production function satisfies

$$F^*(l) = \max_{\phi \in \Phi} F(l, \phi) .$$

Then, by linear homogeneity of F in l (see Lemma 4):

$$\begin{aligned} F^*(\lambda l) &= \max_{\phi \in \Phi} F(\lambda l, \phi) \\ &= \max_{\phi \in \Phi} \lambda F(l, \phi) \\ &= \lambda \max_{\phi \in \Phi} F(l, \phi) \\ &= \lambda F^*(l) \end{aligned}$$

for any $\lambda > 0$. □

B. Proofs for Directed Technical Change

Lemma 1 is a local version of Theorem 5 in Loebbing (2018). Yet it is not strictly covered by the theorem, because, as described in the main text and footnote 13, I use a slightly unusual definition of quasisupermodularity, which allows me to dispense with the lattice structure of Φ. So, I provide a proof for Lemma 1 here. The proof follows closely the proof of Theorem 5 in Loebbing (2018).

Proof of Lemma 1. Take any two labor inputs l and $\tilde{l}$ such that $\tilde{l}$ has greater relative skill supply, that is, $\tilde{l}_\theta/\tilde{l}_{\tilde{\theta}} \geq l_\theta/l_{\tilde{\theta}}$. Since F is linear homogeneous in labor (Lemma 4), wages are independent of the scale of the labor input. So, for the purpose of Lemma 1, we can always scale l up or down such that $F(l,\phi^*(\tilde{l})) = F(\tilde{l},\phi^*(\tilde{l}))$. In words, we scale l such that it is contained in the (exogenous technology) isoquant of F through $(\tilde{l},\phi^*(\tilde{l}))$.

Moreover, by definition of the equilibrium technology ϕ^*, we have $F(l,\phi^*(l)) \geq F(l,\underline{\phi})$ for all $\underline{\phi} \preceq \phi^*(l),\phi^*(\tilde{l})$. Quasisupermodularity then implies that there is a $\overline{\phi} \succeq \phi^*(l),\phi^*(\tilde{l})$ such that $F(l,\overline{\phi}) \geq F(l,\phi^*(\tilde{l}))$.

Now assume, to derive a contradiction, that $\phi^*(\tilde{l}) \not\succeq \phi^*(l)$. Then, $\overline{\phi} \neq \phi^*(\tilde{l})$ and, by uniqueness of $\operatorname{argmax}_{\phi\in\Phi} F(\tilde{l},\phi)$ (Assumption 1), we must have $F(\tilde{l},\phi^*(\tilde{l})) > F(\tilde{l},\overline{\phi})$.

Putting the previous results together, we obtain

$$F(l,\overline{\phi}) \geq F(l,\phi^*(\tilde{l})) = F(\tilde{l},\phi^*(\tilde{l})) > F(\tilde{l},\overline{\phi}) \tag{46}$$

for some $\overline{\phi} \succeq \phi^*(l),\phi^*(\tilde{l})$ and $\underline{\phi} \preceq \phi^*(l),\phi^*(\tilde{l})$. In words, increasing relative skill supply by moving from l to $\tilde{l}$ leaves output unchanged at $\phi^*(\tilde{l})$ but reduces output at $\overline{\phi}$. Intuitively, this is incompatible with $\overline{\phi}$ being more skill-complementary than $\phi^*(\tilde{l})$, which is what we show formally in the following.

To that end, consider a monotonic and differentiable path $l(\tau)$ from l to $\tilde{l}$ such that $l(0) = l$, $l(1) = \tilde{l}$ and $F(l(\tau),\phi^*(\tilde{l})) = F(l,\phi^*(\tilde{l}))$ for all $\tau \in [0,1]$. By monotonicity I mean here that each entry $l_\theta(\tau)$ is monotonic in τ. Applying the mean value theorem, the inequalities in (46) imply that there is a $\tilde{\tau} \in (0,1)$ such that

$$\int_{\underline{\theta}}^{\overline{\theta}} D_{l_\theta} F(l(\tilde{\tau}),\overline{\phi}) \frac{dl_\theta(\tilde{\tau})}{d\tau}\, d\theta < 0\,. \tag{47}$$

Moreover, let *tilde*θ denote a skill level such that $l_\theta \leq \tilde{l}_\theta$ for all $\theta \leq \tilde{\theta}$ and $l_\theta \geq \tilde{l}_\theta$ for all $\theta > \tilde{\theta}$. Such a skill level exists because $\tilde{l}$ has greater relative skill supply than l. Noting that

$$\int_{\underline{\theta}}^{\overline{\theta}} D_{l_\theta} F(l(\tilde{\tau}),\phi^*(\tilde{l})) \frac{dl_\theta(\tilde{\tau})}{d\tau}\, d\theta = 0\,,$$

we can now extend inequality (47) to

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(D_{l_\theta} F(l(\tilde{\tau}),\overline{\phi}) - \frac{D_{l_{\tilde{\theta}}} F(l(\tilde{\tau}),\overline{\phi})}{D_{l_{\tilde{\theta}}} F(l(\tilde{\tau}),\phi^*(\tilde{l}))} D_{l_\theta} F(l(\tilde{\tau}),\phi^*(\tilde{l})) \right) \frac{dl_\theta(\tilde{\tau})}{d\tau}\, d\theta < 0\,. \tag{48}$$

By definition of $\tilde{\theta}$ and monotonicity of $l(\tau)$, we know that $dl_\theta(\tilde{\tau})/d\tau$ is positive for all $\theta > \tilde{\theta}$ and negative for all $\theta \leq \tilde{\theta}$. Moreover, since $\overline{\phi} \succeq \phi^*(\tilde{l})$, the difference

$$D_{l_\theta} F(l(\tilde{\tau}),\overline{\phi}) - \frac{D_{l_{\tilde{\theta}}} F(l(\tilde{\tau}),\overline{\phi})}{D_{l_{\tilde{\theta}}} F(l(\tilde{\tau}),\phi^*(\tilde{l}))} D_{l_\theta} F(l(\tilde{\tau}),\phi^*(\tilde{l}))$$

is also positive for $\theta > \tilde{\theta}$ and negative for $\theta \leq \tilde{\theta}$. This implies that the right-hand side of (48) must be (weakly) positive, a contradiction.

We have hence shown that $\phi^*(\tilde{l}) \succeq \phi^*(l)$, that is, the equilibrium technology is more skill-biased under $\tilde{l}$ than under l. So, the increase in relative skill supply induces skill-biased technical change. The local implication of this global result is Lemma 1. □

C. Proofs and Additional Results for Tax Reforms

In this section I provide proofs and additional results for the tax reform analysis in Section 5. I start with the proof of Lemma 2, which provides alternative characterizations of progressive tax reforms as defined by Definition 1.

Proof of Lemma 2. The strategy of the proof is to show that statement 1 implies statement 2, statement 2 implies statement 3, and statement 3 implies statement 1.
(1 ⇒ 2) Take a function r such that $R_{\tilde{T}}(y) = r(R_T(y))$ for all y. Differentiating both sides with respect to y yields

$$r'(R_T(y))R'_T(y) = R'_{\tilde{T}}(y) \quad \forall y \, ,$$

and after taking logs and rearranging:

$$\log r'(R_T(y)) = \log R'_{\tilde{T}}(y) - \log R'_T(y) \quad \forall y \, .$$

Differentiating again with respect to y and multiplying through by y gives

$$\frac{r''(R_T(y))}{r'(R_T(y))} R'_T(y)y = -(P_{\tilde{T}}(y) - P_T(y)) < 0 \quad \forall y \, ,$$

where the inequality is Definition 1. The assumption that $T'(y) < 1$ and $T'(y) + \mu\tau'(y) < 1$ for all y (see point 6 of Assumption 1) implies that $r'(R_T(y)) > 0$ and $R'_T(y) > 0$, such that $r''(R_T(y)) < 0$ for all y, which is statement 2.
(2 ⇒ 3) Statement 2 implies

$$\frac{R'_{\tilde{T}}(y)}{R'_{\tilde{T}}(\tilde{y})} = \frac{r'(R_T(y))}{r'(R_T(\tilde{y}))} \frac{R'_T(y)}{R'_T(\tilde{y})} \leq \frac{R'_T(y)}{R'_T(\tilde{y})} \quad \forall y \geq \tilde{Y} \, ,$$

because r is concave and R_T strictly increasing. Replacing $R'_{\tilde{T}}(y)$ by $R'_T(y) - \mu\tau'(y)$ and rearranging yields

$$\frac{R'_T(y) - \mu\tau'(y)}{R'_T(y)} \leq \frac{R'_T(\tilde{y}) - \mu\tau'(\tilde{y})}{R'_T(\tilde{y})} \quad \forall y \geq \tilde{y}$$

and hence:

$$\frac{\tau'(y)}{R'_T(y)} \geq \frac{\tau'(\tilde{y})}{R'_T(\tilde{y})} \quad \forall y \geq \tilde{y} \, ,$$

which is statement 3.
(3 ⇒ 1) We can transform statement 3 into

$$\frac{R'_T(y) - \mu\tau'(y)}{R'_T(y)} \leq \frac{R'_T(\tilde{y}) - \mu\tau'(\tilde{y})}{R'_T(\tilde{y})} \quad \forall y \geq \tilde{y} \, .$$

Taking logs and rearranging yields

$$\log R'_{\tilde{T}}(y) - \log R'_{\tilde{T}}(\tilde{y}) \leq \log R'_T(y) - \log R'_T(\tilde{y}) \quad \forall y \geq \tilde{y} \, .$$

Setting $\tilde{y} = y - d$, dividing both sides of the equation by d, and taking the limit as $d \to 0$, we obtain

$$-\frac{1}{y} P_{\tilde{T}}(y) \leq -\frac{1}{y} P_T(y) \quad \forall y$$

and hence

$$P_{\tilde{T}}(y) \geq P_T(y) \quad \forall y \, .$$

□

Next I consider the response of labor inputs to a tax reform.

C.1. Labor Input Responses

First I derive equation (20) by applying the derivative D_τ to labor inputs. In particular, accounting for the general equilibrium contingencies between labor supply and wages, we can write labor supply as $l_\theta(T, w_\theta)$ and wages as $w_\theta(l(T, w_\theta), \phi^*(l(T, w_\theta)))$. Then, using derivatives and elasticities as defined in the main text, it is straightforward to derive equation (20). Starting from equation (20) I prove Lemma 3.

Proof of Lemma 3. **Step 1.** It is easy to see that

$$\hat{l}^{(n)}_{\theta,\tau} = \tilde{TE}^{(n)}_{\theta,\tau} + \tilde{SE}^{(n)}_{\theta,\tau}$$

for all $n \geq 1$. Hence, the two expressions (23) and (24) are equal.
Step 2. Suppose for now that all the series in expressions (23) and (24) converge. Then, take expression (23) and insert it into the fixed point equation (20):

$$\begin{aligned}
\sum_{n=1}^{\infty} \hat{l}^{(n)}_{\theta,\tau} &= -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \epsilon^w_\theta \zeta_{\theta,\theta} \sum_{n=1}^{\infty} \hat{l}^{(n)}_{\theta,\tau} + \epsilon^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \zeta_{\theta,\tilde{\theta}} \sum_{n=1}^{\infty} \hat{l}^{(n)}_{\tilde{\theta},\tau} \, d\tilde{\theta} \\
&= -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \sum_{n=1}^{\infty} \left[\epsilon^w_\theta \zeta_{\theta,\theta} \hat{l}^{(n)}_{\theta,\tau} + \epsilon^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \zeta_{\theta,\tilde{\theta}} \hat{l}^{(n)}_{\tilde{\theta},\tau} \, d\tilde{\theta} \right] \\
&= -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \sum_{n=1}^{\infty} \hat{l}^{(n+1)}_{\theta,\tau} \\
&= -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \sum_{n=2}^{\infty} \hat{l}^{(n)}_{\theta,\tau} \\
&= \sum_{n=1}^{\infty} \hat{l}^{(n)}_{\theta,\tau} \, .
\end{aligned}$$

This proves that, conditional upon convergence of the series, expression (23) solves the fixed point equation (20). By Step 1, then also expression (24) solves the fixed point equation conditional upon convergence.

Step 3. Regarding convergence, consider expression (23) first. Start from the definition of $\hat{l}^{(n)}_{\theta,\tau}$ and take the square of both sides of the equation:

$$\left(\hat{l}^{(n)}_{\theta,\tau}\right)^2 = (\epsilon^w_\theta \zeta_{\theta,\theta})^2 \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 + (\epsilon^w_\theta)^2 \left(\int_{\underline{\theta}}^{\overline{\theta}} \zeta_{\theta,\tilde{\theta}} \hat{l}^{(n-1)}_{\tilde{\theta},\tau}\, d\tilde{\theta}\right)^2 + 2\epsilon^w_\theta \zeta_{\theta,\theta} \hat{l}^{(n-1)}_{\theta,\tau} \epsilon^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \zeta_{\theta,\tilde{\theta}} \hat{l}^{(n-1)}_{\tilde{\theta},\tau}\, d\tilde{\theta}\,.$$

By the Cauchy-Schwarz inequality,

$$\left(\hat{l}^{(n)}_{\theta,\tau}\right)^2 \leq (\epsilon^w_\theta \zeta_{\theta,\theta})^2 \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 + (\epsilon^w_\theta)^2 \int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}$$
$$+ 2\epsilon^w_\theta \zeta_{\theta,\theta} \hat{l}^{(n-1)}_{\theta,\tau} \epsilon^w_\theta \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta}} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}}\,,$$

and after integrating over θ,

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n)}_{\theta,\tau}\right)^2 d\theta \leq \int_{\underline{\theta}}^{\overline{\theta}} (\epsilon^w_\theta \zeta_{\theta,\theta})^2 \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 d\theta + \int_{\underline{\theta}}^{\overline{\theta}} (\epsilon^w_\theta)^2 \int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta}\, d\theta \int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}$$
$$+ 2\int_{\underline{\theta}}^{\overline{\theta}} \epsilon^w_\theta \zeta_{\theta,\theta} \hat{l}^{(n-1)}_{\theta,\tau} \epsilon^w_\theta \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta}\, d\theta} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}}\,.$$

Taking the supremum of $\epsilon^w_\theta \zeta_{\theta,\theta}$ in the first term and applying the Cauchy-Schwarz inequality again to the last term yields:

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n)}_{\theta,\tau}\right)^2 d\theta \leq \sup_{\theta\in\Theta} \left[(\epsilon^w_\theta \zeta_{\theta,\theta})^2\right] \int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 d\theta + \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\epsilon^w_\theta \zeta_{\theta,\tilde{\theta}}\right)^2 d\tilde{\theta}\, d\theta \int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}$$
$$+ 2\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\epsilon^w_\theta \zeta_{\theta,\theta} \epsilon^w_\theta \zeta^2_{\theta,\tilde{\theta}}\right)^2 d\tilde{\theta}\, d\theta} \int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta},\tau}\right)^2 d\tilde{\theta}\,.$$

The coefficients of $\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 d\theta$ on the right-hand side of the inequality amount to

$$\sup_{\theta\in\Theta} (\epsilon^w_\theta \zeta_{\theta,\theta})^2 + \int_{\underline{\theta}}^{\overline{\theta}} (1-\epsilon_\theta \zeta_{\theta,\theta})^2 (\overline{\epsilon}^w_\theta \zeta_{\theta,\tilde{\theta}})^2\, d\tilde{\theta}\, d\theta + 2\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} (\epsilon^w_\theta \zeta_{\theta,\theta})^2 (1-\epsilon_\theta \zeta_{\theta,\theta})^2 (\overline{\epsilon}^w_\theta \zeta_{\theta,\tilde{\theta}})^2\, d\tilde{\theta}\, d\theta}\,,$$

which is strictly smaller than one by condition (22). Hence, the term $\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 d\theta$ is dominated by a geometric sequence converging to zero.
Regarding $\hat{l}^{(n)}_{\theta,\tau}$, the Cauchy-Schwarz inequality implies

$$\hat{l}^{(n)}_{\theta,\tau} \leq \epsilon^w_\theta \zeta_{\theta,\theta} \hat{l}^{(n-1)}_{\theta,\tau} + \epsilon^w_{\theta,\tau} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta}} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta}}\right)^2 d\tilde{\theta}}\,. \tag{49}$$

Suppose now, to derive a contradiction, that $\hat{l}^{(n)}_{\theta,\tau}$ is not dominated by any geometric sequence that converges to zero. Then, for any $c \in (\epsilon^w_\theta \zeta_{\theta,\theta}, 1)$ and for any $N \in \mathbb{N}$, there must exist

$\overline{N}_N > N$ such that

$$\frac{|\hat{l}^{(\overline{N}_N)}_{\theta,\tau}|}{|\hat{l}^{(\overline{N}_N-1)}_{\theta,\tau}|} > c \; .$$

At the same time, since $\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\theta,\tau}\right)^2 d\theta$ is dominated by a geometric sequence converging to zero, we must have

$$\frac{|\epsilon^{w}_{\theta,\tau}\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \zeta^2_{\theta,\tilde{\theta}}\, d\tilde{\theta}}\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(\hat{l}^{(n-1)}_{\tilde{\theta}}\right)^2 d\tilde{\theta}}|}{\hat{l}^{(\overline{N}_N)}_{\theta,\tau}} \to 0 \quad \text{as } N \to \infty \; .$$

But with equation (49) this implies, as $N \to \infty$,

$$\frac{|\hat{l}^{(\overline{N}_N)}_{\theta,\tau}|}{|\hat{l}^{(\overline{N}_N-1)}_{\theta,\tau}|} \to |\epsilon^{w}_{\theta}\zeta_{\theta,\theta}| < c \; ,$$

a contradiction.

So, $\hat{l}^{(n)}_{\theta,\tau}$ is dominated by a geometric sequence converging to zero and the series $\sum_{n=1}^{\infty} \hat{l}^{(n)}_{\theta,\tau}$ indeed exists.

Step 4. For convergence of the series $\tilde{TE}_{\theta,\tau}$ and $\tilde{SE}_{\theta,\tau}$, consider $\tilde{TE}_{\theta,\tau}$ first. Replacing $\zeta_{\theta,\tilde{\theta}}$ by $\rho_{\theta,\tilde{\theta}}$, the reasoning in step 3 implies that $\tilde{TE}_{\theta,\tau}$ converges. Second, note that

$$\tilde{SE}_{\theta,\tau} = \sum_{n=1}^{\infty} \hat{l}^{(n)}_{\theta,\tau} - \tilde{TE}_{\theta,\tau} \; .$$

Since we have already shown that both series on the right-hand side converge, the same must hold for $\tilde{SE}_{\theta,\tau}$.

Step 5. The final step is to prove that, if ϵ^{w}_{θ} is constant in θ and $\epsilon^{R}_{\theta}\tau'(y_\theta(T))/(1-T'(y_\theta(T)))$ increases in θ, the component $\tilde{TE}_{\theta,\tau}$ decreases in θ as well. The proof is by induction.

If $\epsilon^{R}_{\theta}\tau'(y_\theta(T))/(1-T'(y_\theta(T)))$ increases in θ, then by Lemma 1 we have that the term

$$\tilde{TE}^{(1)}_{\theta,\tau} = \epsilon^{w}_{\theta}\rho_{\theta,\theta}(-\epsilon^{R}_{\theta})\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \epsilon^{w}_{\theta}\int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}(-\epsilon^{R}_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta}$$

is decreasing in θ.

Suppose now that $\tilde{TE}^{(n)}_{\theta,\tau}$ decreases in θ. Then, again by Lemma 1,

$$\rho_{\theta,\theta}\tilde{TE}^{(n)}_{\theta,\tau} + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}\tilde{TE}^{(n)}_{\theta,\tau}\, d\tilde{\theta}$$

decreases in θ. If ϵ^{w}_{θ} is constant in θ, the same holds for

$$\epsilon^{w}_{\theta}\rho_{\theta,\theta}\tilde{TE}^{(n)}_{\theta,\tau} + \epsilon^{w}_{\theta}\int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}\tilde{TE}^{(n)}_{\theta,\tau}\, d\tilde{\theta} \; .$$

But this is equal to $\tilde{TE}^{(n+1)}_{\theta,\tau}$. Hence inductively, $\tilde{TE}^{(n)}_{\theta,\tau}$ decreases in θ for all $n \geq 1$. So the sum

$\sum_{n=1} \tilde{TE}^{(n)}_{\theta,\tau}$ decreases in θ as well, which yields the desired result. □

For the special case of CES production introduced in Section 3.4, the labor response to tax reform τ takes a particularly simple form. For later reference, I provide this result in the following lemma.

Lemma 6. *Fix an initial tax T and suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2). Moreover, assume that F and Φ are CES as introduced in Section 3.4 and the elasticity ϵ^w_θ is constant in θ, that is, $\epsilon^w_\theta = \epsilon^w$ for all $\theta \in \Theta$. Then, the effect of tax reform τ on labor inputs can be written as*

$$\hat{l}_{\theta,\tau}(T) = -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} - (\gamma^{CES} + \rho^{CES})\epsilon^w \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T), \phi^*(T))} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta} \,, \quad (50)$$

where

$$\bar{\epsilon}^R_\theta := \frac{\epsilon^R_\theta}{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w} \,.$$

So, $\hat{l}_{\theta,\tau}(T)$ decreases in θ if and only if

$$\bar{\epsilon}^R_{\tilde{\theta}} \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))}$$

increases in θ.
If in addition ϵ^R_θ is constant in θ, $\hat{l}_{\theta,\tau}(T)$ decreases in θ if and only if τ is progressive.

Proof. The fastest way to prove equation (50) is to check that it satisfies the fixed point equation (20). In the CES case and with ϵ^w_θ constant, this equation becomes

$$\hat{l}_{\theta,\tau}(T) = -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \epsilon^w(\gamma^{CES} + \rho^{CES})\hat{l}_{\theta,\tau}(T) - \epsilon^w(\gamma^{CES} + \rho^{CES}) \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T), \phi^*(T))} \hat{l}_{\tilde{\theta},\tau} \, d\tilde{\theta} \,.$$

Inserting equation (50) yields:

$$
\begin{aligned}
\hat{l}_{\theta,\tau}(T) =& -\epsilon^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \epsilon^w(\gamma^{CES}+\rho^{CES})(-\bar{\epsilon}^R_\theta)\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} \\
& - \epsilon^w(\gamma^{CES}+\rho^{CES})(\gamma^{CES}+\rho^{CES})\epsilon^w \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\
& - \epsilon^w(\gamma^{CES}+\rho^{CES}) \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\
& + \epsilon^w(\gamma^{CES}+\rho^{CES}) \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(\gamma^{CES}+\rho^{CES})\epsilon^w \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\hat{\theta}} l_{\hat{\theta}} h_{\hat{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\hat{\theta}})\frac{\tau'(y_{\hat{\theta}}(T))}{1-T'(y_{\hat{\theta}}(T))}\, d\hat{\theta}\, d\tilde{\theta} \\
=& -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} - \epsilon^w(\gamma^{CES}+\rho^{CES}) \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\
& - (\epsilon^w)^2(\gamma^{CES}+\rho^{CES})^2 \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\
& + (\epsilon^w)^2(\gamma^{CES}+\rho^{CES})^2 \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\hat{\theta}} l_{\hat{\theta}} h_{\hat{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\hat{\theta}})\frac{\tau'(y_{\hat{\theta}}(T))}{1-T'(y_{\hat{\theta}}(T))} \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}\, d\tilde{\theta}\, d\hat{\theta} \\
=& -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} - \epsilon^w(\gamma^{CES}+\rho^{CES}) \int_{\underline{\theta}}^{\bar{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta}\,,
\end{aligned}
$$

where the last equality follows from the fact that F is linear homogeneous in l (see Lemma 4) and Euler's homogeneous function theorem. So, equation (50) solves the fixed point equation (20).

The remainder of Lemma 6 then follows from the observation that the second term on the right-hand side of equation (50) is independent of θ and that, in the CES case, $\bar{\epsilon}^R_\theta$ is constant in θ if ϵ^R_θ and ϵ^w_θ are constant in θ. □

C.2. Induced Technical Change

Using the labor input responses from Lemma 3, I prove the results from Section 5.1 on the induced technical change effects of tax reforms.

First I obtain equation (25) by applying the derivative $D_{\phi,\tau}$ to wages. In particular, accounting for the general equilibrium contingencies between wages and labor supply, we can write wages as $w_\theta(l(T, w_\theta), \phi^*(l(T, w_\theta)))$ and labor supply as $l_\theta(T, w_\theta)$. Then, using derivatives and elasticities as defined in the main text, it is straightforward to derive equation (25).

Combining equation (25) with Lemma 3, I prove Proposition 1.

Proof of Proposition 1. Equation (26) is obtained immediately by inserting equation (24) from Lemma 3 into (25).

To sign the slopes of $DE_{\theta,\tau}$ and $TE_{\theta,\tau}$, note that by Lemma 1 the induced technical change effect

$$\rho_{\theta,\theta}\hat{l}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\bar{\theta}} \rho_{\theta,\tilde{\theta}}\hat{l}_{\tilde{\theta},\tau}(T)\, d\tilde{\theta}$$

decreases in θ if $\hat{l}_{\theta,\tau}(T)$ decreases in θ. This immediately implies that $DE_{\theta,\tau}$ decreases in θ if $\epsilon^R_\theta \tau'(y_\theta(T))/(1-T'(y_\theta(T)))$ increases in θ. Moreover, Lemma 3 says that $\tilde{TE}_{\theta,\tau}$ decreases in θ if ϵ^w_θ is constant in θ and $\epsilon^R_\theta \tau'(y_\theta(T))/(1-T'(y_\theta(T)))$ increases in θ. So, under these conditions also $TE_{\theta,\tau}$ decreases in θ. □

Corollary 2 gives the induced technical change effects of reform τ for the special case of a CES production function. I prove Corollary 2 by applying Lemma 6.

Proof of Corollary 2. Since aggregate production F and the equilibrium aggregate production function F^* are linear homogeneous in l (see Lemmas 4 and 5), the induced technical change effects of a proportional change in all types' labor inputs are zero:

$$\rho_{\theta,\theta}\hat{l}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}\hat{l}_{\tilde{\theta},\tau}(T)\, d\tilde{\theta} = 0 \quad \text{for all } \theta$$

if $\hat{l}_{\theta,\tau}(T)$ is constant in θ. Hence, inserting equation (50) into equation (25), the second term of equation (50) vanishes. This leaves

$$\frac{1}{w_\theta} D_{\phi,\tau} w_\theta(T,\phi^*(T)) = \rho^{CES}(-\bar{\epsilon}^R_\theta)\frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} - \rho^{CES}\int_{\underline{\theta}}^{\overline{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T),\phi^*(T))}(-\bar{\epsilon}^R_{\tilde{\theta}})\frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta}\,,$$

which is equation (27).
The second term on the right-hand side of equation (27) is independent of θ. This immediately implies that the relative wage change is decreasing in θ if $\bar{\epsilon}^R_\theta \tau'(y_\theta(T))/(1-T'(y_\theta(T)))$ increases in θ.
Finally, if ϵ^R_θ is constant in θ (in addition to ϵ^w_θ, which is required by Corollary 2 anyway), $\bar{\epsilon}^R_\theta$ is constant in θ as well, and the relative wage change decreases in θ for any progressive reform. □

C.3. Substitution Effects of Tax Reforms

Proposition 1 provides a general formula for the induced technical change effects of tax reforms on wages. Here, I state the counterpart of this proposition for the within-technology substitution effects. The formula has exactly the same structure as the one for the induced technical change effects. The only difference is that here technical change wage elasticities are replaced by substitution elasticities.[35]

Proposition 6. *Fix an initial tax T, suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2), and let conditions (21) and (22) be satisfied.*
Then, the relative effect of the within-technology factor substitution induced by tax reform τ on wages

[35]Note also that in contrast to the induced technical change effects in Proposition 1 we cannot generally sign any component of the within-technology substitution effects induced by a progressive reform τ. This is because the endogenous determination of technology (see equation (7)) lends structure to the induced technical change elasticities which is absent from substitution elasticities.

can be written as

$$\frac{1}{w_\theta} D_\tau w_\theta(T, \phi^*(T)) = \gamma_{\theta,\theta}(-\epsilon^R_\theta) \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}}(-\epsilon^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta}$$
$$+ \gamma_{\theta,\theta} \tilde{TE}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} \tilde{TE}_{\tilde{\theta},\tau}(T) \, d\tilde{\theta}$$
$$+ \gamma_{\theta,\theta} \tilde{SE}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} \tilde{SE}_{\tilde{\theta},\tau}(T) \, d\tilde{\theta} \, , \tag{51}$$

for all $\theta \in \Theta$, where $\tilde{TE}_{\theta,\tau}(T)$ and $\tilde{SE}_{\theta,\tau}(T)$ are defined in Lemma 3.

Proof. Analogously to the induced technical change effects in equation (25), the substitution effects of tax reform τ on wages can be written as

$$\frac{1}{w_\theta} D_\tau w_\theta(T, \phi^*(T)) = \gamma_{\theta,\theta} \hat{l}_{\theta,\tau}(T) + \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} \hat{l}_{\tilde{\theta},\tau}(T) \, d\tilde{\theta} \, . \tag{52}$$

Replacing equation (25) by equation (52), the proof proceeds analogously to the proof of equation (26) in Proposition 1 and is therefore omitted. □

In the CES case, the within-technology substitution effects can be expressed as follows.

Corollary 5. *Fix an initial tax T and suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2). Moreover, assume that F and Φ are CES as introduced in Section 3.4 and the elasticity ϵ^w_θ is constant in θ, that is, $\epsilon^w_\theta = \epsilon^w$ for all $\theta \in \Theta$. Then the relative wage effect of the within-technology factor substitution induced by tax reform τ satisfies*

$$\frac{1}{w_\theta} D_\tau w_\theta(T, \phi^*(T)) = \gamma^{CES}(-\bar{\epsilon}^R_\theta) \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))}$$
$$- \gamma^{CES} \int_{\underline{\theta}}^{\overline{\theta}} \frac{w_{\tilde{\theta}} l_{\tilde{\theta}} h_{\tilde{\theta}}}{F(l(T), \phi^*(T))} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta} \tag{53}$$

for all $\theta \in \Theta$, where

$$\bar{\epsilon}^R_\theta := \frac{\epsilon^R_\theta}{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w} \, .$$

Proof. Again replacing equation (25) by equation (52), the proof of Corollary 5 is analogous to the proof of its counterpart for induced technical change effects, Corollary 2. □

Combining equations (26) and (51), or (27) and (53) for the CES case, we obtain expressions for the total effect of a tax reform on wages. I use these combined equations to assess the wage effects of tax reforms quantitatively in Section 7.

C.4. Alternative Representation of Labor Input Responses

Here I compare equation (24) for the effects of a tax reform on labor inputs (Lemma 3) with an alternative expression for these effects obtained following the iteration approach of Sachs et al. (2019).

For that, I first define elasticities of aggregate labor supply of a given type with respect to the marginal retention rate and the wage. In particular, note that, if all workers of type θ change their labor supply jointly, the wage w_θ will react. The wage response then induces a change in labor supply in addition to the direct response described by the individual labor supply elasticities ϵ^R_θ and ϵ^w_θ. Starting from the individual labor supply elasticities, we can construct elasticities that accounts for this feedback effect as follows:

$$\bar{\epsilon}^R_\theta(T,l,w) := \frac{\epsilon^R_\theta(T,l,w)}{1-(\gamma_{\theta,\theta}+\rho_{\theta,\theta})\epsilon^w_\theta(T,l,w)} \tag{54}$$

and

$$\bar{\epsilon}^w_\theta(T,l,w) := \frac{\epsilon^w_\theta(T,l,w)}{1-(\gamma_{\theta,\theta}+\rho_{\theta,\theta})\epsilon^w_\theta(T,l,w)} \; . \tag{55}$$

Relative to the individual elasticities, the aggregate elasticities are scaled by the feedback from the wage to labor supply. If the own-wage effect $\gamma_{\theta,\theta}+\rho_{\theta,\theta}$ is negative (positive), the individual elasticity is scaled down (up), as an increase in labor supply depresses (raises) the wage, which then counteracts (amplifies) the initial labor supply change.

We can now use the aggregate labor supply elasticities to rearrange equation (20) as follows:

$$\hat{l}_{\theta,\tau} = -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\theta,\tilde{\theta}}+\rho_{\theta,\tilde{\theta}})\hat{l}_{\tilde{\theta},\tau}(T)\, d\tilde{\theta} \; . \tag{56}$$

This fixed point equation can be solved by iteratively inserting the right-hand side of the equation into itself (see the proof below for details).

Lemma 7. *Fix an initial tax T, suppose that workers' second-order conditions hold strictly under T such that the labor supply elasticities ϵ^R_θ and ϵ^w_θ are well defined (see Appendix A.2), and let $(\gamma^{CES}+\rho^{CES})\epsilon^w_\theta < 1$ under T such that the aggregate elasticities $\bar{\epsilon}^R_\theta$ and $\bar{\epsilon}^R_\theta$ are well defined. Moreover, suppose that under T,*[36]

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\bar{\epsilon}^w_\theta(\gamma_{\theta,\tilde{\theta}}+\rho_{\theta,\tilde{\theta}}))^2\, d\tilde{\theta} d\theta < 1 \quad \textit{and} \quad \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\bar{\epsilon}^w_\theta \rho_{\theta,\tilde{\theta}})^2\, d\tilde{\theta} d\theta < 1 \; .$$

Then, the effect of tax reform τ on labor supply can be written as

$$\hat{l}_{\theta,\tau} = -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \bar{\rho}_{\theta,\tilde{\theta}}(-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \bar{\gamma}_{\theta,\tilde{\theta}}(-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \; , \tag{57}$$

[36] These conditions serve the same purpose as conditions (21) and (22) in Lemma (3).

where

$$
\begin{aligned}
\overline{\rho}_{\theta,\tilde{\theta}} &= \sum_{n=1}^{\infty} \rho^{(n)}_{\theta,\tilde{\theta}} \\
\rho^{(1)}_{\theta,\tilde{\theta}} &= \rho_{\theta,\tilde{\theta}} \\
\rho^{(n)}_{\theta,\tilde{\theta}} &= \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(n-1)}_{\theta,\hat{\theta}} \overline{\epsilon}^{w}_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\, d\hat{\theta} \quad \forall n > 1
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{\gamma}_{\theta,\tilde{\theta}} &= \sum_{n=1}^{\infty} \gamma^{(n)}_{\theta,\tilde{\theta}} \\
\gamma^{(1)}_{\theta,\tilde{\theta}} &= \gamma_{\theta,\tilde{\theta}} \\
\gamma^{(n)}_{\theta,\tilde{\theta}} &= \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(n-1)}_{\theta,\hat{\theta}} \overline{\epsilon}^{w}_{\hat{\theta}} (\gamma_{\hat{\theta},\tilde{\theta}} + \rho_{\hat{\theta},\tilde{\theta}})\, d\hat{\theta} + \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(n-1)}_{\theta,\hat{\theta}} \overline{\epsilon}^{w}_{\hat{\theta}} \gamma_{\hat{\theta},\tilde{\theta}}\, d\hat{\theta} \quad \forall n > 1.
\end{aligned}
$$

Proof. Following Sachs et al. (2019), I solve the fixed point equation (56) by iteration. Within the iteration steps, I separate the induced technical change from the (within-technology) substitution effects to obtain a decomposition of the total labor input response along the lines of Lemma 3.

Step 1. The first part of the proof proceeds by induction. We start by substituting equation (56) into itself:

$$
\begin{aligned}
\hat{l}_{\theta,\tau} =& -\overline{\epsilon}^{R}_{\theta} \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\theta,\tilde{\theta}} + \rho_{\theta,\tilde{\theta}}) \left[-\overline{\epsilon}^{R}_{\tilde{\theta}} \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} + \overline{\epsilon}^{w}_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\tilde{\theta},\hat{\theta}} + \rho_{\tilde{\theta},\hat{\theta}}) \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \right] d\tilde{\theta} \\
=& -\overline{\epsilon}^{R}_{\theta} \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} (-\overline{\epsilon}^{R}_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\tilde{\theta},\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\, d\tilde{\theta} \\
&+ \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} (-\overline{\epsilon}^{R}_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\tilde{\theta},\hat{\theta}} + \rho_{\tilde{\theta},\hat{\theta}}) \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\, d\tilde{\theta} \\
&+ \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\tilde{\theta},\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\, d\tilde{\theta}\,.
\end{aligned}
$$

Changing the order of integration in the terms containing $\hat{l}_{\hat{\theta},\tau}$ and summarizing the last two terms, we obtain

$$
\begin{aligned}
\hat{l}_{\theta,\tau} =& -\overline{\epsilon}^{R}_{\theta} \frac{\tau'(y_\theta(T))}{1 - T'(y_\theta(T))} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} (-\overline{\epsilon}^{R}_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} \rho_{\tilde{\theta},\hat{\theta}}\, d\tilde{\theta} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\
&+ \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} (-\overline{\epsilon}^{R}_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\
&+ \overline{\epsilon}^{w}_{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} (\gamma_{\tilde{\theta},\hat{\theta}} + \rho_{\tilde{\theta},\hat{\theta}})\, d\tilde{\theta} + \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}} \overline{\epsilon}^{w}_{\tilde{\theta}} \gamma_{\tilde{\theta},\hat{\theta}}\, d\tilde{\theta} \right] \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\,.
\end{aligned}
$$

Using the definitions of $\rho^{(n)}_{\theta,\tilde{\theta}}$ and $\gamma^{(n)}_{\theta,\tilde{\theta}}$ given in the lemma, this expression can be rewritten as

$$\begin{aligned}\hat{l}_{\theta,\tau} = & -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(2)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(2)}_{\theta,\tilde{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\,.\end{aligned}$$

This constitutes the base case for induction. As an induction hypothesis, suppose now that for any $N \geq 1$ the following holds:

$$\begin{aligned}\hat{l}_{\theta,\tau} = & -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N+1)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\,. \end{aligned} \tag{58}$$

Then, using equation (56) to substitute for $\hat{l}_{\hat{\theta},\tau}$ on the right-hand-side yields

$$\begin{aligned}\hat{l}_{\theta,\tau} = & -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\tilde{\theta},\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\, d\tilde{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N+1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \left[\gamma^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\tilde{\theta},\hat{\theta}} + \rho_{\tilde{\theta},\hat{\theta}}) \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} + \rho^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\tilde{\theta},\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \right] d\tilde{\theta}\,.\end{aligned}$$

Changing again the order of integration in the terms containing $\hat{l}_{\hat{\theta},\tau}$ yields

$$\begin{aligned}\hat{l}_{\theta,\tau} = & -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} \rho_{\tilde{\theta},\hat{\theta}}\, d\tilde{\theta} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N+1)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \left[\int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} (\gamma_{\tilde{\theta},\hat{\theta}} + \rho_{\tilde{\theta},\hat{\theta}})\, d\tilde{\theta} + \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+1)}_{\theta,\tilde{\theta}} \bar{\epsilon}^w_{\tilde{\theta}} \gamma_{\tilde{\theta},\hat{\theta}}\, d\tilde{\theta} \right] \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\ = & -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N+1} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N+2)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta} \\ & + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{N+1} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N+2)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\,.\end{aligned}$$

Hence the induction hypothesis (58) holds for any $N \geq 1$.

Step 2. I take the induction hypothesis from step 1 and let N go to infinity:

$$\hat{l}_{\theta,\tau} = -\bar{\epsilon}^R_\theta \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{\infty} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(N)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}$$
$$+ \bar{\epsilon}^w_\theta \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{\infty} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta} + \bar{\epsilon}^w_\theta \lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(N)}_{\theta,\hat{\theta}} \hat{l}_{\hat{\theta},\tau}(T)\, d\hat{\theta}\,.$$

The goal is to prove that the infinite series are convergent while the limit expressions containing $\hat{l}_{\hat{\theta},\tau}$ vanish on the right-hand side. Let

$$A_n := \int_{\underline{\theta}}^{\overline{\theta}} \rho^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta}$$
$$B_n := \int_{\underline{\theta}}^{\overline{\theta}} \gamma^{(n)}_{\theta,\tilde{\theta}} (-\bar{\epsilon}^R_{\tilde{\theta}}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\, d\tilde{\theta}\,.$$

I start with the series $\sum_{n=1}^{\infty} A_n$. First, using the definition of $\rho^{(n)}_{\theta,\tilde{\theta}}$, the Cauchy-Schwarz inequality implies

$$\left(\rho^{(n)}_{\theta,\tilde{\theta}}\right)^2 = \left(\int_{\underline{\theta}}^{\overline{\theta}} \rho^{(n-1)}_{\theta,\hat{\theta}} \bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\, d\hat{\theta}\right)^2 \le \int_{\underline{\theta}}^{\overline{\theta}} \left(\rho^{(n-1)}_{\theta,\hat{\theta}}\right)^2 d\hat{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\right)^2 d\hat{\theta}\,.$$

Integrating over $\tilde{\theta}$ yields

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\rho^{(n)}_{\theta,\tilde{\theta}}\right)^2 d\tilde{\theta} \le \int_{\underline{\theta}}^{\overline{\theta}} \left(\rho^{(n-1)}_{\theta,\hat{\theta}}\right)^2 d\hat{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\right)^2 d\hat{\theta}\, d\tilde{\theta}\,.$$

Then, applying the inequality iteratively $n-2$ times, we obtain

$$\int_{\underline{\theta}}^{\overline{\theta}} \left(\rho^{(n)}_{\theta,\tilde{\theta}}\right)^2 d\tilde{\theta} \le \int_{\underline{\theta}}^{\overline{\theta}} \left(\rho_{\theta,\hat{\theta}}\right)^2 d\hat{\theta} \left[\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\right)^2 d\hat{\theta}\, d\tilde{\theta}\right]^{n-1}\,.$$

Moreover, again applying the Cauchy-Schwarz inequality, we have

$$A_n^2 \le \int_{\underline{\theta}}^{\overline{\theta}} \left(\rho^{(n)}_{\theta,\tilde{\theta}}\right)^2 d\tilde{\theta} \int_{\underline{\theta}}^{\overline{\theta}} \left(-\bar{\epsilon}^R_{\tilde{\theta}} \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\right)^2 d\tilde{\theta}$$

and hence

$$|A_n| \le \left[\sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\right)^2 d\hat{\theta}\, d\tilde{\theta}}\right]^{n-1} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(\rho_{\theta,\hat{\theta}}\right)^2 d\hat{\theta}} \sqrt{\int_{\underline{\theta}}^{\overline{\theta}} \left(-\bar{\epsilon}^R_{\tilde{\theta}} \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))}\right)^2 d\tilde{\theta}}$$
$$=: \overline{A}_n\,.$$

The sequence $\{\overline{A}_n\}_{n\in\mathbb{N}}$ is geometric. Moreover, since

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}^w_{\hat{\theta}} \rho_{\hat{\theta},\tilde{\theta}}\right)^2 d\hat{\theta}\, d\tilde{\theta} < 1\,,$$

it converges to zero. Hence $\{A_n\}_{n\in\mathbb{N}}$ is dominated in absolute value by a geometric sequence converging to zero. The series $\sum_{n=1}^{\infty} A_n$ is therefore convergent.

For convergence of $\sum_{n=1}^{\infty} B_n$, I show that $\sum_{n=1}(A_n + B_n)$ converges. Convergence of $\sum_{n=1}^{\infty} A_n$ and $\sum_{n=1}^{\infty}(A_n + B_n)$ then immediately implies convergence of $\sum_{n=1}^{\infty} B_n$. By definition we have

$$A_n + B_n = \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\theta,\tilde{\theta}}^{(n)} + \rho_{\theta,\tilde{\theta}}^{(n)})(-\bar{\epsilon}_{\tilde{\theta}}^{R}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta}$$

and

$$\gamma_{\theta,\tilde{\theta}}^{(n)} + \rho_{\theta,\tilde{\theta}}^{(n)} = \int_{\underline{\theta}}^{\overline{\theta}} (\gamma_{\theta,\hat{\theta}}^{(n-1)} + \rho_{\theta,\hat{\theta}}^{(n-1)}) \bar{\epsilon}_{\hat{\theta}}^{w} (\gamma_{\hat{\theta},\tilde{\theta}} + \rho_{\hat{\theta},\tilde{\theta}}) \, d\hat{\theta} \; .$$

Convergence of $\sum_{n=1}^{\infty}(A_n + B_n)$ now follows from exactly the same steps as convergence of $\sum_{n=1}^{\infty} A_n$, with the only difference that $\rho_{\theta,\tilde{\theta}}$ is replaced by $\gamma_{\theta,\tilde{\theta}} + \rho_{\theta,\tilde{\theta}}$ in every step. Following these steps, the condition

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\overline{\theta}} \left(\bar{\epsilon}_{\hat{\theta}}^{w} (\gamma_{\hat{\theta},\tilde{\theta}} + \rho_{\hat{\theta},\tilde{\theta}}) \right)^2 \, d\hat{\theta} \, d\tilde{\theta} < 1$$

implies that the sequence $\{A_n + B_n\}_{n\in\mathbb{N}}$ is dominated in absolute value by a geometric sequence converging to zero, which establishes convergence of $\sum_{n=1}^{\infty}(A_n + B_n)$.

Finally consider the limits

$$\lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\hat{\theta}}^{(N)} \hat{l}_{\hat{\theta},\tau}(T) \, d\hat{\theta} \quad \text{and} \quad \lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\hat{\theta}}^{(N)} \hat{l}_{\hat{\theta},\tau}(T) \, d\hat{\theta} \; .$$

We have already shown that

$$\sum_{n=1}^{\infty} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}^{(n)} (-\bar{\epsilon}_{\tilde{\theta}}^{R}) \frac{\tau'(y_{\tilde{\theta}}(T))}{1 - T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta}$$

converges independently of the specific values of $\bar{\epsilon}_{\tilde{\theta}}^{R}$ and $\tau'(y_{\tilde{\theta}}(T))/(1 - T'(y_{\tilde{\theta}}(T)))$. Thus, by the same reasoning the series

$$\sum_{n=1}^{\infty} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\tilde{\theta}}^{(n)} (-\bar{\epsilon}_{\tilde{\theta}}^{R}) \hat{l}_{\tilde{\theta},\tau} \, d\tilde{\theta}$$

converges. We must therefore have

$$\lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \rho_{\theta,\hat{\theta}}^{(N)} \hat{l}_{\hat{\theta},\tau}(T) \, d\hat{\theta} = 0 \; .$$

Analogous reasoning shows that

$$\lim_{N\to\infty} \int_{\underline{\theta}}^{\overline{\theta}} \gamma_{\theta,\hat{\theta}}^{(N)} \hat{l}_{\hat{\theta},\tau}(T) \, d\hat{\theta} = 0 \; .$$

So, we have shown that, as $N \to \infty$, the induction hypothesis of step 1 becomes

$$\hat{l}_{\theta,\tau} = -\bar{\epsilon}_\theta^R \frac{\tau'(y_\theta(T))}{1-T'(y_\theta(T))} + \bar{\epsilon}_\theta^w \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{\infty} \rho_{\theta,\tilde{\theta}}^{(n)} (-\bar{\epsilon}_{\tilde{\theta}}^R) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta}$$
$$+ \bar{\epsilon}_\theta^w \int_{\underline{\theta}}^{\overline{\theta}} \sum_{n=1}^{\infty} \gamma_{\theta,\tilde{\theta}}^{(n)} (-\bar{\epsilon}_{\tilde{\theta}}^R) \frac{\tau'(y_{\tilde{\theta}}(T))}{1-T'(y_{\tilde{\theta}}(T))} \, d\tilde{\theta} \,,$$

which proves Lemma 7. □

The difference between the expressions provided by Lemma 3 and those given by Lemma 7 is that Lemma 7 uses the aggregate labor supply elasticities $\bar{\epsilon}_\theta^R$ and $\bar{\epsilon}_\theta^w$ whereas Lemma 3 uses individual labor supply elasticities. As can be seen from equations (54) and (55), the aggregate elasticities already contain the own-wage elasticities $\gamma_{\theta,\theta}$ and $\rho_{\theta,\theta}$. This makes it hard to disentangle induced technical change from within-technology substitution effects on the basis of Lemma 7 For example, when decomposing the total labor supply effect given by Lemma 7 along the lines of the decomposition provided in equation (24) of Lemma 3, we would end up with an induced technical change component that still contains within-technology substitution effects via the aggregate labor supply elasticities. Signing the impact of the induced technical change component on relative wages would then require much more demanding restrictions. Specifically, we would have to restrict heterogeneity in own-wage elasticities, which would require restrictions on the aggregate production F, beyond the restrictions already imposed in Section 4.

C.5. Welfare Effects

Next I prove the results on the welfare effects of tax reforms from Section 5.2.

Proof of Proposition 2. Welfare can be written as

$$W(T) = V\left(\{R_T\left(w_\theta(T,\phi^*(T))l_\theta(T,w_\theta)\right) + S\left(l(T,w), w(T,\phi^*(T)), T\right) - v\left(l_\theta(T,w_\theta)\right)\}_{\theta\in\Theta}\right) \,,$$

where

$$S\left(l(T,w), w(T,\phi^*(T)), T\right) = \int_{\underline{\theta}}^{\overline{\theta}} T\left(w_\theta(T,\phi^*(T))l_\theta(T,w_\theta)\right) h_\theta \, d\theta \,.$$

Taking the derivative D_τ yields:

$$\begin{aligned}
D_\tau W(T) = &\int_{\underline{\theta}}^{\overline{\theta}} \left(-g_\theta h_\theta \tau(w_\theta l_\theta) + g_\theta h_\theta \int_{\underline{\theta}}^{\overline{\theta}} h_{\tilde{\theta}} \tau(w_{\tilde{\theta}} l_{\tilde{\theta}})\, d\tilde{\theta} \right) d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta \int_{\underline{\theta}}^{\overline{\theta}} h_{\tilde{\theta}} T'(w_{\tilde{\theta}} l_{\tilde{\theta}}) w_{\tilde{\theta}} (-\epsilon^R_{\tilde{\theta}}) \frac{l_{\tilde{\theta}}}{1 - T'(w_{\tilde{\theta}} l_{\tilde{\theta}})} \tau'(w_{\tilde{\theta}} l_{\tilde{\theta}})\, d\tilde{\theta}\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta (1 - T'(w_\theta l_\theta)) l_\theta D_{\phi,\tau} w_\theta(T, \phi^*(T))\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta \int_{\underline{\theta}}^{\overline{\theta}} h_{\tilde{\theta}} T'(w_{\tilde{\theta}} l_{\tilde{\theta}}) \left(l_{\tilde{\theta}} D_{\phi,\tau} w_{\tilde{\theta}}(T, \phi^*(T)) + w_{\tilde{\theta}} \epsilon^w_{\tilde{\theta}} \frac{l_{\tilde{\theta}}}{w_{\tilde{\theta}}} D_{\phi,\tau} w_{\tilde{\theta}}(T, \phi^*(T)) \right) d\tilde{\theta}\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta (1 - T'(w_\theta l_\theta)) l_\theta D_\tau w_\theta(T, \phi^*(T))\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta \int_{\underline{\theta}}^{\overline{\theta}} h_{\tilde{\theta}} T'(w_{\tilde{\theta}} l_{\tilde{\theta}}) \left(l_{\tilde{\theta}} D_\tau w_{\tilde{\theta}}(T, \phi^*(T)) + w_{\tilde{\theta}} \epsilon^w_{\tilde{\theta}} \frac{l_{\tilde{\theta}}}{w_{\tilde{\theta}}} D_\tau w_{\tilde{\theta}}(T, \phi^*(T)) \right) d\tilde{\theta}\, d\theta \,.
\end{aligned}$$

Using $\int_{\underline{\theta}}^{\overline{\theta}} h_\theta g_\theta \, d\theta = 1$, we can rearrange this expression to obtain

$$\begin{aligned}
D_\tau W(T) = &\int_{\underline{\theta}}^{\overline{\theta}} (1 - g_\theta) \tau(w_\theta l_\theta) h_\theta \, d\theta + \int_{\underline{\theta}}^{\overline{\theta}} T'(w_\theta l_\theta) w_\theta l_\theta (-\epsilon^R_\theta) \frac{\tau'(w_\theta l_\theta)}{1 - T'(w_\theta l_\theta)}\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} \left[g_\theta (1 - T'(w_\theta l_\theta)) + T'(w_\theta l_\theta)(1 + \epsilon^w_\theta) \right] l_\theta D_{\phi,\tau} w_\theta(T, \phi^*(T))\, d\theta \\
&+ \int_{\underline{\theta}}^{\overline{\theta}} \left[g_\theta (1 - T'(w_\theta l_\theta)) + T'(w_\theta l_\theta)(1 + \epsilon^w_\theta) \right] l_\theta D_\tau w_\theta(T, \phi^*(T))\, d\theta \,.
\end{aligned}$$

□

Proof of Proposition 3. Take any initial tax $T \in \mathcal{T}^{ex}$ and any reform τ^{ex} that is progressive and raises welfare when neglecting induced technical change effects, that is, $D^{ex}_{\tau^{ex}} W(T) > 0$ (see equation (28)). The strategy of the proof is to construct another progressive reform τ^{en} that raises welfare when accounting for induced technical change effects, that is, $D_{\tau^{en}} W(T) > 0$. Constructing such a reform proves that $T \in \mathcal{T}$ and hence $\mathcal{T}^{ex} \subseteq \mathcal{T}$.

I will construct the reform τ^{en} such that it exactly replicates the labor input changes that τ^{ex} would induce if there were no induced technical change effects. To move back and forth between induced labor input changes and progressive reforms, I use Lemma 6. In particular, note that Proposition 3 considers reforms of CRP tax schedules when the disutility of labor is iso elastic. Under these conditions the elasticities ϵ^R_θ and ϵ^w_θ are constant in θ. Lemma 6 then says that any progressive reform τ induces labor input responses $\hat{l}_{\theta,\tau}$ that decrease in θ and, conversely, any reform that induces labor input changes that decrease in θ is progressive.

After these preparations, take any $T \in \mathcal{T}^{ex}$ and a progressive reform τ^{ex} that would raise welfare if technology remained constant, that is, $D^{ex}_{\tau^{ex}} W(T) > 0$. We can write welfare as a function of consumption and labor inputs only, that is,

$$\tilde{W}(c, l) := V\left(\{u_\theta(c_\theta, l_\theta)\}_{\theta \in \Theta} \right) \,.$$

Then, the effect of reform τ^{ex} on welfare, ignoring induced technical change effects, is fully determined by the responses of consumption and labor supply to τ^{ex} that we would obtain if technology were fixed. I analyze these responses in the following.

Step 1. Denote the labor input response to τ^{ex} that ignores induced technical change effects by

$$D^{ex}_{\tau^{ex}} l_\theta(T) := D_{\tau^{ex}} l_\theta(T)|_{\rho_{\theta,\tilde{\theta}}=0\ \forall\, \theta,\tilde{\theta}}$$

and similarly the consumption response that ignores induced technical change effects by $D^{ex}_{\tau^{ex}} c_\theta(T)$.

I now characterize the consumption response contingent on the labor input response using incentive compatibility constraints. In particular, at any tax $\tilde{T}$, consumption and labor allocations must satisfy

$$c_{\tilde{\theta}}(\tilde{T}) - v\left(\frac{w_{\tilde{\theta}}(\tilde{T}, \phi^*(\tilde{T}))l_{\tilde{\theta}}(\tilde{T})}{w_\theta(\tilde{T}, \phi^*(\tilde{T}))}\right) \leq c_\theta(\tilde{T}) - v(l_\theta(\tilde{T})) \quad \text{for all } \theta, \tilde{\theta}\ .$$

Via an envelope argument this implies

$$c'_\theta(\tilde{T}) = v'(l_\theta(\tilde{T}))\left[l'_\theta(\tilde{T}) + \hat{w}_\theta(\tilde{T}, \phi^*(\tilde{T}))l_\theta(\tilde{T})\right] \quad \text{for all } \theta\ .$$

Here, $\hat{w}_\theta = w'_\theta / w_\theta$ and the notation $x'_\theta(T)$ is exclusively used to denote differentiation with respect to the type index θ. So $l'_\theta(T)$ is the derivative of $l_\theta(T)$ with respect to θ (and at θ), holding T constant. Integrating over θ, the envelope condition yields:

$$c_\theta(\tilde{T}) = c_{\underline{\theta}}(\tilde{T}) + \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(\tilde{T}))\left[l'_{\tilde{\theta}}(\tilde{T}) + \hat{w}_{\tilde{\theta}}(\tilde{T}, \phi^*(\tilde{T}))l_{\tilde{\theta}}(\tilde{T})\right] d\tilde{\theta} \quad \text{for all } \theta\ . \tag{59}$$

The level $c_{\underline{\theta}}$ is determined via the resource constraint:

$$\int_{\underline{\theta}}^{\overline{\theta}} c_\theta(\tilde{T}) h_\theta \, d\theta = F(l(\tilde{T}), \phi^*(l(\tilde{T})))\ . \tag{60}$$

Using equation (59), the response of consumption to tax reform τ^{ex}, ignoring induced technical change effects, can be expressed as

$$\begin{aligned} D^{ex}_{\tau^{ex}} c_\theta(T) = D^{ex}_{\tau^{ex}} c_{\underline{\theta}}(T) &+ \int_{\underline{\theta}}^{\theta} v''(l_{\tilde{\theta}}(T))\,(D^{ex}_{\tau^{ex}} l_{\tilde{\theta}}(T))\left[l'_{\tilde{\theta}}(T) + \hat{w}_{\tilde{\theta}}(T, \phi^*(T))l_{\tilde{\theta}}(T)\right] d\tilde{\theta} \\ &+ \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(T))\left[D^{ex}_{\tau^{ex}} l'_{\tilde{\theta}}(T) + \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) D^{ex}_{\tau^{ex}} l_{\tilde{\theta}}(T)\right] d\tilde{\theta} \\ &+ \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(T)) l_{\tilde{\theta}}(T) D^{ex}_{\tau^{ex}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T))\, d\tilde{\theta}\ . \end{aligned} \tag{61}$$

Note that the last line contains only the constant-technology effect $D^{ex}_{\tau^{ex}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T))$ but not the induced technical change effect $D^{ex}_{\phi,\tau^{ex}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T))$. The resource constraint (60) implies

$$\int_{\underline{\theta}}^{\overline{\theta}} h_\theta D^{ex}_{\tau^{ex}} c_\theta(T)\, d\theta = \frac{d}{d\mu} F(l(T + \mu\tau^{ex}), \phi^*(l(T)))\bigg|_{\mu=0}\ . \tag{62}$$

Step 2. Suppose now that we can find a reform τ^{en} that replicates the labor input change

$D^{ex}_{\tau^{ex}} l_\theta(T)$ while accounting for induced technical change effects. That is, take τ^{en} such that

$$D_{\tau^{en}} l_\theta(T) = D^{ex}_{\tau^{ex}} l_\theta(T) \quad \text{for all } \theta \,.$$

I verify below that such a reform exists. Again using equation (59), the consumption response to τ^{en}, also accounting for induced technical change effects, can be expressed as

$$\begin{aligned} D_{\tau^{en}} c_\theta(T) = D_{\tau^{en}} c_{\underline{\theta}}(T) &+ \int_{\underline{\theta}}^{\theta} v''(l_{\tilde{\theta}}(T)) \left(D_{\tau^{en}} l_{\tilde{\theta}}(T)\right) \left[l'_{\tilde{\theta}}(T) + \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) l_{\tilde{\theta}}(T)\right] \, d\tilde{\theta} \\ &+ \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(T)) \left[D_{\tau^{en}} l'_{\tilde{\theta}}(T) + \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) D_{\tau^{en}} l_{\tilde{\theta}}(T)\right] \, d\tilde{\theta} \\ &+ \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(T)) l_{\tilde{\theta}}(T) \left[D_{\tau^{en}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) + D_{\phi, \tau^{en}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T))\right] \, d\tilde{\theta} \,. \end{aligned} \tag{63}$$

Note that here the last line contains the total effect of τ^{en} on the wage growth rate $\hat{w}_{\tilde{\theta}}$, that is, the sum of the direct and the induced technical change effect. The resource constraint (60) now implies

$$\int_{\underline{\theta}}^{\overline{\theta}} h_\theta D_{\tau^{en}} c_\theta(T) \, d\theta = \frac{d}{d\mu} F(l(T + \mu\tau^{en}), \phi^*(l(T + \mu\tau^{en}))) \bigg|_{\mu=0} \,. \tag{64}$$

The principle of taxation says that every incentive compatible and resource feasible consumption-labor allocation can be implemented by some tax $\tilde{T}$. By implication, the allocation change $\{D_{\tau^{en}} l_\theta(T), D_{\tau^{en}} c_\theta(T)\}_{\theta \in \Theta}$ can be implemented by some reform $\tilde{\tau}$. Hence, a reform τ^{en} as analyzed above indeed exists.

Step 3. Having characterized the relevant consumption and labor input changes, we can now compare the welfare effect of reform τ^{ex} while ignoring induced technical change effects with the welfare effect of reform τ^{en} while accounting for induced technical change effects. Since the labor input changes are identical in both scenarios, the only difference in the two welfare effects stems from the different consumption responses:

$$D_{\tau^{en}} W(T) - D^{ex}_{\tau^{ex}} W(T) = \int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta \left(D_{\tau^{en}} c_\theta(T) - D^{ex}_{\tau^{ex}} c_\theta(T)\right) \, d\theta \,. \tag{65}$$

From equations (61) and (63), the difference in consumption responses can be expressed as, for every θ,

$$D_{\tau^{en}} c_\theta(T) - D^{ex}_{\tau^{ex}} c_\theta(T) = \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}(T)) l_{\tilde{\theta}}(T) D_{\phi, \tau^{en}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) \, d\tilde{\theta} \,. \tag{66}$$

Here I used that the labor response is the same in both scenarios, such that the constant-technology effect on the wage growth rate is the same as well, that is,

$$D^{ex}_{\tau^{ex}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) = D_{\tau^{en}} \hat{w}_{\tilde{\theta}}(T, \phi^*(T)) \,.$$

Next, consider the induced technology effect on the wage growth rate:

$$\begin{aligned} D_{\phi,\tau^{en}}\hat{w}_{\tilde{\theta}}(T,\phi^*(T)) =& D_{\phi,\tau^{en}}\left[\frac{d}{d\theta}\log\left(w_{\tilde{\theta}}(T,\phi^*(T))\right)\right] \\ =& \frac{d}{d\theta}\left[D_{\phi,\tau^{en}}\log\left(w_{\tilde{\theta}}(T,\phi^*(T))\right)\right] \\ =& \frac{d}{d\theta}\left[\frac{1}{w_{\tilde{\theta}}(T,\phi^*(T))}D_{\phi,\tau^{en}}w_{\tilde{\theta}}(T,\phi^*(T))\right] . \end{aligned}$$

Since τ^{ex} is progressive, the labor response $(1/l_{\tilde{\theta}})D^{ex}_{\tau^{ex}}l_{\tilde{\theta}}(T)$ is decreasing in $\tilde{\theta}$ by Lemma 6. Hence, the identical response $(1/l_{\tilde{\theta}})D_{\tau^{en}}l_{\tilde{\theta}}(T)$ decreases in $\tilde{\theta}$ as well. Then by Lemma 1, the induced technical change effect $(1/w_{\tilde{\theta}})D_{\phi,\tau^{en}}w_{\tilde{\theta}}(T,\phi^*(T))$ must also decrease in $\tilde{\theta}$. We therefore obtain

$$\begin{aligned} 0 \geq& \frac{d}{d\theta}\left[\frac{1}{w_{\tilde{\theta}}(T,\phi^*(T))}D_{\phi,\tau^{en}}w_{\tilde{\theta}}(T,\phi^*(T))\right] \\ =& D_{\phi,\tau^{en}}\hat{w}_{\tilde{\theta}}(T,\phi^*(T)) . \end{aligned}$$

By equation (66), this implies that the consumption difference $D_{\tau^{en}}c_\theta(T) - D^{ex}_{\tau^{ex}}c_\theta(T)$ is decreasing in θ. Moreover, from equations (62) and (64), we have

$$\begin{aligned} \int_{\underline{\theta}}^{\overline{\theta}} h_\theta D_{\tau^{en}}c_\theta(T)\,d\theta &= \left.\frac{d}{d\mu}F(l(T+\mu\tau^{en}),\phi^*(l(T+\mu\tau^{en})))\right|_{\mu=0} \\ &= \left.\frac{d}{d\mu}F(l(T+\mu\tau^{ex}),\phi^*(l(T)))\right|_{\mu=0} \\ &= \int_{\underline{\theta}}^{\overline{\theta}} h_\theta D^{ex}_{\tau^{ex}}c_\theta(T)\,d\theta , \end{aligned}$$

where the second equality uses an envelope argument. By implication:

$$\int_{\underline{\theta}}^{\overline{\theta}} h_\theta\left(D_{\tau^{en}}c_\theta(T) - D^{ex}_{\tau^{ex}}c_\theta(T)\right)\,d\theta .$$

So, inspecting equation (65) reveals that if g_θ were constant in θ, we would have

$$D_{\tau^{en}}W(T) - D^{ex}_{\tau^{ex}}W(T) = 0 .$$

But since both g_θ and $D_{\tau^{en}}c_\theta(T) - D^{ex}_{\tau^{ex}}c_\theta(T)$ are decreasing in θ, we must have

$$\int_{\underline{\theta}}^{\overline{\theta}} g_\theta h_\theta\left(D_{\tau^{en}}c_\theta(T) - D^{ex}_{\tau^{ex}}c_\theta(T)\right)\,d\theta \geq 0$$

and thereby

$$D_{\tau^{en}}W(T) - D^{ex}_{\tau^{ex}}W(T) \geq 0 .$$

So,

$$D_{\tau^{en}}W(T) > 0 .$$

Step 4. Finally, we know that the labor response $(1/l_\theta)D_{\tau^{en}}l_\theta(T)$ decreases in θ. Thus, we can again invoke Lemma 6 to obtain that τ^{en} must be progressive. We have thereby shown that

$$D_{\tau^{en}} W(T) > 0$$

for a progressive reform τ^{en}. So, $T \in \mathcal{T}$.
Since the preceding reasoning applies to any $T \in \mathcal{T}^{ex}$, we have shown that $\mathcal{T}^{ex} \subseteq \mathcal{T}$. □

D. Proofs and Additional Results for Optimal Taxes

This section contains proofs and additional results for the analysis of optimal taxes in Section 6 of the main text. I start by proving Proposition 4 for the general case and then turn to the more detailed analytical results that can be obtained in the CES case.

D.1. General Case

In the general case, optimal taxes are obtained by maximizing wefare $\tilde{W}(c,l)$ subject to the resource constraint (29) and the incentive compatibility constraint (30). The derivation proceeds along the following steps: first eliminate consumption from the welfare maximization problem, then derive first-order conditions, use workers' first-order condition to reintroduce tax rates into the equations, and finally prove the sign conditions for the term TE^*_θ at the bottom and the top of the type distribution using directed technical change theory.
In the first step the following lemma shows how to eliminate consumption from the welfare maximization problem.

Lemma 8. *The pair of consumption and labor inputs (c,l) satisfies the resource and incentive compatibility constraints (29) and (30) if and only if $c = c^*(l)$ where $c^*(l) = \{c^*_\theta(l)\}_{\theta\in\Theta}$ is determined by*

$$c^*_\theta(l) = F(l,\phi^*(l)) - \int_{\underline{\theta}}^{\overline{\theta}} v(l_{\tilde{\theta}})h_{\tilde{\theta}}\,d\tilde{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} v'(l_{\tilde{\theta}})l_{\tilde{\theta}}(1-H_{\tilde{\theta}})\hat{w}_{\tilde{\theta}}\,d\tilde{\theta} + \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}})l_{\tilde{\theta}}\hat{w}_{\tilde{\theta}}\,d\tilde{\theta} + v(l_\theta) \quad (67)$$

for all θ.

Proof. (⇒) I first show that constraints (29) and (30) imply equation (67). For that, write consumption as

$$c_\theta = c_{\underline{\theta}} + \int_{\underline{\theta}}^{\theta} c'_{\tilde{\theta}}\,d\tilde{\theta}\,.$$

By the incentive compatibility constraint (30), this implies for all θ:

$$\begin{aligned} c_\theta &= c_{\underline{\theta}} + \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}})(w'_{\tilde{\theta}}l_{\tilde{\theta}} + w_{\tilde{\theta}}l'_{\tilde{\theta}})\frac{1}{w_{\tilde{\theta}}}\,d\tilde{\theta} \\ &= c_{\underline{\theta}} + \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}})\hat{w}_{\tilde{\theta}}l_{\tilde{\theta}}\,d\tilde{\theta} + v(l_\theta) - v(l_{\underline{\theta}})\,. \end{aligned} \quad (68)$$

Combining this with the resource constraint (29), we obtain the following expression for $c_{\underline{\theta}}$:

$$c_{\underline{\theta}} = F(l, \phi^*(l)) - \int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}) \hat{w}_{\tilde{\theta}} l_{\tilde{\theta}} \, d\tilde{\theta} \, h_\theta \, d\theta - \int_{\underline{\theta}}^{\overline{\theta}} v(l_\theta) h_\theta \, d\theta + v(l_{\underline{\theta}}) \, .$$

Using integration by parts to solve the double integral yields:

$$c_{\underline{\theta}} = F(l, \phi^*(l)) - \int_{\underline{\theta}}^{\overline{\theta}} v'(l_\theta) \hat{w}_\theta l_\theta (1 - H_\theta) \, d\theta - \int_{\underline{\theta}}^{\overline{\theta}} v(l_\theta) h_\theta \, d\theta + v(l_{\underline{\theta}}) \, .$$

Inserting this back into equation (68), we obtain:

$$c_\theta = F(l, \phi^*(l)) - \int_{\underline{\theta}}^{\overline{\theta}} v'(l_{\tilde{\theta}}) \hat{w}_{\tilde{\theta}} l_{\tilde{\theta}} (1 - H_{\tilde{\theta}}) \, d\tilde{\theta} - \int_{\underline{\theta}}^{\overline{\theta}} v(l_{\tilde{\theta}}) h_{\tilde{\theta}} \, d\tilde{\theta} + \int_{\underline{\theta}}^{\theta} v'(l_{\tilde{\theta}}) \hat{w}_{\tilde{\theta}} l_{\tilde{\theta}} \, d\tilde{\theta} + v(l_\theta) \, ,$$

which is equation (67) defining the function c^* above.
($\Leftarrow$) Differentiating c^* with respect to θ shows immediately that equation (67) implies the incentive compatibility constraint (30). Similarly, after multiplying c^*_θ by h_θ and integrating over $[\underline{\theta}, \overline{\theta}]$, standard computations show that

$$\int_{\underline{\theta}}^{\overline{\theta}} c^*_\theta(l) \, d\theta = F(l, \phi^*(l)) \, ,$$

which proves that equation (67) also implies the resource constraint (29). □

Proof of Proposition 4. Lemma 8 provides an equivalent representation of resource and incentive compatibility constraints, which is explicitly solved for c. Hence, instead of maximizing welfare subject to the two constraints, we can study the unconstrained maximization of

$$\hat{W}(l) := \tilde{W}(c^*(l), l)$$

with l being the only choice variable. The first part of the proof now uses the first-order conditions of this unconstrained problem to derive the condition for optimal marginal tax rates provided in Proposition 4.
Part 1. The first-order conditions are given by

$$D_{l_\theta} \hat{W}(l) = 0 \quad \text{for all } \theta \, .$$

We hence study the derivative $D_{l_\theta} \hat{W}(l)$ first. Using the notation for welfare weights introduced in the main text, the derivative can be written as

$$
\begin{aligned}
D_{l_\theta}\hat{W}(l) = {} & w_\theta h_\theta - v'(l_\theta)h_\theta \\
& - \left(v''(l_\theta)l_\theta + v'(l_\theta)\right)(1-H_\theta)\hat{w}_\theta + \tilde{g}_\theta(1-H_\theta)\left(v''(l_\theta)l_\theta + v'(l_\theta)\right) \\
& - \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} v'(l_{\tilde{\theta}})l_{\tilde{\theta}}(1-H_{\tilde{\theta}}) \left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\tilde{\theta} \\
& + \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} g_{\tilde{\theta}}h_{\tilde{\theta}}\int_{\underline{\theta}}^{\tilde{\theta}} v'(l_{\hat{\theta}})l_{\hat{\theta}} \left.\frac{d\hat{w}^*_{\hat{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\hat{\theta}\, d\tilde{\theta}
\end{aligned} \tag{69}
$$

for almost all θ, where the terms in the first two lines were derived following the procedure detailed in Sections A.1 and A.3, which uses continuity of l and $\hat{w}$ in θ.[37] Following the notation introduced in Section 3.2, the expression $d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})/d\mu\big|_{\mu=0}$ denotes the total derivative of $\hat{w}$ in the direction of l_θ, accounting both for the substitution and the induced technical change effects:

$$
\left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} = \left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} + \left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0}.
$$

Using integration by parts to solve the double integral in equation (69), the derivative of the welfare function becomes

$$
\begin{aligned}
D_{l_\theta}\hat{W}(l) = {} & w_\theta h_\theta - v'(l_\theta)h_\theta - (1-\tilde{g}_\theta)\left(v''(l_\theta)l_\theta + v'(l_\theta)\right)(1-H_\theta)\hat{w}_\theta \\
& - \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} v'(l_{\tilde{\theta}})l_{\tilde{\theta}}(1-H_{\tilde{\theta}}) \left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\tilde{\theta} \\
& + \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} \tilde{g}_{\tilde{\theta}}(1-H_{\tilde{\theta}})v'(l_{\tilde{\theta}})l_{\tilde{\theta}} \left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\tilde{\theta} \\
= {} & w_\theta h_\theta - v'(l_\theta)h_\theta - (1-\tilde{g}_\theta)\left(v''(l_\theta)l_\theta + v'(l_\theta)\right)(1-H_\theta)\hat{w}_\theta \\
& - \lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}} (1-\tilde{g}_{\tilde{\theta}})v'(l_{\tilde{\theta}})l_{\tilde{\theta}}(1-H_{\tilde{\theta}}) \left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\tilde{\theta}\,.
\end{aligned} \tag{70}
$$

We now use workers' first-order condition (2) to introduce marginal tax rates into the equation. In particular condition (2) implies

$$
v'(l_\theta)l_\theta = (1-T'(y_\theta))y_\theta \tag{71}
$$

and

$$
v''(l_\theta)l_\theta\hat{w}_\theta + v'(l_\theta)\hat{w}_\theta = \left(1+\frac{1}{e_\theta}\right)(1-T'(y_\theta))w'_\theta\,. \tag{72}
$$

[37]The wage growth function $\hat{w}$ is continuous in θ almost everywhere because l is C^1 almost everywhere by hypothesis of Proposition 4.

Using equations (2), (71), and (72) in equation (70), we obtain

$$D_{l_\theta}\hat{W}(l) = T'(y_\theta)y_\theta h_\theta - (1-T'(y_\theta))\left(1+\frac{1}{e_\theta}\right)(1-\tilde{g}_\theta)(1-H_\theta)w'_\theta l_\theta$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}(1-T'(y_{\tilde{\theta}}))y_{\tilde{\theta}}(1-\tilde{g}_{\tilde{\theta}})(1-H_{\tilde{\theta}})\left.\frac{d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})}{d\mu}\right|_{\mu=0} d\tilde{\theta}\,.$$

Splitting up the total derivative $d\hat{w}^*_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta})/d\mu\Big|_{\mu=0}$ into its substitution and induced technical change components, this becomes

$$D_{l_\theta}\hat{W}(l) = T'(y_\theta)w_\theta h_\theta - (1-T'(y_\theta))\left(1+\frac{1}{e_\theta}\right)(1-\tilde{g}_\theta)(1-H_\theta)w'_\theta$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}(1-T'(y_{\tilde{\theta}}))y_{\tilde{\theta}}(1-\tilde{g}_{\tilde{\theta}})(1-H_{\tilde{\theta}})\left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} d\tilde{\theta}$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}(1-T'(y_{\tilde{\theta}}))y_{\tilde{\theta}}(1-\tilde{g}_{\tilde{\theta}})(1-H_{\tilde{\theta}})\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0} d\tilde{\theta}\,.$$

Equating the derivative to zero, dividing by $1-T'(y_\theta)$ and rearranging yields:

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)(1-\tilde{g}_\theta)\frac{1-H_\theta}{h_\theta}\hat{w}_\theta$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} d\tilde{\theta}$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0} d\tilde{\theta}\,. \tag{73}$$

Finally, let n_w and N_w denote the density and cumulative distribution functions of the distribution of wages and use the change-of-variable $h_\theta = n_{w_\theta}w'_\theta$ to obtain the condition for marginal tax rates from Proposition 4:

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)(1-\tilde{g}_\theta)\frac{1-N_{w_\theta}}{n_{w_\theta}w_\theta}$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} d\tilde{\theta}$$
$$-\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\overline{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0} d\tilde{\theta}\,.$$

Part 2. The second part of the proof is to show that $TE^*_{\underline{\theta}} \leq 0$ and $TE^*_{\overline{\theta}} \geq 0$. We only consider $TE^*_{\overline{\theta}}$ because the proof for $TE^*_{\underline{\theta}}$ works analogously.

Consider first the derivative $d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\overline{\theta}}))/d\mu\Big|_{\mu=0}$. It measures the local induced technical change effect of the labor input change $\tilde{l}_{\Delta,\overline{\theta}}$ (defined in Section 3.2) on relative wages. For $\theta \leq \overline{\theta}-\Delta$, the labor input change is zero by definition. On $\left(\overline{\theta}-\Delta,\overline{\theta}\right]$ it varies in θ

according to

$$\frac{1}{\tilde{l}_{\Delta,\bar{\theta},\theta}}\frac{d\tilde{l}_{\Delta,\bar{\theta},\theta}}{d\theta}=\frac{2\Delta}{2\Delta(\theta-\bar{\theta}+\Delta)}=\frac{1}{\theta-\bar{\theta}+\Delta}\geq\frac{1}{\Delta}\,.$$

Hence, given the optimal labor input l, we can find an $\epsilon > 0$ such that for all $\Delta < \epsilon$ and for all $\theta \in (\bar{\theta}-\Delta, \bar{\theta})$:

$$\frac{1}{\tilde{l}_{\Delta,\bar{\theta},\theta}}\frac{d\tilde{l}_{\Delta,\bar{\theta},\theta}}{d\theta}\geq\frac{1}{l_\theta}\frac{dl_\theta}{d\theta}\,.$$ [38]

So, for $\Delta < \epsilon$, the relative labor input change $\tilde{l}_{\Delta,\bar{\theta},\theta}/l_\theta$ increases in θ. Thus, by Lemma 1, we obtain that

$$\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\bar{\theta}}))}{d\mu}\right|_{\mu=0}\geq 0$$

for all θ if $\Delta < \epsilon$. Hence, for $\Delta < \epsilon$,

$$\int_{\underline{\theta}}^{\bar{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0}d\tilde{\theta}\geq 0$$

and therefore

$$TE^*_{\bar{\theta}}=\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\underline{\theta}}^{\bar{\theta}}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})y_{\tilde{\theta}}\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0}d\tilde{\theta}\geq 0\,.$$

□

D.2. CES Case

To derive expression (32) for optimal tax rates in Proposition 5, I start by specializing the terms TE^*_θ and SE^*_θ to the CES case.

Lemma 9. *Suppose the conditions of Proposition 4 are satisfied and F and Φ take the CES form introduced in Section 3.4. Then, the terms TE^*_θ and SE^*_θ take the following form for almost every θ:*

$$SE^*_\theta=(1-g_\theta)\gamma^{CES}-\left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\gamma^{CES} \tag{74}$$

$$TE^*_\theta=(1-g_\theta)\rho^{CES}-\left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES}\,. \tag{75}$$

Proof. I focus on the expression for TE^*_θ, because the derivation of SE^*_θ is analogous with γ^{CES} in the place of ρ^{CES}.

The central step is to obtain an expression for the derivative of $\hat{w}_{\tilde{\theta}}$ in TE^*_θ. From equation (13) we obtain

$$\hat{w}_{\tilde{\theta}}=\frac{\sigma-1}{\sigma}\hat{\kappa}_{\tilde{\theta}}+\frac{\sigma-1}{\sigma}\hat{\phi}_{\tilde{\theta}}-\frac{1}{\sigma}\hat{l}_{\tilde{\theta}}-\frac{1}{\sigma}\hat{h}_{\tilde{\theta}} \tag{76}$$

and from equation (41):

$$\hat{\phi}^*_{\tilde{\theta}}=\frac{\sigma-1}{(\delta-1)\sigma+1}\left(\hat{\kappa}_{\tilde{\theta}}+\hat{l}_{\tilde{\theta}}+\hat{h}_{\tilde{\theta}}\right)\,. \tag{77}$$

[38]Here we use that by hypothesis l'_θ exists on some right neighborhood of $\bar{\theta}$ and $\limsup_{\theta\to\bar{\theta}} l'_\theta < \infty$.

Hence, the partial effect of the perturbation $\tilde{l}_{\Delta,\theta}$ on $\hat{w}_{\tilde{\theta}}$ is

$$\begin{aligned}\left.\frac{d\hat{w}_{\tilde{\theta}}(l+\mu\tilde{l}_{\Delta,\theta},\phi^*(l))}{d\mu}\right|_{\mu=0} &= -\frac{1}{\sigma}\frac{d}{d\mu}(\widehat{l_{\tilde{\theta}}+\mu\tilde{l}_{\Delta,\theta,\tilde{\theta}}})\Big|_{\mu=0} \\ &= -\frac{1}{\sigma}\frac{d}{d\mu}\frac{l'_{\tilde{\theta}}+\mu\tilde{l}'_{\Delta,\theta,\tilde{\theta}}}{l_{\tilde{\theta}}+\mu\tilde{l}_{\Delta,\theta,\tilde{\theta}}}\Big|_{\mu=0} \\ &= \gamma^{CES}\left(\frac{\tilde{l}'_{\Delta,\theta,\tilde{\theta}}}{l_{\tilde{\theta}}}-\hat{l}_{\tilde{\theta}}\frac{\tilde{l}_{\Delta,\theta,\tilde{\theta}}}{l_{\tilde{\theta}}}\right).\end{aligned}$$

Analogously, the induced technical change effect is given by

$$\begin{aligned}\left.\frac{d\hat{w}_{\tilde{\theta}}(l,\phi^*(l+\mu\tilde{l}_{\Delta,\theta}))}{d\mu}\right|_{\mu=0} &= \frac{(\sigma-1)^2}{(\delta-1)\sigma^2+\sigma}\frac{d}{d\mu}(\widehat{l_{\tilde{\theta}}+\mu\tilde{l}_{\Delta,\theta,\tilde{\theta}}})\Big|_{\mu=0} \\ &= \rho^{CES}\left(\frac{\tilde{l}'_{\Delta,\theta,\tilde{\theta}}}{l_{\tilde{\theta}}}-\hat{l}_{\tilde{\theta}}\frac{\tilde{l}_{\Delta,\theta,\tilde{\theta}}}{l_{\tilde{\theta}}}\right).\end{aligned}$$

Using the last expression and the definition of $\tilde{l}_{\Delta,\theta}$, the term TE^*_θ becomes

$$\begin{aligned}TE^*_\theta = \lim_{\Delta\to 0}\frac{1}{\Delta^2}\int_{\theta-\Delta}^{\theta}&\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})w_{\tilde{\theta}}\rho^{CES}\left(1-\hat{l}_{\tilde{\theta}}(\tilde{\theta}-\theta+\Delta)\right)d\tilde{\theta} \\ &+\frac{1}{\Delta^2}\int_{\theta}^{\theta+\Delta}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})w_{\tilde{\theta}}\rho^{CES}\left(-1-\hat{l}_{\tilde{\theta}}(\theta-\tilde{\theta}+\Delta)\right)d\tilde{\theta}.\end{aligned}$$

Applying L'Hôspital's rule yields:

$$\begin{aligned}TE^*_\theta = \lim_{\Delta\to 0}&\frac{1}{2\Delta}\frac{1-T'(y_{\theta-\Delta})}{1-T'(y_\theta)}\frac{1-H_{\theta-\Delta}}{h_\theta w_\theta}(1-\tilde{g}_{\theta-\Delta})w_{\theta-\Delta}\rho^{CES} \\ &-\frac{1}{2\Delta}\int_{\theta-\Delta}^{\theta}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})w_{\tilde{\theta}}\rho^{CES}\hat{l}_{\tilde{\theta}}\,d\tilde{\theta} \\ &-\frac{1}{2\Delta}\frac{1-T'(y_{\theta+\Delta})}{1-T'(y_\theta)}\frac{1-H_{\theta+\Delta}}{h_\theta w_\theta}(1-\tilde{g}_{\theta+\Delta})w_{\theta+\Delta}\rho^{CES} \\ &-\frac{1}{2\Delta}\int_{\theta}^{\theta+\Delta}\frac{1-T'(y_{\tilde{\theta}})}{1-T'(y_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta w_\theta}(1-\tilde{g}_{\tilde{\theta}})w_{\tilde{\theta}}\rho^{CES}\hat{l}_{\tilde{\theta}}\,d\tilde{\theta}.\end{aligned}$$

Rearranging and replacing marginal retention rates by workers' first-order condition (2), we obtain:

$$\begin{aligned}TE^*_\theta = \lim_{\Delta\to 0}&\frac{1}{2\Delta}\rho^{CES}\left[\frac{v'(l_{\theta-\Delta}}{v'(l_\theta)}\frac{1-H_{\theta-\Delta}}{h_\theta}(1-\tilde{g}_{\theta-\Delta})-\frac{v'(l_{\theta+\Delta}}{v'(l_\theta)}\frac{1-H_{\theta+\Delta}}{h_\theta}(1-\tilde{g}_{\theta+\Delta})\right] \\ &-\frac{1}{2\Delta}\int_{\theta-\Delta}^{\theta}\frac{v'(l_{\tilde{\theta}}}{v'(l_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta}(1-\tilde{g}_{\tilde{\theta}})\rho^{CES}\hat{l}_{\tilde{\theta}}\,d\tilde{\theta} \\ &-\frac{1}{2\Delta}\int_{\theta}^{\theta+\Delta}\frac{v'(l_{\tilde{\theta}}}{v'(l_\theta)}\frac{1-H_{\tilde{\theta}}}{h_\theta}(1-\tilde{g}_{\tilde{\theta}})\rho^{CES}\hat{l}_{\tilde{\theta}}\,d\tilde{\theta}.\end{aligned}$$

Next, we apply L'Hôspital's rule a second time and obtain:

$$TE^*_\theta = -\frac{v''(l_\theta)}{v'(l_\theta)}\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)l'_\theta\rho^{CES} + (1-\tilde{g}_\theta)\rho^{CES} + \frac{1-H_\theta}{h_\theta}\tilde{g}'_\theta\rho^{CES} - \frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES} .$$

Using the definition of the elasticity e_θ yields:

$$TE^*_\theta = -\frac{1}{e_\theta}\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES} + (1-\tilde{g}_\theta)\rho^{CES} + \frac{1-H_\theta}{h_\theta}\tilde{g}'_\theta\rho^{CES} - \frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES} .$$

Finally, it is straightforward to show that

$$\tilde{g}'_\theta = (\tilde{g}_\theta - g_\theta)\frac{h_\theta}{1-H_\theta} .$$

Inserting this into the previous expression for TE^*_θ, we obtain:

$$TE^*_\theta = -\frac{1}{e_\theta}\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES} + (1-\tilde{g}_\theta)\rho^{CES} + (\tilde{g}_\theta - g_\theta)\rho^{CES} - \frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES}$$

and after rearranging:

$$TE^*_\theta = (1-g_\theta)\rho^{CES} - \left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{l}_\theta\rho^{CES} ,$$

which is the desired expression. □

Besides providing an important step in the derivation of equation (32), Lemma 9 allows to revisit the sign of TE^*_θ at the bottom and the top of the type distribution. In the general case, Proposition 4 shows that the induced technical change term is weakly positive at the top and weakly negative at the bottom. For the CES case, Lemma 9 implies[39]

$$TE^*_{\overline{\theta}} = (1-\tilde{g}_\theta)\rho^{CES} > 0$$

and

$$TE^*_{\underline{\theta}} = (1-g_{\underline{\theta}})\rho^{CES} < 0 .$$

Hence, in the CES case the sign restrictions on the induced technical change term hold strictly. Moreover, Lemma 9 implies the opposite signs for the substitution term at the top and bottom types:

$$SE^*_{\overline{\theta}} = (1-\tilde{g}_\theta)\gamma^{CES} > 0$$

and

$$SE^*_{\underline{\theta}} = (1-g_{\underline{\theta}})\gamma^{CES} < 0 .$$

Hence, at the highest and lowest income levels, induced technical change and within-technology substitution effects push optimal marginal tax rates in opposing directions. Adding the two

[39]This again assumes that l'_θ exists on some left (right) neighborhood of $\overline{\theta}$ ($\underline{\theta}$) and $\limsup_{\theta\to\overline{\theta}} l'_\theta < \infty$ ($\liminf_{\theta\to\underline{\theta}} l'_\theta > -\infty$) under the optimal tax, as in the second part of Proposition 4. Moreover, the strict inequalities below require that marginal welfare weights are strictly decreasing at the optimum over parts of type distribution.

up yields:

$$SE^*_{\bar{\theta}} + TE^*_{\bar{\theta}} = (1 - \tilde{g}_{\bar{\theta}})(\gamma^{CES} + \rho^{CES})$$

and

$$SE^*_{\underline{\theta}} + TE^*_{\underline{\theta}} = (1 - g_{\underline{\theta}})(\gamma^{CES} + \rho^{CES}) \ .$$

The total effect of endogenous wage responses therefore depends on whether condition (19) for strong relative bias from Section 4 is satisfied: if and only if $\gamma^{CES} + \rho^{CES} \geq 0$, the adjustment of optimal marginal tax rates for endogeneity of the wage distribution is positive at the top and negative at the bottom.

Since the partial equilibrium term PE^*_θ is zero at the top and bottom types, these insights directly translate into conclusions about the sign of the optimal marginal tax rate at the top and the bottom of the income distribution. The optimal marginal tax is positive at the top if and only if there is strong relative bias of technology, providing a counterpart to the negative marginal top tax result in Stiglitz (1987). Analogously, the optimal marginal tax is negative at the bottom if and only if there is strong bias.

With the expressions from Lemma 9, we are now in a position to derive equation (32).

Proof of Proposition 5. We start with equation (73) from the proof of Proposition 4 and replace SE^*_θ and TE^*_θ by the expressions from Lemma 9. This yields:

$$\frac{T'(y_\theta)}{1 - T'(y_\theta)} = \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta)\hat{w}_\theta \tag{78}$$

$$+ (\gamma^{CES} + \rho^{CES}) \left[(1 - g_\theta) - \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta)\hat{l}_\theta\right]$$

$$= \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta) \left(\hat{w}_\theta - (\gamma^{CES} + \rho^{CES})\hat{l}_\theta\right) \tag{79}$$

$$+ \gamma^{CES}(1 - g_\theta) + \rho^{CES}(1 - g_\theta) \ . \tag{80}$$

The wage growth rate $\hat{w}_\theta$ can be computed from equations (76) and (77) as

$$\hat{w}_\theta = (1 + \gamma^{CES} + \rho^{CES})\hat{\kappa}_\theta + (\gamma^{CES} + \rho^{CES})\hat{h}_\theta + (\gamma^{CES} + \rho^{CES})\hat{l}_\theta \ .$$

Using this in the previous expression for marginal tax rates yields

$$\frac{T'(y_\theta)}{1 - T'(y_\theta)} = \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta) \left[(1 + \gamma^{CES} + \rho^{CES})\hat{\kappa}_\theta + (\gamma^{CES} + \rho^{CES})\hat{h}_\theta\right]$$
$$+ \gamma^{CES}(1 - g_\theta) + \rho^{CES}(1 - g_\theta) \ .$$

Now we use the definition of β,

$$\beta_\theta := \kappa_\theta^{1+\gamma^{CES}+\rho^{CES}} h_\theta^{\gamma^{CES}+\rho^{CES}} ,$$

to note that

$$(1 + \gamma^{CES} + \rho^{CES})\hat{\kappa}_\theta + (\gamma^{CES} + \rho^{CES})\hat{h}_\theta = \hat{\beta}_\theta$$

and hence

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{\beta}_\theta + \gamma^{CES}(1-g_\theta) + \rho^{CES}(1-g_\theta)\,.$$

Finally, with the change-of-variable $h_\theta = b_{\beta_\theta}\beta'_\theta$, we obtain equation (32):

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta}(1-\tilde{g}_\theta) + \gamma^{CES}(1-g_\theta) + \rho^{CES}(1-g_\theta)\,.$$

□

Finally, I provide an alternative expression for optimal marginal tax rates in the CES case, which allows for an interpretation of directed technical change effects via their effect on the aggregate labor supply elasticity $\bar{\epsilon}^R_\theta$ (see Appendix C.4).

Proposition 7. *Suppose the conditions of Proposition 4 are satisfied and F and Φ take the CES form introduced in Section 3.4. Additionally, let* $(\gamma^{CES}+\rho^{CES})\epsilon^w_\theta < 1$ *at the optimal tax T. Then, the conditions for optimal marginal tax rates can be written as*

$$\frac{T'(y)}{1-T'(y)} = \frac{1}{\bar{\epsilon}^R_{\theta_y}}\frac{1-M_y}{m_y y}(1-\tilde{g}_{\theta_y}) + \gamma^{CES}(1-g_{\theta_y}) + \rho^{CES}(1-g_{\theta_y})\,, \tag{81}$$

where

$$\bar{\epsilon}^R_{\theta_y}(T,l,w) := \frac{\epsilon^R_{\theta_y}(T,l,w)}{1-(\gamma_{\theta_y,\theta_y}+\rho_{\theta_y,\theta_y})\epsilon^w_{\theta_y}(T,l,w)}$$

denotes the elasticity of aggregate labor supply of type θ with respect to the marginal retention rate (see Appendix C.4); all variables are evaluated at equilibrium under the optimal tax T; M and m denote the cumulative distribution and the density function of y at the optimum; and θ_y *denotes the type of workers who earn income y at the optimum.*

Proof. To derive equation (81), we start from equation (80) and replace $\hat{l}_\theta$ by $\epsilon^w_\theta\hat{w}_\theta$:

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{w}_\theta\left(1-(\gamma^{CES}+\rho^{CES})\epsilon^w_\theta\right) + \gamma^{CES}(1-g_\theta) + \rho^{CES}(1-g_\theta)\,.$$

Using m_y and M_y to denote density and cumulative distribution function of income, a change-of-variable implies $h_\theta = m_{y_\theta}y' - \theta$. Using this in the previous expression for marginal tax rates, we obtain:

$$\begin{aligned}\frac{T'(y_\theta)}{1-T'(y_\theta)} &= \left(1+\frac{1}{e_\theta}\right)\frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}\frac{1}{\hat{w}_\theta+\hat{l}_\theta}(1-\tilde{g}_\theta)\hat{w}_\theta\left(1-(\gamma^{CES}+\rho^{CES})\epsilon_\theta^w\right)\\ &\quad+\gamma^{CES}(1-g_\theta)+\rho^{CES}(1-g_\theta)\\ &= \left(1+\frac{1}{e_\theta}\right)\frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}\frac{1}{\hat{w}_\theta(1+\epsilon_\theta^w)}(1-\tilde{g}_\theta)\hat{w}_\theta\left(1-(\gamma^{CES}+\rho^{CES})\epsilon_\theta^w\right)\\ &\quad+\gamma^{CES}(1-g_\theta)+\rho^{CES}(1-g_\theta)\\ &= \left(1+\frac{1}{e_\theta}\right)\frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}\frac{1}{1+\epsilon_\theta^w}(1-\tilde{g}_\theta)\left(1-(\gamma^{CES}+\rho^{CES})\epsilon_\theta^w\right)\\ &\quad+\gamma^{CES}(1-g_\theta)+\rho^{CES}(1-g_\theta)\,.\end{aligned}$$

From the expression for ϵ_θ^w and ϵ_θ^R in Section 3.2, it is straightforward to show that

$$\left(1+\frac{1}{e_\theta}\right)\frac{1}{1+\epsilon_\theta^w}=\frac{1}{\epsilon_\theta^R}\,,$$

and hence

$$\frac{T'(y_\theta)}{1-T'(y_\theta)}=\frac{1-(\gamma^{CES}+\rho^{CES})\epsilon_\theta^w}{\epsilon_\theta^R}\frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}(1-\tilde{g}_\theta)+\gamma^{CES}(1-g_\theta)+\rho^{CES}(1-g_\theta)\,.$$

By definition of the aggregate elasticity $\bar{\epsilon}_\theta^R$ (see Appendix C.4), this yields equation (81) from Proposition 5:

$$\frac{T'(y_\theta)}{1-T'(y_\theta)}=\frac{1}{\bar{\epsilon}_\theta^R}\frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}(1-\tilde{g}_\theta)+\gamma^{CES}(1-g_\theta)+\rho^{CES}(1-g_\theta)\,.$$

□

Equation (81) offers an alternative perspective on the role of directed technical change in the CES case. It extends the expression in Proposition 3 of Sachs et al. (2019) to account for directed technical change.

The adjustment term $\rho^{CES}(1-g_{\theta_y})$ is the same as in equation (32) and does not provide any new insights. As described in the main text, it calls for a progressive adjustment of marginal tax rates.

The second place in equation (81) where induced technical change elasticities appear is the aggregate labor supply elasticity $\bar{\epsilon}_\theta^R$. This elasticity measures the response of labor supply of type θ to an increase in the marginal retention rate when taking into account the change in type θ's wage induced by the labor supply response. The change in type θ's wage is modified by directed technical change effects. Specifically, when labor supply of type θ falls due to an increase in the marginal tax rate, directed technical change amplifies the fall in labor supply by reducing the wage of type θ. In this way, directed technical change effects magnify the labor supply response to marginal tax changes, which leads to a downwards adjustment of optimal marginal tax rates.

D.3. Exogenous Technology Planner

As described in the main text, the exogenous technology planner believes that the economy works according to all reduced form equations from Section 3.1 with the exception of the condition for the equilibrium technology (7). Instead of following equation (7), the exogenous technology planner believes that technology remains fixed at its equilibrium value under a given tax $\overline{T}$, $\phi^*(l(\overline{T}))$. The idea is that the planner observes the economy under the tax $\overline{T}$ when computing optimal taxes and believes technology to be exogenous.
The exogenous technology planner's optimal tax $T^{ex}_{\overline{T}}$ then satisfies the conditions provided by the following Lemma.

Lemma 10. *Suppose the conditions of Proposition 4 are satisfied and F and Φ take the CES form introduced in Section 3.4. Suppose equilibrium variables are determined according to the reduced form equations (2), (1), (6), and (8), plus the (exogenous) technology equation*

$$\phi^*(l) = \phi^*(l(\overline{T})) = \operatorname*{argmax}_{\phi \in \Phi} F(l(\overline{T}), \phi) \quad \forall l \,,$$

where $\overline{T}$ is a given initial tax function.
Then, at almost every type θ, the exogenous technology planner's preferred tax $T^{ex}_{\overline{T}}$ satisfies the following conditions.

$$\frac{T^{ex\prime}_{\overline{T}}(y_\theta)}{1 - T^{ex\prime}_{\overline{T}}(y_\theta)} = \left(1 + \frac{1}{e_\theta}\right) \frac{1 - \overline{B}_{\overline{\beta}_\theta}}{\overline{b}_{\overline{\beta}_\theta} \overline{\beta}_\theta} (1 - \tilde{g}_\theta) + \gamma^{CES} (1 - g_\theta) \,,$$

where all variables satisfy the equations listed above under the tax $T^{ex}_{\overline{T}}$; the function $\overline{\beta} : \theta \mapsto \overline{\beta}_\theta$ is given by

$$\overline{\beta}_\theta := \kappa_\theta^{1+\gamma^{CES}} h_\theta^{\gamma^{CES}} \left(\phi^*(\overline{T})\right)^{1+\gamma^{CES}} \qquad \forall \theta \,;$$

and $\overline{B}$ and $\overline{b}$ are the cumulative distribution and the density function of $\overline{\beta}$.

Proof. It can be verified that all steps in the proof of Proposition 4 hold for the case of the exogenous technology planner when imposing

$$\left. \frac{d\hat{w}_{\tilde{\theta}}(l, \phi^*(l + \mu \tilde{l}_{\Delta,\theta}))}{d\mu} \right|_{\mu=0} = 0 \,.$$

With this constraint, we can derive a counterpart to equation (73) for the exogenous technology planner:

$$\frac{T'(y_\theta)}{1 - T'(y_\theta)} = \left(1 + \frac{1}{e_\theta}\right) (1 - \tilde{g}_\theta) \frac{1 - H_\theta}{h_\theta} \hat{w}_\theta - SE^*_\theta \,.$$

Using Lemma 9 to replace SE^*_θ, we obtain:

$$\begin{aligned} \frac{T'(y_\theta)}{1 - T'(y_\theta)} &= \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta) \hat{w}_\theta \\ &\quad + \gamma^{CES} \left[(1 - g_\theta) - \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta) \hat{l}_\theta \right] \\ &= \left(1 + \frac{1}{e_\theta}\right) \frac{1 - H_\theta}{h_\theta} (1 - \tilde{g}_\theta) \left(\hat{w}_\theta - \gamma^{CES} \hat{l}_\theta \right) + \gamma^{CES} (1 - g_\theta) \,. \end{aligned}$$

From the perspective of the exogenous technology planner, the wage growth rate is now given by

$$\hat{w}_\theta = (1+\gamma^{CES})\hat{\kappa}_\theta + (1+\gamma^{CES})\hat{\phi}^*_\theta(\overline{T}) + \gamma^{CES}\hat{h}_\theta + \gamma^{CES}\hat{l}_\theta \ ,$$

where $\hat{\phi}^*_\theta(\overline{T})$ denotes the growth rate of technology that prevails in equilibrium under the initial tax system $\overline{T}$. Using this in the previous expression for the exogenous technology planner's optimal tax rates and applying the definition of $\overline{\beta}$ yields:

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-H_\theta}{h_\theta}(1-\tilde{g}_\theta)\hat{\overline{\beta}}_\theta + \gamma^{CES}(1-g_\theta) \ .$$

With the change of variable $h_\theta = \overline{b}_{\overline{\beta}_\theta}\overline{\beta}'_\theta$, we obtain equation (33),

$$\frac{T'(y_\theta)}{1-T'(y_\theta)} = \left(1+\frac{1}{e_\theta}\right)\frac{1-\overline{B}_{\overline{\beta}_\theta}}{\overline{b}_{\overline{\beta}_\theta}\overline{\beta}_\theta}(1-\tilde{g}_\theta) + \gamma^{CES}(1-g_\theta) \ ,$$

which completes the proof. □

D.4. Comparison between Endogenous and Exogenous Technology Planner

As discussed in the main text, the optimal tax conditions of the endogenous and the exogenous technology planner feature two differences. First, the exogenous technology planner neglects the progressive term $\rho^{CES}(1-g_\theta)$. Second, he uses $\overline{\beta}$ instead of β to measure the degree of exogenous inequality in the economy. Here I show that the exogenous technology planner thereby overestimates exogenous inequality: the function $\overline{\beta}$ progresses at a higher rate in θ than the function β, such that equation (34) holds.

First, let $\overline{l}$ be the equilibrium labor input under the initial tax system $\overline{T}$. Then, by construction the growth rates of β and $\overline{\beta}$ must satisfy

$$\hat{\beta}_\theta = \hat{w}_\theta(\overline{l}, \phi^*(\overline{l})) - (\gamma^{CES}+\rho^{CES})\hat{\overline{l}}_\theta$$

and

$$\hat{\overline{\beta}}_\theta = \hat{w}_\theta(\overline{l}, \phi^*(\overline{l})) - \gamma^{CES}\hat{\overline{l}}_\theta \ .$$

Moreover, if we assume (as is done in the main text) that the rate of progressivity of the initial tax schedule $\overline{T}$ is strictly below one everywhere, then equation (10) implies that $\epsilon^w_\theta > 0$ and hence $\hat{\overline{l}}_\theta = \epsilon^w_\theta \hat{w}_\theta(\overline{l}, \phi^*(\overline{l})) > 0$ for all θ. Combining this with the expressions for $\hat{\beta}_\theta$ and $\hat{\overline{\beta}}_\theta$, we find that $\hat{\beta}_\theta < \hat{\overline{\beta}}_\theta$. Intuitively, since labor supply increases in the skill index under tax $\overline{T}$, technology under this tax must be skill-biased. The exogenous technology planner falsely believes this skill bias to be exogenous and to persist irrespective of changes in labor supply. Thereby, he overestimates the degree of exogenous inequality in the economy.

Now use a change-of-variable to obtain

$$b_{\beta_\theta}\beta_\theta = \frac{f_\theta}{\hat{\beta}_\theta} > \frac{f_\theta}{\hat{\overline{\beta}}_\theta} = \overline{b}_{\overline{\beta}_\theta}\overline{\beta}_\theta \ ,$$

and hence

$$\frac{1-B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta} < \frac{1-\overline{B}_{\overline{\beta}_\theta}}{\overline{b}_{\overline{\beta}_\theta}\overline{\beta}_\theta} ,$$

which proves equation (34) in the main text.

Comparison in the Lower Tail

Proof of Corollary 3. We first show that the two inequalities of Corollary 3 hold at the lowest type $\underline{\theta}$. Since $\tilde{g}_{\underline{\theta}} = 1$, equation (32) from Proposition 5 implies

$$\frac{T'(y_{\underline{\theta}}(T))}{1-T'(y_{\underline{\theta}}(T))} = \gamma^{CES}(1-g_{\underline{\theta}}) + \rho^{CES}(1-g_{\underline{\theta}}) ,$$

while equation (33) from Lemma 10 implies

$$\frac{T^{ex\prime}_{\overline{T}}(y_{\underline{\theta}}(T^{ex}_{\overline{T}}))}{1-T^{ex\prime}_{\overline{T}}(y_{\underline{\theta}}(T^{ex}_{\overline{T}}))} = \gamma^{CES}(1-g_{\underline{\theta}}) .$$

With $\rho^{CES} > 0$ and $g_{\underline{\theta}} > 1$ (because welfare weights are strictly decreasing by hypothesis) the inequality

$$T'(y_{\underline{\theta}}(T)) < T^{ex\prime}_{\overline{T}}(y_{\underline{\theta}}(T^{ex}_{\overline{T}}))$$

follows.

Moreover, if $\gamma^{CES} + \rho^{CES} > 0$, the previous expression for optimal marginal tax rates with endogenous technology effects produces the second inequality

$$T'(y_{\underline{\theta}}(T)) < 0 .$$

Finally, under the conditions of the corollary, the expressions for optimal marginal tax rates in equations (32) and (33) are continuous. Thus, the previous inequalities extend to some right neighborhoods of $\underline{\theta}$, which completes the proof. □

Comparison in the Upper Tail

Proof of Corollary 4. As described in the main text, we trace the Pareto property of the (upper tail of the) income distribution under tax $\overline{T}$ back to the distribution of the exogenous inequality measure β. Inserting this distribution into the optimal tax formula (32) from Proposition 5 then yields the desired result.

First, by two changes-of-variable we obtain

$$\frac{1-B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta} = \frac{1-H_\theta}{h_\theta}\hat{\beta}_\theta = \frac{1-M_{y_\theta}}{m_{y_\theta}y_\theta}\frac{\hat{\beta}_\theta}{\hat{y}_\theta} , \tag{82}$$

where all incomes are assessed at the given tax $\overline{T}$. Now we use

$$\hat{y}_\theta = \hat{w}_\theta + \hat{l}_\theta = (1+\epsilon^w_\theta)\hat{w}_\theta$$

to express the growth rate of income as a function of the growth rate of wages under tax, again assessing all endogenous variables at equilibrium under the given tax $\overline{T}$. For $\hat{\beta}_\theta$ we obtain

$$\hat{\beta}_\theta = \hat{w}_\theta - (\gamma^{CES} + \rho^{CES})\hat{l}_\theta = \left(1 - (\gamma^{CES} + \rho^{CES})\epsilon^w_\theta\right)\hat{w}_\theta \; .$$

It follows that

$$\frac{\hat{\beta}_\theta}{\hat{y}_\theta} = \frac{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w_\theta}{1 + \epsilon^w_\theta}$$

and, with equation (82),

$$\frac{1 - B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta} = \frac{1 - M_{y_\theta}}{m_{y_\theta} y_\theta} \frac{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w_\theta}{1 + \epsilon^w_\theta} \; , \tag{83}$$

where incomes and the labor supply elasticity ϵ^w_θ are assessed under the tax $\overline{T}$. In particular,

$$\epsilon^w_\theta = \frac{(1 - P_{\overline{T}}(y_\theta))e_\theta}{1 + e_\theta P_{\overline{T}}(y_\theta)} \; .$$

Since the tax $\overline{T}$ features a constant top tax rate, we have $\lim_{\theta \to \overline{\theta}} P_{\overline{T}}(y_\theta) = 0$ and hence

$$\lim_{\theta \to \overline{\theta}} \epsilon^w_\theta = e_{\overline{\theta}} = e \; ,$$

where the last equality reflects the assumption that the disutility of labor is iso elastic. Moreover, we know by hypothesis that

$$\lim_{\theta \to \overline{\theta}} \frac{1 - M_{y_\theta}}{m_{y_\theta} y_\theta} = \frac{1}{a} \; .$$

Combining these limits, we obtain

$$\lim_{\theta \to \overline{\theta}} \frac{1 - B_{\beta_\theta}}{b_{\beta_\theta}\beta_\theta} = \lim_{\theta \to \overline{\theta}} \frac{1 - M_{y_\theta}}{m_{y_\theta} y_\theta} \frac{1 - (\gamma^{CES} + \rho^{CES})\epsilon^w_\theta}{1 + \epsilon^w_\theta} = \frac{1 - (\gamma^{CES} + \rho^{CES})e}{a(1 + e)} \; .$$

In words, from the observed Pareto tail of the income distribution under tax $\overline{T}$ we can infer that the exogenous inequality measure β must also have a Pareto tail with tail parameter given by the previous equation. Using this parameter in the optimal tax equation (32) from Proposition 5, we obtain the following expression for the optimal marginal tax rate in the upper tail of the income distribution:

$$\begin{aligned} \lim_{\theta \to \overline{\theta}} \frac{T'(y_\theta)}{1 - T'(y_\theta)} &= \left(1 + \frac{1}{e}\right) \frac{1 - (\gamma^{CES} + \rho^{CES})e}{a(1 + e)}(1 - g^{top}) + \gamma^{CES}(1 - g^{top}) + \rho^{CES}(1 - g^{top}) \\ &= \frac{1 - g^{top}}{ae} + \frac{a - 1}{a}\gamma^{CES}(1 - g^{top}) + \frac{a - 1}{a}\rho^{CES}(1 - g^{top}) \; , \end{aligned}$$

where g^{top} is the asymptotic welfare weight defined in the Corollary. □

To precisely identify the role of endogenously directed technical change, I compare the optimal marginal tax in Corollary 4 to the one computed by the exogenous technology planner. The exogenous technology planner again observes the economy under an initial tax $\overline{T}$ and believes that technology ϕ^* remains fixed at its initial level $\phi^*(\overline{T})$, irrespectively of the tax system. Now, in analogy to the conditions of Corollary 4, the initial tax $\overline{T}$ has a constant marginal top tax rate and the upper tail of the income distribution has a Pareto shape under the initial tax.

Corollary 6. *Suppose the conditions of Proposition 4 are satisfied and F and Φ take the CES form introduced in Section 3.4. Suppose equilibrium variables were determined by the reduced form equations (2), (1), (6), and (8), and by the (exogenous) technology equation*

$$\phi^*(l) = \phi^*(l(\overline{T})) = \operatorname*{argmax}_{\phi\in\Phi} F(l(\overline{T}),\phi) \quad \forall l \,,$$

where $\overline{T}$ is a given initial tax function with $\overline{T}'(y) = \tau^{top}$ for all $y \geq \tilde{y}$ and some threshold $\tilde{y}$. Moreover, assume that under the tax $\overline{T}$ the income distribution satisfies

$$\lim_{\theta\to\bar{\theta}} \frac{1-M_{y_\theta}}{m_{y_\theta} y_\theta} = \frac{1}{a}$$

for some $a > 1$. Finally, let the disutility of labor be iso elastic with $e_\theta = e$ for all θ, and welfare weights satisfy

$$\lim_{\theta\to\bar{\theta}} g_\theta = g^{top}$$

at the exogenous technology planner's preferred tax.
Then, the exogenous technology planner's preferred tax $T^{ex}_{\overline{T}}$ satisfies

$$\lim_{\theta\to\bar{\theta}} \frac{T^{ex\prime}_{\overline{T}}(y_\theta)}{1-T^{ex\prime}_{\overline{T}}(y_\theta)} = \frac{1-g^{top}}{ae} + \frac{a-1}{a}\gamma^{CES}(1-g^{top}) \,.$$

Proof. The corollary can be derived from Lemma 10 in the same way as Corollary 4 is derived from Proposition 5. In particular, consider first the implications of the Pareto shape of the income distribution under tax $\overline{T}$ for the exogenous inequality measure $\overline{\beta}$. Using changes-of-variable, we obtain

$$\frac{1-B_{\overline{\beta}_\theta}}{b_{\overline{\beta}_\theta}\overline{\beta}_\theta} = \frac{1-H_\theta}{h_\theta}\hat{\overline{\beta}}_\theta = \frac{1-M_{y_\theta}}{m_{y_\theta} y_\theta}\frac{\hat{\beta}_\theta}{\hat{y}_\theta} \,, \tag{84}$$

where all incomes are assessed at the given tax $\overline{T}$. The exogenous technology planner's measure of exogenous inequality $\overline{\beta}$ now evolves over the type space according to

$$\hat{\overline{\beta}}_\theta = \hat{w}_\theta - \gamma^{CES}\hat{l}_\theta = \left(1-\gamma^{CES}\epsilon^w_\theta\right)\hat{w}_\theta \,,$$

while

$$\hat{y}_\theta = \hat{w}_\theta + \hat{l}_\theta = (1+\epsilon^w_\theta)\hat{w}_\theta \,,$$

where all endogenous variables are assessed at equilibrium under the given tax $\overline{T}$.[40] Combin-

[40]Note that under the initial tax $\overline{T}$ the exogenous and the endogenous planner agree about the equilibrium and in particular about the equilibrium technology. Hence, there is no need to distinguish the equilibrium values of the endogenous variables under tax $\overline{T}$ as perceived by the exogenous technology planner and their true equilibrium values.

ing the previous expressions, we find that

$$\frac{\hat{\overline{\beta}}_\theta}{\hat{y}_\theta} = \frac{1 - \gamma^{CES} \epsilon_\theta^w}{1 + \epsilon_\theta^w}$$

and, with equation (84),

$$\frac{1 - \overline{B}_{\overline{\beta}_\theta}}{\overline{b}_{\overline{\beta}_\theta} \overline{\beta}_\theta} = \frac{1 - M_{y_\theta}}{m_{y_\theta} y_\theta} \frac{1 - \gamma^{CES} \epsilon_\theta^w}{1 + \epsilon_\theta^w} , \tag{85}$$

where incomes and the labor supply elasticity ϵ_θ^w are assessed under the tax $\overline{T}$. Inserting this expression into equation (33) for optimal marginal tax rates computed by the exogenous technology planner and taking limits yields[41]

$$\begin{aligned} \lim_{\theta \to \overline{\theta}} \frac{T_{\overline{T}}^{ex\prime}(y_\theta)}{1 - T_{\overline{T}}^{ex\prime}(y_\theta)} &= \left(1 + \frac{1}{e}\right) \frac{1 - \gamma^{CES} e}{a(1+e)} (1 - g^{top}) + \gamma^{CES} (1 - g^{top}) \\ &= \frac{1 - g^{top}}{ae} + \frac{a-1}{a} \gamma^{CES} (1 - g^{top}) \, . \end{aligned}$$

□

E. Additional Explanations and Results for the Quantification

In the following I provide details on the calibration described in Section 7.1 of the main text. For the calibration I use the CES version of the model and assume iso-elastic disutility of labor (see Section 3.4).

E.1. Calibration of Directed Technical Change Effects

The parameter ρ^{CES}, which controls the strength of directed technical change effects, is calibrated on the basis of the empirical estimates summarized in Table 1. The long-run estimates in Table 1 (10 years or more) are equated with the sum of within-technology substitution and directed technical change elasticities $\gamma^{CES} + \rho^{CES}$. The short-run estimate (2 years, from Carneiro et al. (2019)) is equated with γ^{CES}, in line with other short-run estimates as discussed in the main text.

Here, I give a brief overview over each of the studies used in Table 1 and explain how I obtain the estimates in the last column of Table 1 from the respective studies.

Carneiro et al. (2019) Carneiro et al. (2019) estimate the responses of relative supply and relative wages of college versus non-college workers to plausibly exogenous college openings in Norwegian municipalities in the 1970s, using synthetic control methods. They find that relative supply in a municipality starts rising shortly after the college opening and follows an upwards trend throughout the observation period of up to 17 years, compared to the synthetic control municipality. The relative wage first declines and then reverses its trend, surpassing

[41] The limit computations are analogous to those in the proof of Corollary 4, so I omit the details here.

the relative wage in the control municipality slightly more than 10 years after the college opening (see Figures 4 and 5 in Carneiro et al. 2019).

The numbers in Table 1 are derived from the estimates presented in Carneiro et al. (2019) as follows. First, measuring relative supply and relative wage changes two years after a college opening, Carneiro et al. (2019) estimate an elasticity of the relative wage with respect to relative supply of −0.549, reported in column 1 of their Table 2. This produces the first row of Table 1 in the present paper.

Second, Figures 4 and 5 imply that after 10 years, relative wages in the treated municipalities and their synthetic controls were equal. Hence, when measured after 10 years, there is a zero effect of the exogenous relative supply increase in the relative wage, leading to the second row of Table 1 in the present paper.

Finally, the third row of Table 1 is obtained from the plots presented in Figure 4 in Carneiro et al. (2019) as follows. The figure shows that after 17 years, the log change in the relative wage, compared to the synthetic control municipality, is

$$\log\left(\frac{w_c^{17}}{w_{nc}^{17}}\right) - \log\left(\frac{w_c^0}{w_{nc}^0}\right) \approx 0.02 \ .$$

At the same time, the log change in the share of college workers in the total workforce was

$$\log\left(\frac{l_c^{17}}{l_c^{17}+l_{nc}^{17}}\right) - \log\left(\frac{l_c^0}{l_c^0+l_{nc}^0}\right) \approx 0.04 \ .$$

To map this change into the change in the ratio of college over non-college workers, I rewrite the log change as follows:

$$\begin{aligned}\log\left(\frac{l_c^{17}}{l_c^{17}+l_{nc}^{17}}\right) - \log\left(\frac{l_c^0}{l_c^0+l_{nc}^0}\right) &= \log(l_c^{17}) - \log(l_{nc}^{17}) - \log(l_c^0) + \log(l_{nc}^0) \\ &\quad - \log(l_c^{17}+l_{nc}^{17}) + \log(l_{nc}^{17}) + \log(l_c^0+l_{nc}^0) - \log(l_{nc}^0) \ .\end{aligned}$$

Carneiro et al. (2019) report that the share of college workers was close to zero in most of the treated municipalities at the beginning of the observation period and still small at the end of the period. Hence, I apply the approximations

$$\log(l_c^{17}+l_{nc}^{17}) - \log(l_{nc}^{17}) \approx \log(l_c^0+l_{nc}^0) - \log(l_{nc}^0) \approx 0$$

to obtain

$$\log\left(\frac{l_c^{17}}{l_c^{17}+l_{nc}^{17}}\right) - \log\left(\frac{l_c^0}{l_c^0+l_{nc}^0}\right) \approx \log(l_c^{17}) - \log(l_{nc}^{17}) - \log(l_c^0) + \log(l_{nc}^0) \approx 0.04 \ .$$

Finally, relating the change in relative supply to the change in relative wages, I obtain

$$\frac{\log\left(\frac{w_c^{17}}{w_{nc}^{17}}\right) - \log\left(\frac{w_c^0}{w_{nc}^0}\right)}{\log\left(\frac{l_c^{17}}{l_{nc}^{17}}\right) - \log\left(\frac{l_c^0}{l_{nc}^0}\right)} \approx 0.05 \ ,$$

which is the estimate used in Table 1.

Note that relative supply did not change instantaneously at the beginning of the observation period but steadily increased throughout (Figure 4 in Carneiro et al. 2019). Hence, part of the relative supply increase occurred only shortly before the relative wage increase is measured at year 17 of the observation period. To the extent that technology adjustments to the more recent part of the rise in relative supply are not yet reflected in the measured increase in relative wages, the above procedure underestimates the actual long-run effect of an exogenous relative supply increase on relative wages.

Lewis (2011) Lewis (2011) uses plausibly exogenous variation in immigrant inflows across US metropolitan areas in the 1980s and 1990s to estimate the relationship between the relative supply of high-school graduates versus high-school dropouts on their relative wages. He studies changes over 10 year intervals, thereby capturing a rather long-run elasticity. He also provides evidence showing that firms' decisions to adopt a range of automation technologies in the manufacturing sector respond to the (exogenous component) of changes in relative supply in the way predicted by theory. This supports the view that the estimated long-run wage elasticity captures directed technical change effects.

In column 1 of Table VIII, Lewis (2011) reports a wage elasticity estimate of -0.136. This is the estimate I use in Table 1.

Dustmann and Glitz (2015) Dustmann and Glitz (2015) exploit the arguably exogenous component of immigration inflows to German regions between 1985 and 1995 to analyze how regions absorb changes in relative skill supply. They decompose the change in relative employment levels between skill groups into a component due to between-firm scale adjustments and within-firm factor intensity adjustments. The latter turns out vastly more important, suggesting that Rybcinsky type output mix adjustments are small. Moreover, they find that relative wages hardly respond to relative supply changes. This leaves technology adjustments biased towards the skill group that becomes more abundant as the main margin of adjustment.

The authors distinguish between workers without postsecondary education (low-skilled), with postsecondary vocational or apprenticeship degrees (medium-skilled), and with college education (high-skilled). Due to extensive right-censoring of wages in the data, they consider their results for college workers less reliable and focus mainly on medium- and low-skilled workers.

For the relative wage of medium- versus low-skilled workers, Dustmann and Glitz (2015) estimate an elasticity with respect to relative supply of -0.091 (row 2, column 4, Table 2) over a period of ten years. This estimate uses data for the tradable goods sector (which includes, but is not limited to, the manufacturing sector). For the non-tradable sector, the authors find a much smaller wage elasticity. Yet, when they pool all industries, results are close to those for the tradable goods sector again (see description on page 727, Dustmann and Glitz 2015). Hence, I use the estimate for the tradable sector in Table 1.

Morrow and Trefler (2017) Morrow and Trefler (2017) start from a detailed neoclassical model of international trade building on Eaton and Kortum (2002). They estimate their model on sectoral factor input and price data for a cross-section of 38 countries in 2006. Country selection is driven by data availability in the World Input Output Database. Labor is partitioned into skilled and unskilled labor. Skilled workers are those with at least some tertiary education, unskilled workers are those without.

In the model, the relative wage between skilled and unskilled workers in each country is determined by the relative labor input and exogenous factor-augmenting productivity. To separately identify factor-augmenting productivity and the elasticity of substitution between labor types, Morrow and Trefler (2017) augment their model's equilibrium conditions by a directed technical change equation similar to equation (41). Unfortunately, their approach requires to fix the technology substitution parameter δ exogenously. In their directed technical change equation, they (implicitly) assume $\delta = 1$. Given a value for δ, the directed technical change equation and the equation for relative wages at given technology identify the elasticity of substitution σ.

In their most elaborate estimation, Morrow and Trefler (2017) find a value for σ of 1.89, which translates into a wage elasticity at exogenous technology (or, short-run wage elasticity) of $-1/1.89 = 0.53$. This is the first value from Morrow and Trefler (2017) I use in Table 1. Combining relative wage and directed technical change equations, the total wage elasticity, including directed technical change effects, is then obtained as $\sigma - 2 = -0.11$, the second estimate from Morrow and Trefler (2017) reported in Table 1.

Relative to the other studies listed in Table 1, a major shortcoming of Morrow and Trefler (2017) is that they do not have a strategy to isolate exogenous variation in factor inputs when estimating their directed technical change equation. Hence, part of the estimated relationship between technology and factor inputs may be driven by reverse causality, which leads to an overestimate of directed technical change effects. On the other hand, the fact that they estimate their model on cross-sectional data may imply underestimation of directed technical change effects, because the observed technology levels may not yet have fully adjusted to the most recent factor input changes.

E.2. Calibration of the Exogenous Technology Parameter

To calibrate the exogenous technology parameter κ, an estimate of the income distribution under the initial tax system $\overline{T}$ is needed. As explained in the main text, the initial tax system is set to approximate the US income tax in 2005. Hence, the income distribution under $\overline{T}$ should approximate the empirical income distribution of the US in 2005.

As is standard in the literature (e.g. Mankiw, Weinzierl and Yagan, 2009; Diamond and Saez, 2011), I approximate the empirical income distribution by merging a lognormal distribution (for the bulk of incomes) and a Pareto distribution (for the upper tail). Since the income distribution enters most of the formulae used in the simulations via its hazard ratio $ym_y/(1 - M_y)$, I directly target the empirical hazard ratio in 2005. In particular, I construct the hazard

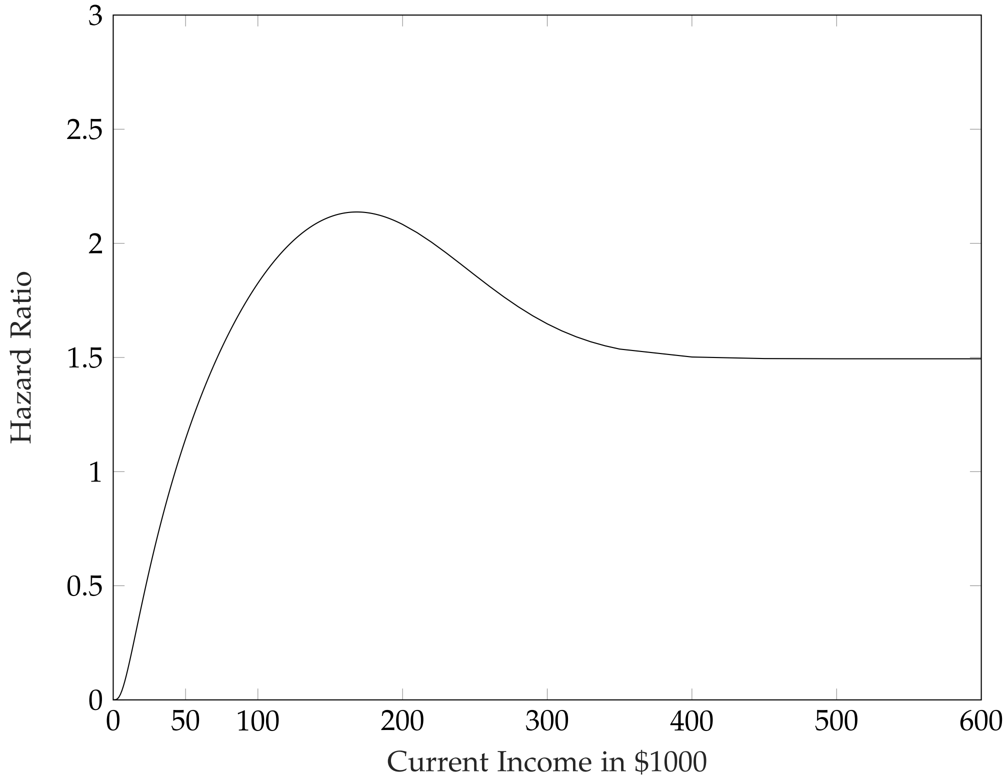

Figure 4. The figure shows the hazard ratio of the income distribution under the initial tax $\overline{T}$ used to calibrate the exogenous technology parameter κ. The construction of the hazard ratio follows the description in the text. The hazard ratio approximates the empirical hazard ratio of the US income distribution in 2005, as depicted, for example, in Figure 2 in Diamond and Saez (2011).

ratio as

$$\frac{ym_y}{1-M_y} = \frac{ym_y^{lognormal}}{1-M_y^{lognormal}}\left(1-\Phi\left(\frac{y-200000}{\sigma^{normal}}\right)\right) + \frac{ym_y^{Pareto}}{1-M_y^{Pareto}}\Phi\left(\frac{y-200000}{\sigma}\right),$$

where the normal distribution used for smoothing has mean 200000, reflecting the region in the income distribution where the transition from lognormal to Pareto occurs. I then choose the parameters of the lognormal and the Pareto distribution to match key properties of the empirical hazard ratio in 2005. The Pareto shape parameter is set to 1.5, which is the hazard ratio of the empirical income distribution for high incomes (see, e.g., Figure 2 in Diamond and Saez 2011). The lognormal mean and variance and the variance σ^{normal} of the smoothing function are set to 10.6, 0.85, and 75000, respectively. These values ensure that the average income matches its empirical counterpart of about \$63k and that the resulting hazard ratio peaks at about \$150k, decreases until about \$350k, and flattens out afterwards, as depicted in Figure 4 (see again Figure 2 in Diamond and Saez 2011 for comparison with the empirical US hazard ratio in 2005).

Given the hazard ratio of incomes, I obtain the cumulative distribution function by solving the corresponding differential equation. Specifically, when k_y denotes the hazard ratio of the income distribution, the cumulative distribution function solves

$$\frac{dM_y}{dy} = \frac{k_y}{y} - \frac{k_y}{y} M_y \; .$$

Finally, the density function of incomes is obtained as the numerical derivative of M.
Since the distribution of types on the type space is uniform, the cumulative distribution function of incomes M returns for each income the type who earns this income under the initial tax $\overline{T}$. Hence, the income function $y : \theta \mapsto y_\theta$ is given by the inverse of M.
Given y_θ, it is straightforward to compute κ_θ from workers' first-order condition and the condition that wages equal marginal products of labor in aggregate production. First, multiplying the first-order condition (2) by l_θ and solving for it yields

$$l_\theta = \left(R'_{\overline{T}}(y_\theta) y_\theta \right)^{\frac{e}{1+e}} \; ,$$

where I used that the disutility of labor is iso elastic in the quantitative analysis. With the estimate of $\overline{T}$ described in the main text, the previous equation allows to compute labor inputs under $\overline{T}$.
For the second step, start from equations (13) and (42), copied here for convenience:

$$\begin{aligned} w_\theta(l, \phi) &= (\kappa_\theta \phi_\theta)^{\frac{\sigma-1}{\sigma}} \, (l_\theta h_\theta)^{-\frac{1}{\sigma}} \, F(l, \phi)^{\frac{1}{\sigma}} \\ \phi^*_\theta &= \overline{C}^{\frac{1}{\delta}} \, (\kappa_\theta h_\theta l_\theta)^{\frac{\sigma-1}{(\delta-1)\sigma+1}} \left[\int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta h_\theta l_\theta)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}} \, d\theta \right]^{-\frac{1}{\delta}} \; . \end{aligned}$$

The total amount of R&D resources $\overline{C}$ is not identified separately from κ, so I normalize it to satisfy

$$\overline{C} = \int_{\underline{\theta}}^{\overline{\theta}} (\kappa_\theta h_\theta l_\theta)^{\frac{(\sigma-1)\delta}{(\delta-1)\sigma+1}} \, d\theta \; .$$

Using this normalization in the above equation for ϕ^*_θ, plugging the equation into the expression for the wage w_θ, multiplying by l_θ, and solving for κ_θ yields:

$$\kappa_\theta = y_\theta^{\frac{1}{1+\gamma^{CES}+\rho^{CES}}} \, l_\theta^{-1} F^{\frac{\gamma^{CES}}{1+\gamma^{CES}+\rho^{CES}}} \; .$$

With $F = \int_{\underline{\theta}}^{\overline{\theta}} y_\theta m_y \, dy$ by Euler's theorem, this allows to compute κ.
Finally, the optimal tax formulae in Proposition 5 and Lemma 10 require the inverse hazard ratio of the exogenous parameters β and $\overline{\beta}$, respectively. In principle, β and $\overline{\beta}$ could be computed from κ and from the equilibrium technology under initial taxes via their definitions. Then, their pdf and cdf, and finally their hazard ratios could be computed. Here, to avoid unnecessary rounds of approximations, I choose a more direct way and compute the inverse hazard ratios of β and $\overline{\beta}$ directly from the hazard ratio of the income distribution, using equations (83) and (85). This ensures that the two hazard ratios inherit their shape directly from the shape of the initial hazard ratio of incomes (which is calibrated to match its empirical counterpart), without numerical differentiation or integration steps and the associated approximation errors in between.