Technology adoption and long discount rates*

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Abstract

Technological change is endogenized by introducing an innovation process driven by R&D investment in a production economy. The model is solved in closed form and reproduces key features of the long-run risk literature: endogenous consumption growth is slightly different from i.i.d., and shocks to realized and expected consumption growth are priced in equilibrium. A zero-coupon bond price formula is derived as an affine combination of discount factors analogous to defaultable bonds with zero recovery, where default in the model corresponds to a new technology being adopted. The far distant future is discounted at the lowest possible adoption-adjusted rate, a result that provides a framework which reconciles cost benefit analysis and risk management in long-term projects like climate mitigation with an arbitrage free equilibrium model.

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1 Introduction

The paper builds on the production economy of Cox, Ingersoll, and Ross (1985) (henceforth CIR), and studies technological changes driven by a R&D-dependent Poisson process which capture an inherent feature of innovation:[1]technological changes arrive sporadically and unexpectedly, and are the ultimate source of changes in output growth (Mokyr (1990)).

The economy is subject to two types of shocks: a Brownian shock that drives short term fluctuations in output and a Poisson shock, that changes the aggregate production function. Both shocks affect the productivity of the entire stock of capital and therefore carry a risk premium in equilibrium. Equilibrium displays key features of the long-run risk model of Bansal and Yaron (2004): shocks to realized and expected consumption growth are risk factors, and due to the nature of technological improvements, consumption growth displays a quasi i.i.d. behavior, with an endogenous slow moving technology dependent component.

Risk premium is characterized by a two-factor C-CAPM. The first factor is standard and reflects the risk premium due to the Brownian shocks to (realized) consumption growth. The second factor is related to changes in technology which shifts both the expectation and the volatility of consumption growth. A sufficient condition that ensures a positive premium due to technological changes is that relative risk aversion coefficient of the representative investor is smaller than one. This observation is reminiscent of the calibration of preferences in Bansal and Yaron (2004), which requires an elasticity of intertemporal substitution (its reciprocal when preferences are CRRA) higher than one in order to match the observed equity premium and risk free rate. In an extension that considers recursive preferences, it is risk aversion that ensures the sign of the risk premium, whereas the elasticity of intertemporal substitution is associated to the desirability to invest on R&D.

Zero bond prices are derived explicitly. The pricing formula is an affine combination of discount factors analogous to defaultable bonds with zero recovery, where default in the model corresponds to a new technology being adopted. It bears resemblance with the pricing rule found in contributions on long-term discounting (Weitzman (1998, 2007, 2013), Gollier and Weitzman (2010) and Brody and Hughston (2018). In this literature, a zero bond represents an aggregate discount function obtained by averaging over the views of agents. This rule is referred to as expected net present value (ENPV).[2] In particular, since the pricing formula is an affine combination of discount factors using a technology adoption-adjusted rate, the far distant future is discounted at the lowest possible rate.

[1]See Aghion and Howitt (1992), Thompson (2001) and Klette and Kortum (2004).

[2]See recent contributions on climate change economics, e.g., Nordhaus (2007), Farmer et al. (2015), Cropper et al. (2014), Giglio et al. (2014), Giglio et al. (2015) and Gollier (2016).

This result provides a framework that reconciles project analysis practices within an arbitrage free equilibrium model,[3] and thus, it has practical implications for climate change economics. As a large fraction of the climate damages generated by greenhouse gases emitted today will not materialize until the distant future, the choice of the rate at which these future damages should be discounted plays a critical role in the determination of the social cost of carbon.

This paper is related to several contributions in the asset pricing literature. Ahn and Thompson (1988) model an economy based on Cox, Ingersoll, and Ross (1985) with jump-diffusion dynamics and focus on the implications for the term structure of interest rates. Lee and Naik (1990) studies jump-diffusions dynamics in an endowment economy framework. The purpose of the paper is broader in the sense that investigates the relationship between technological change, quantities and asset prices. Gârleanu, Panageas, and Yu (2012b) build a general equilibrium model that features time-homogeneous Poisson and Brownian shocks to productivity, a continuum of firms and agents with habit formation preferences. A single factor C-CAPM holds in their economy, as technological changes are not priced in equilibrium. Kogan and Papanikolaou (2014) explore the impact of investment-specific technology shocks on the cross section of stock returns and firms' investment using a production-based asset pricing model where firms acquire new projects exogenously according to a Poisson process with fixed intensity. The focus is on the cross section of stock returns, and aggregate risk premia is exogenous. Other contributions in this literature are Kung and Schmid (2015) and Lin (2011).

The paper is related to the literature on investment under uncertainty and real options, as a technological change decision in the model is a special case of the general irreversible investment problem. The focus as well as the modeling approach differ from earlier contributions. Wang (2001) shows that equilibrium can be viewed as the result of two CIR economies pasted together when the technological growth option is defined in terms of productivity shifts and there are no migration costs. In contrast to Wang (2001), the equilibrium in this paper features multiple technological changes, and optimal policies do not correspond to a sequence of CIR policies. Furthermore, the model shows that it may be optimal to keep investing in R&D since it grants the option to migrate to a better technology in the future. Other contributions in this literature are Grenadier and Weiss (1997) and Doraszelski (2004), who study the adoption of technologies at the firm level, and Pástor and Veronesi (2009), who study an equilibrium model with a single-technology migration decision.

[3] Gollier and Weitzman (2010) discuss the theoretical underpinnings for the ENPV approach which is consistent with utility maximization in the case of logarithmic utility. This paper shows that a similar rule emerges, under a very different context, when technologies are optimally adopted and preferences belong to a much wider class.

The remainder of the paper is organized as follows. Section 2 presents the baseline model, solves the planner's problem and characterizes the optimal consumption and R&D policies. Section 3 characterizes risk premia and the short rate. Section 4 focuses on long rates. Section 5 concludes. All proofs and further results are gathered in the Appendix.

2 The model

This section describes the economic environment and formulate the central planner's problem. It also characterizes the solution and implications of the equilibrium outcome.

2.1 Technology and the planner's problem

2.1.1 Information structure

Consider a continuous time economy on an infinite horizon and assume that the uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that supports a standard Brownian motion Z and counting process N. All random processes are assumed to be adapted with respect to the augmentation of the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

2.1.2 Preferences and endowment

There exists a representative agent with time separable CRRA utility given by $U(t, c) = e^{-\rho t}\frac{c^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ is relative risk aversion (RRA) and $\rho > 0$ is the time discount parameter.[4]

There is a single consumption good, which serves as the numéraire. The representative agent is endowed with initial wealth $K_0 > 0$ and has exclusive access to a risky production technology.

2.1.3 Technology

The risky production technology produces output through the linear return process

$$\frac{dS_t}{S_t} = \mu_n dt + \sigma_n dZ_t, \tag{1}$$

where Z is a standard Brownian motion. There is an arbitrary large number (m) of known technological innovations available, $\{\mu_n, \sigma_n\}_{n=0}^{m}$ and R&D controls the expected timing at which they are adopted. A new technology shifts expected productivity, and hence, a technological change is defined as the following event: the pair (μ_n, σ_n) is replaced by $(\mu_{n+1}, \sigma_{n+1})$ at time τ_{n+1}. This migration is irreversible and costly as it entails a

[4]Appendix B.1 analyzes a similar model with recursive preferences.

proportional cost in capital, denoted by $\{\varphi_n\}_{n=1}^m$, with $\varphi_n \in (0,1)$, associated to the retooling/displacement costs induced by the new technology.[5] Agents cannot leapfrog over technologies.

Let $N = (N_t)_{0\le t<\infty}$ denote the counting process associated with the sequence of technological changes that occur at stopping times $\{\tau_1, \tau_2, \dots\}$, with $\tau_0 = 0$. These stopping times are controlled through R&D expenditure. The process N_t is defined by

$$N_t \equiv \sum_{n\ge 1} 1_{t\ge\tau_n},$$

that is, $N_t = n$ for $t \in [\tau_n, \tau_{n+1})$. The period $[\tau_n, \tau_{n+1})$ is termed epoch n. Note that the production technology in eq. (1) features two sources of uncertainty: (i) the continuous shock in productivity and (ii) the timing of the new technology adoption.

For a given epoch n, the planner's dynamic budget constraint is given by

$$dK_t = K_{t-}\left(\mu_n dt + \sigma_n dZ_t\right) - RD\left(n, K_{t-}, \lambda_t\right) dt - c_t dt - K_{t-}\varphi_{n+1} dN_t, \tag{2}$$

where K denotes the agent's capital (or wealth, as it is used interchangeably), c and RD denote the rate of consumption and R&D expenditure, respectively. The R&D function $RD(\cdot)$ depends on capital and the control variable λ_t, which denotes the intensity at which new technologies are adopted.[6]

Eq. (2) reveals that the output from production is allocated to consumption, reinvested in numéraire good production, or used in the innovation process.

Remark 1. *The (conditional) survival probability of technology n, given by*

$$\mathbb{P}_t\left[\tau_{n+1} > s\right] = \mathbb{E}\left[e^{-\int_t^s \lambda_u du} \middle| S_t, N_t = n\right],$$

is decreasing in R&D expenditure. A higher intensity increases the probability of migrating to a new technology.

R&D function. R&D can be thought of as the cost of all activities utilized for the development of knowledge, products and improvements (e.g., in operational efficiency) that will bring about a technological change. It is defined by

$$RD(n, K, \lambda) \equiv K d_n(\lambda),$$

[5] See Pástor and Veronesi (2009) for a similar specification of migration costs. It can be thought of as reduced form displacement cost, see Gârleanu et al. (2012a).

[6] See also Klette and Kortum (2004) and Thompson (2001) for economies that feature controlled Poisson processes with no aggregate shocks and linear objective functions.

where $d_n(x) : (0, \infty) \to (0, \infty)$ is an increasing and convex function given by

$$d_n(x) = \kappa_n \frac{x^\alpha}{\alpha},$$

with $\kappa_n > 0, \alpha > 1$. The constant κ_n is a technology specific scaling parameter and α determines the curvature of the cost function. This functional form will induce an equilibrium where the amount of R&D depends linearly on the size of the economy, and the expected persistence of a technology depends on the current and future technologies only (and not in size of the economy), a result that has both theoretical and empirical appeal, as argued in Klette and Kortum (2004).[7]

2.1.4 The planner's problem

The planner's problem is stated as follows. The agent maximizes lifetime utility

$$\sup_{(c,\lambda)\in\mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\rho s}\frac{c_s^{1-\gamma}}{1-\gamma}ds\right], \tag{3}$$

subject to the budget constraint in eq. (2). The set $\mathcal{A}$ defines the set of admissible strategies that lead to a strictly positive capital and consumption plans, and nonnegative R&D.

2.2 Solution to the planner's problem

Under the assumption of controlled Poisson arrivals, and the law of iterated expectations, the planner's problem in eq. (3) can be expressed as a sequence of infinite horizon problems for each epoch $n \in \{0, m-1\}$. The value function with technology n in place, denoted $J_n(\cdot)$, is characterized by the integral equation

$$J_n(K_t) \equiv \sup_{(c,\lambda)\in\mathcal{A}} \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s(\rho+\lambda_u)du}\left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda_s J_{n+1}((1-\varphi_{n+1})K_s)\right)ds\right], \tag{4}$$

subject to the budget constraint

$$dK_t = K_t\left(\mu_n dt + \sigma_n dZ_t\right) - \left(RD\left(n, K_t, \lambda_t\right) + c_t\right)dt. \tag{5}$$

Note that the budget constraint in eq. (5) does not contain a jump component, in contrast to eq. (2), and the value function in eq. (4) is characterized by a new reward function that depends on the R&D intensity, consumption and capital and is discounted exponentially at rate $\rho + \lambda$.

[7] Klette and Kortum (2004) make the argument in terms of the amount of R&D spending divided by capital instead of the intensity itself, which in this paper is equivalent.

Remark 2. *Eq.* (4) *reveals the tension generated by the presence of new technologies when compared to eq.* (3)*: a higher R&D spending increases the discount factor from* ρ *to* $\rho + \lambda$ *but also adds the capital-dependent component,* $\lambda J_{n+1}(\cdot)$ *to the objective function.*

Assuming all technologies are adopted, once the last technology available (μ_m, σ_m) is in place, the model is characterized by a simple economy with one risky technology and constant coefficients. The optimal consumption plan of this problem is given by

$$c_t = \Lambda_m K_t,$$

provided

$$\Lambda_m \equiv \frac{1}{\gamma}\left(\rho + \frac{1}{2}(1-\gamma)\gamma\sigma_m^2 - (1-\gamma)\mu_m\right) \tag{6}$$

is strictly positive. The marginal propensity to consume in equation (6) is labeled the *CIR policy*.

Assumption 1. *For a technology* (μ_n, σ_n) *the following parametric constraint holds,* $\Lambda_c(\mu_n, \sigma_n) \equiv (\rho + (1-\gamma)\gamma\sigma_n^2/2 - (1-\gamma)\mu_n)/\gamma > 0$.

The Hamilton-Jacobi-Bellman (HJB) equations that describe the planner's problem are given by

$$\rho J_n(K) = \sup_{c\geq 0, \lambda\geq 0}\left\{\frac{c^{1-\gamma}}{1-\gamma} - (c + RD(n, K, \lambda))J_{n,K}(K) + \lambda\left(J_{n+1}((1-\varphi_{n+1})K) - J_n(K)\right)\right\}$$

$$+\frac{1}{2}\sigma_n^2 K^2 J_{n,KK}(K) + \mu_n K J_{n,K}(K), \tag{7}$$

subject to the transversality conditions

$$\lim_{t\to\infty}\mathbb{E}\left[e^{-\int_0^t(\rho+\lambda_u)du}J_n(K_t)\right] = 0,$$

for $n \in \{0, m-1\}$. To solve for the HJB equations, I substitute the first order conditions with respect to consumption and R&D in eq. (7) and obtain a sequence of nonlinear differential equations. Then conjecture a candidate value function, using the homogeneity of the utility function and the linearity of the budget constraint. As the optimal consumption policy is known in closed form for the last technology (μ_m, σ_m), the sequence of HJB equations generate a recurrence relation that identifies the optimal policies.

Proposition 1 formalizes the above discussion and provides expressions for the value function and the optimal policies.

Proposition 1. *The value function*

$$J_n(K_t) = (\Lambda_n)^{-\gamma} \frac{K_t^{1-\gamma}}{1-\gamma}, \tag{8}$$

solves the HJB equation in epoch n. The optimal consumption and R&D policies are given by

$$c_t = \Lambda_n K_t, \tag{9}$$

$$\lambda_t = \lambda_n \equiv \left(\frac{1}{\kappa_n (1-\gamma)} \left((1-\varphi_{n+1})^{1-\gamma} \left(\frac{\Lambda_n}{\Lambda_{n+1}} \right)^{\gamma} - 1 \right) \right)^{\frac{1}{\alpha-1}}, \tag{10}$$

where the marginal propensity to consume in epoch n, given by the strictly positive constant Λ_n, is determined by the recurrence relation

$$(\alpha - 1)(1/\gamma - 1) d_n (\lambda_n) + \Lambda_n - \Lambda_c (\mu_n, \sigma_n) = 0, \tag{11}$$

for $n \in \{0, m-1\}$ and $\Lambda_m = \Lambda_c (\mu_m, \sigma_m)$.

2.2.1 Characterization of the solution

The value function in eq. (8) inherits the properties of the utility function: it is strictly increasing and concave in capital. The agent chooses a marginal propensity to consume and a R&D intensity that are constant within an epoch but change as soon as a new technology is adopted. The constant Λ_n maps into consumption and R&D the current and future prospects of the economy, represented by the sequence $\{(\mu_k, \sigma_k), \varphi_{k+1}, \kappa_n\}$ and preferences parameters (γ, ρ).

Proposition 1 provides the solution for an economy where technologies induce a positive R&D policy throughout, in accordance to what is observed in the data. Since the consumption policy is obtained in closed form for the last technology, this implies that the recurrence relation in eq. (11) determines the optimal policy down to the initial time, and consequently, the investment policy reads as 'investment in all technologies'. Note, however, that it might be optimal to stop adopting technologies, and thus remain with the current technology forever.

Corollary 1. *The recurrence relation in eq. (11) shows that*

(i) $\Lambda_n = \Lambda_c(\mu_n, \sigma_n)$ *when* $\lambda_n = 0$.

(ii) The investor consumes more (less) marginally than the benchmark economy (CIR) when her RRA is greater (smaller) than one $\Lambda_n > (<) \Lambda_c(\mu_n, \sigma_n), \quad \gamma > (<) 1$.

(iii) R&D investment increases with the gap between optimal and the benchmark consumption policies.

$$\frac{RD_t}{K_t} = \frac{\gamma}{(1-\gamma)(\alpha-1)}\left(\Lambda_c\left(\mu_n,\sigma_n\right) - \frac{c_t}{K_t}\right). \tag{12}$$

There is a negative relationship between consumption and R&D in eq. (12) when $\gamma < 1$, that is, the investor trades off current consumption with future investment and thus future consumption when EIS > 1.

The next proposition presents conditions so that (Λ_n, λ_n) are strictly positive for all technologies. These constraints not only impose a bound on the migration cost but also restrict the path of future technologies and adoption costs.

Proposition 2. $(\Lambda_n, \lambda_n) \in (0,\infty)\times(0,\infty)$ *if and only if*

$$(1-\varphi_{n+1})^{1-1/\gamma}\Lambda_{n+1} > (<)\Lambda_c\left(\mu_n,\sigma_n\right) \quad for \quad \gamma > (<)1. \tag{13}$$

Embedded in eq. (13) is the condition that the sequence of technologies is welfare increasing. Intuition follows from a two-technology economy.

Corollary 2. *Take a two-technology economy characterized by* $\{\mu_0,\sigma_0),(\mu_1,\sigma_1)\}$. *The inequality in eq.* (13) *corresponds to an upper bound on the migration cost,*

$$0 \le \varphi_1 < \varphi^* \equiv \max\left\{0; 1-\left(\Lambda_c(\mu_0,\sigma_0)/\Lambda_c(\mu_1,\sigma_1)\right)^{\gamma/(\gamma-1)}\right\}$$

with $\partial\varphi^*/\partial\mu_0 \le 0, \partial\varphi^*/\partial\sigma_0 \ge 0, \partial\varphi^*/\partial\mu_1 \ge 0, \partial\varphi^*/\partial\sigma_1 \le 0$. *Let* $\varphi_1 = 0$, *the bound in* (13) *corresponds to*

$$\mu_1 - \frac{1}{2}\gamma\sigma_1^2 > \mu_0 - \frac{1}{2}\gamma\sigma_0^2. \tag{14}$$

The corollary reveals that investment in R&D is positive for mean-increasing and variance-decreasing technologies with zero migration costs.[8] If both technologies have the same volatility the inequality is satisfied if and only if $\mu_1 > \mu_0$, whereas if both technologies have the same mean productivity the inequality is satisfied if and only if $\sigma_0^2 > \sigma_1^2$, analogous to the mean-variance preferences in portfolio theory.

Remark 3. *Investment may be positive even though the next technology available is not necessarily better than the current technology in terms of expected productivity. Consider a three-technology economy with average productivity path characterized by* $\mu_2 > \mu_0 > \mu_1$

[8] The results in Corollary 2 is similar to the utility score for government policies used in Pástor and Veronesi (2013).

and $\sigma_i = \sigma, \forall i$. There exists $\mu_2 > \mu_2^*$ so that investing in R&D ($\lambda_i > 0, i = 1, 2$) is optimal because allows the option to migrate from (μ_1, σ) to (μ_2, σ).

2.2.2 Equilibrium dynamics

An application of Itô's lemma shows that the dynamics of consumption growth in epoch n obey

$$\frac{dc_t}{c_{t-}} = (\mu_n - \Lambda_n - d_n(\lambda_n))\, dt + \sigma_n dZ_t + \left(\frac{(1-\varphi_{n+1})\Lambda_{n+1}}{\Lambda_n} - 1\right) dN_t,$$

and reveals that due to the linearity of the production technology, consumption growth exhibits a strong random walk component. The latter is a desirable feature from an empirical standpoint, since variations in consumption are commonly believed to exhibit a stochastic trend. Due to the linear technology, consumption growth volatility inherits a diffusive term (σ_n) and a jump term ($(1-\varphi_{n+1})\Lambda_{n+1}/\Lambda_n - 1$) whose size and frequency depends on the endogenous consumption and R&D policies.

Remark 4. *An agent smooths consumption by minimizing the jump size and by adjusting its R&D policy and hence, decreasing the frequency of jumps. A high RRA (low EIS) translates into a strong desire for smooth consumption paths.*

As in the long-run risk literature,[9] there are two sources of random variation in the aggregate consumption dynamics: (i) high frequency shocks represented by the Brownian Z_t, and (ii) low frequency movements, which are associated to the endogenous technological change process. The conditional expected consumption growth in epoch n, given by

$$\frac{1}{dt}\mathbb{E}_t\left[\frac{dc_t}{c_{t-}}\right] = \mu_n - \Lambda_n - d_n(\lambda_n) + \lambda_n \left(\frac{(1-\varphi_{n+1})\Lambda_{n+1}}{\Lambda_n} - 1\right), \tag{15}$$

shows that the low frequency component depends on parameters that characterize the technology in place plus terms that depend on the consumption-to-capital (Λ_n) and the R&D-to-capital ($d_n(\lambda_n)$) ratios. Future technologies enter equation (15) through the forward looking nature of the consumption and R&D policies, fixed by eq. (11).

[9] Bansal and Yaron (2004) calibrate an exchange economy with exogenous consumption and dividend processes that feature a small but very persistent time varying stochastic component that determines the conditional expected consumption growth, and whose shocks are priced when agents display Epstein-Zin preferences. Unlike an exchange economy, in a production economy shocks to expected consumption growth will be priced in equilibrium with CRRA preferences. The baseline model can be easily extended to a model with recursive preferences, as seen in Appendix B.1.

R&D growth follows a perfectly positively correlated with consumption growth within an epoch, but jumps by a different amount when a new technology is adopted,

$$\frac{dR_t}{R_{t-}} = (\mu_n - \Lambda_n - d_n(\lambda_n))\, dt + \sigma_n dZ_t + \left(\left(\frac{\lambda_{n+1}}{\lambda_n}\right)^{\alpha} - 1\right) dN_t,$$

and thus, as consumption, exhibits a strong random walk component. Finally, The endogenous intensity process λ_t in (10) has dynamics given by

$$\frac{d\lambda_t}{\lambda_{t-}} = \left(\frac{\lambda_{n+1}}{\lambda_n} - 1\right) dN_t.$$

The process is state dependent.[10] and reproduces features of the Hawkes process used in the contagion literature.[11] In particular, technology may induce feedback, meaning that the adoption of a technology may cause a higher rate of adoption of future technologies (self-feeding cascade of jumps), and no additional exogenous state variables are needed to drive the jump intensities, unlike doubly stochastic Poisson processes.

3 Short rate and risk premia

The solution to the representative agent's problem is used to characterize equilibrium prices. The section starts by analyzing the pricing kernel and the short rate, followed by a characterization of the expected excess return of the production technology.

3.1 Market prices of risk

The pricing kernel is characterized by the representative agent's discounted marginal utility (see Duffie (2001)),

$$\xi_t \equiv e^{-\rho t} \left(\frac{c_0}{c_t}\right)^{\gamma}$$

where c_t is the optimal consumption policy in eq. (9). The dynamics of the pricing kernel in epoch n follow a jump-diffusion process given by

$$\frac{d\xi_t}{\xi_{t-}} = -\left(r_n dt + \theta_n^Z dZ_t + \theta_n^N (dN_t - \lambda_n dt)\right), \tag{16}$$

[10] Alternative models generating time variation in the arrival of times are numerous, see e.g., Hawkes (1971), examples in Brémaud (1981), Duffie et al. (2000) and Johannes (2004).

[11] See e.g., Aït-Sahalia et al. (2015) and Fulop et al. (2014).

where r_n is the short interest rate, θ_n^Z is the market price of risk associated to Brownian shocks and θ_n^N is the market price of risk associated to a technological change. The next proposition summarizes these quantities.

Proposition 3. *The market prices of risk in epoch n are given by*

$$\theta_n^Z = \gamma\sigma_n, \tag{17}$$

$$\theta_n^N = 1 - \left(\frac{\Lambda_n}{(1-\varphi_{n+1})\Lambda_{n+1}}\right)^{\gamma} = -\frac{1}{1-\varphi_{n+1}}\left(\varphi_{n+1} + (1-\gamma)\kappa_n\lambda_n^{\alpha-1}\right). \tag{18}$$

The term θ_n^Z is standard and corresponds to the market price of risk of a CIR economy with technology n in place. The term θ_n^N reflects the percentage change in the marginal utility of consumption and its sign depends on the relative size of the consumption-wealth ratios across technologies and the migration cost. It is bounded above by one. It is associated with consumption smoothing across technologies. In particular, if the technology change generates a drop (hike) in the consumption level, θ_n^N is negative (positive), and the technology will carry a positive (negative) premium.

3.2 Interest rate and expected excess returns

As it follows from equation (16), the interest rate in epoch n is determined by the negative expected rate of change of the pricing kernel.

Corollary 3. *The interest rate in epoch n is given by*

$$r_n = \rho + \gamma\frac{1}{dt}\mathbb{E}_t\left[dc^c/c\right] - \frac{1}{2}\gamma(1+\gamma)\sigma_n^2 - \lambda_n\left(\frac{\xi^+}{\xi} - 1\right) \tag{19}$$

$$= \mu_n - \gamma\sigma_n^2 - \frac{\varphi_{n+1}\lambda_n}{1-\varphi_{n+1}}\left(\varphi_{n+1} + (1-\gamma)\kappa_n\lambda_n^{\alpha-1}\right) \tag{20}$$

The term $\frac{1}{dt}\mathbb{E}_t\left[dc^c/c\right] = \mu_n - \Lambda_n - d_n\left(\lambda_n\right)$ is the expected value of the continuous part of consumption growth, and ξ^+ is the value taken by the pricing kernel immediately after a new technology is adopted, so that $\xi^+/\xi = (1-\theta_n^N)$. The first three terms in eq. (19) are standard for a production economy; (i) The risk free rate is higher when the agent is more impatient, (ii) if the agent displays a high degree of risk aversion (i.e., a low elasticity of intertemporal substitution) she will be less willing to postpone consumption, a feature that generates a high interest rate to induce savings. Under good prospects of consumption growth, which depends on the technology in place and the endogenous consumption and R&D policies, the agent smooths out consumption by dissaving, which induces a higher interest rate. (iii) is related to precautionary motives; a prudent agent saves more to minimize the effects of volatile consumption growth, requiring a lower interest rate. Finally, the intuition of the fourth term is related to the fact that the agent uses the money market account as a hedge for the (negative) jump in consumption, inducing a higher demand for the riskless asset and thus, a lower interest rate. In contrast to Wang (2001), the interest rate does not correspond to the CIR economy's interest rate for technology n. The latter follows by setting $\lambda_n = 0$ or $\varphi_{n+1} = 0$ in eq. (20).

Having characterized the interest rate, it is straightforward to obtain a representation for the expected excess return on the production technology. The expected excess return on the risky technology in epoch n is given by

$$\begin{aligned} \mathrm{ep}_n \equiv \mu_n - r_n &= \sigma_n \theta_n^Z - \varphi_{n+1}\lambda_n\theta_n^N \\ &= \gamma\sigma_n^2 + \frac{\varphi_{n+1}\lambda_n}{1-\varphi_{n+1}}\left(\varphi_{n+1} + (1-\gamma)\kappa_n\lambda_n^{\alpha-1}\right) \end{aligned} \tag{21}$$

Note that the risk premium is given by a two-factor C-CAPM. The first term $\sigma_n\theta_n^Z$ is standard and reflects the risk premium due to shocks to (realized) consumption growth. The second term $-\varphi_{n+1}\lambda_n\theta_n^N$ is novel to this paper and relates to changes in technology which shifts both the expectation and the volatility of consumption growth and the frequency of technological changes. It is positive as long as there is a reduction in the level of consumption after a technological change and the migration cost is different from zero and drops to zero when R&D investment is halted.

Corollary 4. *The expected excess return on the risky technology in (21) is a nonlinear function of R&D given by*

$$ep_n(RD_t/K_t) = \gamma\sigma_n^2 + b_{1n}(RD_t/K_t) + b_{2n}\,(RD_t/K_t)^{1/\alpha} \tag{22}$$

where

$$\underset{+/-}{b_{1n}} \equiv \frac{\varphi_{n+1}\alpha(1-\gamma)}{1-\varphi_{n+1}}, \qquad \underset{+}{b_{2n}} \equiv \frac{\alpha\varphi_{n+1}^2}{\kappa_n(1-\varphi_{n+1})}. \tag{23}$$

Inspecting eqs. (22) and (23) shows that RRA smaller than one is a sufficient condition for the equity premium to be increasing in R&D expenditure. Risk premium is increasing in the intensity of R&D, $\frac{\partial \mathrm{ep}_n(RD_t/K_t)}{\partial(RD_t/K_t)} \geq 0$, for all levels of investment when $\gamma < 1$.[12]

[12]This observation is reminiscent of the calibration in Bansal and Yaron (2004), which requires an EIS ($1/\gamma$ in this model) higher than one in order to match the observed equity premium and risk free rate.

4 Long discount rates

Long term rates play a central role in economics and public policy. In project analysis, the rate at which future benefits and costs are discounted often determines whether a project passes the benefit-cost test. This is especially true of project that have long time horizons and have an impact on aggregate consumption. This section sheds light on project evaluation rules within the framework of the model.

4.1 Discounting cash flows

Define by risky cash flow K_T^β at time $T \geq t$, i.e., a capital strip. Its price is given by

$$P(t, T; \beta) = \mathbb{E}_t \left[\xi_{t,T} K_T^\beta \right] \tag{24}$$

where $\xi_{t,T} \equiv \xi_T / \xi_t$. The parameter β controls sensitivity to shocks as in Gollier (2012). The next result provides an explicit form for (24).

Proposition 4. *Take an $m-$technology economy. The price of a capital strip with maturity $T > 0$ is given by*

$$P(0, T; \beta) = K_0^\beta \sum_{n=0}^{m} w_0^n e^{-(r_n - \vartheta_n + \lambda_n^*)T}, \tag{25}$$

where r_n is the short rate in epoch n, $\vartheta_n \equiv \beta \left(\mu_n - d(\lambda_n) - \Lambda_n - 1/2\sigma_n^2 (1 - \beta + 2\gamma) \right)$ and $\lambda_n^* \equiv \lambda_n (1 - \theta_n^N)$ is the risk-neutral intensity of technology adoption. The weights at initial time w_0^n satisfy

$$\sum_{n=0}^{m} w_0^n = 1, \tag{26}$$

and the recurrence relation

$$w_{n-1}^j = \left(1 - \mathbf{1}_{j \neq m} \sum_{u=j+1}^{m} w_j^u \right) \prod_{i=n-1}^{j-1} \frac{\lambda_i^* (1 - \varphi_{i+1})^\beta}{r_i - \vartheta_i + \lambda_i^* - (r_j - \vartheta_j + \lambda_j^*)}, \tag{27}$$
$$n = \{1, \ldots, m\}, \quad j = \{n, \ldots, m\}.$$

The pricing formula described in (25) presents several remarkable features. First, it is very tractable as the constraint in (26) along the recurrence relation in (27) form a well-posed linear system of equations for w_0^n. Second, take a zero bond, which follows by setting $\beta = 0$.

$$B(0, T) = \sum_{n=0}^{m} w_0^n e^{-(r_n + \lambda_n^*)T}. \tag{28}$$

Note that $B_n^d(0,T) \equiv e^{-(r_n+\lambda_n^*)T}$ is analogous to the price of a defaultable zero coupon bond with maturity T and zero recovery in an economy with interest rate r_n and risk-neutral intensity of default λ_n^* (see Duffie and Singleton (1999)). The bond price with endogenous technology adoption is thus an affine combination of pseudo-defaultable zero bond prices, where default corresponds to the event when the technology in place is abandoned and a new technology is adopted.

Third notwithstanding the different economic mechanism, the bond formula in (28) is similar to the pricing rule found in contributions on risk-free long-term discounting referred to as expected net present value (ENPV) (see e.g., Weitzman (1998, 2007, 2013), Gollier and Weitzman (2010) and Brody and Hughston (2018)). In this literature, a zero bond represents an aggregate discount function obtained by averaging over the views of agents. In particular, if e^{-r_iT} is the discount factor of agents of type i for time T, then $B(0,T) = \sum_i p_i e^{-r_iT}$, where $p_i > 0$ is associated with the mass of agents of type i, with $\sum_i p_i = 1$ and the main feature is that the long term discount rate rate is given by the minimal rate in the set $\lim_{T\to\infty} -\frac{1}{T}\log\sum_i p_i e^{-r_iT} = \min\{r_i\}$. Note however that while both pricing formulas involve a weighted logarithmic sum of exponentials, some weights in (25) may be negative.

4.2 Analysis

Take an $m-$technology economy. The term structure of risk adjusted interest rates has an explicit representation given by

$$T \to R(0,T;\beta) \equiv -\frac{1}{T}\log P(0,T;\beta) = -\frac{1}{T}\log K_0^\beta \sum_{n=0}^{m} w_0^n e^{-(r_n-\vartheta_n+\lambda_n^*)T} \tag{29}$$

and define the short and long- ends respectively, $R_0(\beta) \equiv \lim_{T\downarrow 0} R(0,T;\beta)$ and $R_\infty(\beta) \equiv \lim_{T\uparrow\infty} R(0,T;\beta)$. Discount rates for intermediate maturities are not bounded by $R_0(\beta)$ and $R_\infty(\beta)$ as some of the weights w_0^n may be negative.

The next result characterizes the long end.

Proposition 5. *The asymptotic risk adjusted discount rate is given by*

$$R_\infty(\beta) \equiv \min(\{r_n - \vartheta_n + \lambda_n^*\}_{n=0}^m). \tag{30}$$

The result in Proposition 5 resembles that of the EPNV where the far distant future is discounted at the lowest rate. Take $\beta = 0$, the long rate is the minimum of the adoption-adjusted rates $\min(\{r_n + \lambda_n^*\}_{n=0}^m)$. Adoption-adjusted rates are determined by the technology in place, the future sequence of productivity parameters, and preferences. As λ_n^* measures the persistence of technology n under the risk neutral measure, the rule states that one should discount long horizon cash flows under the more persistent technologies (low λ) and lower interest. The term $r_n + \lambda_n^* = \mu_n - \gamma\sigma_n^2 + \lambda_n(1 + \varphi_{n+1} + (1-\gamma)\kappa_n\lambda_n^{\alpha-1})$, reveals that the minimum is attained when the technology in place is low growth and high volatility, and at the same time, there are low persistence (λ_n) and low transition costs (φ_{n+1}).

Comparing long discount rates for safe and risky projects boils down to analyzing the inequality

$$\min(\{r_n - \vartheta_n + \lambda_n^*\}_{n=0}^m) \leq \min(\{r_n + \lambda_n^*\}_{n=0}^m)$$

A sufficient condition the inequality above is verified $\vartheta_n < 0 \iff \mu_n - d(\lambda_n) - \Lambda_n - 1/2\sigma_n^2(1 - \beta + 2\gamma) < 0$.

5 Concluding remarks

This paper builds a stylized production economy model based on Cox, Ingersoll, and Ross (1985) so that technological shifts are driven by a R&D–dependent Poisson process. The model provides a theoretical basis for a long-run risk component in aggregate consumption growth. Risk premium is characterized by a two-factor C-CAPM. The first factor is standard and reflects the risk premium due to the Brownian shocks to (realized) consumption growth. The second factor is related to changes in technology which shifts both the expectation and the volatility of consumption growth. Furthermore, a sufficient condition that ensures a positive premium due to technological change shocks is that relative risk aversion coefficient is smaller than unity.

Pricing formulas are derived for riskless and risky cash flows as an affine combination of discount factors analogous to defaultable bonds with zero recovery, where default in the model corresponds to a new technology being adopted. The far distant future is discounted at the lowest possible adoption-adjusted rate, a result that provides a framework which reconciles cost benefit analysis and risk management in long-term projects with an arbitrage free equilibrium model.

A Proofs

Proof of Proposition 1. I pursue a classical approach to the stochastic control problem which consists on showing that the candidate value function, for a given epoch n satisfies the HJB equation and transversality condition, and that the policies in eqs. (9) and (10) are indeed optimal. Applying Itô's lemma to the discounted candidate value function,

$$V_n(t,K_t) \equiv e^{-\int_0^t(\rho+\lambda_s)ds} J_n(K_t) = e^{-\int_0^t(\rho+\lambda_s)ds} (\Lambda_n)^{-\gamma} \frac{K_t^{1-\gamma}}{1-\gamma},$$

gives

$$\begin{aligned} dV_n(t,K_t) + g_n(t,K_t,c_t,\lambda_t)dt = (1-\gamma)V_n(t,K_t)\left[h_{1,n}(c_t^K) + h_{2,n}(\lambda_t)\right]dt \\ + V_{n,K}(t,K_t)K_t\sigma_n dZ_t, \end{aligned} \tag{31}$$

where the functions $h_{1,n}(x)$ and $h_{2,n}(x)$ are defined by

$$\begin{aligned} h_{1,n}(x) &\equiv (\Lambda_n)^{\gamma}\frac{x^{1-\gamma}}{1-\gamma} - x - \frac{\rho}{1-\gamma} - \frac{1}{2}\gamma\sigma_n^2 + \mu_n, \\ h_{2,n}(x) &\equiv z_n x - \kappa_n \frac{x^{\alpha}}{\alpha}. \end{aligned}$$

and

$$\begin{aligned} c_t^K &\equiv c_t/K_t, \\ g_n(t,K_t,c_t,\lambda_t) &\equiv e^{-\int_0^t(\rho+\lambda_s)ds}\left[U(c_t) + \lambda_t J_{n+1}((1-\varphi_{n+1}))K_t)\right], \\ z_n &\equiv \frac{(1-\varphi_{n+1})^{1-\gamma}}{1-\gamma}\left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^{\gamma} - \frac{1}{1-\gamma}. \end{aligned}$$

Suppose $\Lambda_n > 0$, the function $h_{1,n}(x) : (0,\infty) \to (-\infty,\infty)$ is concave and admits a global maximum at

$$c_t^K = \Lambda_n.$$

Similarly, suppose $z_n > 0$, the function $h_{2,n}(x) : (0,\infty) \to (-\infty,\infty)$ is concave and admits a global maximum at

$$\lambda_t = \left(\frac{z_n}{\kappa_n}\right)^{\frac{1}{\alpha-1}}.$$

These optimality results imply that for any $(c,\lambda) \in \mathcal{A}$,

$$h_{1,n}(c^K) + h_{2,n}(\lambda) \le h_{1,n}(\Lambda_n) + h_{2,n}\left((z_n/\kappa_n)^{\frac{1}{\alpha-1}}\right) = 0$$

where the equality in the second expression corresponds to the recurrence relation in equation (11). This implies that the candidate value function evaluated at the optimum satisfies the HJB equation.
Integrating equation (31) and noting that for an arbitrary pair $(c,\lambda) \in \mathcal{A}$,

$$(1-\gamma)V_n(t,K_t)\left[h_{1,n}(c^K) + h_{2,n}(\lambda)\right] \le 0$$

we obtain

$$V_n(t,K_t) + \int_0^t g_n(s,K_s,c_s,\lambda_s)ds \le V_n(0,K_0) + \int_0^t V_{n,K}(s,K_s)K_s\sigma_n dZ_s \tag{32}$$

where $K_0 = K_{\tau_n+}$, the amount of capital at the beginning of epoch n. Taking expectation in equation (32) and using the fact that the stochastic integral is a martingale for admissible policies, with expected value 0,

$$\mathbb{E}\left[V_n(t,K_t)+\int_0^t g_n(s,K_s,c_s,\lambda_s)ds\right]\le V_n(0,K_0),$$

which holds with equality with controls $c_t=\Lambda_n K_t$ and $\lambda_t=(z_n/\kappa_n)^{\frac{1}{\alpha-1}}$. Furthermore, the dynamics of the candidate value function, when evaluated at the optimal controls, are given by

$$\frac{dV_n(t,K_t)}{V_n(t,K_t)}=-\Upsilon_n dt+(1-\gamma)\sigma_n dZ_t,$$

where the constant Υ_n is defined by

$$\Upsilon_n\equiv\Lambda_n+\lambda_n\left[(1-\varphi_{n+1})^{1-\gamma}\left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^{\gamma}\right].$$

The constant Υ_n is strictly positive, since $\Lambda_n>0$ and $\lambda_n>0$ by assumption, therefore

$$\lim_{t\to\infty}\mathbb{E}\left[V_n(t,K_t)\right]=\lim_{t\to\infty}V_n(0,K_0)e^{-\Upsilon_n t}=0$$

shows that the candidate function is the value function. The last epoch is characterized by a CIR economy with constant coefficients. Using the same argument as in the case of $n<m$, and given a strictly positive constant $\Lambda_c(\mu_m,\sigma_m)$, the optimal consumption policy and value function are given by

$$c_t=\Lambda_c(\mu_m,\sigma_m)K_t,\qquad J_m(K_t)=\Lambda_c(\mu_m,\sigma_m)^{-\gamma}\frac{K_t^{1-\gamma}}{1-\gamma}.$$

□

Proof of Corollary 1. A direct inspection of (11) shows that: Statement (i) follows by setting $\lambda_n=0$, we have $\Lambda_n=\Lambda_c(\mu_n,\sigma_n)$. Statement (ii) follows by noting that the recurrence relation implies the optimal marginal propensity to consume level relative to the CIR policy is determined by the level of the RRA. Statement (iii) follows by rearranging terms in (11). □

Proof of Proposition 2. Assume that $\Lambda_{n+1}>0$. Let $f_n(x)$ and $q_n(x)$ be defined by

$$f_n(x)\equiv\frac{1}{1-\gamma}\left(a_n x^{\gamma}-1\right),$$

with $a_n\equiv\frac{(1-\varphi_{n+1})^{1-\gamma}}{(\Lambda_{n+1})^{\gamma}}$, and

$$q_n(x)\equiv\frac{\alpha-1}{\alpha}(\gamma^{-1}-1)\kappa_n^{\frac{1}{1-\alpha}}f_n(x)^{\frac{\alpha}{\alpha-1}}+x-\Lambda_c\left(\mu_n,\sigma_n\right).$$

Define the set

$$\mathcal{X}=\{x\in(0,\infty):f_n(x)>0\}.\tag{33}$$

The function $q_n(x)$ corresponds to the recurrence relation in equation (11). The marginal propensity to consume in epoch n is determined by

$$\Lambda_n=\{x\in\mathcal{X}:q_n(x)=0\},$$

that is, it corresponds to all the strictly positive roots of the function $q_n(x)$ in the subset which yield a strictly positive intensity. Since the function $q_n(\cdot)$ is monotone increasing in $x\in\mathcal{X}$,

$$\frac{dq_n(x)}{dx}=\kappa_n^{\frac{1}{1-\alpha}}a_n f_n(x)^{\frac{1}{\alpha-1}}x^{\gamma-1}+1>0,$$

parametric restrictions are imposed such that $q_n(x)$ has one positive root in the domain where the intensity is strictly positive. Restrictions are summarized as follows:

1. Let $\gamma > 1$, $f_n(\cdot)$ maps $\mathcal{X} = \left(0, a_n^{-1/\gamma}\right) \to (0, \infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n(0) < 0$ and $q_n\left(a_n^{-1/\gamma}\right) > 0$, that is

$$q_n(0) = -\frac{\alpha - 1}{\alpha\gamma}\kappa_n^{\frac{1}{1-\alpha}}(\gamma - 1)^{\frac{1}{1-\alpha}} - \Lambda_c(\mu_n, \sigma_n) < 0, \tag{34}$$

$$q_n\left(a_n^{-1/\gamma}\right) = (1 - \varphi_{n+1})^{1-1/\gamma}\Lambda_{n+1} - \Lambda_c(\mu_n, \sigma_n) > 0. \tag{35}$$

   Inequality in (34) is satisfied because $\Lambda_c(\mu_n, \sigma_n) > 0, \forall n$ and $\gamma > 1$.

2. Let $\gamma < 1$, $f_n(\cdot)$ maps $\mathcal{X} = \left(a_n^{-1/\gamma}, \infty\right) \to (0, \infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n\left(a_n^{-1/\gamma}\right) < 0$ and $\lim_{x\to\infty} q_n(x) > 0$, that is

$$q_n\left(a_n^{-1/\gamma}\right) = (1 - \varphi_{n+1})^{1-1/\gamma}\Lambda_{n+1} - \Lambda_c(\mu_n, \sigma_n) < 0, \tag{36}$$

$$\lim_{x\to\infty} q_n(x) = \lim_{x\to\infty}\left(\frac{\alpha - 1}{\alpha}(\gamma^{-1} - 1)\kappa_n^{\frac{1}{1-\alpha}} f_n(x)^{\frac{\alpha}{\alpha-1}} + x - \Lambda_c(\mu_n, \sigma_n)\right) = \infty. \tag{37}$$

   Inequality in (37) is satisfied because $\gamma < 1$ and the the fact that $\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{1}{1-\gamma}(a_n x^\gamma - 1) = \infty$. Conditions in (13) correspond to the inequalities in (35) and (36), respectively.

Finally, the equivalence in value functions

$$J_n(x) < J_{n+1}((1 - \varphi_{n+1})x) \iff$$
$$\frac{\Lambda_n^{-\gamma}}{1-\gamma} < (1 - \varphi_{n+1})^{1-\gamma}\frac{\Lambda_{n+1}^{-\gamma}}{1-\gamma} \iff \frac{1}{1-\gamma}\left((1 - \varphi_{n+1})^{1-\gamma}\left(\frac{\Lambda_n}{\Lambda_{n+1}}\right)^\gamma - 1\right) > 0$$

follows from the definition of the set $\mathcal{X}$ in (33).

□

Proof of Corollary 2. Assume a two-technology economy $\{(\mu_0, \sigma_0), (\mu_1, \sigma_1)\}$. This implies that $\Lambda_1 = \Lambda_c(\mu_1, \sigma_1)$. An application of Proposition 2 shows

$$(1 - \varphi_1)^{1-1/\gamma}\Lambda_1 - \Lambda_c(\mu_0, \sigma_0) > (<)0$$
$$\iff \varphi_1 < \varphi^* \equiv \max\left\{0; 1 - (\Lambda_c(\mu_0, \sigma_0)/\Lambda_c(\mu_1, \sigma_1))^{\gamma/(\gamma-1)}\right\}$$

The comparative statics are as follows:

$$\partial\varphi^*/\partial\mu_1 = \frac{2\gamma}{\Lambda_c(\mu_1, \sigma_1))}\left(\frac{\Lambda_c(\mu_0, \sigma_0))}{\Lambda_c(\mu_1, \sigma_1)}\right)^{\gamma/(\gamma-1)} \geq 0$$

$$\partial\varphi^*/\partial\sigma_1 = -\frac{2\gamma^2\sigma_1}{\Lambda_c(\mu_1, \sigma_1))}\left(\frac{\Lambda_c(\mu_0, \sigma_0))}{\Lambda_c(\mu_1, \sigma_1)}\right)^{\gamma/(\gamma-1)} \leq 0$$

$$\partial\varphi^*/\partial\mu_0 = -\frac{2\gamma}{\Lambda_c(\mu_0, \sigma_0))}\left(\frac{\Lambda_c(\mu_0, \sigma_0))}{\Lambda_c(\mu_1, \sigma_1)}\right)^{\gamma/(\gamma-1)} \leq 0$$

$$\partial\varphi^*/\partial\sigma_0 = \frac{2\gamma^2\sigma_0}{\Lambda_c(\mu_0, \sigma_0))}\left(\frac{\Lambda_c(\mu_0, \sigma_0))}{\Lambda_c(\mu_1, \sigma_1)}\right)^{\gamma/(\gamma-1)} \geq 0$$

If $\varphi_1 = 0$, the bounds in (13) correspond to inequality (14). □

Proof of Proposition 3. The market prices of risk follow from an application if Itô's lemma to the definition of the pricing kernel and matching the diffusion and jump components with (16). The market prices of risk are given by equations (17) and (18) where the second equality in equation (18) follows from rearranging terms from the recurrence relation in (11). □

Proof of Corollary 4. The equilibrium expected excess return for a given asset i, with dynamics

$$\frac{dP_t^i}{P_{t-}^i} + \frac{\delta_t^i}{P_{t-}^i} = [\cdot]dt + \sigma_t^i dZ_t^i + \alpha_t^i dN_t$$

is given by a two-factor CCAPM representation

$$\mu_t^i - r_t = -\frac{1}{dt}\mathbb{E}_t\left[\frac{d\xi_t}{\xi_{t-}}\frac{dP_t^i}{P_{t-}^i}\right] = \sigma_t^i \rho_i \theta_t^Z - \alpha_t^i \lambda_n \theta_t^N,$$

with $\rho_i = \frac{1}{dt}\mathbb{E}_t\left[dZ_t^i dZ_t\right]$. The result in (21) follows from identifying the terms loading in (Z, N) in the production technology. Eq. (22) as a function of R&D follows from (22) and the identification of terms. □

Proof of Proposition 4. Let $f(\lambda, \tau) = \lambda e^{-\lambda\tau}$, $F(\lambda, \tau) = e^{-\lambda\tau}$ denote the PDF and CDF of an exponential random variable τ with rate parameter λ, so that

$$\bar{f}_n(u) \equiv \frac{f(\lambda_{n-1}^*, u)}{1 - F(\lambda_{n-1}^*, \tau_{n-1})}, \quad \text{for} \quad n = 1, \ldots, m.$$

$\mathbb{E}^*[\cdot]$ denotes the expectation operator under the risk neutral probability measure $\mathbb{P}^*$, defined by the Radon-Nikodym derivative $d\mathbb{P}^*/d\mathbb{P}|_{\mathcal{F}_t} \equiv e^{\int_0^t r_s ds}\xi_t.$, so that the pricing equation in (24) can be expressed as $P(t, T; \beta) = \mathbb{E}_t^*\left[e^{-\int_t^T r_s ds} K_T^\beta\right]$. The formula in (25) is obtained recursively as follows.
Let $\vartheta_n \equiv \beta\left(\mu_n - d(\lambda_n) - \Lambda_n - 1/2\sigma_n^2(1 - \beta + 2\gamma)\right)$, $\phi_n \equiv r_n - \vartheta_n$ for $n = 0, \ldots, m$.
Set $t = \tau_m$.

$$\begin{aligned}
P(\tau_m, T) &= \mathbb{E}_{\tau_m}^*\left[e^{-r_m(T-\tau_m)} K_T^\beta\right] \\
&= (1 - \varphi_m)^\beta K_{\tau_m}^\beta e^{-(r_m - \vartheta_m)(T-\tau_m)}
\end{aligned}$$

Set $t = \tau_{m-1}$

$$\begin{aligned}
P(\tau_{m-1}, T) &= \mathbb{E}_{\tau_{m-1}}^*\left[e^{-r_{m-1}(T-\tau_{m-1})} K_T^\beta \mathbf{1}_{\tau_m \geq T} + e^{-r_{m-1}(\tau_m - \tau_{m-1})} P(\tau_m, T)\mathbf{1}_{\tau_m < T}\right] \\
&= E_{\tau_{m-1}}^* e^{-r_{m-1}(T-\tau_{m-1})}(1 - \varphi_{m-1})^\beta K_{\tau_{m-1}}^\beta e^{\vartheta_{m-1}(T-\tau_{m-1})}\mathbf{1}_{\tau_m \geq T} \\
&+ \mathbb{E}_{\tau_{m-1}}^* e^{-r_{m-1}(\tau_m-\tau_{m-1})}(1 - \varphi_m)^\beta K_{\tau_m}^\beta e^{-(r_m-\vartheta_m)(T-\tau_m)}\mathbf{1}_{\tau_m < T} \\
&= E_{\tau_{m-1}}^* e^{-r_{m-1}(T-\tau_{m-1})}(1 - \varphi_{m-1})^\beta K_{\tau_{m-1}}^\beta e^{\vartheta_{m-1}(T-\tau_{m-1})}\mathbf{1}_{\tau_m \geq T} \\
&+ E_{\tau_{m-1}}^* e^{-r_{m-1}(\tau_m-\tau_{m-1})}(1 - \varphi_m)^\beta (1 - \varphi_{m-1})^\beta K_{\tau_{m-1}}^\beta e^{\vartheta_{m-1}(\tau_m-\tau_{m-1})} e^{-(r_m-\vartheta_m)(T-\tau_m)}\mathbf{1}_{\tau_m < T}
\end{aligned}$$

regrouping and replacing terms,

$$\begin{aligned}
\frac{P(\tau_{m-1}, T)}{(1 - \varphi_{m-1})^\beta K_{\tau_{m-1}}^\beta} &= e^{-\phi_{m-1}(T-\tau_{m-1})} E_{\tau_{m-1}}^* \mathbf{1}_{\tau_m \geq T} \\
&+ (1 - \varphi_m)^\beta E_{\tau_{m-1}}^* e^{-\phi_{m-1}(\tau_m - \tau_{m-1}) - \phi_m(T-\tau_m)}\mathbf{1}_{\tau_m < T}
\end{aligned}$$

so that

$$\frac{P(\tau_{m-1},T)}{(1-\varphi_{m-1})^{\beta}K^{\beta}_{\tau_{m-1}}} = e^{-\phi_{m-1}(T-\tau_{m-1})}\int_T^{\infty}\bar{f}_m(u)du$$
$$+(1-\varphi_m)^{\beta}\int_{\tau_{m-1}}^{T}e^{-\phi_{m-1}(u-\tau_{m-1})-\phi_m(T-u)}\bar{f}_m(u)du$$
$$=\omega^{m-1}_{m-1}e^{-(\phi_{m-1}+\lambda^*_{m-1})(T-\tau_{m-1})}+\omega^{m}_{m-1}e^{-\phi_m(T-\tau_{m-1})}$$

with

$$\omega^m_{m-1}=\frac{\lambda^*_{m-1}(1-\varphi_m)^{\beta}}{\phi_{m-1}+\lambda^*_{m-1}-\phi_m},$$
$$\omega^{m-1}_{m-1}=1-\omega^m_{m-1}.$$

The general representation of the recursion takes the following form.
Set $t=\tau_{n-1}$,

$$P(\tau_{n-1},T)=\mathbb{E}^*_{\tau_{n-1}}\left[e^{-r_{n-1}(T-\tau_{n-1})}K^{\beta}_T\mathbf{1}_{\tau_n\geq T}+e^{-r_{n-1}(\tau_n-\tau_{n-1})}(1-\varphi_n)^{\beta}P(\tau_n,T)\mathbf{1}_{\tau_n<T}\right]$$

so that

$$P_{\tau_{n-1}}/K^{\beta}_{\tau_{n-1}}=\int_T^{\infty}e^{-\phi_{n-1}(T-\tau_{n-1})}\bar{f}_n(u)du+(1-\varphi_n)^{\beta}\int_{\tau_{n-1}}^{T}e^{-\phi_{n-1}(u-\tau_{n-1})}\tilde{P}(u,T)\bar{f}_n(u)du$$
$$=\sum_{j=n-1}^{m}\omega^j_{n-1}e^{-(\phi_j+\lambda^*_j)(T-\tau_{n-1})}$$

where $\tilde{P}(u,T)=P_u/K^{\beta}_u$ and

$$\omega^j_{n-1}=\left(1-\mathbf{1}_{j\neq m}\sum_{u=j+1}^{m}\omega^u_j\right)\prod_{i=n-1}^{j-1}\frac{\lambda^*_i(1-\varphi_{i+1})^{\beta}}{\phi_i+\lambda^*_i-(\phi_j+\lambda^*_j)}$$
$$n=\{1,\dots,m\},\quad j=\{n,\dots,m\}$$

and the result in the Proposition follows by setting $n=1$.

□

Proof of Proposition 5. Take (25) in the definition of the term structure of spot rates is defined by $T\to R(t,T;\beta)\equiv-\frac{1}{T-t}\log P(t,T;\beta)$. We have

$$-\frac{1}{T}\log P(0,T;\beta)=-\frac{1}{T}\log K^{\beta}_0\sum_{n=0}^{m}w^n_0e^{-(r_n-\vartheta_n+\lambda^*_n)T}$$
$$=-\frac{1}{T}\log K^{\beta}_0-\frac{1}{T}\log\sum_{n=0}^{m}w^n_0e^{-(r_n-\vartheta_n+\lambda^*_n-\ell)T}+\ell \tag{38}$$

for an arbitrary constant ℓ. The latter follows from manipulating $x=\log\sum_{n=0}^{m}w^n_0e^{-(r_n-\vartheta_n+\lambda^*_n)T}$, so that

$$e^{x+\ell T}=\sum_{n=0}^{m}w^n_0e^{-(r_n-\vartheta_n+\lambda^*_n-\ell)T}\iff x=-\ell T+\log\sum_{n=0}^{m}w^n_0e^{-(r_n-\vartheta_n+\lambda^*_n-\ell)T}$$

Fixing $\ell \equiv \min\{r_0 - \vartheta_0 + \lambda_0^*, r_1 - \vartheta_1 + \lambda_1^*, \ldots, r_m\}$ and taking the limit as $T \to \infty$ in (38),

$$\lim_{T\uparrow\infty} (-\frac{1}{T}\log K_0^\beta - \frac{1}{T}\log\sum_{n=0}^{m} w_0^n e^{-(r_n-\vartheta_n+\lambda_n^*-\ell)T} + \ell) = \ell$$

which is what is stated in (30). □

B Supplementary results

B.1 Recursive preferences

The deterministic nature of the new technologies allows for a straightforward extension of the baseline model to an economy where agents have Duffie and Epstein (1992)[13] recursive preferences. These preferences are known for their ability to disentangle risk aversion from the elasticity of intertemporal substitution. I use the normalized aggregator

$$f(c,J) \equiv \frac{1-\gamma}{1-\frac{1}{\psi}} J\left(\left(\frac{c}{((1-\gamma)J)^{\frac{1}{1-\gamma}}}\right)^{1-\frac{1}{\psi}} - \rho\right),$$

where the strictly positive constants (ψ, γ) govern the elasticity of intertemporal substitution and risk aversion, respectively.

It is a well-known result that the optimal consumption policy in an economy with a single technology (μ_n, σ_n), the CIR policy is given by the marginal propensity to consume

$$\Gamma_c(\mu_n,\sigma_n) \equiv \rho\psi + \frac{1}{2}(\psi-1)\gamma\sigma_n^2 + (1-\psi)\mu_n. \tag{39}$$

Note that by letting $\psi^{-1} = \gamma$, the normalized aggregator reduces to the time additive power utility case, $f(c,J) = \frac{c^{1-\gamma}}{1-\gamma} - \rho J$, so that $\Gamma_c(\mu_n,\sigma_n) = \Lambda_c(\mu_n,\sigma_n)$. As in the baseline case, assume that for a technology (μ_n, σ_n) the following parametric constraint holds,

$$\Gamma_c(\mu_n,\sigma_n) > 0, \quad \forall n.$$

The marginal propensity to consume in (39) is decreasing in the productivity parameter μ_m if $\psi > 1$. The optimal policies under recursive preferences are summarized in the following proposition.

Proposition 6. *The optimal consumption and R&D policies are given by*

$$c_t = \Gamma_n K_t, \qquad \lambda_t = \lambda_n \equiv \left(\frac{1}{\kappa_n(1-\gamma)}\left((1-\varphi_{n+1})^{1-\gamma}\left(\frac{\Gamma_n}{\Gamma_{n+1}}\right)^{-\frac{1-\gamma}{1-\psi}} - 1\right)\right)^{\frac{1}{\alpha-1}}, \tag{40}$$

where the marginal propensity to consume in epoch n, given by the strictly positive constant Γ_n, is determined by the recurrence relation

$$(\alpha-1)(\psi-1)d_n(\lambda_n) + \Gamma_n - \Gamma_c(\mu_n,\sigma_n) = 0, \tag{41}$$

for $n \in \{0, m-1\}$ and $\Gamma_m = \Gamma_c(\mu_m,\sigma_m)$.

Proof. Using the same procedure of the baseline economy with power utility, the planner solves in each epoch $n \in \{0, m-1\}$,

$$J_n(K_t) = \sup_{\{c,\lambda\}\in\mathcal{A}} \mathbb{E}_t\left[\int_t^\infty e^{-\int_t^s \lambda_u du}\left(f(c_s, J_n(K_s)) + \lambda_s J_{n+1}((1-\varphi_{n+1})K_s)\right) ds\right],$$

[13] See Ma (2006) for a general description of recursive preferences and HJB equations in economies with jump-diffusion processes.

I proceed as in the proof of propositions 1 and 2. The Hamilton-Jacobi-Bellman (HJB) equations that describe the planner's problem are given by

$$0 = \sup_{c\geq 0,\lambda\geq 0} \{f(c, J_n) - (c + RD\,(n, K, \lambda))J_{n,K}(K) + \lambda\,(J_{n+1}((1-\varphi_{n+1})K) - J_n(K))\} \\ + \frac{1}{2}\sigma_n^2 K^2 J_{n,KK}(K) + \mu_n K J_{n,K}(K),$$

subject to the transversality conditions

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-\int_0^t (\lambda_u)du} J_n(K_t)\right] = 0,$$

for $n \in \{0, m-1\}$. The value function J that solves the HJB in state n is given by

$$J_n(K_t) = \Gamma_n^{\frac{1-\gamma}{1-\psi}} \frac{K^{1-\gamma}}{1-\gamma}.$$

The optimal consumption and intensity rules are given in (40), where Γ_n is determined by the recursion in (41). □

Conditions for $(\Gamma_n, \lambda_n) \in (0,\infty) \times (0,\infty)$ are provided next.

Proposition 7. $(\Gamma_n, \lambda_n) \in (0,\infty) \times (0,\infty)$ *if and only if*

$$(1-\varphi_{n+1})^{1-\psi}\Gamma_{n+1} > (<)\Gamma_c\,(\mu_n, \sigma_n) \quad \textit{for} \quad \psi < (>)1. \tag{42}$$

Proof. I proceed as in the proof of Proposition 2. Assume that $\Gamma_{n+1} > 0$ and $\Gamma_c(\mu_n, \sigma_n) > 0, \quad \forall n$. Let the functions $f_n(x)$ and $q_n(x)$ be defined by

$$f_n(x) \equiv \frac{1}{1-\gamma}\left(a_n x^\phi - 1\right),$$

with $a_n \equiv \frac{(1-\varphi_{n+1})^{1-\gamma}}{(\Gamma_{n+1})^\phi}, \phi \equiv -\frac{1-\gamma}{1-\psi}$ and

$$q_n(x) \equiv \frac{\alpha-1}{\alpha}(\psi-1)\kappa_n^{\frac{1}{1-\alpha}} f_n(x)^{\frac{\alpha}{\alpha-1}} + x - \Gamma_c\,(\mu_n, \sigma_n)\,.$$

Let

$$\mathcal{X} = \{x \in (0,\infty) : f_n(x) > 0\}. \tag{43}$$

The function $q_n(x)$ corresponds to the recurrence relation in equation (41), and is monotone increasing in $x \in \mathcal{X}$,

$$\frac{dq_n(x)}{dx} = \kappa_n^{\frac{1}{1-\alpha}} a_n f_n(x)^{\frac{1}{\alpha-1}} x^{\phi-1} + 1 > 0.$$

The following parametric constraints are necessary and sufficient so that q_n has a unique root in the domain where the intensity is positive.

1. Let $\gamma > 1$, $\psi < 1$ so that $\phi > 0$. $f_n(\cdot)$ maps $\mathcal{X} = \left(0, a_n^{-1/\phi}\right) \to (0,\infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n(0) < 0$ and $q_n\left(a_n^{-1/\phi}\right) > 0$, that is

$$q_n(0) = \frac{\alpha-1}{\alpha}(\psi-1)\kappa_n^{\frac{1}{1-\alpha}}(\gamma-1)^{\frac{\alpha}{1-\alpha}} - \Gamma_c\,(\mu_n, \sigma_n) < 0, \tag{44}$$

$$q_n\left(a_n^{-1/\phi}\right) = (1-\varphi_{n+1})^{1-\psi}\Gamma_{n+1} - \Gamma_c\,(\mu_n, \sigma_n) > 0. \tag{45}$$

   The inequality in (44) follows from the assumption that $\Gamma_c\,(\mu_n, \sigma_n) > 0, \forall n$.

2. Let $\gamma > 1$, $\psi > 1$ so that $\phi < 0$. $f_n(\cdot)$ maps $\mathcal{X} = \left(a_n^{-1/\phi}, \infty\right) \to (0, \infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n\left(a_n^{-1/\phi}\right) < 0$ and $\lim_{x\to\infty} q_n(x) > 0$, that is

$$q_n\left(a_n^{-1/\phi}\right) = (1-\varphi_{n+1})^{1-\psi}\Gamma_{n+1} - \Gamma_c(\mu_n, \sigma_n) < 0, \tag{46}$$

$$\lim_{x\to\infty} q_n(x) = \lim_{x\to\infty}\left(\frac{\alpha-1}{\alpha}(\psi-1)\kappa_n^{\frac{1}{1-\alpha}} f_n(x)^{\frac{\alpha}{\alpha-1}} + x - \Gamma_c(\mu_n, \sigma_n)\right) > 0 \tag{47}$$

The inequality in (47) follows from the fact that $\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{1}{1-\gamma}\left(a_n x^{\phi} - 1\right) = \frac{1}{\gamma-1} > 0$.

3. Let $\gamma < 1$, $\psi < 1$ so that $\phi < 0$. $f_n(\cdot)$ maps $\mathcal{X} = \left(0, a_n^{-1/\phi}\right) \to (0, \infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n(0) < 0$ and $q_n\left(a_n^{-1/\phi}\right) > 0$, that is

$$\lim_{x\to 0} q_n(x) = \lim_{x\to 0}\left(\frac{\alpha-1}{\alpha}(\psi-1)\kappa_n^{\frac{1}{1-\alpha}} f_n(x)^{\frac{\alpha}{\alpha-1}} + x - \Gamma_c(\mu_n, \sigma_n)\right) < 0, \tag{48}$$

$$q_n\left(a_n^{-1/\phi}\right) = (1-\varphi_{n+1})^{1-\psi}\Gamma_{n+1} - \Gamma_c(\mu_n, \sigma_n) > 0 \tag{49}$$

The inequality in (48) follows from the fact that $\lim_{x\to 0} f_n(x) = \lim_{x\to 0} \frac{1}{1-\gamma}\left(a_n x^{\phi} - 1\right) = \frac{1}{\gamma-1} < 0$.

4. Let $\gamma < 1$, $\psi > 1$ so that $\phi > 0$ $f_n(\cdot)$ maps $\mathcal{X} = \left(a_n^{-1/\phi}, \infty\right) \to (0, \infty)$, $q_n(x)$ has one positive root in $\mathcal{X}$ if and only if $q_n\left(a_n^{-1/\phi}\right) < 0$ and $\lim_{x\to\infty} q_n(x) > 0$, that is

$$q_n\left(a_n^{-1/\phi}\right) = (1-\varphi_{n+1})^{1-\psi}\Gamma_{n+1} - \Gamma_c(\mu_n, \sigma_n) < 0, \tag{50}$$

$$\lim_{x\to\infty} q_n(x) = \lim_{x\to\infty}\left(\frac{\alpha-1}{\alpha}(\psi-1)\kappa_n^{\frac{1}{1-\alpha}} f_n(x)^{\frac{\alpha}{\alpha-1}} + x - \Gamma_c(\mu_n, \sigma_n)\right) > 0. \tag{51}$$

The inequality in (51) follows from the fact that $\lim_{x\to\infty} f_n(x) = \lim_{x\to\infty} \frac{1}{1-\gamma}\left(a_n x^{\phi} - 1\right) = \infty$.

Conditions in (42) correspond to the inequalities in (45), (46), (49) and (50) respectively.

Finally, the equivalence

$$J_n(x) < J_{n+1}((1-\varphi_{n+1})x) \iff$$
$$\frac{\Gamma_n^{-\phi}}{1-\gamma} < (1-\varphi_{n+1})^{1-\gamma}\frac{\Gamma_{n+1}^{-\phi}}{1-\gamma} \iff \frac{1}{1-\gamma}\left((1-\varphi_{n+1})^{1-\gamma}\left(\frac{\Gamma_n}{\Gamma_{n+1}}\right)^{\phi} - 1\right) > 0$$

follows from the definition of the set $\mathcal{X}$ in (43)

□

The following result shows the same intuition from Corollary 1 applies to recursive preferences.

Corollary 5. *The recurrence relation in* (41) *shows that*

(i) $\Gamma_n = \Gamma_c(\mu_n, \sigma_n)$ *when* $\lambda_n = 0$.

((ii) The investor consumes more (less) marginally than the benchmark economy when his EIS is smaller (greater) than one

$$\Gamma_n > (<)\Gamma_c(\mu_n, \sigma_n) \qquad \psi < (>)1.$$

(iii) Rewriting (41),

$$\frac{RD_t}{K_t} = \frac{1}{(\psi-1)(\alpha-1)}\left(\Gamma_c(\mu_n, \sigma_n) - \frac{c_t}{K_t}\right) \tag{52}$$

which shows that R&D investment increases with the gap between optimal and the benchmark consumption policies.

Proof. The recurrence relation in (41) shows that: A direct inspection of (11) shows that: Statement (i) follows by setting $\lambda_n = 0$, so that $\Gamma_n = \Gamma_c(\mu_n, \sigma_n)$. Statement (ii) follows by noting that the recurrence relation implies the optimal marginal propensity to consume level relative to the CIR policy is determined by the level of the EIS. Statement (iii) follows by rearranging terms in (41). $\square$

The above results are structurally similar to that presented in Proposition 1. In particular, there is a negative relationship between consumption and R&D in (52) when EIS > 1.

As in the power utility case, it is straightforward to obtain a representation for market risk premia, the interest rate and the expected excess return on the production technology.

Proposition 8. *The interest rate is given by*

$$r_n = \beta_n + \gamma\left(\mu_n - \Gamma_n - d_n(\lambda_n)\right) - \frac{1}{2}\gamma(1+\gamma)\sigma_n^2 - \lambda_n\left((1-\varphi_{n+1})^{-\gamma}\left(\frac{\Gamma_{n+1}}{\Gamma_n}\right)^{\frac{1-\gamma}{1-\psi}} - 1\right) \tag{53}$$

with

$$\beta_n = \frac{(1-\gamma\psi)\Gamma_n - (1-\gamma)\psi\rho}{1-\psi}.$$

The market prices of risk are given by

$$\theta_n^Z = \gamma\sigma_n, \tag{54}$$

$$\theta_n^N = 1 - (1-\varphi_{n+1})^{-\gamma}\left(\frac{\Gamma_{n+1}}{\Gamma_n}\right)^{\frac{1-\gamma}{1-\psi}}. \tag{55}$$

The expected excess return on the risky technology is given by

$$ep_n = \mu_n - r_n = \gamma\sigma_n^2 + \frac{\lambda_n\varphi_{n+1}}{1-\varphi_{n+1}}\left(\varphi_{n+1} + \kappa_n\lambda_n^{\alpha-1}(1-\gamma)\right) \tag{56}$$

Interestingly, note that the expression (56), as in (21), shows that a risk aversion coefficient smaller than one ($\gamma < 1$) is a sufficient condition for a positive premium associated to technological change and thus, the representation in (22) obtains.

Proof. Using the results from Proposition 6, characterizing the market prices of risk and the expected excess return on the technology follows from a direct application of Itô's lemma to the candidate pricing kernel given by

$$\xi_t = e^{\int_0^t f_J(c_s, J_{n,s})ds} f_c(c_t, J_{n,t}).$$

Assume that parameters are such that $\xi_0 = 1$. The dynamics of the pricing kernel are given by

$$\frac{d\xi_t}{\xi_{t-}} = -(r_t dt + \theta_t^c dZ_t + \theta_t^N(dN_t - \lambda_n dt))$$

where the interest rate and the market prices of risk are given by equations (53),(54) and (55). Finally, the equivalence in (56) follows from a direct application of the recursion in (41) on the definition of the equity premium. $\square$

Remark 5. *Since the interest rate and risk neutral intensity for a given epoch n are constant, the term structure of spot rate in the economy with recursive preferences is isomorphic to (29).*

B.2 An economy with infinitely many innovations: an approximation method

I compare the baseline problem with a simplified stationary problem ($m \to \infty$) with mean increasing innovations and exogenous R&D policy. Following an argument similar to the non-stationary case, the planner maximizes now a recursive integral equation

$$J(\mu, K_t) = \sup_{c \in \mathcal{A}(K_0)} E_t \left[\int_t^\infty e^{-(\rho+\lambda)(s-t)} \left(\frac{c_s^{1-\gamma}}{1-\gamma} + \lambda J(\mu(1+\varepsilon), (1-\varphi)K_s) \right) ds \right], \tag{57}$$

subject to the budget constraint

$$dK_t = K_t \left(\mu dt + \sigma dZ_t \right) - \left(K_t d\left(\lambda\right) dt + c_t \right) dt,$$

where $d(\lambda) = \kappa \frac{\lambda^\alpha}{\alpha}$ and $\varphi \in [0, 1)$. Note that the value function in (57) depends on the current mean productivity (μ) and the size of the innovation (ε), yet, in contrast to equation (8) it is independent of the state of technology (n) and the number of innovations available (m). Using standard calculations, the HJB equation associated with the planner's problem is given by

$$\begin{aligned} (\rho + \lambda) J(\mu, K) = {} & \left(\mu - d(\lambda) \right) K J_K(\mu, K) + \frac{1}{2} K^2 \sigma^2 J_{KK}(\mu, K) \\ & + \lambda J((1+\varepsilon)\mu, (1-\varphi)K) + \sup_c \left[U(c) - c J_K(\mu, K) \right]. \end{aligned}$$

As in the non-stationary case, I conjecture a function that exploits the homogeneity of the utility function,

$$J(\mu, K) = \Lambda(\mu)^{-\gamma} \frac{K^{1-\gamma}}{1-\gamma},$$

which implies a consumption policy linear in capital,

$$c_t = \Lambda(\mu) K_t.$$

Substituting these expressions into (**??**) obtains a nonlinear functional equation for the marginal propensity to consume Λ,

$$\lambda (1-\varphi)^{1-\gamma} \Lambda((1+\varepsilon)x)^{-\gamma} - \left(\lambda + (1-\gamma) d(\lambda) + \gamma \Lambda_c(\mu, \sigma) \right) \Lambda(x)^{-\gamma} + \gamma \Lambda(x)^{1-\gamma} = 0. \tag{58}$$

No closed form solution to this equation is known, however, note that by setting $\lambda = 0$ in (58) recovers the CIR policy and use this fact to obtain an approximate solution formulated as a power series representation. Substituting the expression in (59) into equation (58) and expanding the resulting equation in powers of λ gives a sequence of algebraic equations which can be conveniently solved recursively starting from the known function, $\Lambda_c(\mu, \sigma)$

$$\Lambda(\mu) \approx \Lambda_c(\mu, \sigma) + \sum_{\ell=1}^{h} \frac{1}{\ell!} \lambda^\ell g_\ell(\mu), \tag{59}$$

where the terms in the summation are defined by

$$g_\ell(\mu) = \left. \frac{\partial^\ell}{\partial \lambda^\ell} \Lambda(\mu) \right|_{\lambda=0}.$$

This method delivers a good approximation when the intensity parameter λ is expected to be small (e.g., in economies with high κ, α or φ).

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