No-Arbitrage Pricing of GDP-Linked Bonds using Dividend Swaps*

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PRELIMINARY AND INCOMPLETE

Abstract

We use a no-arbitrage term structure model of equity yields computed from the prices of dividend swaps to estimate the possible yields on hypothetical bonds with cash-flows indexed to the level of U.S. GDP. This provides a novel approach for estimating the possible relative cost of conventional and GDP-linked bonds, which is likely to be of interest to issuers considering the case for issuing GDP-linked debt. Our model predicts that U.S. GDP-linked bonds would have yields lower than those on conventional Treasury bonds with the same maturity in our sample from 2010 to 2017. Positive expected future GDP growth lowers the yield on GDP-linked bonds relative to conventional bonds, which more than offsets the estimated GDP risk premium demanded by investors for holding GDP risk.

Keywords: Dynamic term structure model, bond yield, equity yield, risk premia, dividend strips, GDP-linked bonds, spanned macroeconomic factors.

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1 Introduction

Several central banks and other policy institutions have recently expressed an interest in GDP-linked sovereign debt as a potential means of promoting financial stability.[1] While no GDP-linked bonds currently exist, in principle they present a number of attractions. In particular, the repayment burden for a GDP-linked bond would be lower relative to conventional bonds at times when the economy is growing relatively slowly (that is, periods typically associated with lower tax revenues).[2] A greater share of the risk of weak growth outcomes would therefore be borne by the holders of GDP-linked bonds than by holders of conventional bonds. As a consequence, issuers of GDP-linked bonds would likely need to pay an additional risk premium relative to conventional government bonds, in order to compensate investors for the fact that cash-flows from the bond would be smaller in bad times.

In the absence of an existing market for GDP-linked bonds, it is not straightforward to quantify the likely size of this risk premium. Indeed, uncertainty about the size of the risk premium may be one reason why no sovereign has yet issued a bond with payments linked to GDP.[3] Previous attempts to quantify the GDP risk premium have significant limitations.[4] Some studies combine a model of the time-series properties of GDP with assumptions about investors' preferences, and solve for the required risk premium. For example, Barr, Bush and Pienkowski (2014) assume that investors have constant relative risk aversion. However, standard utility functions cannot explain the magnitude and dynamics of risk premiums of existing assets, so we should not attach great weight to their predictions for risk premiums on assets that do not currently exist. Other studies (including Borensztein and Mauro (2004),

[1] See, for example, the German G20 presidency's Compass document, available at https://www.bundesfinanzministerium.de/Content/DE/Standardartikel/Themen/Schlaglichter/G20-2016/g20-compass-for-gdp-linked-bonds.pdf?__blob=publicationFile&v=3

[2] See Benford, Joy and Kruger (2016) for a summary of other potential benefits of GDP-linked bonds.

[3] Borensztein and Mauro (2004) discuss this and other possible explanations for the lack of previous GDP-linked bond issuance.

[4] Some studies have ignored this risk compensation altogether, and instead relied on a risk-neutral framework to estimate a price for GDP-linked debt (see, for example, Chamon and Mauro (2006)).

Kamstra and Shiller (2009) and Bowman and Naylor (2016)) have adopted an approach based on the Capital Asset Pricing Model (CAPM), and use the beta of observed GDP growth with respect to returns on the market portfolio as a proxy for the beta of GDP-linked bonds. Unfortunately, it is by no means obvious why returns on hypothetical GDP-linked bonds and GDP growth itself would have the same covariance with returns on the market portfolio. Moreover, no previous studies that attempt to quantify the GDP risk premium consider how it may vary with the maturity of the GDP-linked bond, which is a crucial practical question in the design of any prospective future instrument.

In this paper, we apply a novel asset pricing approach to estimate U.S. GDP risk premiums using the prices of existing assets. Specifically, we build a no-arbitrage affine term structure model (ATSM), which allows us to estimate predicted yields of GDP-linked bonds from equity yields derived from the prices of S&P 500 dividend swaps. Our approach avoids the limitations of previous studies: it allows a much more flexible specification of investors' preferences, which is known to match risk premiums on existing assets; it avoids any assumptions about the relative covariances of GDP growth and returns on GDP-linked bonds with the market portfolio; and it allows us to study how the GDP risk premium varies with the maturity of the bond. The key assumption underpinning our analysis is that GDP and dividend growth are "spanned" by the cross section of the term structure of equity yields. Formally, the spanning assumption requires that—in the absence of measurement error—we can invert GDP and dividend growth from the term structure of equity yields. In consequence, we can back out the risk-neutral dynamics of GDP growth and dividends from observed equity yields—which allows us to compute predicted yields on GDP-linked bonds.

We find that our model can fit GDP growth precisely, while also reproducing the broad movements in equity yields. The model predicts that the yields on GDP-linked bonds would likely be lower than the yields on conventional bonds. This spread reflects the difference between two partly offsetting factors. First, if GDP growth is expected to be positive, this pushes up the terminal cash-flow on a GDP-linked bond, raises the current price, and lowers the current yield relative to conventional bonds. Second, if GDP growth is expected to be relatively low in bad times (that is, when discount factors are relatively high), investors will demand an additional risk premium for bearing GDP risk. This GDP risk premium pushes down the current prices of GDP-linked bonds and raises the yields relative to conventional bonds. Our model implies that expected GDP growth is about 4.5 percent on average, with the GDP risk premium varying between about 0 and 3 percent (with the magnitude of premia being larger at short maturities). Thus, during our sample GDP-linked bond yields are lower than the yields on conventional bonds.

The remainder of the paper is structured as follows. In Section 2, we describe the basis structure of the hypothetical GDP-linked bonds we consider and provides some basic inuition for the difference in average expected returns between GDP-linked and conventional government bonds. In Section 3, we present a joint no-arbitrage term structure model of GDP-linked bond and equity yields. In Section 4, we explain how we estimate this joint model in the absence of observed GDP-linked bond yields. In Section 5 we present our main results, while in Section 6 we show that an alternative approach using conventional bond yields, rather than equity yields, does not produce plausible estimates of GDP-linked bond yields. In Section 7, we summarize our conclusions.

2 Intuition for the Breakeven Between Conventional and GDP-Linked Bond Yields

In this section we describe the basic structure of the hypothetical GDP-linked bonds we consider and provides some basic inuition for the difference in average expected returns between GDP-linked and conventional government bonds. For both types of bond we take a second-order approximation of the pricing equation. This allows us to decompose the differences in the yields on GDP-linked and conventional bonds with the same maturities into the expected average rate of GDP growth over the lifetime of the bonds and an additional

"GDP risk premium" that compensates investors for GDP risk.

We start by analyzing the prices of conventional (or "nominal") bonds, which make up the majority of government debt in most countries. A k-period nominal (zero-coupon) bond pays one dollar at maturity at time $t+k$. Under the assumption of no arbitrage, the time-t price ($P_{t,n}^{(k)}$) of such a bond is given by

$$P_{t,n}^{(k)} = \mathbb{E}_t \left[\prod_{j=1}^{k} M_{t+j} \right], \tag{1}$$

where M_{t+j} is the nominal stochastic discount factor that discounts dollar cash-flows from time $t+j$ back to $t+j-1$. The yield on a nominal bond is defined as $y_{t,n}^{(k)} = -\frac{1}{k} \log P_{t,n}^{(k)}$. Taking a second-order approximation of equation (1) shows that the yield is approximately given by

$$y_{t,n}^{(k)} \approx -\frac{1}{k} \left\{ \mathbb{E}_t \left[\sum_{j=1}^{k} m_{t+j} \right] + \frac{1}{2} \mathbb{V}_t \left[\sum_{j=1}^{k} m_{t+j} \right] \right\}, \tag{2}$$

where $m_{t+j} = \log M_{t+j}$.[5]

We now turn to GDP-linked bonds. We assume that a hypothetical zero-coupon GDP-linked bond has a dollar cash-flow linked to the growth in nominal GDP between the time of issue and the maturity of the bond. More precisely, we assume that a k-period GDP-linked bond pays $\frac{Y_{t+k}}{Y_t}$ dollars at maturity at time $t+k$, where Y_t is the level of nominal GDP at time t. Under the assumption of no arbitrage, the time-t price ($P_{t,g}^{(k)}$) is given by

$$P_{t,g}^{(k)} = \mathbb{E}_t \left[\prod_{j=1}^{k} M_{t+j} \frac{Y_{t+k}}{Y_t} \right], \tag{3}$$

The yield on a GDP-linked bond is defined as $y_{t,g}^{(k)} = -\frac{1}{k} \log P_{t,g}^{(k)}$. Taking a second-order

[5] These second-order approximations hold exactly when the pricing kernel is log-normal, as in our models.

approximation of equation (3) shows that the yield is approximately given by

$$y_{t,g}^{(k)} \approx -\frac{1}{k}\left\{\mathbb{E}_t\left[\sum_{j=1}^{k}(m_{t+j}+g_{t+j})\right]+\frac{1}{2}\mathbb{V}_t\left[\sum_{j=1}^{k}m_{t+j}\right]+\frac{1}{2}\mathbb{V}_t\left[\sum_{j=1}^{k}g_{t+j}\right]+\mathbb{C}_t\left[\sum_{j=1}^{k}m_{t+j},\sum_{j=1}^{k}g_{t+j}\right]\right\}, \tag{4}$$

where $g_{t+j} = \log Y_{t+j} - \log Y_t$ is the growth rate of nominal GDP.

The difference between the yields on a conventional bond and a GDP-linked bond of the same maturity is therefore given by

$$y_{t,n}^{(k)} - y_{t,g}^{(k)} \approx \frac{1}{k}\left\{\mathbb{E}_t\left[\sum_{j=1}^{k}g_{t+j}\right]+\mathbb{C}_t\left[\sum_{j=1}^{k}m_{t+j},\sum_{j=1}^{k}g_{t+j}\right]+\frac{1}{2}\mathbb{V}_t\left[\sum_{j=1}^{k}g_{t+j}\right]\right\} \tag{5}$$

The first term on the right-hand side of equation (5) is the expected rate of nominal GDP growth. All else equal, the higher expected GDP growth, the higher the expected terminal cash-flow on a GDP-linked bond, the higher the current price of the GDP-linked bond, and the lower the current yield. Thus, if GDP growth is expected to be relative high, the cost today of issuing a GDP-linked bond is relatively low.

The second term on the right-hand side of equation (5) is (the negative of) a risk premium. If GDP growth tends to be relatively low in "bad times" (times when the stochastic discount factor is relatively high and investors put higher value on cash flows), this covariance term will be negative: cash-flows will be lowest at times when they are most valued. In this case, GDP-linked bonds must therefore offer an additional positive premium relative to conventional bonds to compensate investors for bearing this risk. This would push up on the yields of GDP-linked bonds and down on the breakeven rate.

The final term on the right-hand side of equation (5) is a convexity term that arises from working with log prices. In our ATSM, this conditional variance is constant over time and is relatively small compared with the other terms. When we report results from our ATSM below we group (the negative of) the final two terms together as a single "GDP risk premium", reflecting the average expected return on GDP-linked bonds in excess of the nominal yield and expected GDP growth.

3 A Joint Model of GDP-Linked Bond and Equity Yields

In this section, we explain how we compute GDP risk premiums using the prices of existing assets whose payoffs are affected by GDP risk. In Section 3.1 we set out a hypothetical ATSM of GDP-linked bonds, while in Section 3.2 we extend this model to price equity yields, which is the model we take to the data.

3.1 An ATSM of GDP-Linked Bond Yields

We can equivalently write equation (3) as

$$P_{t,g}^{(k)} = \mathbb{E}_t \left[M_{t+1} \exp\left(g_{t+1}\right) P_{t+1,g}^{(k-1)} \right], \tag{6}$$

We assume that the GDP growth rate is an affine function of an $n_x \times 1$ vector of pricing factors ($\mathbf{x}_t$), that is,

$$g_t = g_0 + \mathbf{g}_1' \mathbf{x}_t, \tag{7}$$

where g_0 is a scalar and $\mathbf{g}_1$ is an $n_x \times 1$ vector. We further assume that the stochastic discount factor takes the form

$$M_{t+1} = \exp\left(-r_t - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1}\right), \tag{8}$$

where r_t is the short-term risk-free nominal interest rate, the $n_x \times 1$ vector $\boldsymbol{\lambda}_t$ contains the market prices of risk, and $\boldsymbol{\varepsilon}_{t+1} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{I}\right)$ is an $n_x \times 1$ vector of random shocks. The short rate and market prices of risk are affine functions of the factors, that is,

$$r_t = \rho_0 + \boldsymbol{\rho}_1' \mathbf{x}_t \text{ and} \tag{9}$$

$$\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda}_1 \mathbf{x}_t \tag{10}$$

where ρ_0 is a scalar; $\boldsymbol{\lambda}_0$ and $\mathbf{g}_1$ are $n_x \times 1$ vectors; and $\boldsymbol{\Lambda}_1$ an $n_x \times n_x$ matrix. The factors follow a first-order Gaussian vector autoregression (VAR), that is,

$$\mathbf{x}_{t+1} = \boldsymbol{\mu} + \boldsymbol{\Phi}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}. \tag{11}$$

Under the assumption of no-arbitrage, we can equivalently write the price of a k-period GDP-linked bond as

$$P_{t,g}^{(k)} = \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-r_t + g_{t+1}\right) P_{t+1,g}^{(k-1)} \right], \tag{12}$$

where expectations are formed with respect to the equivalent risk-neutral probability measure, which we denote $\mathbb{Q}$. The above assumptions also imply that the factors follow a first-order VAR under $\mathbb{Q}$, that is,

$$\mathbf{x}_{t+1} = \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}, \tag{13}$$

where $\boldsymbol{\mu}^{\mathbb{Q}} = \boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_0$, $\boldsymbol{\Phi}^{\mathbb{Q}} = \boldsymbol{\Phi} - \boldsymbol{\Sigma}\boldsymbol{\Lambda}_1$, and $\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}} \sim \mathcal{NID}\left(\mathbf{0}, \mathbf{I}\right)$ (see Duffee (2002)). It follows that the yield on a GDP-linked bond is an affine function of the pricing factors, that is,

$$y_{t,g}^{(k)} = -\frac{1}{k}\left(a_{k,g} + \mathbf{b}_{k,g}' \mathbf{x}_t\right), \tag{14}$$

where $a_{n,g}$ and $\mathbf{b}_{n,g}$ follow the recursive equations

$$a_{k,g} = -\rho_0 + g_0 + a_{k-1,g} + \left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)' \boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right) \text{ and} \tag{15}$$

$$\mathbf{b}_{k,g}' = -\boldsymbol{\rho}_1' + \left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)' \boldsymbol{\Phi}^{\mathbb{Q}}, \tag{16}$$

with boundary conditions $a_{0,g} = 0$ and $\mathbf{b}_{0,g} = \mathbf{0}_{n_x \times 1}$. Appendix A provides further details.

3.2 A Joint Model of GDP-Linked Bond Yields and Equity Yields

If we observed yields on a cross-section of GDP-linked bond yields, we could estimate the model set out in Section 3.1 using standard approaches for estimating ATSMs. But the fact that we do not observe GDP-linked bond yields means that we must employ an alternative approach. The option we consider is to use a joint term structure model of GDP-linked bond yields and equity yields.[6] In the joint model, the pricing of GDP-linked bonds is exactly the same as set out in Section 3.1. In the remainder of this section we therefore discuss the pricing of zero-coupon equities, which requires us to make only one additional assumption.

A k-period zero-coupon equity is one which pays a dividend D_{t+k} at maturity[7]. Under the assumption of no-arbitrage, the time-t price of a zero-coupon equity ($P_{t,d}^{(k)}$) scaled by the current level of dividends is given by

$$\frac{P_{t,d}^{(k)}}{D_t} = \mathbb{E}_t \left[\prod_{i=1}^{k} M_{t+i} \frac{D_{t+k}}{D_t} \right] = \mathbb{E}_t \left[M_{t+1} \exp\left(\Delta d_{t+1}\right) \frac{P_{t+1,d}^{(k-1)}}{D_{t+1}} \right], \tag{17}$$

where $\Delta d_{t+1} = \log D_{t+1} - \log D_t$ is the one-period dividend growth rate. Equivalently, under the $\mathbb{Q}$ measure, we can write

$$\frac{P_{t,k}^{(n)}}{D_t} = \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-r_t + \Delta d_{t+1}\right) \frac{P_{t+1,d}^{(k-1)}}{D_{t+1}} \right]. \tag{18}$$

In addition to the assumptions set out in Section 3.1, the dividend growth rate is an affine function of the pricing factors, that is,

[6] Although this specific joint model has not been considered previously, several studies have considered analogous joint models of conventional yields in multiple countries (for example, Anderson, Hammond and Ramezani (2010)) or of nominal and real yields within a single country (for example, D'Amico, Kim and Wei (2014) and Abrahams et al. (2016)). Besides the different type of yields being modelled, the difference between our paper and these previous studies is that we use only data on one class of yields, together with observed macroeconomic data, to infer all of the parameters of the model.

[7] We describe the construction of these zero-coupon equity yields in Section 4.3. The seminal reference is van Binsbergen et al. (2013).

$$\Delta d_t = \delta_0 + \boldsymbol{\delta}_1' \mathbf{x}_t, \tag{19}$$

where δ_0 is a scalar and $\boldsymbol{\delta}_1$ is an $n_x \times 1$ vector. With this one additional assumption, the equity yield—defined as $y_{t,d}^{(k)} = -\frac{1}{k}\left(\log P_{t,d}^{(k)} - \log D_t\right)$ following van Binsbergen et al. (2013)—is an affine function of the factors, that is,

$$y_{t,d}^{(k)} = -\frac{1}{k}\left(a_{k,d} + \mathbf{b}_{k,d}' \mathbf{x}_t\right), \tag{20}$$

where $a_{k,d}$ and $\mathbf{b}_{k,d}$ follow the recursive equations

$$a_{k,d} = -\rho_0 + \delta_0 + a_{k-1,d} + \left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right)' \boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right)' \boldsymbol{\Sigma}\boldsymbol{\Sigma}' \left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right) \text{ and} \tag{21}$$

$$\mathbf{b}_{k,d}' = -\boldsymbol{\rho}_1' + \left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right)' \boldsymbol{\Phi}^{\mathbb{Q}}, \tag{22}$$

with boundary conditions $a_{0,d} = 0$ and $\mathbf{b}_{0,d} = \mathbf{0}_{n_x \times 1}$. Appendix B provides further details.

4 Estimation, Identification and Data

We now turn to the question of how we can estimate this joint model in the absence of data on GDP-linked bond yields. As a preliminary step, in Section 4.1, we show how to eliminate the short-term nominal interest rate r_t from the model, leaving a model that we can estimate without any information from conventional bond yields (which are constrained by the zero-lower bound for much of our sample). In Section 4.2, we show how to estimate the model parameters using only data on equity yields and GDP growth. In Section 4.3, we describe the data set we use to estimate the model.

4.1 Eliminating the Nominal Short Rate from the Model

As currently formulated, both GDP-linked bond yields (in equation (12)) and equity yields (in equation (18)) depend on the yield on a one-period nominal short rate (r_t). However, over most of our sample period the nominal short rate has been at the zero lower bound, which means that a Gaussian model for the short rate is likely to suffer from problems of misspecification (see, for example, Bauer and Rudebusch (2016)). In practice, therefore, we reformulate equations (12)) and ((18) to solve out the nominal short rate. As we show in Appendix B, we can equivalently price zero-coupon equities according to

$$\frac{P_{t,d}^{(k)}}{D_t} = \mathbb{E}_t^{\mathbb{Q}_d}\left[\exp\left(-r_{t,d}\right)\frac{P_{t+1,d}^{(k-1)}}{D_{t+1}}\right] \tag{23}$$

where $\mathbb{Q}_d$ denotes the equivalent probability measure for pricing zero-coupon equities when we discount payoffs using the short-term equity yield

$$r_{t,d} = \rho_{d,0} + \boldsymbol{\rho}_{d,1}'\mathbf{x}_t, \tag{24}$$

with $\rho_{d,0} = \rho_0 - \delta_0 - \boldsymbol{\delta}_1'\boldsymbol{\mu}^{\mathbb{Q}} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1$ and $\boldsymbol{\rho}_{d,1} = \boldsymbol{\rho}_1 - \boldsymbol{\Phi}^{\mathbb{Q}\prime}\boldsymbol{\delta}_1$; and where the factors follow the law of motion under the $\mathbb{Q}_d$ measure

$$\mathbf{x}_{t+1} = \boldsymbol{\mu}^{\mathbb{Q}_d} + \boldsymbol{\Phi}^{\mathbb{Q}_d}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}_d}, \tag{25}$$

with $\boldsymbol{\mu}^{\mathbb{Q}_d} = \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1$ and $\boldsymbol{\Phi}^{\mathbb{Q}_d} = \boldsymbol{\Phi}^{\mathbb{Q}}$. As we show in Appendix C, equity yields are equivalently given by

$$y_{t,d}^{(k)} = -\frac{1}{k}\left(a_{k,d}^{*} + \mathbf{b}_{k,d}^{*\prime}\mathbf{x}_t\right), \tag{26}$$

where $a_{k,d}$ and $\mathbf{b}_{k,d}$ follow the recursive equations

$$a^*_{k,d} = -\rho_{d,0} + a^*_{k-1,d} + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\mu}^{\mathbb{Q}_d} + \frac{1}{2}\mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{b}^*_{k-1,d} \text{ and} \quad (27)$$

$$\mathbf{b}^{*\prime}_{k,d} = -\boldsymbol{\rho}'_{d,1} + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Phi}^{\mathbb{Q}_d}, \quad (28)$$

with boundary conditions $a^*_{0,d} = 0$ and $\mathbf{b}^*_{0,d} = \mathbf{0}_{n_x \times 1}$.

If we price GDP-linked bonds under the $\mathbb{Q}_d$ measure, we have

$$P^{(k)}_{t,g} = \mathbb{E}^{\mathbb{Q}_d}_t \left[\exp\left(-r_{d,t} + g_{t+1}\right) P^{(k-1)}_{t+1,g}\right].$$

The yield on a GDP-linked bond is therefore equivalently given by

$$y^{(k)}_{t,g} = -\frac{1}{k}\left(a^*_{k,g} + \mathbf{b}^{*\prime}_{k,g}\mathbf{x}_t\right), \quad (29)$$

where $a_{k,g}$ and $\mathbf{b}_{k,g}$ follow the recursive equations

$$a^*_{k,g} = -\rho_{d,0} + g_0 + a^*_{k-1,g} + \left(\mathbf{g}_1 + \mathbf{b}^*_{k-1,g}\right)'\boldsymbol{\mu}^{\mathbb{Q}_d} + \frac{1}{2}\left(\mathbf{g}_1 + \mathbf{b}^*_{k-1,g}\right)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\left(\mathbf{g}_1 + \mathbf{b}^*_{k-1,g}\right) \text{ and} \quad (30)$$

$$\mathbf{b}^{*\prime}_{k,g} = -\boldsymbol{\rho}'_{d,1} + \left(\mathbf{g}_1 + \mathbf{b}^*_{k-1,g}\right)'\boldsymbol{\Phi}^{\mathbb{Q}_d} \quad (31)$$

with boundary conditions $a^*_{0,g} = 0$ and $\mathbf{b}^*_{0,g} = \mathbf{0}_{n_x \times 1}$. Thus, we can parameterize a joint model of equity and GDP-linked bond yields in terms of the parameters $\boldsymbol{\Theta} = \left\{\boldsymbol{\mu}, \boldsymbol{\Phi}, \boldsymbol{\mu}^{\mathbb{Q}_d}, \boldsymbol{\Phi}^{\mathbb{Q}_d}, \boldsymbol{\Sigma}, \rho_{d,0}, \boldsymbol{\rho}_{d,1}, g_0, \mathbf{g}_1\right\}$.

As is well-known (see, for example, Dai and Singleton (2000), Joslin, Singleton and Zhu (2011), and Hamilton and Wu (2012)), we must impose additional identifying restrictions on the parameters of an ATSM to ensure identification. Following Hamilton and Wu (2012), we impose the restrictions $\boldsymbol{\mu} = \mathbf{0}$, that $\boldsymbol{\Phi}^{\mathbb{Q}_d}$ is diagonal with $\phi^{\mathbb{Q}_d}_{11} \geq \phi^{\mathbb{Q}_d}_{22} \geq \ldots \phi^{\mathbb{Q}_d}_{n_x n_x}$, and $\boldsymbol{\Sigma} = \mathbf{I}$.

4.2 Maximum-Likelihood Estimator

The standard approach for estimating a joint ATSM of yields on two classes of assets is to use data from both of those asset classes. For example, suppose that we observed a vector of equity yields $\mathbf{y}_{t,d} = \left[y_{t,d}^{(n_{d,1})}, y_{t,d}^{(n_{d,2})}, ..., y_{t,d}^{(n_{d,D})} \right]'$ and a vector of GDP-linked bond yields $\mathbf{y}_{t,g} = \left[y_{t,g}^{(n_{g,1})}, y_{t,g}^{(n_{g,2})}, ..., y_{t,g}^{(n_{g,G})} \right]'$. With the additional assumption that all yields are measured with error, we can compute a maximum likelihood estimator of the free parameters. For example, the measurement equation of the joint model could be given by

$$\begin{bmatrix} \mathbf{y}_{t,d} \\ \mathbf{y}_{t,g} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_d \\ \mathbf{A}_g \end{bmatrix} + \begin{bmatrix} \mathbf{B}_d \\ \mathbf{B}_g \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} \mathbf{w}_{t,d} \\ \mathbf{w}_{t,g} \end{bmatrix}, \tag{32}$$

where the definitions of $\mathbf{A}_d$, $\mathbf{A}_g$, $\mathbf{B}_d$, and $\mathbf{B}_g$ follow from equations (27), (28), (30), and (31); and $\mathbf{w}_{t,d} \sim \mathcal{NID}\left(\mathbf{0}, \sigma_{w_d}^2 \times \mathbf{I}\right)$ and $\mathbf{w}_{t,g} \sim \mathcal{NID}\left(\mathbf{0}, \sigma_{w_g}^2 \times \mathbf{I}\right)$.

However, because we do not observe yields on GDP-linked bonds, we must adopt an alternative approach. We instead include proxies for current GDP growth and expectations of future GDP growth at various horizons in the measurement equation, alongside equity yields, that is,

$$\begin{bmatrix} \mathbf{y}_{t,d} \\ \mathbf{s}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A}_d \\ \mathbf{s}_0 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_d \\ \mathbf{S}_1 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} \mathbf{w}_{d,t} \\ \boldsymbol{\eta}_t \end{bmatrix}. \tag{33}$$

Here $\mathbf{s}_t = \left[g_t, \overline{g}_{t+25,t+36}, \overline{g}_{t+37,t+48}, \overline{g}_{t+49,t+60}, \overline{g}_{t+61,t+120} \right]'$, where $\overline{g}_{t+h,t+i}$ is the expected average growth rate of nominal GDP between periods $t+h$ and $t+i$, that is, $\overline{g}_{t+h,t+i} = \mathbb{E}_t \left[\frac{1}{12} \sum_{m=0}^{11} g_{t+h+m} \right]$. That is, we include average GDP expectations of the following horizons: 3 years ahead, 4 years ahead, 5 years ahead and 6 to 10 years ahead. We assume that that measurement error on $\mathbf{s}_t$ is given by $\boldsymbol{\eta}_t \sim \mathcal{NID}\left(\mathbf{0}, \sigma_\eta^2 \times \mathbf{I}\right)$. Strictly speaking, we could identify the parameters g_0 and $\mathbf{g}_1$ if we only included g_t in the measurement equation alongside equity yields and omitted the surveys of future GDP growth. However, including the expectations of future GDP growth provides additional observations with which to infer

g_0 and $\mathbf{g}_1$, in addition to the $\mathbb{P}$ dynamics of the pricing factors in a short sample (similar to the rationale for including surveys of nominal interest rates in ATSMs of conventional yields proposed by Kim and Orphanides (2012)).

Equations (11) and (33) make up a linear-Gaussian state-space system, so we can estimate the free parameters of the model by maximum likelihood, using the Kalman filter to evaluate the likelihood function. Then with the resulting estimates of the pricing factors and parameters we can compute predicted GDP-linked bond yields according to (29).

It is worth stressing the key assumption that underlies our estimation approach: it implicitly assumes that GDP growth is fully "spanned" by equity yields. To understand the importance of the spanning assumption, suppose that we can partition the factors as $\mathbf{x}_t = \left[\mathbf{x}'_{t,s}, \mathbf{x}'_{t,u}\right]'$, where $\mathbf{x}_{t,s}$ is a vector of $n_x - n_u$ factors spanned by equity yields and $\mathbf{x}_{t,u}$ is a vector of n_u factors unspanned by equity yields. From equation (28), we can see that $\mathbf{x}_{t,u}$ will be unspanned by equity yields if the loadings of the short-term equity yields take the form $\boldsymbol{\rho}_{d,1} = \left[\boldsymbol{\rho}'_{d,1,s}, \mathbf{0}'_{n_u \times 1}\right]'$ and if we can partition $\mathbf{\Phi}^{\mathbb{Q}_d}$ conformably as

$$\mathbf{\Phi}^{\mathbb{Q}_d} = \begin{bmatrix} \mathbf{\Phi}^{\mathbb{Q}_d}_{ss} & \mathbf{0} \\ \mathbf{\Phi}^{\mathbb{Q}_d}_{us} & \mathbf{\Phi}^{\mathbb{Q}_d}_{uu} \end{bmatrix}.$$

Under these conditions, the parameters $\mathbf{\Phi}^{\mathbb{Q}_d}_{us}$ and $\mathbf{\Phi}^{\mathbb{Q}_d}_{uu}$ will be unidentified, that is, we cannot infer the dynamics of $\mathbf{x}_{t,u}$ from equity yields and we cannot use the model to infer GDP-linked bond yields. Implicitly, we are assuming that these zero restrictions do not hold in practice.[8]

[8] As pointed out by Duffee (2011) in the context of ATSMs of conventional bond yields, they are "knife-edge" restrictions: in the absence of measurement error on equity yields, even a tiny non-zero loading of the short-term equity yield on GDP growth would mean that we can fully identify $\mathbf{\Phi}^{\mathbb{Q}_d}$. However, in practice, in the presence of measurement error the loading may be sufficiently small that GDP growth is "partly hidden" (borrowing another term from Duffee (2011))—that is, the dynamics of GDP growth under the $\mathbb{Q}_g$ measure may be estimated extremely imprecisely.

4.3 Data and Number of Pricing Factors

We estimate our model of equity yields using end-month U.S. zero-coupon equity yields with maturities of one, two, three, five, seven, and ten years over a sample period from January 2010 to June 2017.[9] We estimate these zero-coupon equity yields from S&P 500 dividend futures prices provided by Goldman Sachs and dividend indices published by Datastream, following the approach of van Binsbergen et al. (2013). Figure 1 plots equity yields at selected maturities. As shown in Table 1, the average equity yield curve is upward sloping, with standard deviations that decrease with maturity. Equity yields are highly persistent, with first-order autocorrelation coefficients close to one. And, as shown in Table 2, three principal components are sufficient to explain about 99.9 percent of the variation in the cross-section of equity yields, which is similar to the well-known result for conventional bond yields (see, for example, Litterman and Scheinkman (1991)). As is standard in the literature on no-arbitrage models of conventional bonds, we therefore adopt a three-factor specification for our models of equity yields.

<Table 1 about here>

<Figure 1 about here>

<Table 2 about here>

As a proxy for current-month nominal GDP growth, we add together survey-based measures of expected current-quarter real GDP growth and growth in the GDP deflator; specifically, we use the mean expected current-quarter growth rates in real GDP growth and the GDP deflator reported by respondents to monthly Blue Chip Economic Indicators surveys. This survey-based proxy measure of nominal GDP growth has two important advantages relative to official GDP data. First, the survey-based measure is available at a monthly frequency, rather than at the quarter frequency with which official data are published. Second, as shown in Figure 2, the survey-based measure smoothes out much of the high-frequency

[9] There are a small number of missing observations for the one- and ten-year equity yields, which is easily dealt with using the Kalman filter.

volatility in official GDP data while still capturing the low-frequency variation. While the Kalman filter should in principle allow the model to determine how much of the variation in the official data represents genuine low-frequency movements in underlying GDP growth and how much reflects noise in the form of measurement error, preliminary analysis showed that when we attempt to estimate the models using official GDP data (at a quarterly frequency) they tend to suffer from one of two problems. In some cases, the models failed to fit a material part of the variation in observed GDP growth—i.e. they interpreted almost all of the variation in GDP growth, including the low-frequency variation, as measurement error. In turn, this implied that the loadings of GDP growth on the factors in (7) were extremely imprecisely estimated. In other cases, the models devoted a single factor to fitting nominal GDP growth extremely closely—i.e. they interpreted essentially none of the variation as measurement error—with this "GDP growth factor" being only very weakly correlated with variation in observed asset prices. In turn, this implied that the $\mathbb{Q}$ dynamics of this factor were extremely imprecisely estimated. Thus in both cases, predicted GDP-linked bond prices were often very imprecisely estimated and point estimates tended to be highly implausible. In practice, we found that using the smoother survey-based proxy resulted in much more plausible point estimates of GDP risk premia.

<Figure 2 about here>

The survey-based measures of longer-term GDP growth expectations are taken from the semi-annual Blue Chip Economic Indicators long-range surveys of nominal GDP. In March and October of each year, Blue Chip ask survey respondents to report their expectations of nominal GDP growth for future calendar years; we interpolate between the mean responses to compute measures of expectations at fixed horizons. We then interpolate over time in order to obtain a monthly series. While this final step is strictly speaking unnecessary, we found in preliminary analysis that we needed to provide more information in order to ensure that expectations of GDP in months without a survey remained plausible.

5 Results

In this section we present results from the estimation of our joint model of GDP-linked bond yields and equity yields. In Section 5.1 we analyse the model fit of both equity yields and GDP growth. In Section 5.2 we focus on GDP-linked bonds: we report results for their predicted yields, "breakevens" with respect to conventional government bonds, and the decomposition of the breakeven into GDP growth expectations and GDP risk premia.

5.1 The Impact of Including GDP growth

Table 3 shows root mean squared errors (RMSEs) between observed and model-implied equity yields, both for our joint model and for a simpler model estimated only using data on equity yields (that is, without using data on current and expected future GDP growth, and therefore not estimating the parameters g_0 and $\mathbf{g}_1$). A model estimated using only data on equity yields achieves a tight fit to the data, with RMSEs between about 3 and 10 basis points, depending on the maturity. Figure 4 shows our joint model that includes data on current and expected future GDP growth closely matches the proxy for current GDP growth. As we might expect, the inclusion of data on GDP growth—which are additional series to be matched with the same number of factors—worsens the fit of the model to equity yields, with RMSEs for equity yields rising to between about 25 and 50 basis points. Nevertheless, the model can still fit the broad movements in equity yields, as shown in Figure 3, which shows the observed and model-implied equity yields at selected maturities.

<Table 3 around here>

<Figures 3 and 4 about here>

5.2 Predicted GDP-Linked Bond Yields and GDP risk premia

We now turn to the relative cost of borrowing using conventional and GDP-linked bonds, as measured by the breakeven rate between the two yields. To compute the breakeven, we subtract the model-implied GDP-linked bond yield from the observed government bond yield with the same maturity.[10] As described in Section 2, we can decompose the breakeven rate into the expectation of average GDP growth over the life of the underlying bond, and an additional GDP risk premium that compensates investors for exposure to GDP growth risk. Figure 5 shows the evolution of this breakeven rate at selected maturities over the sample. Breakeven rates are positive throughout; that is, predicted GDP-linked bond yields are lower than yields of conventional bonds. In fact, GDP-linked bond yields are mostly negative in our sample. Additionally, Figure 5 also shows that breakeven rates are higher at longer maturities.[11]

<Figure 5 around here>

To decompose the breakevens into expected average GDP growth the the GDP risk premium, we subtract the model-implied expected average GDP growth from the breakeven computed above.[12] Figures 6 and 7 show this decomposition for two- and seven-year GDP-linked bonds respectively. Average GDP growth expectations are sensible, at around 4.5 percent, and are relatively stable over time. If anything, model-implied expected GDP growth displays a slight downward trend over the sample, with the average GDP growth across maturities falling from almost 5% to around 4.2% over the sample. Estimated GDP risk premiums are also positive, but are more volatile over time and vary more signifcantly across maturities. For longer term horizons, probably the most relevant for sovereigns considering the issuance of GDP-linked debt, GDP risk premiums range between 0 and 2 percent over the sample period (Figure 7). At shorter horizons the estimated premia are generally slightly larger in magnitude.

<Figures 6 and 7 around here>

[10] Here, we use the estimated zero-coupon Treasury yields presented by Gurkaynak, Sack and Wright (2007) and updated by the staff of the Board of Governors of the Federal Reserve System.

[11] This is typically true in the sample, independently of the particular maturities displayed in Figure 5.

[12] GDP growth expectations can be easily solved for using Equations 7 and 11.

6 A Model of Conventional and GDP-Linked Bond Yields

Zero-coupon equities are just one asset class that we could have used in order to infer the prices of GDP-linked bonds. Indeed, perhaps the more obvious choice would have been to use conventional government bonds. However, in this section, we show that an analogous joint model of conventional and GDP-linked bonds produces highly implausible GDP-linked bond yields, which is likely a reflection of GDP growth being only very weakly spanned by conventional bond yields (see, for example, Joslin, Priebsch and Singleton (2014) and Bauer and Rudebusch (2017)).

The rest of this section proceeds as follows. In Section 6.1 we set out a joint model of conventional and GDP-linked bond yields and explain how we estimate it using data on observed Treasury yields, GDP growth, and surveys of expected future GDP growth. In Section 6.2 we show that the predictions of the model for GDP-linked bond yields are highly implausible.

6.1 A Joint Model of Conventional and GDP-Linked Bond Yields

A conventional zero-coupon bond is one that pays a fixed nominal cash-flow of \$1 at maturity. Under the assumption of no arbitrage, the time-t price ($P_t^{(n)}$) of an n-period conventional bond is given by

$$P_t^{(n)} = \mathbb{E}_t\left[M_{t+1}P_{t+1}^{(n-1)}\right], \tag{34}$$

or, equivalently, by

$$P_t^{(n)} = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t\right)P_{t+1}^{(n-1)}\right]. \tag{35}$$

Under the assumptions set out in Section 3.1, the yield on an n-period conventional bond, $y_t^{(n)} = -\frac{1}{n}\log P_t^{(n)}$, is an affine function of the pricing factors, that is,

$$y_t^{(n)} = -\frac{1}{n}\left(a_n + \mathbf{b}_n'\mathbf{x}_t\right), \tag{36}$$

where a_n and $\mathbf{b}_n$ follow the recursive equations

$$a_n = a_{n-1} + \mathbf{b}_{n-1}'\boldsymbol{\mu}^{\mathbb{Q}} + \frac{1}{2}\mathbf{b}_{n-1}'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{b}_{n-1} - \rho_0 \text{ and} \tag{37}$$

$$\mathbf{b}_n' = \mathbf{b}_{n-1}'\boldsymbol{\Phi}^{\mathbb{Q}} - \boldsymbol{\rho}_1', \tag{38}$$

with boundary conditions $a_0 = 0$ and $\mathbf{b}_0 = \mathbf{0}_{n_x\times 1}$.

This is an entirely standard ATSM of conventional bond yields (see, for example, Duffee (2002)). We can parameterize the model of conventional yields in terms of the parameters $\left\{\rho_0, \boldsymbol{\rho}_1, \boldsymbol{\mu}^{\mathbb{Q}}, \boldsymbol{\Phi}^{\mathbb{Q}}, \boldsymbol{\Sigma}, \boldsymbol{\mu}, \boldsymbol{\Phi}\right\}$ and estimate these parameters using an $n_y \times 1$ vector of observed yields $\mathbf{y}_t = \left[y_t^{(n_1)}, y_t^{(n_2)}, ..., y_t^{(n_y)}\right]'$. However, if we are to use the model to price GDP-linked bonds we also need estimates of g_0 and $\mathbf{g}_1$. We therefore stack together the conventional bond yields and observed GDP growth into a single measurement equation, that is,

$$\begin{bmatrix} \mathbf{y}_t \\ g_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ g_0 \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{g}_1' \end{bmatrix}\mathbf{x}_t + \begin{bmatrix} \mathbf{w}_t \\ \eta_t \end{bmatrix}, \tag{39}$$

where the definitions of $\mathbf{A}$ and $\mathbf{B}$ follow from equations (37) and (38), respectively, $\mathbf{w}_t \sim \mathcal{NID}\left(\mathbf{0}, \sigma_w^2 \times \mathbf{I}_{n_y}\right)$ is a vector of measurement errors on conventional yields, $\eta_t \sim \mathcal{NID}\left(0, \sigma_\eta^2\right)$ is a measurement error on GDP growth, and the remaining terms are defined as in equation (33).

For identification, we impose the restrictions $\boldsymbol{\mu}^{\mathbb{Q}} = \mathbf{0}$, that $\boldsymbol{\Phi}^{\mathbb{Q}}$ is diagonal with $\phi_{11}^{\mathbb{Q}} \geq \phi_{22}^{\mathbb{Q}} \geq ...\phi_{n_x n_x}^{\mathbb{Q}}$, and $\boldsymbol{\Sigma} = \mathbf{I}$. Noting that equations (39) and (11) make up a linear-Gaussian state-space system, we can estimate the remaining free parameters of the model by maximum likelihood, using the Kalman filter to evaluate the likelihood function. With the resulting estimates of the pricing factors and the parameters, we can compute predicted GDP-linked bond yields according to equation (14).

6.2 Results from the Model Using Conventional Bond Yields

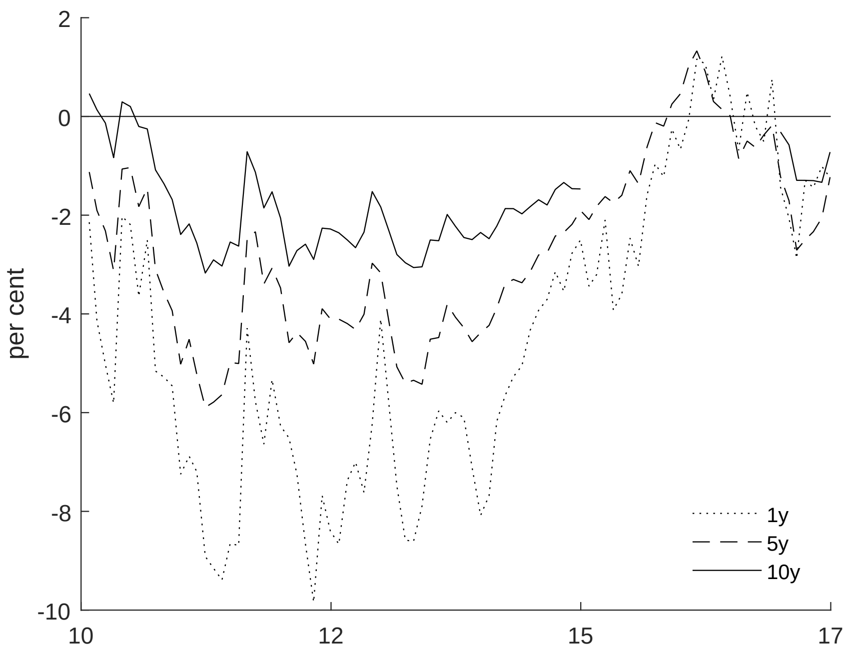
[Placeholder]

7 Conclusions

This paper uses a no-arbitrage term structure model of equity yields computed from the prices of dividend swaps to estimate the possible yields on hypothetical bonds with cash-flows indexed to the level of U.S. GDP. This novel approach for estimating the relative cost of conventional and GDP-linked bonds avoids many of the pitfalls of previous approaches. In particular, our approach uses a flexible specification of investors' preferences which is known to capture the risk premiums on existing assets; it avoids the need to make possibly unrealistic assumptions about the covariance of returns on GDP-linked bonds with a market portfolio; and it provides an estimate of GDP risk premiums that varies with maturity. In short, our model predicts that U.S. GDP-linked bonds would have yields lower than those on conventional Treasury bonds with the same maturity in our sample from 2010 to 2017. Positive expected future GDP growth lowers the yield on GDP-linked bonds relative to conventional bonds, which more than offsets the estimated GDP risk premium demanded by investors for holding GDP risk.

8 Tables and Figures

Figure 1: Equity yields



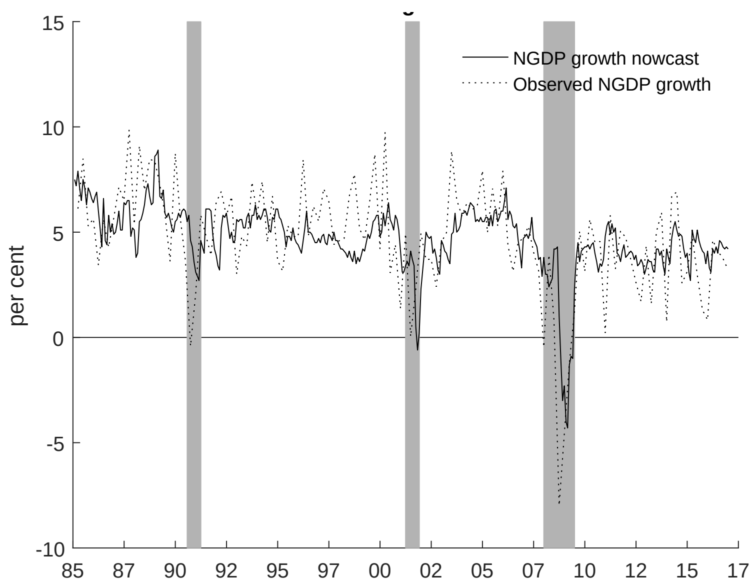
Note. Equity yields are constructed using dividend swaps prices from Goldman Sachs Global Investment Research.

Table 1: Equity yields summary statistics

Maturities	1y	2y	3y	5y	7y	10y
Mean (%)	-4.44	-4.47	-3.77	-2.73	-2.02	-1.81
Stdev (%)	3.01	2.63	2.31	1.81	1.55	0.93
AR(1)	0.92	0.93	0.93	0.94	0.94	0.87

Note: Monthly data on US spot equity yields are constructed using dividend swaps prices from Goldman Sachs Global Investment Research and dividend indices from Datastream, based on the S&P 500. The sample period is from January 2010 to June 2017.

Figure 2: NGDP GROWTH: OBSERVED AND NOWCAST



NOTE. NGDP nowcasts are from Wolters Kluwer Legal and Regulatory Solutions U.S. Blue Chip Economic Indicators. Observed NGDP is from FRED

Table 2: EQUITY YIELDS PRINCIPAL COMPONENT ANALYSIS

Cumulative variance explained by:		
PC1	PC2	PC3
94.54	98.63	99.87

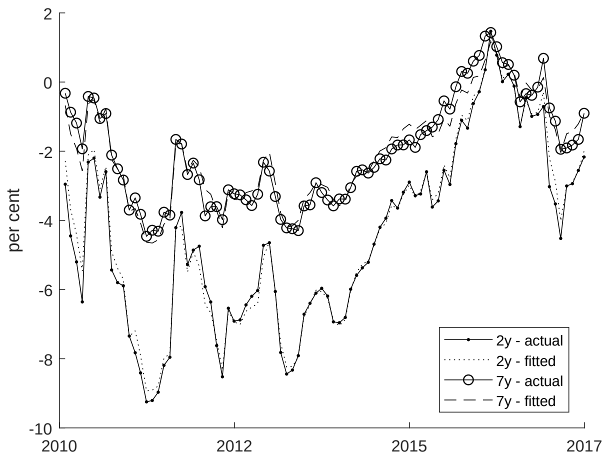
NOTE: Monthly data on US spot equity yields are constructed using dividend swaps prices from Goldman Sachs and dividend indices from Datastream, based on the S&P 500. The sample period is from January 2010 to June 2017.

Table 3: Equity Yields Fitting Errors

Maturities	1y	2y	3y	5y	7y	10y	g_t
Yields only	2.9	10.2	9.3	7.8	5.5	8.6	
With GDP growth	51.5	29.1	27.1	24.8	32.5	40.7	1.1

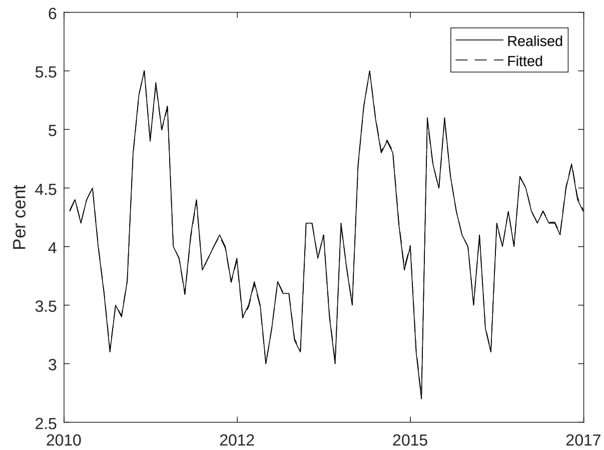
Notes. All units are in basis points. Reported fitting errors are Root Mean Squared Errors (RMSEs)

Figure 3: Fit of Equity Yields



Notes. Observed equity yields are constructed using dividend swaps prices from Goldman Sachs Global Investment Research and dividend indices from Datastream.

Figure 4: Fit of g_t

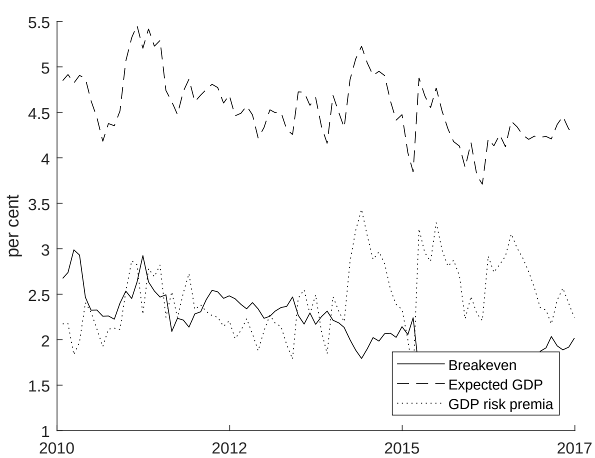


Notes. NGDP nowcasts are from Wolters Kluwer Legal and Regulatory Solutions U.S. Blue Chip Economic Indicators.

Figure 5: GDP-lnked bonds breakevens

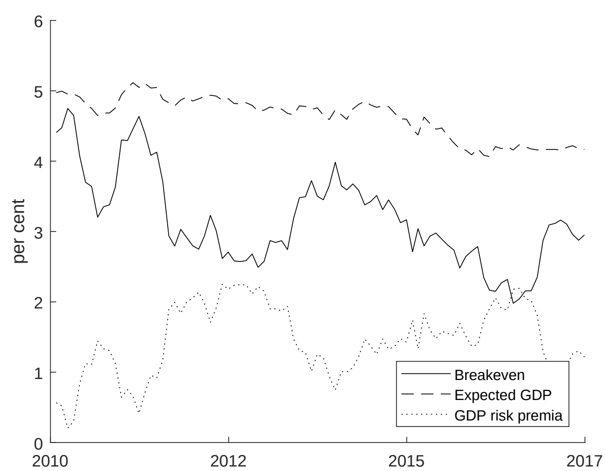
Note Nominal bonds data used to compute breakevens come from the Federal Reserve Board.

Figure 6: 2-Y GDP-LINKERS BREAKEVEN RATE DECOMPOSITION



Notes. Nominal bonds data used to compute breakevens come from the Federal Reserve Board

Figure 7: 7-Y GDP-LINKERS BREAKEVEN RATE DECOMPOSITION



Notes. Nominal bonds data used to compute breakevens come from the Federal Reserve Board

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Appendix A: Solution for GDP-Linked Bond Yields

In this appendix we derive the solution for yields on GDP-linked bonds in equations (14)-(16). We guess that the solution for prices takes the form

$$P_{t,g}^{(k)} = \exp\left(a_{k,g} + \mathbf{b}_{k,g}' \mathbf{x}_t\right).$$

Substituting this equation into equation (12) gives

$$\exp\left(a_{k,g} + \mathbf{b}_{k,g}' \mathbf{x}_t\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t + g_{t+1}\right)\exp\left(a_{k-1,g} + \mathbf{b}_{k-1,g}' \mathbf{x}_{t+1}\right)\right]$$

Taking logs and combining with equations (7), (9), and (13) gives

$$\begin{aligned}
a_{k,g} + \mathbf{b}_{k,g}' \mathbf{x}_t &= \log \mathbb{E}_t^{\mathbb{Q}} \begin{bmatrix} \exp\left(-\rho_0 - \boldsymbol{\rho}_1' \mathbf{x}_t + g_0 + \mathbf{g}_1'\left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right) \\ \exp\left(a_{k-1,g} + \mathbf{b}_{k-1,g}'\left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right) \end{bmatrix} \\
&= -\rho_0 - \boldsymbol{\rho}_1' \mathbf{x}_t + g_0 + a_{k-1,g} + \log \mathbb{E}_t^{\mathbb{Q}} \begin{bmatrix} \exp\left(\mathbf{g}_1'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{g}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \mathbf{g}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right) \\ \exp\left(\mathbf{b}_{k-1,g}'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{b}_{k-1,g}'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \mathbf{b}_{k-1,g}'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right) \end{bmatrix} \\
&= -\rho_0 - \boldsymbol{\rho}_1' \mathbf{x}_t + g_0 + a_{k-1,g} + \mathbf{g}_1'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{g}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \mathbf{b}_{k-1,g}'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{b}_{k-1,g}'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t \\
&\quad + \log \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right] \\
&= -\rho_0 - \boldsymbol{\rho}_1' \mathbf{x}_t + g_0 + a_{k-1,g} + \left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\mu}^{\mathbb{Q}} + \left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t \\
&\quad + \frac{1}{2}\left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\left(\mathbf{g}_1 + \mathbf{b}_{k-1,g}\right).
\end{aligned}$$

Matching coefficients gives equations (15) and (16) in the main text. The boundary conditions $a_{0,g} = 0$ and $\mathbf{b}_{k,g} = \mathbf{0}$ follow from the fact that the time-t price of a zero-period bond paying one dollar at maturity must be one dollar.

Appendix B: Solution for Equity Yields

In this appendix we derive the solution for equity yields in equations (20)-(22). We guess that the solution for prices takes the form

$$\frac{P_{t,d}^{(k)}}{D_t} = \exp\left(a_{k,d} + \mathbf{b}_{k,d}'\mathbf{x}_t\right).$$

Substituting this equation into equation (18) gives

$$\exp\left(a_{k,d} + \mathbf{b}_{k,d}'\mathbf{x}_t\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_t + \Delta d_{t+1}\right)\exp\left(a_{k-1,d} + \mathbf{b}_{k-1,d}'\mathbf{x}_{t+1}\right)\right]$$

Taking logs and combining with equations (19), (9), and (13) gives

$$\begin{aligned}
a_{k,d} + \mathbf{b}_{k,d}'\mathbf{x}_t &= \log \mathbb{E}_t^{\mathbb{Q}}\left[\begin{array}{c}\exp\left(-\rho_0 - \boldsymbol{\rho}_1'\mathbf{x}_t + \delta_0 + \boldsymbol{\delta}_1'\left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right)\\ \exp\left(a_{k-1,d} + \mathbf{b}_{k-1,d}'\left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right)\end{array}\right]\\
&= -\rho_0 - \boldsymbol{\rho}_1'\mathbf{x}_t + \delta_0 + a_{k-1,d} + \log \mathbb{E}_t^{\mathbb{Q}}\left[\begin{array}{c}\exp\left(\boldsymbol{\delta}_1'\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\delta}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\\ \exp\left(\mathbf{b}_{k-1,d}'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{b}_{k-1,d}'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \mathbf{b}_{k-1,d}'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\end{array}\right]\\
&= -\rho_0 - \boldsymbol{\rho}_1'\mathbf{x}_t + \delta_0 + a_{k-1,d} + \boldsymbol{\delta}_1'\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\delta}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \mathbf{b}_{k-1,d}'\boldsymbol{\mu}^{\mathbb{Q}} + \mathbf{b}_{k-1,d}'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t\\
&\quad + \log \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(\left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right)'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}\right)\right]\\
&= -\rho_0 - \boldsymbol{\rho}_1'\mathbf{x}_t + \delta_0 + a_{k-1,g} + \left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\mu}^{\mathbb{Q}} + \left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,g}\right)'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t\\
&\quad + \frac{1}{2}\left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right)'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\left(\boldsymbol{\delta}_1 + \mathbf{b}_{k-1,d}\right).
\end{aligned}$$

Matching coefficients gives equations (**??**) and (**??**) in the main text. The boundary conditions $a_{0,d} = 0$ and $\mathbf{b}_{k,d} = \mathbf{0}$ follow from the fact that the time-t price of a zero-period equity paying D_t dollars at maturity must be D_t dollars.

Appendix C: Alternative Solution for Equity Yields

In this appendix, we show how to derive the equivalent solution of zero-coupon equities provided in Section 3.2.

Note from equations (21) and (22) that we can write the short-term equity yield in the form taken in equation (24), where

$$\begin{aligned} \rho_{d,0} &= \rho_0 - \delta_0 - \boldsymbol{\delta}_1' \boldsymbol{\mu}^{\mathbb{Q}} - \frac{1}{2} \boldsymbol{\delta}_1' \boldsymbol{\Sigma} \boldsymbol{\Sigma}' \boldsymbol{\delta}_1 \text{ and} \quad (40) \\ \boldsymbol{\rho}_{d,1} &= \boldsymbol{\rho}_1 - \boldsymbol{\Phi}^{\mathbb{Q}\prime} \boldsymbol{\delta}_1. \quad (41) \end{aligned}$$

Next, we define $M_{d,t+1} = M_{t+1} \frac{D_{t+1}}{D_t}$. Taking logs and substituting in equations (19) and (8) gives

$$\log M_{d,t+1} = -r_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\varepsilon}_{t+1} + \delta_0 + \boldsymbol{\delta}_1' \mathbf{x}_{t+1}.$$

Substituting in equation (11) gives

$$\log M_{d,t+1} = -r_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\varepsilon}_{t+1} + \delta_0 + \boldsymbol{\delta}_1' \left(\boldsymbol{\mu} + \boldsymbol{\Phi} \mathbf{x}_t + \boldsymbol{\Sigma} \boldsymbol{\varepsilon}_{t+1} \right)$$

Using the mapping between the $\mathbb{P}$ and $\mathbb{Q}$ measures, that is, $\boldsymbol{\mu} = \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Sigma} \boldsymbol{\lambda}_0$ and $\boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathbb{Q}} + \boldsymbol{\Sigma} \boldsymbol{\Lambda}_1$, gives

$$\begin{aligned} \log M_{d,t+1} &= -r_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\varepsilon}_{t+1} + \delta_0 + \boldsymbol{\delta}_1' \left(\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Sigma} \boldsymbol{\lambda}_0 + \left(\boldsymbol{\Phi}^{\mathbb{Q}} + \boldsymbol{\Sigma} \boldsymbol{\Lambda}_1 \right) \mathbf{x}_t + \boldsymbol{\Sigma} \boldsymbol{\varepsilon}_{t+1} \right) \\ &= -r_t - \frac{1}{2} \boldsymbol{\lambda}_t' \boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t' \boldsymbol{\varepsilon}_{t+1} + \delta_0 + \boldsymbol{\delta}_1' \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\delta}_1' \boldsymbol{\Sigma} \boldsymbol{\lambda}_0 + \boldsymbol{\delta}_1' \boldsymbol{\Phi}^{\mathbb{Q}} \mathbf{x}_t + \boldsymbol{\delta}_1' \boldsymbol{\Sigma} \boldsymbol{\Lambda}_1 \mathbf{x}_t + \boldsymbol{\delta}_1' \boldsymbol{\Sigma} \boldsymbol{\varepsilon}_{t+1} \end{aligned}$$

Substituting in the definition of the short rates in equations (9) and (24) gives

$$
\begin{aligned}
\log M_{d,t+1} &= -\rho_0 - \boldsymbol{\rho}_1'\mathbf{x}_t - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1} + \rho_0 - \rho_{d,0} - \boldsymbol{\delta}_1'\boldsymbol{\mu}^{\mathbb{Q}} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_0 \\
&\quad + \boldsymbol{\delta}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -\rho_{d,0} - \left(\boldsymbol{\rho}_{d,1}' + \boldsymbol{\delta}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\right)\mathbf{x}_t - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_0 + \boldsymbol{\delta}_1'\boldsymbol{\Phi}^{\mathbb{Q}}\mathbf{x}_t \\
&\quad + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -\rho_{d,0} - \boldsymbol{\rho}_{d,1}'\mathbf{x}_t - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_0 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -r_{d,t} - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_0 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Lambda}_1\mathbf{x}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}
\end{aligned}
$$

And substituting in the definition of the price of risk $\boldsymbol{\lambda}_t = \boldsymbol{\lambda}_0 + \boldsymbol{\Lambda}_1\mathbf{x}_t$ gives

$$
\log M_{d,t+1} = -r_{d,t} - \frac{1}{2}\boldsymbol{\lambda}_t'\boldsymbol{\lambda}_t - \boldsymbol{\lambda}_t'\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_t + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}
$$

If we further define $\boldsymbol{\lambda}_{d,t} = \boldsymbol{\lambda}_t - \boldsymbol{\Sigma}'\boldsymbol{\delta}_1$ (that is $\boldsymbol{\lambda}_{d,t} = \boldsymbol{\lambda}_{d,0} + \boldsymbol{\Lambda}_{d,1}\mathbf{x}_t$ where $\boldsymbol{\lambda}_{d,0} = \boldsymbol{\lambda}_0 - \boldsymbol{\Sigma}'\boldsymbol{\delta}_1$ and $\boldsymbol{\Lambda}_{d,1} = \boldsymbol{\Lambda}_1$) and substitute this into the previous equation we obtain

$$
\begin{aligned}
\log M_{d,t+1} &= -r_{d,t} - \frac{1}{2}\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right)'\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) - \left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right)'\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 \\
&\quad + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -r_{d,t} - \frac{1}{2}\left(\boldsymbol{\lambda}_{d,t}' + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\right)\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) - \left(\boldsymbol{\lambda}_{d,t}' + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\right)\boldsymbol{\varepsilon}_{t+1} - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 \\
&\quad + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -r_{d,t} - \frac{1}{2}\left(\boldsymbol{\lambda}_{d,t}'\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\left(\boldsymbol{\lambda}_{d,t} + \boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right)\right) - \boldsymbol{\lambda}_{d,t}'\boldsymbol{\varepsilon}_{t+1} - \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&\quad - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_{d,t} + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -r_{d,t} - \frac{1}{2}\left(\boldsymbol{\lambda}_{d,t}'\boldsymbol{\lambda}_{d,t} + \boldsymbol{\lambda}_{d,t}'\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_{d,t} + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1\right) - \boldsymbol{\lambda}_{d,t}'\boldsymbol{\varepsilon}_{t+1} - \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&\quad - \frac{1}{2}\boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\lambda}_{d,t} + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1 + \boldsymbol{\delta}_1'\boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1} \\
&= -r_{d,t} - \frac{1}{2}\boldsymbol{\lambda}_{d,t}'\boldsymbol{\lambda}_{d,t} - \boldsymbol{\lambda}_{d,t}'\boldsymbol{\varepsilon}_{t+1}.
\end{aligned}
$$

Thus, $M_{d,t+1}$ takes an analogous form to M_{t+1}. We can therefore equivalently price zero-coupon equities according to

$$\frac{P_{t,d}^{(n)}}{D_t} = \mathbb{E}_t^{\mathbb{Q}_d}\left[\exp\left(-r_{d,t}\right)\frac{P_{t+1,d}^{(n-1)}}{D_{t+1}}\right], \tag{42}$$

where the factors follow the law of motion in equation (25) under the probability measure $\mathbb{Q}_d$, with

$$\begin{aligned}
\boldsymbol{\mu}^{\mathbb{Q}_d} &= \boldsymbol{\mu} - \boldsymbol{\Sigma}\boldsymbol{\lambda}_{0,d} \\
&= \boldsymbol{\mu}^{\mathbb{Q}} + \boldsymbol{\Sigma}\left(\boldsymbol{\lambda}_0 - \boldsymbol{\lambda}_{0,d}\right) \\
&= \boldsymbol{\mu}^{\mathbb{Q}} - \boldsymbol{\Sigma}\boldsymbol{\Sigma}'\boldsymbol{\delta}_1
\end{aligned}$$

and

$$\begin{aligned}
\boldsymbol{\Phi}^{\mathbb{Q}_d} &= \boldsymbol{\Phi} - \boldsymbol{\Sigma}\boldsymbol{\Lambda}_{1,d} \\
&= \boldsymbol{\Phi}^{\mathbb{Q}} + \boldsymbol{\Sigma}\left(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_{1,d}\right) \\
&= \boldsymbol{\Phi}^{\mathbb{Q}}.
\end{aligned}$$

Finally, we guess the solution for equity yields takes the form

$$\frac{P_{t,d}^{(k)}}{D_t} = \exp\left(a_{k,d}^{*} + \mathbf{b}_{k,d}^{*\prime}\mathbf{x}_t\right).$$

Substituting this into equation (42) gives

$$\exp\left(a_{k,d}^{*} + \mathbf{b}_{k,d}^{*\prime}\mathbf{x}_t\right) = \mathbb{E}_t^{\mathbb{Q}}\left[\exp\left(-r_{d,t}\right)\exp\left(a_{k-1,d}^{*} + \mathbf{b}_{k-1,d}^{*\prime}\mathbf{x}_{t+1}\right)\right]$$

Taking logs and combining with equations (24) and (25) gives

$$\begin{aligned}
a_{k,d} + \mathbf{b}'_{k,d}\mathbf{x}_t &= \log \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(-\rho_{d,0} - \boldsymbol{\rho}'_{d,1}\mathbf{x}_t\right) \exp\left(a^*_{k-1,d} + \mathbf{b}^{*\prime}_{k-1,d} \left(\boldsymbol{\mu}^{\mathbb{Q}_d} + \boldsymbol{\Phi}^{\mathbb{Q}_d}\mathbf{x}_t + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}^{\mathbb{Q}_d}_{t+1} \right)\right)\right] \\
&= -\rho_0 - \boldsymbol{\rho}'_1\mathbf{x}_t + a^*_{k-1,d} + \log \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(\mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\mu}^{\mathbb{Q}_d} + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Phi}^{\mathbb{Q}_d}\mathbf{x}_t + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Sigma}\boldsymbol{\varepsilon}^{\mathbb{Q}_d}_{t+1} \right)\right] \\
&= -\rho_0 - \boldsymbol{\rho}'_1\mathbf{x}_t + a^*_{k-1,d} + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\mu}^{\mathbb{Q}_d} + \mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Phi}^{\mathbb{Q}_d}\mathbf{x}_t + \log \mathbb{E}_t^{\mathbb{Q}} \left[\exp\left(\mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Sigma}\boldsymbol{\varepsilon}^{\mathbb{Q}_d}_{t+1} \right)\right] \\
&= -\rho_0 - \boldsymbol{\rho}'_1\mathbf{x}_t + a^*_{k-1,g} + \mathbf{b}^{*\prime}_{k-1,g}\boldsymbol{\mu}^{\mathbb{Q}_d} + \mathbf{b}^{*\prime}_{k-1,g}\boldsymbol{\Phi}^{\mathbb{Q}_d}\mathbf{x}_t + \frac{1}{2}\mathbf{b}^{*\prime}_{k-1,d}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{b}^*_{k-1,d}.
\end{aligned}$$

Matching coefficients gives equations (26)-(28) in the main text.