Non-cooperative games with prospect theory
players and dominated strategies

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Abstract

We investigate a framework for non-cooperative games in normal form where players have behavioral preferences following Prospect Theory (PT) or Cumulative Prospect Theory (CPT). On theoretical grounds CPT is usually considered to be the superior model, since it normally does not violate first order stochastic dominance in lottery choices. We find, however, that CPT when applied to games may select purely dominated strategies, while PT does not. For both models we also characterize the cases where mixed dominated strategies are preserved and where violations may occur.

Keywords: Prospect theory, framing, reference dependent utility, rank dependent probability weighting, Nash equilibrium, stochastic dominance, dominance of strategies

JEL classification: C70, C73, D81.

1 Introduction

1.1 Games, Utilities and Prospects

The study of games as a cohesive theory has been brought into life by the work of von Neumann starting in 1928 and culminating in his seminal book with Morgenstern (v. Neumann 1928, v. Neumann & Morgenstern 1944). It has since then been widely applied as a prescriptive, but also as a descriptive model. Von Neumann and Morgenstern already observed that a study of games, strategies and outcomes, is only meaningful once we have understood the preferences that the players have on the possible outcomes of a game. They therefore started their considerations of games with a chapter on Expected Utility Theory (EUT), formalizing the ideas which Bernoulli had developed more than 200 years earlier (Bernoulli 1738), and thereby breaking the ground for yet another important foundation of modern economic theory.

The role of this decision model as a normative theory is uncontroversial. Recent years, however, have seen substantial progress on the understanding

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of differences between the actual, sometimes irrational decisions of individuals, and rational decisions according to EUT. There are now several models to describe decisions under risk, e.g. Anticipated Utility as introduced by Quiggin (1982), Rank-Dependent Expected Utility (RDU) as presented in Quiggin (1993), Prospect Theory (PT), as developed by Kahneman & Tversky (1979) and Cumulative Prospect Theory (CPT), an extension of PT Tversky & Kahneman (1992). For the latter two models Daniel Kahneman was awarded with the Nobel Prize in economics in 2002. Since PT and especially CPT are nowadays the most frequently used behavioral decision models, we concentrate on them and describe in the following their key ideas and features.

**Prospect Theory** modifies classical EUT in several ways:

1. Unlike EUT, not the final wealth is evaluated, but the payoffs are framed as gains and losses with respect to a reference point; they are called ‘prospects’. Typically, subjects have been found to be risk-averse in gains and risk-seeking in losses.

2. Close to the reference point, losses loom larger than gains, hence the marginal utility in losses is larger than in gains.

3. Small probabilities are over-weighted and moderate to large probabilities are under-weighted.\(^{1}\)

Mathematically, the first two features are reflected in a two-part S-shaped value function \(v\) (which replaces the usual utility function) – concave in gains and convex in losses and with larger slope for losses than for gains at the reference point. The prototypical example has been given in Tversky & Kahneman (1992) for \(\alpha, \beta \in (0, 1)\) and \(\lambda > 1:\)

\[
v(x) := \begin{cases} 
x^\alpha, & \text{if } x \geq 0 \\
-\lambda \cdot (-x)^\beta, & \text{if } x < 0,
\end{cases}
\]

where the reference point is normalized to 0 and \(x \in \mathbb{R}\) is wealth.

The third feature is captured by using the so-called *probability weighting function* \(\omega(\cdot)\) to transform the probability distribution. The original example of Tversky & Kahneman (1992) is given by

\[
w(p) := \frac{p^\gamma}{(p^\gamma + (1-p)\gamma)^\frac{1}{\gamma}}
\]

with \(\gamma < 1\). As this probability weighting function is decreasing in \(p\) whenever \(p^\gamma \cdot (\gamma - 1) + (1-p)^\gamma \cdot (\gamma + \frac{\gamma - 1}{\gamma}) < 0\), we assume \(\gamma > 0.3\) (see Rieger & Wang 2006). In the classical form of PT, the function \(\omega\) is applied directly to the probabilities of the different outcomes, resulting in an over-weighting of small probabilities, regardless of their associated outcome. Kahneman & Tversky (1979) suggest to define the ex-ante utility of PT-agents for a lottery \(\{x_i : p_i\}_{i=1}^n\), with \(n\) outcomes \(x_i\) and probabilities \(p_i\) as

\[
V^{PT}(p) = \sum_{i=1}^n \omega(p_i) \cdot v(x_i),
\]

\(^{1}\)In the case of CPT, this is strictly spoken only true for probabilities of the worst and best returns.
1 INTRODUCTION

Figure 1: A value function $v(\cdot)$ as defined in (1) with $\alpha = \beta = \frac{1}{2}$ and $\lambda = 2$.

Figure 2: Left: A probability weighting function $\omega(\cdot)$ as defined in (2) with $\gamma = \frac{1}{2}$. Right: The probability weighting function is decreasing in $p$ for pairs $(p, \gamma)$ in the gray area.

This functional form can explicitly be found in Lopes & Oden (1999), while Wakker (1989) considers $\sum_{i=1}^{n} \omega(p_i) \cdot x_i$.

The updated version of Prospect Theory, **Cumulative Prospect Theory**, arranges the set of monetary outcomes in an increasing order and divides it into gains and losses with respect to a reference point. An agent with CPT-preferences weights the cumulative probabilities of losses and the decumulative probabilities of gains. The subjective probability of an outcome corresponds to the marginal contribution of that outcome to the (de-)cumulative probability weight. The result is that only low probability events with extreme outcomes are over-weighted. CPT helped in recent years to explain various effects in decision theory, economics and finance.
1.2 Why Prospect Theory Changes Game Theory

When we play a game, payoffs are often given in monetary (or similar) amounts. As von Neumann and Morgenstern noticed, game theory has therefore to take into account the players’ preferences on sure and uncertain monetary outcomes of the game. This is why their famous work starts with a chapter on the notion of utility in which they present their formalization of EUT (v. Neumann & Morgenstern 1944). It is therefore not too surprising that this theory is somehow tailor-made for the application to game theory. In fact, the payoffs of a game are usually already defined to be given in utility units. Nevertheless it is an important and nontrivial feature that after a simple transformation of monetary outcomes into utility outcomes for each player, in the further analysis of the transformed game no additional considerations regarding the players’ preferences have to be made.

This seamless interplay between EUT and game theory, however, obscures the fact that there is indeed something nontrivial in this connection, and that therefore changes might be necessary when using a different preference model in the study of games. This might explain why the problem of applying game theory in a PT setting has not been widely studied so far, although it seems very natural to substitute this successful behavioral model into the study of games. Shalev (2000) introduces one aspect of PT, reference dependent loss aversion, into game theory and provides some existence results. The other aspect, probability weighting, is less popular in the game theoretic literature. A widely accepted perception is that as PT-agents who overweight small probabilities may prefer first order stochastically dominated lotteries (see our example 3 in section 3 or Quiggin 1982) and thus are ‘leaving money on the table’ and it is regarded ‘pointless to go through the trouble of modeling sophisticated strategic behavior’. Instead, Quiggin (1993)’s model of RDU preserves first order stochastic dominance. RDU is equivalent to CPT, if all payments are larger than the reference point. For arbitrary payments Baucells & Heukamp (2006) characterize CPT-agents who do not prefer stochastically dominated strategies. In this light it seems natural to study strategic choices of CPT-agents rather than PT-agents in normal form games. In this paper we offer an analytical framework for both concepts – PT and CPT – for strategic behavior in normal form games. We show that, in contrast to the results on lotteries, PT-agents do never select strategies that are strictly dominated by pure strategies. We show further, that if a PT-agent prefers a strategy which is strictly dominated by a mixed strategy, this is due to risk aversion, i.e. there also exists an EUT-agent who prefers the strictly dominated strategy. Again, in contrast to the perceived conceptual advantage of CPT we provide a simple example in which CPT-agents prefer a strategy which is strictly dominated by a pure strategy. We characterize the cases in which such preferences do not occur.

Non-linear weighting functions provide the ground for a second conceptual problem, which occurs not only in PT but also in CPT: agents do not need to be capable of consistently reducing compound lotteries. In the context of game theory, Dekel et al. (1991) focus on this problem and distinguish between two

\begin{footnotesize}
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\item[2] We are grateful to receive this and similar comments from anonymous referees. It might, however, be interesting to consider Birnbaum & McIntosh (1996), Birnbaum & Martin (2003) and Birnbaum (2005), who provide empirical evidence that in some situations choices of stochastically dominated lotteries regularly occur in human decision making.
\end{itemize}
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perceptual hypotheses: players perceive themselves as moving first or second. We follow the first hypothesis which implies that the agents do not weight their own probabilities with which they choose a strategy. If the own probability is also weighted, further conditions must be imposed to provide existence of strategic equilibria: Chen & Neilson (1999) show that in games with continuous strategy spaces the payoff function must be quasiconvex in the probabilities. Ritzberger (1996) shows that the probability weighting function must be concave to provide existence in finite games. Fershtman et al. (1991) consider inefficient delays in reaching agreements by general deviations from EUT. Our approach in modeling non-cooperative games for CPT-agents is closest to Ritzberger (1996).

We depart from there by defining the concept for gains and losses, by including a generalized (fixed) reference point, and most importantly by assuming that agents do not weight their own probabilities. The latter ensures existence of equilibrium without further assumptions on the curvature of the utility function or the probability weighting function. A previous version of our approach has also been applied in Rieger (2014).


Why is it non-trivial to bring game theory and CPT together? The naïve approach to incorporate PT and CPT into an analysis of a game would be to transform the monetary outcomes via the value function and to transform actual probabilities for chance moves into experienced probabilities by applying the probability weighting functions. This procedure mimics the method one successfully applied when dealing with EUT. Nevertheless in the case of PT and CPT, there are two difficulties in this approach which pose interesting problems, namely:

1. If we believe that the reference point is not exogenously given, but also depends on endogenous factors such as the payoffs of the opponents, the reference point itself is endogenous and the transformation of the monetary outcomes via the value function is not as harmless as it seems, since the reference point has to be chosen first.

2. The probabilities of chance moves are not the only probabilities that ought to be transformed by the probability weighting function, since the existence of mixed strategy Nash equilibria complicates considerations: the objective probability with which one player chooses a particular strategy will be transformed to a (different) subjective probability by another player who will choose his own strategy according to this subjective probability, rather than to the objective probability. This leads to an interesting interplay between the weighting functions of the players.  

3In Metzger & Rieger (2009), we show equilibrium existence for games of gains with two players.

4Here one has to be careful to transform the game into a form where the probabilities of the chance moves can be weighted separately for each player, since their probability weighting functions might differ.

5To be more precise, one should not speak about subjective or weighted probabilities, since PT is not about misestimation of probabilities, but states that decisions are made as if the underlying probabilities were misestimated or weighted. Mathematically, however, this results in the same formula, and so we keep for simplicity the slight abuse of language and talk about subjective or weighted probabilities as if they had a real meaning in PT and were not only auxiliary quantities.
We devote section 2 to a discussion of these points (including an overview of previous research on these questions). We provide a model for PT- and CPT-preferences in normal form games, prove equilibrium existence for fixed reference points and given an example for non-existence. In section 3 we prove results on the choice of dominated strategies and section 4 concludes.

2 Model

2.1 Probability Weighting Functions

Consider a probability space \((P, \mathcal{P}, \pi)\), where \((P, \mathcal{P})\) is a measurable space and \(\pi\) is a probability measure. A probability weighting function \(\omega : [0, 1] \rightarrow [0, 1]\) maps probabilities to real numbers. We assume that

- \(\omega(0) = 0, \omega(1) = 1\),
- \(\omega\) is continuous, strictly monotonic and differentiable,
- there exists a unique fixed point \(\bar{p} \in (0, 1)\),
- \(\omega(p) > p\) for all \(0 < p < \bar{p}\) and \(\omega(p) < p\) for all \(1 > p > \bar{p}\).

Clearly, the weighting function defined in (2) satisfies these properties.

2.2 Value Functions, Reference Points and Expectations

We assume that for any of the three concepts EUT, PT and CPT all agents have the same preference over deterministic monetary payments: each agent prefers to receive more money rather than less. Therefore we can identify agent \(i\)'s utility by any monotonic transformation of the monetary amount, or simply the amount itself. When it comes to preferences over stochastic payments, we deploy cardinal utility functions, where the curvature of the function over the set of deterministic payments captures the risk preferences of the agent. We assume that for PT and CPT agents also some reference point \(r \in \mathbb{R}\) matters. We denote utility functions over deterministic payments by small letters and utility functions over stochastic outcomes by capital letters. For EUT agents we use the terms Bernoulli utility \(u : \mathbb{R} \rightarrow \mathbb{R}\) and expected utility \(EU : \Delta(\mathbb{R}) \rightarrow \mathbb{R}\) and for PT and CPT agents we use the terms value function \(v : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) and ex-ante utility \(V : \Delta(\mathbb{R}) \times \mathbb{R} \rightarrow \mathbb{R}\). For any value function \(v(\cdot, \cdot)\) and reference point \(r \in \mathbb{R}\) we assume the following for all \(r \in \mathbb{R}\):

- \(v(x, r)\) is continuous and differentiable in \(x \forall x \in \mathbb{R}\)
- \(v(x, r) > v(y, r) \iff x > y\)
- \(v(x, r)\) is concave in \(x\) for all \(x \geq r\)
- \(v(x, r)\) is convex in \(x\) for all \(x < r\)

A monetary outcome \(x \geq r\) is considered as a gain and \(x < r\) is considered as a loss. The set of assumptions on \(v\) imply that PT and CPT agents are risk averse in gains and risk loving in losses. Whenever the reference point is a constant, we refer to the context as ‘fixed frames’.
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2.3 Monetary Games in Strategic Form

We consider finite games in normal form \((N, S, x)\), where the three elements have the following meaning: \(N = \{1, \ldots, n\}\) is the finite set of players and \(S = \times_{i=1}^n S_i\) is the finite set of pure strategy combinations with \(S_i\) being the set of pure strategies of player \(i \in N\). We denote the set of pure strategies of all players except \(i\) by \(S_{-i} = \times_{j \neq i} S_j\) with typical element \(s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)\).

\(x : S \rightarrow \mathbb{R}^n\) consists of \(n\) functions \(x_i : S \rightarrow \mathbb{R}\), where \(x_i(s)\) is the monetary payoff to player \(i\) given the pure strategy \(s \in S\).6 In the setup without mixed strategies, EUT, PT and CPT all induce the same predictions. One of the most interesting effects of PT and CPT on the analysis of games is the interplay between probability weighting and mixed strategies. Denote by \(\Delta_i\) the set of mixed strategies of player \(i\), by \(\Delta_{-i} = \times_{j \neq i} \Delta_j\) the set of mixed strategies of all players except \(i\) and by \(\Delta = \times_{i=1}^n \Delta_i\) the set of mixed strategies of all players. Define further \(\sigma_{-i}(A) = \prod_{s_{-i} \in A, j \neq i} \sigma_{ij}(s_j)\) the set of mixed strategies of all players. We will discuss later, in Sec. 2.4, how different framing affects the following considerations.

The ex-ante utility of player \(i\) for the stochastic monetary payment induced by the pure strategy \(s_i\), and the mixed strategy \(\sigma_{-i} \in \Delta_{-i}\) depends on the underlying decision model. In the case of EUT the reference point is irrelevant and this utility becomes

\[
EU_i(\sigma, \sigma_{-i}) = \sum_{s \in S} u_i(x(s)) \cdot \sigma_i(s_i) \cdot \sigma_{-i}(s_{-i}) .
\]

Using PT and CPT, we need to decide about the framing a player uses and whether the probability weighting occurs on the individual or on the joint strategies of the opponents. Let us for the moment assume fixed framing and denote by \(r_i \in \mathbb{R}\) the reference point of player \(i\). In a context of flexible framing, player \(i\)'s frame would be a function \(\rho_i : \Delta \rightarrow \mathbb{R}\).7

In the case of PT we allow for two alternative variants of probability weighting, our results apply for both of them. In the first variant agent \(i\) weights the individual probabilities with which the other players choose their mixed strategies. Player \(i\) perceives the probability of state \(s_{-i}\) as \(\prod_{j \neq i} \omega_i(\sigma_j(s_j))\). In the second variant we assume that the agent weights the joint probability over the outcomes in \(s_{-i}\), hence the probability is perceived as \(\omega_i\left(\prod_{j \neq i} \sigma_j(s_j)\right)\). Whenever the results do not depend on the probability-variant we write \(\omega_i(\sigma_{-i}(s_{-i}))\).

In the case of CPT we exclusively use the second variant. The agents use a reference point dependent value function \(v_i : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}\) as defined in section

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6 We assume that player \(i\) cares only for his own payment \(x_i\), where \(i\) strictly prefers \(x_i\) to \(y_i\) whenever \(x_i > y_i\).

7 We will discuss later, in Sec. 2.4, how different framing affects the following considerations.
2.2. The ex-ante utility of a PT-agent then becomes

\[ V^\text{PT}_i(\sigma_i, \sigma_{-i}, r_i) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot \sum_{s_{-i} \in S_{-i}} v_i(x_i(s_i, s_{-i}), r_i) \cdot w_i(\sigma_{-i}(s_{-i})) \]  \hspace{1cm} (4)

In the case of CPT the reference point not only influences the value attached to the monetary payoff but also the weight which is associated with this value. If \( x_i \) is a gain, the agent weights the probability with which at least \( x_i \) is obtained while if \( x_i \) is a loss, the agent weights the probability with which at most \( x_i \) realizes. The weight associated with the value function consists of the marginal contribution to this probability. We firstly need to rank the possible outcomes, before we can compute the probability weighting. For each player \( i \in N \) define the function \( l_i : S \rightarrow \{1, \ldots, |S_{-i}|\} \) such that for each \( s_i \in S_i \):

\[ l_i(s_i, s_{-i}) < l_i(s_i, \tilde{s}_{-i}) \Rightarrow x_i(s_i, s_{-i}) \leq x_i(s_i, \tilde{s}_{-i}) \]

and \( l_i(s_i, \cdot) \) is a bijection for each \( s_i \in S_i \).

Given the index function \( l \) define for each \( s \in S \) and \( i \in N \) the set \( S^R_i(s) = \{\tilde{s}_{-i} \in S_{-i} : l_i(s_i, \tilde{s}_{-i}) R l_i(s_i, s_{-i})\} \), where \( R \in \{<, \leq, \geq, >\} \). Now define for given reference point \( r_i \in \mathbb{R} \) and mixed strategy \( \sigma_{-i} \in \Delta_{-i} \) the function \( \psi_i(\cdot|r_i, \sigma_{-i}) : S \rightarrow \mathbb{R}_+ \) as

\[
\psi_i(s|r_i, \sigma_{-i}) = \begin{cases} 
\omega_i(\sigma_{-i}(S^<_{-i}(s))) - \omega_i(\sigma_{-i}(S^\leq_{-i}(s))) & \text{if } x_i(s) < r_i \\
\omega_i(\sigma_{-i}(S^\geq_{-i}(s))) - \omega_i(\sigma_{-i}(S^>(s))) & \text{if } x_i(s) \geq r_i
\end{cases}
\]  \hspace{1cm} (5)

Finally, define the ex-ante utility of a CPT agent as

\[ V^\text{CPT}_i(\sigma) = \sum_{s \in S} \sigma_i(s_i) \cdot \psi_i(s|r_i, \sigma_{-i}) \cdot v_i(x_i(s), r_i) \]  \hspace{1cm} (6)

Our model is similar to the ad hoc-formulation of Goeree et al. (2003) who apply probability weighting to a 2 \times 2 game.

**Example 1 (rank dependent probability)** Tom chooses between \( L \) and \( R \) with the probabilities \( \sigma_2(L) \) and \( \sigma_2(R) \) and Sally chooses between \( T \), \( M \) and \( B \).

The table below lists Sally’s monetary payoffs, her reference point is \( r = 7 \):

<table>
<thead>
<tr>
<th></th>
<th>( L )</th>
<th>( R )</th>
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</thead>
<tbody>
<tr>
<td>( T )</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>( M )</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>( B )</td>
<td>6</td>
<td>7</td>
</tr>
</tbody>
</table>

The payoffs implied by \( M \) induce the index function \( l \) with values \( l_1(M, R) = 1 \) and \( l_1(M, L) = 2 \). As both outcomes induced by \( M \) are gains, she calculates the probabilities that she receives at least as much as and strictly more than
the respective gain. Her ex-ante utility derived from the choice $M$ is given by

$$V_1^{CPT}(M, \sigma_2, r) = \left( \frac{\text{perceived prob}\{l_1(M,s_2) = 1\}}{\text{prob}\{l_1(M,s_2) \leq 1\}} \cdot \omega\left( \frac{\sigma_2(L)}{\text{prob}\{l_1(M,s_2) = 1\}} \right) - \omega\left( \frac{\sigma_2(0)}{\text{prob}\{l_1(M,s_2) < 1\}} \right) \right) \cdot v_1(7, r) + \left( \frac{\text{perceived prob}\{l_1(M,s_2) = 2\}}{\text{prob}\{l_1(M,s_2) \geq 2\}} \cdot \omega\left( \frac{\sigma_2(L)}{\text{prob}\{l_1(M,s_2) = 2\}} \right) - \omega\left( \frac{\sigma_2(0)}{\text{prob}\{l_1(M,s_2) < 2\}} \right) \right) \cdot v_1(8, r).$$

The index function $l$ which is associated with $T$ may have the values $l_1(T, L) = 1$ and $l_1(T, R) = 2$. Sally considers these payoffs as losses and therefore calculates the probabilities that she receives at most as much as and strictly less than the respective loss. Her ex-ante utility induced by $T$ is

$$V_1^{CPT}(T, \sigma_2) = \left( \frac{\text{perceived prob}\{l_1(T,s_2) = 1\}}{\text{prob}\{l_1(T,s_2) \leq 1\}} \cdot \omega\left( \frac{\sigma_2(L)}{\text{prob}\{l_1(T,s_2) = 1\}} \right) - \omega\left( \frac{\sigma_2(0)}{\text{prob}\{l_1(T,s_2) < 1\}} \right) \right) \cdot v_1(5, r) + \left( \frac{\text{perceived prob}\{l_1(T,s_2) = 2\}}{\text{prob}\{l_1(T,s_2) \geq 2\}} \cdot \omega\left( \frac{\sigma_2(L) + \sigma_2(R)}{\text{prob}\{l_1(T,s_2) = 2\}} \right) - \omega\left( \frac{\sigma_2(L)}{\text{prob}\{l_1(T,s_2) < 2\}} \right) \right) \cdot v_1(5, r).$$

The pair $(B, L)$ induces a loss and the pair $(B, R)$ induces a gain. The values of the associated index function are given by $l_1(B, L) = 1$ and $l_1(B, R) = 2$. Therefore, Sally’s ex-ante utility is given by

$$V_1^{CPT}(B, \sigma_2) = \left( \frac{\text{perceived prob}\{l_1(B,s_2) = 1\}}{\text{prob}\{l_1(B,s_2) \leq 1\}} \cdot \omega\left( \frac{\sigma_2(L)}{\text{prob}\{l_1(B,s_2) = 1\}} \right) - \omega\left( \frac{\sigma_2(0)}{\text{prob}\{l_1(B,s_2) < 1\}} \right) \right) \cdot v_1(6, r) + \left( \frac{\text{perceived prob}\{l_1(B,s_2) = 2\}}{\text{prob}\{l_1(B,s_2) \geq 2\}} \cdot \omega\left( \frac{\sigma_2(R)}{\text{prob}\{l_1(B,s_2) = 2\}} \right) - \omega\left( \frac{\sigma_2(0)}{\text{prob}\{l_1(B,s_2) < 2\}} \right) \right) v_1(7, r).$$

### 2.4 Equilibrium

A Nash equilibrium (Nash 1950, 1951) requires that all players choose their strategies optimally given their beliefs and that all beliefs are consistent with the choices of the opponents. It is the latter point which is violated if players non-linearly weight the probabilities. For this reason we provide the following definitions for PT and CPT:

**Definition 1 (PT- & CPT-equilibrium)**

We call a strategy $\tilde{\sigma} \in \Delta$ a PT-equilibrium given reference point $r \in \mathbb{R}^n$ if for all $i = 1, \ldots, n$ and all $\sigma_i \in \Delta_i$ we have $V_i^{PT}(\tilde{\sigma}, r_i) \geq V_i^{PT}(\sigma_i, \sigma_{-i}, r_i)$. Analogously, we say that $\tilde{\sigma} \in \Delta$ is a CPT-equilibrium given reference point $r \in \mathbb{R}^n$ if for $i = 1, \ldots, n$ and all $\sigma_i \in \Delta_i$ we have $V_i^{CPT}(\tilde{\sigma}, r_i) \geq V_i^{CPT}(\sigma_i, \sigma_{-i}, r_i)$.

Before we state existence, let us briefly compare our model to existing models of reference dependent utilities in normal form games.
Cheng & Zhu (1995) show existence of a mixed strategy equilibrium for 2×2-games and ex-ante utility functions that are quadratic in the probabilities. We consider arbitrary finite games in normal form and allow for but do not require that the ex-ante utility is quadratic in the probabilities.

Besides Tversky & Kahneman (1992), Ritzberger (1996) is the blueprint for our CPT-model. While Ritzberger (1996) restricts attention to monetary games of gains, we allow for general (fixed) reference points and regular games. With example 6 we show that CPT-agents behave qualitatively different in regular games than in games of losses or games of gains. Yet the crucial difference to our model is our assumption that players choose own probabilities and observe the probabilities chosen by other players. This corresponds to the first perceptual hypothesis in Dekel et al. (1991) and is also supported by the ideas of Goeree et al. (2003) who distinguish between the probabilities the player decides and the ones that he merely reacts to. Keskin (2014) and Keskin (2015) also follows this approach. In our model, own probabilities do not enter the weighting function. Ritzberger (1996) shows that if players have concave probability weighting functions, an equilibrium exists and if players have convex probability weighting functions, no mixed strategy equilibria exist. The class of weighting functions as defined in 2.1 does not contain concave weighting functions. The following example exhibits a CPT-equilibrium in mixed strategies with a probability weighting function that is locally convex at the equilibrium probabilities: consider the doubly symmetric 2×2-game with $S_i = \{a, b\}$. Let $x_i(a, a) = x_i(b, b) = 0$, $x_i(a, b) = x_i(b, a) = 1$, reference points $r_i = 0$, $v_i(\cdot)$ as defined in (1) and $w_i(\cdot)$ as defined in (2) for $i = 1, 2$. We have $V_i^{\text{CPT}}(a, \sigma_i(a)) = w_i(\sigma_i(a))$ and $V_i^{\text{CPT}}(b, \sigma_i(b)) = w_i(\sigma_i(b))$, hence $\sigma_i = \left(\frac{1}{2}, \frac{1}{2}\right)$ is a CPT-equilibrium and $w_i(\pi)$ is locally convex at $\pi = \frac{1}{2}$.

Chen & Neilson (1999) consider monetary games with continuous strategy spaces for non-expected utility players. They show that an equilibrium (in pure strategies) exists, if the ex-ante utility function is quasiconvex. We do not require quasiconvexity for equilibrium existence. Here is a simple example with a non-quasiconvex ex-ante utility: player 2 may choose between left and right inducing a loss -3 and a gain -2. Given a mixed strategy $\sigma_2$ and weighting function $\omega_2(\cdot)$ the ex-ante utility is $V_i^{\text{CPT}}(s_1, \sigma_2) = \omega_i(\sigma_2(L)) - 3 + \omega_i(\sigma_2(R)) - 2$. For $\omega(\cdot)$ defined as in (2) with $\gamma = \frac{1}{2}$ the reader can verify that $V_i^{\text{CPT}}(s_1, \left(\frac{1}{2}, \frac{1}{2}\right)) = -\frac{3}{2} > -2 = V_i^{\text{CPT}}(s_1, R) > -3 = V_i^{\text{CPT}}(s_1, L)$.

Eichberger & Kelsey (2000) show equilibrium-existence if the ex-ante utility can be expressed as a Choquet integral with respect to some capacity $\nu$ and if that capacity is convex. We provide an example of a non-convex capacity implied by some probability weighting function and mixed strategy $\sigma$. Clearly, the function $\omega_i \circ \sigma_{-i} : S_{-i} \to \mathbb{R}$ is a capacity: $\omega_i(\sigma_{-i}(\emptyset)) = 0$, $\omega_i(\sigma_{-i}(S_{-i})) = 1$ and $A \subseteq B \subseteq S_{-i} \Rightarrow \omega_i(\sigma_{-i}(A)) \leq \omega(\sigma_{-i}(B))$. Consider two non-empty sets $A, B \subset S_{-i}$, $A \cap B = \emptyset$ with $0 < \sigma_{-i}(A), \sigma_{-i}(B) < \bar{p} \leq \sigma_{-i}(A) + \sigma_{-i}(B)$, where $\bar{p} = \omega_i(\bar{p})$. Non-convexity follows by $\sigma_{-i}(A) + \sigma_{-i}(B) = \sigma_{-i}(A \cup B)$ and $\sigma_{-i}(A \cap B) = 0$:

$$\omega_i(\sigma_{-i}(A)) + \omega_i(\sigma_{-i}(B)) > \sigma_{-i}(A) + \sigma_{-i}(B) > \omega_i(\sigma_{-i}(A \cup B)) + \omega_i(\sigma_{-i}(A \cap B))$$.

We conclude that the simple model which we provide here fills a gap in the literature. Furthermore, as the own probabilities enter the ex-ante utility function linearly, the existence proof uses standard arguments.
Proposition 1 (existence) Every finite monetary game with fixed reference points admits a PT- and a CPT-equilibrium.

Proof: The arguments for PT and CPT are identical and we suppress the superscripts PT and CPT for this proof. For any given $r_i \in \mathbb{R}$ and for each $i \in \mathcal{N}$ and $\sigma_{-i} \in \Delta_{-i}$ the ex-ante utility function $V_i(\sigma_i, \sigma_{-i}, r_i)$ is linear in $\sigma_i(s_i)$ for each $s_i \in S_i$ and $\Delta_i$ is compact, hence the set $BR_i(\sigma_{-i}, r_i) := \{ \sigma_i \in \Delta_i : V_i(\sigma_i, \sigma_{-i}, r_i) \geq V_i(\tilde{\sigma}_i, \sigma_{-i}, r_i) \}$ is nonempty, compact and convex valued. We assume in section 2.1 that $\omega_i(p)$ is continuous in $p$, therefore $V_i(\sigma_i, \sigma_{-i}, r_i)$ is continuous in $\sigma_{-i}$ for each $\sigma_{-i} \in \Delta_{-i}$. To see this, note that $\sigma_{-i}$ enters $V_i^{PT}(\cdot)$ via $\omega(\cdot)$ in the form of $\sigma_j(s_j)$ or $\sigma_{-i}(s_{-i}) = \prod_{j \neq i} \sigma_j(s_j)$. For CPT observe that $\sigma_{-i}$ enters $V_i^{CPT}$ via $\omega(\cdot)$ in the form of $\sigma_{-i}(A) = \sum_{s_{-i} \in A} \prod_{j \neq i} \sigma_j(s_j)$ for given subsets $A \subseteq S_{-i}$. The set $\Delta_{-i}$ is compact. By Berge’s Maximum Theorem the correspondence $BR_i(\sigma_{-i}, r_i)$ is upper hemicontinuous and by Kakutani’s Fixed Point Theorem there exists some $\hat{\sigma}_i \in \Delta$ such that $\hat{\sigma}_i \in BR_i(\hat{\sigma}_{-i}, r_i)$ for all $i \in \mathcal{N}$. □

2.5 Effects of Framing

One might wonder whether it is possible to extend the existence result of Prop. 1 to cases with general frames. This is certainly an interesting question, since in many applications the reference point is not fixed, e.g., it might change depending on previous payoffs. But even if we do not study sequential- but simultaneous games, it is not always clear how the reference point should be chosen. Keskin (2014) provides results on endogenous reference points, where – given some mixed strategy profile – the reference point is roughly the conditional ex-ante value obtained by the mixed strategy profile. Hereby, the reference point is uniquely defined for each profile of mixed strategies. Keskin uses this fact to show equilibrium existence. On the other hand there is evidence that suggests that in the same real-life decision different people might select different frames (“self-framing”), see Wang & Fischbeck (2004). In the case of games, the choice of the frame might depend on the situation, e.g., on the payoff of other players. But shouldn’t all these considerations be irrelevant to obtain existence at least in simple normal form games? It is interesting to see that this is not the case: even in the simplest possible setting of a $2 \times 2$ game the framing effect can play a decisive role, as the following example demonstrates, which also shows that Proposition 1 can in fact not be generalized to arbitrary frames.

Example 2 (non-existence with flexible framing) This example exploits the dependence of the value function on the reference point. It is not relevant whether the agents use PT or CPT. Let $\mathcal{L} = \{0 : \frac{1}{2}, 2 : \frac{1}{2}\}$ denote a lottery with 50% chance each to win a monetary amount of 2 or nothing. Consider the two players Sally and Tom who do not weight probabilities. Let the monetary outcomes of the game be given by the matrix below, where Sally chooses rows and Tom chooses columns. Assume that Tom has a reference point $r \in \mathbb{R}$ and is risk-averse in gains with respect to $r$ and risk loving in losses with respect to $r$. Assume that Sally is risk neutral in gains and in losses.
Sally chooses rows, Tom chooses columns.

Tom prefers the certain outcome 1 to the lottery $L$, if he is risk averse and he doesn’t if he is risk loving. Tom’s risk preference in turn depends on his reference point. If the reference point is greater or equal to 2, the outcomes 0, 1, and 2 are perceived as losses, Tom is risk loving and prefers the lottery to its expected value. If the reference point is lower or equal to zero, the outcomes 0, 1 and 2 are perceived as gains, Tom is risk averse and prefers the certain outcome to the lottery. The subtle detail of this example is that we allow Tom’s reference point to depend on Sally’s choice. Note that in equilibrium, players know the choices of their opponents. We assume that Tom’s reference point is equal to 0 if Sally chooses $s_2$ with probability one and that it is equal to 2 whenever Sally chooses $s_1$ with positive probability.

Sally’s best response to Tom’s mixed strategy is $s_1$, if Tom chooses $t_1$ with probability greater than $\frac{1}{2}$ and $s_2$, if Tom chooses $t_1$ with probability lower than $\frac{1}{2}$. If Tom chooses both of his strategies with equal probability, all of Sally’s strategies are best responses. As argued above, Tom’s best response to Sally’s strategy depends on whether Sally uses strategy $s_1$ with positive probability or not. If Sally places probability one on her strategy $s_2$, Tom’s best response is $t_1$. If Sally places positive probability on her strategy $s_1$, Tom’s best response is $t_2$. The graph of Tom’s best response correspondence is not closed, and hence the usual application of Kakutani’s fixed point theorem as in Nash (1950) is not valid. There exists no equilibrium for this game, given the described framing of the two players! This is in stark contrast to the standard case where Nash equi-
libria always exist for finite games and underlines the importance of assuming fixed frames in Prop. 1.

Considering this, we will concentrate in this article on situations where the choice of the reference point is fixed, and therefore existence of equilibria is guaranteed.

3 Stochastic Dominance and Dominated Strategies

Two our knowledge, Bawa (1975) is the first who uses first order stochastic dominance due to Quirk & Saposnik (1962) as a selection criterion for decisions under uncertainty. For two finite lotteries \( A \sim \{a_i, \alpha_i\}_{i=1}^n \) and \( B \sim \{b_j, \beta_j\}_{j=1}^m \) with outcomes \( a_i \) and \( b_j \) and probabilities \( \alpha_i \) and \( \beta_j \), lottery \( A \) (first order) stochastically dominates lottery \( B \) if \( \text{Prob}(a \leq \bar{x}|A) \leq \text{Prob}(b \leq \bar{x}|B) \) for all \( \bar{x} \in \mathbb{R} \) with strict inequality for at least one \( \bar{x} \). Intuitively, the probability of observing an outcome of at least \( \bar{x} \) is higher under lottery \( A \) than under lottery \( B \) for any value of \( \bar{x} \). Bawa (1975) shows that any decision maker whose utility function is increasing and differentiable and who assesses the uncertainty correctly chooses the first order stochastic dominant lottery. While in our setting agents have increasing utility functions, they fail to have an unbiased perception of stochastic environments. Hence, an agent that uses PT does not necessarily prefer a stochastically dominant lottery \( A \) over lottery \( B \), as the following example which we adapt from Quiggin (1982) illustrates:

Example 3 (stochastic dominance and PT) For lottery \( A \) let the valuation of outcomes be \( a_i = 1 - i \cdot \epsilon \) and the probabilities be \( \alpha_i = \frac{1}{n} \) for \( i = 1, \ldots, n \) and let \( B \) be associated with the outcome \( b_1 = 1 \) which occurs with probability 1. Let the reference point be equal to zero. Then

\[
V^{PT}(A) = \sum_{i=1}^{n} (1 - i \cdot \epsilon) \cdot w\left(\frac{1}{n}\right) = n \cdot w\left(\frac{1}{n}\right) \cdot \left(1 - \epsilon \cdot \frac{n + 1}{2}\right)
\]

and

\[
V^{PT}(B) = 1.
\]

For any \( \epsilon > 0 \), lottery \( B \) stochastically dominates \( A \). For \( \epsilon \) small enough, \( V^{PT}(A) > V^{PT}(B) \) as small probabilities are over-weighted: \( w\left(\frac{1}{n}\right) > \frac{1}{n} \).

This seemingly artificial result is implied by the inability of a PT-agent to categorize a class of similar outcomes and evaluate the probability of the whole category. In example 3, the agent does not categorize all outcomes under lottery \( A \) as one event, which would be plausible for \( \epsilon \) close to zero. In general, there is a bias in favor of the lottery with a larger number of outcomes, because more outcomes imply lower probabilities which are over-weighted by PT-agents. In fact, this examples demonstrates that PT does not only violate stochastic dominance, but even the “in-betweenness axiom”, that states that the certainty equivalent of a lottery should be between the smallest and largest possible outcomes. For further discussion of stochastic dominance in the original formulation of PT and in the variant by Karmarkar (1978) we refer to Rieger & Wang (2008).
In models of interaction, for each choice $s_i$ the opponent’s mixed strategy $\sigma_{-i} \in \Delta_{-i}$ induces a lottery on the set of outcomes $\{s_i\} \times S_{-i}$. Note that given $\sigma_{-i}$ two different lotteries, one for strategy $\hat{s}_i$ and one for strategy $s_i$, say, have the same number of outcomes and have equal probability distributions. When choosing among two different lotteries, the agent faces the same distribution on the set of outcomes while he may attach different values to the outcomes. Suppose now that an agent who chooses strategy $\sigma_i \in \Delta_i$ is better off in the monetary game $(X, S, x)$ than choosing some strategy $\hat{s}_i \in S_i$ no matter what the others do. Rapoport (1966) coined the term ‘Dominating Strategy’. Formally, strategy $\sigma_i \in \Delta_i$ strictly dominates strategy $\hat{s}_i \in S_i$ if for all $s_{-i} \in S_{-i}$ we have

$$\sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i) > v_i(x_i(\hat{s}_i, s_{-i}), r_i).$$

**Proposition 2** If $\sigma_i \in \Delta_i$ strictly dominates $\hat{s}_i \in S_i$, then

$$V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \forall \sigma_{-i} \in \Delta_{-i}.$$  

**Proof:**

$$V_i^{PT}(\sigma_i, \sigma_{-i}) = \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{-i}), r_i)$$

$$> \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot v_i(x_i(\hat{s}_i, s_{-i}), r_i) = V_i^{PT}(\hat{s}_i, \sigma_{-i}).$$

□

The following results disentangle the effects due to risk aversion, which also occur in EUT, and the effects due to probability weighting. We say that strategy $\sigma_i \in \Delta_i$ strictly dominates strategy $\hat{s}_i \in S_i$ in monetary terms if for all $s_{-i} \in S_{-i}$ we have

$$x_i(\sigma_i, s_{-i}) > x_i(\hat{s}_i, s_{-i}).$$

The following result follows immediately:

**Proposition 3 (PT: pure monetary domination)**

If $s_i \in S_i$ strictly dominates $\hat{s}_i \in S_i$ in monetary terms, then

$$V_i^{PT}(s_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \forall \sigma_{-i} \in \Delta_{-i}.$$  

**Proof:**

$$V_i^{PT}(s_i, \sigma_{-i}) - V_i^{PT}(\hat{s}_i, \sigma_{-i})$$

$$= \sum_{s_{-i} \in S_{-i}} (v_i(x_i(s_i, s_{-i}), r_i) - v_i(x_i(\hat{s}_i, s_{-i}), r_i)) \cdot \omega_i(\sigma_{-i}(s_{-i})).$$

As $x_i(s_i, s_{-i}) > x_i(\hat{s}_i, s_{-i}) \forall s_{-i} \in S_{-i}$ and $v_i(x, r_i)$ is strictly monotonic in $x$ we have that $v_i(x_i(s_i, s_{-i}), r_i) > v_i(x_i(\hat{s}_i, s_{-i}), r_i)$ for each $s_{-i} \in S_{-i}$ which establishes the result.

□

If the choice $\sigma_{-i}$ of the other players induces two monetary lotteries which are associated with $s_i$ and $\hat{s}_i$, one first order stochastically dominating the other, we
have shown that there is no value function \( v(\cdot) \), probability weighting function \( \omega(\cdot) \) and reference point \( r \) as defined in section 2 such that a PT-agent prefers the dominated lottery, if the lotteries are generated via a monetary game. Note that choosing \( \hat{s}_i \) would imply to relinquish some amount of money with certainty.

If \( \hat{s}_i \in S_i \) is dominated by \( \sigma_i \in \Delta_i \) in monetary terms, agent \( i \) looses money in expectation but not necessarily in every possible outcome \( s_{-i} \in S_{-i} \). In this case agent \( i \) may prefer \( \hat{s}_i \) over \( \sigma_i \), as the following example illustrates:

**Example 4 (PT: preference for dominated strategy)** Consider the following monetary game of gains \( (r_1 = 0, \text{the payoffs of player 2 are not relevant}), \) the value function as defined in (1) with \( \alpha = \frac{1}{2} \) and any probability weighting function as defined in section 2.1.

\[
\begin{array}{c|cc}
 & L & R \\
T & 3 & 0 \\
M & 1 & 1 \\
B & 0 & 3 \\
\end{array}
\]

The mixed strategy \( \sigma_1 = \left( \frac{1}{2}, 0, \frac{1}{2} \right) \) strictly dominates the pure strategy \( M \) but
\[
V_i^{PT}(\sigma_1, \sigma_2) = \frac{1}{2} \cdot \sqrt{3} \cdot (\omega_1(\sigma_2(L)) + \omega_1(\sigma_2(R))) < \omega_1(\sigma_2(L)) + \omega_1(\sigma_2(R)) = V_i^{PT}(M, \sigma_2).
\]

The following proposition implies that the preference for a dominated strategy generally is not caused by the probability weighting function but by the risk preferences of the agents – which of course is not due to PT as EUT also allows for risk aversion.

**Proposition 4 (PT: mixed monetary domination with few risk aversion)**
If \( \sigma_i \in \Delta_i \) strictly dominates \( \hat{s}_i \in S_i \) in monetary terms and, given reference point \( r_i \in \mathbb{R} \), the value function \( v_i(\cdot, r_i) \) satisfies
\[
v_i(\lambda \cdot x_i(s'), (1 - \lambda) \cdot x_i(s''), r_i) < \lambda \cdot v_i(x_i(s'), r_i) + (1 - \lambda) \cdot v_i(x_i(s''), r_i) + \min_{s_{-i}} v_i(x_i(\sigma_i, s_{-i}), r_i) - v_i(x_i(\hat{s}_i, s_{-i}), r_i)
\]
for all \( \lambda \in (0, 1) \) and all \( s', s'' \in S \), then
\[
V_i^{PT}(\sigma_i, \sigma_{-i}) > V_i^{PT}(\hat{s}_i, \sigma_{-i}) \quad \forall \sigma_{-i} \in \Delta_{-i}.
\]

**Proof**:
For all \( \sigma_{-i} \in \Delta_{-i} \) we have
\[
V_i^{PT}(\sigma_i, \sigma_{-i}) > \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot (v_i(x_i(\sigma_i, s_{-i}), r_i) - \bar{\epsilon})
\]
\[
\geq \sum_{s_{-i} \in S_{-i}} \omega_i(\sigma_{-i}(s_{-i})) \cdot v_i(x_i(\hat{s}_i, s_{-i}), r_i) = V_i^{PT}(\hat{s}_i, \sigma_{-i}),
\]
where \( \bar{\epsilon} := \min_{s_{-i}} v_i(x_i(\sigma_i, s_{-i}), r_i) - v_i(x_i(\hat{s}_i, s_{-i}), r_i). \)

Proposition 4 states that a PT-agent does not forgo expected money, if the agent exhibits few enough risk-aversion. The following corollary exploits the fact that PT-agents are risk-loving in losses:
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Corollary 1 (PT: mixed monetary domination in losses) If $x_i(s) < r_i \forall s \in S$ and $\sigma_i$ strictly dominates $\hat{s}_i$ in monetary terms, then

$$V_i^{PT}(\sigma_i, \sigma_{\neg i}) > V_i^{PT}(\hat{s}_i, \sigma_{\neg i}) \forall \sigma_{\neg i} \in \Delta_{\neg i}.$$  

**Proof:** As $v_i(x_i(s), r_i)$ is convex for all $x < r_i$, it holds that $v_i(\lambda \cdot x_i(s'), r_i) + (1 - \lambda) \cdot v_i(x_i(s''), r_i) < \lambda \cdot v_i(x_i(s'), r_i) + (1 - \lambda) \cdot v_i(x_i(s''), r_i) \forall \lambda \in (0, 1)$ and $s', s'' \in S$ and Proposition 4 applies. 

In any regular game or any game of gains there are ample possibilities that PT-agents choose a pure strategy which is strictly dominated by a mixed strategy in monetary terms. Is this fact sufficient to reject the application of PT-preferences in non-cooperative game theory? We believe that this stance is unsustainable because it would also argue against EUT as Proposition 5 points out:

**Proposition 5 (PT: mixed monetary domination and EUT)** If $\sigma_i \in \Delta_i$ strictly dominates $\hat{s}_i \in S_i$ in monetary terms and for a given mixed strategy $\sigma_{\neg i} \in \Delta_{\neg i}$ some PT-agent prefers $\hat{s}_i$ to $\sigma_i$, then there exists some pure strategy $\tilde{s}_{\neg i} \in S_{\neg i}$ such that some EUT-agent prefers $\tilde{s}_i$ to $\sigma_i$ given $\tilde{s}_{\neg i}$.

**Proof:** We have

$$\sum_{s_{\neg i} \in S_{\neg i}} \omega_i(\sigma_{\neg i}(s_{\neg i})) \sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, s_{\neg i}), r_i) \geq \sum_{s_{\neg i} \in S_{\neg i}} \omega_i(\sigma_{\neg i}(s_{\neg i})) \cdot v_i(x_i(s_i, s_{\neg i}), r_i).$$

As $\omega_i(\sigma_{\neg i}(s_{\neg i})) \geq 0 \forall s_{\neg i} \in S_{\neg i}$, there exists some $\tilde{s}_{\neg i} \in S_{\neg i}$ with $\sigma_{\neg i}(s_{\neg i}) > 0$ and $\sum_{s_i \in S_i} \sigma_i(s_i) \cdot v_i(x_i(s_i, \tilde{s}_{\neg i}), r_i) < v_i(x_i(s_i, s_{\neg i}), r_i).$

How does strict dominance of strategies relate to stochastic dominance of a lottery? As strict dominance of strategies is a dominance relation restricted to lotteries that are induced by the strategy choice in a monetary game it implies 'state-wise dominance' which is stronger than stochastic dominance. Hence strict dominance implies stochastic dominance but not vice versa.

We turn now to the analysis of the choices of CPT-agents. Let us firstly consider two pure strategies $s_i, \hat{s}_i \in S_i$ for which $s_i$ strictly dominates $\hat{s}_i$, that is $v_i(x_i(s_i, s_{\neg i}), r_i) > v_i(x_i(\hat{s}_i, s_{\neg i}), r_i) \forall s_{\neg i} \in S_{\neg i}$. The argument why the CPT-agent $i$ prefers $s_i$ over $\hat{s}_i$ is non-trivial because – given reference point $r_i \in \mathbb{R}$ and mixed strategy $\sigma_{\neg i} \in \Delta_{\neg i}$ – the perceived probability $\psi_i(s_i, s_{\neg i}|r_i, \sigma_{\neg i})$ that $(s_i, s_{\neg i})$ occurs is rank dependent and does not need to coincide with the rank dependent perceived probability $\psi_i(\hat{s}_i, s_{\neg i}|r_i, \sigma_{\neg i})$ that $(\hat{s}_i, s_{\neg i})$ occurs. Firstly, we rewrite (6) to show that CPT-agents do not prefer first order stochastically dominated perceived lotteries. Secondly, we show that if a pure strategy strictly dominates another pure strategy and the game is not regular, then the perceived lotteries can be ordered via first order stochastic dominance.

Given a mixed strategy $\sigma \in \Delta$ use $\psi(|r_i, \sigma_{\neg i})$ from (5) to define the cumulative distribution function $F(|\sigma) : \mathbb{R} \rightarrow [0, 1]$, where for $x \in \mathbb{R}$

$$F(x|\sigma) = \sum_{s \in S : x_i(s) \leq x} \sigma_i(s_i) \cdot \psi(s|r_i, \sigma_{\neg i}) / \sum_{s \in S} \sigma_i(s_i) \cdot \psi(s|r_i, \sigma_{\neg i}).$$
max

note that if

\( x \Psi x \) STOCHASTIC DOMINANCE AND DOMINATED STRATEGIES

\( \psi \) \( \Psi \) \( v \) \( \gamma \) < \( F \) 

Proof :

The proof is the by now well known application of Theorem 1 in Bawa (1975)

\( \hat{F} \) \( \hat{F} \)

Corollary 2 (Bawa (1975), Theorem 1)

\( x \) finite in \( v \) \( \omega \) \( \omega \) \( \omega \) \( \omega \) \( v \) \( \omega \) \( \omega \) \( \omega \) \( \omega \) \( \omega \)

we state the following corollary given that

Proposition 6 (CPT: pure dominance in monetary games)

extremely useful in the light of the next statement:

\( \bullet \) \( s \) \( \psi \) \( s \) \( \Psi \) \( \Psi \) \( \Psi \)

Note that if \( \omega_i \) is defined according to (2), then \( \omega_i(p) + \omega_i(1-p) < 1 \) for all \( \gamma < 1 \) and \( p \in (0,1) \). Nevertheless, for \( x_i(s) < r_i \forall s \in S \) or \( x_i(s) \geq r_i \forall s \in S \) we state the following corollary given that \( v_i(x, r_i) \) is strictly increasing and finite in \( x \) for any \( x, r_i \in \mathbb{R} \):

Corollary 2 (Bawa (1975), Theorem 1) If the monetary game is a game of losses or a game of gains, a CPT-agent prefers \( s_i \) over \( \hat{s}_i \) whenever \( F(x|s_i, \sigma_{-i}) \leq F(x|\hat{s}_i, \sigma_{-i}) \forall x \in \mathbb{R} \) and \( < \) for some \( x \in \mathbb{R} \).

The proof is the by now well known application of Theorem 1 in Bawa (1975) given that \( \Psi_i(s_i, \sigma_{-i}|r_i) \) and \( \Psi_i(\hat{s}_i, \sigma_{-i}|r_i) \) are equal to one. The corollary is extremely useful in the light of the next statement:

Proposition 6 (CPT: pure dominance in monetary games) If \( s_i \in S_i \) strictly dominates \( \hat{s}_i \in S_i \), and either

\( \bullet \) \( x_i(\hat{s}_i, s_{-i}) \geq r_i \forall s_{-i} \in S_{-i} \) or

\( \bullet \) \( x_i(s_i, s_{-i}) < r_i \forall s_{-i} \in S_{-i} \) or

\( \bullet \) \( \Psi(s_i, \sigma_{-i}|r_i) \) and \( \Psi(\hat{s}_i, \sigma_{-i}|r_i) \) are close to 1,

then \( F(|s_i, \sigma_{-i}) \) first order stochastically dominates \( F(|\hat{s}_i, \sigma_{-i}) \) for all \( \sigma_{-i} \in \Delta_{-i} \).

Proof : For \( x \in \mathbb{R} \) we have

\( F(x|\hat{s}_i, \sigma_{-i}) = \)

\( \left\{ \begin{array}{ll}
\omega_i(\sigma_{-i}(\{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) \leq x\})/\Psi_i(\hat{s}_i, \sigma_{-i}|r_i) & \text{if } x < r_i \\
1 - \omega_i(\sigma_{-i}(\{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) > x\})/\Psi_i(\hat{s}_i, \sigma_{-i}|r_i) & \text{if } x \geq r_i \\
\end{array} \right. \)

Clearly, \( F(x|s_i, \sigma_{-i}) \leq F(x|\hat{s}_i, \sigma_{-i}) \) for all \( x \leq \min\{x_i(s_i, s_{-i})\} \) and \( x \geq \max\{x_i(\hat{s}_i, \sigma_{-i})\} \). To analyze the case \( \min\{x_i(s_i, \sigma_{-i})\} < x < \max\{x_i(\hat{s}_i, \sigma_{-i})\} \)

note that if \( x_i(\hat{s}_i, s_{-i}) \geq r_i \forall s_{-i} \) or if \( x_i(s_i, s_{-i}) < r_i \forall s_{-i} \) we have \( \Psi_i(s_i, \sigma_{-i}) = \Psi_i(\hat{s}_i, \sigma_{-i}) = 1 \forall \sigma_{-i} \in \Delta_{-i} \). If \( \min\{x_i(s_i, s_{-i})\} < r_i < \max\{x_i(\hat{s}_i, s_{-i})\} \), we need to assume that \( \Psi_i(s_i, \sigma_{-i}) = \Psi_i(\hat{s}_i, \sigma_{-i}) \approx 1 \forall \sigma_{-i} \in \Delta_{-i} \). Then,
for $\min\{x_i(s_i, s_{-i})\} < x < \min\{\max\{x_i(\hat{s}_i, s_{-i})\}, r_i\}$ we have $F(x|s_i, \sigma_{-i}) \leq F(x|\hat{s}_i, \sigma_{-i})$ whenever

$$\omega_i(\sigma_{-i} (\{s_{-i} \in S_{-i} : x_i(s_i, s_{-i}) \leq x\})) \leq \omega_i (\sigma_{-i} (\{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) \leq x\}))$$

and for $\max\{\min\{x_i(s_i, s_{-i})\}, r_i\} < x < \max\{x_i(\hat{s}_i, s_{-i})\}$ we have $F(x|s_i, \sigma_{-i}) \leq F(x|\hat{s}_i, \sigma_{-i})$ whenever

$$\omega_i(\sigma_{-i} (\{s_{-i} \in S_{-i} : x_i(s_i, s_{-i}) > x\})) \geq \omega_i (\sigma_{-i} (\{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) > x\})).$$

As $s_i$ strictly dominates $\hat{s}_i$ implies that $\{s_{-i} \in S_{-i} : x_i(s_i, s_{-i}) \leq x\} \subseteq \{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) \leq x\}$ with a strict inclusion for each $x \in \{x_i(\hat{s}_i, s_{-i})\}$, $s_{-i} \in S_{-i}$. Similarly, $\{s_{-i} \in S_{-i} : x_i(\hat{s}_i, s_{-i}) > x\} \subseteq \{s_{-i} \in S_{-i} : x_i(s_i, s_{-i}) > x\}$ with a strict inclusion for each $x \in \{x_i(s_i, s_{-i})\}$. The monotonicity of $\omega_i(\cdot)$ establishes the result.

Proposition 6 (together with corollary 2) gives us the clear result that any CPT-agent prefers pure strategy $s_i$ over pure strategy $\hat{s}_i$ if $s_i$ strictly dominates $\hat{s}_i$ in monetary terms, if the game is a game of losses or a game of gains, or if the perceived probabilities $\Psi_i(s_i, \sigma_{-i}|r_i)$ and $\Psi_i(\hat{s}_i, \sigma_{-i}|r_i)$ sum up sufficiently close to one. We provide counter-examples for the case of mixed dominated and the case of pure domination in regular games.

In proposition 2 we showed that if a mixed strategy $\sigma_i$ strictly dominates a pure strategy $\hat{s}_i$, then PT-agents prefer $\sigma_i$ over $\hat{s}_i$. In proposition 4 we showed the stronger result that if a mixed strategy $\sigma_i$ strictly dominates a pure strategy $\hat{s}_i$ in monetary terms, then a PT-agent who exhibits sufficiently few risk aversion prefers $\sigma_i$ over $\hat{s}_i$. Surprisingly, not even the weaker of the two results holds for CPT-agents!

Example 5 (preference for dominated strategy continued) Consider the monetary game of example 4 with $r_i = 0$ and either consider an almost risk neutral CPT-agent or suppose that the payoff matrix reflects valuations $v_i$ rather than monetary payoffs $x_i$. Clearly, the mixed strategy $\sigma_1 = (\frac{1}{2}, 0, \frac{1}{2})$ strictly dominates the pure strategy $\hat{s}_1 = M$, that is $\frac{1}{2} \cdot \sigma_2(L) + \frac{3}{2} \cdot \sigma_2(R) = 3 > \frac{4}{2} > 1 \forall \sigma_2 \in \Delta_2$. The mixed strategy $\tilde{\sigma} \in \Delta$ induces the following cumulative distribution functions:

$$F(x|\tilde{\sigma}) =
\begin{cases}
0 & \text{if } x < 0 \\
\tilde{\sigma}_1(T) \cdot (1 - \omega_1(\tilde{\sigma}_2(L))) + \tilde{\sigma}_1(B) \cdot (1 - \omega_1(\tilde{\sigma}_2(R))) & \text{if } 0 \leq x < 1 \\
1 - \tilde{\sigma}_1(T) \cdot \omega_1(\tilde{\sigma}_2(L)) - \tilde{\sigma}_1(B) \cdot \omega_1(\tilde{\sigma}_2(R)) & \text{if } 1 \leq x < 3 \\
1 & \text{if } 3 \leq x
\end{cases}$$

As $\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) < 2$ for any $\tilde{\sigma}_2$ we have that $F(x|\hat{s}_1, \tilde{\sigma}_2)$ is not stochastically dominated by $F(x|\sigma_1, \tilde{\sigma}_2)$ for any $\tilde{\sigma}_2 \in \Delta_2$ and we cannot apply corollary 2 to infer that the player prefers $\sigma_1$ over $\hat{s}_1$. In fact, the opposite can be true:

$$V_{1CPT}^C(\sigma_1, \tilde{\sigma}_2) = \frac{3}{2} \cdot (\omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)))$$

$$V_{1CPT}^C(\hat{s}_1, \tilde{\sigma}_2) = 1$$
3 STOCHASTIC DOMINANCE AND DOMINATED STRATEGIES

Hence whenever \( \omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) < \frac{2}{3} \), player 1 prefers \( \hat{s}_1 \) over \( \sigma_1 \)!

If \( \omega_1(\cdot) \) is defined according to (2) and \( \tilde{\sigma}_2(L) = \tilde{\sigma}_2(R) = \frac{1}{2} \) and \( \gamma = \frac{1}{3} \) then \( \omega_1(\tilde{\sigma}_2(L)) + \omega_1(\tilde{\sigma}_2(R)) = \left( \frac{1}{2} \right)^\frac{1}{3} < \frac{2}{3} \). Note that this surprising result is not due to risk aversion but due the way in which CPT-agents rank probabilities.

This counterexample reveals that the assumption of rank dependent probability weighting may imply irrational choices when it comes to mixed domination in strategic games. In the light of this example and propositions 2 and 4 it seems that PT has a conceptual advantage over CPT in the study of strategic interactions. The arguments against CPT become even stronger with the following counter example:

Example 6 (CPT: pure domination in regular games) Suppose Sally from example 1 has the weighting function \( w(p) = \frac{\sqrt{p}}{\sqrt{1-p}} \). Strategy \( B \) strictly dominates strategy \( T \) (in terms of \( v(\cdot) \) and in monetary terms). With \( v_1(x, r) = x \) we know from example 1 that

\[
V_{CPT}^1(T, \sigma_2) = 5 \\
V_{CPT}^1(B, \sigma_2) = \frac{\sqrt{\sigma_2(L)}}{\sqrt{\sigma_2(L)} + \sqrt{\sigma_2(R)}}^2 \cdot 6 + \frac{\sqrt{\sigma_2(R)}}{\sqrt{\sigma_2(L)} + \sqrt{\sigma_2(R)}}^2 \cdot 7
\]

In particular, for \( \sigma_2(L) = \sigma_2(R) = \frac{1}{2} \) we have \( V_{CPT}^1(B, (\frac{1}{2}, \frac{1}{2})) = \frac{13}{2\sqrt{2}} < 5 \).
4 Conclusions

We analyze decisions of agents who use Prospect Theory or Cumulative Prospect Theory (Kahneman & Tversky 1979, Tversky & Kahneman 1992) when they face strategic interaction. These agents differ from traditional expected utility maximizers with respect to two dimensions of irrationality: Firstly, they are risk averse in gains and risk loving in losses. Secondly, those agents overestimate small probabilities and underestimate large probabilities.

(Cumulative) Prospect Theory describes behavior under uncertainty which usually is modeled as a choice among various exogenous lotteries. In our strategic setting – normal form games – a lottery arises from the potentially mixed expectation on the choice of the strategic opponents. If not the choice itself, its expectation certainly fails to be independent of the own decision. Hence, when analyzing solution concepts of game theory we have to admit endogenous lotteries. In this setting, we analyze irrational choices in the sense of Expected Utility Theory and identify the effects caused by probability misestimation.

An immediate finding is that pure best replies are equivalent for EUT-and (C)PT-agents. This implies that purely dominant strategies or pure Nash Equilibria are invariant with respect to any monotone value function or probability weighting function. When it comes to mixed strategies, less properties carry over from Expected Utility Theory to (Cumulative) Prospect Theory. We give examples in which the set of best replies to some beliefs is not invariant with respect to the probability misestimation. While a dominated strategy is dominated for agents who maximize according to Prospect Theory in any case, this does not need to be the case for agents who apply Cumulative Prospect Theory, if the dominating strategy is mixed. This is a striking result, since previously CPT was thought to be the “mathematically” superior theory as compared to PT, since CPT does not violate first order stochastic dominance, but PT does. Our analysis demonstrates that, when only considering games, PT might be the mathematically preferable theory.

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