Measuring the frequency dynamics of financial connectedness
and systemic risk*

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Abstract

Financial risk management has generally focused on aggregate connectedness, overlooking its
short-term, medium-term, and long-term sources. We argue that the frequency dynamics is
insightful for studying the connectedness as shocks with heterogeneous frequency responses
create linkages with various degrees of persistence. Such connections are important for un-
derstanding possible sources of systemic risk, but remain hidden when aggregate measures
of connectedness are used. To estimate short-term, medium-term, and long-term financial
connectedness, we propose a general framework based on spectral representation of variance
decompositions. In an empirical application we document rich dynamics of connectedness
in US market risk with short-term connections due to contemporaneous correlations, as well
as significant weekly, monthly, and yearly connections playing role. Hence financial market
clears part of the information although permanent changes in investors’ expectations having
longer-term responses are non-negligible.

Keywords: Connectedness, frequency, spectral analysis, systemic risk

JEL: C18; C58; G10

1 Introduction

Economic markets grew in size and tangled extraordinarily during the last decades. This
evolution not only caused the change in magnitude of connections but also a change in the
structure of the markets and their connections. With the surge, the importance of evaluation
of connections among different parts of markets grew, and understanding connectedness became
central to many areas of research such as risk management, portfolio allocation, and busines-
cycle analysis. Being painfully aware of the unsuitability of the standard correlation-based
measures, academics have concentrated on the development of more general frameworks. The
developed frameworks, however, still overlook several fundamental properties of connectedness,

*For estimation of the frequency dependent connectedness measures introduced by this paper, we pro-
vide the package \texttt{frequencyConnectedness} in R software. The package is available on \url{https://github.com/}
tomaskrehlik/frequencyConnectedness

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hence possible sources of systemic risk. In this work, we argue that for understanding the sources, it is crucial to understand the frequency dynamics of the connectedness as shocks to economic activity impact variables at different frequencies with different strengths. This is especially important in financial markets, where shocks due to changes in investor’s expectations will have rich impact at different timescales. To consider the long-term, medium-term, and short-term frequency responses of shocks, we propose a general framework which will allow us to measure the financial connectedness at any interesting frequency bands.

The main reason why we should believe that agents operate on different investment horizons is in formation of their preferences. Ortu et al. (2013) disaggregate consumption growth into cyclical components classified by their level of persistence and develop an asset pricing model where consumption responds to shocks of heterogeneous durations affecting asset returns through heterogeneous preference choices in stochastic discount factor model. With this respect, authors extend growing literature on consumption-based asset pricing models in which long-run risk in consumption growth is priced (Bansal and Yaron, 2004). In one of earlier contributions, Cogley (2001) decomposes approximation errors in stochastic discount factor models by frequency and applies the frequency decomposition to a number of consumption based discount factor models. In line with this research, Bandi and Tamoni (2015) argue that consumption growth should be separated into a variety of cyclical components, as investors may not focus on very high-frequency components of consumption representing short-term noise, but lower frequency components of consumption growth with heterogeneous periodicities instead. In a financial system, asset prices driven by consumption growth with different cyclical components will naturally generate shocks with heterogeneous responses, and thus various sources of connectedness creating short-term, medium-term, and long-term systemic risk. In turn, when studying connectedness, we should focus on linkages with various degree of persistence underlying systemic risk.

The importance of distinction between short-term and long-term parts of system became evident even earlier with the dawn of co-integration (Engle and Granger, 1987). Assuming and leveraging co-integration in the system, subsequent literature builds a preliminary notion of disentangling short-term from long-term movements in connectedness (Gonzalo and Ng, 2001; Blanchard and Quah, 1989; Quah, 1992). Given the decomposition to the long-term common stochastic trend and deviations from trend, one can move the projection in such a way that error to one series will be a shock to long-term trend and the other will be a shock to the deviation from the trend. A shock with strong long-term effect will have high power at low frequencies and in case it transmits to other variables, it points to long-term connectedness. For example, in case of stock markets, long-term spillovers may be attributed to permanent changes in expectations about future dividends (Balke and Wohar, 2002). In case shocks with heterogeneous frequency responses due to differing expectations about future propagate to other institutions, financial system becomes interconnected with different sources of linkages. To capture this connectedness, we propose a general framework for decomposing the connectedness to any frequency band of interest. Similarly to Dew-Becker and Giglio (2013), who set asset pricing into the frequency domain, we view the frequency domain as a natural place for measuring the connectedness.

As noted by Diebold and Yilmaz (2009, 2012), and later Diebold and Yilmaz (2014), variance decompositions from approximating models are convenient framework for empirical measurement of connectedness. Precisely, Diebold and Yilmaz (2009) define the measures based on assessing shares of forecast error variation in one variable due to shock arising in another variable in the system. To identify uncorrelated structural shocks from reduced-form shocks, Diebold and Yilmaz (2012) use the generalized variance decomposition of Pesaran and Shin (1998) which moreover allows to define directional connectedness. This approach quickly be-
came popular and recognized by researchers due to its universality.

Being interested in frequency origins of connectedness in variables, one may think about using different forecast horizons of variance decomposition. Staying in time domain, heterogeneous frequency responses of shocks impacting future uncertainty with different strength will stay hidden, as the effects are simply aggregated through frequencies. To see this, let us consider two examples of a system of bivariate autoregressive process with opposite signs of coefficients. The positive coefficients in the first example will create large connectedness driven by low frequencies of the cross-spectral density. With increasing forecast horizon of variance decompositions, one will measure higher connectedness in the process. In the second example, the negative coefficients of the same magnitude will create equal connectedness as in the first case at all forecasting horizons, although connections come solely from the high frequencies due to anti-persistent nature of the process. Hence simply assessing connectedness at different horizons to capture the heterogeneous frequency responses due to differing expectations of investors is not enough.

Instead of assessing overall error variation in a variable \( a \) due to shock arising in a variable \( b \), we are hence interested in assessing shares of forecast error variation in a variable \( a \) due to shock to a variable \( b \) at a specific frequency band. This is a natural step to take, as it will show the long-term, medium-term, and short-term impacts of a shock, which can conveniently be summed to total aggregate effect, if needed. For the purpose of frequency dependent measurement, we define spectral representation of generalized forecast error variance decomposition. To achieve this we work with its Fourier transforms of the impulse response functions—frequency responses. In the frequency domain, we are simply interested in the portion of forecast error variance at a given frequency band that is attributed to shocks in another variable. Our work is inspired by previous research of Geweke (1982, 1984, 1986), and Stiassny (1996) who use related measures in more restrictive environments.

In addition to introducing the frequency dynamics into the measurement of connectedness, we also study how cross-sectional correlations impact the connectedness. Higher contemporaneous correlation does not necessarily need to indicate connectedness in a sense literature tries to measure it. A good example is recent crisis of 2007–2008, when stock markets recorded strong cross-sectional correlations biasing the contagion effects estimated by many researchers [Forbes and Rigobon, 2002; Bekaert et al., 2005].

The paper starts with theoretical discussion that is followed by a relevant application on financial data that helps us to show the usefulness of the framework and guide a user to apply the introduced methods appropriately. Concretely, we study an important problem of connectedness in financial system, which directly translates to a mesure of systemic risk. We use the spectral representations of variance decompositions locally to recover the time-frequency dynamics of connectedness in the US stock market, and we document rich dynamics in frequency responses of shocks in volatilities. Dynamics of connectedness is mainly driven by frequencies from one day up to one month. While short-term connections driven mainly by contemporaneous correlations play role, we document significant weekly, monthly, and yearly as well. Hence financial market seems to clear part of the information although permanent changes in investors’ expectations having longer-term responses play significant role.

## 2 Measuring connectedness in frequency domain

As argued by [Diebold and Yilmaz, 2014], variance decompositions can intuitively be used for measurement of connectedness between economic variables. A natural way to measure its
frequency dynamics is to consider the spectral representation of variance decompositions based on frequency responses of shocks instead of impulse responses of shocks. Frequency domain—being natural place to study the long-term, medium-term, or short-term connectedness—shifts our focus from assessing shares of variances due to shocks to other variables to assessing shares of spectra due to shocks to other variables. Stiassny (1996) introduced a first notion of spectral representation for variance decompositions, although in restrictive setting. In our work we define general spectral representation of variance decompositions and we show how we can use them for defining the frequency dependent connectedness measures.

The spectral representations of variance decompositions can also be viewed as more general way of measuring causality in frequency domain. Geweke (1982) proposes a frequency domain decomposition of the usual likelihood ratio test statistic for Granger causality, and Dufour and Renault (1998); Breitung and Candelon (2006); Yamada and Yanfeng (2014) provide a formal framework for testing causality on various frequencies. Geweke (1984); Granger (1969) develop a multivariate extensions, but all the analysis is done using partial cross-spectra and is therefore silent on indirect causality chains. Hence, this part of literature is also part of our motivation to propose a more general framework.

Before defining the connectedness measures in frequency domain, we briefly discuss the notion of measuring connectedness introduced by Diebold and Yilmaz (2012) using the generalized forecast error variance decompositions (GFEVD), as we build on these ideas in frequency domain later in the text.

2.1 Measuring connectedness with variance decompositions

An intuitive way to measure the connectedness between variables is to consider an estimate of vector auto-regressive (VAR) process and its forecast error variance decomposition (FEVD). Formally, let us have the $n$-variate process $x_t = (x_{t,1}, \ldots, x_{t,n})$ described by the structural VAR($p$) at $t = 1, \ldots, T$ as

$$
\Phi(L)x_t = \epsilon_t,
$$

where $\Phi(L) = \sum_h \Phi_h L^h$ is an $n \times n$ $p$-th order lag-polynomial and $\epsilon_t$ is white-noise generated by (possibly non-diagonal) covariance matrix $\Sigma$. Assuming that the roots of $|\Phi(z)|$ lie outside the unit-circle, the VAR process has the following MA($\infty$) representation

$$
x_t = \Psi(L)\epsilon_t,
$$

where $\Psi(L)$ is an $n \times n$ matrix of infinite lag polynomials.

The usual FEVD derived directly from impulse responses of the system does not have straightforward interpretation as the shocks to variables are not identified. A shock to variable $j$ does not necessarily appear alone, i.e. orthogonally to shocks to other variables. Hence, to identify the shocks and derive meaningful FEVD we need to employ an identification scheme. Diebold and Yilmaz (2009) use the standard Sims’ recursive identification where the errors are standardized by Cholesky decomposition of the covariance matrix. The standardization matrix from Cholesky decomposition, however, depends on the ordering of the variables in the VAR system implying dependence of any measures devised using this identification scheme on reordering of variables. For this reason, Diebold and Yilmaz (2012) use the generalized VAR setting (Pesaran and Shin 1998) that mitigates the issue by imposing additional assumption of normality of the shocks. Convenient feature of this identification scheme is the possibility to consider a directional connectedness measures. In our work, we stay within the generalized
framework, although the spectral representation can be analogously defined to any other identification scheme and approximating model that can produce impulse response functions. With this respect the next sections present general results.

Generalized FEVD can be written in the form\footnote{Note to notation: \((A)_{j,k}\) denotes the \(j\)th row and \(k\)th column of matrix \(A\) denoted in bold. \((A)_{\cdot k}\) denotes the full \(j\)th row, likewise for columns.} (for detailed derivation of the formula see Appendix A)
\[
(\theta^H)_{j,k} = \frac{\sigma_{kk}^{-1} \sum_{h=0}^{H} ((\Psi_h \Sigma)_{j,k})^2}{\sum_{h=0}^{H} \sum_{j,j} (\Psi_h \Sigma \Psi'_h)_{j,j}},
\]
where \(\Psi_h\) is an \(n \times n\) matrix of coefficients corresponding to lag \(h\), and \(\sigma_{kk} = (\Sigma)_{k,k}\). The \((\theta^H)_{j,k}\) denotes the contribution of the \(k\)th variable of the system to the variance of forecast error of the element \(j\). Due to one of the notable implications of the generalized VAR framework the effects do not add up to one within columns by definition. To standardize the effects we define
\[
(\bar{\theta}^H)_{j,k} = \frac{(\theta^H)_{j,k}}{\sum_k (\theta^H)_{j,k}}.
\]

The connectedness measure is then defined as the share of variances in the forecasts contributed by other than own errors, or equally as ratio of the sum of the off-diagonal elements to sum of the whole matrix (Diebold and Yilmaz 2012)
\[
C^H = \frac{\sum_{j \neq k} (\bar{\theta}^H)_{j,k}}{\sum_{j,k} (\bar{\theta}^H)_{j,k}} = 100 \left(1 - \frac{\text{Tr} \{\bar{\theta}^H\}}{\sum (\bar{\theta}^H)_{j,k}}\right),
\]
where \(\text{Tr} \{\cdot\}\) is the trace operator. Hence, the connectedness is the relative contribution to the forecast variances from the other variables in the system. Note that although \(C^H\) measures connectedness of whole system, directional connectedness measures can be conveniently defined within this framework along the lines of exposition.

2.2 Spectral representation for variance decompositions and connectedness measures

Generalized forecast error variance decompositions (GFEVD) are central to measuring connectedness, hence to define frequency dependent measures, we need to consider its spectral counterpart. As can be noted from Equation 1 connectedness measure is based on impulse response functions \(\Psi_j\) defined in time domain. As a building block of the presented theory, we consider a frequency response function, \(\Psi(e^{-i\omega}) = \sum_h e^{-i\omega h} \Psi_h\), which can be simply obtained from Fourier transform of the coefficients \(\Psi\), with \(i = \sqrt{-1}\). A spectral density of \(x_t\) at frequency \(\omega\) can then be conveniently defined as a Fourier transform of MA(\(\infty\)) filtered series as
\[
S_x(\omega) = \sum_{h=-\infty}^{\infty} E(x_t x_{t-h}') e^{-i\omega h} = \Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega})
\]
The power spectrum \(S_x(\omega)\) describes how the variance of the \(x_t\) is distributed over the frequency components \(\omega\). Using the spectral representation for covariance, i.e. \(E(x_t x_{t-h}') = \int_{-\pi}^{\pi} S_x(\omega) d\omega\), following definition naturally introduces the frequency domain counterparts of variance decomposition.

(1)
Definition 2.1. Generalized causation spectrum over frequencies $\omega \in (-\pi, \pi)$ is defined as

$$(f(\omega))_{j,k} \equiv \frac{\sigma_{kk}^{-1} \left| (\Psi(e^{-i\omega}) \Sigma)_{j,k} \right|^2}{(\Psi(e^{-i\omega})\Sigma'\Psi'(e^{+i\omega}))_{j,j}},$$

where $\Psi(e^{-i\omega}) = \sum_h e^{-ih\omega}\Psi_h$ is the Fourier transform of the impulse response $\Psi$.

It is important to note that $(f(\omega))_{j,k}$ represents the portion of the spectrum of $j$th variable at frequency $\omega$ due to shocks in $k$th variable. In a sense, we can interpret the quantity as a within-frequency causation, as denominator holds spectrum of the $j$th variable (on-diagonal element of cross-spectral density of $x_t$) at given frequency $\omega$. Based on this notion, we will define a within frequency connectedness measures. The generalized causation spectrum can also be related to standard coherence measure, but only one way relation is taken into account in here. Thus the word causation as introduced in [Stiassny (1996)] is justified, as the weighing by respective variances and covariances brings restriction and identification assumptions of the generalized VAR and hence we can interpret the measure causally within the validity of the invoked assumptions on the system. The quantity we consider is different from that of [Stiassny (1996)], which is reflected in the word generalized.

To obtain a natural decomposition of original GFEVD to frequencies, we can simply weight the $(f(\omega))_{j,k}$ by the frequency share of variance of the $j$th variable. The weighting function can be defined as

$$\Gamma_j(\omega) = \frac{\left(\Psi(e^{-i\omega})\Sigma'\Psi'(e^{+i\omega})\right)_{j,j}}{2\pi \int_{-\pi}^{\pi} \left(\Psi(e^{-i\lambda})\Sigma'\Psi'(e^{+i\lambda})\right)_{j,j} d\lambda},$$

and represents the power of $j$th variable at given frequency, which sums through frequencies to a constant value of $2\pi$. Note that while the Fourier transform of the impulse response is in general a complex valued quantity, the generalized causation spectrum is the squared modulus of the weighted complex numbers, hence producing a real quantity.

The following proposition formalizes the discussion, and is central to the development of the connectedness measures in frequency domain.

Proposition 2.1. Suppose $x_t$ is wide-sense stationary with $\sigma_{kk}^{-1} \sum_{h=0}^{\infty} |(\Psi_h \Sigma)_{j,k}| < +\infty, \forall j, k$. Then

$$(\theta)_{j,k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_j(\omega) (f(\omega))_{j,k} d\omega.$$
that we define as the amount of forecast error variance created on a convex set of frequencies. The quantity is then given by integrating only over the desired frequencies $\omega \in (a,b)$.

Formally, let us have a frequency band $d = (a, b) : a, b \in (-\pi, \pi), a < b$. The generalised FEVD on frequency band $d$ is defined as

$$
(\theta_d)_{j,k} = \frac{1}{2\pi} \int_d \Gamma_j(\omega) \left( \hat{f}(\omega) \right)_{j,k} d\omega.
$$

(3)

Because the introduced relationship is an identity and the integral is linear operator, sum over disjoint intervals covering the whole range $(-\pi, \pi)$ will recover the original GFEVD. The following remark formalizes the fact.

**Remark 2.1.** Denote by $d_s$ an interval on the real line from the set of intervals $D$ that form a partition of the interval $(-\pi, \pi)$, such that $\cap d_s \in D d_s = \emptyset$, and $\cup d_s \in D d_s = (-\pi, \pi)$. Due to the linearity of integral and the construction of $d_s$ we have

$$(\theta_\infty)_{j,k} = \sum_{d_s \in D} (\theta_{ds})_{j,k}.$$  

Using the spectral representation of GFEVD, it is straightforward to define connectedness measures on a given frequency band that mimic the measures defined in (2).

**Definition 2.2.** Let us define scaled GFEVD on the frequency band $d = (a, b) : a, b \in (-\pi, \pi), a < b$ as

$$(\tilde{\theta}_d)_{j,k} = (\theta_{ds})_{j,k} \sum_k (\theta_\infty)_{j,k},$$

where $\theta_d$ and $\theta_\infty$ are defined as by Equation 3 and Proposition 2.1

- The within connectedness on the frequency band $d$ is then defined as

$$C_d^{W} = 100 \left( 1 - \frac{\text{Tr}\left\{ \tilde{\theta}_d \right\} }{\sum (\tilde{\theta}_d)_{j,k}} \right).$$

- The frequency connectedness on the frequency band $d$ is then defined as

$$C_d^{F} = 100 \left( \frac{\sum (\tilde{\theta}_d)_{j,k} - \text{Tr}\left\{ \tilde{\theta}_d \right\} }{\sum (\theta_\infty)_{j,k}} - \frac{\sum (\theta_\infty)_{j,k}}{\sum (\theta_\infty)_{j,k}} \right) = C_d^{W} \frac{\sum (\tilde{\theta}_d)_{j,k}}{\sum (\theta_\infty)_{j,k}}.$$

The Definition 2.2 works with two notions: the frequency connectedness and the within connectedness. The within connectedness gives us the connectedness effect that happens within the frequency band and is weighted by the power of the series on the given frequency band exclusively. On the other hand the frequency connectedness decomposes the overall connectedness defined in the Equation 2 into distinct parts that in sum give the original connectedness measure $C_\infty$. The following remark formalizes the notion of reconstruction of the overall connectedness.

**Remark 2.2 (Reconstruction of frequency connectedness).** Denote by $d_s$ an interval on the real line from the set of intervals $D$ that form a partition of the interval $(-\pi, \pi)$, such that $\cap d_s \in D d_s = \emptyset$, and $\cup d_s \in D d_s = (-\pi, \pi)$. We then have that

$$C_\infty = \sum_{d_s \in D} C_{d_s}^{F}.$$  

(4)
To illustrate the difference between the frequency and within connectedness, recall that the typical spectral shape of economic variables has the most power concentrated on low frequencies (long-term movements or trend). Hence, we could decompose the connectedness into two parts, the one that covers long-term movements and the one that covers the short-term movements. Suppose that 90% of the spectral density is concentrated in the long-term movements. Now, suppose that the connectedness on high frequencies is high, say 80%, and low on long-term, say 10%. These numbers represent the within connectedness. The total connectedness will be much closer to 10%, because the short-term connectedness of size 80% will be down-weighted by the very low amount of spectral density on the short-term frequencies. Otherwise said, even thought the short-term activities are very connected because the small share of variance on the short-term frequencies, this connection becomes negligible in the system connectedness. This can be seen clearly in the simulations in the following section.

We conclude the theoretical section with remark showing that the two concepts of within and frequency connectedness coincide when whole frequency band $d = (-\pi, \pi)$ is considered, i.e. the connectedness measure due to Diebold and Yilmaz (2012) is both within connectedness and frequency connectedness.

**Remark 2.3.** Let us have $d = (-\pi, \pi)$. We then have
\[ C_d^F = C_d^W. \] (5)

**Proof.** See Appendix.

### 2.3 Estimation of connectedness in frequency domain

The previous theory is devised in terms of theoretical quantities and the method of estimation deserves to follow. Primarily, the standard VAR framework is not the only way how to obtain the estimates of the behavior of the system. More advanced and often more suited approaches can be devised in many cases. For example, estimators that use shrinkage or Bayesian approaches might be a viable alternative in many cases. In our application, we restrict ourselves to use of the standard VAR and leave other methods for future investigation.

Having the estimates, two main issues remain. The first is the approximation of the MA($\infty$) representation of the series and the second is the estimation of the theoretical spectral densities.

The MA($\infty$) representation is used to compute the forecast error variance decomposition that relates percentage of mean squared error (MSE) of forecasts of variable $k$ due to shocks to variable $j$. Because the computation of these theoretical quantities is based on an infinite process, we make it feasible by a finite MA($H$) approximation. This is possible mainly due to the fact that if the system is stable, there must exist $H$, such that
\[
MSE(x_{k,H}) - MSE(x_{k,H+1}) < \epsilon,
\]
meaning that the error due to approximation disappears with growing $H$. Reverse would mean that innovations would have permanent impact. Hence, we can in principle use two ways to approximate the decomposition. First, we can use an ad hoc selected $H$ that is beyond doubt high enough. Or second, we can use a measure of similarity of matrices to choose the appropriate $H$ during the approximation. The $\hat{\Psi}_h$ coefficients are then computed through standard recursive scheme $\hat{\Psi}_0 = I$, $\hat{\Psi}_h = \sum_{j=1}^{\max\{h,p\}} \Phi(j)\hat{\Psi}_{h-1}$, where $p$ is the order of VAR and $h \in \{1, \ldots, H\}$. Here we note that by studying the quantities in the frequency domain,
$H$ serves only as an approximation factor, and has no interpretation as in the time domain. In the applications, we advice to set the $H$ high enough to obtain better approximation of the quantities at all frequencies, especially when small frequency bands are of interest.

Secondly, the spectral quantities are estimated using standard discrete Fourier transforms. The following definition specifies accurately the used estimates of the quantities.

**Definition 2.3.** The cross-spectral density on the interval $d = (a, b) : a, b \in (-\pi, \pi), a < b$

\[
\int_d \Psi(e^{-i\omega}) \Sigma \Psi'(e^{+i\omega}) d\omega
\]

is estimated as follows

\[
\sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega),
\]

for $\omega \in \left\{ \left\lfloor \frac{aH}{2\pi} \right\rfloor, \ldots, \left\lfloor \frac{bH}{2\pi} \right\rfloor \right\}$ where

\[
\hat{\Psi}(\omega) = \sum_{h=0}^{H-1} \hat{\Psi}_h e^{-2i\pi\omega/H},
\]

and $\hat{\Sigma} = \hat{\epsilon}'\hat{\epsilon}/(T - z)$, where $z$ is correction for loss of degrees of freedom and depends on VAR specification.

The decomposition of the impulse response function at the given frequency band is then estimated as

\[
\hat{\Psi}(d) = \sum_{\omega} \hat{\Psi}(\omega), \text{ for } \omega \in \left\{ \left\lfloor \frac{aH}{2\pi} \right\rfloor, \ldots, \left\lfloor \frac{bH}{2\pi} \right\rfloor \right\}.
\]

Using this definition the estimate of the generalized causation spectrum over the interval $d$ is then defined as follows

\[
\left( \hat{f}(\omega) \right)_{j,k} = \frac{\hat{\sigma}_{kk}^{-1} \left( \left( \hat{\Psi}(\omega) \hat{\Sigma} \right)_{j,k} \right)^2}{\left( \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega) \right)_{j,j}}.
\]

By employing the estimate of weighting function

\[
\hat{\Gamma}_j(\omega) = \frac{\left( \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega) \right)_{j,j}}{(\Omega)_{j,j}},
\]

where $\Omega = \sum_{\omega} \hat{\Psi}(\omega) \hat{\Sigma} \hat{\Psi}'(\omega)$, we estimate the decomposed GFEVD to a frequency band as

\[
\hat{\theta}_{j,k}(d) = \sum_{\omega} \hat{\Gamma}_j(\omega) \left( \hat{f}(\omega) \right)_{j,k}, \text{ for } \omega \in \left\{ \left\lfloor \frac{aH}{2\pi} \right\rfloor, \ldots, \left\lfloor \frac{bH}{2\pi} \right\rfloor \right\}.
\]

Then the connectedness measures $\hat{C}^W$ and $\hat{C}^F$ at a given frequency band of interest can be readily derived by plugging the $\hat{\theta}_{j,k}(d)$ estimate into the Definition 2.2.

\[\text{The whole estimation is done using the package frequencyConnectedness in R software. The package is available on } \text{https://github.com/tomaskrehlik/frequencyConnectedness}\]
Connectedness Connectedness without correlation

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Table 1: Simulation results. The first three columns describe parameters for the simulation as described in Equation 6. We set $\beta = \beta_1 = \beta_2$. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is sample standard deviation.

3 Generating the frequency dependent connectedness

To motivate the usefulness of the proposed measures, we study the processes that generate frequency dependent connectedness by simulations. We look at connectedness that are induced through cross-sectional correlations or interactions between bivariate autoregressive (AR) processes. We illustrate the emergence of connectedness and their spectral footprints through change in coefficients in the simplest bivariate VAR(1) case. Suppose the simplest case that nevertheless illustrates the mechanics generating the data from the following equations

$$
y_{1,t} = \beta_1 y_{1,t-1} + s y_{2,t-1} + \epsilon_{1,t},
$$

$$
y_{2,t} = s y_{1,t-1} + \beta_2 y_{2,t-1} + \epsilon_{2,t},
$$

where $(\epsilon_{1,t}, \epsilon_{2,t}) \sim N(0, \Sigma)$ with $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$.

By altering the true coefficients generating the data, we study several cases with known values of theoretical connectedness estimates. We start with a symmetric processes with $\beta = \beta_1 = \beta_2$ with three important cases generating distinctly connected variables $y_{1,t}$ and $y_{2,t}$. First case is the $\beta = \beta_1 = \beta_2 = 0$, when we have two independent processes, which have connectedness zero at all frequencies. Secondly, we study the connectedness of two symmetrically connected AR processes with the parameter $\beta = \beta_1 = \beta_2 = 0.9$ and $s = 0.09$ or $\beta = \beta_1 = \beta_2 = -0.9$ and $s = -0.09$ generating equal total connectedness with different sources from low and high frequencies of cross-spectral densities for positive and negative values of coefficients respectively.

In addition to motivating the importance of frequency dynamics of connectedness, we also show the importance of cross-sectional correlations, which translates to all frequencies, and may bias the connectedness measures. Hence for all cases, we consider two extremes of cross-sectional dependence: no correlation $\rho = 0$ and correlation of $\rho = 0.9$. To show how the cross-sectional correlations impact the connectedness measures, we compute the measures with additional step in estimation, considering only diagonals of the covariance matrix of residuals, hence removing the cross-sectional dependence. In this way, we disentangle the influence of correlations from the true dynamics. In the text we always present only the estimates on the simulated data, and save the true values of measures in the Table 4 in the appendix.

The Table I shows the results. We can see that the system connectedness of two unconnected and uncorrelated processes is practically zero on both total and all spectral parts. In case of

3 Other more elaborate simulation scenarios are outlined in the R package provided with the paper.
connectedness without correlation

\[ \beta_1 \beta_2 \]

\[ s \rho \]

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Table 2: Simulation results. The first three columns describe parameters for the simulation as described in Equation 6. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is simple sample standard deviation.

correlated noises, the total connectedness with estimated correlation matrix is estimated around 45 with equal footprint on all the scales. Considering only diagonal elements from estimated covariance matrix of residuals, and hence removing the cross-sectional dependence correctly estimates the connectedness zero at all frequencies.

In the case with AR coefficient equal to 0.9, the uncorrelated case shows that the connection between the processes is on the long-term part (as is expected due to the spectral density of the underlying process). On the other hand, introducing correlation increases the total connectedness and most of all obfuscates the source of the dynamics. Considering only diagonals of covariance matrix of residuals, we can see that the correlation in the estimated covariance matrix correctly exposes the underlying dynamics. The remaining case with coefficient equal to -0.9 is very similar to the previous case with the difference that the spectral mass is concentrated on the short frequencies. Otherwise the qualitative results remain the same.

It is important to note that while coefficients with opposite signs of 0.9 and -0.9 generate the time series with equal connectedness, its source is from different parts of spectra. This example motivates the usefulness of our measures, which are able to locate precisely the part of cross-spectra generating the connectedness.

Next we move to the case where the two processes are not symmetric. With the simulation, we want to illustrate two important cases, how the connectedness arises. First, let us keep the parameter \( s \) that governs the connection of the two processes through lagged observation constant and change the spectral structure of the processes through the coefficient \( \beta_2 \). The Table 2 shows that the connectedness in this case is arising due to the increase in spectral similarity of the processes in question. One could take parallel from physics and state if the two processes (time-series) can resonate, even a relatively small interaction coefficient is capable to create strong connectedness.

On the other hand, keeping the structure of the processes constant and increasing the parameter of interconnection increases the connectedness as is documented by Table 3.

This simulation suggests a possible sources of connectedness and motivates the usefulness of our measures. The role of covariance among the processes can be studied through exclusion of the covariance terms, the role of similarity can be examined through individual spectral densities, however, as mentioned most of the economic series have similar spectral densities (Granger 1966). Our measures estimate the rich dynamics precisely.
4 Connectedness of risk in major assets of the US market

The question how stock markets are connected has been studied by the literature extensively in past decades. From studies focusing on causality effects, comovement, spillovers, connectedness, and systemic risk, researchers primarily try to answer the question using methods measuring the aggregate effects. In this section, we argue that it is important to measure frequency sources of connectedness, as shocks to volatility will impact future uncertainty differently. For example, fundamental changes in investors’ expectations will impact the market in long term. These expectations are then transmitted to surrounding assets in the portfolio differently than shocks having short-term impact. In a financial system, asset prices driven by consumption growth with different cyclical components will naturally generate shocks with heterogeneous responses, and thus various sources of connectedness creating short-term, medium-term, and long-term systemic risk.

The early literature measuring the connectedness of stock markets was predominantly interested in contagion effects in market prices during crises. In already seminal paper, Forbes and Rigobon (2002) have, however, shown that if we account for volatilities of the price processes, the contagion effects disappear. This led to a rather strong statement of no-contagion, and interdependence among the markets remained the main effect of interest. Tse and Tsui (2002) concentrated on investigating the connection in the multivariate GARCH framework. They report high cross-correlations on Forex market, national stock market, and Hang Seng sectoral indices. Bae et al. (2003) investigate the co-incidence of the extreme returns across markets and connect this measure by extreme value theory. They evaluate the contagion effects among various parts of the world, such as Latin America, Asia, and the United States, finding high coincidence of negative returns across markets. Engle et al. (2012) provide exhaustive review of empirical literature on volatility spillovers.

A broader picture was later provided by Diebold and Yilmaz (2009) who explicitly investigated volatilities and returns separately, and uncovered contagion effects in volatilities. In the same paper, authors side-stepped the controversial topic of contagion, which had already been tied predominantly to financial crises and introduce the concept of spillovers that refer to varying interdependency between the markets. Borrowing from both contagion and interdependence notions, Diebold and Yilmaz (2009) define a rigorous framework for measuring spillovers of returns and volatility across markets, coined a connectedness in their subsequent work (Diebold and Yilmaz, 2014). The methodology has been successfully used to measure connectedness effects in the literature by hundreds of studies in few years. Still the literature is silent about origins of the connectedness in stock markets.

4.1 Data

Volatility, as one of the mostly studied quantities in the financial literature, is also largely perceived as a risk measure. Hence, considering the connectedness of volatility, we investigate the question how the risk in markets is connected at different frequencies. We study the intra-market connectedness of seven major stocks representing largest sectors within the US economy. Although the companies might live in the same economy, they are generally interconnected by different aspects of it and as argued earlier, these aspects may have different impact at frequencies.

Sectors are defined in accordance with the Global Industry Classification Standard (GICS) and in a similar manner as in Beber et al. (2011).
Concretely, we investigate the connectedness between financial, technology, consumer staples that are pro-cyclical, consumer staples that are counter-cyclical, communication, energy, and health sectors. From each sector, we select the most liquid stock to represent the sector\(^5\) namely, Bank of America Corporation (BAC), Microsoft Corporation (MSFT), Walt Disney Company (DIS), Coca-Cola Company (KO), AT&T (T), Exxon Mobil Corporation (XOM), and Pfizer (PFE) representing the sectors in the same order as named earlier in this paragraph.

For the computation of volatility, we restrict the analysis to daily logarithmic realized volatility computed using 5-minute returns\(^6\) during the 9:30 a.m. to 4:00 p.m. business hours of the New York Stock Exchange (NYSE). The data are time-synchronized by the same time-stamps, eliminating transactions executed on Saturdays and Sundays, U.S. federal holidays, December 24 to 26, and December 31 to January 2, because of the low activity on these days, which could lead to estimation bias. The data span years 2005 to 2016 providing sample of 2826 trading days. The descriptive statistics of the data can be found in the appendix Table 5. The period under study is informative in terms of market development, sentiment, and expectations since we cover the 2007–2008 financial crisis and its aftermath years. The original raw data were obtained from the TICK Data.\(^7\)

### 4.2 Time-frequency systemic risk measurement

One of the issues that has recently gained importance in volatility modelling is giving up the assumption of global stationarity of the data (Stărică and Granger, 2005; Engle and Rangel, 2005) and focusing on local stationarity instead. When studying connectedness of market risk using variance decompositions, it is important to face nonstationarity of realized volatilities as zero frequency may dominate the rest of the frequencies in case we study unconditional connectedness. The discussion gains importance when studying frequency dynamics, as applying

\(^5\)We chose the stocks in order to best capture the total capitalization of the sector. Note that this is an approximate extent that varies over time.

\(^6\)Realized volatility for a given day is computed as sum of squared intra-day returns, hence we are using the standard simplest measure.

\(^7\)https://www.tickdata.com/
our measures blindly to the nonstationary data would result to false inferences.

Giving up the assumption of global stationarity of the data, we assume the dynamics come from shifts in the unconditional variance of returns. This leads us to convenient approximation of nonstationary data locally by stationary models. In essence, our approach is closely related to the one taken by Stărică and Granger (2005), although we study multivariate system with quite different tools. We use the spectral representations of variance decompositions to recover the time-frequency dynamics of connectedness with moving window of approximately one year (250 trading days), where we confirm stationarity of volatility. Vector auto-regression with two lags including constant and trend on the logarithm of volatilities is used to capture the dynamics in the window.

Focusing on locally stationary structure of the data, we do not report unconditional frequency connectedness table as commonly done in the literature. Instead, we study the time-frequency dynamics of connectedness. Figure reports the time dynamics of the total connectedness of system as measured by time domain variance decompositions in the left part. One can quickly infer that the connectedness was rather low during first two year period increasing dramatically during the 2007–2008 crisis, and varying in the aftermath of the crisis considerably. Right plot of the Figure presents the decomposition of the total connectedness into frequency bands up to one week, one week to one month, one month to one quarter, and one quarter to one year computed as \( C_{d_4} \) on the bands corresponding to \( d_1 \in [1, 5], d_2 \in (5, 20], d_3 \in (20, 60], \) and \( d_4 \in (60, 250] \) days. Note the lowest frequency is bounded at each time point by the window length.

The decomposition shows rich time-frequency dynamics of connections. Focusing on the frequency dynamics, largest portion of connections is created on a band from one week up to one month, although higher frequencies up to one week play similar role in connectedness. The most interesting observations can be made when considering time dynamics of the frequency connections, as we cannot see any clear pattern of some frequency band dominating all others. Instead, we infer rich time-frequency dynamics. While connectedness has been driven mostly by information up to one month (\( d_1 \) and \( d_2 \)) during the first three years of the sample, the structure changed dramatically during the year 2008, and this change lasted until the end of

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\(^*\)We have experimented with different lag lengths with no changes in results. This only confirms the appropriateness of the approach, as large changes in time-frequency dynamics due to different lags in the approximating VAR model would point to nonstationarities within windows, where larger number of lags would be approximating the information in the low frequencies. In some sense, this analysis serves as a robustness check. We make these results available upon request.
the year 2012. During this period, we observe rich dynamics with lower frequencies playing role in connectedness. Before this dynamics returns to the connectedness structure again during the year 2015, there is two year change with higher frequencies driving the connectedness.

The Figure 2 shows the time-frequency dynamics from a different point of view, which serves as a helpful complementary visualization. In this figure, frequency bands form coloured ribbons where colour shows the strength of the connection, while horizontal axis still holds time. One can view this representation as looking at the three dimensional space of connectedness at time and frequency domains from top, where the third axis showing the strength of connectedness at each time-frequency point is highlighted by color. The heat map representation is useful as one can more clearly see the decomposition of the connectedness into time-frequency space.

The Figure 3 in the Appendix shows bigger format picture with (5%, 95%) confidence band of the decompoistion estimated by bootstrap. The confidence band is sufficiently tight around the estimate to be informative, and confirms statistically significant short-term, medium-term, and long-term connections. The confidence bands for other decompositions are also contained therein and will not be refered to.

Economically, periods with connectedness being created in high frequencies are periods when stock markets seem to process information rapidly, and a shock to one asset in the system will have impact mainly in a short-term (with frequency response at high frequencies). In case the connections come from the opposite part of the cross-spectral density, lower frequencies, it points us to the belief that shocks are being transmitted for longer periods (with frequency response at low frequencies mainly). This behavior may be attributed to fundamental changes in investor’s expectations, which impact the market in longer term. These expectations are then transmitted to surrounding assets in the portfolio. In a financial system where asset prices are driven by consumption growth with different cyclical components, shocks with heterogeneous responses create linkages with various degree of persistence, hence, various sources of connectedness and systemic risk. Our results point us to the fact that when evaluating systemic risk of a financial system, we should pay attention to short-term, medium-term, as well as long term linkages, as they all play important role in the system. Before making further conclusions about the nature of connectedness in US stock market, we look deeper into its sources.

Until now, we have focused on the decomposition of the connections to frequency bands, guaranteeing they will always sum to total connectedness. The frequency components are in fact within spillovers, or causation spectra at frequency band weighted by the variance share at the given band. Hence in case low frequencies hold large amount of information, it will overweight other frequencies. While the frequency decomposition considering power of shocks is important for relative comparison, it is useful to look at unweighted connections as well. Ignoring information outside the considered band, connections within frequency bands can be understood as pure unweighted connections. The Figure 3 shows the within sectoral connectedness of the market in left plot. All frequencies share very similar time dynamics, hence the rich time-frequency decomposition found in previous part is mainly driven by power of frequency responses, as expected.

The main reason why we look at the pure within connectedness is to study the effect of cross-sectional dependence on the connectedness. When using variance decompositions, we are mainly interested in finding causal effects, but these can be biased due to strong contemporaneous relations. To find if there is such a bias in the connections we measure, we adjust the correlation matrix of VAR residuals by the cross-sectional correlations.

The right plot from the Figure 3 shows within connectedness adjusted for this correlation effect. Strikingly, the structure changes dramatically, pointing us to the result that the high frequency connectedness is mainly driven by cross-sectional correlations, while connectedness at
lower frequencies is not affected so heavily, mainly during the crisis. One can infer that increase of system connectedness during the crisis is mainly created by increase in contemporaneous short-term correlations, and causal longer term connectedness.

While studying connectedness of the whole system, the time-frequency dynamics of directional connectedness including pairwise connections, influences “from”, and “to” the considered stocks may be of interest. Inevitably, reporting all these interesting quantities would substantially inflate this already pregnant text. The main purpose of our work is to introduce the quantities and their proper usage in different situations, hence we leave their full usage to all applications which are sure to come in near future.\footnote{The time-frequency quantities from directional connectedness table are easily computable using the package we provide to the paper.}

5 Conclusion

In this work, we contribute to the understanding of connectedness between economic variables by proposing to measure its frequency dynamics. Based on the spectral representations of variance decompositions and connectedness measures, we provide a general framework for disentangling the sources of connectedness between economic variables. As shocks to economic activity impact variables at different frequencies with different strength, we view the frequency domain as a natural place for measuring the connectedness between economic variables.

As noted by Diebold and Yilmaz (2009, 2012), and later Diebold and Yilmaz (2014), variance decompositions from approximating models are convenient framework for empirical measurement of connectedness. Diebold and Yilmaz define the measures based on assessing shares of forecast error variation in one variable due to shock arising in another variable in the system. Focusing on frequency responses of shocks instead, we are interested in the portion of the spectrum as counterpart of variance at given frequency band that is attributed to shocks in another variable. Moreover, we elaborate on the role of correlation of the residuals in the magnitude and spectral shape of the connectedness.

In the empirical part, we investigate connectedness of stock market risk. We approximate the data locally and obtain rich time-frequency dynamics of connectedness. We conclude that

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**Figure 3:** Dynamic within connectedness of the US market risk on frequency bands. Left plot presents the relative connectedness within the frequency band, $C_{d_s}^W$ with $d_1 \in [1,5]$ days in solid bold, $d_2 \in (5,20]$ solid, $d_3 \in (20,60]$ dotted bold, and $d_4 \in (60,250]$ dotted lines. Right plot presents relative connectedness within the frequency band without the effect of cross-sectional correlations.
the dynamics is mainly driven by frequencies from one day up to one month, although this does not hold in the period of turmoil with high levels of uncertainty. In these periods the total connectedness increases, and the increase is mainly due to contemporaneous correlations at short-term, and causal longer term connectedness. Economically, periods with connectedness being created in high frequencies are periods when stock markets seem to process information rapidly, and a shock to one asset in the system will have impact mainly in a short-term. In case the connections come from the lower frequencies, it points us to the belief that shocks are being transmitted for longer periods. This behavior may be attributed to changes in investor’s expectations, which impact the market in longer term. These expectations are then transmitted to surrounding assets in the portfolio. The results underline the importance of proper measurement of dynamics across time and frequencies and emphasize the important role of cross-sectional correlation in the connectedness origins.

The frequency based approach opens new fascinating routes in understanding connectedness of economic variables with important implications for measurement of systemic risk. The further research applying our measures to wider areas of interest will be important in uncovering the connections of assets within market, or industry, connections across asset classes, international markets, and provide grounds for further research in risk management, portfolio allocation, or business cycle analysis where understanding origins of connections is essential.
References


A Derivation of the GFEVD

Let us have the MA(∞) representation of the GVAR model (details in (Pesaran and Shin 1998, Dees et al. 2007)) given as

\[ x_t = \Psi(L)\epsilon_t, \]  

(7)

with the covariance matrix of the errors \( \Sigma \). Because the errors are assumed to be serially uncorrelated, the total covariance matrix of the forecast error conditional at the information in \( t - 1 \) is

\[ \Omega_H = \sum_{h=0}^{H} \Psi_h \Sigma \Psi_h'. \]

(8)

Next we define the covariance matrix of the forecast error conditional on knowledge of today’s shock and future expected shocks to \( j \)-th equation. Starting from the conditional forecasting error,

\[ \gamma^k_t(H) = \sum_{h=0}^{H} \Psi_h [\epsilon_{t+h-H} - E(\epsilon_{t+h-H}|\epsilon_{k,t+H-h})], \]

(9)

assuming normal distribution, we have

\[ \gamma^k_t(H) = \sum_{h=0}^{H} \Psi_h [\epsilon_{t+h-H} - \sigma_{kk}^{-1}(\Sigma)_{j,k}\epsilon_{k,t+H-h}]. \]

(10)

Finally the covariance matrix is

\[ \Omega^k_H = \sum_{h=0}^{H} \Psi_h \Sigma \Psi_h' - \sigma_{kk}^{-1} \sum_{h=0}^{H} \Psi_h (\Sigma)_{j,k} (\Sigma)_{j,k}' \Psi_h'. \]

(11)

Then

\[ \Delta_{(j)kH} = (\Omega_{H} - \Omega^k_{H})_{j,j} = \sigma_{kk}^{-1} \sum_{h=0}^{H} ((\Psi_h \Sigma)_{j,k})^2 \]

(12)

is the unscaled \( H \)-step ahead forecast error variance of \( k \)-th component with respect to \( j \)-th component. Scaling the equation yields the desired

\[ (\theta_H)_{j,k} = \frac{\sigma_{kk}^{-1} \sum_{h=0}^{H} ((\Psi_h \Sigma)_{j,k})^2}{\sum_{h=0}^{H} (\Psi_h \Sigma \Psi_h')_{j,j}} \]

(13)
B Proofs

Proposition 2.1. To prove the equality we need the following:

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \Gamma_j(\omega) (f(\omega))_{j,k} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\omega}) \Sigma \Psi'(e^{i\omega}))_{j,j} \sigma_{kk}^{-1} \left| \left( \Psi(e^{-i\omega}) \Sigma \right)_{j,k} \right|^2 d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\omega}) \Sigma \Psi'(e^{i\omega}))_{j,j} d\lambda \left( \sigma_{kk}^{-1} \left| \left( \Psi(e^{-i\omega}) \Sigma \right)_{j,k} \right|^2 \right) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{kk}^{-1} \left| \left( \Psi(e^{-i\omega}) \Sigma \right)_{j,k} \right|^2 d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\omega}) \Sigma \Psi'(e^{i\omega}))_{j,j} d\lambda
\]

\[
= \sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left( (\Psi_h \Sigma)_{j,k} \right)^2
\]

\[
= \left( \theta_{\infty} \right)_{j,k}
\]

Hence, the proof essentially simplifies to proving two things

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \sigma_{kk}^{-1} \left| \left( \Psi(e^{-i\omega}) \Sigma \right)_{j,k} \right|^2 d\omega = \sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left( (\Psi_h \Sigma)_{j,k} \right)^2
\]

(14)

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} (\Psi(e^{-i\omega}) \Sigma \Psi'(e^{i\omega}))_{j,j} d\lambda = \left( \sum_{h=0}^{\infty} (\Psi_h \Sigma \Psi_h') \right)_{k,k}
\]

(15)

(16)

For the following steps we will leverage the standard integral

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(u-v)} d\omega = \begin{cases} 
1 & \text{for } u = v \\
0 & \text{for } u \neq v.
\end{cases}
\]

(17)

This integral is mostly useful in cases when we have series \( \sum_{h=0}^{\infty} \phi_h \psi_h \) and we want to arrive to spectral representation. Note that \( \sum_{h=0}^{\infty} \phi_h \psi_h = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \phi_u \psi_v e^{i\omega(u-v)} d\omega \). Leveraging this knowledge we prove the Equation 15.
\[
\sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left( \Psi_h^* \Sigma \right)_{j,k}^2 = \sigma_{kk}^{-1} \sum_{h=0}^{\infty} \left( \sum_{z=1}^{n} \left( \Psi_h \right)_{j,z} \left( \Sigma \right)_{z,k} \right)^2 \\
= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \left( \sum_{x=1}^{n} \left( \Psi_u e^{i\omega u} \right)_{j,x} \left( \Sigma \right)_{x,k} \right) \left( \sum_{y=1}^{n} \left( \Psi_v e^{-i\omega v} \right)_{j,y} \left( \Sigma \right)_{y,k} \right) e^{i\omega(u-v)} d\omega \\
= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \left( \sum_{x=1}^{n} \left( \Psi_u e^{i\omega u} \right)_{j,x} \left( \Sigma \right)_{x,k} \right) \left( \sum_{y=1}^{n} \left( \Psi_v e^{-i\omega v} \right)_{j,y} \left( \Sigma \right)_{y,k} \right) d\omega \\
= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{x=1}^{n} \left( \Psi \left( e^{i\omega} \right) \right)_{j,x} \left( \Sigma \right)_{x,k} \right) \left( \sum_{y=1}^{n} \left( \Psi \left( e^{-i\omega} \right) \right)_{j,y} \left( \Sigma \right)_{y,k} \right) d\omega \\
= \sigma_{kk}^{-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \Psi \left( e^{-i\omega} \right) \right|_{j,k}^2 d\omega
\]

(18)

We use the switch to the spectral representation of the MA coefficients in the second step. The rest is manipulation with the last step invoking the definition of modulus squared of a complex number to be defined as \(|z|^2 = zz^*\). Note that we can use this simplification without loss of generality, because the \(MA(\infty)\) representation that is described by the coefficients \(\Psi_h\) has always symmetric spectrum.

Next we concentrate on the Equation [16] leveraging similar steps and the positive semidefiniteness of the matrix \(\Sigma\) that ascertains that there exists \(P\) such that \(\Sigma = PP'\).

\[
\sum_{h=0}^{\infty} \left( \Psi_h \Sigma \Psi_h' \right) = \sum_{h=0}^{\infty} \left( \Psi_h P \right) \left( \Psi_h P \right)' \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \left( \Psi_u e^{i\omega u} P \right) \left( \Psi_v e^{-i\omega v} P \right)' d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{u=0}^{\infty} \left( \Psi_u e^{i\omega u} P \right) \sum_{v=0}^{\infty} \left( \Psi_v e^{-i\omega v} P \right)' d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \Psi \left( e^{i\omega} \right) P \right) \left( \Psi \left( e^{-i\omega} \right) P \right)' d\omega \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \Psi \left( e^{i\omega} \right) \Sigma \Psi' \left( e^{-i\omega} \right) \right) d\omega
\]

(19)

This completes the proof. \(\square\)
Remark 2.2. Using the Remark 2.1 and appropriate substitutions, we have:

\[
\sum_{d_z \in D} C^F_{d_z} = \sum_{d_z \in D} \left( \frac{\sum (\tilde{\theta}_{d_z})_{j,k}}{\sum (\tilde{\theta}_\infty)_{j,k}} - \frac{\text{Tr}\{\tilde{\theta}_{d_z}\}}{\sum (\tilde{\theta}_\infty)_{j,k}} \right) = 1 - \frac{\sum_{d_z \in D} \text{Tr}\{\tilde{\theta}_{d_z}\}}{\sum (\tilde{\theta}_\infty)_{j,k}} = 1 - \sum_{d_z \in D} \text{Tr}\{\tilde{\theta}_{d_z}\}/\sum (\tilde{\theta}_\infty)_{j,k} = C_\infty
\]  

(20)

where the next to last equality follows from the linearity of the trace operator.

Remark 2.3. Using the definition of the connectedness, we have

\[
C^W_{(-\pi,\pi)} = C_\infty
\]  

(21)

\[
C^F_{(-\pi,\pi)} = \frac{(\bar{\theta}_{(-\pi,\pi)})_{j,k}}{n} - \frac{\text{Tr}\{\tilde{\theta}_\infty\}}{\sum (\tilde{\theta}_\infty)_{j,k}} = \frac{n}{n} - \frac{\text{Tr}\{\tilde{\theta}_\infty\}}{\sum (\tilde{\theta}_\infty)_{j,k}} = 1 - \frac{\text{Tr}\{\tilde{\theta}_\infty\}}{\sum (\tilde{\theta}_\infty)_{j,k}} = C_\infty
\]  

(22)

\square
### Table 3: Simulation results. The first three columns describe parameters for the simulation as described in equation 6. The results are based on 100 simulations of VAR with the specified parameters of length 1000 with a burnout period of 100. The estimate is computed as mean of the 100 observations and the standard error is simple sample standard deviation. The numbers are multiplied by hundred.
### Table 4: The true values for connectedness in the VAR settings.

<table>
<thead>
<tr>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\rho$</th>
<th>Connectedness</th>
<th>$(\pi/2, \pi)$</th>
<th>$(\pi/4, \pi/2)$</th>
<th>$(0, \pi/4)$</th>
</tr>
</thead>
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<tr>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
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<td>44.75</td>
<td>44.75</td>
<td>44.75</td>
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<td>0.9</td>
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<td>40.50</td>
<td>0.30</td>
<td>0.90</td>
<td>41.15</td>
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<td>0.9</td>
<td>0.0</td>
<td>49.47</td>
<td>44.25</td>
<td>44.41</td>
<td>49.51</td>
</tr>
<tr>
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<td>-0.9</td>
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<td>40.50</td>
<td>40.77</td>
<td>0.34</td>
<td>0.24</td>
</tr>
<tr>
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<td>-0.9</td>
<td>0.0</td>
<td>41.28</td>
<td>41.01</td>
<td>45.22</td>
<td>45.22</td>
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<td>0.4</td>
<td>0.0</td>
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<td>0.88</td>
<td>7.48</td>
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<td>0.32</td>
<td>0.80</td>
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<td>44.25</td>
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<tr>
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<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
<td>0.45</td>
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<td>0.0</td>
<td>44.76</td>
<td>44.26</td>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
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<td>3.33</td>
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<td>3.33</td>
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<td>23.08</td>
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<td>23.08</td>
</tr>
<tr>
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<td>48.51</td>
<td>45.13</td>
<td>38.84</td>
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### Table 5: Descriptive statistics of the BPV realised volatilities of the sample.

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<th>BAC</th>
<th>DIS</th>
<th>KO</th>
<th>MSFT</th>
<th>PFE</th>
<th>T</th>
<th>XOM</th>
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</thead>
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<td>Mean</td>
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<td>0.012</td>
<td>0.009</td>
<td>0.012</td>
<td>0.012</td>
<td>0.011</td>
<td>0.012</td>
</tr>
<tr>
<td>Median</td>
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<td>0.010</td>
<td>0.008</td>
<td>0.010</td>
<td>0.010</td>
<td>0.009</td>
<td>0.010</td>
</tr>
<tr>
<td>Standard Dev.</td>
<td>0.018</td>
<td>0.007</td>
<td>0.005</td>
<td>0.006</td>
<td>0.006</td>
<td>0.007</td>
<td>0.007</td>
</tr>
</tbody>
</table>

25
Figure 4: The decomposition of spillovers with cross-sectional dependence. The individual lines represent spillover measures at a given frequency band, more concretely: a) connectedness from one day to one week, b) connectedness from one week to one month, c) connectedness from one month to one quarter, and d) connectedness from one quarter to one year. Shaded area represents the space between 5% and 95% quantiles of the bootstrapped measure.