Variation margins, fire sales, and information-constrained optimality

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Abstract

In order to share risk, protection buyers trade derivatives with protection sellers. Protection sellers’ actions affect the riskiness of their assets and therefore counterparty risk. Because these actions are unobservable there is moral hazard, which hinders risk sharing. To mitigate this problem, privately optimal derivative contracts involve variation margins. When margins are called, protection sellers must liquidate some assets, depressing asset prices. This tightens the incentive constraints of other protection sellers, reducing their ability to provide insurance. Despite such externalities, equilibrium is information-constrained efficient as investors, who benefit from fire sales in which they buy underpriced assets, find it optimal to supply insurance against the risk of fire sales.

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1 Introduction

In the aftermath of the crisis regulators have been calling for higher margins in derivative contracts, while cautioning that margins could lead to inefficient fire-sales and negative pecuniary externalities.\textsuperscript{1} We offer a general equilibrium analysis of this issue, investigating whether privately optimal variation margins are socially optimal.\textsuperscript{2}

Our model features three types of agents: protection buyers, protection sellers and investors. Protection sellers are, e.g., investment banks or specialised insurance companies. Protection buyers, e.g., commercial banks, share risk with protection sellers by trading derivative contracts. Protection sellers are endowed with risky assets, backing the contractual payments implied by their derivative positions. But they have limited liability, which creates counterparty risk, as protection sellers default when contractual payment exceed the value of their assets.\textsuperscript{3} Protection sellers can, however, take risk-management actions to limit the risk of their assets. The key friction in our model is that such actions are unobservable, i.e., there is moral hazard.\textsuperscript{4} Finally, investors, e.g., sovereign funds, or sovereign funds can hold the risky assets of protection sellers, but are less efficient at doing so, e.g., because they are less able to manage and bear risk. We also assume there is a publicly observable signal on the future value of protection buyers’ risky assets. For example, if protection buyers are commercial banks hedging the risk on their real estate loans, the signal can be provided by a retail estate market index. The signal occurs after initial contracting, but before protection sellers’ risk management.

\textsuperscript{1}E.g., the Committee on the Global Financial System (2010) or the European Systemic Risk Board (2017).
\textsuperscript{2}McDonald and Paulson (2015) offer a very informative discussion of variation margins.
\textsuperscript{3}Fleming and Sarkar (2014) discuss counterparty default in the Lehman bankruptcy.
\textsuperscript{4}In our baseline, protection sellers must exert costly effort to limit downward risk, as in Holmström and Tirole (1997). We also consider a second specification in which protection sellers can engage in risk-shifting as in Jensen and Meckling (1976). In both cases, the undesirable action hurts protection buyers by increasing the probability of counterparty default.
Our first contribution is to characterize the information-constrained optimum, i.e., the second best. It is the set of consumptions and asset allocations, contingent on all publicly observable information, that maximizes a weighted average of the three types of agents’ expected utilities, subject to incentive, participation, and resource constraints.

The second best has two key characteristics. First, moral hazard prevents perfect risk sharing. Protection sellers’ incentive constraints limit their ability to provide insurance, driving a wedge between their marginal rate of substitution and that of protection buyers and investors. Second, following a bad signal (which in the example of real estate loans mentioned above, could be a drop in the real estate index) a transfer of assets from protection sellers to investors can be optimal. When the signal reveals bad news about the protection buyers’ assets, it becomes more likely that consumption will be transferred from protection sellers to protection buyers. This undermines the protection sellers’ incentives to take actions limiting the risk of their assets. To alleviate moral hazard, and thus improve risk sharing, it is useful to reduce the amount of risky assets left under the control of protection sellers. But transferring assets to investors is costly, because they are less efficient at holding them. In the second best, the marginal cost of inefficient asset allocation is equal to the marginal benefit of better risk sharing.

Our second contribution is to analyze market equilibrium. Market participants can write and trade contracts contingent on all observable variables, so there is no exogenous market incompleteness. Yet, incentive constraints limit the amount of insurance that protection sellers can credibly promise, which generates endogenous market incompleteness. We show that privately optimal contracts between protection buyers and protection sellers involve derivative contracts (e.g., Credit Default Swaps insuring protection buyers against the default risk of their loans) and variation margins.\(^5\) After a bad signal variation margins are called,

\(^5\)We build on the partial equilibrium analysis of risk-sharing under moral hazard in Biais, Heider and
requesting protection sellers to deposit cash on their margin account. To do so, protection sellers must liquidate risky assets. Reducing the amount of risky assets under the control of protection sellers alleviates moral hazard, but implies selling risky assets to investors who are less efficient at holding them. Thus, variation margins trigger price drops, which can be interpreted as fire sales.\textsuperscript{6}

Fire sales generate externalities, but do they create inefficiencies? One might be tempted to think so, because a protection seller’s incentive constraint depend on market prices: What protection sellers can promise to pay without jeopardizing their incentives to take appropriate risk management actions (cash on margin account, plus pledgeable income from risky assets under management), must exceed the liability from the derivative contract. When one protection seller liquidates assets, this contributes to depressing the price at which all protection sellers liquidate their assets, reducing cash proceeds deposited on margin accounts and tightening incentive constraints.\textsuperscript{7}

Our third contribution, however, is to establish that market equilibrium, and the corresponding fire sales, are information-constrained efficient. The intuition is the following. Protection buyers are hurt by fire sales because they obtain less insurance from protection sellers, but investors benefit from the fire sale because they can buy underpriced assets.\textsuperscript{8} Since protection buyers and investors have opposite exposures to fire sale risk, they benefit from insuring one another. In equilibrium they exploit this risk sharing opportunity until their marginal rates of substitution are equalised, just as in the second best. Still, and as in Hoerova (2017). In a similar framework, Bolton Oehmke (2015) analyse whether, under moral hazard, derivatives should be privileged in bankruptcy.

\textsuperscript{6}Bian et al. (2017) document margin-induced fire sales triggering price drops in equity markets. Ellul et al. (2011) find that fire sales of downgraded corporate bonds by insurance companies trigger price declines. Merrill et al. (2014) document fire sales of residential mortgage-backed securities (RMBS). While in our model fire sales are due to moral hazard, in Dow and Han (2017) they reflect adverse selection.

\textsuperscript{7}Chernenko and Sundaram (2017) find that mutual funds belonging to the same fund family try to mitigate fire-sale externalities on the other funds of the family by holding back on asset sales.

\textsuperscript{8}Meier and Servaes (2015) show that firms buying distressed assets in fire sales earn excess returns.
the second best, the marginal rate of substitution of protection buyers differs from that of protection sellers and investors, because incentive constraints prevent perfect risk sharing.

Our paper is related to the literature on equilibrium inefficiency in incomplete markets, e.g., Stiglitz (1982), Greenwald and Stiglitz (1986), Geanakoplos and Polemarchakis (1986), Gromb and Vayanos (2002) and Lorenzoni (2008). In Gromb and Vayanos (2002) financially constrained arbitrageurs supply insurance to hedgers. When arbitrageurs suffer losses, their leverage constraints tighten, and they have to liquidate their positions. Markets are incomplete because hedgers cannot directly trade with one another. In Lorenzoni (2008) entrepreneurs borrow to fund investment projects. Financial constraints compel entrepreneurs to sell assets after negative shocks. Markets are incomplete because entrepreneurs cannot insure against these shocks. Both in Gromb and Vayanos (2002) and in Lorenzoni (2008), the combination of financial constraints and market incompleteness generates pecuniary externalities, leading to equilibrium constrained inefficiency. The key role played by exogenous market incompleteness for equilibrium inefficiency is manifest in Davila and Korinek’s (2017) analysis of economies with financial frictions, in which when there is no exogenous market incompleteness, agents perfectly share risk. These results stand in contrast with ours: In contrast with Gromb and Vayanos (2002) and Lorenzoni (2008), we find that equilibrium is constrained efficient. In contrast with Davila and Korinek (2017), we find that, even with a complete set of contingent contracts, risk sharing is imperfect. These differences arise because in our model constraints do not reflect exogenous market incompleteness, but moral hazard generating endogenous market incompleteness.

On the other hand, our analysis is in line with Prescott and Townsend (1984) and Alvarez and Jermann (2000) who find that, with optimal contracting and complete markets, equilibrium is constrained efficient.\(^9\) The major difference between our setting and theirs concerns

the timing of events. In our setting, unlike in theirs, after initial contracting but before the final resolution of uncertainty, a signal is observed and interim trades can take place, followed by agents’ effort decisions. It is at this interim point that variation margins are called and firesales can occur, affecting prices in the incentive constraint. There is no such mechanism, and in particular no price in the incentive constraint in Prescott and Townsend (1984) and Alvarez and Jermann (2000). Our result that equilibrium is constrained efficient in spite of prices in the incentive constraints is one of the distinguishing features of our model relative to that literature.

While we study risk sharing, Kuong (2016) and Acharya and Viswanathan (2011) study lending. Kuong (2016) obtains self-fulfilling fire sales: When creditors expect low collateral liquidation value, they require high debt yield. In response, borrowers engage in risk taking. Eventually, this leads to a large number of defaults, triggering many collateral liquidations, leading to low collateral value. A major difference between our paper and those of Kuong (2016) and Acharya and Viswanathan (2011) is that we conduct a normative analysis, characterising the second best and comparing it to market equilibrium.

In Kuong (2016) and Fostel and Geneakoplos (2015), margins work through their effect on the structure of liabilities: larger margins correspond to lower leverage. Thus, these papers offer a liability-side view of margins. In contrast, in our analysis, margins work through their effect on the structure of assets: larger margins correspond to more pledgeable assets. Thus, we offer an asset-side view of margins.\footnote{For example, in Kuong (2016) larger margins reduce required debt service, while in Fostel and Geneakoplos (2015) lower assets’ minimum payoffs constraint leverage to be lower, raising the percentage of the asset purchase that must be financed by equity.}

Section 2 presents the model and its mapping to real markets and institutions. Section 3 analyses the first best, Section 4 the second best, and Section 5 market equilibrium. Section \footnote{In Section 7 we elaborate on the distinction between the asset- and liability-side views of margins, comparing the effects of margins on balance sheet in the two settings.}
analyses the case in which protection sellers can engage in risk shifting. Section 7 presents implications of our analysis.

2 Economic environment

2.1 Model

There are three dates: time 0, 1 and 2, one consumption good, and assets generating consumption good at time 2.

Agents and endowments: There is a unit-mass continuum of protection buyers, each with utility \( u \), increasing and concave, and endowed at time 0 with one unit of a risky asset, paying \( \tilde{\theta} \) at time 2. There is also a unit-mass continuum of investors each with utility \( v \), increasing and concave, and endowed at time 0 with \( e \) units of a safe asset, each paying 1 unit of consumption good at time 2. Finally, there is a unit-mass continuum of risk-neutral protection sellers, each endowed with one unit of a productive asset, paying \( \tilde{R} \) at time 2.

Assets payoffs: The payoff of the protection buyers’ asset at time 2, \( \tilde{\theta} \), can be \( \tilde{\theta} \) with probability \( \pi \), or \( \theta \) with probability \( 1 - \pi \). While that payoff is exogenous, the payoff of a protection seller’s asset depends on his action. The action a protection seller is not observable, which coupled with limited liability, creates a moral hazard problem.

The main specification of the moral hazard problem we consider is as in Holmstrom and Tirole (1997). Each unit of the protection sellers’ asset yields \( R \) at time 2 for sure if protection sellers exert risk-management effort, at cost \( \psi \) per unit, at time 1. When consuming \( c_S \) units of the consumption good and exerting effort over \( y \) units of the asset, a protection seller obtains utility \( c_S - y\psi \). If a protection seller does not exert risk-management
effort, his asset’s payoff is $R$ with probability $\mu$ and 0 with probability $1 - \mu$. We assume $R - \psi > \mu R$, so that protection seller’s effort is efficient. Following Holmström and Tirole (1997), pledgeable income, i.e., the part of the physical return that can be promised without jeopardising incentives, is

$$P \equiv R - \frac{\psi}{1 - \mu} > 0. \quad (1)$$

In Section 6, we consider an alternative specification of the unobservable action problem: risk-shifting, à la Jensen and Meckling (1976). In both specifications, the undesirable action (not exerting costly effort, or risk-shifting) hurts the protection buyer by increasing downside risk on the protection seller’s assets and, correspondingly, counterparty risk for the protection buyer. We show that similar economic mechanisms are at play in the two specifications.

**Signals:** At time 1 an advanced signal $\tilde{s}$ about $\tilde{\theta}$ is publicly observed. When the final realisation of $\tilde{\theta}$ is $\bar{\theta}$, the signal is $\tilde{s}$ with probability $\lambda > 1/2$ and $\bar{s}$ with probability $1 - \lambda$. When the final realisation is $\bar{\theta}$, it is $\bar{s}$ with probability $1 - \lambda$ and $\tilde{s}$ with probability $\lambda$.

**Asset transfers:** Effort takes place at time 1, after the signal is publicly observed. Before effort is exerted (but after observing the public signal), a fraction $\alpha$ of the productive asset can be transferred from protection sellers to investors. This is costly because investors are less efficient than protection sellers at managing assets. Investors’ per-unit cost of managing the asset is larger than that of protection sellers: $\psi_I(\alpha) > \psi$, $\forall \alpha$. When consuming $c_I$ units of the consumption good and exerting effort over $\alpha$ units of asset, an investor obtains utility $\nu(c_I - \alpha \psi_I(\alpha))$. We assume $\psi_I' \geq 0$ and $\psi_I'' \geq 0$. Thus, investors’ marginal cost, $\psi_I(\alpha) + \alpha \psi_I'$, is increasing. Yet, we assume it is efficient that investors exert effort even when
holding all of the asset: $R - \psi_I(1) \geq \mu R$. We also maintain the following assumption:

$$\psi_I(1) + \psi_I'(1) > \frac{\psi}{1 - \mu} > \psi_I(0).$$

As will be seen below, the right inequality in (2) allows for asset transfers, by making them not too inefficient when $\alpha$ is close to 0. The left inequality in (2) precludes full transfer of assets ($\alpha = 1$) because this would be too inefficient.\textsuperscript{12}

**Risk-sharing and moral hazard:** Risk-averse protection buyers seek insurance against the risk $\tilde{\theta}$. They can turn to protection sellers or to investors, facing the following trade-off. On the one hand, protection sellers are efficient providers of insurance, as they are risk-neutral, but they have a moral-hazard problem. On the other hand, investors are less efficient at managing the productive asset and at providing insurance since they are risk-averse. Risk aversion, however, suppresses the moral hazard problem when $v(0)$ is sufficiently low, which we hereafter assume.\textsuperscript{13} Thus, while we need to impose incentive-compatibility constraints for protection sellers, we do not need to do so for investors.

**Sequence of events:** Summarising, the sequence of events is as follows:

- At time 0, agents receive their endowments.
- At time 1, first the signal $s$ is observed, then a fraction $\alpha(s)$ of the productive asset can be transferred from protection sellers to investors, and then holders of the productive asset decide whether to exert effort or not.

\textsuperscript{12}In general, assets could also be transferred to protection buyers. For simplicity we assume this is not possible as protection buyers do not have the technology to manage those assets.

\textsuperscript{13}If $v(0)$ is low enough, threatening risk-averse investors to give them 0 consumption when the asset yields 0 is enough to induce effort (making the zero return on investors’ assets an out-of-equilibrium event).
• At time 2, assets’ payoffs are realised and publicly observed, and consumption takes place.

For given effort decisions, $\tilde{\theta}$ and $\tilde{R}$ are independent. So there is no exogenous correlation between the valuations of the two assets. In spite of this simplifying assumption, we show below that moral hazard creates endogenous positive correlation.

### 2.2 Mapping the model to real markets and institutions

Protection buyers can be commercial banks seeking to insure risk, while protection sellers can be investment banks or specialised firms providing this insurance. Prior to the 2007-09 crisis, banks frequently bought protection against credit-related losses on corporate loans and mortgages. Under the first Basel Agreement, buying protection enabled them to reduce or eliminate regulatory capital requirements. Out of $533$ billion (net notional amount) of credit default swaps sold by AIG at year-end 2007, 71% were categorized as such “Regulatory Capital” contracts (see Harrington, 2009).

Protection sellers take decisions that increase or reduce the riskiness of their assets. In the case of a loan portfolio, the due diligence effort reducing downside risk corresponds to the screening and monitoring of loans. Lack of screening and monitoring leads to a higher risk of losses. For example, the report of the Financial Crisis Inquiry Commission (2011) states that “investors relied blindly on credit rating agencies as their arbiters of risk instead of doing their own due diligence” and “... Merrill Lynch’s top management realized that the company held $55$ billion in “super-senior” and supposedly “super-safe” mortgage-related securities that resulted in billions of dollars in losses”.

Alternatively, the sellers’ assets can include financial securities and portfolio positions, whose risk is affected by the protection seller’s management of collateral, liquidity, and
exposures. For example, as part of its securities-lending activity, AIG received cash-collateral from its counterparties. Instead of holding this collateral in safe and liquid assets, such as Treasury bonds, AIG bought risky illiquid instruments, such as Residential Mortgages Backed Securities. As the value of these securities dropped, this resulted in approximately $21 billion of losses for the company in 2008 (see McDonald and Paulson, 2015). More generally, several institutions failed to manage emerging risks, ignored warnings from their risk managers and increased their mortgage exposures even as the US housing market had begun to show signs of weakening (see Financial Crisis Inquiry Commission, 2011).

Consistent with our assumption that lack of proper risk-management effort increases downside risk, Ellul and Yerramilli (2013) document that banks with a weaker risk-management function at the onset of the financial crisis had higher tail risk and higher nonperforming loans during the financial-crisis years.

3 First best

The first best obtains when effort is observable. In that case, effort is always requested by the planner and exerted by the agents. Hence, the protection sellers’ assets always yield \( R \). The state variables, on which decisions and consumptions are contingent, are the realisations of the protection buyers’ asset \((\theta)\) and the signal \((s)\).

The social planner chooses the consumptions of protection buyers \((c_B(\theta, s))\), protection sellers \((c_S(\theta, s))\) and investors \((c_I(\theta, s))\), as well as the fraction of protection sellers’ assets transferred to investors \((\alpha(s))\), to maximise the expected utility of protection buyers and investors (with respective Pareto weights \(\omega_B\) and \(\omega_I\)):

\[
\omega_B E[u(c_B(\tilde{\theta}, \tilde{s}))] + \omega_I E[v(c_I(\tilde{\theta}, \tilde{s}) - \alpha(\tilde{s})\psi_I(\alpha))],
\] (3)
We assume that the social planner places no weight on protection sellers, i.e., $\omega_S = 0$. Correspondingly, when analysing the market equilibrium, we will assume zero bargaining power for the protection sellers.\textsuperscript{14} The constraints are the participation constraint of protection buyers,

$$E[u(c_B(\tilde{\theta}, \tilde{s}))] \geq E[u(\tilde{\theta})],$$

the participation constraint of investors,

$$E[v(c_I(\tilde{\theta}, \tilde{s}) - \alpha(\tilde{s})\psi_I(\alpha))] \geq v(1),$$

the participation constraint of protection sellers,

$$E[c_S(\tilde{\theta}, \tilde{s}) - (1 - \alpha(\tilde{s}))\psi] \geq R - \psi,$$

the budget constraints in each state,

$$c_B(\theta, s) + c_I(\theta, s) + c_S(\theta, s) \leq \theta + 1 + R, \quad \forall (\theta, s),$$

and the constraint that $\alpha(s)$ must be between 0 and 1. The participation constraints reflect the respective autarky payoffs of protection buyers ($E[u(\tilde{\theta})]$), investors ($v(1)$) and protection sellers ($R - \psi$). Our first proposition states the solution of the first-best problem.

**Proposition 1** In the first best, there is no transfer of the productive asset, $\alpha(s) = 0, \forall s$, and protection buyers and investors receive constant consumption, $c_B(\theta, s) = c_B, c_I(\theta, s) = c_I$.

\textsuperscript{14}When effort is unobservable, protection sellers are agents, while protection buyers are principals. Our assumption that protection buyers have all the bargaining power is in line with the principal-agent literature, in which the principal makes a take-it-or-leave-it offer to the agent. Our assumption that $\omega_s = 0$ implies that, in Section 4, we only characterise a subset of the information-constrained Pareto frontier. This does not affect qualitatively our welfare analysis, as we show in Section 5 that the equilibrium implements a point on that subset of the information-constrained Pareto frontier.
Their total consumption is

\[ c_B + c_I = E[\tilde{\theta}] + 1, \]  

while protection sellers’ consumption is

\[ c_S(\theta, s) = \theta - E[\tilde{\theta}] + R, \quad \forall(\theta, s). \]  

In the first best, the productive asset is held entirely by its most efficient holders, the protection sellers, i.e., \( \alpha(s) = 0 \). Moreover, the risk-neutral protection sellers fully insure the risk-averse agents, whose consumption is equal to the expected value of their endowment. Hence, marginal rates of substitution between consumption in different states are equal to one for all agents. Figure 1 illustrates the market implementation of the first best.

Figure 1: Market implementation of the first best
4 Second best

Now turn to the second best, when protection sellers’ effort is unobservable. The social planner still chooses consumptions and asset transfers to maximise his objective function (3) under participation constraints (4), (5), and (6), and budget constraints (7). In addition, the planner is constrained by the protection sellers’ incentive-compatibility condition. In what follows, we assume that the first-best allocation is not feasible in this more constrained, second-best problem:

\[ P < E[\tilde{\theta}] - E[\tilde{\theta}|s]. \] (10)

If (10) did not hold, the pledgeable return would be sufficiently large for protection sellers to credibly promise full insurance despite the moral-hazard problem.

4.1 Incentive compatibility

Protection sellers decide on effort after the realisation of the signal \( s \). The incentive constraint is that they prefer effort to shirking:

\[ E[c_S(\tilde{\theta}, s) - (1 - \alpha(s))\psi|s] \geq \mu E[c_S(\tilde{\theta}, s)|s]. \]

The left-hand side is protection sellers’ (on-equilibrium-path) expected consumption net of the cost of effort. The right-hand side is their (off-equilibrium-path) expected consumption when shirking. Under shirking, the asset yields \( R \) only with probability \( \mu \). In this case, protection sellers still receive the same expected consumption as under effort. But with probability \( 1 - \mu \), the asset returns zero. In order to relax the incentive-compatibility constraint, the planner optimally allocates zero consumption to limited-liability protection sellers in the (out-of-equilibrium) event of a zero asset return.
The incentive-compatibility constraint rewrites as

\[ E[c_S(\tilde{\theta}, s)|s] \geq (1 - \alpha(s)) \frac{\psi}{1 - \mu}. \]  \hspace{1cm} (11)

The left-hand side of (11) is the expected consumption of protection sellers after observing the signal \( s \). The right-hand side is the incentive-adjusted cost of managing the fraction of assets protection sellers still control after a possible transfer. Transferring assets from protection sellers to investors relaxes the incentive constraint.

4.2 Risk-sharing in the second best

Our first result is that protection buyers and investors are exposed only to the risk associated with the signal \( s \). Correspondingly we write their respective consumptions as \((c_B(\bar{s}), c_B(s))\) and \((c_I(\bar{s}), c_I(s))\).

**Lemma 1** The consumption of protection buyers and investors depends only on the realisation of the signal \( s \), but not on the realisation \( \theta \) of protection buyers’ assets.

The incentive constraint (11) implies that only the expected consumption of protection sellers conditional on the signal matters for incentives. For a given \( E[c_S(\tilde{\theta}, s)|s] \), the split between \( c_S(\bar{s}, s) \) and \( c_S(\bar{\theta}, s) \) does not affect the incentive constraint or the participation constraint of protection sellers. Hence, it is optimal to set \( c_S(\bar{s}, s) \) and \( c_S(\bar{\theta}, s) \) to fully insure protection buyers conditional on the realisation of \( s \), by equalising their marginal utility in states \((\bar{s}, s)\) and \((\bar{\theta}, s)\). Similarly, it is optimal to equalise investors’ marginal utility in these two states. The next two lemmas further characterize the second-best outcome.

**Lemma 2** In the second best, the resource constraint as well as the participation constraint of protection sellers bind. Moreover, one, and only one, of the two incentive-compatibility
conditions (after $\bar{s}$ or $s$) binds.

Lemma 3 After a good signal, the incentive-compatibility constraint of protection sellers is slack and there is no asset transfer, $\alpha(\bar{s}) = 0$. After a bad signal, the incentive-compatibility constraint of protection sellers binds. Moreover, the consumption of protection buyers is larger after a good signal than after a bad signal, $c_B(\bar{s}) > c_B(s)$.

After a good signal, protection sellers’ expected consumption is large, which relaxes their incentive constraint. As a result, there is no need to transfer protection sellers’ assets to less efficient investors ($\alpha(\bar{s}) = 0$).

After a bad signal, the opposite happens. Protection sellers’ expected consumption is low, which tightens their incentive constraint. Because of the binding incentive constraint, protection buyers cannot be fully insured and remain exposed to signal risk ($c_B(\bar{s}) > c_B(s)$).

Combining Lemmas 1, 2, we obtain the next proposition, characterising the consumption of protection buyers and investors for a given level of asset transfers after a bad signal $\alpha(s)$.

Proposition 2 After $s$, the total consumption of protection buyers and investors is

$$c_B(s) + c_I(s) = 1 + E[\bar{\theta}|s] + \alpha(s)R + (1 - \alpha(s))R,$$

while after $\bar{s}$ it is

$$c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\bar{\theta}|\bar{s}] - \frac{P_r[s]}{P_r[\bar{s}]}[\alpha(s)(R - \psi) + (1 - \alpha(s))P].$$

Signal risk is perfectly shared between protection buyers and investors

$$\frac{v'(c_I(s) - \alpha(s)\psi_I(s))}{v'(c_I(\bar{s}))} = \frac{u'(c_B(s))}{u'(c_B(\bar{s}))}.$$

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Consumption is split between protection buyers and investors according to their Pareto weights:

\[
\frac{u'(c_B(s))}{u'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I + \lambda_I}{\omega_B + \lambda_B},
\]

(15)

where \(\lambda_B\) and \(\lambda_I\) are the respective Lagrange multipliers of the participation constraint of protection buyers (4) and investors (5).

Since there is there is no incentive problem between protection buyers and investors, they share signal risk perfectly, as reflected in equation (14). While in the first best, the joint consumption of protection buyers and investors is given by the unconditional expectation of their joint endowment, reflecting full insurance, the second best involves the conditional expectation of their joint endowment. The second best also involves pledgeable income, which without asset transfers is just \(\mathcal{P}\), and with assets transfers is increased by \(\alpha(\bar{s})(R - \mathcal{P})\).

4.3 Optimal asset transfers after bad news

To complete the analysis of the second best, the next proposition characterises asset transfers after bad news.

**Proposition 3** If

\[
\frac{u'(c_B(\bar{s}))}{u'(c_B(\bar{s}))} \big|_{\alpha(\bar{s})=0} > \frac{\psi}{1-\mu - \psi I(0)},
\]

(16)

then, in the second best, the asset transfer is interior, \(\alpha(\bar{s}) \in (0,1)\), and such that

\[
\frac{u'(c_B(\bar{s}))}{u'(c_B(\bar{s}))} = \frac{\psi}{1-\mu} - \psi
\]

(17)

where \(c_B(s)\) and \(c_B(\bar{s})\) are as given in Proposition 2. Otherwise, \(\alpha(s) = 0\).
To interpret the left-hand sides of (16) and (17), recall that Lemma 3 implies there is signal risk: \( c_B(\bar{s}) < c_B(s) \), which, in turn, implies the marginal rate of substitution, \( \frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \), is larger than 1. The worse the insurance, the larger this marginal rate of substitution. Thus, the marginal rate of substitution on the left-hand-sides of (16) and (17) reflects the marginal benefit of an increase in insurance.

While the left-hand sides of (16) and (17) reflect the preferences of protection buyers, the right-hand sides of (16) and (17) reflect the technology and incentives of those who hold the productive asset. The denominator of the right-hand side of (16) and (17) is the wedge between the productive asset’s marginal pledgeable income when it is held by investors \( (R - (\psi_I(\alpha(s)) + \alpha(s)\psi_I'(\alpha(s))) + \alpha(s)\psi_I''(\alpha(s))) \) and its counterpart when it is held by protection sellers (\( \mathcal{P} \)). Thus, it measures how much more income one can pledge by transferring the productive asset from protection sellers to investors. The numerator is a similar wedge between the pledgeable income in the first best \( (R - \psi) \) and its counterpart under moral hazard (\( \mathcal{P} \)). Thus, the right-hand-sides of (16) and (17) can be interpreted as a marginal rate of transformation, reflecting the marginal cost of an increase in incentive-compatible insurance.

Condition (16) means that, at \( \alpha(s) = 0 \), the marginal benefit of a small asset transfer exceeds its marginal cost. Since \( \psi_I' \geq 0 \) and \( \psi_I'' \geq 0 \), the marginal cost of effort for investors \( \psi_I(\alpha(s)) + \alpha(s)\psi_I'(\alpha(s)) \) is increasing. So the right-hand-side of (17) is increasing, and takes its minimum value at \( \alpha(s) = 0 \), as in the right-hand-side of (16). Furthermore, by (2), there exists threshold \( \hat{\alpha} < 1 \) at which the right-hand-side of (17) goes to infinity. Hence, under (16), there exists an interior value of \( \alpha(s) \in (0, \hat{\alpha}) \) for which the marginal benefit of additional insurance is equal to its marginal cost. This pins down the optimal asset transfer after bad news in the second best.

Figure 2 illustrates the interaction of protection buyers, protections sellers, and investors in the second best when there are asset transfers after bad news.
5 Market equilibrium

We do not rule out trading of any contract based on the publicly observed variables \((s, R, \theta)\), but, as will be clear below, the following three markets are sufficient:

Market for insurance against the realization of \(\tilde{\theta}\): Protection buyers and protection sellers participate in this market at time 0. In line with our simplifying assumption that the social planner places no weight on protection sellers, we assume protection buyers have all the bargaining power, so protection sellers are held to their reservation utility. Each protection buyer is matched with one protection seller and makes an exclusive take-it-or-leave-it offer.\(^{15}\) The offer includes time 2 transfers, \(\tau(\theta, s, R)\) and variation margins. A positive transfer \(\tau(\theta, s, R) > 0\) denotes a payment from the seller to the buyer and vice versa.

\(^{15}\)For an analysis of issues arising with non-exclusive contracting, see Acharya and Bisin (2014).
Market for protection sellers’ assets: In the previous section we showed that the second best can involve asset transfers from protection sellers to investors after bad news. Therefore, we allow the protection buyer to request his counterparty to sell a fraction \( \alpha_S \geq 0 \) of his assets after a bad signal. The asset sale occurs at time 1, after the realisation of the signal, and before effort is exerted. The price is denoted by \( p \). While supply \( \alpha_S \) stems from protection sellers, demand \( \alpha_I \) stems from investors. All market participants are competitive.

The proceeds from the variation margin call, \( \alpha_S p \), belong to protection sellers but are put on an escrow account, in which they are ring-fenced from moral hazard and can be used to pay protection buyers at time 2.

Market for insurance against signal risk: When effort is observable, this market is not needed, as protection sellers can fully insure protection buyers against the risk associated with their endowment \( \tilde{\theta} \). With full insurance protection buyers are not exposed to signal risk. In contrast, when moral hazard limits the extent to which protection sellers can insure protection buyers, leaving them exposed to signal risk. This opens the scope for signal risk-sharing between protection buyers and investors. The corresponding market is held at time 0, and enables participants to exchange consumption after a bad signal against consumption after a good signal. Owners of one unit of the contract receive \( q \) units of consumption good after a bad signal and pay 1 unit of consumption good after a good signal. We denote protection buyers’ demand by \( x_B \) and investors’ supply by \( x_I \).

Equilibrium: Equilibrium consists of transfers \( \tau(\theta, s, R) \), prices \( (p, q) \), and trades \( (\alpha_S, \alpha_I) \) and \( (x_B, x_I) \), such that all participants behave optimally and markets clear: \( \alpha_I = \alpha_S \) and \( x^d = x^s \). To solve for equilibrium, we take the following steps. First, we derive incentive and participation constraints. Second, we characterize investors’ trading decisions, \( \alpha_I \) and \( x_I \), for given prices. Third, we analyse contracting between protection buyers and protection sellers.
Fourth, we impose market clearing.

5.1 Protection sellers’ incentive and participation constraints

**Incentive compatibility:** As in the second best, the incentive-compatibility condition of protection sellers after a good signal is slack. After a bad signal ($s$), the incentive-compatibility condition under which protection sellers exert effort is

$$
(1 - \alpha_S)(R - \psi) + \alpha_S p - E[\tau(\tilde{\theta}, \tilde{s}, R)]|s| \geq \mu(\alpha_S R + \alpha_S p - E[\tau(\tilde{\theta}, \tilde{s}, R)]|s|) + (1 - \mu)E[\max[\alpha_S p - \tau(\tilde{\theta}, \tilde{s}, 0), 0]|s|].
$$

(18)

The left-hand side of (18) is the expected gain of protection sellers on the equilibrium path: They exert effort and obtain $R - \psi$ for each of the $1 - \alpha_S$ units of the productive asset they keep. In addition, protection sellers own the proceeds from the asset sale, $\alpha_S p$, deposited in the margin account. Finally, the expected net payment by protection sellers to protection buyers is $E[\tau(\tilde{\theta}, \tilde{s}, R)]|s|$. The right-hand side of (18) is the expected profit of protection sellers if they deviate and do not exert effort. In that case, with probability $\mu$, protection sellers’ productive assets still generate $R$, and their expected gain is the same as on the equilibrium path, except that the cost of effort, $(1 - \alpha_S)\psi$, is not incurred. With probability $1 - \mu$, the productive assets held by protection sellers generate no output. In that case, because of limited liability, protection sellers cannot pay more than $\alpha_S p$. Hence their gain is $\max[\alpha_S p - \tau(\theta, s, 0), 0]$. It is optimal to set $\tau(\theta, s, 0) = \alpha_S p$. This relaxes the incentive constraint by reducing the right-hand side of (18), and does not affect the rest of the analysis because transfers $\tau(\theta, s, 0)$.
only occur off the equilibrium path. Protection sellers’ incentive constraint thus reduces to

\[
\alpha_S p + (1 - \alpha_S)P \geq E[\tau(\tilde{\theta}, \tilde{s})|\tilde{s}] 
\]

where we write \(\tau(\tilde{\theta}, \tilde{s}, R) = \tau(\tilde{\theta}, \tilde{s})\) to simplify the notation. The right-hand side of (19) is how much protection sellers expect to pay protection buyers, which can be interpreted as the implicit debt of protection sellers. The left-hand side of (19) is how much protection sellers can credibly pledge to pay, i.e., the sum of i) proceeds from asset sales, deposited on margin account \((\alpha_S p)\), which are fully pledgeable, and ii) pledgeable part \((P)\) of output \((R)\) obtained on the \(1 - \alpha_S\) units of productive asset kept by protection sellers.

**Participation constraint:** A protection seller accepts the contract if it gives him equilibrium expected gains no smaller than his autarkry payoff, i.e.,

\[
E[-\tau(\tilde{\theta}, \tilde{s})] \geq \Pr[\tilde{s}]\alpha_S(R - \psi - p)
\]

Expectation net payments to protections sellers must be larger than the expected opportunity cost of selling a fraction \(\alpha_S\) of the asset at price \(p\) after a bad signal.

### 5.2 Investors’ optimal trades

When selling \(x_I\) units of the insurance contract against signal risk and buying \(\alpha_I\) units of protection sellers’ assets, investors obtain time 2 consumption equal to \(e + x_I\) after a good signal and \(e - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)\) after a bad signal. Their expected utility is

\[
\Pr[\tilde{s}]v(e + x_I) + \Pr[\tilde{s}]v(e - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)).
\]
Investors’ supply of insurance against signal risk: At time 0, investors choose $x_I$ to maximise (21). The first-order condition is

$$\Pr[s]v'(e + x_I) = \Pr[\bar{s}]q v'(e - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)),$$  \hspace{1cm} (22)

which implies that $x_I$ decreases in $q$.\footnote{The left-hand side of (22) is decreasing in $x_I$, while the right-hand side is increasing in $x_I$. Their intersection pins down the optimal supply of insurance by investors, $x_I$. Now, the right-hand side is increasing in $q$. Thus, an increase in $q$ shifts up the right-hand side of (22), which leads to an intersection between the right- and the left-hand sides of (22) at a lower value of $x_I$.}

Equation (22) rewrites as

$$q = \frac{\Pr[\bar{s}]v'(e + x_I)}{\Pr[s]v'(e - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p))},$$  \hspace{1cm} (23)

which states that the price of insurance against signal risk is equal to the probability-weighted marginal rate of substitution between consumption after good and bad news.

Investors’ demand for protection sellers’ assets: At time 1, after a bad signal, investors choose $\alpha_I$ to maximise their utility $v(e - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p))$. When $p \geq R - \psi_I(0)$, the price of the asset is so high that investors’ demand is 0. Otherwise, their demand is pinned down by the first-order condition:

$$p = R - \left[\psi_I(\alpha_I) + \alpha_I \psi_I'(\alpha_I)\right],$$  \hspace{1cm} (24)

which states that the price is equal to the marginal valuation of the investor for the asset. Because the marginal cost $\psi_I(\alpha_I) + \alpha_I \psi_I'(\alpha_I)$ is increasing, (24) implies that investors’ demand for the asset is decreasing in $p$.\footnote{Increasing marginal cost also implies the second-order condition holds.}

\footnote{The second-order condition $\Pr[\bar{s}]v''(1 + x_I) + q^2 \Pr[s]v''(1 - qx_I + \alpha_I(R - \psi_I(\alpha_I) - p)) < 0$ holds by the concavity of the utility function.}
5.3 Contracting between protection buyers and sellers

Protection buyers’ privately-optimal contract specifies transfers \( \tau(\theta, s) \) and asset sale \( \alpha_S \). Moreover, protection sellers demands \( x_B \) units of the insurance against signal risk, paying \( qx_B \) after bad news and \(-x_B\) after good news. Correspondingly, the consumption of protection buyers at time 2 is \( \theta + \tau(\theta, \bar{s}) - x_B \) after a good signal and \( \theta + \tau(\theta, s) + qx_B \) after a bad signal. They choose \( x_B, \tau(\theta, s) \) (for all \( \theta \in \{\bar{\theta}, \bar{\theta}\} \) and \( s \in \{\bar{s}, \bar{s}\} \)), as well as \( \alpha_S \in [0, 1] \) to maximise

\[
Pr[\bar{s}]E[u(\bar{\theta} + \tau(\bar{\theta}, \bar{s}) - x_B)|\bar{s}] + Pr[s]E[u(\bar{\theta} + \tau(\bar{\theta}, \bar{s}) + qx_B)|s], \tag{25}
\]

subject to the protection seller’s incentive and participation constraints, (19) and (20). The next lemma states protection buyers’ consumption as a function of \( \alpha_S \).

Lemma 4 In equilibrium, in the privately-optimal contract between protection buyers and protection sellers, protection sellers’ participation and incentive constraints bind. Moreover, protection buyers receive full insurance conditional on the signal, i.e., for a given realisation of the signal, their consumption does not depend on the realisation of \( \tilde{\theta} \):

\[
c_B(\bar{\theta}, \bar{s}) = c_B(\bar{\theta}, \bar{s}) = E[\tilde{\theta}|\bar{s}] = \frac{Pr[s]}{Pr[\bar{s}]}[\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}] - x_B, \tag{26}
\]

\[
c_B(\bar{\theta}, s) = c_B(\bar{\theta}, s) = E[\tilde{\theta}|s] + \alpha_Sp + (1 - \alpha_S)\mathcal{P} + qx_B. \tag{27}
\]

Lemma 4 is similar to Lemma 1. Both in the second best and in the market equilibrium, protection buyers are fully insured conditional on the signal, and the economic intuition is the same in the two cases. The next lemma states what fraction of their assets protection sellers are required to sell after bad news.
Lemma 5  When

\[ p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\tilde{\theta}|\tilde{s}] - \frac{Pr[s]}{Pr[\bar{s}]} \mathcal{P} - x_B)}{u'(E[\tilde{\theta}|\tilde{s}] + \mathcal{P} + qx_B)} \]  \tag{28}

then \( \alpha_S = 0 \), otherwise \( \alpha_S \) is strictly positive and such that

\[ \frac{u'(E[\tilde{\theta}|\tilde{s}] + \alpha_s p + (1 - \alpha_s) \mathcal{P} + qx_B)}{u'(E[\tilde{\theta}|\tilde{s}] - \frac{Pr[s]}{Pr[\bar{s}]}[\alpha_s (R - \psi) + (1 - \alpha_s) \mathcal{P}] - x_B)} = \frac{\lambda_1}{(p - \mathcal{P}) Pr[s] \lambda_S} + \frac{\psi}{1 - \mu - \psi}, \tag{29} \]

where \( \lambda_1 \) is the Lagrange multiplier of the constraint \( \alpha_S \leq 1 \).

Equation (29) is similar to (17). In both cases, the left-hand side is protection buyers’ marginal rate of substitution between consumption after a bad signal and after a good signal, and therefore reflects the marginal benefit of an increase in insurance. In (17), the right-hand side involves the ratio

\[ \frac{\psi}{1 - \mu - \psi} - (\psi_1(\alpha(s)) + \alpha(s) \psi'_1(\alpha(s))) \]

while in (29) the corresponding ratio is

\[ \frac{\psi}{1 - \mu - \psi} - \frac{\psi}{p - \mathcal{P}}. \]

In both ratios the denominator reflects how much more income can pledged for insurance by transferring assets from protection sellers to investors.

5.4 Equilibrium

Equilibrium in the market for insurance against signal risk: Taking the first-order condition with respect to \( x_B \) in (25), \( x_B \) such that

\[ q = \frac{Pr[s]}{Pr[\bar{s}]} \frac{u'(\theta + \tau(\theta, \bar{s}) - x_B)}{u'(\theta + \tau(\theta, \bar{s}) + qx_B)}. \tag{30} \]
Since the right-hand side of (30) is increasing in $x_B$, (30) implies $x_B$ is increasing in $q$, while (23) implies that $x_I$ decreases in $q$. At equilibrium, $q$ is such that $x_B = x_I$. Combining (23) and (30), we obtain our next proposition:

**Proposition 4** Equilibrium in the market for insurance against signal risk involves price $q^*$ and trading volume $x^*$ such that

$$q^* = \frac{Pr[\bar{s}]}{Pr[s]} \frac{v'(e + x^*)}{v'(e - q^*x^* + \alpha_I(R - \psi_I(\alpha_I) - p))} = \frac{Pr[\bar{s}]}{Pr[s]} \frac{u'(\theta + \tau(\theta, \bar{s}) - x^*)}{u'(\theta + \tau(\theta, \bar{s}) + q^*x^*)}. \quad (31)$$

Equation (31) states that in equilibrium, the marginal rates of substitution between consumption after a bad signal and after a good signal is equated among protection buyers and investors, i.e., they share risk optimally, as in the second best (see Proposition 2). Moreover, this marginal rate of substitution (weighted by the probabilities of a good and a bad signal) is equal to the price of insurance against signal risk.

As long as protection buyers are exposed to signal risk, we have

$$\frac{u'(\theta + \tau(\theta, \bar{s}) - x^*)}{u'(\theta + \tau(\theta, \bar{s}) + q^*x^*)} < 1,$$

which, combined with (31), implies that insurance against signal risk is not actuarially fair. Investors who supply protection buyers with insurance against a bad signal earn profits on average. This, in turn, means that investors’ equilibrium supply is strictly positive. Thus, the market for insurance against signal risk is active, i.e., $x^* > 0$. Protection buyers - who cannot get full insurance from protection sellers because of moral hazard - demand strictly positive amount of additional insurance from investors.

**Equilibrium in the market for protection sellers’ assets:** Given equilibrium $(q^*, x^*)$ in the market for insurance against signal risk, equilibrium in the market for protection sell-
ers’ assets is defined by a price $p^*$ and a trading volume $\alpha^*$, such that the market clears, i.e., $\alpha_s(p^*) = \alpha_I(p^*) = \alpha^*$. The next proposition characterises equilibrium in the market for protection sellers’ assets.

**Proposition 5** If
\[
\frac{u'(E[\tilde{\theta}|s] + \mathcal{P} + qx^*)}{u'(E[\tilde{\theta}|s] - \frac{P[s]}{P[s]} \mathcal{P} - x^*)} > \frac{\psi}{1-\mu} - \psi, \quad (32)
\]
the equilibrium level of asset sales $\alpha^*$ is strictly positive and such that
\[
\frac{u'(E[\tilde{\theta}|\bar{s}] - P[s] \mathcal{P} - x^*)}{u'(E[\tilde{\theta}|\bar{s}] - \frac{P[s]}{P[s]} [\alpha^*(R - \psi) + (1 - \alpha^*) \mathcal{P}] - x^*)} = \frac{\psi}{1-\mu} - \psi - (\psi I(\alpha^*) + \alpha^* \psi_I'(\alpha^*)) , \quad (33)
\]
while the market price of protection sellers’ assets is:
\[
p^* = R - [\psi I(\alpha^*) + \alpha I \psi_I'(\alpha^*)]. \quad (34)
\]

Otherwise, if (32) does not hold, there are no asset sales in equilibrium, i.e., $\alpha^* = 0$.

### 5.5 Equilibrium constrained efficiency

Comparing Lemma 4 and Propositions 4 and 5 to Propositions 2 and 3, we obtain the following welfare theorem.

**Proposition 6** Market equilibrium is information-constrained Pareto efficient.

It is striking that, in spite of moral hazard, the market equilibrium is second best, all the more so as the price in the incentive constraint (19) induces pecuniary externalities.

Proposition 6 reflects the presence of two countervailing pecuniary externalities. When one protection buyer demands larger margins, this depresses the price, which tightens the
incentive constraint of all protection sellers. This negative pecuniary externality tends to increase the amount of signal risk protection buyers must bear.

There is, however, a countervailing, stabilising effect. The decline in the price increases the profits of investors after a bad signal. Thus, while a negative signal is a negative shock for protection buyers, it is a positive shock for investors. This creates scope for risk-sharing gains from trade between investors and protection buyers. When the market is complete, investors and protection buyers fully exploit this risk-sharing opportunity until their marginal rates of substitution are equalised, exactly as in the second best.

5.6 Contracting between investors and protection sellers

Above, we analysed the supply of insurance against signal risk by investors to protection buyers. Similar outcomes could be obtained with protection sellers buying insurance against signal risk from investors. The payment protection sellers would thus receive after bad news would be deposited on the margin account, and therefore added to the payment credibly promised to protection buyers. The payment protection buyers would pay to investors after good news would be deducted to the transfer they would make to protection buyers. Therefore this transaction would leave the incentive and participation constraints of protection sellers unaffected. Moreover it would leave the consumption of the protection buyer after good and bad news equal to its value in equations (26) and (27) respectively. So equilibrium could as well be implemented with contracting between protection sellers and investors. That implementation, however, would be more involved than that studied above: Contracting between protection sellers and investors would have to be observable by protection buyers, and contractible. Otherwise protection sellers could be tempted not to purchase insurance against signal risk, which would increase counterparty risk for protection buyers.
6 Risk shifting

So far, the protection seller had to exert unobservable risk-management effort to improve asset returns in the sense of first-order stochastic dominance, as in Holmstrom and Tirole (1997). We now consider another type of unobservable action: “risk-shifting” à la Jensen Meckling (1976). To do so, we follow Biais and Casamatta (1999). We assume there is one competitive protection seller, whose per-unit asset return \( \tilde{R} \) can be high \((H)\), medium \((0 < M < H)\), or 0 (in which case the limited liability protection seller defaults on any obligation). Without risk shifting, the probability of the high return is \( \kappa \), and the probability of the medium return is \( 1 - \kappa \). In that case, the expected return on asset is \( E(\tilde{R}) = \kappa H + (1 - \kappa)M \). Risk-shifting increases the probability of 0 return to \( b \in (0, 1) \) and the probability of \( H \) to \( \kappa + a \in (\kappa, 1) \). Correspondingly, risk-shifting reduces the probability of \( M \) to \( 1 - (\kappa + a + b) \geq 0 \). Risk-shifting generates second-order stochastic dominance, i.e., the expected return under risk-shifting is \( \hat{E}(\tilde{R}) = (\kappa + a)H + (1 - (\kappa + a + b))M < E(\tilde{R}) \), that is, \( (a + b)M > aH \).

In this section, we analyse market equilibrium with optimal contracts under risk-shifting.\(^{19}\) We follow exactly the same steps as above in Section 5, and show that risk-shifting involves the same economic forces as unobservable costly effort.

6.1 Protection sellers’s incentive and participation constraints

As before, after good news the incentive constraint of the protection seller is slack and there is no margin call. Also as before, it is optimal to set \( \tau(\theta, g, 0) = \alpha S p \). Correspondingly, the

\(^{19}\)For brevity, we only study equilibrium and omit the planner’s problem. Also for brevity, we do not consider derivatives contingent on \( \tilde{R} \). While this restriction matters for efficiency, it does not affect qualitatively the analysis in this section.
protection seller’s incentive constraint after a bad signal at $t = 1$ is

$$
\alpha_s p + (1 - \alpha_S)E(\bar{R}) - \kappa E[\tau(\bar{\theta}, \bar{s}, H)] - (1 - \kappa) E[\tau(\bar{\theta}, \bar{s}, M)] \\
\geq (1 - b)\alpha_s p + (1 - \alpha_S)\hat{E}(\bar{R}) - (\kappa + a) E[\tau(\bar{\theta}, \bar{s}, H)] - (1 - (\kappa + a + b)) E[\tau(\bar{\theta}, \bar{s}, M)],
$$

that is,

$$
 b\alpha_s p + (1 - \alpha_S) \left[E(\bar{R}) - \hat{E}(\bar{R})\right] \geq (a + b) E[\tau(\bar{\theta}, \bar{s}, M)] - a E[\tau(\bar{\theta}, \bar{s}, H)].
$$

The left-hand side is the protection seller’s expected loss from risk shifting. With probability $b$ the protection seller loses the cash deposited on the margin $\alpha_s p$. Furthermore, expected returns decline by $E(\bar{R}) - \hat{E}(\bar{R})$ on the fraction $1 - \alpha_S$ of assets the seller still manages. The right-hand side is the seller’s expected gain from risk shifting. She has to make the transfers in case of return $M$ less often and those in case of return $H$ more often.

The incentive condition rewrites as

$$
\alpha_s p + (1 - \alpha_S)\hat{P} \geq \left(\frac{a + b}{b}\right) E[\tau(\bar{\theta}, \bar{s}, M)] - \frac{a}{b} E[\tau(\bar{\theta}, \bar{s}, H)],
$$

(35)

where $\hat{P} = \frac{E(\bar{R}) - \hat{E}(\bar{R})}{b}$ is the counterpart of the pledgeable income in the costly risk-management effort model. Thus, (35) has the same structure as the incentive constraint in the unobservable costly effort case (19). In particular, in both cases margins relax incentive constraints if the market price is larger than the pledgeable income.

At $t = 0$, the protection seller’s participation constraint is

$$
Pr[\bar{s}] \left(E(\bar{R}) - E[\tau(\theta, \bar{s}, \bar{R})]\right) + Pr[\bar{\theta}] \left(\alpha_s p + (1 - \alpha_S)E(\bar{R}) - E[\tau(\bar{\theta}, \bar{s}, \bar{R})]\right) \geq E(\bar{R}),
$$
that is,

$$- E[\tau(\tilde{\theta}, \tilde{s}, \tilde{R})] \geq \Pr[\tilde{s}] \alpha_S \left[ E(\tilde{R}) - p \right].$$  \hspace{1cm} (36)

The left-hand side is the expected transfer to the protection seller, while the right-hand side is the expected opportunity cost of liquidating a fraction $\alpha_S$ of the asset in the market at price $p$ instead of keeping them and earning expected return $E(\tilde{R})$. The participation constraint (36) mirrors that with costly unobservable effort, (20). In particular, in both cases margin calls tighten the participation constraint of the protection seller when $p < E(\tilde{R})$.

Thus, with risk-shifting as with unobservable effort, optimal margin calls trade-off the benefit of relaxing the incentive constraint against the cost of tightening the participation constraint.

### 6.2 Investor’s optimal trades

The investor chooses $x_I$ to maximize

$$\Pr[\tilde{s}]v(e + x_I) + \Pr[\tilde{s}]E[v(e - qx_I + \alpha_I(\tilde{R} - p))].$$  \hspace{1cm} (37)

This is similar to equation (21) except that now, when buying the protection seller’s asset, investors bear the risk of $\tilde{R}$. The first order condition yields

$$q = \frac{\Pr[\tilde{s}]}{\Pr[\tilde{s}]} \frac{v'(e + x_I)}{E[v(e - qx_I + \alpha_I(\tilde{R} - p))]},$$  \hspace{1cm} (38)

which is the analogue of equation (23).

After a bad signal, investors choose $\alpha_I$ to maximise $E[v(e - qx_I + \alpha_I(\tilde{R} - p))]$. The
derivative of the objective with respect to $\alpha_I$ is:

$$\kappa(H - p)v'(e - qx_I + \alpha_I(H - p)) + (1 - \kappa)(M - p)v'(e - qx_I + \alpha_I(M - p)).$$

If $p \leq M$ the derivative is positive for all $\alpha_I$, and the investor’s demand infinite, which cannot be the case in equilibrium. Moreover, as the investor is risk-averse, his demand is 0 as long as the price is strictly above $E(\tilde{R})$. For $p \in (M, E(\tilde{R}))$, there is an interior solution given by the following inverse demand function

$$p = \kappa(\alpha_I)H + (1 - \kappa(\alpha_I))M,$$

where

$$\kappa(\alpha_I) = \frac{v'(e - qx_I + \alpha_I(H - p))}{E[v'(e - qx_I + \alpha_I(\tilde{R} - p))]}, \quad \kappa \in (0, \kappa),$$

can be interpreted as a risk-adjusted probability. Under regularity conditions (satisfied for log or exponential utilities) stated in the Supplementary Appendix, $\kappa(\alpha_I)$ decreases in $\alpha_I$. As risk averse investors hold more of the risky asset, they are less willing to pay for it. Equation (39) is similar to equation (24). In both cases, the price of the asset is equal to its expected return minus a discount that increases in $\alpha_I$. In (24) it reflects the cost of effort, while in (39) it reflects the cost of bearing risk.

### 6.3 Contracts and equilibrium

The protection buyer chooses, at $t = 0$, the amount of insurance purchased from investors, the transfers from the protection sellers $\tau(\theta, s, R)$ and the margin call after bad news $\alpha_S$, to maximise her expected utility, subject to the incentive (35) and participation (36) constraints of the protection seller. In the Supplementary Appendix we derive the following results:
The participation and incentive constraints of the protection seller bind, and, for a given realisation of the protection seller’s asset return and the signal, the protection buyer is insured against the residual risk on \( \theta \). These results are similar to those obtained for costly unobservable effort (see Lemma 4).

After good news, the protection buyer is not exposed to the risk of the protection seller’s asset, but after bad news, transfers to the protection buyer are larger after \( H \) than after \( M \), in order to deter risk shifting. The latter effect did not arise with costly unobservable effort.

Finally, the first order condition with respect to the margin call \( \alpha_S \) yields, for an interior solution
\[
\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} = 1 + \frac{E(\tilde{R}) - pa + b}{p - \tilde{p}} \frac{1}{b(1 - \kappa)}.
\]

Equation (40) is similar to Equation (29) in Lemma 5. In both equations the left-hand side reflects the cost of imperfect insurance associated with signal risk, while the right-hand side reflects the cost of fire sales, which is particularly large when the price \( p \) is low. Equation (40) also shows that, as long as \( p < E(\tilde{R}) \), the marginal rate of substitution of the protection buyer is strictly larger than one. In contrast with the first best, under risk-shifting there is imperfect insurance.

The first order condition of the protection buyers with respect to the amount of insurance against signal risk, \( x_B \), yields
\[
q = \frac{\Pr[s]}{\Pr[\bar{s}]} \frac{u'(\theta + \tau(\theta, s) - x_B)}{\kappa u'(\theta + \tau(\theta, s, H) + qx_B) + (1 - \kappa)u'(\theta + \tau(\theta, s, M) + qx_B)}.
\]

Equating (41) and (38), yields the market clearing amount of insurance against signal risk, \( x^* \), as in Proposition 4. With risk shifting, as with unobservable costly effort, equilibrium in the market for insurance against signal risk entails equalisation of the marginal rates of substitution of protection buyers and investors. Finally, to obtain the condition characterising
the market clearing margin call $\alpha^*$ (similarly to in Proposition 5), substitute (39) in (41).

7 Implications

**Economic balance sheet:** The variation-margin call requests that protection sellers deposit safe assets in a margin account. These safe assets, however, are still owned by protection sellers. Therefore the margin deposit still shows up on the asset side of their balance sheet. This is in line with the remark of McDonald and Paulson (2015, page 92) that the “transfer of funds based on a market value change is classified as a change in collateral and not as payment.”

Figure 3 shows the balance sheet of protection sellers at time 1. This is not an accounting balance sheet, but an economic one, showing the value of the assets and liabilities of protection sellers, including those corresponding to derivative positions, implied by our theoretical analysis. After good news, the total derivative position of the protection seller is expected to make a positive profit. This “marked-to-market” net derivative position is on the asset side of the balance sheet of protection sellers along with the portfolio of loans under management (which will, on the equilibrium path, return $R$ at time 2 since the protection seller will exert effort). On the other side of the balance sheet, there is just the equity of protection sellers. After bad news, the derivative position of protection sellers is expected to be loss-making, which triggers the variation-margin call. Correspondingly, on the asset side of the balance sheet, there is the margin deposit ($\alpha p$) and the downsized portfolio of loans, $(1 - \alpha)R$, while on the other side of the balance sheet, there is the liability corresponding to the net value of the derivative position $\alpha p + (1 - \alpha)P$ and the lower equity of protection sellers.
Variation margins vs solvency regulation: In line with stylised facts, variation margins in our model are called after a drop in equity capital, due to derivative losses. The ensuing asset sales could be compared to those triggered by solvency regulation, whose goal is to reduce leverage. In our analysis, however, variation margin calls increase leverage: After bad news, the total value of assets (margin deposits plus risky assets under management) is $R - \alpha(R - p)$. This is decreasing in $\alpha$, reflecting that assets under management are more profitable than cash in the margin account. The value of liabilities, i.e., the expected loss on the derivative position, is $\mathcal{P} + \alpha(p - \mathcal{P})$. This is increasing in $\alpha$, reflecting that larger variation margins enable to promise more insurance ex ante, generating larger liabilities after bad news. Thus equity (assets minus liabilities) decreases in $\alpha$. This striking difference between the consequences of variation margins and those of solvency regulation underscores the difference between our asset-side view of variation margins, which change the structure of assets to enhance pledgeability, and the liability side approach of solvency regulation, which
changes the structure of liabilities to reduce leverage.\footnote{Fostel and Geneakoplos (2015), who take a liability-side approach of margins, also find opposite results to ours regarding the link between margins and equity ratios. In their analysis, as the minimum payoff or the asset purchased decreases, leverage constraints tighten, and larger initial margins are requested. Thus larger initial margins correspond to lower leverage.}

**Contagion between asset classes:** Bad news ($\delta$) lower the conditional expectation of the final value of protection buyers’ assets (to $E[\tilde{\theta}|s]$). In the first best, there is no simultaneous change in the valuation of protection sellers’ assets. The situation is different with moral hazard in which, after bad news, variation margin calls lead to asset sales lowering the price of protection sellers’ assets. Thus, moral hazard generates endogenous correlation between protection buyers’ and protection sellers’ assets. This can be interpreted as contagion after bad news, and is in line with the empirically observed increase in correlation during bear markets (see, e.g., Ang and Chen, 2002).

**Price of protection:** An increase in variation margin calls $\alpha$ generates productive inefficiencies by reducing the return on protection sellers’ assets. The break-even constraint of protection sellers then requires them to make lower average transfers to protection buyers. Empirically, this means that the price of protection, proxied for example by CDS spreads, should increase when protection sellers expect larger margin calls, e.g., after a negative shock to their balance sheets. This matches the recent finding by Siriwardane (2018) that negative shocks to the capital of protection sellers in the CDS market increase the cost of insurance they provide. Our model further predicts that the effect Siriwardane (2018) documents should be more pronounced when the asset-price impact of margin calls is large.

**Policy:** Regulators are concerned by margin-induced fire sales (see, e.g., ESRB 2017, p.5). As noted by Meier and Servaes (2015), however, buyers of underpriced assets benefit
from fire sales. Meier and Servaes (2015) argue that a welfare analysis should weigh these benefits against the losses of the sellers. Our comparison of market equilibrium and second best considers both the ex-ante as well as the ex-post consequences of fire sales. Ex post, during the fire sale, the profits of asset buyers are the mirror image of the losses of asset sellers. Ex ante, before the fire sale, what matters for welfare is the way in which those profits and losses are taken into account. When all market participants rationally anticipate the risk of fire sales, efficient insurance against that risk is supplied and equilibrium is constrained efficient.

This points to a form of complementarity between markets. To efficiently share protection buyers’ risk, we need both i) insurance markets against the final value of protection buyers’ assets, and ii) insurance markets against the interim risk of fire sales. In practice, margin calls are often triggered by drops in the market valuation of the insured asset. In that case, the above mentioned final and interim risks correspond to different maturities of derivatives with the same underlying asset. When overseeing the development of derivatives markets, regulators and market organisers should therefore ensure that the set of maturities and risks traded be comprehensive enough. They should also make sure all market participants are aware of the risk of fire sales and can sell insurance against that risk.
References


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Appendix

A Proofs

Proof of Proposition 1

The Lagrangian is:
\[ L_{FB}(c_B(\theta, s), c_S(\theta, s), c_I(\theta, s), \alpha(s)) = \omega_B E[u(c_B(\theta, s))] + \omega_I E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] \]
\[ + \lambda_B [E[u(c_B(\theta, s))] - E[u(\theta)]] \]
\[ + \lambda_I [E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] - v(1)] \]
\[ + \lambda_S [E[c_S(\theta, s) - (1 - \alpha(s))\psi] - (R - \psi)] \]
\[ + \sum_{\theta, s} \lambda(\theta, s)[\theta + 1 + R - (c_B(\theta, s) + c_S(\theta, s) + c_I(\theta, s))] \]
\[ + \sum_s (\lambda_I(s)[1 - \alpha(s)] + \lambda_0(s)[\alpha(s)]) \]

First-order conditions with respect to \( c_B(\theta, s), c_I(\theta, s), c_S(\theta, s) \) and \( \alpha(s) \) are
\[ (\omega_B + \lambda_B)\Pr[\theta, s]u'(\theta, s) = \lambda(\theta, s), \] (A.1)
\[ (\omega_I + \lambda_I)\Pr[\theta, s]v'(\theta, s) = \lambda(\theta, s), \] (A.2)
\[ \lambda_S\Pr[\theta, s] = \lambda(\theta, s), \] (A.3)

and
\[ -(\omega_I + \lambda_I)\Pr[s]E[v'(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(\psi_I + \alpha(s)\psi_I')]|s] + \lambda_S\Pr[s]\psi = \lambda_1(s) - \lambda_0(s), \] (A.4)
respectively (where, in (A.4), we have used \( \Pr[\theta, s] = \Pr[\theta|s]\Pr[s] \)). The second-order conditions with respect to \( c_B(\theta, s), c_I(\theta, s) \) and \( c_S(\theta, s) \) hold because of decreasing marginal utilities. The second-order condition with respect to \( \alpha \) is:
\[ -(\omega_I + \lambda_I)\Pr[s]E[-v''(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(\psi_I + \alpha(s)\psi_I')^2 \]
\[ + v'(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))(2\psi_I' + \alpha(s)\psi_I'')|s] \leq 0, \] (A.5)
which holds since \( \psi_I'' \geq 0 \) and \( v'' < 0 \).

Equations (A.1), (A.2) and (A.3) imply
\[ (\omega_B + \lambda_B)u'(\theta, s) = (\omega_I + \lambda_I)v'(\theta, s) = \lambda_S \quad \forall(\theta, s). \] (A.6)

Because neither \( \omega_B, \omega_I, \lambda_B, \lambda_I, \) nor \( \lambda_S \) depend on the state \( (\theta, s) \), equation (A.6) implies that
the marginal utilities of buyers and investors are constant across states. Hence, \( c_B(\theta, s) = c_B \) and \( c_I(\theta, s) = c_I \).

The resource constraints bind, \( \lambda(\theta, s) > 0 \). Suppose not. Because \( v', u' > 0, Pr[\theta, s] > 0 \), this implies \( \omega_B + \lambda_B = 0 \) and \( \omega_I + \lambda_I = 0 \), and hence, \( \omega_B = \omega_I = 0 \). But because we also have \( \omega_S = 0 \) (by assumption), the planner’s objective would then become zero.

The participation constraint of the sophisticated investors binds, \( \lambda_S > 0 \). Because \( Pr[\theta, s] > 0 \), this is immediate once \( \lambda(\theta, s) > 0 \).

There is no asset transfer in any state, \( \alpha(s) = 0 \). Suppose there were positive asset transfers, i.e., \( \alpha(s) > 0 \). Using the second equality in (A.6), dividing by \( \lambda_S Pr[s] \), and rearranging, the first-order condition with respect to \( \alpha(s) \) becomes

\[
\psi - \psi_I(\alpha(s)) = \frac{\lambda_1}{\lambda_S Pr[s]} + \alpha(s)\psi_I'.
\]

Given that \( \lambda_S > 0 \), \( \psi_I' \geq 0 \) and \( \psi < \psi_I(\alpha(s)) \) when \( \alpha(s) > 0 \), this is a contradiction: the left-hand side is negative while the right-hand side is weakly positive.

Given constant consumption for buyers and investors, and the binding resource constraints, we have

\[
c_B + c_I + c_S(\theta, s) = \theta + 1 + R \quad \forall(\theta, s).
\]

Using this to substitute for \( c_S(\theta, s) \) in the binding participation constraint of investors, together with \( \alpha(s) = 0 \), we have

\[
c_B + c_I = E[\tilde{\theta}] + 1.
\]

QED

**Proof of Lemma 1**

The Lagrangian of the second-best maximisation problem is

\[
L_{SB}(c_B(\theta, s), c_S(\theta, s), c_I(\theta, s), \alpha(s)) = \omega_B E[u(c_B(\theta, s))] + \omega_I E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] \\
+ \sum_s \lambda_I c(s) \left[ E[c_S(\theta, s) | s] - \frac{(1 - \alpha(s))\psi}{1 - \mu} \right] + \lambda_B [E[u(c_B(\theta, s)) - E[u(\theta)]] + \lambda_I [E[v(c_I(\theta, s) - \alpha(s)\psi_I(\alpha))] - v(1)] + \lambda_S [E[c_S(\theta, s) - (1 - \alpha(s))\psi] - (R - \psi)] \\
+ \sum_{\theta, s} \lambda(\theta, s)[\theta + 1 + R - (c_B(\theta, s) + c_S(\theta, s) + c_I(\theta, s))] \\
+ \sum_s (\lambda_1(s)[1 - \alpha(s)] + \lambda_0(s)[\alpha(s)])
\].

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First-order conditions with respect to $c_B(\theta, s)$ and $c_I(\theta, s)$ are the same as in the first best, (A.1) and (A.2), respectively. The first-order conditions with respect to $c_S(\theta, s)$ and $\alpha(s)$ are altered, to take into account the incentive constraint, and write

$$\lambda_{IC(s)} \Pr[\theta | s] + \lambda_S \Pr[\theta, s] = \lambda(\theta, s)$$  \hspace{1cm} (A.7)

and

$$-(\omega_I + \lambda_I) \Pr[s] E[v'(\theta, s)|s](\psi_I(\alpha) + \alpha(s) \psi_I') + \lambda_{IC(s)} \frac{\psi}{1 - \mu} + \lambda_S \Pr[s] \psi = \lambda_1(s) - \lambda_0(s),$$  \hspace{1cm} (A.8)

respectively. The second-order conditions are as in the first best.

The first-order conditions with respect to $c_B(\theta, s)$ and $c_S(\theta, s)$, (A.1) and (A.7), respectively imply

$$u'(\theta, s) = \frac{1}{\omega_B + \lambda_B} \left( \lambda_{IC(s)} \frac{1}{\Pr[s]} + \lambda_S \right)$$  \hspace{1cm} (A.9)

while the first-order conditions with respect to $c_I(\theta, s)$ and $c_S(\theta, s)$, (A.2) and (A.7), respectively imply

$$v'(\theta, s) = \frac{1}{\omega_I + \lambda_I} \left( \lambda_{IC(s)} \frac{1}{\Pr[s]} + \lambda_S \right).$$  \hspace{1cm} (A.10)

Because their right-hand sides are independent of $\theta$, (A.9) and (A.10) imply that, for a given realisation of the signal $s$, the marginal utility of consumption of the protection buyers and investors is the same in $(\bar{\theta}, s)$ and $(\tilde{\theta}, s)$.

QED

**Proof of Lemma 2**

First, we prove that the resource constraints bind, $\lambda(\theta, s) > 0$. Suppose not. Because $v', u' > 0$, $\Pr[\theta, s] > 0$, by (A.1) and (A.2), this implies $\omega_B + \lambda_B = 0$ and $\omega_I + \lambda_B = 0$, and hence, $\omega_B = \omega_I = 0$, a contradiction.

Second, we prove that the participation constraint of the protection seller binds. Suppose not, $\lambda_S = 0$. Then, using $\lambda(\theta, s) > 0$ in (A.7) yields $\lambda_{IC(s)} > 0$ for all $(\theta, s)$, i.e., both incentive constraints bind. From the binding incentive constraints, we have $E[c_S(\theta, s)|s] = \frac{1 - \alpha(s)}{1 - \mu} \psi$ and hence,

$$E[c_S(\theta, s)] = \Pr[\tilde{s}] \frac{1 - \alpha(s)}{1 - \mu} \psi + \Pr[\bar{s}] \frac{1 - \alpha(s)}{1 - \mu} \psi = (1 - E[\alpha(s)]) \frac{\psi}{1 - \mu}$$  \hspace{1cm} (A.11)
Substituting this into the slack participation constraint of the protection seller yields
\[
(1 - E[\alpha(s)]) (1 - \psi) (1 - \mu) - (1 - E[\alpha(s)]) \psi > R - \psi
\]
and, after some rearranging,
\[
-E[\alpha(s)] \psi \frac{\mu}{1-\mu} > R - \frac{\psi}{1-\mu},
\]
which contradicts the assumption that \( P > 0 \).

Third, we prove that one of the two incentive constraints (or both) must bind. If not, then the first-best allocation would solve the second best problem. Now, with the seller’s first-best consumption (9) and \( \alpha(s) = 0 \), the incentive constraint after a bad signal becomes
\[
\Pr(\bar{\theta} | s) (\bar{\theta} - E[\bar{\theta}] + R) + \Pr(\bar{\theta} | s) (\bar{\theta} - E[\bar{\theta}] + R) \geq \frac{\psi}{1-\mu},
\]
i.e.,
\[
E[\bar{\theta} | s] - E[\bar{\theta}] + R \geq \frac{\psi}{1-\mu},
\]
which violates assumption (10).

Fourth, we prove that both ICs cannot bind at the same time. Suppose they do. Then, we have again (A.11), which after substituting the binding participation constraint of the sophisticated investor and rearranging yields
\[
-E[\alpha(s)] \psi \frac{\mu}{1-\mu} = R - \frac{\psi}{1-\mu},
\]
which contradicts the assumption that \( P > 0 \).

QED

**Proof of Lemma 3**

First, we prove that when the incentive-compatibility condition in state \( s \) is slack, then \( \alpha(s) = 0 \). Suppose not, i.e., \( \alpha(s) > 0 \) and \( \lambda_{IC(s)} = 0 \). Then, using (A.2) and (A.7), (A.8) becomes
\[
-\lambda_S Pr[s] (\psi_I(\alpha(s)) + \alpha(s) \psi_I'(\alpha(s))) + \lambda_S Pr[s] \psi = \lambda_1(s).
\]
Dividing by \( \lambda_S Pr[s] > 0 \) and rearranging yields
\[
\psi - \psi_I(\alpha(s)) = \frac{\lambda_1(s)}{\lambda_S(s) Pr[s]} + \alpha(s) \psi_I'(\alpha(s)).
\]

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Given that $\psi' \geq 0$ and $\psi < \psi_I$ when $\alpha(s) > 0$, we obtain the desired contradiction. The left-hand side is negative while the right-hand side is weakly positive.

Second, we prove that the incentive-compatibility condition after a bad signal binds. Suppose not, $\lambda_{IC(s)} = 0$, and only the incentive constraint after the good signal binds. Now, given that the incentive constraint after a bad signal is slack, so that $\alpha(\bar{s}) = 0$, and the incentive constraint after a good signal binds, we have

$$E[c_S(\theta, s)|\bar{s}] = \frac{(1 - \alpha(\bar{s}))\psi}{1 - \mu} = \frac{\psi}{1 - \mu} - \frac{\alpha(\bar{s})\psi}{1 - \mu}$$

which implies that

$$E[c_S(\theta, s)|\bar{s}] - E[c_S(\theta, s)|\bar{s}] > 0.$$ (A.12)

Next, from the binding resource constraints and full risk-sharing conditional on the signal, we have

$$c_S(\theta, s) = \theta + 1 + R - (c_B(s) + c_I(s))$$ (A.13)

and hence

$$E[c_S(\theta, s)|\bar{s}] = E[\theta|\bar{s}] + 1 + R - (c_B(\bar{s}) + c_I(\bar{s}))$$ (A.14)

$$E[c_S(\theta, s)|s] = E[\theta|s] + 1 + R - (c_B(s) + c_I(s))$$ (A.15)

so that

$$E[c_S(\theta, s)|s] - E[c_S(\theta, s)|\bar{s}] = E[\theta|s] - E[\theta|\bar{s}] - [c_B(\bar{s}) - c_B(s) + c_I(s) - c_I(\bar{s})].$$ (A.16)

To obtain that expression in (A.16) is weakly negative, so that we have the contradiction to (A.12), the term in squared brackets with the consumptions must be weakly positive (because the signal is (weakly) informative, we have $E[\theta|s] - E[\theta|\bar{s}] \leq 0$). From (A.1), (A.7) and the slack incentive constraint after a bad signal, we have

$$(\omega_B + \lambda_B)u'(\theta, \bar{s}) = \lambda_S + \frac{\lambda_{IC(s)}}{\Pr[\bar{s}]}$$

$$(\omega_B + \lambda_B)u'(\theta, \bar{s}) = \lambda_S$$

Together with full risk-sharing conditional on the signal, this implies that

$$c_B(\bar{s}) \geq c_B(s).$$

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The same type of argument also establishes that
\[ c_I(\bar{s}) \geq c_I(s). \]
Hence, the term in squared brackets in (A.16) is (weakly) positive, which yields the desired contradiction.

Third, we analyse the ranking of the consumptions of the protection buyers after bad and good signals. Combining (A.2) with (A.7), and using the fact that there is full risk-sharing conditional on the signal and that only the incentive constraint after the bad signal binds, we obtain:

\[
(\omega_B + \lambda_B) \Pr[\theta, \bar{s}] u'(c_B(\bar{s})) = \lambda_S \Pr[\theta, \bar{s}]
\]
\[
(\omega_B + \lambda_B) \Pr[\theta, s] u'(c_B(s)) = \lambda_{IC(\bar{s})} \Pr[\theta|\bar{s}] + \lambda_S \Pr[\theta, s]
\]
so that
\[
u'(c_B(s)) = 1 + \frac{\lambda_{IC(\bar{s})}}{\Pr[s]\lambda_S}. \tag{A.17}
\]
Because \(\lambda_{IC(\bar{s})} > 0\) and \(\lambda_S > 0\), we have imperfect risk-sharing across signals with
\[ c_B(s) < c_B(\bar{s}). \]

QED

**Proof of Proposition 3**

First, we write down more precisely the first-order optimality condition with respect to \(\alpha(s)\). Using (A.2) and Lemma 1, the derivative of the Lagrangian with respect to \(\alpha(s)\) is

\[
\frac{\partial L_{SB}}{\partial \alpha(s)} = -\lambda(\theta, s) \Pr[s] \frac{1}{\Pr[\theta, s]} (\psi_I(\alpha(s)) + \alpha(s) \psi'_I(\alpha(s))) + \lambda_{IC(s)} \frac{\psi}{1 - \mu} + \lambda_S \Pr[\bar{s}] \psi - (\lambda_1(s) - \lambda_0(s)).
\]

Using (A.7), this rewrites as

\[
\frac{\partial L_{SB}}{\partial \alpha(s)} = -(\lambda_S \Pr[\bar{s}] + \lambda_{IC(s)}) (\psi_I(\alpha(s)) + \alpha(s) \psi'_I(\alpha(s))) + \lambda_{IC(\bar{s})} \frac{\psi}{1 - \mu} + \lambda_S \Pr[\bar{s}] \psi - (\lambda_1(s) - \lambda_0(s)).
\]

Collecting terms,

\[
\frac{\partial L_{SB}}{\partial \alpha(s)} = \lambda_{IC(s)} \left[ \frac{\psi}{1 - \mu} - (\psi_I(\alpha(s)) + \alpha(s) \psi'_I(\alpha(s))) \right] + \lambda_S \Pr[\bar{s}] \left[ \psi - (\psi_I(\alpha(s)) + \alpha(s) \psi'_I(\alpha(s))) \right] - (\lambda_1(s) - \lambda_0(s)). \tag{A.18}
\]

Second, we show that under (16) there must be some asset transfer, i.e., \(\alpha(s) > 0\).
Suppose not, i.e., suppose we have \( \alpha(s) = 0 \). Then, \( \lambda_1(s) = 0 \) and, by (A.18), the optimality condition such that \( \alpha(s) = 0 \), \( \frac{\partial L_S}{\partial \alpha(s)} \leq 0 \), writes as

\[
\lambda_{IC(s)} \left[ \frac{\psi}{1 - \mu} - \psi_I(0) \right] + [\psi - \psi_I(0)] \leq - \frac{\lambda_0(s)}{\lambda_S \Pr[\bar{s}]}.
\]  

(A.19)

Now, (A.17) yields

\[
\frac{\lambda_{IC(s)}}{\Pr[\bar{s}] \lambda_S} = \frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \bigg|_{\alpha(s) = 0} - 1.
\]  

(A.20)

Substituting into (A.19) yields

\[
\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \bigg|_{\alpha(s) = 0} = - \frac{1}{\frac{\psi}{1 - \mu} - \psi_I(0)} - \frac{\lambda_0(s)}{\lambda_S \Pr[\bar{s}] \left[ \frac{\psi}{1 - \mu} - \psi_I(0) \right]},
\]  

(A.21)

which contradicts (16), since the latter states that

\[
\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \bigg|_{\alpha(s) = 0} > \frac{\psi}{1 - \mu} - \psi_I(0).
\]

Third, we characterise asset transfers when they are interior, i.e., when \( \alpha(s) \in (0, 1) \). In that case, (A.18) and (A.17) imply

\[
\left[ \frac{u'(c_B(s))}{u'(c_B(\bar{s}))} - 1 \right] + \frac{\psi - (\psi_I(\alpha(s)) + \alpha(s) \psi_I'(\alpha(s)))}{\frac{\psi}{1 - \mu} - (\psi_I(\alpha(s)) + \alpha(s) \psi_I'(\alpha(s)))} = 0
\]

or, equivalently,

\[
\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} = \frac{\psi}{\frac{\psi}{1 - \mu} - (\psi_I(\alpha(s)) + \alpha(s) \psi_I'(\alpha(s)))},
\]

where \( c_B(s) \) and \( c_B(\bar{s}) \) are as given in Proposition 2.

QED
Proof of Lemma 4

First, we write down the Lagrangian of the protection buyer and use it to show that the participation constraint of protection sellers bind. The Lagrangian is:

\[
L(\tau(\theta, s), \alpha_S, x_B) = \Pr[\bar{s}]E[u(\theta + \tau(\theta, s) - x_B)|\bar{s}] + \Pr[s]E[u(\theta + \tau(\theta, s) + qx_B)|s] + \lambda_{IC}[\alpha_S p + (1 - \alpha_S)\mathcal{P} - E[\tau(\theta, s)|s]]
\]

\[
+ \lambda_S[\Pr[\bar{s]}(R - \psi) + \Pr[s](1 - \alpha_S)(R - \psi) + \alpha_S p) - E[\tau(\theta, s)] - (R - \psi)]
\]

\[
+ \lambda_1[1 - \alpha_S] - \lambda_0 \alpha_S.
\]

The first-order conditions of (A.22) with respect to \(\tau(\theta, \bar{s})\) and \(\tau(\theta, s)\) are:

\[
\Pr[\bar{s}]\Pr[\theta|\bar{s}]u'(\theta, \bar{s}) = \lambda_S \Pr[\theta, \bar{s}] \quad \forall \theta
\]

\[
\Pr[s]\Pr[\theta|s]u'(\theta, s) = \lambda_S \Pr[\theta, s] + \lambda_{IC}\Pr[\theta|s] \quad \forall \theta
\]

which simplify to

\[
u'(\theta, \bar{s}) = \lambda_S \quad \forall \theta \quad (A.23)
\]

\[
u'(\theta, \bar{s}) = \lambda_S + \frac{\lambda_{IC}}{\Pr[\bar{s}]} \quad \forall \theta. \quad (A.24)
\]

(A.23) implies that \(\lambda_S > 0\), i.e., the participation constraint of protection sellers binds.

Second, we use the first-order conditions with respect to \(\tau(\theta, \bar{s})\) and \(\tau(\theta, s)\) to show that the protection buyer is fully insured conditional on the signal. Because the right-hand sides of (A.23) and (A.24) do not depend on \(\theta\), we have \(u'(\theta, \bar{s}) = u'(\theta, \bar{s}), \forall \theta\), i.e,

\[
\bar{\theta} + \tau(\bar{\theta}, \bar{s}) = \bar{\theta} + \tau(\bar{\theta}, \bar{s}) \quad (A.25)
\]

\[
\bar{\theta} + \tau(\bar{\theta}, s) = \bar{\theta} + \tau(\bar{\theta}, s) \quad (A.26)
\]

Thus, conditional on the realisation of the signal \(s\), the protection buyer is fully insured against remaining \(\theta\)-risk.

Third, we prove by contradiction that the incentive-compatibility condition of the protection seller binds. To do so, we proceed in two steps.

The first step is to prove that, if the incentive-compatibility condition of the protection seller was slack, there would be no asset sale in equilibrium. This first step proceeds by contradiction. Suppose \(\lambda_{IC} = 0\) and \(\alpha_S = \alpha_I > 0\). Consider the first-order condition of the Lagrangian (A.22) with respect to \(\alpha_S\), when \(\alpha_S > 0\) (and hence, \(\lambda_0 = 0\)) and \(\lambda_{IC} = 0\):

\[
- \lambda_S \Pr[s](R - \psi - p^*) = \lambda_1,
\]

where \(p^*\) is the equilibrium price in the asset market. From the investors' demand for the productive asset, we know that \(\alpha_I > 0\) requires \(p^* < R - \psi_I(0)\). Because \(\psi_I(0) \geq \psi\) by
assumption, the left-hand side of (A.27) is strictly negative, which contradicts the fact that
the right-hand side is weakly positive.

The second step is to prove that slack protection seller’s incentive constraint would con-
tradict our assumption that \( \mathcal{P} < E[\tilde{\theta}] - E[\tilde{\theta}|s] \). Suppose \( \lambda_{IC} = 0 \). Equations (A.23) and
(A.24) imply full insurance, \( \tau(\theta, s) = \tau(\theta, \tilde{s}) \equiv \tau(\theta) \) for all \( \theta \), and \( \tilde{\theta} + \tau(\tilde{\theta}) = \bar{\theta} + \tau(\bar{\theta}) \).
Using that \( \alpha_S = 0 \) and there is full insurance when \( \lambda_{IC} = 0 \), and substituting the binding
participation constraint, we obtain \( \tau(\tilde{\theta}) = -(1 - \pi)(\tilde{\theta} - \theta) \) and \( \tau(\bar{\theta}) = \pi(\bar{\theta} - \theta) \). Using this
in the slack incentive constraint yields

\[
\mathcal{P} > \Pr[\tilde{\theta}|s](-1)(1 - \pi)(\tilde{\theta} - \theta) + (1 - \Pr[\tilde{\theta}|s])\pi(\tilde{\theta} - \theta)
\]

\[
= (\pi - \Pr[\tilde{\theta}|s])(\tilde{\theta} - \theta)
\]

\[
= E[\tilde{\theta}] - E[\tilde{\theta}|s],
\]

a contradiction.

Fourth, we compute the transfers. The binding incentive and participation constraints imply

\[
E[\tau(\theta, s)|s] = -\frac{\Pr[s]}{\Pr[\bar{s}]}\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}, \tag{A.28}
\]

\[
E[\tau(\theta, s)|\bar{s}] = \alpha_S p + (1 - \alpha_S)\mathcal{P}. \tag{A.29}
\]

Equations (A.28) and (A.29), together with full insurance conditional on the signal, (A.25)
and (A.26), yield the set of transfers, for a given \( \alpha_S \):

\[
\tau^*(\tilde{\theta}, s) = -\Pr[\tilde{\theta}|s](\tilde{\theta} - \theta) + \alpha_S p + (1 - \alpha_S)\mathcal{P},
\]

\[
\tau^*(\bar{\theta}, s) = \Pr[\tilde{\theta}|s](\tilde{\theta} - \theta) + \alpha_S p + (1 - \alpha_S)\mathcal{P},
\]

\[
\tau^*(\tilde{\theta}, s) = -\Pr[\tilde{\theta}|s](\tilde{\theta} - \theta) - \frac{\Pr[s]}{\Pr[\bar{s}]}\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P},
\]

\[
\tau^*(\bar{\theta}, s) = \Pr[\tilde{\theta}|s](\tilde{\theta} - \theta) - \frac{\Pr[s]}{\Pr[\bar{s}]}\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}.
\]

QED

**Proof of Lemma 5**

The first-order condition of the Lagrangian (A.22) with respect to \( \alpha_S \) is

\[
\lambda_{IC}(p - \mathcal{P}) - \lambda_p \Pr[s](R - \psi - p) = \lambda_1 - \lambda_0. \tag{A.30}
\]
From (A.23) and (A.24) we have
\[
\frac{u'(\theta, s)}{u'(\theta, s)} = 1 + \frac{\lambda_{IC}}{Pr[s] \lambda_s} > 1, \tag{A.31}
\]
where the inequality follows from the binding incentive constraint stated in Lemma 4.

Combining (A.30) and (A.31), and using the consumptions in Lemma 4, we obtain
\[
\frac{u'(E[\theta|s] + \alpha_S p + (1 - \alpha_S)\mathcal{P} + qx^d)}{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}(\alpha_S(R - \psi) + (1 - \alpha_S)\mathcal{P}) - x^d)} = \frac{\lambda_1 - \lambda_0}{(p - \mathcal{P})Pr[s] \lambda_s} + \frac{R - \psi - \mathcal{P}}{p - \mathcal{P}}. \tag{A.32}
\]

Next, we show that when \( p > \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)} \) then \( \alpha_S > 0 \). In that case, (A.32) with \( \lambda_0 = 0 \) yields (29). Suppose not, \( \alpha_S = 0 \), so that \( \lambda_0 > 0 \) and \( \lambda_1 = 0 \). Then solving (A.32) with \( \alpha_S = 0 \) for \( p \) yields
\[
p = \mathcal{P} + \frac{R - \psi - \mathcal{P}}{\frac{u'(E[\theta|s] + \mathcal{P} + qx^d)}{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})Pr[s] \lambda_s}}.
\]

This contradicts the assumption that \( p > \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)} \), because
\[
\frac{u'(E[\theta|s] + \mathcal{P} + qx^d)}{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})Pr[s] \lambda_s} < (R - \psi - \mathcal{P}) \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)},
\]
as
\[
1 < \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)} \left[ \frac{u'(E[\theta|s] + \mathcal{P} + qx^d)}{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)} + \frac{\lambda_0}{(p - \mathcal{P})Pr[s] \lambda_s} \right],
\]
due to
\[
1 < 1 + \frac{\lambda_0}{(p - \mathcal{P})Pr[s] \lambda_s} \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)}.
\]

Finally, we show that when \( p \leq \mathcal{P} + (R - \psi - \mathcal{P}) \frac{u'(E[\theta|s] - \frac{Pr[s]}{Pr[t]}\mathcal{P} - x^d)}{u'(E[\theta|s] + \mathcal{P} + qx^d)} \), then \( \alpha_S = 0 \). To do
Lemma 5 states that if

\[ \text{Proof of Proposition 5} \]

the right-hand side of (A.32) goes to infinity, contradiction since the left-hand side is finite.

\[ \alpha \]

while otherwise

\[ \in \]

\[ \alpha \]

This price decreases when

\[ \alpha \]

because

\[ \in \]

\[ \alpha \]

Hence, the right-hand side is strictly negative while the left-hand side is strictly positive.

Second, when \( P < p \leq P + (R - \psi - P) \) \( \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} P - x^d)}{u'(E[\theta | \bar{s}] + P + qx^d)} \), then \( \alpha = 0 \). Suppose not, \( \alpha > 0 \) and hence, \( \lambda_0 = 0 \). Then, solving (A.32) for \( p \) yields

\[
p = \frac{\lambda_1}{Pr[\bar{s}]\lambda_S} \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} \alpha_S(R - \psi) + (1 - \alpha_S)P - x^d)}{u'(E[\theta | \bar{s}] + \alpha_SP + (1 - \alpha_S)P + qx^d)} + P + (R - \psi - P) \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} \alpha_S(R - \psi) + (1 - \alpha_S)P - x^d)}{u'(E[\theta | \bar{s}] + \alpha_SP + (1 - \alpha_S)P + qx^d)}.
\]

This price decreases when \( \alpha_S \) decreases (since the ratio of marginal utilities is strictly increasing in \( \alpha_S \)). Yet, with \( \alpha_S > 0 \), the price will always be larger than the largest price allowed in the starting condition

\[
p = P + (R - \psi - P) \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} P - x^d)}{u'(E[\theta | \bar{s}] + P + qx^d)}
\]

because \( \lambda_1 \geq 0 \) and

\[
\frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} \alpha_S(R - \psi) + (1 - \alpha_S)P - x^d)}{u'(E[\theta | \bar{s}] + \alpha_SP + (1 - \alpha_S)P + qx^d)} > \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} P - x^d)}{u'(E[\theta | \bar{s}] + P + qx^d)}
\]

when \( \alpha_S > 0 \).

Third, when \( P = P \), then \( \alpha = 0 \). Suppose not, \( \alpha_S > 0 \) and hence, \( \lambda_0 = 0 \). As \( p \to P \), the right-hand side of (A.32) goes to infinity, contradiction since the left-hand side is finite.

QED

**Proof of Proposition 5**

Lemma 5 states that if

\[
p \leq P + (R - \psi - P) \frac{u'(E[\theta | \bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]} P - x_B)}{u'(E[\theta | \bar{s}] + P + qx_B)},
\]

then \( \alpha = 0 \), otherwise \( \alpha > 0 \), where \( \alpha_S \) is given by (29).

Moreover, the above analysis of investors trades showed that if \( p < R - \psi (0) \) then \( \alpha_I > 0 \), while otherwise \( \alpha_I = 0 \). So two cases must be distinguished.
If
\[
\frac{u'(E[\tilde{\theta}|s] - \frac{Pr[s]}{Pr[\tilde{\theta}|s]} \mathcal{P} - x_B)}{u'(E[\tilde{\theta}|s] + \mathcal{P} + qx_B)} > \frac{\psi}{1-\mu} - \psi
\]
then \( \alpha^* = 0 \) and \( p^* \) is any price in \([R - \psi I(0), \hat{p}(x^d, q)]\).

Otherwise, there exists \((p^*, \alpha^*)\) such that \(\alpha S(p^*) = \alpha I(p^*) = \alpha^* > 0\). A sufficient condition for \(\alpha^* < 1\) is provided by (2), which implies \(\psi I(1) + \psi' > \frac{\psi}{1-\mu}\). To see this, proceed by contradiction and suppose \(\alpha^* = 1\). Then, (24) implies the price is \(p^* = R - (\psi I(1) + \psi')\). Substituting into (29)
\[
\frac{u'(E[\tilde{\theta}|s] + \alpha S \mathcal{P} + (1 - \alpha S) \mathcal{P} + qx^d)}{u'(E[\tilde{\theta}|s] - \frac{Pr[s]}{Pr[\tilde{\theta}|s]} \alpha S (R - \psi) + (1 - \alpha S) \mathcal{P} - x^d)} = \frac{\lambda_1}{\left(\frac{\psi}{1-\mu} - (\psi I(1) + \psi')\right) Pr[s]} \lambda S Pr[s] + \frac{R - \psi - \mathcal{P}}{\frac{\psi}{1-\mu} - (\psi I(1) + \psi')}
\]
The left-hand side is strictly positive but if \(\psi I(1) + \psi' > \frac{\psi}{1-\mu}\), the right-hand side is strictly negative, so we have a contradiction.

In the second case, the price \(p^*\) is obtained by applying (24). Substituting this price into (29) while setting \(\lambda_1 = 0\) yields (33).

QED

**Proof of Proposition 6**

To prove Proposition 6, we first recall the equilibrium conditions, then we recall the second-best conditions, and finally we show that for any allocation that satisfies the equilibrium conditions there exists a set of Pareto weights such that this allocation satisfies the conditions for second-best optimality.

**Equilibrium allocation:** Substituting equilibrium prices and trades \(\alpha^*, p^*, x^*, \) and \(q^*\) into (26) and (27), equilibrium protection buyers’ consumption is
\[
c_B(\tilde{\theta}, s) = c_B(\hat{\theta}, \bar{s}) = E[\tilde{\theta}|s] - \frac{Pr[s]}{Pr[\tilde{\theta}|s]} [\alpha^*(R - \psi) + (1 - \alpha^*) \mathcal{P}] - x^*, \quad (A.33)
\]
\[
c_B(\bar{\theta}, \bar{s}) = c_B(\hat{\theta}, \bar{s}) = E[\bar{\theta}|s] + \alpha^* p^* + (1 - \alpha^*) \mathcal{P} + q^* x^*. \quad (A.34)
\]
Similarly, substituting \(\alpha^*, p^*, x^*, \) and \(q^*\) into investors’ consumptions
\[
c_I(\tilde{\theta}, s) = c_I(\hat{\theta}, s) = 1 + x^*, \quad (A.35)
\]
\[
c_I(\bar{\theta}, s) = c_I(\hat{\theta}, s) = 1 - q^* x^* + \alpha^*(R - p^*). \quad (A.36)
\]
Substituting $\alpha^*, p^*, x^*, q^*$, (A.33) and (A.34) into (31), marginal rates of substitution between consumption after good news and after bad news are equalised for protection buyers and investors.

$$\frac{v'(c_I(\theta, s) - \alpha^*\psi_I(\alpha^*))}{v'(c_I(\theta, \bar{s})))} = \frac{u'(c_B(\theta, s))}{u'(c_B(\theta, \bar{s}))}. \tag{A.37}$$

Substituting (A.33) and (A.34) into condition (32), the condition writes as

$$\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \bigg|_{\alpha = 0} > \frac{\psi}{1-\mu} - \psi. \tag{A.38}$$

When that condition does not hold, $\alpha^* = 0$. When it holds, substituting $\alpha^*, p^*, x^*, q^*$, into (33), the marginal rate of substitution between consumption after bad news and consumption after good news is equal to what we interpreted, in the discussion of equation (17) in Proposition 3, as the marginal cost of insurance:

$$\frac{u'(c_B(\theta, s))}{u'(c_B(\theta, \bar{s}))} = \frac{\psi}{1-\mu} - \psi. \tag{A.39}$$

**Second best allocation:** Equations (12) and (13) state the total consumption of protection buyers and investors, after bad news and after good news, in the second best:

$$c_B(s) + c_I(s) = 1 + E[\hat{\theta}|s] + \alpha(s)R + (1 - \alpha(s))P, \tag{A.40}$$

$$c_B(\bar{s}) + c_I(\bar{s}) = 1 + E[\hat{\theta}|\bar{s}] - \frac{\text{Pr}[s]}{\text{Pr}[\bar{s}]}[\alpha(s)(R - \psi) + (1 - \alpha(s))P]. \tag{A.41}$$

Equation (14) states that in the second best marginal rates of substitution are equalised between protection buyers and investors:

$$\frac{v'(c_I(s) - \alpha(s)\psi_I(s))}{v'(c_I(\bar{s}) - \alpha(\bar{s})\psi_I(\bar{s}))} = \frac{u'(c_B(s))}{u'(c_B(\bar{s}))}. \tag{A.42}$$

Inequality (16) states the condition under which asset transfers are strictly positive in the second best:

$$\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} \bigg|_{\alpha(s)=0} > \frac{\psi}{1-\mu} - \psi; \tag{A.43}$$

if that condition does not hold, then there are no asset transfers in the second best.

Equation (17) gives the interior asset transfer:

$$\frac{u'(c_B(s))}{u'(c_B(\bar{s}))} = \frac{\psi}{1-\mu} - \psi. \tag{A.44}$$
Finally, equation (15) states how total consumption is split between protection buyers and investors as a function of their Pareto weights:

$$\frac{u'(c_B(s))}{u'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I + \lambda_I}{\omega_B + \lambda_B}. \quad (A.45)$$

Investors’ and protection buyers’ consumptions and asset transfers such that (A.40), (A.41), (A.42), (A.43), (A.44) and (A.45) hold are second best.

Comparing second best and equilibrium allocations: Consider an equilibrium allocation

$$\mathcal{E} = \{c_I(\theta, s), c_B(\theta, s), \alpha^*\}.$$ 

It is such that i) (A.33) to (A.36) hold and ii) if (A.38) holds, then (A.39) holds.

Equilibrium is information-constrained Pareto efficient if $\mathcal{E}$ satisfies the second-best optimality conditions, (A.40) to (A.45). Out of these six conditions, 5 are obviously satisfied:

Adding (A.33) to (A.35), and (A.34) to (A.36), in equilibrium the total consumption of protection buyers and investors is

$$c_B(\theta, \bar{s}) + c_I(\theta, \bar{s}) = 1 + E[\tilde{\theta}|\bar{s}] - \frac{\Pr[s]}{\Pr[\bar{s}]}[\alpha^*(R - \psi) + (1 - \alpha^*)\mathcal{P}], \forall \theta, \quad (A.46)$$

after good news and

$$c_B(\theta, s) + c_I(\theta, s) = 1 + E[\tilde{\theta}|s] + \alpha^*R + (1 - \alpha^*)\mathcal{P}, \forall \theta, \quad (A.47)$$

after bad news. (A.47) is equivalent to (A.40), while (A.46) is equivalent to (A.41).

Equation (A.37) shows that in equilibrium the MRS of protection buyers and investors are equalised, exactly as requested in the second best, in (A.42).

Third, (A.38) is equivalent to (A.43), and (A.39) is equivalent to (A.44).

So, it only remains to check that $\mathcal{E}$ satisfies (A.45). To do so, we need to show that there are Pareto weights $\omega_I$ and $\omega_B$ such that (A.45) holds for the consumptions in $\mathcal{E}$. Now, investors are strictly better off when participating in the market equilibrium than in autarky, since they strictly prefer to trade in the market for insurance against signal risk. Protection buyers also are strictly better off since they can, at least, extract all the surplus from contracting with protection sellers with $\alpha = 0$. Consequently, the participation constraints of protection buyers and investors are slack, implying $\lambda_I = \lambda_B = 0$. Hence, (A.45) holds for the consumptions in $\mathcal{E}$ if and only if there exist Pareto weights $\omega_I$ and $\omega_B$ such that

$$\frac{u'(c_B(s))}{u'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))} = \frac{\omega_I}{\omega_B}. \quad (A.45)$$
This is always the case. To see this, pick an arbitrary $\omega_B$, then set

$$\omega_I = \omega_B \frac{u'(c_B(s))}{v'(c_I(s) - \alpha(s)\psi_I(\alpha(s)))}.$$ 

QED
Supplementary Appendix  
(not for publication)

This supplementary appendix presents a number of additional results. First, we present an example where risk averse buyers and investors have power utility. This example illustrates the second best allocation. Second, we present the proofs of the risk-shifting model of Section 6 of the paper. Finally, we show the constrained inefficiency of the market outcome when buyers cannot share the signal risk with investors.

Power utility

To illustrate our results, assume

\[ u(x) = v(x) = \frac{x^{1-\gamma}}{1-\gamma}, \quad \text{(S.1)} \]

and

\[ \psi_I = \psi + \delta_0 + \delta_1 \alpha, \quad \text{(S.2)} \]

and denote the Pareto weight of investors by \( \omega > 0 \) and that of protection buyers by \( 1 - \omega \). We assume an interior solution, i.e., the participation constraints of protection buyers (4) and investors (5) are slack, and margins will be used ((16) holds).

After good news, condition (15) writes

\[ \left( \frac{c_I(\bar{s})}{c_B(\bar{s})} \right)^\gamma = \frac{\omega}{1-\omega} \]

so that

\[ c_I(\bar{s}) = \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} c_B(\bar{s}). \]

Correspondingly,

\[ c_I(\bar{s}) + c_B(\bar{s}) = \left[ 1 + \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} \right] c_B(\bar{s}). \quad \text{(S.3)} \]

Similarly, after bad news, condition (15) yields

\[ c_I(\bar{s}) + c_B(\bar{s}) = \left[ 1 + \left( \frac{\omega}{1-\omega} \right)^{\frac{1}{\gamma}} \right] c_B(\bar{s}). \quad \text{(S.4)} \]
Substituting (S.3) and (S.4) into (12) and (13), we obtain

\[ c_B(\bar{s}) = \frac{1 + E[\theta|\bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]}[\alpha(s)(R - \psi) + (1 - \alpha(s))\mathcal{P}]}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{y}}}, \]

\[ c_B(s) = \frac{1 + E[\theta|s] + \alpha(s)R + (1 - \alpha(s))\mathcal{P}}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{y}}}. \]

With (S.1) and (S.2), the condition on the optimal asset transfer in the second best (17) is

\[ \frac{c_B(\bar{s})}{c_B(s)} = \left(\frac{\omega}{1-\mu} - \psi\right) \left(\frac{1}{1-\mu} - (\psi + \delta_0 + 2\delta_1\alpha(s))\right)^{\frac{1}{y}}, \]

which becomes

\[
\frac{1 + E[\theta|\bar{s}] - \frac{Pr[s]}{Pr[\bar{s}]}[\alpha(s)(R - \psi) + (1 - \alpha(s))\mathcal{P}]}{1 + E[\theta|\bar{s}] + \alpha(s)R + (1 - \alpha(s))\mathcal{P}} = \left(\frac{\omega}{1-\mu} - \psi\right) \left(\frac{1}{1-\mu} - (\psi + \delta_0 + 2\delta_1\alpha(s))\right)^{\frac{1}{y}}, \quad \text{(S.5)}
\]

after substituting the above consumptions of protections buyers and investors.

With power utility, the protection buyers’ share of the total consumption of protection buyers and investors is

\[
\frac{1}{1 + \left(\frac{\omega}{1-\omega}\right)^{\frac{1}{y}}},
\]

after both signals. Moreover, asset transfers are independent from the Pareto weights, i.e., there is a separation between production (asset transfers) and allocation decisions. The former set the level of asset transfers that maximises the sum of the protection buyers’ and investors’ consumptions independently of \(\omega\). The latter allocate total consumption as a function of the Pareto weight for investors, \(\omega\).
Analysis of the set-up with risk-shifting

Optimal transfers

The Lagrangian for the protection buyer’s problem is

\[ L(\tau(\theta, s, R), \alpha_s, x_B) = \Pr[\bar{s}]E[u(\bar{\theta} + \tau(\bar{\theta}, \bar{s}, \bar{R}) - x_B)|\bar{s}] \] (S.6)

\[ + \Pr[\bar{s}]E[u(\bar{\theta} + \tau(\bar{\theta}, \bar{s}, \bar{R}) + qx_B)|\bar{s}] \] 

\[ + \lambda_{IC} \left[ \alpha_s p + (1 - \alpha_s)\hat{P} - \frac{a + b}{b} E[\tau(\bar{\theta}, \bar{s}, \bar{M})|\bar{s}] + \frac{a}{b} E[\tau(\bar{\theta}, \bar{s}, \bar{H})|\bar{s}] \right] \] 

\[ + \lambda_s \left[ \Pr[\bar{s}]\alpha_s (p - (\kappa H + (1 - \kappa)M)) - E[\tau(\bar{\theta}, \bar{s}, \bar{R})] \right] \] 

\[ + \lambda_1 [1 - \alpha_s] - \lambda_0 \alpha_s. \]

The first-order conditions of (S.6) with respect to \( \tau(\theta, \bar{s}, H) \) and \( \tau(\theta, \bar{s}, H) \) are:

\[ u'(\theta, \bar{s}, H) = \lambda_S \quad \forall \theta \quad \text{(S.7)} \]

\[ u'(\theta, \bar{s}, H) = \lambda_S - \frac{\lambda_{IC}}{\Pr[\bar{s}]} \frac{1}{\kappa} \frac{a}{b} \quad \forall \theta, \text{ (S.8)} \]

and with respect to \( \tau(\theta, \bar{s}, M) \) and \( \tau(\theta, \bar{s}, M) \) they are:

\[ u'(\theta, \bar{s}, M) = \lambda_S \quad \forall \theta \quad \text{(S.9)} \]

\[ u'(\theta, \bar{s}, M) = \lambda_S + \frac{\lambda_{IC}}{\Pr[\bar{s}]} \frac{1}{1 - \kappa} \frac{a + b}{b} \quad \forall \theta, \text{ (S.10)} \]

Participation constraint binds since the first-order conditions in case of a good signal imply that \( \lambda_S > 0 \).

There is full risk-sharing conditional on the signal and conditional on the realization of \( \tilde{R} \):

\[ \tilde{\theta} + \tau(\tilde{\theta}, s, R) = \theta + \tau(\theta, s, R) \quad \forall (s, R) \]

Moreover, after a good signal, there is also insurance of the remaining \( \tilde{R} \)-risk so that

\[ \tau(\theta, \bar{s}, H) = \tau(\theta, \bar{s}, M) \quad \forall \theta \]

Furthermore, if the incentive constraint binds (we show below that this is the case), \( \lambda_{IC} > 0 \), it is optimal for the buyer to bear some \( \tilde{R} \)-risk:

\[ u'(\theta, \bar{s}, M) - u'(\theta, \bar{s}, H) = \frac{\lambda_{IC}}{\Pr[\bar{s}]} \frac{1}{b} \left( \frac{a + b}{1 - \kappa} + \frac{a}{\kappa} \right) \quad \forall \theta, \text{ (S.11)} \]

which implies that \( \tau(\theta, \bar{s}, H) > \tau(\theta, \bar{s}, M) \forall \theta \). In order to incentivize the protection seller
not to shift risk after a bad signal, it is optimal that the seller makes a larger transfer to the buyer when the return $H$ occurs.

**Optimal $\alpha_S$**

The first-order condition of (S.6) with respect to $\alpha_S$ is

$$\lambda_{IC}(p - \bar{P}) = \lambda_S \Pr[\bar{s}] (\kappa H + (1 - \kappa) M - p) + \lambda_1 - \lambda_0.$$  \hspace{1cm} (S.12)

Combining (S.9) and (S.9) to get

$$u'(\theta, s, M) - u'(\theta, \bar{s}, M) = \frac{\lambda_{IC} \Pr[\bar{s}]}{1 - \kappa} \frac{a + b}{b},$$  \hspace{1cm} (S.13)

and using (S.9), we can substitute for $\lambda_S$ and $\lambda_{IC}$ in (S.12) to get:

$$\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} = 1 + \frac{\kappa H + (1 - \kappa) M - p}{p - \bar{P}} \frac{1}{1 - \kappa} \frac{a + b}{b} + \frac{\lambda_1 - \lambda_0}{u'(\theta, \bar{s}, M) \Pr[\bar{s}]} \frac{1}{p - \bar{P}} \frac{a + b}{1 - \kappa}.$$  \hspace{1cm} (S.14)

Note that, by (S.13), if $\lambda_{IC} > 0$, then $\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} > 1$ and there is a risk-sharing wedge as the binding incentive constraint after a bad signal prevents equalization of consumptions after a good and a bad signal in state $M$.

**Binding IC**

We now prove that $\lambda_{IC} > 0$ and the incentive-compatibility condition of the seller binds. We prove the claim by contradiction, and proceed in two steps.

First, we claim that if the incentive-compatibility condition of the protection seller was slack, then $\alpha_S = 0$. Suppose not, i.e., $\lambda_{IC} = 0$ and $\alpha_S = \alpha_I > 0$. Then, the first-order condition (S.12) simplifies to:

$$-\lambda_S \Pr[\bar{s}] (\kappa H + (1 - \kappa) M - p^*) = \lambda_1.$$  \hspace{1cm} (S.15)

where $p^*$ is the equilibrium price in the asset market. From the investors’ demand for the productive asset, we know that $\alpha_I > 0$ requires $p^* < \kappa H + (1 - \kappa) M$. It follows that the left-hand side of (S.15) is strictly negative, which contradicts $\lambda_1 \geq 0$.

Second, we claim that if the incentive-compatibility condition of the protection seller was slack, then the first-best would be feasible, contradicting our assumption $\bar{P} < E[\theta] - E[\bar{\theta}|\bar{s}]$. Suppose $\lambda_{IC} = 0$. First-order conditions for transfers imply full insurance. Using $\alpha_S = 0$, full insurance, and substituting the binding participation constraint, we obtain $\tau(\bar{\theta}) = -(1-$
\( \pi(\bar{\theta} - \theta) \) and \( \tau(\bar{\theta}) = \pi(\bar{\theta} - \theta) \). Using this in the slack incentive constraint yields

\[
\hat{P} > E[\bar{\theta}] - E[\bar{\theta}|s],
\]
a contradiction.

**Computing transfers**

Combining the equations characterizing the optimal transfers (S.7)-(S.10), we obtain

\[
\frac{u'(\theta, \bar{s}, \bar{R}) - u'(\theta, \bar{s}, H)}{u'(\theta, \bar{s}, M) - u'(\theta, \bar{s}, \bar{R})} = \frac{a}{a+b} \frac{1 - \kappa}{\kappa}, \tag{S.16}
\]

The binding participation and incentive constraints yield:

\[
-E[\tau(\bar{\theta}, \bar{s}, \bar{R})] = \Pr[\bar{s}]\alpha S [\kappa H + (1 - \kappa)M - p], \tag{S.17}
\]
\[
\alpha_s p + (1 - \alpha_S) \hat{P} = \left(\frac{a + b}{b}\right) E[\tau(\bar{\theta}, \bar{s}, M)] - \frac{a}{b} E[\tau(\bar{\theta}, \bar{s}, H)]. \tag{S.18}
\]

Given full risk-sharing conditional on the signal and conditional on the realization of \( \bar{R} \), we have, after a good signal:

\[
\tau(\bar{\theta}, \bar{s}, M) = \tau(\bar{\theta}, \bar{s}, H) = \tau(\bar{\theta}, \bar{s}, H) + (\bar{\theta} - \bar{\theta}) = \tau(\bar{\theta}, \bar{s}, M) + (\bar{\theta} - \bar{\theta})
\]
and, after a bad signal,

\[
\tau(\bar{\theta}, \bar{s}, H) = \tau(\bar{\theta}, \bar{s}, H) + (\bar{\theta} - \bar{\theta})
\]
\[
\tau(\bar{\theta}, \bar{s}, M) = \tau(\bar{\theta}, \bar{s}, M) + (\bar{\theta} - \bar{\theta}).
\]

We have three transfers, \( \tau(\bar{\theta}, \bar{s}, M) \), \( \tau(\bar{\theta}, \bar{s}, H) \) and \( \tau(\bar{\theta}, \bar{s}, M) \), and three equations (S.16)-(S.18) to determine them.

The following two equations express the transfers \( \tau(\bar{\theta}, \bar{s}, M) \) and \( \tau(\bar{\theta}, \bar{s}, M) \) as a function of \( \tau(\bar{\theta}, \bar{s}, H) \), for a given \( \alpha_S \):

\[
\tau(\bar{\theta}, \bar{s}, M) = \frac{\Pr[\bar{s}]}{\Pr[\bar{s}]} \left[ \alpha_s (p - (\kappa H + (1 - \kappa)M)) - (1 - \kappa) \frac{b}{a+b} \left(\alpha_s p + (1 - \alpha_S) \hat{P}\right) \right.
\]
\[
- \tau^*(\bar{\theta}, \bar{s}, H) \left( \kappa + (1 - \kappa) \frac{a}{a+b} \right) \right] + \frac{\pi(\bar{\theta} - \bar{\theta})}{\Pr[\bar{s}]} \left[ 1 - \Pr[\bar{s}] (1 - \kappa) \frac{b}{a+b} \right]
\]
\[
\tau(\bar{\theta}, \bar{s}, H) = \frac{b}{a+b} \left[ \alpha_s p + (1 - \alpha_S) \hat{P} + \frac{a}{b} \tau(\bar{\theta}, \bar{s}, H) + \pi(\bar{\theta} - \bar{\theta}) \right].
\]
Supply in the asset market

By (S.14), we have that

$$\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} - 1 = \left[ \frac{\kappa H + (1 - \kappa)M - p}{p - \hat{P}} + \frac{\lambda_1 - \lambda_0}{\lambda S \Pr[s](p - \hat{P})} \right] \frac{1}{1 - \kappa} \frac{a + b}{b} \quad (S.19)$$

First note that if $p > \kappa H + (1 - \kappa)M$ and $p > \hat{P}$, we must have that $\lambda_1 > 0$ and $\alpha_S = 1$. It remains to characterize the supply for $p \leq \kappa H + (1 - \kappa)M$.

We now show that if $p > \kappa H + (1 - \kappa)M + \hat{P} u' \left( \frac{\theta, s, M}{\theta, \bar{s}, M} \right) - 1$,

$$p > \frac{\kappa H + (1 - \kappa)M + \hat{P} \frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} - 1}{1 + \frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} - 1},$$

then the optimal $\alpha_S > 0$. Suppose not, $\alpha_S = 0$, so that $\lambda_0 \geq 0$ and $\lambda_1 = 0$. Then, using $\alpha_S = 0$ and $\lambda_1 = 0$ in (S.19) yields

$$\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0} - 1 = \left[ \frac{\kappa H + (1 - \kappa)M - p}{p - \hat{P}} - \frac{\lambda_0}{\lambda S \Pr[s](p - \hat{P})} \right] \frac{1}{1 - \kappa} \frac{a + b}{b}.$$

Note that for $\alpha_S = 0$, we have that $\tau(\theta, s, M)$ and $\tau(\theta, \bar{s}, M)$ are independent of $p$. Therefore, also $\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0}$ is independent of $p$. Moreover, by (S.13), we know that $\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} > 1$.

Since $\lambda_0 \geq 0$, we have that

$$\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0} - 1 \leq \frac{\kappa H + (1 - \kappa)M - p}{p - \hat{P}} \frac{1}{1 - \kappa} \frac{a + b}{b},$$

so that

$$(p - \hat{P}) \frac{\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0} - 1}{\frac{1}{1 - \kappa} \frac{a + b}{b}} \leq \kappa H + (1 - \kappa)M - p$$

or, equivalently,

$$p \left( 1 + \frac{\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0} - 1}{\frac{1}{1 - \kappa} \frac{a + b}{b}} \right) \leq \kappa H + (1 - \kappa)M + \hat{P} \frac{\frac{u'(\theta, s, M)}{u'(\theta, \bar{s}, M)} \bigg|_{\alpha_S = 0} - 1}{\frac{1}{1 - \kappa} \frac{a + b}{b}}.$$
Since the expression in brackets on the left-hand-side is positive, we get
\[
p \leq \frac{\kappa H + (1 - \kappa) M + \hat{P} \frac{u'(\theta, s, M)}{u'(\theta, s, M)_{\alpha_S = 0} - 1}}{1 + \frac{u'(\theta, s, M)_{\alpha_S = 0} - 1}{1 - \alpha_S - b}}\]

a contradiction.

We now show that if
\[
p \leq \kappa H + (1 - \kappa) M + \hat{P} \frac{u'(\theta, s, M)}{u'(\theta, s, M)_{\alpha_S = 0} - 1}
\]

\[
1 + \frac{u'(\theta, s, M)_{\alpha_S = 0} - 1}{1 - \alpha_S - b},
\]

then the optimal \(\alpha_S = 0\). To do so, we proceed in three steps, corresponding to different values of \(p\).

First, when \(p < \hat{P}\), \(\alpha_S = 0\). Suppose not, \(\alpha_S > 0\) and hence \(\lambda_0 = 0, \lambda_1 \geq 0\). Then, the first term in the squared brackets on the right-hand side of (S.19) is negative and the second term is weakly negative. Hence, the right-hand side is strictly negative while the left-hand side is strictly positive. Intuitively, \(p < \hat{P}\) means that the market price is lower than the pledgeable income so that a margin cannot relax the incentive constraint (35).

Second, when \(\hat{P} < p \leq \kappa H + (1 - \kappa) M + \hat{P} \frac{u'(\theta, s, M)_{\alpha_S = 0} - 1}{1 + \frac{u'(\theta, s, M)_{\alpha_S = 0} - 1}{1 - \alpha_S - b}}\), then \(\alpha_S = 0\). Suppose not, \(\alpha_S > 0\) and hence, \(\lambda_0 = 0\). Then, using (S.19), we get

\[
\frac{u'(\theta, s, M)}{u'(\theta, s, M) - 1} = \left[\frac{\kappa H + (1 - \kappa) M - p}{p - \hat{P}} + \frac{\lambda_1}{\lambda_S \Pr[s](p - \hat{P})} \right] \frac{1}{1 - \alpha_S - b} \frac{a + b}{b}.
\]

Since \(\lambda_1 \geq 0\), we have

\[
\frac{u'(\theta, s, M)}{u'(\theta, s, M) - 1} \geq \frac{\kappa H + (1 - \kappa) M - p}{p - \hat{P}} \frac{1}{1 - \alpha_S - b} \frac{a + b}{b}
\]

so that

\[
p \geq \frac{\kappa H + (1 - \kappa) M + \hat{P} \frac{u'(\theta, s, M) - 1}{1 + \frac{u'(\theta, s, M) - 1}{1 - \alpha_S - b}}}{1 + \frac{u'(\theta, s, M) - 1}{1 - \alpha_S - b}}.
\]

(S.20)
Note that for \( \alpha_S > 0 \),

\[
\frac{\kappa H + (1 - \kappa)M + \hat{\mathcal{P}} \frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\alpha_S} \frac{1}{\nu''(\theta,s,M)} - 1}{1 + \frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\nu''(\theta,s,M)} - 1} > \frac{\kappa H + (1 - \kappa)M + \hat{\mathcal{P}} \frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\alpha_S} \frac{1}{\nu''(\theta,s,M)} - 1}{1 + \frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\nu''(\theta,s,M)} - 1}. \tag{S.21}
\]

To see this, first note that for \( \alpha_S > 0 \),

\[
\frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\nu''(\theta,s,M)} - 1 < \frac{\nu'(\theta,s,M)}{\nu'(\theta,s,M) - 1} \frac{1}{1 - \kappa} \frac{1}{\nu''(\theta,s,M)} - 1 \tag{\text{\textsuperscript{(S.20)}}}
\]

the sign of the derivative is given by

\[
\hat{\mathcal{P}} - [\kappa H + (1 - \kappa)M],
\]

which is negative since we have \( \hat{\mathcal{P}} < p \leq \kappa H + (1 - \kappa)M \). Combining (S.20) and (S.21) yields a contradiction.

Third, when \( p = \hat{\mathcal{P}} \), then \( \alpha_S = 0 \). Suppose not, \( \alpha_S > 0 \) and hence, \( \lambda_0 = 0 \). As \( p \to \hat{\mathcal{P}} \), the right-hand side of (S.19) goes to infinity, contradiction since the left-hand side is finite.

**Demand in the asset market**

The investor demand for assets is downward-sloping. To see that, note that

\[
\frac{dp}{d\alpha_I} = \frac{(\kappa H - p)\nu_H'' + (1 - \kappa)M(M - p)\nu_M'')(\kappa \nu_H' + (1 - \kappa)\nu_M')}{(\kappa \nu_H' + (1 - \kappa)\nu_M')^2} - \frac{(\kappa \nu_H' H + (1 - \kappa)\nu_M' M)(\kappa(H - p)\nu_H'' + (1 - \kappa)(M - p)\nu_M'')}{(\kappa \nu_H' + (1 - \kappa)\nu_M')^2}
\]

Considering only the numerator, factoring out terms, and simplifying

\[
\kappa(1 - \kappa)(H - M)((H - p)\nu_H'' \nu_M' - (M - p)\nu_M'' \nu_H')
\]

So we have a downward sloping demand curve if and only if

\[
(H - p) \left( -\frac{\nu''}{\nu_H'} \right) > (M - p) \left( -\frac{\nu''}{\nu_M'} \right),
\]

i.e., if the coefficient of absolute risk aversion is not decreasing too fast. The condition holds for CARA obviously, but also for log-utility. (Note that in the above condition, we can focus on the case where \( M < p < H \), see the discussion above.)
Incomplete markets

In the analysis investors participate in the market at time 1, buying in the fire sale triggered by a bad signal, and also at time 0, providing insurance against bad news at time 1. It is possible, however, that in practice investors are not fully aware at time 0 of the possible occurrence of fire sales at time 1. In that case, they would not provide insurance against signal risk at time 0, which constitutes a form of market incompleteness.

To analyse that case, we study the market equilibrium with the constraint \( x_B = x_I = 0 \). Proceeding along similar lines as with complete markets, one can show that equilibrium in this incomplete-market case is as follows. In the privately-optimal contract between protection buyers and protection sellers, protection sellers’ participation and incentive constraints bind, and protection buyers receive full insurance conditional on the signal. The consumption of protection buyers after bad news and after good news is as in Lemma 4, except that \( x_B \) is set to 0. Similarly, the condition for asset sales is the same as (32), with \( x^* = 0 \). When condition (32) does not hold, there are no asset sales in the incomplete-market case, \( \alpha^{IM} = 0 \). When condition (32) holds, \( \alpha^{IM} \) is strictly positive and pinned down by the same equation as (33) with \( x^* \) set to 0. Finally, the equilibrium price is as in (34) and investors’ consumption net of cost is equal to \( 1 + \alpha^{IM}(R - \psi_I(\alpha^{IM}) - p^{IM}) \) after a bad signal and equation to 1 after a good signal.

The main difference between this incomplete-market setting and the complete-market setting is that protection buyers and investors cannot share risk and end up with different marginal rates of substitution. Correspondingly, equilibrium with incomplete markets is Pareto dominated by equilibrium with complete markets.

What are the consequences of market participants’ inability to trade insurance against signal risk at time 0? In that case, protection buyers cannot purchase insurance against signal risk from investors. To make up for that risk exposure, protection buyers request larger variation-margin calls from protection sellers in order to increase the amount of insurance they can obtain from them. This leads to our next result.

**Proposition S.1** Either \( \alpha^{IM} = \alpha^* = 0 \), so that there are no variation-margin calls irrespective of whether the market is complete or not, or \( \alpha^{IM} > 0 \), in which case margin calls are larger and the price of protection sellers’ assets is lower when markets are incomplete than when they are complete.

**Proof:** First, consider the case in which that \( \alpha^{IM} = 0 \). In that case, we have

\[
\frac{u'(E[\hat{\theta}|\hat{s}] - \psi_I(\alpha^{IM}) - p^{IM})}{u'(E[\hat{\theta}|\hat{s}] - \psi_I(\alpha^{IM}) - p^{IM})} < \frac{u'(E[\hat{\theta}|\hat{s}] + P + qx^*)}{u'(E[\hat{\theta}|\hat{s}] + P + qx^*)} < \frac{\psi}{1-\mu} - \psi
\]

where the first inequality follows from \( x^* > 0 \) and the fact that \( u' \) is decreasing. By Proposition 5, we have that \( \alpha^* = 0 \). Therefore \( \alpha^* = \alpha^{IM} = 0 \), and correspondingly \( p^* = p^{IM} \).
Second, consider the case in which $\alpha^{IM} > 0$. Since equilibrium price decreases in $\alpha$, it suffices to prove that $\alpha^{IM} > \alpha^*$. There are two possibilities: Either $\alpha^* = 0$, implying that $\alpha^{IM} > \alpha^*$, or $\alpha^* > 0$. In the latter case, $\alpha^*$ is the root of

$$u'(E[\tilde{\theta}|s] + \alpha(R - [\psi_I(\alpha) + \alpha\psi'_I(\alpha)]) + (1 - \alpha)P + q^*x^*) = \frac{\psi^{I}}{1-\mu} - \psi$$

while $\alpha^{IM}$ is the root of

$$u'(E[\tilde{\theta}|s] + \alpha(R - [\psi_I(\alpha) + \alpha\psi'_I(\alpha)]) + (1 - \alpha)P) = \frac{\psi^{IM}}{1-\mu} - (\psi_I(\alpha) + \alpha\psi'_I(\alpha)).$$

The two equations are very similar. They have the same right-hand side, which is an increasing function of $\alpha$. The equilibrium $\alpha$ is such that this right-hand side intersects the left-hand side, (S.22) for complete markets and (S.23) for incomplete markets, respectively. Note further that the left-hand side of (S.22) is lower than the left-hand side of (S.23). Consequently, the intersection of the left- and right-hand sides occurs for lower $\alpha$ in (S.22) than in (S.23). Hence $\alpha^* < \alpha^{IM}$.

QED

Proposition S.1 implies that, because of market incompleteness, variation-margin calls are inefficiently high and the price for protection sellers’ assets inefficiently low. That is, market incompleteness leads to inefficient fire sales.

In our simple general equilibrium context, there are interactions between markets. The ability of protection buyers and investors to trade insurance against signal risk in one market reduces the need to sell protection sellers’ assets in another market. When the insurance market does not exist, this depresses prices in the asset market.

While fire sales are a symptom of the inefficiency induced by market incompleteness, they are not a necessary condition for inefficiency. Even when $\alpha^{IM} = 0$, market incompleteness prevents the equalisation of the marginal rates of substitution of protection buyers and investors, leading to an information-constrained inefficient allocation of risks.