A Buyout Option Alleviates Implicit Collusion in Uniform-Price Auctions

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Abstract
Theoretical research shows that in uniform-price auctions, a bidder will optimally submit bids below his valuations on all units except the first unit, which is referred to in the literature as demand reduction. Such demand reduction can be interpreted as implicit collusion among bidders because it causes low revenues. In this paper, we investigate the potential of a buyout option for alleviating implicit collusion. We employ a two-stage model to study a uniform-price auction with a buyout option. We focus on the extreme case in which two bidders with two-unit demand submit a “single-unit bid,” which yields a revenue of zero to the seller when there is no buyout option. Our main result is that the seller obtains a positive expected revenue unless the buyout price is high. We find that a bidder will exercise a buyout option even though he is risk neutral; that is, auction aversion is fully endogenous, in contradiction to the findings of previous work.

1 Introduction
A variety of objects are sold through uniform-price auctions (UPAs). Examples include Treasury bills, FCC spectrum licenses, and electricity supply. The UPA is one of several natural extensions of a second-price auction (SPA) to an environment of multi-unit objects, but the analysis of SPA cannot be applied to UPA. As Noussair (1995) and Engelbrecht-Wiggans and Kahn (1998) point out, bidders optimally employ bidding strategies that exhibit demand reduction in any symmetric equilibrium. That is, in UPA, a bidder will submit bids below his valuations on all units except the first unit. In particular, demand reduction is viewed as implicitly collusive when revenues are extremely low. Wilson (1979) was the first to point out the low-price equilibrium that arises in UPA. This theoretical prediction has been verified in experimental research. Engelmann and Grimm (2009) conducted a series of laboratory experiments to compare the performance of several formats of multi-unit auction, including UPA. They report that the sealed-bid UPA exhibits the worst performance in terms of both revenue and efficiency. The inferiority of uniform-price auctions has been found in many experimental studies.¹

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¹See, for example, List and Lucking-Reiley (2000), Kagel and Levin (2001), and Porter (2005).
Implicit collusion in UPA is of more than academic interest. Ausubel and Cramton (1996) find demand reduction in those FCC spectrum auctions that exhibit similarity with UPA. Wolfram (1998) finds evidence of demand reduction in the UPA-format electricity auctions in England and Wales wherein two major firms compete. The format was changed from UPA to a discriminatory auction in 2001, and the underlying force behind the reform may have been the low revenues caused by implicit collusion.

How can we alleviate implicit collusion? Clearly, one approach is to conduct repeated SPAs. In reality, however, this may not be possible because of time constraints. In this paper, we investigate the potential of using a buyout option to alleviate implicit collusion.2

In online auctions such as on eBay and Yahoo, a seller is allowed to indicate a buyout price (also called the “buy price”) and give bidders a buyout option. If a bidder exercises the buyout option, he immediately obtains the object at the buyout price. In Yahoo Japan auctions, we can find various objects sold through multi-unit auctions with a buyout option. Figure 1 shows an example. In this auction, three identical units of Italian wine are being auctioned with a buyout option available at a price of JPY 15,000. Bidders can immediately purchase up to three units by paying this buyout price for each. If no bidder exercises the buyout option, the available units will be awarded to the highest bids at the market clearing price. Thus, this is a UPA with a buyout option.3

![Figure 1: Uniform-price auction with a buyout option](image)

We employ a two-stage model to study UPA with a buyout option in the independent private valuation environment. In the first stage, a bidder, randomly chosen out of two bidders, decides whether to exercise the buyout option. If he either rejects the buyout price or exercises the buyout option, the available units will be awarded to the highest bids at the market clearing price. Thus, this is a UPA with a buyout option.3

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2The literature suggests ways of remedieng underpricing in UPA. A seller with divisible goods can improve revenues by requiring bidders to submit discrete bid schedules rather than demand functions (Kremer and Nyborg, 2004a, 2004b) or by deciding how many units to sell after the bidding (McAdams, 2007). In UPA with indivisible goods, a change of pricing rules can serve this purpose. Burkett and Woodward (2018) show that the expected revenue improves dramatically if the hammer price is determined as the lowest accepted bid rather than the highest rejected bid. Moreover, Bresky (2013) argues that a positive reserve price can improve both revenue and efficiency.

3Within divisible goods, Treasury auctions might also fit this paradigm. See Back and Zender (1993), Wang and Zender (2002), Holmberg (2009), and Pycia and Woodward (2017), for instance.
option for one unit, both bidders go on to compete via bidding in the second stage. The pricing rule takes the highest losing bid as the final price. Our model is similar to the one introduced by Ivanova-Stenzel and Kröger (2008), wherein bidders with single-unit demand compete in SPA with a buyout option.

We do not aim to provide a comprehensive analysis of the buyout option in UPA. Rather, we illustrate the potential of using a buyout option to alleviate implicit collusion. To this end, we focus on the extreme case in which a seller suffers the most. What is the worst outcome for a seller that can arise in equilibrium without pre-auction purchase? Theoretically, a “zero-revenue equilibrium” is realized if two bidders with two-unit demand submit a “single-unit bid,” wherein they acquire the objects without making any payment (Engelbrecht-Wiggans and Kahn, 1998; Ausubel et al., 2014). Moreover, Ausubel et al. (2014) demonstrate that a zero-revenue equilibrium is unique within the class of symmetric monotonic equilibria.

Our goal is to suggest that bidders will exercise the buyout option with a positive probability, which will result in positive expected revenue. As we show later, a bidder optimally employs a monotone threshold strategy for deciding how many units to buy. That is, given a buyout price, the bidder with the higher valuation exercises the buyout option for more units. Our finding is that the seller obtains a positive expected revenue unless the buyout price is high.

To the best of our knowledge, this paper is the first to analyze UPA with a buyout option. Moreover, the literature on how to improve revenues in UPA has not yet addressed the question of pre-auction buyout options (see, e.g., Kremer and Nyborg, 2004a, 2004b; McAdams, 2007; and Burkett and Woodward, 2018). We believe that our analysis is the first step toward a deeper understanding of this issue with respect to UPA.

This paper also contributes to the buyout price literature. Within the existing two-stage models, a seller can potentially benefit by giving bidders a buyout option in single-unit auctions if bidders are risk-averse (Budish and Takeyama, 2001; Reynolds and Wooders, 2009) or impatient (Mathews, 2004). To an impatient bidder, a buyout option is obviously beneficial, as the utility is discounted as time passes. A risk-averse bidder, on the other hand, wishes to purchase an object regardless of the buyout price in order to avoid variations in the hammer price; that is, the buyout option functions as insurance for him. In other words, risk-neutral bidders have no interest in relatively high buyout prices that could enhance revenues for a seller.

In our framework, however, it can be shown that risk-neutral bidders will exercise the buyout option for a buyout price that enhances revenues. That is, auction aversion is fully endogenous. This is in sharp contrast to previous work that generates interesting results by giving bidders an exogenous reason to avoid auction participation. The underlying reason is that a bidder with a buyout option
faces a trade-off between (i) obtaining two units of the object by paying the buyout price and (ii) obtaining one unit without making any payment. The former is more attractive than the latter if the buyout price is low relative to the bidder’s valuation(s).

In the following section, we present our two-stage model. Our model is similar to the one introduced by Ivanova-Stenzel and Kröger (2008) for a single-unit auction, but ours considers a multi-unit auction. In section 3, we investigate an optimal buyout strategy that a bidder follows to exercise a buyout option. In section 4, we provide comparative statics analysis; then, we show that a buyout price yields positive expected revenue unless it is high. In section 5, we discuss several issues. We provide concluding remarks in section 6. All of the proofs appear in the Appendix.

2 The model

We consider a multi-unit UPA with a buyout option in which a seller sells two units of identical indivisible objects. The pricing rule takes the highest losing bid as the price to be paid. There are two risk-neutral bidders, bidder 1 and bidder 2, who demand two units of the objects. Bidder $i$ values the object at $x^i$ for the first unit and $kx^i$ for the second unit, with $k \in (0, 1]$, which is common to both bidders. Note that $k = 1$ corresponds to flat demand. The valuation $x^i \in [0, 1]$ is an independent draw according to the c.d.f. $F(\cdot)$; that is, bidders have independent private values (IPVs). We let $f(\cdot) = F'(\cdot)$ denote the density function and assume $f(x) > 0$ for any $x \in [0, 1]$. Hereafter, we refer to the valuations for the first and second units as the first-unit valuation and second-unit valuation, respectively. The seller is assumed to value the objects at zero.

We employ a two-stage model. In the first stage, a buyout option is awarded to either bidder 1 or bidder 2 with equal probability. Without loss of generality, we assume that bidder 1 obtains the buyout option. Then, bidder 1 offers to buy a quantity $q \in \{0, 1, 2\}$ given an exogenous buyout price. We can interpret $q > 0$ as meaning that bidder 1 exercises the buyout option. The process ends after the first stage if bidder 1 chooses $q = 2$ because nothing remains for sale. In the second stage, if any, bidders compete by bidding. If both units of the objects remain, the second stage is a UPA. Bidder $i$ submits bids $(p^1_i, p^2_i)$, where $p^\ell_i$ represents bidder $i$’s bid for the $\ell$th unit of the objects, and then each object is auctioned off for the third-highest bid (i.e., the highest rejected bid). If only one unit remains, on the other hand, the second stage is equivalent to a single-unit SPA, wherein the object is auctioned off for the losing bid. We assume that no reserve price is set.

In this paper, we restrict our attention to the equilibrium in which bidder $i$ submits a single-unit bid if both units of the objects remain in the second stage. We define a single-unit bid as a couplet consisting of a bid of the upper-bound valuation for the first unit and a bid of zero for the second unit (i.e., $(p^1_i, p^2_i) = (1, 0)$). Without a buyout option, the single-unit bid yields a revenue of zero because
the highest rejected bid is zero. Note that bidder $i$ cannot profitably deviate when his opponent submits a single-unit bid. If the second stage is an SPA, bidder $i$ submits sincerely his valuation. Understanding the consequent outcomes, bidder 1 optimally decides how many units, $q$, to purchase for the buyout price $b$ in the first stage.

### 3 Buyout strategy

In this section, we investigate bidder 1’s buyout strategy $Q(x)$. He chooses $q$ so as to maximize the (expected) payoff, denoted by $U(q, x; b)$.

First, we calculate the payoff for the case of bidder 1 choosing $q = 2$. He obtains two units of the objects at price $b$ with certainty; thus, the payoff is given by

$$U(2, x; b) = (1 + k)x - 2b.$$ 

Next, we calculate the expected payoff for the case of bidder 1 choosing $q = 1$. He obtains one unit of the object at price $b$ in the first stage, and then the game moves on to the second stage, SPA, wherein both bidders sincerely submit their own valuations. Bidder 1 wins the auction if his second-unit valuation exceeds his opponent’s first-unit valuation. Thus, bidder 1 obtains the expected payoff

$$U(1, x; b) = x - b + \int_0^{kx} (kx - y)dF(y)$$

$$= x - b + \int_0^{kx} F(y)dy,$$

where $y$ denotes bidder 2’s first-unit valuation.

Finally, we calculate the expected payoff for the case of bidder 1 choosing $q = 0$. Because the second stage is UPA, each of the bidders is awarded one unit of the objects at a price of zero, by the assumption that both of them submit single-unit bid $(p_i^1, p_i^2) = (1, 0)$. Thus, bidder 1’s payoff is given by

$$U(0, x; b) = x.$$ 

Before we characterize an equilibrium, we show the following lemmas regarding bidder 1’s buyout strategy.

**Lemma 1.** In any equilibrium, bidder 1 follows a monotone threshold strategy in the first stage.

Lemma 1 implies that, given a buyout price, bidder 1, having the higher valuation, has an incentive to purchase more units. This is a straightforward consequence of the supermodularity of the bidder’s payoff function.
Lemma 2. In any equilibrium, given $b > 0$, bidder 1 chooses $q = 0$ with a positive probability. On the other hand, bidder 1 chooses $q = 2$ with probability one for $b = 0$ unless his valuations are zero.

Clearly, bidder 1 has no incentive to exercise the buyout option if his first-unit valuation is below the buyout price (i.e., $x < b$). The first statement of lemma 2 says that this situation happens with a positive probability because of the full-support assumption on $F(\cdot)$. The second statement is clear. In the remainder of this paper, we focus on $b > 0$. Lemmas 1 and 2 jointly induce lemma 3.

Lemma 3. In any equilibrium, bidder 1’s buyout strategy is given by one of the following (step) functions: For some thresholds $t$, $\hat{t}$, and $\tilde{t}$,

1. $Q(x) = 0$ for any $x \in [0, 1]$,  
2. $Q(x) = \begin{cases} 1 & \text{if } x \in [t, 1] \\ 0 & \text{if } x \in [0, t) \end{cases}$  
3. $Q(x) = \begin{cases} 2 & \text{if } x \in [t, 1] \\ 0 & \text{if } x \in [0, t) \end{cases}$  
4. $Q(x) = \begin{cases} 2 & \text{if } x \in [\hat{t}, 1] \\ 1 & \text{if } x \in [t, \hat{t}] \\ 0 & \text{if } x \in [0, t) \end{cases}$

These thresholds depend on $b$ and $k$. In what follows, we restrict our attention to investigating these threshold values. For convenience, we define the following two functions:

5. $P(x) \equiv \int_0^x F(y) \, dy = x F(x) - \int_0^x y F'(y) \, dy$,  
6. $L(x) \equiv x - P(x)$.

Equation (5) represents the expected payoff that a bidder with a valuation $x$ obtains from participating in an SPA. Equation (6), on the other hand, represents the expected “loss” to him in the sense that he must give up a surplus of $x - P(x)$ in the SPA. The loss emerges because he potentially loses a chance to obtain the object, and he has to pay money when winning the auction. We summarize characteristics regarding these functions as the following lemmas.

Lemma 4.

1. $P(0) = 0$. $P(\cdot)$ is increasing and convex in $x$; $P^{-1}(0) = 0$. $P^{-1}(\cdot)$ is increasing and concave in $x$.

2. $L(0) = 0$. $L(\cdot)$ is increasing and concave in $x$; $L^{-1}(0) = 0$. $L^{-1}(\cdot)$ is increasing and convex in $x$. 

6
3. There exists a unique \( x^* > 0 \) that satisfies \( P(x^*) = L(x^*) = x^*/2 \). Moreover, \( L(x) > x/2 > P(x) \) for any \( x \in (0, x^*) \), and \( P(x) > x/2 > L(x) \) for any \( x > x^* \).

As we see below, \( x^* \) plays an important role for understanding what equilibrium will arise. This crucial value \( x^* \) depends solely on the shape of \( F(\cdot) \), not on \( b \) and \( k \). As the following lemma shows, \( x^* \) increases with the likelihood of a high valuation.

**Lemma 5.** Let \( F(x) \) and \( G(x) \) be distribution functions. Let \( x^*_F \) and \( x^*_G \) be valuations that satisfy \( P(x^*_j) = x^*_j/2, \ j = F, G, \) given \( F(x) \) and \( G(x) \), respectively. If \( F(x) < G(x) \) for any \( x \in (0,1) \), then \( x^*_F > x^*_G \).

### 3.1 Without interior threshold

In this subsection, we characterize an equilibrium with equation (1). In this equilibrium, bidder 1 cannot profitably deviate to choose \( q > 0 \); that is, \( U(0,x;b) \geq U(2,x;b) \) and \( U(0,x;b) \geq U(1,x;b) \) should hold for any \( x \in [0,1] \). These two formulas yield the following proposition.

**Proposition 1.** There exists an equilibrium in which bidder 1 chooses \( q = 0 \) for any \( x \in [0,1] \) (i.e., he follows equation (1)) if and only if \( b \geq \max\{k/2, P(k)\} \).

Proposition 1 says that bidder 1 never exercises the buyout option if \( b \) is large and \( k \) is small. Clearly, it is not attractive to purchase the object(s) at a high price \( b \). In order to understand the impact of \( k \), suppose that \( k \) is close to 0 (i.e., bidder 1 demands approximately a single unit). In this case, he has no incentive to purchase two units. Moreover, he finds it unprofitable to exercise the buyout option for one unit because he is awarded the object at a price of zero in the second stage.

### 3.2 With interior threshold(s)

First, we consider an equilibrium with equation (2). In this equilibrium, bidder 1 should be indifferent between choosing \( q = 1 \) and \( q = 0 \) if his first-unit valuation \( x \) is equal to threshold \( t \). Thus, threshold \( t \) satisfies \( U(1,t;b) = U(0,t;b) \). Moreover, bidder 1 cannot benefit from choosing \( q = 2 \). Consequently, we obtain the following proposition.

**Proposition 2.** There exists an equilibrium in which bidder 1 chooses either \( q = 1 \) or \( q = 0 \) following equation (2) with threshold \( t = P^{-1}(b)/k \) if and only if \( x^*/2 < L(k) \leq b < P(k) \). The threshold increases with \( b \) and decreases as \( k \) increases.

Proposition 2 implies that a buyout price should be “intermediate.” This is intuitive because bidder 1 could profitably purchase two units, rather than one unit, for a low buyout price, whereas he could benefit by discarding a buyout option when a buyout price is high. Moreover, proposition
2 shows the lower bound of $k$ for the equilibrium to exist. In fact, this equilibrium cannot exist for $k \leq x^*$ because this condition yields $L(k) \geq P(k)$. The second sentence implies that bidder 1 is less likely to exercise the buyout option as the buyout price increases and the second-unit valuation decreases. This finding is also intuitive.

Second, we consider an equilibrium with equation (3). Bidder 1 should be indifferent between choosing $q = 2$ and $q = 0$ if his first-unit valuation $x$ is equal to threshold $t$. Thus, threshold $t$ satisfies $U(2,t;b) = U(0,t;b)$. Moreover, in this equilibrium, bidder 1 cannot benefit from choosing $q = 1$. Consequently, we obtain the following proposition.

**Proposition 3.** There exists an equilibrium in which bidder 1 chooses either $q = 2$ or $q = 0$ following equation (3) with threshold $t = 2b/k$ if and only if $0 < b < k/2$ and $b \leq x^*/2$. The threshold increases with $b$ and decreases as $k$ increases.

Proposition 3 says that this equilibrium is more likely to exist with higher $k$ and lower $b$. Bidder 1 enjoys the buyout option more at a lower buyout price. Moreover, given that he purchases two units for the buyout price, his payoff increases as his second-unit valuation increases. Clearly, the conditions in proposition 3 show this incentive of bidder 1.

Finally, we consider an equilibrium with equation (4). Bidder 1 should be indifferent between choosing $q = 2$ and $q = 1$ if his first-unit valuation $x$ is equal to threshold $\tilde{t}$. Thus, $\tilde{t}$ satisfies $U(2,\tilde{t};b) = U(1,\tilde{t};b)$. Similarly, 1 should be indifferent between choosing $q = 1$ and $q = 0$ if his first-unit valuation $x$ is equal to threshold $\ell$. Thus, $\ell$ satisfies $U(1,\ell;b) = U(0,\ell;b)$. Note that lemma 1 implies that (i) $U(0,x;b) > U(1,x;b) > U(2,x;b)$ for $x \in [0,\ell)$, (ii) $U(1,x;b) > U(0,x;b)$ and $U(1,x;b) > U(2,x;b)$ for $x \in (\ell,\tilde{t})$, and (iii) $U(2,x;b) > U(1,x;b) > U(0,x;b)$ for $x \in (\ell,1]$. Therefore, an equilibrium with equation (4) exists if and only if $0 < \ell < \tilde{t} < 1$. Consequently, we obtain the following proposition.

**Proposition 4.** There exists an equilibrium in which bidder 1 chooses $q = 2$, $q = 1$, or $q = 0$ following equation (4) with thresholds $\ell = P^{-1}(b)/k$ and $\tilde{t} = L^{-1}(b)/k$ if and only if $x^* < b < L(k)$. These thresholds increase with $b$ and decrease as $k$ increases.

Proposition 4 implies that the equilibrium requires intermediate buyout prices. The reason is essentially the same as that for proposition 2; that is, this condition gives an appropriate incentive to bidder 1 to choose $q = 1$. An increase in $k$ expands the range of the buyout price for which the equilibrium exists. With large $k$, bidder 1 is more likely to have an incentive to choose $q = 2$. 
4 Analysis

In this section, by using the findings in section 3, we analyze how \(b\) and \(k\) affect the existence of the equilibrium. As shown in the following proposition, an equilibrium is unique if it exists.

**Proposition 5.** There exists a unique equilibrium.

1. Suppose \(k > x^*\). In equilibrium, bidder 1 chooses (i) \(q = 0\) for \(b \geq P(k)\), (ii) either \(q = 1\) or \(q = 0\) for \(b \in [L(k), P(k)]\), (iii) \(q = 2, \ q = 1,\) or \(q = 0\) for \(b \in (x^*, L(k))\) if \(L(k) > x^*\), and (iv) either \(q = 2\) or \(q = 0\) for \(b \in (0, x^*/2]\).

2. Suppose \(k \leq x^*\). In equilibrium, bidder 1 chooses (i) \(q = 0\) for \(b \geq k/2\) and (ii) either \(q = 2\) or \(q = 0\) for \(b \in (0, k/2)\).

The thresholds are given by propositions 2, 3, and 4.

Proposition 5 clarifies the impact of the buyout price on bidder 1’s optimal buyout decision. We can easily observe that with a lower buyout price, bidder 1 exercises the buyout option for more units. Moreover, as proposition 5 shows, the buyout decision is diverse when \(k \geq x^*\). It is worth noting three comments on proposition 5.

First, the range of \(b\) in proposition 5 has no overlap; thus, the issue of multiple equilibria does not arise in our model. On the other hand, if \(k > x^*\), no equilibrium exists for \(b \in (x^*/2, \min\{x^*, L(k)\})\]. This range expands as \(x^*\) decreases (i.e., the bidder’s valuation becomes low on average). We will discuss this point in section 5.

Second, as \(x^*\) declines, the buyout decision becomes diverse for \(k > x^*\). As shown in lemma 5, this is the case when the bidder’s valuations are low with high likelihood. The underlying intuition is simple. Bidder 1 has an incentive to participate in the auction by choosing \(q = 1\) when (i) his own second-unit valuation is high (i.e., high \(k\)) and (ii) bidder 2’s first-unit valuation is on average low. In particular, the equilibrium with two thresholds (i.e., the one in proposition 4) cannot exist unless \(x^*\) is sufficiently small. In other words, the incentive of bidder 1 to choose \(q = 1\) is vulnerable to an increase in \(x^*\) (and \(b\)), and then he could profitably deviate to choose \(q = 2\) or \(q = 0\).

For an intuitive understanding of the impact of \(x^*\) on the equilibrium, we consider an example with the specified distribution function \(F(x) = x^\alpha\), where \(\alpha > 0\) represents the degree to which the bidder’s valuation is likely to be high. (The higher the value of \(\alpha\), the higher the likelihood of a high valuation.) Figures 2 and 3 illustrate the cases with \(\alpha = 0.25\) and \(\alpha = 0.5\), respectively. In these figures, we obtain \(x^* = 0.153\) and 0.563, respectively. As we can see in the figures, the equilibrium with two thresholds disappears in the case with \(\alpha = 0.5\).
Third, when \( x^* \geq 1 \), any \( k \in (0,1] \) satisfies \( k \leq x^* \), and thus bidder 1 never chooses \( q = 1 \) (proposition 5.2). Moreover, an equilibrium exists for any \((b,k) \in [0,1] \times [0,1]\). By using the same example as above, we obtain figure 4, which illustrates the case with \( \alpha = 1 \).

The following proposition, which is immediate from proposition 5, expresses the main point of this work, which is that the buyout option can alleviate implicit collusion in UPA.

**Proposition 6.** The positive expected revenue is realized in equilibrium if the buyout price satisfies

(i) \( b \in (0,k/2) \) if \( k \leq x^* \),
(ii) \( b \in (0,x^*/2] \) or \( b \in (x^*,P(k)) \) if \( x^* < k \) and \( x^* < L(k) \), and
(iii) \( b \in (0,x^*/2] \) or \( b \in [L(k),P(k)) \) if \( L(k) \leq x^* < k \).
5 Discussion

5.1 Second-stage zero-revenue equilibrium

We focus on a single-unit bid of \((p_1^i, p_2^i) = (1, 0)\), which yields zero revenue in the second stage when bidder 1 chooses \(q = 0\). However, there is another form of single-unit bidding: The bidder submits his truthful valuation for the first unit and zero for the second unit. In fact, without a buyout option, there exists a zero-revenue equilibrium in which bidder \(i\) with valuations \((x^i, kx^i)\) submits \((p_1^i, p_2^i) = (x^i, 0)\), \(i = 1, 2\). Moreover, bidding \((x^i, 0)\) is a weakly dominant strategy.

In this subsection, we discuss how the equilibrium will change if the bidder with valuations \((x^i, kx^i)\) submits \((x^i, 0)\), rather than \((1, 0)\), in UPA. Suppose that bidder 1 chooses \(q = 0\), which yields the UPA, if \(x^1 < t\). Then, conditional on being in the UPA, bidder 2 correctly understands that his opponent’s first-unit valuation is \(x^1 \in [0, t)\).

On the one hand, suppose that bidder 2 has valuations \((x^2, kx^2)\) that satisfy \(x^2 \geq kx^2 \geq t\). By bidding \((p_1^2, p_2^2) = (x^2, t)\), bidder 2 wins two units and then obtains the expected payoff of \((1+k)x^2 - 2t\). Thus, if \(kx^2 > 2t\), bidder 2 can profitably change his bids from \((x^2, 0)\) to \((x^2, t)\). On the other hand, suppose that bidder 2’s valuations satisfy \(x^2 \geq t > kx^2\). Similarly, bidder 2 can benefit by bidding \((p_1^2, p_2^2) = (x^2, p)\) with \(p \in (0, kx^2)\), rather than bidding \((p_1^2, p_2^2) = (x^2, 0)\), if

\[
\frac{1}{F(t)} \left( \int_0^p [(1+k)x^2 - 2x^1]dF(x^1) + \int_t^p (x^2 - p)dF(x^1) \right) > x^2.
\]

The first and second terms of the left-hand side are the expected payoffs when bidder 2 is awarded two units and one unit, respectively. In both cases, for a lower \(t\), bidder 2 is likely to have an incentive to deviate from choosing bid \((x^2, 0)\). Thus, in equilibrium, \(t\) should be high so that bidder 2 optimally submits \((p_1^2, p_2^2) = (x^2, 0)\).

This result implies a lower likelihood that bidder 1 will exercise the buyout option if bidder \(i\) submits single-unit bid \((x^i, 0)\) in the UPA. Importantly, however, our main result essentially remains unchanged. The buyout option can alleviate implicit collusion because bidder 1 has an incentive to exercise the buyout option at sufficiently low buyout prices.

5.2 Other bidding strategies

Although in sections 3 and 4 we restricted our attention to a specific equilibrium, there exist other equilibria that are induced by different bidding strategies. In what follows, we discuss such equilibria by separately considering bidding strategies in SPA and UPA.

First, we consider bidding strategies in the SPA that is realized after bidder 1 chooses \(q = 1\). We let \(H(p)\) denote the distribution function of a bid \(p\) that is induced by bidder 2’s bidding strategy.
Then, by choosing \( q = 1 \) with valuations \((x, kx)\), bidder 1 obtains the expected payoff
\[
U(1, x; b) = x - b + \int_0^{kx} H(p)dp.
\]
If bidder 2’s bid is on average low (e.g., bidder 2 submits \( p = 0 \) with probability one), bidder 1 has a stronger incentive to choose \( q = 1 \) than that given by propositions 2 and 4. However, this bid cannot constitute an equilibrium, because bidder 2 can benefit by submitting a truthful bid in the SPA. On the other hand, if bidder 2’s bid is on average high, the equilibria described in propositions 2 and 4 are less likely to exist. For instance, bidder 1 never chooses \( q = 1 \) if he understands that his opponent will submit \( p = 1 \) in the subsequent stage of the game. Note that this situation is equivalent to the case with \( x^* \geq 1 \). In summary, by replacing \( F(\cdot) \) with \( H(\cdot) \), we can duplicate the analysis given in sections 3 and 4.

Second, we consider bidding strategies in the UPA that is realized after bidder 1 chooses \( q = 0 \). It is optimal for bidders to submit a single-unit bid when they will share the objects. Thus, we focus on the situation in which either bidder 1 or bidder 2 alone obtains two units on the equilibrium paths.

1. Bidders 1 and 2 submit \((p_1^1, p_2^1) = (1, 1)\) and \((p_1^2, p_2^2) = (0, 0)\), respectively. Then, bidder 1 obtains two units without making any payment. Thus, for any \( b > 0 \), bidder 1 has no incentive to choose \( q > 0 \) in the first stage. In equilibrium, a buyout option cannot improve the revenue.

2. Bidders 1 and 2 submit \((p_1^1, p_2^1) = (0, 0)\) and \((p_1^2, p_2^2) = (1, 1)\), respectively. Then, bidder 1 obtains nothing in the UPA, and thus he has no incentive to choose \( q = 0 \). Thus, bidder 1 chooses \( q > 0 \) unless the buyout price is high.

Note that in both cases, depending on \( F(\cdot) \), there may exist other pairs of bidding strategies that yield the same outcome. On the other hand, there exist strategies that make bidder 1 avoid choosing \( q = 0 \); that is, UPA is off the equilibrium paths. These strategies increase bidder 1’s incentive to exercise the buyout option. In other words, our main result remains unchanged for a large set of bidding strategies.

### 5.3 Endogenous buyout price

Although we assume an exogenous buyout price in this paper, we can alternatively employ a model with a seller who selects an optimal buyout price. If we assume that the seller has no private information that is relevant to a bidder’s payoff, the subsequent game after the seller chooses buyout price \( b \) remains unchanged; that is, proposition 5 describes the subsequent game. Thus, the seller chooses to offer \( b \) in order to maximize the expected revenue. One important problem is that if distribution function \( F(\cdot) \) induces \( x^* < 1 \) (i.e., the bidder’s valuations are likely to be low), it is impossible for the seller to calculate the expected revenue because no equilibrium exists. Note, however, that any buyout price
under the conditions of proposition 6 yields a positive expected revenue with certainty. Therefore, the strategic seller benefits by giving bidders a buyout option, whereas zero revenue can arise from an injudiciously selected buyout price. This claim strengthens this paper’s central result.

5.4 Simultaneous buyout decisions by both bidders

We can consider an alternative model wherein both bidders 1 and 2 have a buyout option and they simultaneously decide quantities to purchase for a given buyout price in the first stage (i.e., bidder \( i \) chooses \( q^i \)). In this alternative model, we assume that each offer is with equal likelihood awarded a right of purchase when demand exceeds supply (i.e., \( q^1 + q^2 > 2 \)). That is,

- if \( q^1 = q^2 = 2 \), then each individual bidder will receive two units with a probability of 1/6, one unit with a probability of 4/6, and nothing with a probability of 1/6; and
- if \( q^i = 2 \) and \( q^j = 1 \), then bidder \( i \) will receive two units with a probability of 1/3 and one unit with a probability of 2/3, whereas bidder \( j \) will receive one unit with a probability of 2/3 and nothing with a probability of 1/3.

The auction moves on to the second stage if and only if \( q^1 + q^2 < 2 \). This modification does not essentially change our central result, but it is more difficult to explicitly calculate threshold(s) for buyout strategies. The bidders have a stronger incentive to choose \( q = 2 \) in equilibrium because they face the possibility that the auction will end at the first stage by their opponent’s choosing \( q = 2 \). However, it is ambiguous whether the bidder has a stronger incentive to choose \( q = 1 \). This is because, as we saw in section 4, the decision to choose \( q = 1 \) is sensitive to \( F(\cdot), k, \) and \( b \).

5.5 More bidders and more units of the objects

It may be of interest to generalize our analysis in two directions: \( N > 2 \) bidders and \( K > 2 \) units of identical objects. When two or more bidders with two-unit demand compete for two objects, the single-unit bid, in which the first-unit bid is either an extremely high bid or the bidder’s actual valuation, does not construct an equilibrium in UPA without a buyout option. Thus, the zero-revenue equilibrium disappears, and then the worst outcome that can arise in equilibrium is ambiguous. In general, however, demand reduction remains in any equilibria. The empirical research supports this prediction, although demand reduction may weaken as the auction comes to involve many more bidders (Engelbrecht-Wiggans et al., 2006). On the other hand, as noted in List and Lucking-Reiley (2000), demand reduction may intensify when the auction involves many more objects. As long as demand reduction exists in UPA, a bidder with a buyout option faces a trade-off between buying the object(s) for a relatively high buyout price and rejecting the buyout option in expectation of a low price, if not
zero, in the auction. Therefore, we can expect that in general, a seller will benefit by giving a buyout option to bidders, although its revenue-enhancing effect may be weakened.

6 Concluding remarks

The literature of multi-unit auctions points out the implicit collusion among bidders in uniform-price auctions. Theoretically, there can exist an equilibrium in which bidders obtain objects for no payment. In this paper, we focus on such an extreme case and illustrate the potential of a buyout option for alleviating implicit collusion. A bidder will wish to exercise the buyout option even if he is risk neutral, resulting in positive expected revenue. Although our analysis is not comprehensive, the result obtained in this paper provides an insight for auctioneers and auction designers. We close this paper with the following comments.

First, it would be interesting to compare the performance of different methods for improving revenues, including the buyout option, reserve price (Bresky, 2013), and change of pricing rules (Burkett and Woodward, 2018). An experimental approach seems suitable for this purpose. In fact, the design used by Ivanova-Stenzel and Kröger (2008), who experimentally study SPA with a buyout price, can be applied directly to the experimental design of UPA with a buyout price.

Second, a buyout option will probably perform better in multi-unit auctions than in single-unit ones. In single-unit auctions, a buyout option can harm the seller if the buyout price is inappropriately low because she loses a potential gain that she would obtain without the presence of the buyout option. As Chen et al. (2013) discuss, a seller needs sufficient information about the distribution of valuation and the degree of risk aversion. However, this does not matter in uniform-price auctions, in which low revenues are indeed realized. In other words, the introduction of a buyout option is less “risky” in multi-unit uniform-price auctions than in single-unit auctions.

Apart from these topics, more work remains for future research. We await further understanding of multi-unit auctions with a buyout option from future research.

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4 See also footnote 2.
5 Moreover, we already have important benchmark studies conducting a series of experiments wherein two bidders with two-unit demand compete in multi-unit auctions. See List and Lucking-Reiley (2000) and Engelmann and Grimm (2009), for instance.
References


Appendix

Proof of lemma 1

Proof. It suffices to show that all of $U(2, x; b) - U(1, x; b)$, $U(2, x; b) - U(0, x; b)$, and $U(1, x; b) - U(0, x; b)$ are increasing with $x$. By differentiating each of them with respect to $x$, we obtain

$$\frac{\partial}{\partial x}(U(2, x; b) - U(1, x; b)) = k(1 - F(x)) \geq 0,$$

$$\frac{\partial}{\partial x}(U(2, x; b) - U(0, x; b)) = k \geq 0,$$

$$\frac{\partial}{\partial x}(U(1, x; b) - U(0, x; b)) = kF(x) \geq 0.$$

The strict inequality holds for $x \in (0, 1)$. □

Proof of lemma 4

Proof. 1. $P'(x) = F(x)$, and $P''(x) = f(x)$. Thus, $P(x)$ is increasing and convex in $x$. Because $P'(x) > 0$ for $x \in (0,1]$, $P^{-1}(x)$ is well defined for $x \geq 0$. Thus, it is increasing and concave in $x$.

2. $L'(x) = 1 - F(x)$, and $L''(x) = -f(x)$. Thus, $L(x)$ is increasing and concave in $x$. Because $L'(x) > 0$ for $x \in (0,1]$, $L^{-1}(x)$ is well defined for $x \geq 0$. Thus, it is increasing and convex in $x$.

3. $P(x) = L(x)$ if and only if $x = 2P(x) - 2L(x)$. Thus, $P(x^*) = L(x^*) = x^*/2$ by $x^* > 0$. The convexity of $P(\cdot)$ and the concavity of $L(\cdot)$ ensure that $x^*$ is unique, that $L(x) > x/2 > P(x)$ for $x \in (0, x^*)$, and that $P(x) > x/2 > L(x)$ for $x > x^*$. □

Proof of lemma 5

Proof. Suppose that $F(x) < G(x)$ for any $x \in (0,1)$. Then we obtain $x_G^*/2 = P(x_G^*) = \int_0^{x_G^*} G(y)dy > \int_0^{x_G^*} F(y)dy$. Thus, $x_G^* \in (0, x_F^*)$ follows from lemma 4.3. □

Proof of proposition 1

Proof. By lemma 1, $U(0, x; b) \geq U(2, x; b)$ for any $x \in [0,1]$ if and only if $U(0, 1; b) \geq U(2, 1; b)$, or, equivalently, $b \geq k/2$. Similarly, $U(0, x; b) \geq U(1, x; b)$ for any $x \in [0,1]$ if and only if $U(0, 1; b) \geq U(1, 1; b)$, or, equivalently, $b \geq P(k)$. By combining these two formulas, we obtain $b \geq \max\{k/2, P(k)\}$. □

Proof of proposition 2

Proof. The equilibrium exists if and only if (i) $0 < t < 1$, (ii) $U(0, x; b) \geq U(2, x; b)$ for any $x \in [0,t)$, and (iii) $U(1, x; b) \geq U(2, x; b)$ for any $x \in [t,1]$. 

18
First, \( U(1, t; b) = U(0, t; b) \) gives \( t = P^{-1}(b)/k \). It is clear that \( t \) increases with \( b \) and decreases as \( k \) increases. By lemma 4.1, \( 0 < t < 1 \) if and only if \( 0 < b < P(k) \). Second, by lemma 1, \( U(0, x; b) \geq U(2, x; b) \) for any \( x \in [0, t) \) if and only if \( U(0, t; b) > U(2, t; b) \). Thus, we obtain

\[
U(0, t; b) > U(2, t; b) \iff b > \frac{kt}{2} = \frac{P^{-1}(b)}{2}
\]

\[
\iff b < P(2b)
\]

\[
\iff b > \frac{x^*}{2}.
\]

The penultimate and last equivalences follow from lemmas 4.1 and 4.3, respectively. Third, by lemma 1, \( U(1, x; b) \geq U(2, x; b) \) for any \( x \in [t, 1] \) if and only if \( U(1, 1; b) \geq U(2, 1; b) \), which is equivalent to \( b \geq L(k) > 0 \). These formulas lead to \( L(k) \leq b < P(k) \) and \( b > x^*/2 \).

We consider whether these two conditions are compatible for some \( b > 0 \). Clearly, it is impossible if \( x^*/2 \geq P(k) \). Thus, suppose \( P(k) > x^*/2 \), which is equivalent to \( k > x^* \) by lemma 4.3. Also, by lemma 4.3, \( k > x^* \) if and only if \( x^*/2 > L(k) \). Therefore, we obtain \( x^*/2 < L(k) \leq b < P(k) \). \( \square \)

**Proof of proposition 3**

**Proof.** The equilibrium exists if and only if (i) \( 0 < t < 1 \), (ii) \( U(0, x; b) \geq U(1, x; b) \) for any \( x \in [0, t) \), and (iii) \( U(2, x; b) \geq U(1, x; b) \) for any \( x \in [t, 1] \). First, \( U(2, t; b) = U(0, t; b) \) gives \( t = 2b/k \). It is clear that \( t \) increases with \( b \) and decreases as \( k \) increases. We observe that \( 0 < t < 1 \) if and only if \( 0 < b < k/2 \). Second, by lemma 1, \( U(0, x; b) \geq U(1, x; b) \) for any \( x \in [0, t) \) if and only if \( U(0, t; b) > U(1, t; b) \), which is equivalent to \( t \geq P^{-1}(b)/k \). Third, by lemma 1, \( U(2, x; b) \geq U(1, x; b) \) for any \( x \in [t, 1] \) if and only if \( U(2, t; b) \geq U(1, t; b) \), which is equivalent to \( t \geq L^{-1}(b)/k \). By combining the second and third formulas, we obtain

\[
\frac{L^{-1}(b)}{k} \leq t = \frac{2b}{k} \leq \frac{P^{-1}(b)}{k} \iff P(2b) \leq b \leq L(2b)
\]

\[
\iff b \leq \frac{x^*}{2}.
\]

These two equivalences follow from lemmas 4.1, 4.2, and 4.3. \( \square \)

**Proof of proposition 4**

**Proof.** The equilibrium exists if and only if \( 0 < \bar{t} < \bar{\bar{t}} < 1 \). \( U(1, \bar{t}; b) = U(0, \bar{t}; b) \) gives \( b = P(k\bar{t}) \), which is equivalent to \( \bar{t} = P^{-1}(b)/k \). Similarly, \( U(2, \bar{\bar{t}}; b) = U(1, \bar{\bar{t}}; b) \) gives \( b = L(k\bar{\bar{t}}) \), which is equivalent to \( \bar{\bar{t}} = L^{-1}(b)/k \). Clearly, \( \bar{t} > 0 \) if and only if \( P^{-1}(b) > 0 \), which is equivalent to \( b > 0 \). Similarly, \( \bar{\bar{t}} < 1 \) if and only if \( L^{-1}(b) < k \), which is equivalent to \( b < L(k) \). Moreover, we obtain

\[
\bar{\bar{t}} > \bar{t} \iff L^{-1}(b) > P^{-1}(b)
\]

\[
\iff P(b) > L(b)
\]

\[
\iff b > x^*.
\]

19
The last equivalence follows from lemma 4.3. Therefore, we obtain $x^* < b < L(k)$. 

**Proof of proposition 5**

*Proof.* It suffices to show the range of the buyout price for each equilibrium to exist. First, suppose $k > x^*$. (i) Because $k > x^*$ if and only if $P(k) > k/2$ by lemma 4.3, there exists a buyout price that satisfies $b \geq P(k)$. (ii) $k > x^*$ implies $L(k) > x^*/2$, because $2L(k) > 2L(x^*) = x^*$ by lemmas 4.2 and 4.3. Moreover, by lemma 4.3, $k > x^*$ if and only if $P(k) > L(k)$. Thus, there exists a buyout price that satisfies $x^*/2 < L(k) \leq b < P(k)$. (iii) Clearly, $L(k) > x^*$ if there exists a buyout price that satisfies $x^* < b < L(k)$. Note that $k > x^*$ implies $L(k) > x^*/2$ by lemmas 4.2 and 4.3. (iv) Because $k > x^*$ if and only if $k/2 > x^*/2$, there exists a buyout price that satisfies $0 < b < x^*/2$.

Second, suppose $k \leq x^*$. (i) Because $k \leq x^*$ if and only if $P(k) \leq k/2$ by lemma 4.3, there exists a buyout price that satisfies $b \geq k/2$. (ii) Because $k \leq x^*$ if and only if $k/2 \leq x^*/2$, there exists a buyout price that satisfies $0 < b < k/2$. Because $k \leq x^*$ implies $P(k) \leq L(k) \leq L(x^*) = x^*/2 < x^*$ by lemmas 4.1–4.3, there exists no buyout price for the equilibria with equations (5) and (7) (i.e., propositions 2 and 4) to exist. 

□