Bidding Behavior in Share Auctions with Reserved Reopenings

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Abstract

In this paper I build an equilibrium model for Treasury auctions with topups. Topups are non-competitive placements, taking place after the main (competitive) auction, where admitted bidders have the right (but not the obligation) to buy an additional amount of securities, proportional to the amount won in the competitive round, for a price set on the main auction date. The objective of the analysis is to assess how the introduction of this supplementary placement affects bidding behavior inside the main auction. For the model specification, I assume that strategies in the main auction are left-continuous step functions and I analyze three different payment rules for the reopening auction design. I use a local perturbation method to derive the analogous of first order conditions in a non-continuous setting (Kastl, 2010; 2012). I will show that in equilibrium topups may impact on bidding inside the main auction through two opposite channels. The first channel is a quantity effect, that operates when the bid is marginal, and leads bidders to ask for higher amounts to extract more surplus in the non-competitive placement. The second channel is a price effect and induces bidders to request smaller amounts to diminish the topup price. How important the first effect is relative to the second one depends critically on the payment rule in the reopening auction and on the probability that the bid will be marginal. In particular, more aggressive bidding is expected when the topup price is less dependent on bidders’ strategies inside the main auction and the probability that the bid will be marginal is high. The empirical analysis is conducted on Italian Treasury auctions data for short-term notes.

Keywords: Treasury Auctions, Topups, Stepwise Demand Functions.

JEL Classification: D44

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I. Introduction

The crisis that erupted in 2008 has left a legacy of high public debt and elevated sovereign risk in many countries. The surge in borrowing needs and growing concerns over the threat of sovereign defaults have worsened issuance conditions, with several countries reporting weak demand in primary markets and liquidity pressures and/or distortions in secondary markets.

This situation has prompted debt management offices in many countries to review existing issuance policies and procedures.\(^1\) Although not a substitute for sound fiscal policy, skillful adjustments of the debt management strategy may in fact reduce government borrowing costs and partially address market absorption problems.

In this context, auction theory may offer an important contribution. Giving insight on the bidders’ strategic behavior within different auction designs, it may help the sovereign to choose the type of placement that, while minimizing the Treasury’s financing costs, it encourages the broadest participation.

The focus of Treasury auction theory has long been on the relative merits and flaws of two formats, discriminatory and uniform. However, the inconclusive results relative to a revenue ranking of these two mechanisms has led some authors to conclude that the analysis of the stand-alone auction format cannot lead to a complete understanding of the Treasury auction environment.\(^2\) Thus, recent literature has dedicated attention to other elements of the overall placement procedure which can also be critical to the auction outcomes, such as the adoption of a primary dealership system,\(^3\) the role of the strategic interplay between the forward market for the securities to be auctioned (also called when-issued market) and the auction,\(^4\) or the possibility for the seller to adjust the total quantity sold after receiving the bids.\(^5\)

My paper attempts to offer a contribution to this new line of research by examining the consequences of introducing systematic topups (also called supplementary or reserved reopening auctions) after the main auction (also called ordinary or base auction).

Topups are non-competitive placements where admitted participants have the right (but not the obligation) to buy additional quantities of Government securities at a price settled on the date of the base competitive auction. Some authors have compared this right to an European call option where the strike price is equal to

\(^{1}\) For a review of recent changes in issuing procedures and instruments in OECD countries, see Blomenstein [2009] and OECD [2018].

\(^{2}\) See for example Wang and Zender [2002].

\(^{3}\) See Pacini [2009] and Sareen [2011]. Both the authors use a reduced form approach to analyze the role of primary dealership systems in determining auctions outcomes. Pacini analyzes the 2004 auction results of a cross section of European countries, while Sareen examines the bidding behavior inside Canada Treasury auctions in the period 1992-2003.

\(^{4}\) See Bikhchandani et al. [2000] and Nyborg and Strebuleav [2004].

\(^{5}\) See Back and Zender [2000] and McAdams [2007].
the price settled on the main auction date. Bidders’ allocations in the topups are typically a function of the shares won in the main auction. From the issuer’s point of view, the topup mechanism is expected to increase participation in the main auction, by attracting those investors that would otherwise not participate, and to incentivize more aggressive bidding.

Systematic post-auction topups are a common feature of institutional settings which operate under a primary dealership system. In these cases, the access to reopening auctions is restricted to Primary Dealers (hereafter referred to as PDs) as a special privilege for their market-making role.

To date, academic literature has only dedicated limited attention to reopening auctions. While some authors intuited that the presence of post-auction non-competitive rounds changes bidders’ maximization problem and influences bidding in the main auction itself, the relative irrelevance of non-competitive bidding in the empirical setting under analysis or the underlying complications connected to modeling a placement design with reopening have impeded the development of an equilibrium model which includes topup auctions.

This notwithstanding, there are a few interesting empirical works, which have shaded some light on the role played by reopening auctions. In particular, Coluzzi [2007], who analyzes reopenings auctions for the Italian medium- to long-term bonds, estimates that, despite its very short life, the option implicitly granted in the reopening has a value significantly different from zero for any security under exam. The author also argues that the phenomenon of overpricing present in the Italian primary market could be partially explained by the operators’ aggressive bidding in the auction to gain the right to participate to the reopening session. Instead, Marszalec [2009], who studies the Polish Treasury bill auction system, suggests that the introduction of topups should impact on bidding inside the main auction through two opposite channels. On the one side, there is an incentive to place lower value bids, to depress the price to be paid in the reopening auction; on

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6See Coluzzi [2007] and Cardozo [2010].
7Speaking about the introduction of topup auctions in the placement procedure, Ms. Jo Whelan, Deputy-Chief Executive of the U.K. Debt Management Office, observed: “what we have tried to do . . . is to introduce a mechanism to incentivize bidders at auctions by allowing them the option to take up to 10 percent of their allocation at the clearing price on the same day but a few hours later . . . We need have, if you like, eager bidders at the auction and the top-up facility is one way of trying to achieve that”. See House of Commons – Treasury Committee [2010].
8See Pacini [2007], Table 1.
9The reserved reopening auctions analyzed in this paper must not be confused with the practice of reopening the auction of a debt instrument already outstanding and traded on the secondary market. See Scalia [1998].
10See Février et al. [2002].
11See Marszalec [2009].
12This value results to be higher for coupon bonds than for zero coupon bonds, and it is increasing in the original maturity of the security. See Coluzzi [2007].
the other there is an incentive to bid more aggressively, to gain a larger share of the reopening supply. The author, however, can’t say a final word on which of the two effects would prevail because he is not able to find a solution to the optimization program which includes reopening auctions.

In a recent empirical work, [Marszalec 2017] finds that ignoring the top-up auctions when modeling the main auction may indeed introduce a bias in valuation estimates. In particular, his results seem to suggest that, in the context of the Polish institutional setting, where the topup price is set equal to the average winning price in the main (discriminatory) auction, the price effect, which creates an incentive to shade the bid, may be dominant and lead to bid shading.

Finally, [Cardozo 2013], analyzing the Colombian Treasury auction system, where the topup price is set equal to the average secondary market price of the bond on the auction date, finds that the overpricing of auctioned bond compared to the secondary market is higher for bonds with higher bidders’ participation in the topup auction. The author suggests that this could be driven by the fact that bidders incorporate the price of the option implicitly granted in the reopening auction in their bidding strategy for the main auction and pay more for the options that have a higher probability of being exercised.

My contribution to this literature is to build a theoretical model for Treasury auctions with reopening. It should be pointed out that the objective of the paper is not to study the topup mechanism per se, but to examine how the introduction of this post-auction placement impacts on the bidding behavior inside the main competitive auction. Thus, while introducing reopening auctions in the model design, my focus will remain on the main auction.

For the model specification, I assume that the main auction uses a discriminatory payment rule and that strategies are left-continuous step functions, as in [Kastl 2012]. As regards topups, I assume that they are exclusively reserved to PDs who won a share in the main auction and analyze bidding behavior under three different pricing rules. The assumption of discontinuous strategy functions in the main auction makes the theory closer to real-world auctions, where bid-functions are sets of quantity/price pairs, but complicates the solution of the model due to the lack of differentiability. In order to find the analogous of the first order conditions, I will adopt an extended version of the method proposed by [Kastl 2011, 2012] based on local perturbations.

Concerning the valuation structure, the complex interplay between the issuer and the primary dealers, based on a set of requirements and privileges, seems to induce private value inside the main auction and common value inside the reopening. This valuation structure makes the identification of the structural variables of the model impracticable without adding further restrictions and/or parametric assumptions. Consequently, I will employ a reduced form model, that adopts a difference-indifference regression, to test the impact of reserved reopenings on bidders’ strategic behavior inside the main auction.
This empirical strategy is made possible by the specific characteristics of the dataset used for estimation. In detail, I use the Italian Treasury auction data for twelve-month bills. For this maturity the Italian Treasury introduced topups reserved to PDs (specialists in Treasury securities) in February 2009. This policy change offers a “natural experiment” environment to test empirically the impact of reopenings on the main auction outcomes.

The contribution of the paper is twofold. Under the theoretical point of view, it contributes to the debate on the optimal auction mechanism testing whether reopening auctions impact on the bidding behavior inside the main auction. I will show that, while the presence of systematic post-auction reopenings introduces incentives that act in opposite directions, as argued by Marszalec [2009], the incentive to bid more aggressively is higher for bids that have a higher likelihood to be marginal and when the topup price doesn’t depend on bidders’ strategies inside the main auction.

Under the empirical point of view, to my knowledge this is the first study that analyzes bidding behavior in Italian Treasury auctions. Given that the Italian Treasury securities primary market is one of the largest in the world, any mispricing and/or infraction of competition inside the auctions may imply a large transfer of wealth. The paper gives also insights on the role played by PDs in Italian Treasury auctions. Thanks to the availability of a very detailed data set, my work offers an extensive examination of the strategic behavior of PDs inside the auction from which is possible to derive policy advice.

The rest of the paper is structured as follows: Section II reviews the literature on Treasury auctions and discusses the main open issues; Section III describes the framework for share auctions with reserved reopenings; Section IV presents the model; Section V describes the auction mechanism for Italian Treasury Bills and the data set; Section VI discusses the the estimation method and results. Lastly, Section VII summarizes the main findings and suggests further research paths. Some lengthy proofs are put in the appendix.

II. RELATED LITERATURE

There is a long-standing debate regarding the auction design that a sovereign should use to issue debt instruments. Most of the debate has focused on the comparative performance of two formats: discriminatory and uniform.

While the discriminatory (also called pay-as-bid or multiple-price) auction is the traditional and, until now, most common auction format for the placement of Treasury securities [Bartolini and Cottarelli 1997; Brenner et al. 2009; Sareen 2004], the

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13Every year the Italian government auctions about 450,000 million Euros of Treasury securities to finance the public debt.
merits of this type of auction have been questioned starting with the Sixties, when Milton Friedman argued that the U.S. Treasury could reduce the cost of financing the debt by switching to the uniform price auction format [Friedman 1960].

In both discriminatory and uniform auctions, individual bidders submit collections of bids (demand schedules) and the securities are awarded in the order of descending price until supply is exhausted. However, uniform auctions differ from discriminatory in that the winning competitive bidders pay the same price, equal to the lowest winning bid, rather than their bid price. The argument in favor of this format is that, reducing the winner’s curse relative to discriminatory auctions, uniform auctions would lessen bidders’ incentive to hide their true valuations and would lead them to bid truthfully.14 In addition, being strategically simpler, this format would reduce bid preparation costs and encourage more bidders to participate. Consequently, the uniform price mechanism should result in lesser collusion and higher revenues for the Treasury.

The theoretical literature after Friedman advanced arguments both in favor and against uniform auctions. Several economists supported this format on the basis of the comparison with single-unit auction models: they viewed the uniform price auction as a multiunit extension of the second-price-sealed-bid auction where bidders are induced to bid sincerely [Bikhchandani and Huang 1992; Milgrom 1989]. However, models by [Wilson 1979] and [Back and Zender 1993], which explicitly incorporate bidders’ demand functions in the models, reached the opposite conclusion. These authors showed that uniform auction are susceptible to arbitrary large underpricing, as a result of the market power which arises endogenously in equilibrium, and that discriminatory auctions can have a better revenue-raising performance.15

More recent literature has shown that the results based on the single auction cannot be generalized to auctions where bidders desire more than one unit. The intuition that this comparison was deceitful aroused already in [Vickrey 1961], but

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14Treasury auctions models are frequently considered common value models, where the value of the good for sale is the same to everybody but unknown at the time of bidding. In these models the winners are the bidders who made the highest guesses on the common value. This means that, conditionally on winning, the estimate is upward biased (winner’s curse). This creates an incentive for bidders to place price bids inferior to what they believe is the common value (practice known as bid shading). In uniform auctions where all winners pay the clearing price, bidders should be relatively unconcerned with the winner’s curse and have less incentives to shade their bids.

15Back and Zender [2000] noticed that equilibria with bad outcomes from the seller’s point of view “can be supported in a uniform-price auction because bidders are concerned with a single point on their demand curves, the one corresponding to the stop-out price. The rest of a bidder’s demand curve is left unrestricted and can be used to inhibit competition from other bidders. In a discriminatory auction, this is not the case. Each price bid by any bidder (above the stop-out price) will be paid by that bidder. This means each bidder has a direct interest in his entire demand curve. The use of a discriminatory auction effectively acts as a restriction on the set of strategies that may be played in equilibria.”
it is only at the end of the Nineties that Ausubel and Crampton [1998] showed “under general circumstances, that a bidder who desires more than one unit in a uniform price auction has an incentive to shade her bid”. The authors concluded that a theoretical ranking of formats may be impossible and that the superiority of an auction mechanism can only be determined empirically on a case-by-case basis.

Empirical research started to show large interest in auctioning systems in the Nineties, when the U.S. Treasury launched the so-called “Treasury experiment”. In this period, several works were dedicated to evaluating the revenue-raising abilities of the discriminatory and uniform auction formats by comparing the level of price determined in the auction relative to benchmarks, such as contemporaneous prices in the when-issued or secondary markets. The authors found evidence of under-pricing in auction outcomes of both discriminatory and uniform auctions, although under-pricing appeared to be slightly higher in discriminatory auctions.

More recent empirical findings [Cardozo, 2010; OECD, 2018; Pacini, 2007] show a reversal in Treasury auctions outcomes with a strong evidence of overpricing in both the auctions formats. This phenomenon, particularly evident in European

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16 In November 1992 the U.S. Treasury, that had been using the discriminatory auction format for the issuance of bills since 1929 and for the issuance of all the other bonds and notes since the Seventies, started to run uniform-price auctions for the placement of 2-year and 5-year notes as an experiment. This change was prompted by Salomon Brothers’ violations of the restrictions on the maximum amount of a security that a single firm can purchase in an auction. The Treasury’s purpose in conducting uniform-price auctions was to determine whether the uniform-price auction technique could reduce the Treasury’s financing costs compared with multiple-price auctions, by encouraging more aggressive bidding by participants, and whether it could broaden participation and reduce concentration of securities on original issue. Although the experiment yielded non-conclusive results, in 1998 the Treasury finally switched to the uniform price auction format for all issues.

17 Since prices of securities auctioned by different methods frequently cannot be directly compared, many empirical works conduct the comparison on the auction mark-up, computed as difference between the price in auction and the price in the “when-issued market”, where the price in auction is the lowest winning price bid in uniform price auction or the average winning price in discriminatory auctions while the price in the “when-issued market” refers to the average transaction price in “when-issued market” in the time period from shortly before the auction up until the time of the auction. The “when-issued market” is a market for forward contracts where transactions on securities are made “conditionally” because securities have been authorized, but not yet issued.


19 This phenomenon is at odds with the standard multi-auction theory based on common value and private information. In this setting the issuer should expect revenues inferior than the expected value of the securities being auctioned (common value), since bidders can extract informational rents from their private information. However, overpricing may be compatible with models based on private value and private information, as the bidders’ private values of the auctioned security may be superior to the value prevailing in the secondary market. Overpricing can also be justified by some factors external to the auction mechanism per se, such as the interplay between the when-issued market and the auction. For example, Bikchandani et al. [2000] argue that overpricing is
Bidding Behavior in Share Auctions with Reserved Reopening

Treasury auctions, may be positive for the issuer as it permits to meet the financial needs of government at a lower price; however it doesn’t eliminate the concern that market manipulability may undermine the pricing process and can turn harmful in the longer-term, as it may discourage participation (particularly of small investors) in Treasury auctions.20

The inconclusive results relative to a revenue ranking of the uniform and discriminatory auctions has led the most recent literature on multiunit auctions to move in two different directions. On the one side, theoretical research has extended the analysis to the performances of other multiunit auction formats21 or it has examined how other features of the overall placement procedure, exogenous to the auction mechanism per se, can impact on pricing inside the auction.22 On the other side, the pioneering work of Paarsch [1992] and the subsequent developments in the estimation of structural models for auctions23 have completely revolutionized the empirical research in Treasury auctions. Differently from previous empirical works, which adopted a reduced-form approach to examine how bids, or bid summary statistics, varied with auction-specific and bidder-specific variables, the structural econometric literature adopt bidders’ optimality conditions to estimate structural parameters.

My paper inserts itself in this recent literature. In particular, I will analyze how a feature of the overall placement procedure exogenous to the auction mechanism per se, namely the introduction of reopening auctions reserved to PDs, may impact on the bidding behavior inside the main auction.

III. FRAMEWORK FOR SHARE AUCTIONS WITH REOPENING

III.1. The Mechanism

An auction mechanism with reopening is a placement design where the sovereign sells a particular issuance of debt securities in two sessions.

In \( t_0 \) the Treasury announces the specifics of the whole placement (e.g. offering amounts inside the main and topup auctions, term and type of security, CUSIP the cost of revealing positive information too early by trading in when-issued markets before the auction.

20For a descriptive analysis of the causes and risks of overpricing see Pacini [2009].

21See for example Armantier and Sbai [2006] on the relative performance of different auction mechanisms, including the Spanish format.

22For example, Back and Zender [2000] show that the seller can reduce equilibrium under-pricing by choosing the supply ex post to maximize revenue, while Sareen [2011] and Pacini [2009] examine the consequences of a Primary Dealership system on the auction results.

23See in particular the work of Guerre et al. [2000] and Li et al. [2002]. For multi-unit auctions, see the work of F´evrier et al. [2002], Hortaçsu [2002], Wolak [2003], Armantier and Sbai [2006], Chapman et al. [2006], and Kastl [2011, 2012].
Figure 1: Timeline of the placement procedure

number, and issue, maturity, and settlement dates). The first round is a competitive auction ("ordinary" or "main" or "base" auction). Bidders participating in the main auction must place their demand schedules, in the form of a collection of price/quantity bids, inside the bid window \([t_0, t_{MA}]\), where \(t_{MA}\) is the "cut-off time". In the main auction securities are awarded in the order of descending price until supply is exhausted. The pricing rule (e.g. discriminatory pricing or uniform pricing) determines how much the winning bidders must pay. The results of the main auction are communicated in \(t_{MA}\).

The second round is a non-competitive placement ("supplementary auction" or "secondary auction" or "reopening auction" or "topup"), where participants have the right (but not the obligation) to buy an additional amount of securities. This right can be exercised at the time \(t_{RA}\) (with \(t_{RA} > t_{MA}\)). Differently from the competitive session, in the reopening auctions bids consist only in an amount of the securities the bidder wants. The topup price is identical for all bidders and corresponds to a price set in \(t_{MA}\) based on a specific payment rule. The settlement of the securities awarded both in the ordinary and supplementary auctions take places in \(t_{SE} > t_{RA}\).

Besides these common features, topups specifications differ widely among coun-
Bidding Behavior in Share Auctions with Reserved Reopening

tries.

- Depending upon the institutional setting, reopening auctions can be performed systematically or at discretion of the Treasury,\textsuperscript{24} and the time when the reopening auction takes place varies between a few hours and a few days after the main auction.

- The amount of securities offered in the topup auction can be a fixed share of the amount supplied in the ordinary auction or can vary at discretion of the Treasury.\textsuperscript{25}

- Participation in the reopening session can be restricted in different ways. Typically only the winning bidders in the first round are admitted to topups, but in most institutional settings participation is restricted further to winning PDs.\textsuperscript{26}

- The price paid by bidders in the reopening auction is determined in $t_{MA}$ and can be derived from the secondary market,\textsuperscript{27} or, more frequently, it is a function of the price settled in the main auction.\textsuperscript{28}

- Allocation rules for the topup phase differs among countries.\textsuperscript{29}

\textsuperscript{24}Considering the cases of Italy, Colombia, and Poland (countries for which studies on reopenings are available) we have the following: the Italian Government runs reopening auctions for medium-and long-term bonds and for six- and 12-months bills on a regular basis (see Coluzzi [2007]); the Colombian Government performs the reopening auction only if the coverage of the main auction is at least 1.2 (see Cardozo [2010]); and the Polish Government runs topup auctions for 52-week bills and 2-years bonds on a discretionary basis (see Marszałek [2009]).

\textsuperscript{25}In Italy the amount offered is equal to 10 percent of the ordinary issue (25 percent for the first tranche of medium- and long-term bonds). In Colombia the supply for the reopening is 80 percent of the ordinary auction supply if the coverage of this auction was at least 2, 55 percent if the coverage was between 1.2 and 2. In Poland the Treasury has discretion to offer up to 20 percent of the amount issued in the main auction.

\textsuperscript{26}See Pacini [2009].

\textsuperscript{27}For instance, in the Colombian treasury auctions, the price that bidders have to pay in the topup is the average secondary market price of the bond on the auction date, as calculated and published by the Colombian Securities and Stock Exchange. See Cardozo [2010].

\textsuperscript{28}In this case, the price paid in the reopening usually depends on the format of the ordinary auction. Typically for uniform-price auctions the price is set equal to the stop-out price of the ordinary auction, while in discriminatory auctions the price is set equal to the weighted average winning price.

\textsuperscript{29}In the reopening each bidder has a preliminary allocation which is typically a function of the share obtained in the ordinary auction. However, the actual share may differ from this initial allocation, depending on other parameters. The preliminary allocation, as well as the rule to determine the actual allocation may vary significantly among countries. For example, in Italy the preliminary allocation is a share of the topup supply equal to the share obtained by the bidder in the last three ordinary auctions. In Poland the preliminary allocation depends only on the share obtained in the last ordinary auction. In Colombia the Treasury offers three different amounts in the reopening: the preliminary allocation for the first supply is proportional to what the bidders
In the paper I will build an equilibrium model for Treasury auctions with reopenings which has the following design:

**Feature 1.** The main auction is a competitive placement, based on a specified pricing rule (e.g. discriminatory or uniform pricing).

**Feature 2.** Topups take place regularly after the main auction.

**Feature 3.** The amount of securities offered in the topup auction is a fixed share \( \delta \) of the amount allocated in the ordinary auction.

**Feature 4.** Participation in the reopening is reserved to PDs who won in the main auction.

**Feature 5.** In the reopening auction, each participant receives a share of the topup supply equal to the share won in the ordinary auction.\(^{30}\)

**Feature 6.** The topup price, equal for all admitted bidders, is set after the main auction based on a specified payment rule.

**Feature 7.** The placement design is public information at the moment agents present their bids for the main auction.

It is important to disentangle the implications of these features. Features \(^{2}\) and \(^{7}\) imply that the bidders participating in the base auction at time \( t_{MA} \) know that an additional amount will be made available for purchase inside the reopening placement in \( t_{RA} \).\(^{31}\) They also know that the price of the securities in the topup is set after the main auction based on a payment rule that is common information (Features \(^{6}\) and \(^{7}\)), and that admitted participants will receive a share of securities equal to the share won in the main auction (Features \(^{5}\) and \(^{7}\)). Finally, while the whole information on the placement mechanism is common knowledge to all bidders (Feature \(^{7}\)), the part relative to the reopening mechanism interests only the PDs, as they are the only bidders admitted to the topups (Feature \(^{4}\)).

In the paper, I will use Feature \(^{4}\) to introduce asymmetry in the bidders’ behavior. In particular, I will assume that, when preparing their strategies for the main auction, non-PDs will use only the piece of information on the payment rule in the main auction, the preliminary allocations for the other two supplies is restricted to the PDs that are in the top positions of the PD program for the year.\(^{30}\) Thus, a PD winning \( x \) percent of the total amount of securities allocated in the main auction will receive \( x \) percent of the topup supply.\(^{31}\) It is worth highlighting the importance of assumption 2. Reopenings are expected to impact on agents’ bidding inside the main auction because bidders know with certainty that the reopening auction will take place after the main auction and that their bids in the ordinary auction will impact on the reopening allocation and, possibly, price. Instead, if the Treasury would run reopening auctions on a discretionary basis, the bidding behavior inside the main auction is likely to be less affected (or not affected at all).
Bidding Behavior in Share Auctions with Reserved Reopening

I will consider three different pricing rules for the reopening auction to analyze how this feature of the auction design impact on bidders’ strategic behavior inside the main auction (see Section IV.2).

III.2. Deciding Between the Common and Private Value Paradigms

Treasury auctions have been traditionally modeled as common value (CV) auctions, where the good for sale has the same value to every participant although unknown at the moment of the auction. However, many recent works have highlighted that there may also be “private value” (PV) aspects to the Treasury auctions, and that in some cases these can prevail.\(^{32}\)

In general, it may be stated that whether PV or CV is more appropriate depends on the institutional setting, the profile of investors, and on whether bidders intend to hold the securities, resell them to their clients, or trade them on the secondary market. Also methodological aspects may influence authors’ choice of the information structure. In this respect, the PV paradigm offers an attractive framework for empirical research because the identification and estimation for structural econometric models is very simple, requiring only observation of the bids presented by the agents participating in the auction. This also explains why in the last few years there has been a surge of structural econometric works on Treasury auctions based on the PV assumption.\(^{33}\)

Looking at the institutional setting of the Italian Treasury auction system, my empirical setting, the PV component is likely dominant in the ordinary auction, while the CV specification seems to fit better the reopening session.

The choice of the PV paradigm for the main auction is motivated by the institutional characteristics of the Italian primary market, based on a formal system of primary dealership. This system involves a composite system of requirements and obligations to maintain the status of PD (or to gain the status in the case the operator is not a PD). The most significant of these obligations requires that each PD (or aspiring PD) must win a share not inferior to the 3 percent of the total volume issued by the Treasury during the year. Each bidder’s primary market share is calculated

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32While recognizing the existence of a CV component in Turkish Treasury auctions, Hortaçsu [2002] states that the PV paradigm can be a good benchmark specification as bidding behavior appears to be related to the inventory/liquidity needs that each bidder faces. Hortaçsu and Kastl [2008] argue that for short term debt the private valuation component is probably more important, because most investors hold these papers in their portfolios until maturity so that there is almost no resale.

considering only the quantities she has obtained inside ordinary auction sessions.\footnote{Individual market shares are calculated by the Treasury as a weighted average of the bidders’ allocations along the year. Each maturity has a different weight in contributing to the 3 percent requirement, with weights being higher the longer is the residual maturity of the security. For example, the weight for 3-months bills is 0.25, while the weight for 30-years bonds is 13. This system of weights is motivated by the Treasury’s desire to extend the debt maturity profile.}

This institutional setting is likely to induce PV inside the main auction, as each PD (or aspirant PD) participating in the auction will value the securities for sale accordingly to her own status in fulfilling the 3 percent primary market share requirement. Thus, it is likely that PDs that have obtained large shares in previous auctions will bid less aggressively in the current auction than PDs that have obtained only small allocations. It is worth mentioning that the exact state of each bidder is private information because individual allocations are not made public by the Treasury. Relative to non-PDs, it must be noted that, while some bidder can be interested in gaining the PD status, and thus she will value the auctioned securities according to her status in fulfilling the 3 percent obligation, most non-PD financial operators participating in the auction are too small to aim at gaining the PD status.\footnote{In addition to the primary market requirements, PDs must contribute to the efficiency of the secondary market and must satisfy organizational requirements relative to minimum capital and organizational structure.}

In those cases PV remains a valid paradigm if non-PD operators buy the notes to meet liquidity or collateral requirements\footnote{Over the last few years, with the unfolding of the financial crisis, the collateral motivation has likely become increasingly important for banks (PDs and non-PDs) that have difficulties accessing the wholesale funding markets and must resort to the liquidity facilities of the ECB.} or fill customer orders. The PV assumption is backed by the empirical finding of systematic overpricing in Italian Treasury auctions.\footnote{See Coluzzi [2007]; Pacini [2009].}

This evidence would be difficult to justify if the CV component were dominant: if the security for sale is valued the same by any bidder and this value is the resale price in the secondary market after the auction, the systematic bias in the auction determined price would be puzzling. Instead, the phenomenon of overpricing is not at odds with a PV specification: agents bid above the expected resale market price to maintain (or to gain) the PD status.

Moving to the reopening session, the main argument in support of the CV paradigm is connected to the fact that, differently from the main auction, participation in the topup doesn’t come with strings attached. The requirements and obligations to maintain (or gain) the PD status regard only the bidding behavior inside the main auction. On the contrary, the reopening auction is a privilege granted to PDs for their market-maker role. In this context, it is reasonable to assume that the CV component is the most important and that PDs admitted to the non-competitive session will exercise their right to buy only when they can profit from it, i.e. when the reopening price is inferior to the price of the security in the secondary market (or CV). The CV hypothesis finds strong support in my data set, as in most of the...
reopening auctions (80 percent) either all PDs exercise their right to buy in the reopening auction or none exercises it (See Table 2), confirming that bidders attribute the same value to the securities for sale, equal to the resale price on the secondary market.

For the sake of completeness, it must be noted that, while the PV and CV components are likely dominant in the main auction and reopening auction respectively, in both the rounds the real valuation of the securities is likely a combination of private and common value considerations. In particular, in the main auction bidders likely take also into account the price on the secondary (or when-issued) market when they present their bids, as confirmed by the empirical evidence that bidders tend to concentrate the submission of their bids in the very last minutes before the cut-off time, clearly trying to extract the most updated signal relative to the common value of the securities from the when-issued or secondary market. In the reopening auction private value considerations may step in when a bidder isn’t able to obtain an allocation inside the main auction sufficient to cover the orders done by her clients and must resort to the reopening auction even if the selling price is not convenient. In my data set the evidence of the private value component in the reopening auction may be found in the presence of topups with partial exercise. Nevertheless, the dominance of the CV component finds support in the fact that reopening auctions with partial exercise are much less numerous than topups with full or null exercise (20 against 80 percent, see Table 2). In addition, PV considerations likely intervene only ex post, i.e. after the main auction results are announced, when the bidder learns that her main auction allocation is not sufficient to cover her clients’ orders.

IV. Model Specification for Auctions with Reopening

In this section I build an equilibrium model for auctions with reopening. My analysis is based on the divisible good auction model of Kastl [2011, 2012], where bidders are restricted to use step-functions as their bid functions. I will extend Kastl’s model by introducing the reopening auction (RA), in order to study its impact on the bidding behavior inside the main auction (MA).

IV.1. The Main Auction (Kastl, 2012)

After the Treasury communicates the specifics of the MA auction, bidders decide and submit their bids in the time interval $[t_0, t_{MA}]$ (bid window).

As specified above, I will assume that two classes of potential bidders participate in the MA: the PDs, admitted to the reopening session whenever they win a share in the MA, and the non-PDs, not allowed participating in the non-competitive round.
Assumption 1 (Heterogeneity). Let PD and NPD be the two groups of bidders participating in the MA auction, respectively composed of $N_{PD}$ and $N_{NPD}$ agents (with $N_{PD} + N_{NPD} = N$). Bidders within a given group are symmetric, but bidders are asymmetric across groups. The number of bidders and their division between the two groups are common knowledge.

Prior to the MA, each bidder receives a private signal $S_i$ which conveys information about the value of the securities.

Assumption 2 (Private Information). Private signals $S_i = (S_{i, MA}, S_{i, RA})$ (with $i = 1, \ldots, N$) are two-dimensional random vectors such that:

1) $\forall i \in \{1, \ldots, N\}$ $S_i$ satisfies: i) $S_{i, MA}: \Omega_{S_{i, MA}} \rightarrow \mathbb{R}^+$, ii) $S_{i, RA}: \Omega_{S_{i, RA}} \rightarrow \mathbb{R}^+$, iii) $S_{i, MA} \perp S_{j, RA}$ $\forall j \in \{1, \ldots, N\}$; 2) Signal components $S_{i, MA}$ are i.i.d. within each group $g \in \{PD, NPD\}$ with a symmetric continuous distribution function $F^g_{S_{i, MA}}(S_{i, MA}) = F^g_{S_{i, MA}}(S_{i, MA})$ and density function $f^g_{S_{i, MA}}(S_{i, MA}) = f^g_{S_{i, MA}}(S_{i, MA})$.

The component $S_{i, MA}$ of the signal gives bidder $i$ private information on the underlying private value of the auctioned securities in the MA, while the component $S_{i, RA}$, assumed to be independent of $S_{i, MA}$, conveys information on the common value of the securities in the reopening session.

Assumption 3 (Random Supply). Supply $Q$ is a random variable distributed on $[Q, \bar{Q}]$, with strictly positive density conditional on $S_i$, $\forall i$.

Assumption 3 implies that the distribution of the market clearing quantity allocated to each bidder type has no mass points at any quantity in the support of that bidders demand. This assumption will be used to rule out cutting-edge solutions of the maximization problem.

After receiving the signal, agents decide their strategies for the MA. Following Kastl, I assume that bidders submit discrete bid-points rather than continuous demand functions and that, as in most institutional settings, each bidder $i$ is constrained to submit up to a maximum of $M \geq 1$ bid-points.
**Assumption 4** (Step-wise bid functions). An admissible bid-point \((q, p)\) is a share-price pair where \(q \in [0, Q]\) and \(p \in [0, \bar{p}]\). A pure strategy for agent \(i\) is any map which associates a \(M_i\)-tuple of admissible bids to a realization \(s_i\) of the private signal \(S_i\). Bid-points inside each strategy satisfy: i) \(q_{i,m} < q_{i,m+1}\) and ii) \(p_{i,m} > p_{i,m+1}\), where \(m = 1, \ldots, M_i\) and \(M_i \in \{1, \ldots, M\}\).

A strategy for bidder \(i\) can thus be summarized by a \(M_i\)-dimensional vector of share-price pairs, where bids are sorted by decreasing (increasing) prices (shares). This defines a weakly decreasing left-continuous step-wise demand function, where the bid point \((q_{i,m}, p_{i,m})\) marks the point in the share-price space where the \(m\)-th step of the function finishes, and the difference \(q_{i,m} - q_{i,m-1}\) defines the marginal share that bidder \(i\) demands at price \(p_{i,m}\) (i.e. the length of the step). It is also possible to express bidders’ strategies through a bidding function \(y_i(\cdot|s_i)\), where \(y_i(p|s_i)\) specifies bidder’s \(i\) cumulated demand at prices greater than or equal to \(p\), when the realization of the private signal is \(s_i\).

When the bid window is closed, the issuer sorts the price-quantity bids in descending order of price and determines the market clearing price \(P^c\) at which total quantity demanded equals the supply of securities. Bids above (or equal to) the market clearing price win, and pay the price they indicate for that quantity.

The specification with step-wise demand functions permits to take into account the phenomenon of rationing, which is neglected in continuous models, making theory closer to real-world auction practice. In this setting, all bids at prices above the clearing price are served in full, while marginal bids are normally rationed. I will assume that the rationing scheme adopted by issuer is pro-rata on the margin, as in most institutional settings.\(^{42}\)

**Assumption 5** (Pro-rata rationing). When \(\sum_{i=1}^{N} y_i(P^c|s_i) > Q\), the issuer serve the marginal bids adopting a standard pro-rata rationing rule, so that the marginal requested shares will be adjusted according to the rationing coefficient calculated as follows:

\[
\rho = \frac{Q - \lim_{p \downarrow P^c} \sum_{i=1}^{N} y_i(p|s_i)}{\sum_{i=1}^{N} y_i(P^c|s_i) - \lim_{p \downarrow P^c} \sum_{i=1}^{N} y_i(p|s_i)}
\]

maximum \(M\) they are allowed to, and that the number of bids may differ among bidders, Kastl assumes that for each bid point bidders face a cost of submission and that this cost varies across bidders. For example it is reasonable to assume that the cost is lower for large operators. This hypothesis is compatible with my data set where PD operators tend to place the maximum number of bids they are allowed to (i.e. 3), while non-PD operators place frequently only one bid (see Section VI.1).

\(^{42}\)As in Kastl, in the paper I will distinguish two different situations: (i) only one bidder is marginal, and (ii) multiple bidders are marginal (i.e. there is a tie).
In cases of full auction coverage, non-rationing solutions may occur only if the total demand at the clearing price $P^c$ is exactly equal to one (i.e., equal to the total supply). While this event is very unlikely with discrete bid functions,\footnote{For example, rationing occurs in any auction in my data set.} it must be taken into account in the model, as it may give rise to multiple equilibrium clearing prices. This happens each time the aggregate demand function and the residual supply have a common vertical segment in correspondence of $q = Q$ in the share/price space. To rule out uncertainty about the equilibrium, I assume, as in Kastl (2012), and consistently with most real-world auction procedures, that, when multiple equilibria arise, the auctioneer selects the equilibrium that maximizes her revenue inside the MA, i.e., the equilibrium with the highest price that clears the market.

**Assumption 6** (Highest clearing price). If in any MA $\exists p_{lo}, p_{up} \mid \forall p \in [p_{lo}, p_{up}]: \sum_{i=1}^{N} y_i(p|s_i) = Q$, then $P^c = \arg\max_{p \in [p_{lo}, p_{up}]} \left( \sum_{i=1}^{N} y_i(p|s_i) = Q \right)$.

The final assumption regards valuations. As specified in the previous section, I assume PV inside the MA as bidders value the securities for sale accordingly to their own status in fulfilling the requirements to maintain (or gain) the PD status.

**Assumption 7** (Private Values). Winning the quantity $q$ of securities is valued by agent $i$ of type $s_i$ according to a marginal valuation function $v_i(q, s_i^{MA})$, with $v_i(\cdot, \cdot)$ bounded, strictly increasing in $s_i^{MA}$ and weakly decreasing and continuous in $q$.

Given Assumptions 1-7, the expected utility inside the MA for a bidder $i$ belonging to group $g$, who receives the signal $s_i$ and employs a strategy $y_i(\cdot|s_i)$, can be written as:

$$
\Pi_{i,g}^{MA}(s_i) = \mathbb{E}_{S_{-i}|S_i=s_i} \left[ \int_{0}^{Q_i^e(S, y(\cdot|S))} v_i(u, s_i^{MA}) du - \sum_{m=1}^{M} \mathbb{I}_{\{Q_i^e(S, y(\cdot|S)) > q_i^m\}}(q_i^m - q_i^{m-1}) \cdot P_i^m - \sum_{m=1}^{M} \mathbb{I}_{\{q_i^m \geq Q_i^e(S, y(\cdot|S))\}}(Q_i^e(S, y(\cdot|S)) - q_i^{m-1}) \cdot P_i^m \right]
$$

where $q_i^0 = 0$ and $\mathbb{I}_{\{A\}}$ is the indicator function of the event $A$. The random variable $Q_i^e(S, y(\cdot|S))$ is the total share bidder $i$ obtains in the MA if the state is $S$ and bidders use the strategies specified in the vector $y(\cdot|S) = (y_1(\cdot|S_1), \ldots, y_N(\cdot|S_N))$.

To lighten notation, henceforth I will drop $(S, y(\cdot|S))$ from the argument of $Q_i^e$.

**IV.2. The Reopening Auction**

In $t_{MA}$, when the bid window for the MA is closed, the issuer communicates the results of the first round and makes available for purchase an additional fixed amount $\delta \cdot \sum_{i=1}^{N} y_i(P^c|s_i)$ of securities, with $\delta \in [0, 1]$. Admitted bidders can exercise their right to buy at time $t_{RA}$, with $t_0 < t_{MA} < t_{RA}$.
As anticipated in Section III.1, the admission to the RA is a privilege granted only to PDs who won in the MA (Feature A of the auction design).

**Assumption 8** (Restricted Admission). Each bidder \( i \) belonging to the PD group and won a share \( Q^c_i > 0 \) in the MA is admitted to the RA.

In addition, as illustrated in Section III.1, I assume that both allocations and prices in the RA are determined in \( t_{MA} \), when the MA results are communicated (Features 5 and 6 of the auction design).

**Assumption 9** (Placement without quantity and price uncertainty). RA allocations and price are determined and communicated in \( t_{MA} \). Each admitted bidder \( i \) who exercises her right to buy in the RA in \( t_{RA} \) will be allocated a share \( \frac{\delta Q^c_i}{\sum_{j=1}^{N} Q^c_j} \) and will pay the unit price \( P^{RA} \).

Assumption 9 implies that the RA is a placement without uncertainty, where both allocations and prices are known to bidders when they decide whether to take part in the secondary auction. In this context, the action set for each admitted bidder turns out to be very simple: it is a choice between participating in the RA or not. In the language of option theory, it is a choice between exercising the right to buy or not.

**Assumption 10** (Action Set: All or Nothing). Inside the RA an admissible bid point belongs to the set \( \{(0,0), \left( \frac{\delta Q^c_i}{\sum_{j=1}^{N} Q^c_j}, P^{RA} \right) \} \).

As regards valuations inside the RA, I assume CV, as anticipated in section III.2. The main argument in favor of this choice is that, differently from the MA, in the RA agents’ bidding behavior is not subject to requirements and obligations to maintain (or gain) the PD status. In this context, admitted agents are likely to use “economic considerations” to value the securities for sale. In particular, in the case of active and liquid secondary markets, the common value is likely to be equal to the secondary-market price of the security at the time \( t_{RA} \).

As the focus of the analysis is the bidding behavior inside the MA, and not the RA per se, I need to derive agent \( i \)'s expected payoff from the RA at the time she presents her bids for the MA, i.e. inside the bid window \( [t_0, t_{MA}] \), when the common value is unknown and bidder \( i \) infers it from the private signal \( S_{RA}^i \).

**Assumption 11** (Common Value). Let \( v_{t_{RA}} \) to be the true common marginal value of the securities at the time of the RA. This value is a realization at the time \( t_{RA} \) of the stochastic process \( \{V_t\}_{t \geq t_0} \), where \( V_{t_0} \) is the initial state of the process, known to all bidders. \( V_t \) has a continuous marginal distribution function at time \( t \) equal to \( F_{V_t}(v_t) \) corresponding density \( f_{V_t}(v_t) \). Conditional on \( V_{t_{RA}} \) the signal component \( S_{t_{RA}}^i \) with \( i = 1, \ldots, N \) is distributed according to a continuous distribution function \( F_{S_{t_{RA}}^i \mid V_{t_{RA}}}(s_{t_{RA}}^i \mid v_{t_{RA}}) \) with corresponding density \( f_{S_{t_{RA}}^i \mid V_{t_{RA}}}(s_{t_{RA}}^i \mid v_{t_{RA}}) \).
From assumptions 9-11 it follows that at the time of presenting her bids for the MA, bidder \(i\)'s expected payoff from the RA is:

\[
E \Pi_{RA}^i(s_i) = E \left[ V_{tRA}, S_i^{-i} \right| s_i \left\{ V_{tRA} > P_{RA} \right\} \delta \cdot Q_c^i \sum_{j=1}^N Q_{c_j} \left( V_{tRA} - P_{RA} \right) \right] \tag{2}
\]

where \(i\) belongs to the class of PDs and \(1_{\{A\}}\) is the indicator function of the event A. It is of interest to note that, while expression (2) resembles closely the analytical expression of the expected value of a standard European call option with strike price equal to the topup price \(P_{RA}\), it differs in so far that \(P_{RA}\) is unknown at the time agents submit their bids for the MA.\(^{44}\) Indeed, inside the bid window \([t_0, t_{MA}]\), both the average winning price \(P_{RA}\) and agent \(i\)'s assigned quantity \(Q_c^i\) are random variables.

It remains to define the payment rule inside the topup auction. I will consider three different payment rules to analyze how this feature of the topup design impacts on bidder’s strategic behavior inside the main auction.

**Assumption 12** (Payment Rule 1: Price in the Secondary Market at \(t_{MA}\)). Based on this payment rule, the topup price is set equal to the price in the secondary market at the time when the MA results are communicated.

\[P_{RA} = V_{tMA}\]

This payment rule implies that the RA price, equal for all bidders, doesn’t depend on bidders’ strategic behavior inside the MA.

**Assumption 12-bis** (Payment Rule 2: Lowest Winning Price). Based on this payment rule, the topup price is set equal to the lowest winning price in the MA (corresponding to the clearing price).

\[P_{RA} = P_c = \arg\max_{\{p_j^k, j\in(1,\ldots,N), k\in(1,\ldots,M_j)\}} \sum_{i=1}^N y_i(p_j^k|s_i) \geq 1\]

This payment rule implies that the RA price, equal for all bidders, doesn’t depend on individual bidders’ strategies inside the MA but only on the lowest winning price. This topup payment rule is common in auction designs with reopening that adopt an uniform payment rule in the MA.

\(^{44}\)It must be noted that even in \(t_{MA}\), when the MA results are communicated, the strike \(P_{RA}\) should not be seen as constant. Indeed, while its nominal value is fixed, the real value is dependent (in the long run) on the interest rate. Nevertheless, given the short life of the option connected to the RA, I assume that after the communication of the MA results bidders consider \(P_{RA}\) as a fixed parameter.
Assumption 12-ter (Payment Rule 3: Average Winning Price). Based on this payment rule, the topup price is set equal to the average winning price in the MA, defined as:

\[
P = \frac{\sum_{j=1}^{N} \sum_{m=1}^{M_j} 1\{(p^m_j \geq P_c)\} (q^m_j - q^{m-1}_j) \cdot p^m_j}{\sum_{j=1}^{N} \sum_{m=1}^{M_j} 1\{(p^m_j \geq P_c)\} q^m_j}
\]

if \( \sum_{j=1}^{N} y_j(P^c|s_j) \leq Q \)

\[
P = \frac{\sum_{j=1}^{N} \sum_{m=1}^{M_j} 1\{(p^m_j > P_c)\} (q^m_j - q^{m-1}_j) \cdot p^m_j + \left(Q - \lim_{p \downarrow P_c} \sum_{j=1}^{N} y_j(p|s_j)\right) \cdot P_c}{Q}
\]

if \( \sum_{j=1}^{N} y_j(P^c|s_j) > Q \)

where \( q^0_j = 0.45 \).

This payment rule implies that the RA price, equal for all bidders, depends on individual bidders’ strategies inside the MA because any winning bid contributes to determine the average winning price. This topup payment rule is common in auction designs with reopening that adopt a discriminatory payment rule in the MA.

IV.2.1 Equilibrium

In this section I will set up the maximization program for the two classes of agents participating in the MA and I will characterize the Bayesian-Nash equilibria for the auction model with reopenings.

As non-PD bidders are admitted only to the MA, I assume that the expected payoff inside the MA represents the objective function for a generic agent \( i \) belonging to this group. Instead, given that PD bidders are admitted also to the RA, it is reasonable to assume that they take into account that their strategic decisions inside the MA impact on their payoff inside the RA (see Assumption 8, related to the admission to the reopening auction, Assumption 9, related to the allocation in the secondary auction, and, depending on the payment rule, Assumption 12-bis or 12-ter, related to the topup price). Thus, I assume that PDs select their strategies in the MA in order to maximize the payoff from the overall placement procedure, which includes both the MA and the RA.

To derive the equilibrium for this asymmetric model, I will consider a representative bidder in each of the two groups and derive the analogous of the first order

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45 In case of rationing \( \left( \sum_{j=1}^{N} y_j(P^c|s_j) > 1 \right) \), the formula to compute the average winning price \( P \) multiplies the clearing price by the remaining supply available at that price and not by the marginal bid quantities, as the latter are not served in full.
conditions defining the best response of a representative player \( i \) in group \( g \) to her opponents strategy profile.

**Definition 1.** An equilibrium for the auction model with reopening is a collection of \( N^{NPD} \) functions of the type \( y_{i}^{NPD}(\cdot | S) \) and \( N^{PD} \) functions of the type \( y_{i}^{PD}(\cdot | S) \), such that for each bidder \( i \) belonging to the class of non-PDs \( y_{i}^{NPD}(\cdot | S) \) satisfies:

\[
y_{i}^{NPD}(\cdot | S) \in \arg\max \left( E_{i,NPD}(s_{i}) \right)
\]

and for each bidder \( i \) belonging to the class of PDs \( y_{i}^{PD}(\cdot | S) \) satisfies:\textsuperscript{46}

\[
y_{i}^{PD}(\cdot | S) \in \arg\max \left( E_{i,PD}(s_{i}) + E_{i,RA}(s_{i}) \right)
\]

where \( y_{i}^{NPD}(\cdot | S) \) and \( y_{i}^{PD}(\cdot | S) \) are formulated inside the time interval \([t_{0}, t_{MA}]\), the MA results are communicated in \( t_{MA} \), and the RA takes place in \( t_{RA} \).

Finding a solution for this optimization problem is complicated by the fact that bidders submit step-wise bid functions and therefore the payoff function is not differentiable. To circumvent this difficulty and derive the optimality conditions in a discriminatory auction, \[\text{Kastl} \ 2012\] proposes a solution method which focuses on bidder \( i \)’s strategic decision on where to locate the \( m \)-th step of her demand function and it is based on the condition that rules out profitable perturbations of the control variable \( q_{i}^{m} \). To derive the analogous of first order conditions for the auction model with reopenings I adopt an extension of this method.

**Proposition 1.** Under Assumptions 1-7, in any Bayesian-Nash equilibrium, every step \( m \) of the strategy \( y_{i}^{NPD}(\cdot | S) \) in the support of \( i \) must satisfy:

\[
Pr\left(p_{i}^{m+1} < p^{c} < p_{i}^{m}\right)\left[v_{i}(q_{i}^{m}, s_{i}^{MA}) - p_{i}^{m}\right] = Pr\left(p^{c} \leq p_{i}^{m+1}\right)\left(p_{i}^{m} - p_{i}^{m+1}\right)
\]

if \( m \in \{1, \cdots, M_{i} - 1\} \)

\[
v_{i}(q_{i}^{m}, s_{i}^{MA}) = p_{i}^{m}
\]

if \( m = M_{i} \)

where expressions \(3\) and \(3-\text{bis}\) are derived in \[\text{Kastl} \ 2012\].

\textsuperscript{46}The RA expected payoff is not discounted because both the MA and the RA are settled at the same time, in \( t_{SE} > t_{RA} \).
The optimality conditions for PDs depend on the RA payment rule.

**Proposition 2.** Under Assumptions 1-11, and 12, in any Bayesian Nash equilibrium, every step \( m \) of the strategy \( y^P_M(\cdot;S) \) in the support of \( i \) must satisfy:

\[
Pr\left(p_i^{m+1} < P^c < p_i^m\right) \\
\cdot \left[ v_i(q_i^m, s_{iMA}^i) - p_i^m \right] + \delta \cdot E_{V_{iRA}, V_{iMA}} \left( V_{iRA} - V_{iMA}; V_{iRA} > V_{iMA} \mid s_i \right) \\
= Pr(P^c \leq p_i^{m+1}) \cdot (p_i^m - p_i^{m+1}) \quad \text{if } m \in \{1, \cdots, M_i - 1\},
\]

where expression (4) and (4-bis) are derived in Appendix A.1.

**Proposition 3.** Under Assumptions 1-11 and 12-bis, in any Bayesian Nash equilibrium, every step \( m \) of the strategy \( y^P_M(\cdot;S) \) in the support of \( i \) must satisfy:

\[
Pr\left(p_i^{m+1} < P^c < p_i^m\right) \\
\cdot \left[ v_i(q_i^m, s_{iMA}^i) - p_i^m \right] + \delta \cdot E_{V_{iRA}, V_{iMA}} \left( V_{iRA} - V_{iMA}; V_{iRA} > V_{iMA} \mid s_i \right) \\
= Pr(P^c \leq p_i^{m+1}) \cdot (p_i^m - p_i^{m+1}) \quad \text{if } m \in \{1, \cdots, M_i - 1\},
\]

where expressions (5) and (5-bis) are derived in Appendix A.2.
Proposition 4. Under Assumptions 1-11, and 12-ter, in any Bayesian Nash equilibrium, every step $m$ of the strategy $y^{PD}(\cdot|S)$ in the support of $i$ must satisfy:

$$\Pr\left(p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j\right)$$

$$\cdot \left[ v(q^m_i, s_i^{MA}) - p^m_i \right] + \frac{\partial}{\partial q^m_i} \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ \mathbb{E}_{V_{RA}} \left( V_{RA} - \mathbb{P}_i ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j \right]$$

$$+ q^m_i \cdot \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ \mathbb{E}_{V_{RA}} \left( V_{RA} ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j \right]$$

$$= \Pr\left( P^c \leq p^{m+1}_i \right) \cdot \left[ q^m_i - p^{m+1}_i \right] - \frac{\partial}{\partial q^m_i} \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ Q^R_i \cdot \mathbb{E}_{V_{RA}} \left( V_{RA} ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i \right] \text{ if } m \in \{1, \ldots, M_i - 1\},$$

$$v(q^m_i, s_i^{MA}) + \frac{\partial}{\partial q^m_i} \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ \mathbb{E}_{V_{RA}} \left( V_{RA} ; V_{RA} - \mathbb{P}_i ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j \right]$$

$$+ q^m_i \cdot \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ \mathbb{E}_{V_{RA}} \left( V_{RA} ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j \right]$$

$$- q^m_i \cdot \left[ \mathbb{E}_{S_{-i} \mid s_i}\left[ \mathbb{E}_{V_{RA}} \left( V_{RA} ; V_{RA} > \mathbb{P}_i | s_i \right) \right] p^{m+1}_i < P^c < p^m_i \cap \sum_j y_j(P^c|s_i) > \sum_j Q^c_j \right]$$

$$= p^m_i \text{ if } m = M_i,$$

(6-bis)

where expression (6) and (6-bis) are derived in Appendix A.3.

The optimality conditions in Propositions 1-4 do not take into consideration the case of ties (i.e. the possibility that more than one bidder is rationed at the clearing price $P^c$). This simplifies significantly the optimality conditions because the terms with a rationed quantity vanish. Indeed, ignoring ties, when agent $i$ is rationed she is the only one to be rationed, and in this case a small perturbation of $q^m_i$ doesn’t change the allocation (and thus the payoff). Empirically, the exclusion of ties can be justified if ties occur with zero probability or if bidders ignore the effect of their demand on the quantity they are allocated in the case of a tie. In the empirical setting I study, while ties occur in virtually any auction of my dataset, from the perspective of an individual bidder the ex ante probability of tying at a step is rather low.
In the rest of this section I will provide the economic intuition behind the optimality conditions. To begin with, it is worth noting that a perturbation of $q^m_i$ impacts on the expected payoff of a generic bidder $i$, belonging either to the class of PDs or to the class of non-PDs, only in states where $P^c \leq p^m_i$ (because for states where the clearing price is higher than $p^m_i$, the $m$-th bid is not served). Abstracting from this common feature, the cost/benefit assessment of quantity shading differs in the two classes of operators.

For non-PDs the cost of shading $q^m_i$ is associated with losing the MA surplus $v_i(q^m_i, s^{MA}_i) - p^m_i$ on the shaded unit, which happens only in states where the $m$-th bid is marginal (i.e. $P^c \in [p^{m+1}_i, p^m_i]$), while the benefit is connected to saving on the difference between $p^m_i$ and $p^{m+1}_i$ on the shaded unit in states where the $m$-th bid is inframarginal (i.e. $P^c \leq p^{m+1}_i$).

For PDs the analysis of costs and benefits is more complex, as, in addition to the cost and benefit inside the MA (equivalent to those of non-PDs), shading the quantity $q^m_i$ provokes also changes in the RA expected payoff. As intuied by Marszalec [2009], it is possible to disentangle a quantity and, depending on the payment rule, a price effect associated with the shading.

The quantity effect reflects the impact of the perturbation of $q^m_i$ on agent $i$’s allocation inside the RA. This quantity effect is independent of the payment rule and entails a cost corresponding to losing the expected surplus $E_{s_i \in V_{RA}|s_i}(V^R_{t_{RA}} - P^R_{t_{RA}})$ on the shaded quantity in states where the $m$-th bid is marginal in the MA.

The price effect reflects the impact of quantity shading on the RA price and depends on the payment rule.

1. Under payment rule in Assumption [12] the RA price does not depend on bidders’ strategic behavior in the MA and consequently the price effect is null.

2. Under payment rule in Assumption [12-bis] the price effect, as the quantity effect, manifests only in states where the $m$-th bid is marginal in the MA because only in those states a perturbation of $q^m_i$ impacts on the clearing price $P^C$. This effect can be divided in two parts. The first is associated with the increase in the expected value of the option implicitly granted by the RA

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47In order to compute the ex ante likelihood of tying I used the formula proposed by [Kastl 2011]:

$$\frac{100 \cdot \text{avg # of tying bidders}}{\text{avg # of participating bidders}} \cdot \frac{\text{avg # of steps}}{\text{avg # of steps}}.$$  

48Indeed, in states where the $m$-th bid is not marginal, the shaded unit will be served in the $m+1$-th step, where the price is lower.
auction, the second with a decline in the expected cost of the RA allocation. Both these effects correspond to a benefit associated with shading $q^m_i$.

3. Under payment rule in Assumption 12-ter, the price effect manifests in any state where $P^c < p^m_i$, given that any winning bid contributes to determine the average winning price and thus the RA price. In all those states, shading the quantity $q^m_i$ provokes the increase in the expected value of the option implicitly granted by the RA auction and a decline in the expected cost of the RA allocation. Both these effects correspond to a benefit associated with shading $q^m_i$.

On the basis of the above considerations, it is possible to conclude that the overall net impact of the RA on bidding inside the MA depends on two factors.

1. The (marginal versus inframarginal) state of the bid. In states where the $m$-th bid is marginal, the quantity effect, which associates shading with a cost and thus incentives more aggressive bidding, is always present, independently of the payment rule. The price effect, instead, that associates shading with a benefit and thus spurs less aggressive bidding, may or may not be present, depending on how the RA price is formulated. In states where the $m$-th bid is inframarginal, instead, the introduction of the RA either does not impact on bidding or induces the bidder to request a smaller amount to diminish the RA price, depending on the RA payment rule. Consequently, while the importance of the quantity effect relative to the price effect can be determined only empirically on a case-by-case basis, it is a positive function of the ratio of the probability that the $m$-th bid is marginal over the probability that the bid is inframarginal. Let

$$\lambda_m = \frac{Pr(p_{m+1}^{i+1} < P^c < p^m_i)}{Pr(P^c \leq p^{m+1}_i)}$$

denotes this ratio at the $m$-th step, from Assumptions 4 and 6, it follows that $\lambda_{m+1} > \lambda_m$ for any $m \in \{1, \ldots, M_i - 1\}$. Thus, bidder $i$ has increasing incentives to bid more aggressively as $m$ increases (i.e. in lower steps of her demand function).

2. The payment rule. How the RA price is determined influences critically bidding inside the MA. In particular, the less the RA price is dependent on bidders’ strategies, the higher is the incentive to bid more aggressively because the quantity effect dominates. Consequently, the more aggressive bidding should be expected, ceteris paribus, under payment rule 12 and the least aggressive bidding under payment rule 12-ter.
V. THE AUCTION FRAMEWORK FOR ITALIAN TREASURY BILLS

V.1. The Auction Mechanism

The Italian government issues bills (*Buoni Ordinari del Tesoro* - BOTs) with three standard maturities: three-, six-, and twelve-months. The bills are all sold via a discriminatory auction mechanism (*asta competitiva*), but while six- and twelve-month BOTs are issued at regular intervals —with the twelve-month BOTs issued in the second week of every month, and the six-month auctioned in the fourth week—, three-month BOTs, as well as BOTs with atypical maturity, are issued periodically to fund short-term cash needs of the Treasury. The auctions are conducted by Bank of Italy and only authorized dealers can participate in the auction (Section V.2). Reopenings reserved to PDs (called *Specialists in Government securities*) are envisaged only for six- and, starting with February 2009, twelve-months bills. In the reopening auction the amount supplied by the Treasury is the 10 percent of the face value offered in the base auction.

The auction mechanism that I describe in this section refers to the mechanism in place during the period of observation (January 2007-April 2010).

Every participant can submit a maximum of 3 bids before 11:00 am of the auction day (*cut-off time*). Each bid is a price-quantity pair, where prices must be specified with 3-decimal precision (i.e. the minimum increment is 0.001) The minimum quantity a bidder can request is 1.5 million Euros, whereas the maximum admissible requested amount is equal to the quantity offered by the Treasury during the auction. The first bids to be accepted are those with the highest price and then all the others are allocated in descending order until the amount of accepted bids reaches the amount tendered by the Treasury. Given that the auction is discriminatory, every accepted bid is settled at the requested price. If the total requested quantity is greater than the amount offered, the bids made at the lowest accepted price will be allocated *pro-quota*.

During the period of observation, the allocation mechanism underwent some modifications. An important change was the introduction of reopening auctions for twelve-month BOTs in February 2009 reserved to PDs. Another reform was introduced in April 2009, requiring, in conformity with current international practice, bids to be expressed as yield-quantity pairs rather than price-quantity pairs.

V.2. Bidders

Banks and investment firms that intend to participate in the Italian Treasury auctions must satisfy certain legal, technical, and operational requirements and

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49Starting with May 2010 the number of bids has been extended to 5.
50Operators must be authorized to exercise at least one of the activities indicated in Article 1.5 of Legislative Decree 58 of 24 February 1998 (the "Consolidated Law on Financial Intermediation").
must sign the Agreement with the Bank of Italy.

While all the registered firms are allowed to participate in the auctions, activity inside the auction has been increasingly dominated by PDs. These, active on the Italian regulated government bond wholesale secondary market Mercato Telematico dei titoli di Stato (MTS), are selected by the Department of the Treasury (DoT) in accordance with Ministerial Decree no. 219 of 13 May 1999, in order to guarantee a high level of efficiency and transparency to the Italian Government securities market. To obtain a PD status, a bidding institution must apply for a license from the DoT, and satisfy certain requirements. The status of PD implies several obligations and privileges. The most important obligation (necessary condition for retaining the status of PD) is attaining the minimum annual primary market share of 3 percent of the bonds issued by the Ministry of the Economy and Finance on the primary market. Between the privileges there is the exclusive access to reopenings following the auctions of medium-long-term bonds and six- and twelve-month BOTs.

VI. Empirical Analysis of the Impact of Reopening Auctions on Bidders’ Strategic Behavior

VI.1. Data

My data set contains information on 331 competitive auctions of Italian Treasury bills and covers the period January 2007-April 2010. The sample includes 29 three-month, 40 six-month, and 40 twelve-month BOT auctions, as well as 23 auctions of bills with atypical maturity (See Table 1). The bid-to-cover ratio averages 1.5 for six- and twelve-months BOT auctions and 2 for three-months BOT auctions and auctions for bills with atypical maturity, and it is superior to 1 in all the competitive placements in the sample, which means that no auctions have been under-subscribed. The data set contains all the bids associated with these auctions.

On average, Specialists submit three bids while non-Specialists two. In most of the cases, Specialists submit more than one bid (above 95 percent of the cases

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In addition, for the purposes of participating in tenders, the intermediary must be able to send and receive messages on the National Interbank Network, while, for the purposes of settling the items assigned in the auction, the intermediary must be a participant in Monte Titoli SpA system or must appoint an intermediary (the “settlement agent”) and have the latter complete the form accepting the engagement. Finally, in order to verify the correct management of the messages on the Network, each intermediary intending to participate in tenders must perform tests with the Bank of Italy.

51 This is a regulated market pursuant to Section 66 of the Legislative Decree N.58, 24/2/1998, and operated by MTS S.p.A. under the supervision of Bank of Italy and CONSOB.

52 Over the years obligations have become increasingly stringent, and include qualitative aspects - such as, for example, the use at each auction of all three available bids and their relative distribution - as well as quantitative criteria aimed at discouraging bidding behaviors that result to be distortive of auction outcomes.
for all maturities), non-Specialists, instead, submit one bid in half of the cases (57 percent). The thrust of this finding is that Specialists submit a demand schedule as a rule, differently from non-Specialists. The distribution of the number of dealers who actually participate in the auctions is rather stable across auctions and is dominated by the participation of Specialists.

The dataset contains also information on 55 topup auctions, of which 40 associated with six-month and 15 with twelve-month placements. In most of the reopening auctions (80 percent of the cases) either all PDs exercise their right to buy or none exercise it (See Table 2). This finding supports the assumption of common value inside the reopening auction (Assumption 11). In addition, several of the auctions with partial exercise were almost fully covered, likely reflecting cases where the when-issued or secondary market price was in line with the topup price.

VI.2. Estimation Method and Results

Recent empirical literature on auctions adopts the structural econometric approach, which assumes that observed bids are generated from the equilibrium conditions of the auction model under consideration. This approach implies addressing the problem of identification, which consists –given the equilibrium conditions– in being able to uniquely recover the latent structural variables from the observables.

In the model with reopening auctions the identification problem is complicated by the coexistence of common and idiosyncratic components in PD evaluations. In this context, a fully non-parametric identification approach, commonly used in PV settings, is impracticable without adding further restrictions to the model. Indeed, even recovering the distribution of bids from the observable, a non-parametric method would leave the model unidentified as there are a number of private marginal valuation functions \( v_i(q_m^i, s_{MA}^i) \) and distribution functions \( F_{S_{RA|i}, V_{RA|i}} \) that could satisfy PD optimality condition. In addition, it must be noted that, while generally non-parametric identification is considered preferable because free, by definition, from misspecification errors, in the case of the model with reopening auctions, the level of indeterminacy is so high that non-parametric identification would be possible only adding very restrictive assumptions that could hamper severely the realism of the model.
Table 1: Italian Treasury Bill Auctions: Summary Statistics of Competitive Placements

<table>
<thead>
<tr>
<th>Number of Auctions</th>
<th>3-Month Bills</th>
<th>6-Month Bills</th>
<th>12-Month Bills</th>
<th>Atypical Maturity Bills</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Count</td>
<td>%</td>
<td>Count</td>
<td>%</td>
</tr>
<tr>
<td>Total</td>
<td>29</td>
<td></td>
<td>40</td>
<td></td>
</tr>
<tr>
<td>Totally Served</td>
<td>28</td>
<td>100.0%</td>
<td>40</td>
<td>100.0%</td>
</tr>
<tr>
<td>Partially Served</td>
<td>0</td>
<td>0.0%</td>
<td>0</td>
<td>0.0%</td>
</tr>
<tr>
<td>Bid-to-Cover Ratio</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td></td>
<td>2.0</td>
<td>0.36</td>
<td>1.4</td>
<td>0.19</td>
</tr>
<tr>
<td>Number of Bidders</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>Specialists</td>
<td>21.8</td>
<td>2.1</td>
<td>21.5</td>
<td>1.4</td>
</tr>
<tr>
<td>Non-Specialists</td>
<td>8.5</td>
<td>1.8</td>
<td>8.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Number of Bids per Bidder</td>
<td>Mean</td>
<td>Std. Dev.</td>
<td>Mean</td>
<td>Std. Dev.</td>
</tr>
<tr>
<td>Specialists</td>
<td>3.0</td>
<td>0.3</td>
<td>3.0</td>
<td>0.2</td>
</tr>
<tr>
<td>Non-Specialists</td>
<td>1.9</td>
<td>0.9</td>
<td>2.0</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Table 2: Italian Treasury Bill Auctions: Participation in Topup Auctions

<table>
<thead>
<tr>
<th></th>
<th>6-Month Bills</th>
<th>12-Month Bills</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Count</td>
<td>%</td>
</tr>
<tr>
<td>Full Exercise</td>
<td>16</td>
<td>40.0%</td>
</tr>
<tr>
<td>No Exercise</td>
<td>17</td>
<td>42.5%</td>
</tr>
<tr>
<td>Partial Exercise</td>
<td>7</td>
<td>17.5%</td>
</tr>
</tbody>
</table>
Based on this considerations, I leave the issue of identification of the model to future research and in this paper I use a reduced form approach to evaluate the effects of topups on bidders’ strategic behavior inside the MA. As estimation method, I adopt a difference-in-difference (DID) approach, which is typically used to estimate the effect of a specific intervention or treatment (in our case the introduction of topup auctions) by comparing the changes in outcomes over time between a population that is exposed to the intervention (in our case PDs) and a population that is not (in our case non-PDs). This approach suits well the Italian Treasury bill auction dataset, given that the introduction of reopening auctions reserved to PD for twelve-month bills in February 2009 offers a “natural experiment” environment to evaluate the effects of topups on bidders’ strategic behavior inside the MA.

I limit the analysis to three-bid demand schedules and apply the DID method to both cumulated quantity and price bids ($q^m_i$ and $p^m_i$, respectively) by running the following regressions:

\[
q^m_i = \alpha_0^m + \alpha_1^m \cdot D_T + \alpha_2^m \cdot D_{PD} + \alpha_3^m \cdot (D_T \cdot D_{PD}) \\
p^m_i = \beta_0^m + \beta_1^m \cdot D_T + \beta_2^m \cdot D_{PD} + \beta_3^m \cdot (D_T \cdot D_{PD})
\]

with $i \in \{1, \cdots, N\}$ and $m \in \{1, 2, 3\}$, where $D_T$ is a time dummy equal to 0 before February 2009 and to 1 otherwise, $D_{PD}$ is a state dummy equal to 0 for non-PDs and 1 for PDs, $e D_T \cdot D_{PD}$ is the interaction of the two dummies. The time dummy controls for biases from comparisons over time in the PD group that could be the result of trends due to causes unrelated to the introduction of reopening auctions. The status dummy removes biases in post-intervention period comparisons between PDs and non-PDs that could be the result of permanent differences between those groups. Finally, the interaction is the DID term, which identifies the impact of RA on quantity and prices. In this regression, $\alpha_0^m$ and $\beta_0^m$ are the average quantity and price for bid $m$, $\alpha_1^m$ and $\beta_1^m$ represent the time trends in the non-PD group, $\alpha_2^m$ and $\beta_2^m$ represent the differences between PDs and non-PDs before the introduction of the RA, and $\alpha_3^m$ and $\beta_3^m$ represent the difference in the changes over time to be attributed to the introduction of the RA.

The DID relies on the assumption that in absence of treatment, the unobserved differences between treatment and control groups are the same overtime. To meet this condition, I control from fluctuations in market conditions that could impact on the common value of auctioned bills by dividing price bids by the average price in the secondary market on the day of the auction.

Table 3 reports regressions’ estimates. This analysis permits to identify two main findings:

1. The introduction of RA has had a positive impact on PDs’ requested cumulated quantities, as shown by a positive significant coefficient for the DID term for all bids in the demand schedule. This impact is higher the the higher is
the probability that the bid is marginal, as illustrated by a higher DID term coefficient for the second bid compared to the first (0.31 against 0.18) and for the third bid compared to the second (0.40 compared to 0.31).

2. The introduction of RA has had a negative impact on PDs’ price bids different from the first, as shown by a negative significant coefficient for the second and third bids in the demand schedule. This impact is higher the the higher is the probability that the bid is marginal, as illustrated by a more negative DID term coefficient for the third bid compared to the second (-0.24 compared to -0.18).

These findings are in line with the theoretical model, which predicts higher requested quantities the higher is the likelihood that the bid is marginal. In addition, estimation results confirm that the RA payment rule has a role in determining bidding behavior inside the MA. In particular, the payment rule adopted in the context of the Italian Treasury BOT auctions, which sets the RA price equal to the average winning price in the MA, incentives PD to shade price bids to reduce the RA price.

To analyze whether the quantity or price effect is dominant, I apply the DID method to the cumulated value of bids by running the following regression.

\[ v_{i}^{m} = \sum_{j=1}^{m} d_{i}^{j} \cdot p_{i}^{j} = \gamma_{0}^{m} + \gamma_{1}^{m} \cdot D_{T} + \gamma_{2}^{m} \cdot D_{PD} + \gamma_{3}^{m} \cdot (D_{T} \cdot D_{PD}) \]

Table 4 shows that the introduction of RA has had a positive impact on PDs’ requested value for all bids, as illustrated by a positive significant DID term coefficient. This analysis permits to conclude that despite a RA payment rule that introduces incentives to shade price bids, overall the introduction of RA for twelve-month BOT auctions has prompted more aggressive bidding.
Table 3: DID Estimates Applied to Cumulative Quantity and Price Bids

<table>
<thead>
<tr>
<th>First Bid</th>
<th>Second Bid</th>
<th>Third Bid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dependent Variable: ( p^1_i )</td>
<td>Dependent Variable: ( q^1_i )</td>
<td>Dependent Variable: ( p^3_i )</td>
</tr>
<tr>
<td>( \text{Coefficient} )</td>
<td>( \text{Std. Error} )</td>
<td>( \text{t-Statistic} )</td>
</tr>
<tr>
<td>TIME Dummy</td>
<td>-0.004</td>
<td>0.008</td>
</tr>
<tr>
<td>PD Dummy</td>
<td>0.032***</td>
<td>0.004***</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>-0.00114</td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.041214</td>
<td></td>
</tr>
<tr>
<td>Log likelihood</td>
<td>1630.357</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dependent Variable: ( q^2_i )</th>
<th>Dependent Variable: ( q^3_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Coefficient} )</td>
<td>( \text{Std. Error} )</td>
</tr>
<tr>
<td>TIME Dummy</td>
<td>0.025**</td>
</tr>
<tr>
<td>PD Dummy</td>
<td>0.032***</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>0.187514</td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.035129</td>
</tr>
<tr>
<td>Log likelihood</td>
<td>1753.325</td>
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<tr>
<td>Prob(F-statistic)</td>
<td>0.000000</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Dependent Variable: ( q^4_i )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{Coefficient} )</td>
</tr>
<tr>
<td>TIME Dummy</td>
</tr>
<tr>
<td>PD Dummy</td>
</tr>
<tr>
<td>ID Term</td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
</tr>
<tr>
<td>S.E. of regression</td>
</tr>
<tr>
<td>Log likelihood</td>
</tr>
<tr>
<td>Prob(F-statistic)</td>
</tr>
</tbody>
</table>
### Table 4: DID Estimates Applied to Cumulative Value Bids

<table>
<thead>
<tr>
<th></th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>t-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>First Bid</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TIME Dummy</td>
<td>-0.009*</td>
<td>0.005</td>
<td>-1.690</td>
<td>0.091</td>
</tr>
<tr>
<td>PD Dummy</td>
<td>0.015***</td>
<td>0.002</td>
<td>6.406</td>
<td>0.000</td>
</tr>
<tr>
<td>DID Term</td>
<td>0.018***</td>
<td>0.005</td>
<td>3.402</td>
<td>0.001</td>
</tr>
<tr>
<td>C</td>
<td>0.011***</td>
<td>0.002</td>
<td>4.757</td>
<td>0.000</td>
</tr>
<tr>
<td>Adjusted R-squared</td>
<td>0.120</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S.E. of regression</td>
<td>0.020</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Log likelihood</td>
<td>2279.201</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Prob(F-statistic)</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Second Bid</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TIME Dummy</td>
<td>-0.013*</td>
<td>0.007</td>
<td>-1.956</td>
<td>0.051</td>
</tr>
<tr>
<td>PD Dummy</td>
<td>0.024***</td>
<td>0.003</td>
<td>7.465</td>
<td>0.000</td>
</tr>
<tr>
<td>DID Term</td>
<td>0.032***</td>
<td>0.007</td>
<td>4.457</td>
<td>0.000</td>
</tr>
<tr>
<td>C</td>
<td>0.017</td>
<td>0.003</td>
<td>5.831</td>
<td>0.000</td>
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<tr>
<td>Adjusted R-squared</td>
<td>0.185</td>
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<td>S.E. of regression</td>
<td>0.027</td>
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<tr>
<td>Log likelihood</td>
<td>2014.002</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Prob(F-statistic)</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Third Bid</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TIME Dummy</td>
<td>-0.018**</td>
<td>0.009</td>
<td>-2.056</td>
<td>0.040</td>
</tr>
<tr>
<td>PD Dummy</td>
<td>0.038***</td>
<td>0.004</td>
<td>9.006</td>
<td>0.000</td>
</tr>
<tr>
<td>DID Term</td>
<td>0.041***</td>
<td>0.009</td>
<td>4.421</td>
<td>0.000</td>
</tr>
<tr>
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<td>6.650</td>
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<td>Adjusted R-squared</td>
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<tr>
<td>S.E. of regression</td>
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<td>Prob(F-statistic)</td>
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### VII. Conclusions

The paper builds a model of Treasury auctions where the Sovereign sells a particular issuance of debt securities in two sessions. The first session is a competitive placement (main auction) while the second is a non-competitive placement (reopening auction), where admitted bidders have the right to buy an additional amount of securities, proportional to the amount won in the main auction, for a price set on the competitive auction date. The equilibrium characterization is provided under a discriminatory auction mechanism for the competitive placement and three different payment rules for the non-competitive round.

I show that the introduction of the non-competitive placement impacts on bidding
in the competitive placement through two channels that act in opposite direction: a quantity effect, which operates when the bid is marginal, which leads bidders to ask for higher amount to extract more surplus in the the second round, and a price effect, depending on the reopening auction payment rule, which induces bidders to request smaller amounts to reduce the price to be paid in the non-competitive round. While the importance of the quantity effect relative to the price effect can be determined only empirically on a case-by-case basis, more aggressive bidding is expected when the probability that the bid is marginal is higher (i.e. in lower steps of bidder \textit{i}'s demand function) and the less the reopening auction price is dependent on bidders’ strategies inside the main auction.

The empirical analysis is conducted using a reduced form approach that employs a difference-in-difference method applied to Italian twelve-month Treasury bill auctions, where the main auction uses a discriminatory price mechanism and the reopening price is set equal to the average winning price in the main auction. The analysis confirms that the introduction of reopening auctions in 2009 has introduced incentives that act in two opposite directions but the incentive to bid more aggressively appears to be dominant.

The identification of the model with reopening auctions and the adoption of a structural econometric approach are left for future research.
REFERENCES


A. Derivation of Optimality Conditions

The optimality conditions for a generic bidder belonging to the class of non-PDs [expression (3) and (3-bis)] are derived in Kastl [2012] from a method that permits the author to circumvent the difficulties related to the differentiation of the piece-wise objective function.

The method focuses on bidder \( i \)'s strategic decision on where to locate the \( m \)-th step of her demand function (with \( m = 1, \ldots, M_i \)) and is based on the condition that rules out profitable local perturbations of \( q^m_i \). The procedure consists of two steps. In the first step, for each bid-point \((q^m_i, p^m_i)\) presented by agent \( i \), Kastl creates a partition of the state space into cells such as that within each cell bidder \( i \)'s payoff function is continuous and differentiable with respect to the control variable \( q^m_i \); then, by applying the law of total probability, he rewrites the expected payoff function as a weighted average of the payoffs within each cell. In the second step, Kastl identifies the optimality conditions, studying the effects of an infinitesimal perturbation of \( q^m_i \) on agent’s \( i \) payoff in each cell of the partition.

In the rest of the Appendix, I will extend Kastl’s method to derive the optimality conditions for a generic agent belonging to the class of PDs under different payment rules in the RA. Because PDs’ optimality conditions for the MA are equivalent to non-PDs’, I will focus only on the RA payoff function.

RA allocations (Assumption 9) and prices (under payment rules in Assumptions 12-bis and 12-ter) depend on the MA outcome. Therefore, I define a partition of the possible MA outcomes in a way that within each cell of the partition the continuity and differentiability of the RA payoff function under the different payment rules is granted. I use the partition in Kastl [2012] as a starting point. This partition ensures the continuity and differentiability of the RA payoff function under the payment rules in Assumptions 12 and 12-bis and can be used to derive the optimality conditions 4, 4-bis, 5 and 5-bis. However, under the payment rule in Assumption 12-ter, this partition ensures only the continuity of bidder \( i \)'s total allocation \( Q^c_i \) within each cell. To guarantee the continuity of the topup price (corresponding to the average winning price \( \bar{p} \) under Assumption 12-ter), I will need to introduce some corrections to make the partition finer (see Section A.3).

The partition in Kastl [2012] divides the sample space \( \Omega_{S_{-i}} \) for the random vector \( S_{-i} \) in five subsets corresponding to the following events:

1. \( (q^m_i, p^m_i) \) is not winning. The subset of \( \Omega_{S_{-i}} \) corresponding to this event is:

\[
\Omega^1_{S_{-i}}(q^m_i) = \left\{ s_{-i} : P^c > p^m_i \cap Q^c_i < q^m_i \right\}
\]

This set includes all the \( s_{-i} \) at which the residual supply cuts the step-wise demand function of bidder \( i \) at a price \( P^c > p^m_i \). In this case the bid \((q^m_i, p^m_i)\)
is not served.

2. \((q^m_i, p^m_i)\) is agent \(i\)'s lowest winning (or marginal) bid and it is rationed. The subset of \(\Omega_{S_{-i}}\) corresponding to this event is:

\[
\Omega^2_{S_{-i}}(q^m_i) = \left\{ s_{-i} : P^c = p^m_i \cap Q^c_i = Q^{\text{RAT}}_i < q^m_i \right\}
\]

This set includes all the \(s_{-i}\) at which the residual supply cuts the step-wise demand function of bidder \(i\) at the price \(P^c = p^m_i\) but bidder \(i\) is rationed on the marginal requested quantity \(q^m_i - q^{m-1}_i\).

3. \((q^m_i, p^m_i)\) is agent \(i\)'s lowest winning (or marginal) bid and it is served in full. The subset of \(\Omega_{S_{-i}}\) corresponding to this event is:

\[
\Omega^3_{S_{-i}}(q^m_i) = \left\{ s_{-i} : p^{m+1}_i < P^c \leq p^m_i \cap Q^c_i = q^m_i \right\}
\]

This set includes all the \(s_{-i}\) at which the residual supply cuts the step-wise demand function of bidder \(i\) at the price \(P^c \in [p^{m+1}_i, p^m_i]\) and the marginal requested quantity \(q^m_i - q^{m-1}_i\) is served in full.

4. \((q^m_i, p^m_i)\) is agent \(i\)'s second lowest winning bid and there is a tie at \(p^{m+1}_i\). The subset of \(\Omega_{S_{-i}}\) corresponding to this event is:

\[
\Omega^4_{S_{-i}}(q^m_i) = \left\{ s_{-i} : P^c = p^{m+1}_i \cap Q^c_i = Q^{\text{RAT}}_i < q^{m+1}_i \cap \text{there is a tie at } p^{m+1}_i \right\}
\]

This set includes all the \(s_{-i}\) at which the residual supply cuts the step-wise demand function of bidder \(i\) at the price \(P^c = p^{m+1}_i\) and multiple bidders are marginal at \(p^{m+1}_i\). In this case the bid \((q^m_i, p^m_i)\) is served in full while the marginal requested quantity \(q^{m+1}_i - q^m_i\) is rationed pro-rata.

5. \((q^m_i, p^m_i)\) is winning but it is not agent \(i\)'s marginal bid and there is not a tie at \(p^{m+1}_i\). The subset of \(\Omega_{S_{-i}}\) corresponding to this event is:

\[
\Omega^5_{S_{-i}}(q^m_i) = \left\{ s_{-i} : P^c \leq p^{m+1}_i \cap Q^c_i > q^m_i \cap \text{there is not a tie at } p^{m+1}_i \right\}
\]

This set includes all the \(s_{-i}\) at which the residual supply cuts the step-wise demand function of bidder \(i\) at a price \(P^c \leq p^{m+1}_i\) and there is not a tie at \(p^{m+1}_i\). The bid \((q^m_i, p^m_i)\) is served in full.

It is relevant to highlight that subsets \(\Omega_1-\Omega_4\) correspond to states \(s_{-i}\) where the bid-to-cover ratio is equal or superior to 1, which implies that \(\sum_{j=1}^N Q^c_j = Q\).
those cases the RA allocation is equal to $\delta \cdot Q^*_i$ (Assumption 9). States in subset $\Omega_5$, instead, may be associated with a bid-to-cover ratio that may or may not be equal or superior to 1.

Using this partition of the set $\Omega_{S_{-i}}$, it is possible to rewrite agent’s $i$ expected RA payoff associated with bid point $(q^m_i, p^m_i)$ (with $m = 1, \ldots, M_i - 1$) as a weighted average of the expected payoffs within each cell of the partition:

$$
\Pi^{RA}_i(s_i | q^m_i) = \sum_{j=1}^{5} P(\Omega^j) \cdot E_{S_{-i} | s_i} \left[ E_{V_{tRA}} \left( \Pi^{RA}_i | S_{-i}, s_i \right) : \Omega^j \right] 
$$

where the first equality derives from the application of the law of total probability and the law of iterated expectations, with the external expectation taken with respect to $S_{-i}$ and the inner expectation with respect to $V_{tRA}$, and the second equality follows by the Bayes’s rule (where I adopt the notation $E(X; E)$ to denote $\int_E x dx$). To simplify notation I have dropped $(q^m_i)$ from the argument of $\Omega_j$ ($j = 1, \ldots, 5$).

Now, in order to find agent $i$’s optimality conditions inside the RA, Kastl [2012] perturbs the $m$-th step to $\bar{q} = q^m_i - \epsilon$ and studies the effects of this deviation on the agent’s expected payoff. This perturbation is likely to provoke some reshuffling of states $s_{-i}$ between the subsets $\Omega_{S_{-i}}$. In particular, it is possible to define the following sets of potential transferred states: the set $\omega^{2,3}(\bar{q})$ which includes all the states $s_{-i}$ transferred from $\Omega^2_{S_{-i}}(q^m_i)$ to $\Omega^3_{S_{-i}}(\bar{q})$, the set $\omega^{2,4,5}(\bar{q})$ which includes all the states $s_{-i}$ transferred from $\Omega^2_{S_{-i}}(q^m_i)$ to $\Omega^4_{S_{-i}}(\bar{q}) \cup \Omega^5_{S_{-i}}(\bar{q})$, and the sets $\omega^{3,4,5}(\bar{q})$ which includes all the states $s_{-i}$ transferred from $\Omega^3_{S_{-i}}(q^m_i)$ to $\Omega^4_{S_{-i}}(\bar{q}) \cup \Omega^5_{S_{-i}}(\bar{q})$. The set $\Omega^1_{S_{-i}}$ is not impacted by the perturbation. The partition of $\Omega_{S_{-i}}$ based on the perturbed bid $(\bar{q}, p^m_i)$ satisfies:

$$
\begin{align*}
\Omega^1_{S_{-i}}(\bar{q}) &= \Omega^1_{S_{-i}}(q^m_i) \\
\Omega^2_{S_{-i}}(\bar{q}) &= \Omega^2_{S_{-i}}(q^m_i) \setminus \omega^{2,3}(\bar{q}) \setminus \omega^{2,4,5}(\bar{q}) \\
\Omega^3_{S_{-i}}(\bar{q}) &= \Omega^3_{S_{-i}}(q^m_i) \cup \omega^{2,3}(\bar{q}) \setminus \omega^{3,4,5}(\bar{q}) \\
\Omega^4_{S_{-i}}(\bar{q}) \cup \Omega^5_{S_{-i}}(\bar{q}) &= \Omega^4_{S_{-i}}(q^m_i) \cup \Omega^5_{S_{-i}}(q^m_i) \cup \omega^{2,4,5}(\bar{q}) \cup \omega^{3,4,5}(\bar{q})
\end{align*}
$$

Using the relations (iv) between the original partition and the perturbed one, it is possible to find the condition that rules out profitable perturbations of $q^m_i$, overcoming the difficulty that $Q^*_i$ and $P^{RA}$ are not continuous over the cells of the partition.\footnote{For details see Kastl [2011], Appendix A page 1007.}

In particular, for all those states $s_{-i}$ that don’t change cell of the partition after

\footnote{From the Bayes’ rule it follows that $E(X; E) = Pr(E) \cdot E(X | E)$.}
the perturbation, the payoff function is continuous and differentiable with respect to \( q_i^m \). The continuity of the payoff function is, however, not preserved in those states \( s_{-i} \) that change cell of the partition when the perturbation occurs.

This notwithstanding, it is possible to make some reasonable assumptions on the behavior of these sets of transferred states when \( \epsilon \to 0 \) (i.e. when the perturbation of \( q_i^m \) is infinitesimal) in a way that guarantees that continuity is preserved. \(^{55}\) Kastl [2012] assumes that \( \lim_{\epsilon \to 0} Pr(\omega^2) = \lim_{\epsilon \to 0} Pr(\omega^3) = 0 \) (i.e. after the infinitesimal perturbation, the allocation \( \hat{q} \) remains with probability 1 in \( \Omega^2 \), and \( \Omega^3 \) respectively), while \( \lim_{\epsilon \to 0} Pr(\omega^3) > 0 \). The latter case coincides with knife-edge solutions where \( \hat{q} \) is completely served but the clearing price \( P^c \) remains equal to \( p_i^m \).

Keeping these observations in mind, the effect of an infinitesimal perturbation in \( q_i^m \) on agent \( i \)'s expected payoff inside the RA under payment rules in Assumptions [12 and [12-bis] can be expressed through the following limit:

\[
\lim_{\epsilon \to 0} \frac{E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(q_i^m, s_i)|s_i) \right] - E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(q_i^m, s_i)|s_i) \right]}{\epsilon} = \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^2) \right] \right] \\
= \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i) - \Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^2) \right] \right] \\
+ \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(q_i^m, s_i) - \Pi_{tRA}(q_i^m, s_i)|s_i); \Omega^3) \right] \right] \\
+ \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i) - \Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^4) \right] \right] \\
- \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i) - \Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^3) \right] \right] \\
+ \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^2) \right] \right] \\
+ \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^3) \right] \right] \\
+ \lim_{\epsilon \to 0} \left[ E_{s_{-i}|s_i} \left[ E_{V_{i,R_A}}(\Pi_{tRA}(Q_{tRA}(q_i^m, s_i)|s_i); \Omega^4) \right] \right] \\
\text{(8)}
\]

where the first equality follows by the fact that perturbing the \( m \)-step \( q_i^m \) does

\(^{55}\)In this case continuity is preserved because after the infinitesimal perturbation the allocation remains inside the step corresponding to \( p_i^m \) and within each step the payoff function is continuous.
not alter the payoff function in sets $\Omega^1$ and $\Omega^5$ and the probabilities of sets $\Omega_j$. The second equality follows by the application of the relations (10)-(11) between the original partition and the perturbed one.

A.1. Proof of Propositions 2

Using expression (7), the RA payoff function under the payment rule in Assumption 12 can be written as:

$$
\Pi_i^{RA}(s_i|q^m_i) = \delta \cdot E_{V_{i,MA},V_{i,RA}} \left[ I_{\{V_{i,RA} > V_{i,MA}\}} \left( V_{i,RA} - V_{i,MA} \right) | s_i \right] \cdot \left[ f(q_1^i, \cdots, q_{m-1}^i) \right] + E_{S_{-i}|s_i} \left[ Q_i^{RA}(q^m_i); \Omega^2 \right] + E_{S_{-i}|s_i} \left[ \delta \cdot Q_i^{RA}(q^m_i); \Omega^2 \right] + E_{S_{-i}|s_i} \left[ Q_i^{RA}(q^m_i+1); \Omega^4 \right] + E_{S_{-i}|s_i} \left[ Q_i^c; \Omega^5 \right]
$$

where $f(q_1^i, \cdots, q_{m-1}^i|\Omega^1)$ is a function which doesn’t depend on $q^m_i$, $Q_i^{RA}(q^j_i)$ is the total share bidder $i$ obtains in the MA when the marginal bid $q^j_i - q^{j-1}_i$ is rationed (with $j = 1, \cdots, M_i - 1$, and $q^0_i = 0$) and $Q_i^c$ is the bidder $i$’s MA allocation.

In [9], the expected payoff depends on the partition of the state space through the RA allocation but not the price. The allocation depends on $q^m_i$ in sets $\Omega^2$, $\Omega^3$, and $\Omega^4$ and is a random variable in all states $S_{-i}$ but in $\Omega^5$. The RA price doesn’t depend on $q^m_i$ and is a random variable in all states $S_{-i}$.

Using expression (8) to study the effect of an infinitesimal perturbation in $q^m_i$ on agent $i$’s expected payoff in the RA [expression (9)] and referring to the derivative definition, the limit can be re-written as:

$$
\delta \cdot E_{V_{i,MA},V_{i,RA}} \left( V_{i,RA} - V_{i,MA}; V_{i,RA} > V_{i,MA} | s_i \right) \cdot \left[ E_{S_{-i}|s_i} \left( \frac{\partial Q_i^{RA}(q^m_i)}{\partial q^m_i}; \Omega^2 \right) \right] + Pr(\Omega^3 \cup \omega^{2-3}) + E_{S_{-i}|s_i} \left( \frac{\partial Q_i^{RA}(q^m_i+1)}{\partial q^m_i}; \Omega^4 \right)
$$

where I used the definition of indicator function$^{56}$ and the limit assumptions on the sets $\omega^{2-3}$, $\omega^{2-4-5}$, and $\omega^{3-4-5}$.

As specified in paragraph [IV.2.1] I will ignore the case of ties. Consequently, the

---

$^{56}$Invoking the definition of indicator function, the expected utility of a bidder $i$ belonging to the class of PD in the RA can be rewritten as:

$$
\Pi_i^{RA}(s_i) = E_{S_{-i}|s_i} \left[ E_{V_{i,MA},V_{i,RA}} \left[ I_{\{V_{i,RA} > V_{i,MA}\}} \delta \cdot Q_i^c; \left( V_{i,RA} - V_{i,MA} \right) | s_i \right] \right] = \delta \cdot E_{V_{i,MA},V_{i,RA}} \left[ V_{i,RA} - V_{i,MA}; V_{i,RA} > V_{i,MA} | s_i \right] \cdot E_{S_{-i}|s_i} \left( Q_i^c \right)
$$
terms involving the derivative of the rationed quantity vanish, because if agent $i$ is
the only agent to be rationed then $\frac{\partial q_i^{RAT}}{\partial q_i} = 0$. Now, putting together the derivative of
the expected payoff inside the MA with respect to $q_i^m$ [expression 6] with the
derivative inside the RA [expression (10)], rewriting the $\Omega^j$ with the corresponding
states of market clearing prices, noting that under Assumption 3 $Pr(P^C = p_i^m \cap Q_i^c =
q_i^m) = 0$, and rearranging terms, I obtain the analogous of the FOCs for a generic
PD:

$$Pr\left(p_i^{m+1} < P^c < p_i^m\right)$$

$$\cdot \left[v_i(q_i^m, s_i^{MA}) - p_i^m\right] + \delta \cdot E_{V_i^{MA},V_i^{RA}}(V_{t^{RA}} - V_{t^{MA}}; V_{t^{RA}} > V_{t^{MA}} | s_i)$$

$$= Pr(P^c \leq p_i^{m+1}) \cdot (p_i^m - p_i^{m+1})$$

which corresponds to expression 4 in the main text.

Finally at the last step $\overline{M}_i$, there is not next step so the optimality conditions are:

$$v_i(q_i^m, s_i^{MA}) + \delta \cdot E_{V_i^{MA},V_i^{RA}}(V_{t^{RA}} - V_{t^{MA}}; V_{t^{RA}} > V_{t^{MA}} | s_i)$$

$$= p_i^m \quad \text{if } m = \overline{M}_i,$$

which corresponds to expression 4-bis in the main text. QED

A.2. Proof of Propositions 2bis

Using expression 7 I can rewrite the RA payoff function under the payment rule
in Assumption 12-bis as:

$$\begin{align*}
\Pi_i^{RA}(s_i | q_i^m) &= \delta \cdot f(q_1^1, \ldots, q_i^{m-1}) \\
&+ E_{V_i^{RA}} \left[I_{\{V_{t^{RA}} > p_i^m\}} \left(V_{t^{RA}} - p_i^m\right) \mid s_i\right] \cdot E_{S_{-i} \mid s_i} \left[Q_i^{RAT}(q_i^m) ; \Omega^2\right] \\
&+ q_i^m \cdot E_{S_{-i} \mid s_i} \left[\left[\left. E_{V_i^{RA}} \left[I_{\{V_{t^{RA}} > P^c\}} \left(V_{t^{RA}} - P^c\right) \mid s_i\right) \right] \left[Q_i^{RAT}(q_i^m) ; \Omega^3\right]\right] \\
&+ E_{V_i^{RA}} \left[I_{\{V_{t^{RA}} > p_i^{m+1}\}} \left(V_{t^{RA}} - p_i^{m+1}\right) \mid s_i\right] \cdot E_{S_{-i} \mid s_i} \left[Q_i^{RAT}(q_i^{m+1}) ; \Omega^4\right] \\
&+ E_{S_{-i} \mid s_i} \left[\left[\left. E_{V_i^{RA}} \left[I_{\{V_{t^{RA}} > P^c\}} \left(V_{t^{RA}} - P^c\right) \mid s_i\right) \right] \left[Q_i^{RAT}(q_i^m) ; \Omega^5\right]\right] \right]
\end{align*}$$

\footnote{Instead in presence of a tie (i.e. more than one bidder rationed at the clearing price) a perturbation of $q_i^m$ impacts on the rationed quantity because it changes the rationing coefficient $\rho$ (see Assumption 3).}
where \( f(q_i^1, \ldots, q_i^{m-1}|\Omega^1) \) is a function which doesn’t depend on \( q_i^m \); \( Q_{i}^{\text{RAT}}(q_i^j) \) is the total share bidder \( i \) obtains in the MA when the marginal bid \( q_i^j - q_i^{j-1} \) is rationed (with \( j = 1, \cdots, M_i - 1 \), and \( q_i^0 = 0 \)) and \( Q_i^c \) is the bidder \( i \)'s MA allocation. In (11), the expected payoff depends on the partition of the state space through both the RA allocation and price. The RA allocation depends on \( q_i^m \) in sets \( \Omega^2, \Omega^3 \), and \( \Omega^4 \), while the RA price depends on \( q_i^m \) in sets \( \Omega^3 \). The RA allocation is a random variable in sets \( \Omega^2, \Omega^4 \), and \( \Omega^5 \), while the RA price is a random variable in sets \( \Omega^3 \), and \( \Omega^5 \).

Using expression (9) to study the effect of an infinitesimal perturbation in \( q_i^m \) on agent \( i \)'s expected payoff in the RA [expression (11)] and referring to the derivative definition, the limit can be re-written as:

\[
\delta \cdot \left[ \mathbb{E}_{V_{tRA}}(V_{tRA} - p_{t}^m; V_{tRA} > p_{t}^m|s_i) \cdot \mathbb{E}_{S_{-i}|s_i} \left( \frac{\partial Q_{i}^{\text{RAT}}(q_i^m)}{\partial q_i^m}; \Omega^2 \right) \right. \\
+ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{E}_{V_{tRA}}(V_{tRA} - P^c; V_{tRA} > P^c|S_{-i}, s_i); \Omega^3 \cup \omega^{2\rightarrow3} \right] \\
\left. + q_i^m \cdot \frac{\partial}{\partial q_i^m} \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{E}_{V_{tRA}}(V_{tRA}; V_{tRA} > P^c|S_{-i}, s_i); \Omega^3 \cup \omega^{2\rightarrow3} \right] \\
- q_i^m \cdot \frac{\partial}{\partial q_i^m} \mathbb{E}_{S_{-i}|s_i} \left[ P^c \cdot \mathbb{P}_{V_{tRA}|s_i} \left( V_{tRA} > P^c|S_{-i}, s_i\right); \Omega^3 \cup \omega^{2\rightarrow3} \right] \\
+ \mathbb{E}_{V_{tRA}}(V_{tRA} - p_{t}^{m+1}; V_{tRA} > p_{t}^{m+1}|s_i) \cdot \mathbb{E}_{S_{-i}|s_i} \left( \frac{\partial Q_{i}^{\text{RAT}}(q_i^{m+1})}{\partial q_i^{m+1}}; \Omega^4 \right) \right]
\]

(12)

where I used the definition of indicator function and the limit assumptions on the sets \( \omega^{2\rightarrow3}, \omega^{2\rightarrow4\rightarrow5}, \omega^{3\rightarrow4\rightarrow5} \).

Ignoring the case of ties and putting together the derivative of the expected payoff inside the MA with respect to \( q_i^m \) [expression (9)] with the derivative inside the RA [expression (12)], rewriting the \( \Omega^j \) with the corresponding states of market clearing prices, noting that under Assumption 3 \( \Pr(P^C = p_{t}^m \cap Q_i^c = q_i^m) = 0 \), and rearranging terms, I obtain the analogous of the FOCs for a generic PD:
\[ Pr\left(p_{i}^{m+1} < Pc < p_{i}^{m}\right) \]

\[
\cdot \left[ v_{i}(q_{i}^{m}, s_{i}^{MA}) - p_{i}^{m}\right] + \delta \cdot \left( E_{S_{i}^{-}|s_{i}} \left[ E_{V_{RA}} \left( V_{RA} - Pc; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right]
\]

\[
+ q_{i}^{m} \cdot \frac{\partial}{\partial q_{i}^{m}} \left( E_{S_{i}^{-}|s_{i}} \left[ E_{V_{RA}} \left( V_{RA}; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right)
\]

\[
- q_{i}^{m} \cdot \frac{\partial}{\partial q_{i}^{m}} \left( E_{S_{i}^{-}|s_{i}} \left[ Pc \cdot P_{V_{RA}} \mid s_{i} \left( V_{RA}; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right)
\]

\[ = Pr(Pc \leq p_{i}^{m+1}) \cdot (p_{i}^{m} - p_{i}^{m+1}) \]

which corresponds to expression (5). QED

Finally at the last step $M_{i}$, there is not next step so the optimality conditions are:

\[ v_{i}(q_{i}^{m}, s_{i}^{MA}) + \delta \cdot \left( E_{S_{i}^{-}|s_{i}} \left[ E_{V_{RA}} \left( V_{RA} - Pc; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right]
\]

\[
+ q_{i}^{m} \cdot \frac{\partial}{\partial q_{i}^{m}} \left( E_{S_{i}^{-}|s_{i}} \left[ E_{V_{RA}} \left( V_{RA}; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right)
\]

\[
- q_{i}^{m} \cdot \frac{\partial}{\partial q_{i}^{m}} \left( E_{S_{i}^{-}|s_{i}} \left[ Pc \cdot P_{V_{RA}} \mid s_{i} \left( V_{RA}; V_{RA} > Pc \mid s_{i}\right) \mid p_{i}^{m+1} < Pc < p_{i}^{m}\right]\right)
\]

\[ = p_{i}^{m} \text{ if } m = M_{i}, \quad (5\text{-bis}) \]

which corresponds to expression (5-bis). QED

A.3. Proof of Proposition 2ter

As anticipated above, the partition of $\Omega_{S_{i}}$, introduced in Section A doesn’t permit to apply the local perturbation method to the RA in case of the payment rule in Assumption 12-ter, as the continuity and differentiability of the RA payoff function with respect to $q_{i}^{m}$ is not preserved in some cells of the partition. In particular, in cell $\Omega_{S_{i}}^{3}$ there is need to separate the states where there is rationing at the clearing price from the states where no agent is rationed, as the expression of the average winning
price is a different function of $q_i^m$ in the two cases (Assumption 9). I therefore create a finer partition of $\Omega_{S-i}^3$ defined, as follows.

1. ({$q_i^m, p_i^m$}) is agent $i$'s lowest winning bid and no agent is rationed

This event is split satisfies the following condition:

$$\Omega_{S-i}^{3,1} = \{s_{-i} : p_i^{m+1} < p_i^m \cap q_i^m \cap y_j(Pc|s_i) = Q_j^c \forall j \in \{1, \cdots , N\}\}$$

The subset $\Omega_{S-i}^{3,1}$ includes all the $s_{-i}$ at which the residual supply cuts the step-wise demand function of bidder $i$ at the price $p_i^{m+1} < P_c \leq p_i^m$ and the marginal requested quantities of all bidders are served in full.

2. ({$q_i^m, p_i^m$}) is agent $i$'s lowest winning bid and it is served in full but other bidders are rationed

This event satisfies the following condition:

$$\Omega_{S-i}^{3,2} = \{s_{-i} : p_i^{m+1} < P_c < p_i^m \cap Q_i^c = q_i^m \cap \sum y_j(Pc|s_i) > \sum Q_j^c\}$$

The subset $\Omega_{S-i}^{3,2}$ includes all the $s_{-i}$ at which the residual supply cuts the step-wise demand function of bidder $i$ at the price $p_i^{m+1} < P_c < p_i^m$, and while $q_i^m$ is served in full, other agents are rationed at $P_c$.

The subsets $\Omega_{S-i}^{3,1}$ and $\Omega_{S-i}^{3,2}$ satisfy:

$$\Omega_{S-i}^3(q_i^m) = \Omega_{S-i}^{3,1}(q_i^m) \cup \Omega_{S-i}^{3,2}(q_i^m)$$

Using the modified partition of the set $\Omega_{S-i}$ and applying the law of total probability, it is possible to rewrite agent’s $i$ expected payoff in the RA auction as a weighted average of the expected payoffs within each cell of the partition:

---

58 In states where $(q_i^m, p_i^m)$ is agent $i$’s marginal bid, it is served in full and there is rationing at the clearing price $p_i^{m+1} < P_c < p_i^m$, the quantity $q_i^m$ affects the remaining supply available at the clearing price $(1 - \lim_{m \to \infty} \sum_{j \neq i} y_j(p(s_j) - q_i^m))$, and thus the average winning price, differently from states where $(q_i^m, p_i^m)$ is agent $i$’s lowest bid, it is served in full and there is not rationing.
\[ \mathbb{E}_{i}^{RA}(s_{i}|q_{i}^{m}) = \delta \cdot \left[ f(q_{i}^{1}, \cdots, q_{i}^{m-1}) + \mathbb{E}_{s_{-i}|s_{i}} \left[ \mathbb{E}_{V_{i}} \left[ \mathbb{I}_{\{V_{i} > P\}} \left( V_{i} - P \right) \right] \cdot Q_{i}^{RA}(q_{i}^{m}); \Omega \right] + q_{i}^{m} \cdot \mathbb{E}_{s_{-i}|s_{i}} \left[ \mathbb{E}_{V_{i}} \left[ \mathbb{I}_{\{V_{i} > P\}} \left( V_{i} - P \right) \right] \cdot Q_{i}^{RA}(q_{i}^{m+1}); \Omega^{2} \right] + q_{i}^{m} \cdot \mathbb{E}_{s_{-i}|s_{i}} \left[ \mathbb{E}_{V_{i}} \left[ \mathbb{I}_{\{V_{i} > P\}} \left( V_{i} - P \right) \right] \cdot Q_{i}^{RA}(q_{i}^{m+1}); \Omega^{3} \right] + \mathbb{E}_{s_{-i}|s_{i}} \left[ \mathbb{E}_{V_{i}} \left[ \mathbb{I}_{\{V_{i} > P\}} \left( V_{i} - P \right) \right] \cdot Q_{i}^{RA}(q_{i}^{m+1}); \Omega^{3} \right] + \mathbb{E}_{s_{-i}|s_{i}} \left[ \mathbb{E}_{V_{i}} \left[ \mathbb{I}_{\{V_{i} > P\}} \left( V_{i} - P \right) \right] \cdot Q_{i}^{C}; \Omega^{5} \right] \right] \] (13)

In (13), the expected payoff depends on the partition of the state space through both the RA allocation and price. The RA allocation depend on \( q_{i}^{m} \) in sets \( \Omega^{2}, \Omega^{3.1}, \Omega^{3.2}, \) and \( \Omega^{4} \), the RA price depends on \( q_{i}^{m} \) in sets \( \Omega^{3.1}, \Omega^{3.2}, \Omega^{4}, \) and \( \Omega^{5} \). The RA price \( P \) is a random variable in all the sets,\(^59\) the allocation is a random variable only in the states where the requested quantity is rationed.

Now, in order to find agent \( i \)'s optimality conditions inside the RA, I perturb the \( m \)-th step to \( \hat{q} = q_{i}^{m} - \epsilon \) and study the effects of this deviation from \( (q^{m}, p^{m}) \) on the agent’s expected payoff. This perturbation is likely to provoke some reshuffling of states \( s_{-i} \) between the subsets \( \Omega_{S_{-i}}^{5} \). In particular, it is possible to define the following sets of potential transferred states: the sets \( \omega_{2 \rightarrow 3.1}(\hat{q}) \) and \( \omega_{2 \rightarrow 3.2}(\hat{q}) \), which includes all the states \( s_{-i} \) transferred from \( \Omega_{S_{-i}}^{2}(q_{i}^{m}) \) to \( \Omega_{S_{-i}}^{3.1}(\hat{q}) \) and \( \Omega_{S_{-i}}^{3.2}(\hat{q}) \), respectively, the set \( \omega_{2 \rightarrow 4 \rightarrow 5}(\hat{q}) \) which includes all the states \( s_{-i} \) transferred from \( \Omega_{S_{-i}}^{2}(q_{i}^{m}) \) to \( \Omega_{S_{-i}}^{4}(\hat{q}) \cup \Omega_{S_{-i}}^{5}(\hat{q}) \), the set \( \omega_{3.1 \rightarrow 3.2}(\hat{q}) \), which include all the states \( s_{-i} \) transferred from \( \Omega_{S_{-i}}^{3.1}(q_{i}^{m}) \) to \( \Omega_{S_{-i}}^{3.2}(\hat{q}) \), and the sets \( \Omega_{S_{-i}}^{3.1 \rightarrow 4 \rightarrow 5}(\hat{q}) \) and \( \Omega_{S_{-i}}^{3.2 \rightarrow 4 \rightarrow 5}(\hat{q}) \), which includes all the states \( s_{-i} \) transferred from \( \Omega_{S_{-i}}^{3.1}(q_{i}^{m}) \) and \( \Omega_{S_{-i}}^{3.2}(q_{i}^{m}) \), respectively, to \( \Omega_{S_{-i}}^{4}(\hat{q}) \cup \Omega_{S_{-i}}^{5}(\hat{q}) \). The set \( \Omega_{S_{-i}}^{1} \) is not impacted by the perturbation. The partition of \( \Omega_{S_{-i}} \) based on the perturbed bid \( (\hat{q}, p^{m}) \) satisfies:

\(^{59}\)While \( P \) is a random variable in all the sets, the clearing price \( P^{c} \) is a random variable only in sets \( \Omega^{1}, \Omega^{3.1}, \Omega^{3.2}, \) and \( \Omega^{5} \).
Using the relations (a)-(e) between the original partition and the perturbed one, it is possible to find the condition that rules out profitable perturbations of $q^m_i$, overcoming the difficulty that $Q^c_i$ and $P^c$ are not continuous in $q^m_i$.

For this purpose, I assume that: $\lim_{\epsilon \to 0} Pr(\omega^2 \to 3,1) = \lim_{\epsilon \to 0} Pr(\omega^2 \to 4,5) = \lim_{\epsilon \to 0} Pr(\omega^3,1 \to 3,2) = \lim_{\epsilon \to 0} Pr(\omega^3,2 \to 4,5) = 0$ (i.e. after the infinitesimal perturbation, the allocation $\hat{q}$ remains with probability 1 in $\Omega^2$, $\Omega^3$, and $\Omega^3,2$, respectively), while $\lim_{\epsilon \to 0} Pr(\omega^2 \to 3,1) > 0$. In the latter case I assume a knife-edge solution where $\hat{q}$ is completely served but the clearing price $P^c$ remains equal to $p^m_i$.  

Using the derivative definition and the limit assumptions on the sets $\omega^2 \to 3,1$, $\omega^2 \to 4,5$, $\omega^3,1 \to 3,2$, $\omega^3,1 \to 3,4$, and $\omega^3,2 \to 4,5$, the effect of an infinitesimal perturbation in $q^m_i$ on agent $i$’s expected payoff in the RA [expression (11)] can be studied through the following expression:

\[
\begin{align*}
\Omega^1_{S_i}(\hat{q}) &= \Omega^1_{S_i}(q^m_i) \\
\Omega^2_{S_i}(\hat{q}) &= \Omega^2_{S_i}(q^m_i) \setminus \omega^2 \to 3,1(\hat{q}) \setminus \omega^2 \to 4,5(\hat{q}) \\
\Omega^3_{S_i}(\hat{q}) &= \Omega^3_{S_i}(q^m_i) \cup \omega^2 \to 3,1(\hat{q}) \setminus \omega^3,1 \to 3,2(\hat{q}) \setminus \omega^3,1 \to 4,5(\hat{q}) \\
\Omega^4_{S_i}(\hat{q}) &= \Omega^4_{S_i}(q^m_i) \cup \omega^2 \to 4,5(\hat{q}) \cup \omega^3,1 \to 4,5(\hat{q}) \cup \omega^3,2 \to 4,5(\hat{q}) \\
\Omega^5_{S_i}(\hat{q}) &= \Omega^5_{S_i}(q^m_i) \cup \omega^2 \to 4,5(\hat{q}) \cup \omega^3,1 \to 4,5(\hat{q}) \cup \omega^3,2 \to 4,5(\hat{q}) \\
\end{align*}
\]

\(^{60}\)In this case continuity is preserved because after the infinitesimal perturbation the allocation remains inside the step corresponding to $p^m_i$ and within each step the payoff function is continuous.
\[
\delta \cdot \left[ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{E}_{V_{RA}} \left( V_{RA} - \mathcal{P}; V_{RA} > p_{i}^m | s_{-i}, s_i \right) \cdot \frac{\partial Q_{RAT}(q_{i}^m)}{\partial q_{i}^m} ; \Omega^2 \right] \right] \\
+ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{E}_{V_{RA}} \left( V_{RA} - \mathcal{P}; V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^{3,1} \cup \omega^{2-3,1} \cup \Omega^{3,2} \right] \\
+ q_{i}^m \cdot \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{E}_{V_{RA}} \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^{3,1} \cup \omega^{2-3,1} \right] \right]}{\partial q_{i}^m} \\
- q_{i}^m \cdot \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{P} \cdot \mathbb{P}_{V_{RA}} \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^{3,2} \right] \right]}{\partial q_{i}^m} \\
+ q_{i}^m \cdot \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ \mathbb{P} \cdot \mathbb{P}_{V_{RA}} \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^{3,2} \right] \right]}{\partial q_{i}^m} \\
- q_{i}^m \cdot \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ Q_{i}^{RAT}(q_{i}^{m+1}) \cdot \mathbb{E}_{V_{RA}} \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^4 \right] \right]}{\partial q_{i}^m} \\
+ \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ Q_{i}^{RAT}(q_{i}^{m+1}) \cdot \mathbb{P} \cdot \mathbb{P}_{S_{-i}, V_{RA}} | s_i \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^4 \right] \right]}{\partial q_{i}^m} \\
- \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ Q_{i}^{R} \cdot \mathbb{P} \cdot \mathbb{P}_{S_{-i}, V_{RA}} | s_i \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^5 \right] \right]}{\partial q_{i}^m} \\
+ \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ Q_{i}^{R} \cdot \mathbb{P} \cdot \mathbb{P}_{S_{-i}, V_{RA}} | s_i \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^5 \right] \right]}{\partial q_{i}^m} \\
- \frac{\partial \left[ \mathbb{E}_{S_{-i}|s_i} \left[ Q_{i}^{R} \cdot \mathbb{P} \cdot \mathbb{P}_{V_{RA}} | s_i \left( V_{RA} > \mathcal{P} | s_{-i}, s_i \right) ; \Omega^5 \right] \right]}{\partial q_{i}^m} \right]
\]

where I used the definition of indicator function.

Ignoring the case of ties and putting together the derivative of the expected payoff inside the MA with respect to \( q_{i}^m \) [expression (3)] with the derivative inside the RA [expression (12)], rewriting the \( \Omega^j \) with the corresponding states of market clearing prices, noting that under Assumption 3 \( P_{T} \left( P_{C} = p_{j}^m \cap \sum_j y_j | P_{C} = \sum_j Q_{j}^c \right) = 0 \) \((\forall j \in \{1, \ldots, N\} \text{ and } \forall k \in \{1, \ldots, M_j\})\), and rearranging terms, I obtain the analogous of the FOCs for a generic PD:
\[
P_r(p_i^{m+1} < p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j)
\]
\[
= P_r(p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j)
\]
\[
\cdot \left[ v_i(q_i^m, s_i^{MA}) - p_i^m \right] + \delta \left( g_{s_{-i}|s_i} \left[ E_{V_{iRA}} \left( V_{iRA} - P; V_{iRA} > P \right| s_i \right) \right] 
\]
\[
\cdot \left[ p_i^{m+1} < p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right]
\]
\[
+ q_i^m \cdot \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)
\]
\[
- \frac{\partial \left( g_{s_{-i}|s_i} \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \right) \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)}{\partial q_i^m}
\]
\[
= P_r(p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j)
\]
\[
\cdot \left[ v_i(q_i^m, s_i^{MA}) - p_i^m \right] + \delta \left( g_{s_{-i}|s_i} \left[ E_{V_{iRA}} \left( V_{iRA} - P; V_{iRA} > P \right| s_i \right) \right] 
\]
\[
\cdot \left[ p_i^{m+1} < p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right]
\]
\[
+ q_i^m \cdot \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)
\]
\[
- q_i^m \cdot \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)
\]
\[
= P_r(p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j)
\]
\[
\cdot \left[ v_i(q_i^m, s_i^{MA}) - p_i^m \right] + \delta \left( g_{s_{-i}|s_i} \left[ E_{V_{iRA}} \left( V_{iRA} - P; V_{iRA} > P \right| s_i \right) \right] 
\]
\[
\cdot \left[ p_i^{m+1} < p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right]
\]
\[
+ q_i^m \cdot \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)
\]
\[
- q_i^m \cdot \left[ Q^R - E_{V_{iRA}} \left( V_{iRA} > P \right| s_i, s_{-i}, s_i \right) \right] \left( p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j \right)
\]
\[
= P_r(p^c < p_i^m \cap \sum_j y_j(P^e|s_i) > \sum_j Q_j)
\]

which corresponds to expression \(6-\text{bis}\) in the main text. QED