A Simple Theory of Crowdfunding Dynamics*

Matthew Ellman† and Michele Fabi‡

Incomplete draft - please do not distribute

January 2019

Abstract

This paper develops a dynamic game using endogenous inspection costs to explain empirically salient bidding profiles in crowdfunding. In our baseline, bidders, who arrive at constant exogenous rate must pay an inspection cost if they want to learn if they like the product to be produced. With a fixed inspection cost, the funding profile is always weakly decreasing. We provide sufficient conditions for time-increasing and decreasing funding profiles. We can explain the U-shape or bathtub funding profiles documented in much of the recent empirical literature in two ways: (1) given a distribution of costs that imply an increasing profile for a constant arrival rate, adding a group of close contacts who can bid on the project at the start, the average profile is U-shaped; (2) allowing bidders to choose in which period they bid, the least-cost bidders bunch at the beginning while higher cost bidders delay their inspection until it becomes clearer whether the project will succeed. We also study how conditioning on project success or failure affects the funding profile, pertinent to data sources that restrict to successful projects. A continuous version of the model provides similar results.

---

*We thank Sjaak Hurkens and Antonio Miralles for valuable discussions and gratefully acknowledge financial support from the 2016 FBVVA grant “Innovación e Información en la Economía Digital,” the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563), the Ministry of Science, Innovation and Universities (ECO2014-59959, Ellman; FPI fellowship 914886-79792965, Fabi) and the Generalitat de Catalunya (2017 SGR 1136).

†Institute of Economic Analysis, IAE-CSIC, and Barcelona GSE (matthew.ellman@iae.csic.es)

‡Universitat Autònoma de Barcelona, IDEA-UAB, and Barcelona GSE (michele.fabi@e-campus.uab.cat)
1 Introduction

The dynamics of pledging to a crowdfunding campaign has recently received much attention thanks to the growing wealth of empirical data, yet no solid theory has yet been developed that can simultaneously explain key stylized facts. This paper provides a new theoretical approach by endogenizing inspection. Bidders’ inspection costs are an immediate flip-side of crowdfunding’s defining feature, that entrepreneurs solicit bids from buyers before producing their product. Entrepreneurs describe their project in detail but it remains harder for buyers to evaluate than when the product already exists. Modelling the resulting inspection costs generates sharp predictions for pledging dynamics. Buyers typically will not pledge without first inspecting and they are only willing to inspect when the project has a reasonable chance of success. In turn, this success rate depends on crowdfunding entrepreneurs’ central new strategic choice – the funding threshold. Buyers avoid wasting time inspecting projects with excessive thresholds. Dynamics follow from how success rates vary over time with the shrinking gap between a project’s fixed threshold and its rising funding level. We track this gap and model how buyers’ inspection and bidding decisions respond to it.

Our analysis focuses on the widespread case of All-or-Nothing, reward-based crowdfunding. In this paradigm, the entrepreneur pitches a project to collect at least a minimal amount of money, the threshold, from a large crowd of funders during a specified funding period, called the campaign. Bidders or funders pledge to buy the entrepreneur’s product or reward when the campaign succeeds in reaching its target – a minimum threshold on aggregate funds pledged. When the campaign fails, the product is not delivered and all bids are returned. The entrepreneur can set her threshold and minimum price for the product to guarantee she at least covers her production costs or she can set her threshold strategically to raise profits or audience. In either case, production is determined by the Aggregate Funding Threshold (AFT) rule, also known as a provision point mechanism.

In practice, entrepreneurs provide a menu of rewards depending on how much a funder bids to pay, but we focus on the simplest case where the crowdfunding campaign is described by: (i) a funding threshold, (ii) a funding period and (iii) a single price.\(^1\) The crowdfunding threshold plays a crucial role in determining the bidding profile or evolution of average bidding over the funding period.

We characterize the qualitative features of bidding profiles for a range of inspection cost distributions and the impact of changing the crowdfunding threshold. We provide sufficient conditions for time-increasing and decreasing funding profiles. We outline a campaign’s dynamic profile by tracking the distribution of the funding gap over time. Our algorithmic solution computes the exact expectations and quantiles for the key equi-

\(^1\)We represent the price as a discount on the regular price offered to buyers who buy ex-post, that is, after crowdfunding, in the event that the campaign is successful.
librium variables. We also study profiles conditioned on either success or failure by restricting to the relevant specific paths.

For ex-ante homogenous bidders, that is, fixed inspection costs, we find that the bidding profile is always (weakly) decreasing. Either the campaign never gets started in that bidders never inspect, or it is initially “alive” but has an increasing risk of “dying” in the sense that (all) bidders cease to inspect. A project dies if the gap falls too slowly as the campaign progresses: as the time left for further bids shrinks, the gap needs to be lower if the project is to have a chance of success; dying is of course an absorbing state.

A uniform distribution also generates a decreasing bidding profile losing momentum has a stronger negative effect on “popularity” or the proportion of bidders who inspect and hence on the bidding rate than does sustained momentum have a positive effect. We add an atom of bidders with zero inspection cost, representing a group of informed bidders or close contacts of the entrepreneur. This mitigates the decreasing profile but does not generate an increasing profile.

However, making the cost distribution convex by assuming a linear increasing density (a quadratic cumulative distribution function), we obtain an increasing profile when the atom at zero is large enough. Then there is always somebody willing to inspect even when the expected success rate is low. Making the atom large enough ensures the predominance of those paths where momentum is sustained and bidders with high inspection costs eventually become willing to bid.

To probe further, we next investigate when the direction of the bidding profile changes on conditioning on campaign success or failure. We provide a numerical analysis for binary inspection cost distributions. When the threshold is relatively high, the profile of successful projects tends to increase due to the gradual rise of the expected success rate, while that of failed tends to decrease for the opposite effect. When the threshold is relative low, successful and failed projects experience a gradual (expected) reduction in collected bids because of the probability associated to paths where the project loses its popularity. We introduce a simple and tractable continuous time extension of the model to track how the slope of the average bid rate varies with time.

We are also interested in understanding when the U-shape (or bathtub) profile of successful campaigns arises; in other words, when bidding is concentrated at the start and at the end of the funding cycle. Currently several complementary theories can provide a plausible explanation for the emergence of the U-shape. The first relies on a common value, endogenous timing model: bidders receive signals of different value and precision and decide whether and when to bid. Early bidding is caused by those who receive positive and precise signals; late bidding is driven by those who receive imprecise signals and wait the campaign to progress and infer the quality of the product looking at past bidding. If the campaign is likely to be successful, many bids are collected, and the posterior on project’s quality increases up to the point that can lead to the formation of cascades. On
the other hand, bidders who receive negative and precise signals never bid. An alternative approach is based on a private values, endogenous timing and endogenous inspection model. Allowing bidders to choose in which period they inspect (and consequently bid), the least-cost bidders bunch at the beginning self selecting themselves as promoters of the campaign while higher-cost bidders delay their inspection until it becomes clearer whether the project will succeed becoming followers. Another compelling alternative relies on a model of advertising and strategic prominence determined by the entrepreneur and the platform. High and low momentum phases may reflect different intensities of attention that the project receives. The initial peak of the U-shape occurs because of a higher intensity of promotion by the entrepreneur; with a strong enough kick-start, the size of the supporting community may grow over time leading pledges to gradually increases.

This paper focuses on the prior step of understanding the role of the threshold: can we explain the observed contribution patterns using only the forces present in a private value setting? In such a setting, the driving force is the crowdfunding threshold. We home in on that precise issue. To our knowledge, we propose the only available model that focuses just on the threshold impact on the success (and failure) rate, and in turn on the dynamic bidding profile. We abstracts from common value effects and mix rigid (i.e. time constrained) bidders arriving at an exogenous rate with flexible (unconstrained) bidders that follow the campaign along its funding period and choose when to inspect.

We will show how the threshold alone can generate a range of dynamic effects that will continue to be relevant in richer models. Advertising and platform promotion strategies represent an unquestionably important part of the puzzle, but to derive the optimal promotion strategies, we first want to know how the dynamics of campaigns respond to an exogenous bidder arrival rate. Other forces, such as word-of-mouth learning and common value effects, can have strong influence on the observed funding patterns, but we have shown that the current model is simple enough to account for the most salient facts, generating a U-shaped profile, as well as increasing and decreasing patterns, depending on the project’s starting condition and the distribution of inspection costs. Since the threshold matters even without any of these additional effects, we start from there, providing a parsimonious model that can fill a basic gap of this literature. So we start from here.

The rest of the paper is organized as follows: The literature review is discussed in Section 2. In Section 3 we present the model. In Section 4 we provide a sufficient condition for determining the slope of the bidding profile. In Section 6 we develop a continuous version of the model and introduce endogenous movers.
2 Literature review

The literature on crowdfunding models began with Belleflamme et al. (2014), shortly followed by others (Ellman and Hurkens, 2016; Sahm, 2016; Strausz, 2016; Chang, 2016; Chemla and Tinn, 2016). Parker (2014) present a simple model of information cascades to show that the presence of uninformed investors not always dampens the success of good projects, and that a vast presence informed funders can lead to an inefficient allocation of funds towards a narrow set of projects.

Empirical studies proliferated along with theoretical analyses. The following paper provide evidence on the contribution patterns more commonly observed. In particular, a group of authors acknowledges the evidence of the U-shape and other momentum effects (Kuppuswamy and Bayus, 2015; Mollick, 2014; Belleflamme et al., 2015; Cordova et al., 2015; Colombo et al., 2015). Babich et al. (2017) and Vismara (2016) are also noteworthy for their contributions to the study of equity-based crowdfunding.

Most close to our work are those that model explicitly the dynamics of pledging. The following papers present two-period models. Ellman and Hurkens (2016) develop a two-period model assuming common value. Tucker and Zhang (2011) develop a two-period model where purchasing choices are determined by the horizontal matching between buyers’ preferences and sellers offers. Their results are experimentally tested in wedding markets. Zhang and Liu (2012) shows that herding is rational in that popular p2p loans are repaid more often and presents a simple two-period model. (Cason and Zubrickas, 2018) studies theoretically and experimentally a dynamic donation game assuming refund bonuses. In comparison to our model, bidders can pledge more than once. (Deb et al., 2018) assume one exogenous bidder per period and an endogenous-moving donor, who is the key strategic player.

The following authors present games where time runs for more than two period or in continuous time. Alaei et al. (2016) is a solid operation research paper that studies optimal crowdfunding design. It assumes bidders are forward-looking and have private binary values and face an opportunity cost when the project fails. They show that projects either succeed or fail and collect a very limited portion of the required funds. This result is driven by cascades as, which are incorporated in the model using anticipating random walks. It also shows that there is a cutoff threshold above which the project fails almost surely if surpassed. On the optimization side, even though the model has no closed form solution, the policy recommendation that can be grasped is to set low thresholds and relatively high prices. Zhang et al. (2017) studies optimal revenue management choosing campaigns’ price and duration. The authors assume continuous time and fixed horizon. Bidders have private values and are divided in two categories: ordinary (early) bidders, who arrive sequentially at an increasing time rate and bid if their valuation is high enough, regardless of the amount collected; herding bidders, who pledge at T only if the project
is successful and their valuation is high enough. This modelling choices are consistent
with the evidence on a final peak of contributions. In our model, also the initial peak is
obtained by introducing endogenous movers.

Chakraborty and Swinney (2016) studies contributors behavior and optimal campaign
design in a two-period model. Bidders choose endogenously whether and when to con-
tribute. The authors assume all bidders have the same valuation and hassle cost from
contributing, and each either high or low waiting cost. This heterogeneity divides bid-
ders into early and late movers. Equilibria with delays occurs when hassle costs are high
relative to waiting costs, but the entrepreneur can improve this situation by setting a low
price before success and triggering a higher price afterwards.

Cason and Zubrickas (2018) studies theoretically and experimentally the impact of
refund bonuses on the provision of a threshold public good. They find that refund bonuses
help funders to coordinate and rule out bad equilibria. They also find that refund bonuses
reduce momentum as they make profitable to miss the threshold.

Asami (2018) presents a dynamic model of common value crowdfunding. Assumes
continuous time and exogenous movers, binary actions and binary signal. Bidders arrival
rate is uniform over the funding horizon and the size of the population is uncertain. If
the population size is exponentially distributed, multiple equilibria exist where the bidding
pattern is either flat, with initial and final peaks only, or U-shaped.

Deb et al. (2018) explains how donations can gauge projects’ success in the presence of
heterogeneous backers. Donors, who only care about the probability of project’s success,
contribute at the beginning and at the end of the campaign, while buyers contribute
steadily during the campaign. This equilibrium is consistent with the observed data on
Kickstarter. When common value components are introduced, the impact of donations
on future fund-raising is reduced as they do not signal project’s quality. On the empirical
part of the analysis, the authors find evidence for the U-shaped pattern. Also, they
find that pledges decrease when the goal is met, finding this effect sharp from donors
and softer for buyers. This effect is not produced by our model, but, as Williams at al.
explain, it can be generated by a shock in backers’ arrival rate or by the presence of low
price, capacity constrained rewards before the goal is met, and high price unconstrained
rewards after the goal is met.

Liu (2018) provides a rationale and test empirically a theory of endogenous formation
of leaders and followers in models of learning and collective action. In the context of
crowdfunding, they explain that the combination of the two problems considered provides
the incentive to confident agents, who receive a high signals, to endorse the venture
pledging early, while confident agents wait and support the venture conditionally high
enough back-loading. Due to this mechanism, the eventually successful ventures are
expected to collect substantial lead investment, while failed projects die after receiving
little early support. To develop this intuition, they build a 2-period, common value model
and assume endogenous timing of actions. In the empirical side of their paper they also report evidence for the U-shape using Kickstarter data.

Du et al. (2017) focus on project promotion through stimulus policies (seeding via free samples, product upgrades and other time limited offers). As in our setting, bidders decide whether to bid or not upon arrival based on the success rate.

Our article is also loosely related to models of bargaining or irreversible investment in continuous time with endogenous move order. Zhang (1997) study investment cascades over a finite horizon in continuous time. In their set-up, the first contributors with high precision signals revel all information; all other bidders pledge right afterwards because there is no value in waiting. On the other hand, Ma and Manove (1993) claim the opposite result in a model of bargaining with deadline and strategic delay, assuming bidders have imperfect control over the time they pledge because of a buffering period. In their model, successful offers are accepted in proximity of the deadline.

3 Model

A crowdfunding campaign aims at collecting $K$ bids of fixed amount within $T$ funding rounds to launch production. Each bidder’s type is represented by the pair $(v_i, c_i)$, where $v_i$ is bidder $i$’s utility enjoyed from having the good and $c_i$ represents his inspection cost. Bidders do not know their tastes for the good when the campaign starts but can inspect and perfectly learn them. During the funding period, bidders arrive at a rate of $\lambda$ per round, which we initially set to 1. A bidder arrival represents the event that a bidder’s attention is drawn to the project, and we assume that bidders are so busy that they only pay attention in that single round. So they either inspect or bid on the project in the round of their arrival, or they never inspect or bid. We refer to this type of bidders as exo-movers since the time they act is exogenous.

Each bidder can bid to buy the good at a discount $d$ on the ex-post price; here we normalize $d$ to 1. Any such bids are held escrow until the campaign ends. If the campaign succeeds, all bids are paid and the good is produced and delivered to bidders. If it fails, the good is not produced and all bids are returned. We denote the event of success by $\mathcal{S}$, which occurs if the funding gap $g$ goes from $K$ to 0 before time $T$ runs out, and the complementary event of failure $\mathcal{F}$ if the gap ends at $g_0 > 0$. While active, a campaign’s possible states are indexed by $(r, g) \in \{0, \ldots, T\} \times \{K - T, \ldots, K\}$ where $r = T - t + 1$ denotes the number of remaining rounds before the campaign ends, and $g$ is the gap between bids collected so far and the threshold $K$. For simplicity, we refer to $g_r$ as the gap at the start of round $r$. We define $P_{(r,g)}$ as the probability of reaching state $(r,g)$ from the initial state; that is the probability of transiting from $(T,K)$ to $(r,g)$, and $Q_{(r,g_r)(s,g_s)}$
as the probability of transiting across two given states: from \((r, g_r)\) to \((s, g_s)\).\(^2\) In this notation, \((T, K)\) denotes the starting state. We use the backward-moving time index \(r\) and gap \(g\) rather than bids collected so far to facilitate our backward-induction analysis below. An important variable is the probability that the project succeeds given it is currently in state \((r, g)\); we denote this success rate by \(S_{r,g}\).

Bidders share a common prior belief that any individual \(i\) will enjoy utility \(v_i = \nu\) with probability \(q \in (0,1)\), and \(v_i = 0\) with probability \(1 - q\); each individual’s taste for the good is an independent draw from the same Bernoulli distribution. However, each bidder can perfectly learn his type \(v_i\) by paying an inspection cost \(c_i\), representing his opportunity cost of devoting time to assess his private value for the crowd-funded good. Inspection costs \(c_i\)’s are i.i.d. draws from the c.d.f. \(F(.)\) with support contained in \((-\infty, +\infty)\). Bidders with negative inspection costs are simply bidders who like to inspect.

Given that inspecting the project without bidding is a dominated strategy, bidders choose among three actions: Check (inspect and bid if \(v_i = \nu\)); Blind bid (bid without inspecting), and Abstain from bidding; we denote them by \(C, B\) and \(A\). Under binary valuations, action \(C\) captures the immediate optimal response to learning after inspection. Inspection costs are the key determinants of bid dynamics.

For simplicity, we suppose that the expected regular ex-post rent is fixed during the campaign, so that ex-post purchases generate no consumer rent.\(^3\) As a consequence, a bidder who buys ex-post would have to pay \(\nu\), while an ex-ante bidder pays only \(\nu - 1\). Thus, the utility from composite action \(C\) is given by

\[
U_{r,g}^C = qS_{r-1,g-1} - c_i
\]

While the utility from playing \(B\) is given by

\[
U_{r,g}^B = S_{r-1,g-1} [q + (1 - q)(1 - \nu)]
\]

Clearly, playing \(A\) gives zero utility. To restrict the analysis to cases where bidders never play \(B\), we assume:\(^4\)

**Assumption 1** (No blind bidders). \(q < 1 - 1/\nu\)

Under this assumption, bidders only contemplate playing \(C\) or \(A\). Note also that \(F(q)\) is the fraction of arrivals that can potentially be willing to inspect and therefore bid.

---

\(^2\)There are no strategic moves in round \(r = 0\), which represents the end of the campaign.

\(^3\)This is valid for the case of binary valuations that we assume. In a model where ex-post rents depends on the campaigns’ outcome this, as for example the case where prices vary according to the amount of funds collected, this would not hold.

\(^4\)An alternative normalization is to set \(\nu = 1\). In this case assumption 1 becomes \(q < 1 - d\).
To simplify, we assume that the arrival rate is 1. In this way, this allows us to extend the model to continuous time where the probability of exact simultaneous arrivals falls to zero.

Finally, we assume bidders always play C when indifferent; that is, we resolve indifference in favour of the inquisitive choice.

In a given state at most one bid is collected with \( p_{r,g} \). In that case the gap drops by one, so \( gr_{r-1} = g - 1 \); if not, \( gr_{r-1} = gr_r \). Thus, the expected next round gap is \( E_{r,g} gr_{r-1} = gr_r - p_{r,g} \). Average bids are defined as

\[
A_r = E_T \{ gr_r - gr_{r-1} \} \quad (3)
\]

\[
A^S_r = E_T \{ gr_r - gr_{r-1} \mid g_0 \leq 0 \} \quad (4)
\]

\[
A^F_r = E_T \{ gr_r - gr_{r-1} \mid g_0 > 0 \} \quad (5)
\]

The construction of the average bid rate, aggregate and conditional on the campaign’s outcome, is based on the iterative definitions of the success rate and the probabilities of transaction across funding states. The success rate evolves according to the following expression:

\[
\begin{aligned}
S_{r,g} &= p_{r,g} S_{r-1,g-1} + (1 - p_{r,g}) S_{r-1,g} \\
S_{0,g} &= 1 \text{ if } g \leq 0 \\
S_{0,g} &= 0 \text{ if } g > 0
\end{aligned}
\quad (6)
\]

Given the initial state \((T, K)\), expression 6 pins-down \( S_{r,g} \) for every state. On the other hand, the funding state evolves according to a Markovian process defined by

\[
Q_{(r,g_{r})(r-1,g_{r-1})} = \begin{cases} 
  p_{r,g} & \text{if } g_r = g_{r-1} - 1 \\
  1 - p_{r,g} & \text{if } g_r = g_{r-1} \\
  0 & \text{if } g_r \notin \{g_{r-1} - 1, g_{r-1}\}
\end{cases}
\quad (7)
\]

and letting \( G_r = \{K - r + 1, K - r, \ldots, K\} \) denote the set of feasible gaps, we compute

\[
\mathcal{P}_{(r,g)} = \sum_{g_{r+1} \in G_{r+1}} \mathcal{P}_{(r+1,g_{r+1})} Q_{(r+1,g_{r+1})(r,g)}
\quad (8)
\]

Since the initial state is given, we can construct \( \mathcal{P}_{(r,g)} \) by iterating the previous recursion until the initial state is reached.

With these elements we can compute average bids. Since at most one bid is collected in each round, we only need to compute the probability mass of collecting it. To get averages without conditioning on outcome, we compute first \( A_{r,g} \): the average bid at \((r, g)\), or in other words, in round \( r \) given gap \( g \), which is simply given by \( p_{r,g} \). Then \( A_r \)
is computed by taking expectations over possible $g$’s.\(^5\)

\[ A_r = \sum_{g \in G_r} \mathcal{P}_{(r,g)} p_{r,g} \]  

(9)

To compute averages conditional on $S$, first calculate the average bids at $r$, for a given $g$, conditional on the gap being on successful trajectories, which is given by $A^S_{r,g} = p_{r,g} S_{r-1,g-1}$. Then, taking expectations over $g$ along potentially successful paths and conditioning on success we get

\[ A^S_r = \frac{1}{S_{T,K}} \sum_{g \in G^S_r} \mathcal{P}_{(r,g)} [p_{r,g} S_{r-1,g-1}] \]  

(10)

Finally, to get averages conditional on $F$, we first compute $A^F_{r,g} = p_{r,g} (1 - S_{r-1,g-1})$, that is, average bids, in round $r$ for a given $g$, given that the gap is on a trajectory that will lead to failure. Then, taking expectation over $g$ along paths that can lead to fail and conditioning on failure we get

\[ A^F_r = \frac{1}{1 - S_{T,K}} \sum_{g \in G^F_r} \mathcal{P}_{(r,g)} [p_{r,g} (1 - S_{r-1,g-1})] \]  

(11)

### 4 Basic equilibrium properties

Using the model defined in section 3, we analyse the bidding profile that emerge from the equilibrium bidding strategies under several distributional assumptions.

First of all, since one bidder at most arrives each round, the game has a unique PBE in threshold strategies: bidders play C if their inspection cost is below a given threshold $\hat{c}_{r,g}$.\(^6\)

**Proposition 1** (Existence and uniqueness of the equilibrium). *The game defined in section 3 has a unique PBE such that bidders play C if and only if $c_i \leq \hat{c}_{r,g}$, where*

\[ \hat{c}_{r,g} \triangleq q S_{r-1,g-1} \]  

(12)

*and $S_{r-1,g-1}$ are given by the Eq. (6)*

**Proof of Proposition 1.** We prove this by backward induction, starting from the final action period $r = 1$. $U_C^{r=1} = q S_{0,1-1} - c_i$ which equals $q - c_i$ if $g \leq 1$ and $-c_i$ if $g > 1$. So $\hat{c}_{1,g\leq1} = q$ and $\hat{c}_{1,g>1} = 0$. Also, $S_{1,g} = 0$, for $g > 1$, $S_{1,1} = F(q)$, $S_{1,g} = 1$ for $g < 1$.

\(^5\)Even though in expressions 9, 10, and 11 the sum is over a subsets of $G$, we could, in principle, sum over $G$ itself without affecting the result, since gaps that are not feasible or impossible given the conditioning outcome occur with probability 0.

\(^6\)Assuming multiple arrivals the game can have a multiple equilibria due to possible coordination failure.
Suppose that for all later rounds \( r - 1, r - 2, \ldots, 1 \), bidders play \( C \) if \( c_i \leq \hat{c}_{r-1,g_{r-1}} \) for all possible \( g_{r-1}, \ldots, g_1 \). Then in round \( r \), we have \( U^C_{r,g} = qS_{r-1,g_{r-1}} - c_i \). By the inductive hypothesis, we have \( S_{r-1,g_{r-1}} = qF(\hat{c}_{r-1,g_{r-1}})S_{r-2,g_{r-2}} + (1 - qF(\hat{c}_{r-1,g_{r-1}}))S_{r-2,g_{r-1}} \), which is constant w.r.t. \( c_i \). Thus, since \( i \)'s utility function is linear in \( c_i \), he finds optimal to play \( C \) if and only if \( c_i \leq \hat{c}_{r,g} = qS_{r-1,g_{r-1}} \). ■

As a consequence of Proposition 1, the probability of collecting a bid at a given state (or heat) is equivalent to the probability that the bidder arriving at \( r \) likes the good and has low enough inspection cost relative to the state-specific threshold on \( c_i \).

\[
p_{r,g} = qF(\hat{c}_{r,g}) = qF(qS_{r-1,g_{r-1}})
\]

(13)

Also, note that Eq. (6) links the cost thresholds among adjacent states \( \hat{c}_{r,g} \), \( \hat{c}_{r-1,g_{r-1}} \) and \( \hat{c}_{r-1,g} \):

\[
\hat{c}_{r,g} = p_{r-1,g_{r-1}}\hat{c}_{r-1,g_{r-1}} + (1 - p_{r-1,g_{r-1}})\hat{c}_{r-1,g}
\]

(14)

In equilibrium, expressions 9, 10, 11 become

\[
\begin{align*}
A_r &= \sum_{g \in G_r} P_{(r,g)} [qF(qS_{r-1,g_{r-1}})] \\
A^S_r &= \frac{1}{S_{T,K}} \sum_{g \in G^S_r} P_{(r,g)} [qF(qS_{r-1,g_{r-1}})S_{r-1,g_{r-1}}] \\
A^F_r &= \frac{1}{1 - S_{T,K}} \sum_{g \in G^F_r} P_{(r,g)} [qF(qS_{r-1,g_{r-1}})(1 - S_{r-1,g_{r-1}})]
\end{align*}
\]

Also, note that Eq. (6) links the cost thresholds among adjacent states \( \hat{c}_{r,g} \), \( \hat{c}_{r-1,g_{r-1}} \) and \( \hat{c}_{r-1,g} \):

(15)

(16)

(17)

Table 1 shows how the success rate can be computed using Proposition 1 in a simple \( K = 2, T = 3 \) game.

<table>
<thead>
<tr>
<th>( S_{r,g} )</th>
<th>( g = 1 )</th>
<th>( g = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r = 1 )</td>
<td>( qF(q) )</td>
<td>0</td>
</tr>
<tr>
<td>( r = 2 )</td>
<td>( qF(q) + (1 - qF(q))S_{1,1} )</td>
<td>( qF(qS_{1,1})S_{1,1} )</td>
</tr>
<tr>
<td>( r = 3 )</td>
<td>( - )</td>
<td>( qF(qS_{2,1})S_{2,1} + (1 - qF(qS_{2,1}))S_{2,2} )</td>
</tr>
</tbody>
</table>

Table 1: Success rate at each state assuming \( K = 2 \) and \( T = 3 \)

Before examining the dynamic bidding profile, we provide an auxiliary result related to the structure of \( S_{r,g} \). In the following lemma, we show that the success rate at a given \( (r, g) \) is decreasing in the gap \( g \) still to be met and increasing in the remaining time \( r \); also, it rises if the gap falls maximally as time rises.

**Lemma 1** (Success rate). The success rate \( S_{r,g} \) satisfies the following properties:

i (Vertical Monotonicity): For any given \( r \in \{0, T\} \), \( S_{r,g} \) is (weakly) decreasing in \( g \);
ii (Horizontal Monotonicity): For any given \( g \in \{K-T,K\} \), \( S_{r,g} \) is (weakly) increasing in \( r \).

iii (Diagonal Monotonicity): \( S_{r-1,g-1} \geq S_{r,g} \) for all \( r,g \)

Proof of Lemma 1. For proving part (i), define the mapping \( S_r(g): g \mapsto S_{r,g} \) for \( g \in \{K-T,K\} \). The claim holds if \( S_r(\cdot) \) is non-increasing for all \( r \). We proceed by induction on \( r \). Base case: \( r = 0 \). Using expression 6, it is obvious that \( S_0(\cdot) \) is decreasing. Induction: Assume \( S_{r-1}(\cdot) \) is decreasing. Using expression 6 and expanding \( p_{r,g} \) we obtain

\[
S_r(g) = S_{r-1}(g) + qF(qS_{r-1}(g-1)) \cdot [S_{r-1}(g-1) - S_{r-1}(g)]
\]

Subtracting \( S_r(g-1) \) from \( S_r(g) \) and rearranging the expression we obtain

\[
S_r(g) - S_r(g-1) = [S_{r-1}(g) - S_{r-1}(g-1)] \cdot (1 - qF(qS_{r-1}(g-1))) + [S_{r-1}(g-1) - S_{r-1}(g-2)] \cdot qF(qS_{r-1}(g-2))
\]

which is non-positive for all \( g \) by the inductive hypothesis.

For proving part (ii), define the mapping \( S_g(r): r \mapsto S_{r,g} \) for \( r \in \{0,\ldots,T\} \). Our claim holds if \( S_g(\cdot) \) is non-decreasing for all \( g \). We proceed by construction. If \( g \leq 0 \) we have \( S_{g\leq0}(r) = 1 \) for all \( r \), which is constant. For other values of \( g \), manipulate expression 18 to obtain

\[
S_g(r) - S_g(r-1) = qF(qS_{g-1}(r-1)) \cdot [S_{g-1}(r-1) - S_{g}(r-1)]
\]

From part (i) of the lemma, we know that \( S_{g-1}(r-1) - S_{g}(r-1) \) is non-negative, hence the whole expression is non-negative for any given \( g \). We can conclude that \( S_g(\cdot) \) is non-decreasing.

For part (iii), note that \( S_{r,g} = p_{r,g}S_{r-1,g-1} + (1-p_{r,g})S_{r-1,g} < S_{r-1,g} \) by part (ii)

Part (i) is intuitive: a lower gap implies a higher chance of success because fewer bids are required to reach the funding goal. For part (ii), more time to attract bids to fill a given gap also raises the chances of success: \( S_{r,g} \) rises with \( r \), the number of remaining rounds. For part (iii), note that the success rate necessarily increases when the best bidding outcome arises in a given time interval.

5 Bid profiles

In this section we analyse the general properties of bidding profiles. We start by presenting the following general rules. If \( T = K \), there is only one possible path to success: a bid in

\footnote{Part (iii) of lemma 1 does not hold if we assume multiple arrivals or endogenous movers.}
Figure 1: Monotonicity of \( S_{r,g} \) in the \((g,t)\) space

every period. So conditioning on success, the bid profile is flat at unity: \( A^S_r = 1 \) for all \( r \).
Also, if \( K = 1 \), then clearly \( A^F_r = 0 \) for all \( r \). No threshold always implies a flat profile.
If \( K > T \), then the campaign can never succeed, so bidding only comes from bidders with
negative or null inspection cost, and \( A^S \) is undefined.

The following proposition presents a sufficient condition to determine funding profiles’
monotonicity:

**Proposition 2** (Monotonicity of the Bid Profile). The direction of aggregate and conditional bid profile is determined by the differences

\[
\Delta_{r,g} = p_{r,g}(1 - p_{r-1,g-1}) - (1 - p_{r,g})p_{r-1,g}
\]

(21)

If \( \Delta_{r,g} \geq 0 \) for all \( r, g \), then aggregate and conditional profiles are (weakly) decreasing,
while if \( \Delta_{r,g} \leq 0 \) for all \( r, g \), all profiles are (weakly) increasing.

Intuition: Suppose one bid arises at \( r \) or \( r - 1 \). We want to understand when it is more
likely to occur sooner than later. The two possible paths the gap can follow correspond
to the dashed and dotted lines between nodes \((r,g)\) and \((r-2,g-1)\) presented in Fig. 2.
The high road has no bid at \( r \) with probability \( 1 - p_{r,g} \) followed by a bid lowering \( g \) to
\( g - 1 \) at \( r - 1 \) with probability \( p_{r-1,g} \), so the high road (or delayed bid path) occurs with
probability \( (1 - p_{r,g})p_{r-1,g} \). The low road has a bid at \( r \) with probability \( p_{r,g} \), lowering
the gap form \( g \) to \( g - 1 \), and no bid at \( r - 1 \), with probability \( 1 - p_{r-1,g-1} \). Thus the low
road (or early bid path) is followed with probability \( p_{r,g}(1 - p_{r-1,g-1}) \). If the delayed bid
path is more likely than the early bid path, then bidding profile rises.
Figure 2: Graphical representation of Proposition 2

(i) Cooling and heating effect along adjacent states

(ii) Conditional on $S$

(iii) Conditional on $F$
Note that if two or no bids are collected the resulting profile is flat, so we can focus on the cases considered. Moreover, since the state is governed by a Markovian process, what happens between \((r, g)\) and \((r - 2, g - 1)\) is independent of the path from \((T, K)\) to \((r, g)\), and from \((r, g)\) to the final states.

**Proof of Proposition 2.** We prove by considering an exhaustive set of possibilities; we show it holds for all \(r \in \{T, \ldots, 2\}\). At a given state \((r, g)\), the heat is \(p_{r,g}\). In the following round, \(g_{r-1} = g\) with probability \(1 - p_{r,g}\) and \(g_{r-1} = g - 1\) with probability \(p_{r,g}\). In these adjacent states the heat is \(p_{r-1,g-1}\) and \(p_{r-1,g}\), so using the law of total probability we determine the probability of collecting a bid in round \(r - 1\) conditional on the campaign being at state \((r, g)\):

\[
p_{r,g}p_{r-1,g-1} + (1 - p_{r,g})p_{r-1,g-1}
\]

Taking expectations over possible realizations of \(g\) according to expression 9, we obtain

\[
A_r - A_{r-1} = \sum_{g \in G_r} \mathcal{P}_{(r,g)} \left[ p_{r,g}(1 - p_{r-1,g-1}) - (1 - p_{r,g})p_{r-1,g} \right]
\]

It can be easily seen that condition 2 is sufficient to determine the sign of \(A_r - A_{r-1}\).

We analyse now successful projects. Using expression 6, we expand \(S_{r-1,g-1}\) to obtain

\[
S_{r-1,g-1} = p_{r-1,g-1}S_{r-2,g-2} + (1 - p_{r-1,g-1})S_{r-2,g-1}
\]

Substituting expression 24 into 10

\[
A^S_r = \sum_{g \in G^S_r} \mathcal{P}_{(r,g)} \left[ S_{r-2,g-2} (p_{r,g}p_{r-1,g-1} + p_{r,g}(1 - p_{r-1,g-1})) \right] / S_{T,K}
\]

Also, combining expressions 10 evaluated at \(r - 1\) and using 8

\[
A^S_{r-1} = \sum_{g \in G^S_r} \mathcal{P}_{(r,g)} \left[ S_{r-2,g-2} (p_{r,g}p_{r-1,g-1} + (1 - p_{r,g})p_{r-1,g}) \right] / S_{T,K}
\]

so again taking differences

\[
A^S_r - A^S_{r-1} = \sum_{g \in G^S_r} \mathcal{P}_{(r,g)} \left[ S_{r-2,g-2} (p_{r,g}(1 - p_{r-1,g-1}) - (1 - p_{r,g})p_{r-1,g}) \right] / S_{T,K}
\]

Condition 2 is sufficient for determining the sign of \(A^S_r - A^S_{r-1}\).

Finally, for failing projects, we use expression 6 to obtain

\[
(1 - S_{r-1,g-1}) = p_{r-1,g-1}(1 - S_{r-2,g-2}) + (1 - p_{r-1,g-1})(1 - S_{r-2,g-1})
\]
computing the difference between expression 11 at \( r \) after substituting 28 and 11 at \( r - 1 \)

\[
A_r^F - A_{r-1}^F = \sum_{g \in G_r^F} P_{(r,g)} [(1 - S_{r-2,g-2})(p_{r,g}(1-p_{r-1,g-1}) - (1-p_{r,g})p_{r-1,g})] / (1 - S_{T,K})
\]  

(29)

Again, condition 2 determines the sign of the \( A_r^F - A_{r-1}^F \). ■

Proposition 2 provides a set of sufficient conditions for increasing and another set for decreasing. These are far from being necessary conditions: if it holds, determines the direction of the bidding profile, regardless of the campaign’s outcome. However, note that not all monotonic profiles have to satisfy Proposition 2. Indeed, expressions 9 - 11, require that inequality 2 holds on average on different sets of \( g \) with no weighting coefficient if aggregate, and weighted by success and failure rate if conditional. Thus, since averages are taking over different subset of gaps, nothing in principle prevents the profile to change its slope depending on the conditioning outcome.

In the remaining part of the current section, we present how proposition 2 helps us in determining the bidding profile under different assumptions on \( F \), solve numerically the model under different cost distributions.

5.1 Homogeneous inspection costs

Before assuming a distribution of inspection costs, we consider the case where all bidders have the same \( c_i = c \) for \( c \in (0, +\infty) \). In this setting the campaign has two heat levels, which we refer to as hot and frozen. In hot states \( (r,g) \in H \), the campaign is alive and receives an additional bid with probability \( q \). In frozen states \( (r,g) \in F \), the campaign is dead since the gap is too high to induce bidders to play \( C \).

\[
p_{r,g} = \begin{cases} q & \text{if } (r,g) \in H \\ 0 & \text{if } (r,g) \in F \end{cases}
\]  

(30)

The state-space is divided into hot and frozen regions by the wall of death \( \hat{g} = (\hat{g}_1 \hat{g}_2 \cdots \hat{g}_T) \), a vector of gap thresholds \( \hat{g}_r \). Formally,

\[
\hat{g}_r = \max \{ g \in G_r \cup \{ +\infty, -\infty \} | \hat{c}_{r,g} \geq c \}
\]  

(31)

Clearly if \( c > q \) there is no gap such that bidders inspect, thus \( \hat{g}_r = -\infty \) for all \( r \); on the other hand, if \( c = 0 \), bidders always inspect and so \( \hat{g}_r = +\infty \) for all \( r \). Using \( \hat{g}_r \) we

---

\(^8\)This is equivalent to assume that inspection costs are drawn from a degenerate distribution such that \( F(c) = 1 \) for \( c_i \geq c \) and \( F(c_i) = 0 \) for \( c_i < c \).
identify hot and frozen regions and the resulting heat

\[ H : \{(r, g) : g \leq \hat{g}_r\} \]
\[ F : \{(r, g) : g > \hat{g}_r\} \]

The following lemma proves the basic properties of the wall of death; that is, each \( \hat{g}_r \) is decreasing in time (increasing in \( r \)) and makes at most a unit jump at the following round.

**Lemma 2 (Wall of death).** The wall of death \( \hat{g} = (\hat{g}_1 \hat{g}_2 \cdots \hat{g}_T) \) satisfies the following properties:

i. \( \hat{g}_r \) is increasing in \( r \).

ii. \( \hat{g}_r - \hat{g}_{r-1} \leq 1 \)

**Proof of Lemma 2.** We prove part (i) by contradiction that \( \hat{g}_r < \hat{g}_{r-1} \). Since \( \hat{g}_r \) is integer, \( \hat{g}_r - 1 = \hat{g}_r + k \) for some \( k \geq 1 \) without loss of generality.

By definition, \( \hat{g}_r \) has to satisfy

\[ S_{r, \hat{g}_r} \geq \frac{c}{q} > S_{r, \hat{g}_r+1} \]
\[ S_{r-1, \hat{g}_{r-1}} \geq \frac{c}{q} > S_{r-1, \hat{g}_{r-1}+1} \]

But since \( S_{r,g} \) is non-decreasing in \( r \) and by Eq. (32) \( S_{r-1, \hat{g}_r+1} < c/q \), and since \( S_{r,g} \) is non-increasing in \( g \), then \( S_{r-1, \hat{g}_r+k} < c/q \) for any integer \( k \geq 1 \); a contradiction.

Part (ii) is an immediate consequence of the martingale property of \( S_{r,g} \). By Eq. (31), \( S_{r, \hat{g}_r} \geq c/q \). If \( \hat{g}_{r-1} < \hat{g}_r - 1 \) then \( S_{r-1, \hat{g}_{r-1}} < c/q \) but also \( S_{r, \hat{g}_r} = S_{r-1, \hat{g}_{r-1}} + p_{r, \hat{g}_r} (S_{r-1, \hat{g}_{r-1} - 1} - S_{r-1, \hat{g}_{r-1}}) < c/q \), a contradiction. \( \blacksquare \)

Letting \( \pi_r = \sum_{g=K-T+r}^{g_r} P_{(r,g)} \) denote the probability that the campaign is hot in round \( r \), we express expected bids at round \( r \) as

\[ A_r = \sum_{g=K-T+r}^{g_r} P_{(r,g)} q = \pi_r q \quad (33) \]

where

\[ P_{(r,g)} = P_{(r+1,g+1)} p_{r+1,g} + P_{(r+1,g)} (1 - p_{r+1,g}) \quad (34) \]

Combining Lemma 2 and Eq. (34), we express Eq. (33) as

\[ A_{r-1} = \begin{cases} 
\pi_r q = A_r & \text{if } \hat{g}_{r-1} = \hat{g}_r \\
(\pi_r - P_{(r,\hat{g}_r)} (1-q)) q = A_r - P_{(r,\hat{g}_r)} (1-q) q & \text{if } \hat{g}_r = \hat{g}_{r+1} - 1 \end{cases} \quad (35) \]
Figure 3: The Wall of Failure

\[ c \approx 0 \]  \hspace{1cm} \[ c = .5 \]  \hspace{1cm} \[ c = .75 \]

\[ \hat{g}_r \text{ for different values of } c. \text{ Other parameters: } T = 100, K = 40, q = .75 \]

it is easy to note that \( \pi_r \) and \( A_r \) are decreasing in time (increasing in \( r \)). If the campaign starts frozen it is dead from the start. If it starts hot, it can becomes frozen and die.

**Proposition 3.** If \( c_i = c \) for all \( i \), then the profile is decreasing over time.

**Proof of Proposition 3.** We apply Proposition 2 and study the sign of \( \Delta_{r,g} \). By Lemma 2, \( \hat{g}_{r-1} \in \{ \hat{g}_r, \hat{g}_r - 1 \} \). In round \( r \) where \( \hat{g}_{r-1} = \hat{g}_r \), then \( \Delta_{r,g} = 0 \) for all \( g \).

At \( r \) such that \( \hat{g}_{r-1} = \hat{g}_r - 1 \), \( \Delta_{r,g} = 0 \) if \( g \leq \hat{g}_r - 1 \) and \( g > \hat{g}_r + 1 \) and if \( g = \hat{g}_r \), \( \Delta_{r,g} = q(1 - q) > 0 \). Thus \( \Delta_{r,g} \geq 0 \) at all \((r,g)\) and we can conclude that the profile is decreasing over time.

**5.2 Binary inspection cost**

In this section we assume that inspection costs are Bernoulli distributed; that is, each bidder’s inspection cost is either \( c^L \) or \( c^H \), with probability \( z \) on \( c^L \). The state-space is now partitioned into three heat regions where the heat is given by

\[
p_{r,g} = \begin{cases} 
q & \text{if } (r,g) \in H \\
zq & \text{if } (r,g) \in C \\
0 & \text{if } (r,g) \in F
\end{cases}
\]

(36)

Similarly to the setting presented in Section 5.1, heat regions are delimited the wall of cold \( \hat{g}^H = (\hat{g}_1^H, \hat{g}_2^H, \ldots, \hat{g}_T^H) \) and the wall of death \( \hat{g}^L = (\hat{g}_1^L, \hat{g}_2^L, \ldots, \hat{g}_T^L) \). Both walls satisfy the property presented in Lemma 2.
Figure 4: Dynamic profile under homogeneous inspection costs.

(i) Expected bids

(ii) Heat map

(iii) Expected gap

(iv) Gap quartiles

\[
H = \{(r, g) : g \leq \hat{g}_r^H\},
C = \{(r, g) : \hat{g}_r^H < g \leq \hat{g}_r^L\},
F = \{(r, g) : g > \hat{g}_r^L\} \tag{37}
\]

Letting \(\pi_r\) and \(\rho_r\) denote the probability that the campaign is hot and cold in round \(r\), we express expected bids in round \(r\) as

\[
A_r = q(\pi_r + z\rho_r) \tag{38}
\]
and using Lemma 2 we relate recursively $A_{r+1}$ with $A_r$.

\[
A_{r-1} = \begin{cases}
q \left( \pi_r + P_{(r,\hat{g}^H_{r-1})}zq + z \left( \rho_r - P_{(r,\hat{g}^H_{r-1})}zq \right) \right) \\
= A_r + zqP_{(r,\hat{g}^H_{r-1})}q(1-z) \\
\text{if } \hat{g}^H_{r-1} = \hat{g}^H_r \text{ and } \hat{g}^L_{r-1} = \hat{g}^L_r \\
q \left( \pi_r - P_{(r,\hat{g}^H_{r-1})}(1-q) + z \left( \rho_r + P_{(r,\hat{g}^H_{r-1})}(1-q) \right) \right) \\
= A_{r-1} - (1-q)P_{(r,\hat{g}^H_{r-1})}q(1-z) \\
\text{if } \hat{g}^H_{r-1} = \hat{g}^H_r - 1 \text{ and } \hat{g}^L_{r-1} = \hat{g}^L_r \\
q \left( \pi_r - P_{(r,\hat{g}^H_{r-1})}(1-q) + z \left( \rho_r + P_{(r,\hat{g}^H_{r-1})}(1-q) - P_{(r,\hat{g}^L_{r-1})}(1-zq) \right) \right) \\
= A_{r-1} - (1-q)P_{(r,\hat{g}^H_{r-1})}q(1-z) - (1-zq)P_{(r,\hat{g}^L_{r-1})}zq \\
\text{if } \hat{g}^H_{r-1} = \hat{g}^H_r - 1 \text{ and } \hat{g}^L_{r-1} = \hat{g}^L_r - 1
\end{cases}
\]

When $c_L = 0$, then $\hat{g}^L_r = +\infty$ for all $r$ and there are no frozen states, so $\rho_r = 1 - \pi_r$ for all $r$. In this case, Equations (38) and (39) become

\[
A_r = q(z + \pi_r(1-z))
\]

\[
A_{r-1} = \begin{cases}
q \left( z + (\pi_r + P_{(r,\hat{g}^H_{r-1})}zq) \right) (1-z) \quad \text{if } \hat{g}_{r-1} = \hat{g}_r \\
= A_r + zqP_{(r,\hat{g}^H_{r-1})}q(1-z) \\
q \left( z + (\pi_r - P_{(r,\hat{g}^H_{r-1})}(1-q)) \right) (1-z) \quad \text{if } \hat{g}_{r-1} = \hat{g}_r - 1 \\
= A_r - (1-q)P_{(r,\hat{g}^H_{r-1})}q(1-z)
\end{cases}
\]

On the other hand, when $c^L > q$, then $\hat{g}^L_r = -\infty$ for all $r$, so this setting becomes equivalent to the one presented in Section 5.1.

### 5.2.1 Cold start

We now investigate which bid profiles can arise in the binary cost set-up. We classify bid profiles according to the starting heat, i.e. whether it is hot or cold at the initial state.
(T, K).

We first analyse the case of K = 2 using Proposition 2 whenever possible and otherwise providing a closed-form solution. More general cases will be analysed numerically.

The starting heat is determined by whether H-types and L-types are willing to play C in the first round. Note that after collecting one bid the campaign gets hot, so if an H-type plays C at r while \( g_r = 2 \) his utility is

\[
U^C_{r,2}(c^H) = qS_{r-1,1} - c^H \\
= q (1 - (1 - q)^{r-1}) - c^H
\]  

(42)

Since \( S_{r,g} \) is increasing in \( r \), this will be positive for \( r \geq \hat{r}^H \) and negative otherwise. So \( \hat{r}^H = \lceil \tilde{r}^H \rceil \), where \( \tilde{r}^H \) is the value of \( r \) that makes \( IC_{(r,g),c}^{C,A} \) binding.

\[
q \left( 1 - (1 - q)^{\tilde{r}^H-1} \right) = c^H \\
\Leftrightarrow \tilde{r}^H = 1 + \log_{1-q} \left( 1 - \frac{c^H}{q} \right)
\]

and so

\[
\hat{r}^H \triangleq [\tilde{r}^H] = 1 + \left\lceil \log_{1-q} \left( 1 - \frac{c^H}{q} \right) \right\rceil
\]  

(43)

similarly, the time threshold for the L-types is

\[
\hat{r}^L \triangleq [\tilde{r}^L] = 1 + \left\lceil \log_{1-q} \left( 1 - \frac{c^L}{q} \right) \right\rceil \leq \hat{r}^H
\]  

(44)

The campaign starts hot when \( \hat{r}^H \geq T \), and starts cold when \( \hat{r}^H < T \leq \hat{r}^L \).

If the campaign starts cold and there are no frozen states, i.e. \( c^L = 0 \), it gradually heats up regardless of whether it succeeds or fails. When we introduce frozen states (\( c^L > 0 \)), then profiles are non monotone. For \( r < \hat{r} \), successful campaigns jumps downward at \( \hat{r} \). Conditional on success is then constant as always hot, while the aggregate profile is decreasing due to the probability of freezing.

We can apply Proposition 2 to study the monotonicity regions of the bid profile.

**Lemma 3** (Cold Start). Assuming \( K = 2 \) and cold start, the bid profile is increasing if \( c^L = 0 \). If \( c^L > 0 \), the profile is increasing from \( r = T \) to \( r = \hat{r} \) and constant afterwards.

At \( \hat{r} \) all profiles make a downward jump.

---

\(^9\)For \( K > 2 \), we have thresholds \( \hat{r}^H_g, \hat{r}^L_g \) for \( g \in \{1, 2, \ldots, K\} \) with \( \hat{r}^H_1 = 1 \) and \( \hat{r}^H_g \triangleq [\tilde{r}^H_g] \) where \( \tilde{r}^H_g : qS_{r-1,g-1} = c^H \). \( \tilde{r}^H_g \) denotes the last round where H-types play C when the gap is \( g \). \( \hat{r}^L_g \) is defined in the same way.
Proof of Lemma 3. We apply Proposition 2. for all $r < \hat{r}$

$$
\Delta_{r,2} = \begin{cases} 
q(1-q) - (1-q)q \leq 0 & \text{for } r > \hat{r} \\
q(1-q) \geq 0 & \text{at } r = \hat{r} \\
0 & \text{for } r < \hat{r}
\end{cases}
$$

$\Delta_{r,1} = 0$ for all $r$

Figures ?? and ?? represent the antithetical cases of cold start with progressive heating and hot start with progressive cooling. In both cases, $\hat{g}_r$ is plotted over-imposed to the gap-quantiles in the top-right panels, while bottom-right panels reflect the fact that heat is binary.

What is the impact of increasing $K$ on the curvature of bidding profiles under hot and cold start? In the hot start case represented in ?? an upward shift in $K$ makes all profiles steeper since it becomes more likely enter in cold states early; aggregate and failed profiles drop and successful profiles need compensate by accumulating more early. Conversely, in the cold start case of ??, the effect is the opposite: profiles are less steep since it take more time to reach hot states.

5.3 Uniform distribution with atom at 0

We consider now the case where $c_i$’s is 0 with probability $\alpha < 1$ and is uniformly distributed with mass $f$ over $[0, 1/f]$ with probability $1 - \alpha$. The atom $\alpha$ accounts for the mass of bidders who have null or negative inspection.\(^{10}\) In this setting, $F(c) = \alpha + (1 - \alpha)fc$ for $c \in [0, 1/f]$, $F(c) = 1$ for $c \geq 1/f$, $F(c) = \alpha$ for $c \leq 0$. In this case the probability of bidding is given by

$$
pr_{r,g} = q[\alpha + (1 - \alpha)f \min\{qS_{r-1,g-1}, 1/f\}] 
$$

and average bids are given by

$$
A_r = \sum_{g \in G_r} p_{(r,g)}[q(\alpha + (1 - \alpha)f \min\{S_{r-1,g-1}, 1/f\})] 
$$

$$
A^S_r = \frac{1}{S_{T,K}} \sum_{g \in G^S_r} p_{(r,g)}[q(\alpha + (1 - \alpha)f \min\{S_{r-1,g-1}, 1/f\}) S_{r-1,g-1}] 
$$

$$
A^F_r = \frac{1}{1 - S_{T,K}} \sum_{g \in G^F_r} p_{(r,g)}[q(\alpha + (1 - \alpha)f \min\{S_{r-1,g-1}, 1/f\})(1 - S_{r-1,g-1})] 
$$

\(^{10}\)An atom at 0 produces the same effect on bidding profiles as a positive mass on negative inspection costs. If $c_i$’s are uniformly distributed on $[\xi, \tau]$ with $\xi < 0$, then $F(c_i) = \xi/(\xi + \tau) + \min\{c_i, \tau\}/(\tau + \xi)$ for $c_i \geq 0$. By letting $\alpha = \xi/(\xi + \tau)$ and $f = 1/\tau$ we obtain the same distribution.
Using proposition 2, we can easily see that the bidding profile is always downward sloping.

**Proposition 4** (Dynamic profile under uniformly distributed inspection costs with atom at 0). If \( c_i \) are 0 with probability \( \alpha \) and are uniformly distributed with mass \( f \) over \([0, 1/f]\) with probability \( 1 - \alpha \), then the dynamic profile is weakly decreasing.

**Proof of Proposition 4.** We apply Proposition 2, we know that the profile is decreasing if the following inequality holds:

\[
\frac{\min\{S_{r-1,g-1}, 1/fq\} - \min\{S_{r-2,g-1}, 1/fq\}}{\min\{S_{r-2,g-2}, 1/fq\} - \min\{S_{r-2,g-1}, 1/fq\}} \geq q[\alpha + (1 - \alpha)f_q \min\{S_{r-1,g-1}, 1/fq\}] \tag{49}
\]

From lemma 1, \( S_{r-2,g-2} \geq S_{r-1,g-1} \geq S_{r-2,g-1} \); thus, depending on \( fq \), we distinguish four cases:

i. \( fq \leq 1/S_{r-2,g-2} \)

ii. \( 1/S_{r-2,g-2} \leq fq \leq 1/S_{r-1,g-1} \)

iii. \( 1/S_{r-1,g-1} \leq fq \leq 1/S_{r-2,g-1} \)

iv. \( fq \geq 1/S_{r-2,g-1} \)

Case (i): Eq. (49) becomes

\[
S_{r-1,g-1} - S_{r-2,g-1} \geq q[\alpha + (1 - \alpha)f_q S_{r-1,g-1}](S_{r-2,g-2} - S_{r-2,g-1})
\]

From the definition of \( S_{r,g} \), Eq. (6), evaluated at \((r - 1, g - 1)\) we obtain

\[
S_{r-1,g-1} - S_{r-2,g-1} = q[\alpha + (1 - \alpha)f_q S_{r-2,g-2}](S_{r-2,g-2} - S_{r-2,g-1})
\]

and since \( S_{r-2,g-2} \geq S_{r-1,g-1} \) by lemma 1, we can conclude that the profile is decreasing in time.

Case (ii): \( \min\{S_{r-2,g-2}, 1/fq\} = 1/fq \). Eq. (49) becomes

\[
S_{r-1,g-1} - S_{r-2,g-1} \geq q[\alpha + (1 - \alpha)f_q S_{r-1,g-1}]
\]

and Eq. (6) evaluated at \((r - 1, g - 1)\) is given by

\[
S_{r-1,g-1} - S_{r-2,g-1} = q(S_{r-2,g-2} - S_{r-2,g-1})
\]

and since \( S_{r-2,g-2} \geq 1/fq \) by assumption, we conclude that the profile is decreasing.

Case (iii): \( \min\{S_{r-2,g-2}, 1/fq\} = \min\{S_{r-2,g-2}, 1/fq\} = 1/fq \). Eq. (49) reduces to

\[
1 \geq q
\]

which holds since \( q \in (0, 1) \).
Case (iv): Bidders are always willing to inspect; thus $p_{r,g} = q$ for all $r, g$. As a result, condition 2 holds with equality and bidding profiles are flat.

We conclude that the bidding profile resulting from a uniform $[0, 1/f]$ distribution of costs are weakly decreasing in time. ■

Figure 5: Dynamic profile under uniformly distributed inspection costs.

Parameters $T = 100$, $K = 39$, $f = .1$, $\alpha = 0$, $q = .75$. 
5.4 Linear p.d.f.

One way to obtain an increasing profile is by assuming a quadratic distribution with atom at 0. Specifically, we assume the following mixture distribution function:

\[
F(c) = \begin{cases} 
0 & c < 0 \\
\alpha & c = 0 \\
\alpha + (1 - \alpha)c^2 & c \in (0, 1] \\
1 & c > 1 
\end{cases}
\]

This functional form reflects the mix between two populations of bidders: a fraction \( \alpha \) of the population are informed bidders with 0 inspection cost, while a fraction \( 1 - \alpha \) are uninformed bidders with inspection costs distributed according to \( F(c) = c^2 \) over \([0, 1]\). In the next proposition, we will show mathematically that increasing the density of informed bidders can shift the profile from non-increasing to non-decreasing.

**Proposition 5.** *Dynamic profiles under a convex quadratic c.d.f. with an atom at zero.*

If a fraction \( \alpha \) of bidders have zero cost and a fraction \( 1 - \alpha \) have quadratically distributed inspection costs on \([0, 1]\), then there exist values \( \underline{\alpha} \) and \( \bar{\alpha} \) such that

(i) if \( \alpha \leq \underline{\alpha} \), then all profiles are non-increasing

(ii) if \( \alpha \geq \bar{\alpha} \), then all profiles are non-decreasing

Figure 6: Increasing and decreasing profiles assuming quadratic distribution

(i) \( K = 48, \alpha = .5 \)  
(ii) \( K = 30, \alpha = .15 \)

In both panels \( T = 100 \) and \( q = .75 \). Expected bids are scaled up by a factor of 100.
Figure 7: Increasing profile under quadratic distribution of inspection costs.

(i) Expected bids
(ii) Heat map
(iii) Expected gap
(iv) Gap quartiles

Parameters $T = 100$, $K = 39$, $f = .1$, $\alpha = 0$, $q = .75$.

6 Continuous time

In this section, we consider the continuous limit of the model presented in Section 3. To obtain the limiting results, every round is split into $n$ sub-rounds of length $1/n$ and the arrival rate is scaled accordingly, i.e. $\lambda = \lambda/n$, but for simplicity we normalize it to 1. In this way time can be treated as continuous. To distinguish this and the discrete setting, we denote the deadline as $\tau$.

The heat in the infinitesimal interval $[r, r - dr]$ is again $p_{r,g} = qF(\hat{c}_{r,g})$. and the derivative of the success rate with respect to $r$ is given by
\[ \dot{S}_{r,g} \triangleq \lim_{dr \to 0} \frac{S_{r+dr,g} - S_{r,g}}{dr} \] (50)

The continuous limit of the model preserves its Markovian structure. Since the campaign fails if \( g_0 > 0 \) and succeeds if \( g_0 \leq 0 \), the success rate is determined by the following recursive formulae:

\[
\begin{cases}
\dot{S}_{r,g} = p_{r,g} (S_{r,g-1} - S_{r,g}) \\
S_{0,g} = 0 \text{ if } g > 0 \\
S_{0,g} = 1 \text{ if } g \leq 0
\end{cases}
\] (51)

Moreover, aggregate and conditional bid rates are determined according to Eq. (9), Eq. (10) and Eq. (11).

The continuous extension developed in the current section allows provides a simple and tractable way to determine the campaign’s dynamic profile assuming the binary inspection cost distribution presented in Section 5.2.

In this setting, the starting heat is determined by the condition that makes bidders’ indifferent to play A or C. Specifically, assume bidder \( i \) arrives when the campaign starts. His pay-off from playing C is

\[ qS_{r,1} - c_i \] (52)

so he is willing do so only if \( S_{r,1} \) is high enough.

Also, the probability of success on a path that reaches a gap of 1 at time \( r \) is given by\(^{11}\):

\[ Q_{(r,1)(0,1)} = 1 - \lim_{n \to +\infty} \left( 1 - q \frac{1}{n} \right)^{rn} = 1 - e^{-qr} \] (53)

Since the campaign becomes hot after receiving one bid \( S_{r,1} = 1 - e^{-qr} \), and so \( i \) plays C if

\[ c_i \leq \hat{c}_{r,2} = q \left( 1 - e^{-qr} \right) \] (54)

or equivalently if length \( \tau \) of the funding period is large enough to convince bidder \( i \) to play C:

\[ \tau \geq \hat{\tau}(c_i) \triangleq -\frac{1}{q} \ln \left( 1 - \frac{c_i}{q} \right) \] (55)

Therefore, in a binary cost setting \( c \) and \( c' > c \) and two corresponding thresholds on the campaign length, which for brevity we call \( \tau \) and \( \tau' \). The heat regions vary according to the following cases: if \( c = c' > 0 \) (and \( \tau = \tau' \)), the project is either alive and hot or dead and frozen; if \( \tau \geq \hat{\tau} \) the campaign starts, otherwise it is dead at its origin. if \( c = 0 \)

\(^{11}\)the limiting result is achieved by noticing that

\[ \left( 1 - q \frac{1}{n} \right)^{rn} = \left( 1 - q \frac{r}{m} \right)^{m} \]

which converges to \( e^{-qr} \) as \( n \) tends to infinity.
and \( c' > 0 \), then we have hot and cold states; a hot start occurs when \( \tau \geq \hat{\tau}' \) and a cold start occurs in the opposite case. If \( c \neq c' \) and \( c, c' > 0 \), then we have temperate, hot, and frozen (or dead) states. The starting heat is frozen if \( \hat{\tau} < \tau << \hat{\tau}' \), temperate if \( \tau < \hat{\tau} < \hat{\tau}' \), and hot if \( \tau < \hat{\tau}' < \hat{\tau} \).

To illustrate the forces driving the bidding patterns computed from the discrete model in a simple and traceable way, we characterize the equilibrium bidding profiles assuming \( K = 2 \), avoiding to cover the obvious cases where \( c, c' > q \), or \( c = 0, c' > q \), or \( c, c' = 0 \) where aggregate and conditional bid rates are flat, either at 0 or at some positive constant.

### 6.0.1 Homogeneous inspection costs

If inspection costs are homogeneous and the project does not start dead, then according to the reasoning developed in Section 5.1, there will be a switching time \( \hat{\tau} \) that, if surpassed, determines the campaign’s death if a critical mass of bids is not reached. As a consequence, bidding profiles will have a discontinuity at \( r = \hat{\tau} \). To find \( \hat{\tau} \), we solve the indifference condition between \( A \) and \( C \).

\[
\hat{r} : qS_{\hat{r},1} = c
\]

so that \( \hat{\tau} \) is given by

\[
\hat{\tau} = -\frac{1}{q} \ln(1 - \frac{q}{c})
\]

so a campaign dies if it fails to collect at least one bid before \( \hat{r} \), or given it is alive at \( \hat{\tau} \), if it fails to collect at least one bid before \( r = 0 \). The probability that the project is still alive after \( \hat{\tau} \) (for \( r < \hat{\tau} \)) is given by

\[
P(\hat{\tau}, 2) = e^{-q(r-\hat{\tau})}
\]

We can easily compute the failure rates:

\[
F_{r,2} = \begin{cases} 
Q_{(r,2)(\hat{r},2)} + Q_{(r,2)(\hat{r},1)} Q_{(\hat{r},1)(0,1)} & \text{for } r \geq \hat{\tau} \\
1 & \text{for } r < \hat{\tau}
\end{cases}
\]

since \( p_{r,g} = q \) for \( r, g \in H \), then the probability of collecting one bid in the time interval \([r, \hat{\tau}]\) is equal to \( \text{Pois}(1; q(r - \hat{\tau})) \). Evaluating all the \( Q_{(r,g)(r',g')} \) and \( S_{r,2} \) in the previous expression we get

\[
F_{r,2} = \begin{cases} 
e^{-q(r-\hat{\tau})} + q(r - \hat{\tau})e^{-qr} & \text{for } r \geq \hat{\tau} \\
1 & \text{for } r < \hat{\tau}
\end{cases}
\]

and clearly

\[
F_{r,1} = Q_{(r,1)(0,1)} = e^{-qr}
\]
To compute the ex-ante success rates $S_{\tau,2}$ we simply take the complement of $F_{\tau,2}$ from Eq. (63).

We are ready to compute the aggregate and conditional expected bid rates. The aggregate bid rate is initially (for $r > \hat{r}$) $q$. At $r = \hat{r}$, it drops by a factor of $(1 - e^{-q(\tau - \hat{r})})$ reflecting probability that the gap did not fall:

$$A_r = \begin{cases} 
q & \text{for } r \geq \hat{r} \\
q (1 - P_{(\hat{r},2)}) = q(1 - e^{-q(\tau - \hat{r})}) & \text{for } r < \hat{r}
\end{cases} \tag{58}$$

$A_r^S$ is instead given by

$$A_r^S = \begin{cases} 
P_{(r,2)}qS_{r,1} + (1 - P_{(r,2)})qS_{r,2} & \text{if } r \geq \hat{r} \\
(1 - P_{(\hat{r},2)}) qS_{r,2} & \text{if } r < \hat{r}
\end{cases} \tag{59}$$

which is identical to

$$A_r^S = \begin{cases} 
q [1 - e^{-qr}] & \text{if } r \geq \hat{r} \\
1 - e^{-q(\tau - \hat{r})} - q(\tau - \hat{r})e^{-q\tau} & \text{if } r < \hat{r}
\end{cases} \tag{60}$$

As in the aggregate case, we can see that the expected bid rate of succeeding campaigns is piecewise-constant with a discontinuity at $\hat{r}$. The intuition behind this pattern is the following: initially, the expected bid rate of succeeding campaign is constant because all campaigns can succeed and, conditional on bidding, success is achieved if at least one additional bid is collected; but, at $\hat{r}$, the only campaigns that can succeed are those with a gap of 1 or lower, so aggregate expected bids drop by a factor of $(1 - e^{-q(\tau - \hat{r})})$ and remain constant afterwards.

Failing projects on the other hand have a positive expected bid rate only if $g_r = 2$. If that case before $\hat{r}$, conditional on bidding, the project fails if no other bid is collected; while from $\hat{r}$ onwards, the bid rate is obviously 0 since the project is frozen.

$$A_r^F = \begin{cases} 
P_{(r,2)}q(1 - S_{r,1}) = qe^{-q\tau} & \text{if } r \geq \hat{r} \\
0 & \text{if } r < \hat{r}
\end{cases} \tag{61}$$

So we can conclude that under homogeneous inspection costs expected bid rates follow a piecewise-constant pattern with a discontinuity at $\hat{r}$ when they jump downward. Notice that, coherently with the analysis developed in Section 5.1, all profiles are overall non-
increasing.

6.0.2 Cold start

When $\hat{\tau}^L \leq \tau < \hat{\tau}^H$ the campaign starts cold. As already discussed in Section 5.2.1 if $c^L = 0$ the campaign experiences at most one change of heat, from cold to hot. In order to compute the expected bid rates, we first determine the failure rate $1 - S_{r,g}$, and as a consequence the success rate as the probability of the complementary event. To do so note that if $g_r = 2$ the project fails with either $g_0 = 2$ or $g_0 = 1$, thus with probability $Q_{(r,2)(0,2)} + Q_{(r,2)(0,1)}$. Since the project is cold as long as no bids is collected

$$Q_{(r,2)(0,2)} = e^{-qz\tau}$$

(62)

$Q_{(r,2)(0,1)}$ is the probability that the campaign follows a path where one bid is collected. This occurs at $r_1 \triangleq \max\{r : g_r \leq 1\}$. At this time, the probability of failing is

$$Q_{(r_1,1)(0,1)} = e^{-q^r_1}$$

(63)

Since the time density of $\hat{r}_1$ is Erlang($z, q, 1$), we obtain $Q_{(r,2)(0,1)}$ integrating-out $\hat{r}_1$ from Eq. (69)

$$Q_{(r,2)(0,1)} = \int_0^r Q_{(r_2)(s,2)}(qz)Q_{(s,1)(0,1)}ds$$

(64)

$$\quad = \int_0^r e^{-qz(r-s)}qze^{-qs}ds$$

(65)

$$\quad = \frac{z}{1-z} (e^{-qzr} - e^{-qr})$$

(66)

Combining Eq. (68) and Eq. (72) we obtain

$$1 - S_{r,2} = \frac{e^{-qzr} - ze^{-qr}}{1-z}$$

(67)

and of course $S_{r,2} = 1 - Q_{(r,2)(0,2)} + Q_{(r,2)(0,1)}$. $\mathcal{F}_{r,1}$ is given by $Q_{(r,1)(0,1)}$, thus

$$1 - S_{r,1} = Q_{(r,1)(0,1)} = e^{-qr}$$

(68)

At this point, we can compute conditional and aggregate average bids. Since the bid rate is $q$ when the campaign is hot and $zq$ when cold, the aggregate expected bid rate is given by

$$A_r = zqP_{(r,2)} + q\left(1 - P_{(r,2)}\right)$$

$$= q \left(1 - (1 - z)e^{-qz(r-r)}\right)$$

(69)
While conditional expected bid rates, take also into account the probability of succeeding or failing after bidding:

\[ A_r^S = \frac{zqP_{(r,2)}S_{r,1} + q(1 - P_{(r,2)})}{S_{r,2}} \]

\[ = \frac{zqe^{-qz(r-r)}(1 - e^{-qr}) + q(1 - e^{-qz(r-r)})}{S_{r,2}} \]  

\[ A_r^F = \frac{zqP_{(r,2)}(1 - S_{r,1})}{1 - S_{r,2}} \]

\[ = \frac{zqe^{-qz(r-r)}(e^{-qr})}{1 - S_{r,2}} \]  

Now we can study the monotonicity of the expected bid rates by looking at how their slopes vary. It is clear from expression 75 that the aggregate average is increasing in time (decreasing in \( r \)), as for the expected bid rate of succeeding campaigns. By looking at expression 77, we can see that the sign of the slope \( A_r^S \) is not obvious. One the one hand, as time passes (and \( r \) reduces), it becomes more likely that the campaign becomes hot, but it also becomes lower the probability of collecting additional bids before the deadline and thus \( S_{r,1} \). To determine formally which effect prevails, we compute the backward-time derivative of the numerator of Eq. (77).

\[ S_{r,2}A_r^S = \frac{d}{dr}\left[ 1 + P_{r,2}(zS_{r,1} - 1) \right] \]

The sign of the slope is determined by the sign of

\[ z\left( \dot{P}_{(r,2)}S_{r,1} + P_{(r,2)}S_{r,1} \right) - \dot{P}_{(r,2)} < 0 \]

(72)

We can see that it is negative by substituting

\[ \dot{P}_{r,2} = qzP_{r,2} \] and \( \dot{S}_{r,1} = q(1 - S_{r,1}) \)

and noticing that Eq. (78) becomes equivalent to

\[ - S_{r,1}(1 - z) < 0 \]

(73)

thus we can see that the heating force prevails and the expected bid rate of succeeding projects is increasing in time (decreasing in \( r \)).

We apply the same procedure to study the monotonicity of failing campaigns. The

\[ A_r^S = \frac{A_r^F(1 - S_{r,2})}{S_{r,2}} \]

An alternative procedure to construct the average of successful project is
sign of the derivative with respect to $r$ is determined by the following condition:

$$\dot{P}_{(r,2)}(1 - S_{r,1}) - P_{(r,2)}\dot{S}_{r,1} = qzP_{(r,2)}(1 - S_{r,1}) - P_{(r,2)}q(1 - S_{r,1}) < 0$$  \hspace{1cm} (74)$$

so also the expected bid rate of failing project is increasing in time (decreasing in $r$). Also this time we see the contrast between two opposite forces: on the one hand, the expected bid rate is positive only if the gap is 2, and the probability of this event reduces with time (increases with $r$); on the other hand, for a given gap, it is more likely to fail closer to the deadline, so late bidding have a higher weight than early bidding on the conditional expectation. The increase of the conditional expected bid rate in time is determined by the prevalence of second force.

6.0.3 Hot start

If instead we assume $\tau < \hat{\tau}$, the campaign starts hot, and as in Section 5.1 the campaign goes cold if no bid is collected up to time $\hat{\tau}$. The difference with the case of homogeneous inspection costs is that now a cold campaign can still receive bids from informed bidders. In this setting, the campaign can experience at most two changes of heat: either becoming and staying cold, or doing in-and-out from the cold region.

To compute the failure rate before the switching time (for $r \geq \hat{\tau}$), we sum the probability that the project fails with $\hat{g}_0 = 2$, to the probability that the project fails with $\hat{g}_0 = 1$. The second event can occur if ether $g_{\hat{r}} = 1$ and $g_0 = 1$, or when $g_{\hat{r}} = 2$ and $g_0 = 1$; that is

$$F_{r,2} = Q_{(r,2)(\hat{r},2)}Q_{(\hat{r},2)(0,2)} + \left( Q_{(r,2)(\hat{r},1)}Q_{(\hat{r},1)(0,1)} + Q_{(r,2)(\hat{r},2)}Q_{(\hat{r},2)(0,1)} \right)$$  \hspace{1cm} (75)$$

$$= e^{-q(\hat{r}-\hat{r})}e^{-q\hat{r}} + q(r - \hat{r})e^{-q(r-\hat{r})}e^{-q\hat{r}} + e^{-q(r-\hat{r})} \int_{0}^{\hat{r}} e^{-zq(\hat{r}-s)}qze^{-qs}ds$$  \hspace{1cm} (76)$$

While after the switching time (for $r \leq \hat{r}$), the campaign fails by either staying cold until the end, with probability $e^{-qr}$, or becoming hot but missing to meet the funding goal, with probability $\int_{0}^{\hat{r}} e^{-qz(r-s)}qze^{-qs}ds$. Summarizing:

$$F_{r,2} = \begin{cases} 
\frac{e^{-q(\hat{r}-\hat{r})}e^{-q\hat{r}} + q(r - \hat{r})e^{-q(r-\hat{r})}e^{-q\hat{r}} + e^{-q(r-\hat{r})} \int_{0}^{\hat{r}} e^{-zq(\hat{r}-s)}qze^{-qs}ds}{32} & \text{for } r \geq \hat{r} \\
\frac{e^{-qr} + \int_{0}^{\hat{r}} e^{-qz(r-s)}qze^{-qs}ds}{32} & \text{for } r < \hat{r}
\end{cases}$$  \hspace{1cm} (77)$$

On the other hand, if $g_r = 1$, then the campaign fails by missing to collect any bid while hot.

$$F_{r,1} = e^{-qr}$$  \hspace{1cm} (78)$$

Success rate can be computed as the complement of Eq. (83) and Eq. (84).
Expected aggregate and conditional bid rates are given by:

\[
A_r = \begin{cases} 
q & \text{for } r \geq \hat{r} \\
qzP_{(r,2)} + q(1 - P_{(r,2)}) & \text{for } r < \hat{r}
\end{cases}
\]  

(79)

\[
A^S_r = \begin{cases} 
qS_{r,1} & \text{for } r \geq \hat{r} \\
qzP_{r,2}S_{r,1} + q(1 - P_{r,2}) & \text{for } r < \hat{r}
\end{cases}
\]  

(80)

\[
A^F_r = \begin{cases} 
\frac{q(1 - S_{r,1})}{1 - S_{r,2}} & \text{for } r \geq \hat{r} \\
\frac{P_{r,2}qz(1 - S_{r,1})}{1 - S_{r,2}} & \text{for } r < \hat{r}
\end{cases}
\]  

(81)

The slope of the aggregate expected bid rate is easy to determine. Eq. (85) is constant for \( r \geq \hat{r} \) while for \( r < \hat{r} \) the slope can be computed as

\[
\dot{A}_r = -q(1 - z)\dot{P}_{(r,2)} \leq 0
\]  

(82)

since

\[
\dot{P}_{(r,2)} = \frac{d}{dr}Q_{(r,2)}(\hat{r},2)Q_{(r,2),(r,2)} = \frac{d}{dr}e^{-q[(\tau - \hat{r}) + z(\hat{r} - r)]} = qzP_{(r,2)} \geq 0
\]  

(83)

The expected aggregate bid rate increasing in time (decreasing in \( r \)) due to the higher probability that the campaign becomes hot.

Now we compute and compare the slope of the conditional expected bid rates. Since the denominator of all the previous expressions is constant, we check how the slopes vary with time by taking the derivative of the numerators with respect to \( r \). The slope of Eq. (86) for \( r \geq \hat{r} \) is clearly proportional to

\[
\dot{S}_{r,1} = -q(1 - e^{-\hat{r}z}) = qe^{-\hat{r}z} = q(1 - S_{r,1}) \geq 0
\]  

(84)

so it is decreasing in time (increasing in \( r \)). Since collecting early bids makes the campaign hot, and hot campaigns are more likely to be successful. When \( r \in [\hat{r}, 0] \), how the direction of change of \( A^S_r \) is ambiguous. Indeed, as time passes, the gap is less likely to be 2, which implies higher expected bids; but, it is also less likely to observe bidding since early bidding is prevailing for successful projects. The expected bid rate of successful project is expected to increase with time if

\[
\frac{d}{dr}(P_{(r,2)}S_{r,1}) - \dot{P}_{(r,2)} \leq 0
\]
Using the chain rule, the previous expression is equivalent to

\[ \dot{P}_{r,2}S_{r,1} + \dot{S}_{r,1}P_{r,2} \leq P_{r,2} \]  

(85)

Using Eq. (89) and Eq. (90), we write it explicitly as

\[ q[1 - S_{r,1}(1 - z)] \leq 1 \]  

(86)

which always holds; so, the bid rate of successful campaigns is expected to decrease when \( r > \hat{r} \), drop at \( r = \hat{r} \) and increase when \( r < \hat{r} \). Note that the motive behind the initial decreasing region relies only on threshold effects, so provides another interpretation of the empirical evidence: the initial peak of the U-shape might be driven by threshold effects, while the last peak by the expected increase in heat.

For failing campaigns, we can easily determine from Eq. (87) that \( \dot{A}_{r,\hat{r}} \leq 0 \) for \( r \geq \hat{r} \), so the expected bid rate is increasing in time (decreasing in \( r \)) as failing implies higher probability of late bidding. For \( r \leq \hat{r} \), whether the slope is positive or negative depends on the opposite of condition Eq. (92). The idea is that, as time passes, chances of having a positive bid rate decrease, but among failed projects, profiles with late bidding are prevailing as we saw before. Since the slope goes in the opposite direction than successful campaigns, we can conclude that the profile is decreasing in time (increasing in \( r \)).

### 6.0.4 Temperate start

In this section we three heat regions with temperate start: the project starts temperate, but as time passes, it can either go on fire or become frozen depending on how rapidly or slowly the gap falls, so a campaign experiences at least one change of heat.

To compute the probability of success or failure, we first calculate the probability that the campaign collects 1 or 0 bids. If the \( g_r = 2 \) for \( r < \hat{r} \), then no additional bid can be collected. Thus, for \( r \geq \hat{r} \),

\[ Q_{(r,2)(0,1)} = \int_{\hat{r}}^{r} Q_{(r,2)(s,2)} zqQ_{(s,1)(0,1)} ds \]

\[ = \int_{\hat{r}}^{r} e^{-qs(r-s)}zq e^{-qs} ds \]

\[ = zq e^{-qz} \int_{\hat{r}}^{r} e^{sq(z-1)} ds \]

\[ = \left. \frac{z}{z-1} e^{-qs+sq(z-1)} \right|_{\hat{r}}^{r} = \frac{z}{z-1} e^{-qz+(r-\hat{r})q(z-1)} \]

\[ Q_{(r,2)(0,2)} = e^{-qs(r-\hat{r})} \text{ for } r \geq \hat{r} \]

and obviously \( Q_{(r,2)(0,1)} = 0 \) and \( Q_{(r,2)(0,2)} = 1 \) for \( r < \hat{r} \). Since as before \( F_{r,g} = Q_{(r,2)(0,1)} + \)
\( Q(r,2) = 1 \) for \( r < \hat{r} \) and \( F_{r,2} = e^{-qz(r-\hat{r})} + \frac{z}{z_1} e^{-qz+(r-\hat{r})q(z-1)} \) for \( r \geq \hat{r} \).

Average bids rates are

\[
A_r = \begin{cases} 
P(r,2)zq + (1 - P(r,2))q & \text{if } r \geq \hat{r} \\
(1 - P(r,2))q & \text{if } r < \hat{r}
\end{cases}
\] (87)

\[
A_r^S = \begin{cases} 
\frac{P(r,2)zS_{1,r} + (1 - P(r,2))q}{S_{r,2}} & \text{if } r \geq \hat{r} \\
\frac{(1 - P(\hat{r},2))q}{S_{r,2}} & \text{if } r < \hat{r}
\end{cases}
\] (88)

\[
A_r^F = \begin{cases} 
\frac{P(r,2)q(1 - S_{r,1})}{(1 - S_{r,2})} & \text{if } r \geq \hat{r} \\
0 & \text{if } r < \hat{r}
\end{cases}
\] (89)

Before the switching time \( \hat{r} \) (for \( r \geq \hat{r} \)), the sign of the slope of the expected aggregate bid rate is negative, so expected aggregate bids increase with time due to the higher probability that the campaign goes on fire. Indeed,

\[
\dot{A}_r = \frac{d}{dr} \left[ q \left( 1 - P(r,2) (1-z) \right) \right] < 0
\]

Since obviously

\[
\dot{P}_{r,2} = \frac{d}{dr} e^{-qz(\tau-r)} > 0
\] (90)

At \( \hat{r} \), the expected bid rate jumps downwards by \( P(r,2)zq/S_{r,2} \), reflecting campaigns with \( g_\hat{r} = 2 \) that fail. After the switching time (For \( r < \hat{r} \), \( A_r = 0 \), so the expected aggregate bid rate is constant and equal to the bid rate when the campaign is on fire scaled by the probability of collecting at least one bid before period \( \hat{r} \).

Now we analyse the slope of conditional bid rates. As in ?? and Section 6.0.3, we determine the sign of the slope by looking at the numerators of expressions Eq. (94) and Eq. (95):

\[
S_{r,2} \dot{A}_r^S = \frac{d}{dr} \left[ q \left( 1 - P(r,2) (1-z) \right) \right] \] (91)

which is positive if

\[
z \frac{d}{dr} (P_{r,2}S_{r,1}) - \dot{P}_{r,2} \leq 0
\]

using Eq. (89) and Eq. (59), we find that the previous expression is equivalent to

\[
z (S_{r,1} + z(1 - S_{r,1})) \leq 1
\] (92)
so we can see that for $r \geq \hat{r}$ the expected bid rate of successful campaigns is expected to increase with time (decrease with $r$). At $\hat{r}$, the expected bid rate jumps downward accounting for campaigns on paths where $g_r = 2$ that fail, and remains constant afterwards.

Finally, we check the slope of the expected bid rate of failing projects. From (95) we see clearly see that for $r < \hat{r}$, that the opposite of Eq. (91) holds, so the slope is positive, and the expected conditional bid rate is decreasing in time (increasing in $r$). at $\hat{r}$, failing campaigns either go frozen or survive and remain temperate but fail to collect any additional bid.

The following lemma summarizes all the finding of Section 6:

**Lemma 4** (Dynamic profile of the campaign in the continuous time model). Using the continuous time approximation of the model and assuming $c_i \in c, c'$ with $c' \geq c$ and $q \geq c, c' \geq 0$, the model predicts the following bidding patterns in a $K = 2$ example:

- if $c = c'$, so that inspection costs are homogeneous, then aggregate and conditional expected bid rates are piecewise-linear with a discontinuous drop at $\hat{\tau} = -1/q \ln(1 - q/c)$.

- if $c = 0, c' > 0$ and $\tau < \hat{\tau}' = -1/q \ln(1 - q/c')$, the campaign starts cold, and all profiles are continuous and increasing in time.

- if $c = 0, c' > 0$ and $\tau \geq \hat{\tau}' = -1/q \ln(1 - q/c')$, the campaign starts hot. Aggregate and conditional expected bid rates are discontinuous at $\hat{\tau}$. $A_r$ is initially constant (and positive) and increasing after $\hat{\tau}$. $A_F$ is initially decreasing and increasing after $\hat{\tau}$.

- if $0 < c < c'$ and $\hat{\tau}' < \tau < \hat{\tau}$, the campaign starts temperate. Expected bid rates are discontinuous at $\hat{\tau}$: $A_r$ and $A_F$ are increasing before $\hat{\tau}$ and constant (and positive) afterwards, while $A_F$ is decreasing before and zero after $\hat{\tau}$.

7 Endogenous movers

In this section, we relax the assumption that all bidders act only upon arrival by introducing a group of endogenous movers. We call them “endos” for short. They follow the campaign from the moment it initiates up to its end. We formally let them differ form exo-movers by replacing action Abstain, with action Delay, which means, wait till the next round and choose an action again. Thus, endo-movers choose among actions B,C and D.\(^{13}\) We break a tie on actions C and D by assuming that C is played.

**Assumption 2** (Tie-breaking rule for endo-movers). If $C \sim_i D$, then $a_i = C$.

\(^{13}\)Endo-movers abstain from bidding by always playing D.
Bidders have a preference for checking the project early because in that way they can devote their energy to other fruitful activities, or because they face a small waiting cost. Under assumption 1, endo-movers’ strategy specifies an action \( a_{r,g} \in \{C, D\} \) at each \( r, g \). Introducing endo-movers to the model creates an additional strategic element: \( C \) actions are strategic complements; \( D \) actions are strategic substitutes.

Allowing endogenous timing complicates the analysis considerably since the current funding state is generally not a sufficient statistic for summarizing bidders’ information sets. Bidders have to form a belief on the composition of the group of other bidders waiting to play \( C \) at a future state, and on the group of those who, having learned that they do not like the project’s product, play \( D \) until the campaign ends.

### 7.0.1 The role of friends

For the moment, we assume it is common knowledge that \( m \) informed endo-movers (with zero inspection cost) follow the campaign from its inception. They can be thought of as a group of friends or commercial partners of the entrepreneur. Friends are informed about the project and, in case they like it, they place their bids so as to help raise its chance of success. The resulting game can still be solved by adding little to the analysis developed in Section 4. Since multiple bids can take place in a given round, bidders form beliefs on the number of bids collected from other simultaneous bidders. From the point of view of a given friend \( i \), we let \( b_{-i} \in \{0, (m-1)+1\} \) denote the bids collected from other bidders (\( m-1 \) friends and one exo). We show that such bidders optimally concentrate their bids at the start, thereby generating the initial peak of the U-shape.

The following lemma states that friends never choose to delay their bids.

**Lemma 5** (Friends’ equilibrium play). Friend’s strategies are such that \( a_{r,g} = C \) for all \( r, g \).

**Proof of Lemma 5.** Since the game is finite, we apply the one-shot deviation principle. Suppose friend \( i \) deviates to play action \( D \) in some state \( (r, g) \) before reverting to the strategy \( C \) at \( r-1 \). His equilibrium payoff at \( (r, g) \) is \( U_{r,g}^C = \sum_{b_{-i}=0}^{m} p(b_{-i}) S_{r-1,g-1-b_{-i}} \), where \( p(b_{-i}) \) is the probability of collecting \( b_{-i} \) bids from other simultaneous bidders. On the other hand, his payoff from deviating to action \( D \) is given by \( U_{r,g}^D = \sum_{b=0}^{m} p(b_{-i}) S_{r-1,g-b_{-i}} \). By Lemma 1, \( S_{r,g} \) is weakly decreasing in \( g \); thus \( S_{r-1,g-b_{-i}} \leq S_{r-1,g-1-b_{-i}} \) for all \( r, g \) and \( b_{-i} \). If \( U_{r,g}^D = U_{r,g}^C \), by Assumption 2, \( a_{r,g} = C \); a contradiction. If \( U_{r,g}^D < U_{r,g}^C \), we also have \( U_{r,g}^D \geq U_{r,g}^C \) since equilibrium actions are optimal; a contradiction. We can conclude that \( D \) is never played in equilibrium.

This result is due to the strategic complementarity of actions \( C \): collecting more bids leads to a higher success rate, which makes other bidders more willing to inspect. Since friends maximize the success rate, they have the highest impact on future exo-movers by placing their bids as soon as the campaign starts.
8 Conclusion

We have presented a highly parsimonious model of crowdfunding that is able to account for a variety of observed momentum effects in the dynamics of funding, including the much-commented U-shape profile. Of course, reality is substantially more complex than our baseline model, which focuses on pure private values and bidders exogenously constrained to either bid in the same period that they become aware of the project or fully abstain from bidding. However, as with our endogenous bidder extension, it is clear that the insights from our basic set-up readily extend to cover richer scenarios and to generate additional effects. Moreover, the basic model’s simplicity is a virtue, since it allowed us to understand the driving dynamic forces despite the complexity of the challenge faced by bidders. They must anticipate how project success chances depend on the inspection calculus of all bidders who have not yet moved. Leaving aside the “friends” who enjoy looking at the entrepreneur’s project, each bidder is only willing to dedicate any time or effort to inspecting a project if learning his type is useful. That in turn requires the project to have a chance of success. As a result, the distribution of inspection costs and bidder knowledge about those costs take centre-stage.

Crowdfunding campaigns have three key features: the threshold, the price (more generally, minimum price) and the duration. The difference between this threshold and the current funding aggregate, when divided by the price, determines the funding gap that later bidders must bridge before the deadline is reached (duration completed) in order for the project to succeed and for any benefits to accrue. However, it is not just the initial gap that matters, because bidders arrive over time and if the gap is too high relative to time remaining, bidders may stop inspecting and the project may fizzle out. With a distribution of inspection costs, some bidders are relevant while others are not, in that their inspection-costs are prohibitive and they do not even consider bidding. Overall, projects are more likely to succeed the lower is the initial funding gap (threshold divided by price) relative to project duration and the inspection-cost adjusted bidder arrival rate.

Our study is incomplete but we can summarize some preliminary results.

For relatively high initial gaps, projects start in a cold state in that only lowest cost bidders are willing to inspect. If there is a sufficient mass of bidders with no cost of inspection, such projects are likely to end up in a hot state, and this draws in a greater rate of inspection and hence bids among later arrivals. The bidding rate profile is therefore increasing over time. Conversely, when the gap is initially relatively low, the project starts in a hot-state, but projects face a risk of falling out of their hot state. Such projects tend to exhibit a decreasing funding profile. Conditioning on project success and failure moderates these effects but they remain relevant.

Including the set of endogenously timed arrivals of friends and close contacts then allows us to explain the U-shaped profile in cases where projects start in a cold state.
Loosely, given that project success rates on Kickstarter are generally under 50%, it is likely that a majority indeed begin in a cold state. So our model is broadly consistent with the U-shaped dynamics that initially motivated our study. In later drafts of this paper, we will conduct empirical tests to probe our understanding further.

The distribution of inspection costs also plays a major role. If there is an atom of this distribution at zero (or equivalently, if inspection costs can be negative in that bidders enjoy inspecting), then projects can more easily get hot after a cold start. If there is no such atom, the possibility for projects to grow over time is too small and the U-shape is not expected.

As we noted in the introduction, it will be important to allow for common value effects and advertising to account for the full wealth of possible dynamics. Common values readily generate positive and negative cascades but combining with the threshold effect in our dynamic framework is far from trivial.

Advertising and project promotion effects are highly compatible with our model. Depending on the technology of advertising, it is likely that a lot could be gained from strategically-timed promotions that raise bidder arrival rates precisely when projects are at risk of going cold and dying out. Such an analysis would have to pay attention to bidders’ awareness of the promotion strategies in use.

Our analysis is also relevant for entrepreneurs who need to evaluate how their strategic choices affect project success prospects. They know that raising price lowers the chance of a success. Our analysis shows that the resulting trade-off is significantly more complicated than simply assessing the initial gap ($\lceil T/p \rceil$), as sufficed in the static context with a single price. Dividing the gap by time available ($T/pr$) is a natural first look in a setting where bidders arrive over the duration of the project, but inspection costs can relegate many potential bidders into mere irrelevancies who pass over the project if the gap when they come across the project is too high relative to the remaining time available on the campaign clock.

References


