Merger Analysis in Multiproduct-Firm Oligopoly with Network Externalities

Preliminary and Incomplete

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Using an aggregative-games approach, I present a model of multiproduct-firm oligopoly with firm-level direct and indirect network externalities and analyze the impacts of network effects on the merger policy. First, focusing on the direct network externalities, I find a string of results. Any acquisition of a sufficiently small firm is CS-increasing without any technological synergies. In an originally symmetric industry, a merger between two firms is CS-increasing without technological synergies if and only if the magnitude of the direct network effects is above certain threshold, which decreases with the number of firms. I also examine how direct network externalities influence the technological synergies required for mergers between two firms to be CS-increasing and find that (1) the required synergies for mergers between weak firms decrease with the magnitude of network externalities, (2) the synergies required for acquisition of weak firms by strong firms decrease with the magnitude of network effects as long as their joint market shares are not too large, and (3) the synergies required for mergers between two strong firms that leads to extreme concentration increase with the magnitude of network effects. Finally, I present a merger analysis in two-sided markets that is characterized by indirect network externalities and discuss how the firm size of merging party on one side affects the merger policy on the other side.

Keywords: Aggregative games; merger policy; network effects; two-sided markets

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1. Introduction

In the digital economy, firms that operate platforms enable the interaction among participants, generating the direct and indirect network externalities. While beneficial themselves, the network externalities inherent to these businesses give rise to the positive-feedback effects, leading consumers to concentrate on a few number of dominant platforms. Due to this “winner-takes-all” feature, competition authorities in many countries raise serious concerns about the potential harm to the consumers through a persistent dominance of particular platforms.

One particular interest of competition authorities is how to conduct merger (or divestiture, if any) policy when the merging entities operate platforms. Many scholars argue that the merger policy should be adjusted according to the emergence of multi-sided platforms. Relative to its importance, however, there is little knowledge on how the scrutiny of merger review should vary with the degree of network externalities and two-sidedness. The aim of this study is to enrich the theoretical guidance to the merger policy in the presence of network externalities.

There are several opposing viewpoints on the social desirability of mergers in the presence of network externalities. On one hand, the consolidation of two networks generates the greater compatibility and thus benefits consumers. On the other hand, such network expansion enlarged the market power of merged entity, which may increase markups and make small firms difficult to expand their networks. While the former effect works in favor of the merger, the latter would work against the merger. The relative importance of these two effects should depend on the sizes of merging entities and the magnitude of merger-specific efficiency gains, among others. Furthermore when the merger involves multi-sided platforms, the matter is even more complex. The merger between multi-sided platforms accompanies the change in revenue structure, which leads to the change in price structure. This change in price structure should be taken into account in addition to the aforementioned effects of network externalities.

There is no comprehensive theory of merger policy that addresses the issues mentioned above. The aim of this study is to develop a theoretical framework to address these issues. In this study, I consider how the merger policy should be conducted by a consumer-surplus oriented competition authority when the industry is characterized by network externalities.

Taking a view that the objective of competition authorities can be approximated by consumer surplus (Whinston (2007)), I examine how the merger policy by a consumer-surplus oriented competition authority should be conducted in multiproduct-firm oligopoly with network effects. Specifically, I introduce the firm-level network externalities into the framework of multiproduct-firm oligopoly by Nocke and Schutz (2018a), and examine the impacts of network externalities on the merger policies.

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1Examples of such platform businesses include credit cards, operating systems, video game consoles, instant-messaging, social-networking services, media, etc. See Evans and Schmalensee (2016) for the review of multi-sided platform businesses. Direct network externalities typically stem from the nature of communication among network participants (instant-messaging, social-networking), the peer-effect in the usage of products (Bapna and Umyarov (2015)), values generated from use-generated contents (Zhang, Evgeniou, Padmanabhan and Richard (2012)), tariff-mediated network externalities (Laffont, Rey and Tirole (1998)), and higher-quality services through data accumulation (Belleflamme and Peitz (2018)).

2The concern about the market power leveraged by network externalities is not an entirely new issue. For example, in AOL/Time-Warner merger case, FTC and FCC raised the concern that in the instant-messaging service where AOL originally had a dominant position in the market, the merged entity would obtain even more dominant position by leveraging the network externalities and the rich contents provided by Time-Warner. See Faulhaber (2002) for a review.

3See Evans and Noel (2008) for example.
The presence of direct network effects affects the merger policy in two ways. On the one hand, network effects generate demand-side synergies through mergers, which increases the consumer surplus from increased concentration. On the other hand, network effects increase the market power of the merged entity by leveraging the expanded network size, which increases the markups and hurts the consumer surplus. Whether the greater network effects are associated with more stringent merger policy depends on which effect dominates. I identify several situations where one effect dominates the other.

First, I find that in the presence of network effects, any acquisition of a sufficiently small firm is CS-increasing without technological synergies. This reflects the fact that the gain from an access to the product offered by a small acquired firm always dominates the marginal increase in the market power of the acquiring firm.

Next, I find that a merger between two firms in a symmetric oligopoly is CS-increasing without technological synergies if and only if the strength of the magnitude of network effects is above certain threshold, which decreases with the number of firms. This result highlights the trade-off between network expansion and market power in terms of merger policy. The merger improves the consumer surplus if the consumer gain from network expansion is greater than the increase in the market power, which is likely to hold when the strength of network effects is large. Further, the gain from network expansion relative to the loss from increased market power becomes greater as the number of firm increases. This decreases the strength of network effects to justify the merger between two firms.

Finally, I examine how the required technological synergies for mergers between two firms to be CS-increasing changing with the strength of network effects and find that (1) the required synergies for mergers between weak firms decrease with the magnitude of network effects, (2) the synergies required for acquisitions of weak firm by strong firms decrease with the magnitude of network effects as long as their joint market shares are not too large, and (3) the synergies required for mergers between two strong firms that leads to extreme concentration increase with the magnitude of network effects.

After the analysis of merger policy in the presence of direct network externalities, I briefly analyze the merger policy in two-sided markets, which is characterized by indirect network externalities. Assuming that only consumers on one side gains from indirect network externalities, I obtain some limit results. First, for any merger between two firms, there is a unique pair of technological synergies on two sides of the market. Then, I show that the effect of a greater productivity/quality of a merging firm on one side may leads to both the stricter and looser requirement for the technological synergies. The condition under which the greater value on one side is associated with the stricter requirement for technological synergies depends on the configuration of the pre-merger markets shares of merging entities.

Putting my findings together, the presence of network effects may serve as a justification of mergers that do not lead to extreme concentration. However, the network effects make the merger that leads to extreme concentration more likely to hurt consumers. Thus, the relation between merger policy and the network depends on the market shares of merging firms.

The remainder of the paper is organized as follows. Section 2 reviews the related literature. Section 3 presents a workhorse of multiproduct-firm oligopoly with direct and indirect network externalities. Section 4 and Section 5 then analyze the impacts of direct and indirect network externalities on merger policies, respectively. Section 6 concludes.
2. Related Literature

This study is related to three strands of interrelated literature: competition with network effects, competition in two-sided markets, and welfare effects of horizontal mergers. I review these strands of literature and explain the contribution of the present study to each of them.

2.1. Competition with Network Externalities

The first strand of literature is the competition with network externalities. (Katz and Shapiro (1985), Farrell and Saloner (1986), Farrell and Klemperer (2007), Cabral (2011)). This literature examines how the network externalities affect the consumer network decisions (Farrell and Saloner (1986)), its impact on the firms’ pricing in static (Katz and Shapiro (1985)) and dynamic environments (Farrell and Klemperer (2007), Cabral (2011)). My framework belong to the static model of price competition and illustrates how the impact of merger changes according to the value of network externalities and the merging entities’ sizes.

2.2. Competition in Two-Sided Markets

The second strand of literature is the competition in two-sided markets (Caillaud and Jullien (2003), Rochet and Tirole (2003), Rochet and Tirole (2006), Armstrong (2006), Weyl (2010)). This literature focus on the determinants of price structure in the presence of indirect network externalities. Recent studies started to analyze the impacts of mergers between two-sided platforms on the welfare of consumers (Correia-da Silva, Jullien, Lefouili and Pinho (2019), Anderson, Foros and Kind (2019), Anderson and Peitz (2015)). Anderson and Peitz (2015) adopts an aggregative-game model to analyze the competition among media platforms in competitive-bottleneck model and show that there is often conflicts of interests between consumers and advertisers on the effects of merger between media platforms. Anderson et al. (2019) analyze the effects of consumer multihoming on the competition between media platforms, and show that multihoming flips the side of the market on which platforms compete, which alters the side that is affected by the merger between platforms. Finally, Correia-da Silva et al. (2019) presents a simple model of Cournot competition among homogeneous platforms, and show that an average-marginal-cost-preserving merger harms consumers on both sides when the total network effects are small, while it benefits consumers on both sided when the total network effects are large. I contribute to this literature by analyzing the effects of the change in the pricing structure after a merger on the consumer surplus on each sides.

My main modelling framework is based on the multinomial logit discrete choice model, which is not typically used in theoretical analysis of two-sided markets, but frequently adopted in empirical frameworks (Ohashi (2003), Rysman (2004), Clements and Ohashi (2005), Lee (2013), Jeziorski (2014)).

2.3. Welfare Effects of Horizontal Mergers

The last strand of literature is the traditional merger analysis. (Farrell and Shapiro (1990), Cunningham, Ederer and Ma (2018), Whinston (2007), Nocke and Schutz (2018b), Nocke and Schutz (2018a), Faulhaber (2002)). In particular, the general framework of my model is an extension of Nocke and Schutz (2018a). I contribute to this literature by examining how the
results in standard setting is preserved or altered by an introduction of direct and indirect network externalities.

3. General Framework of Multiproduct-Firm Oligopoly with Network Externalities

I first present a general framework which our specific analyses belong to. There is an industry with two sides of markets \( A, B \) with a set \( \mathcal{N}^J \) of imperfectly substitutable products on side \( J \in \{A, B\} \), produced by a set of firms \( \mathcal{F} \). Each firm \( f \) produces the set \( \mathcal{N}^J_f \) of products on side \( J \).

**Consumers** Consumers on side \( J \) derive firm-level network externalities from the purchase of each product \( i \in \mathcal{N}^J_f \), which depends on the number of consumers who purchase the products of firm \( f \). Specifically, each consumer on side \( J \in \{A, B\} \) yields the indirect subutility from the purchase of product \( i \in \mathcal{N}^J_f \) by

\[
\log z^J_i(p_i) + \alpha_J \log n^J_f + \beta_J \log n^I_f + \epsilon_i,
\]

where \( \log z^J_i(p_i) \) is the stand-alone indirect subutility from product \( i \) at price \( p_i \), \( n^J_f \) and \( n^I_f \) are the amounts of consumers on side \( J \) and side \( I \) who purchase products provided by firm \( f \). \( \alpha_J \in (0, 1) \) represents the magnitude of direct network externalities, \( \beta_J \in (0, 1) \) represents the magnitude of indirect network externalities, and \( \epsilon_i \) is an idiosyncratic taste shock that follows i.i.d. type-I extreme value distributions. I assume that \( (1 - \alpha_A)(1 - \alpha_B) - \beta_A\beta_B > 0 \), which means that network effects are not too strong. In the later analysis, I adopt a multinomial logit specification where

\[
h^J_i(p_i) = \exp \left( \frac{a_i - p_i}{\lambda^J} \right)
\]

and a CES-type specification where

\[
h^J_i(p_i) = \begin{cases} a_ip_i^{1-\sigma^J} & \text{if } p_i > 0 \\ +\infty & \text{if } p_i \leq 0 \end{cases}
\]

with \( \sigma^J > 1 \).

Given the network sizes \( (n^A_f, n^B_f) \) for each \( f \) and prices \( p := (p_i)_{i \in \mathcal{N}^A \cup \mathcal{N}^B} \), consumers choose one product to purchase and the amount of the purchase. This formulation can be seen as an extension of the discrete/continuous choice microfoundation of the demand system in Nocke and Schutz (2018b). I assume that there is no outside option so that all consumers on side \( J \) purchase some product in the set \( \mathcal{N}^J \). I further assume that consumers single-home, that is, each consumer chooses only one product to purchase. From the above utility specification, the corresponding demand system is derived as follows. First, define the firm-level and industry-level aggregators

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4I mean by the word “CES-type” that, if \( \alpha_J = \beta_J = 0 \), \( J \in \{A, B\} \), the demand system obtained from indirect subutility \( h^J_i(p_i) = a_ip_i^{1-\sigma^J} \) corresponds with that of CES demand function.
The network share of products produced by firm $f$ is given by

$$H_f^I(p_f') = \sum_{i \in \mathcal{N}_f'} h_i^f(p_i), \quad \text{where } p_f' := (p_i)_{i \in \mathcal{N}_f'}.$$  

Next, I derive demand for each product conditional on the purchase. Applying Roy’s identity, the conditional demand function for product $i$ conditional on the purchase is given by $-(h_i^f)'(p_i)/H_i^f(p_i)$. I assume that consumers form the correct expectation that all firm have positive network shares. I call the network choice of consumers based on such expectation as an interior equilibrium. The consumer choice probability $s_i^J$ of product $i \in \mathcal{N}_f'$ given the network shares $(n_i^f, n_i^{f'})_{f' \in F}$, $J = A, B$ is given by

$$s_i^J = \frac{h_i^f(p_i) \left( n_i^f \right)^{\alpha_J} \left( n_i^{f'} \right)^{\beta_J}}{\sum_{f' \in F} \sum_{j \in \mathcal{N}_f'} h_j^f(p_j) \left( n_j^f \right)^{\alpha_J} \left( n_j^{f'} \right)^{\beta_J}}. \quad (2)$$

The network share $n_i^f$ of firm $f$ on side $J$ is given by the sum of the choice probability of products produced by firm $f'$ on side $J$:  

$$n_i^f = \sum_{i \in \mathcal{N}_f'} s_i^f. \quad (3)$$

From equations (2) and (3), the share of product $i \in \mathcal{N}_f'$ in the set of products sold by firm $f$ is given by

$$s_i^f = \frac{h_i^f(p_i)}{H_i^f(p_f')}. \quad (4)$$

As derived in the Appendix A.1, the network share $n_i^f$ of firm $f$ on side $J$ in the interior consumer equilibrium is given by

$$n_i^f(p) = \frac{1}{H^I(p)} \left( (H_i^I(p_f'))^{\frac{1}{1-\alpha_J}} \left( H_i^I(p_f') \right)^{\frac{\beta_J}{1-\alpha_J}} \left( H_i^I(p_f') \right)^{\frac{\beta_J}{1-\alpha_J}} \right). \quad (5)$$

Combining equations (5) and (4), the probability that product $i \in \mathcal{N}_f'$ is purchased by a consumer is given by the equation

$$s_i^f(p) = n_i^f(p) \frac{h_i^f(p_i)}{H_i^f(p_f')} \quad (6)$$

Finally, the demand for the product $i \in \mathcal{N}_f'$ given the profile of prices $p$ has the following form.

$$D_i^f(p) = D_i^f \left( p_i, H_i^I(p_f'), H_i^I(p_f'), H_i^I(p) \right)$$

$$= s_i^f(p) \times \frac{-(h_i^f)'(p_i)}{h_i^f(p_i)} \quad (7)$$

$$= -\left( H_i^I(p_f') \right)^{\frac{1}{1-\alpha_J}} \left( H_i^I(p_f') \right)^{\frac{\beta_J}{1-\alpha_J}} \left( H_i^I(p_f') \right)^{\frac{\beta_J}{1-\alpha_J}} \frac{h_i^f(p_i)}{H^I(p)}$$
With CES-type demand and negative price, we cannot use Roy’s identity to derive demand. To allow for the demand for negative prices, we assume that \( D_f^i(p) = \lim_{p_i \to 0} D_f^i(p) = +\infty \) for \( p_i < 0 \).

**Firm pricing** Each product \( i \in N^J \) has a constant marginal cost \( c_i \) of production. Given this demand system, the profit function of each firm \( f \in F \) is given by

\[
\Pi_f(p_f, H^A(p), H^B(p)) = \Pi^A_f + \Pi^B_f
\]

where

\[
\Pi^I_f = \sum_{i \in N^I_f} D_f^i(p_i, H_f^i(p_f^i), H_f^i(p_f^i), H_f^i(p)) (p_i - c_i).
\]

The pricing game consists of the demand systems \((h^I_f)_{i \in N}, \alpha^J, \beta^J \in \{A, B\} \) the set of firms \( F \), and a profile of marginal costs \((c_i)_{i \in N^J}, J \in \{A, B\} \). In the pricing game, firms simultaneously set the prices of their products. I call a Nash equilibrium of this pricing game as a pricing equilibrium. In the following analysis, I often suppress the arguments of functions to save the space.

As formally derived in Appendix [A.1], the first-order condition for the profit-maximization of each firm \( f \) is given by

\[
-(h^I_f)'' (p_i - c_i) = 1 - \frac{1}{n^J_f} \frac{\{(1 - \alpha_f)\alpha_f + \beta_f^J \beta_f \Pi^I_f + \beta_f I \Pi^I_f}{(1 - \alpha_f)(1 - \alpha_f) - \beta_f^J \beta_f} + \frac{(1 - \alpha_f)}{(1 - \alpha_f)(1 - \alpha_f) - \beta_f^J \beta_f} \left( \Pi^I_f + \frac{\beta_f I n^I_f}{1 - \alpha_f n^I_f} \Pi^I_f \right)
\]

network-externality terms cannibalization terms

\[
=: \mu^J_f
\]

As Nocke and Schütz (2018a) do, I call \( \mu^J_f \) as the \( \iota \)-markup of firm \( f \) on side \( J \). This \( \iota \)-markup summarizes the pricing incentive of each firm.

Let me explain how the \( \iota \)-markup of each firm is determined in the equation (10). the first term in the second line of the equation (10) is the baseline \( \iota \)-markup, which would be set under the monopolistic competition. The second term is the downward-pricing pressure due to the direct and indirect network externalities. The third term is the upward-pricing pressure due to the cannibalization effects under oligopoly. The relative magnitudes of the second and the last terms on each side determine the pricing level and the pricing structure of each firm.

We have \( -(h^I_f)''/(h^I_f)' = 1/\lambda^J \) and thus \( p_i = c_i + \lambda^J \mu^J_f \) in the case of multinomial logit demand, and \( -(h^I_f)''/(h^I_f)' = \sigma^J/p_i \) and thus \( p_i = c_i/(1 - \mu^J_f/\sigma^J) \) in the case of CES-type demand. Using these functional forms, the formula for the subaggregators and profit functions are given by

\[
H_f^J = \begin{cases} 
T_f^J \exp(-\mu^J_f) & \text{in the case of MNL-type demand,} \\
T_f^J \left( 1 - \frac{\mu^J_f}{\sigma^J} \right)^{\sigma^J-1} & \text{in the case of CES-type demand,}
\end{cases}
\]
and

\[ \Pi'_f = \begin{cases} \frac{n'_f \mu'_f}{\sigma_f - 1} n'_f \mu'_f & \text{in the case of MNL-type demand,} \\ \frac{n'_f \mu'_f}{\sigma_f - 1} n'_f \mu'_f & \text{in the case of CES-type demand,} \end{cases} \]  

(12)

where \( T'_J := \sum_{i \in N'_J} \exp \left( \frac{a_i - c_i}{\lambda} \right) \) for MNL and \( T'_J := \sum_{i \in N'_J} a_i c_i^{1-\sigma} \) for CES-type demand. \( T_f \) is the “type” of firm \( f \) that corresponds with the value of the subaggregator of firm \( f \) when it engages in the marginal cost pricing. The property that all the pricing information is summarized by unidimensional type \( T_f \) is called as the “type-aggregation property” (Nocke and Schutz (2018a)). This property greatly simplifies the analysis of merger policy. As a result, the \( \iota \)-markup \( \mu'_f \) and the network share \( n'_f \) depends only on \( T_A, T_B, H_A, \) and \( H_B \). Whenever they are unique, I write the \( \iota \)-markups \( \mu'_f \) and market shares \( n'_f \) under each firm’s optimal pricing as functions

\[ \mu'_f = m' \left( T'_f, T'_f, H'_f, H'_I \right), \]  

(13)

and

\[ n'_f = N' \left( T'_f, T'_f, H'_f, H'_I \right), \]  

(14)

for \( J \in \{A, B\} \).

**Equilibrium and consumer surplus** Given the above market share functions, the equilibrium condition for the aggregators \((H^A, H^B)\) is

\[ \sum_{f \in N'_J} N' \left( T'_f, T'_f, H'_f, H'_I \right) = 1, \quad J \in \{A, B\}, \quad I \neq J. \]  

(15)

Finally, the consumer surplus on side \( J \) is given by

\[ CS^J = \log \left( \sum_{f \in F} \frac{H'_f^{1-\alpha_J} \beta_I^{1-\alpha_I} \beta_J^{1-\alpha_J} \beta_I^{1-\alpha_I}}{(H'_f)^{\alpha_J}(H'_I)^{\beta_J}} \right) \]  

\[ = (1 - \alpha_J) \log H'_f - \beta_J \log H'_I \]  

(16)

Note that in the presence of indirect network externalities \( \beta_J > 0 \), the consumer surplus on side \( J \) is decreasing in the value of aggregator \( H'_I \) on side \( I \neq J \) while it increases with the value of the aggregator on the same side \( H'_I \). This is because the stronger competition on other side shrinks the market share and thereby resulting in the fragmentation of networks.

The general equilibrium characterization and the analysis of merger policy are not easy to conduct. Thus, to obtain the clear-cut insights, I divide the analysis of merger policy into two special cases where there is only direct network externalities and where there is only indirect network externalities. For each special case, I separately prove the existence and the uniqueness of the equilibrium.

**Proposition 1.** The following statements hold:

1. For any pair of aggregators \((H^A, H^B)\), each firm’s pricing is characterized by the first-order condition [10].

2. If \( \beta_A = \beta_B = 0 \), then there is unique pricing equilibrium.
3. If the demand system is given by MNL-type demand, $\alpha_A = \alpha_B = 0$, and either $\beta_A$ or $\beta_B$ is close to 0, then there is unique equilibrium.

Proof. See Appendix A.2. □

The first part of Proposition 1 show that in any equilibrium, firms’ pricing is characterized by the first-order condition (10). The second part guarantees the existence and the uniqueness of equilibrium in special cases I focus on the subsequent analyses.

Based on this proposition, I proceed to the analysis of several special cases where there either direct or indirect network externalities.

4. Merger with Direct Network Externalities

In this section, focus on a special case where there is only direct network externalities, that is, $\beta_A = \beta_B = 0$. In this case, we can rewrite the first-order condition as

$$1 - \frac{1}{1 - \alpha J} \left( \frac{\alpha J}{n_f^J} - 1 \right) \Pi_f^J = \mu_f J, \quad J \in \{A, B\}. \tag{17}$$

Since there is no interaction between two sides, I focus on one side of the market and drop the scripts of sides. Let $\alpha$ be the magnitude of the direct network externalities. Plugging equations (11) and (12) into equation (10), we obtain the first-order conditions

$$1 = \frac{\mu_f}{1 - \alpha J} \left( 1 - \gamma(T_f) \exp \left( -\frac{\mu_f}{1 - \alpha} \right) \right) \tag{FOC-MNL}$$

and

$$1 = \frac{\mu_f}{\sigma(1 - \alpha J)} \left( \sigma - \alpha - (\sigma - 1) \frac{\gamma(T_f)}{H} \left( 1 - \frac{\mu_f}{\sigma} \right)^{\frac{\sigma - 1}{\sigma - \alpha}} \right) \tag{FOC-CES}$$

where $\gamma(x) = x^{\frac{1}{1 - \alpha}}$.

Solving the equation (FOC-MNL) for MNL and (FOC-CES) for CES-type demands, we can simplify the $\iota$-markup function $m_J^J$ in equation (13) as $m_J^J = m(\gamma(T_f)/H, \alpha)$. Using this $\iota$-markup function, we can further simplify the network share function $N_J^J$ in equation (14) into the function $N(\gamma(T_f)/H, \alpha)$ such that

$$N(\gamma(T_f)/H, \alpha) = \frac{\gamma(T_f)}{H} \exp \left( -m \left( \frac{\gamma(T_f)}{H}, \alpha \right) \right) \tag{Share-MNL}$$

and

$$N(\gamma(T_f)/H, \alpha) = \frac{\gamma(T_f)}{H} \left( 1 - m \left( \frac{\gamma(T_f)}{H}, \alpha \right) \right)^{\frac{\sigma - 1}{\sigma - \alpha}} \tag{Share-CES}$$

respectively.

The equilibrium condition for the aggregator $H$ is thus written as

$$\sum_{f \in F} N(\gamma(T_f)/H, \alpha) = 1. \tag{18}$$
Solving this equation, we obtain the equilibrium value of the aggregator, \( H^* \).

Based on the analysis above, I proceed to analyze the conditions under which a merger between two firms improves the consumer surplus. I say that a merger is CS-increasing if the merger increases the consumer surplus. The consumer surplus under any equilibrium with aggregator \( H^* \) is given by \((1 - \alpha) \log H^*\). Thus, the change in consumer surplus from any particular merger corresponds with the change in the value of aggregator \( H^* \).

Consider a merger between two firms \( f \) and \( g \) with \( T_f \) and \( T_g \). Suppose that a firm that generated from the merger between two firms \( f \) and \( g \) gains the technological synergy \( \Delta \). Then, the merger between two firms is CS-increasing if and only if

\[
N \left( \frac{\gamma(T_f + T_g + \Delta)}{H^*}, \alpha \right) \geq N \left( \frac{\gamma(T_f)}{H^*}, \alpha \right) + N \left( \frac{\gamma(T_g)}{H^*}, \alpha \right).
\]

\[ (19) \]

4.1. Merger Involving a Small Firm

First, I show that in the presence of firm-level network effects, an acquisition of a sufficiently small firm is always CS-increasing.

**Proposition 2.** (Merger involving a small firm) Consider a merger between firm \( f \) and firm \( g \). If either \( T_f \) or \( T_g \) is sufficiently small, then the merger is CS-increasing without technological synergies.

**Proof.** In Appendix A.3.

An intuition is in order. A small firm suffers from its disadvantage in the network share. When this firm is acquired by a large firm, it can leverage the network effects and attract more consumers, which also benefits consumers. Further, the contribution of this small firm to the market power of the merged entity is marginal. Thus, in total, the former benefit from an increase in the access to the products of the small firm dominates, and the consumer surplus increases.

4.2. Merger from Symmetric Oligopoly

Next, I examine the welfare effects of mergers from an originally symmetric oligopoly. The symmetric environment is well suited to illustrate the tradeoff between the benefit from network expansion and the cost of increased market power in a simplest way.

Consider a symmetric oligopoly with \(|\mathcal{F}|\) firms with the same types \( T \). In the symmetric oligopoly, the equilibrium market share of each firm is given by \(1/|\mathcal{F}|\). Thus, we have the value of the equilibrium aggregator by solving

\[
N \left( \frac{\gamma(T)}{H^*}, \alpha \right) = \frac{1}{|\mathcal{F}|}
\]

Based on this value of the equilibrium aggregator \( H^* \), a merger between two firms improves the consumer surplus if and only if

\[
N \left( \frac{\gamma(2T)}{H^*}, \alpha \right) \geq \frac{2}{|\mathcal{F}|}
\]

\[ (20) \]

The right-hand side of the above inequality does not depend on the value of \( \alpha \). Further, as I show, the market share of the merged entity increases with \( \alpha \). These jointly imply that the merger is CS-increasing if \( \alpha \) is above certain threshold. The next proposition formalizes this discussion.
Proposition 3. (Merger from symmetric oligopoly) Suppose that $T^f = T$ for all $f \in \mathcal{F}$. Then, the required synergy for a merger between two firms is decreasing in $\alpha$. Furthermore, there exists $\hat{\alpha}$ such that if $\alpha > \hat{\alpha}$, the merger is CS-increasing without synergies. $\hat{\alpha}$ is decreasing in $|\mathcal{F}|$.

Proof. In Appendix [A.4].

As mentioned earlier, an increase in the magnitude of network effects increases the equilibrium market share of the merged entity, which gives the justification of the merger. This monotonicity implies that when the magnitude of network effects is greater than some critical value, the merger between two firms improves the consumer surplus. I further show that, as the number of firms increases, the critical magnitude of network effects becomes smaller. This is because the increase in the market power due to the merger becomes smaller, which decreases the magnitude of the network effects required to offset the consumer harm due to the market power.

4.3. Technological Synergies and Network Effects

In the previous analysis, I have assumed that there is no technological synergies through mergers among firms. In this section, I analyze the amount of technological synergies required for a merger to be CS-increasing. In the course of the analysis, I investigate how the required technological synergies change according to the magnitude of network effects.

For the tractability of analysis, in this subsection I confine my analysis to the MNL demand system. I first introduce some terminologies to describe the firms’ sizes, based on the relation between network effects and market shares. The MNL demand system has an advantage in that the market share function $N((\gamma(T^f)/H,\alpha))$ depends on $\alpha$ only through the value of $\gamma(T^f)/H$. Thus, we can write the network share function as $N((\gamma(T^f)/H,\alpha)) = N_0((\gamma(T^f)/H))$ where $N_0(x) := N(x, 0)$. Using the implicit function theorem, we have

$$\frac{d}{d\alpha} \left( \frac{\gamma(T^f)}{H^*} \right) = \frac{1}{(1 - \alpha)^2} \left( \frac{\gamma(T^f)}{H^*} \right) \left( \sum_{f' \in \mathcal{F}} (\log T^f - \log T^{f'}) N_0^\prime \left( \frac{\gamma(T^{f'})}{H^*} \right) \right)$$

From this equation, the following proposition is obtained.

Proposition 4. (Network effects and market share.) For any type profile $T := \{T^f\}_{f \in \mathcal{F}}$, there is a threshold value $T^*$ such that

$$\frac{d}{d\alpha} \left( \frac{\gamma(T^f)}{H^*} \right) \geq 0$$

if and only if $T^f \geq T^*$. As a result, $N_0(\gamma(T^f)/H^*)$ increases with $\alpha$ if and only if $T^f \geq T^*$. Furthermore, as long as the equilibrium market share $N^f$ is not greater than $\bar{N}$ for each $f \in \mathcal{F}$, an increase in $\alpha$ increases the difference between the market share of the largest firm and the other firms.

This result is a familiar positive-feedback effects of network externalities. Network effects expand the market shares of firms with greater market shares and shrink the market share of firms with smaller market share. The threshold type $T^*$ stands for the critical value defining the direction in which the positive feedback effects influence the market shares.
I call the firms with $T_f > T^*$ as strong firms and the firms with $T_f < T^*$ as weak firms. In this section, I analyze the relation between the technological synergies required for mergers to improve consumer surplus and the magnitude of network effects.

Now based on the definition of strong firms and weak firms, I analyze the impacts of network effects on the criteria for approving mergers between firms. From the condition (19) and the fact that the network share function is increasing in the type, the merger between two firms is CS-increasing if and only if

$$\Delta > \hat{\Delta}$$

such that

$$N \left( \frac{\gamma(T_f + T_g + \hat{\Delta})}{H^*}, \alpha \right) = N \left( \frac{\gamma(T_f)}{H^*}, \alpha \right) + N \left( \frac{\gamma(T_g)}{H^*}, \alpha \right)$$

(23)

I call the value $\hat{\Delta}$ as the technological synergy required for a merger between two firms.

First, I show that the greater the magnitude of the network effects is, the less technological synergy is required for a merger between weak firms.

**Proposition 5.** (Merger between weak firms) Consider a merger between weak firms $f$ and $g$. The required technological synergy $\hat{\Delta}$ decreases with $\alpha$.

**Proof.** In Appendix A.5 □

The intuition behind this proposition is similar to the intuition behind Proposition 1. Weak firms fail to leverage network effects due to their small market shares. The greater the magnitude of the network effects is, the more serious this the failure of the weak firms to internalize the network externalities. In such cases, the merger softens this failure to internalize network externalities. Therefore, the merger between weak firms is more desirable as the magnitude of network effects grows.

Similar result partially extends to the case where one firm in the merging party is strong and the other is weak.

**Proposition 6.** (Merger between strong and weak firms) Consider a merger between a strong firm $f$ and a weak firm $g$ with network shares $N_f$ and $N_g$. Suppose that $N_f + N_g < N$. Then, the required technological synergy $\hat{\Delta}$ decreases with $\alpha$.

**Proof.** In Appendix A.5 □

While the impacts of network effects on the mergers involving weak firms and mergers in symmetric oligopoly is straightforward, their impacts on mergers between strong firms are ambiguous. When both firms in a merging party are strong, the demand-side synergy effect may not be enough to compensate an increase in markups due to the concentration. The following proposition illustrates such scenario in some limit cases where the merging parties are so large that the merged entity can virtually monopolize the whole market.

**Proposition 7.** (Merger that leads to extreme concentration) Consider a merger between two strong firms $f$ and $g$. Suppose that $N_f + N_g$ is sufficiently large, that is, $N_f + N_g$ is close to 1. Then the required synergy $\hat{\Delta}$ increases with $\alpha$.

**Proof.** In Appendix A.6 □

When the joint market share of two firms is too large, the merged entity has an extremely strong market power. An increase in the consumer gain from a further increase in the magnitude of network effects is more offset by an increase in the markup of the merged entity. This is why when the joint market share is sufficiently large, a greater magnitude network effects leads to more stringent merger policy toward mergers between large firms.
5. Merger with Indirect Network Externalities

In this section, I focus on the merger in the presence of indirect network externalities, with typical interest in two-sided markets. To this end, I assume that $\alpha_A = \alpha_B = 0$. Further, for an analytical tractability, I assume that $\beta_B = 0$ and confine the attention to MNL demand system. Let $\beta_A = \beta$. Substituting equation (12) into (10) and rearranging, we obtain the $\iota$-markup $\mu_f^A$ and $\mu_f^B$ by solving the equation (10): 

$$1 - \left(1 - n_f^A \right) \mu_f^A = 0,$$  \hspace{1cm} (24) 

$$1 - \left(1 - n_f^B \right) \mu_f^B - \beta_A \frac{n_f^A}{n_f^B} = 0.$$  \hspace{1cm} (25) 

Let $m^A(T_f^A, T_f^B, H^A, H^B)$ and $m^B(T_f^A, T_f^B, H^A, H^B)$ be the solution to this system of equations. Then, we can further define 

$$N^A \left( T_f^A, T_f^B, H^A, H^B \right) = \frac{T_f^A}{H^A} \exp \left( -m^A(T_f^A, T_f^B, H^A, H^B) - \beta m^B(T_f^A, T_f^B, H^A, H^B) \right)$$  \hspace{1cm} (26) 

$$N^B \left( T_f^A, T_f^B, H^A, H^B \right) = \frac{T_f^B}{H^B} \exp \left( -\mu_f^B(T_f^A, T_f^B, H^A, H^B) \right)$$  \hspace{1cm} (27) 

The equilibrium condition for $H^A, H^B$ is that 

$$\sum_{f \in F} N^J \left( T_f^A, T_f^B, H^A, H^B \right) = 1, \text{ for } J \in \{A, B\}.$$  \hspace{1cm} (28) 

The following lemma illustrates the relation between the type of a firm and its $\iota$-markups and market shares.

**Lemma 1.** Suppose that the demand system is MNL-type and $\alpha_A = \alpha_B = \beta_B = 0$. Then, the following statements hold:

1. $m^A$ is increasing in $T_f^A$ and $T_f^B$.
2. $m^B$ is decreasing in $T_f^A$ and increasing in $T_f^B$.
3. Both functions $N^A$ and $N^B$ are increasing in $T_f^A$ and $T_f^B$.

**Proof.** In Appendix [A.7] \hspace{1cm} $\square$

This proposition characterizes how the pricing and the market share of each firm on one side react to its productivity on the other side. The first two results show how the pricing of each firm changes according to its type. A greater value of $T_f^A$ or $T_f^B$ means that the firm can collect a large number of consumers on side $A$, which leads the firm to charge a higher price on side $A$, which leads to the first part of Lemma [1]. Similarly, a greater value of $T_f^B$ mean that the firm can collect a large number of consumers on side $B$, which leads the firm to charge a higher price on side $B$. By contrast, a greater value of $T_f^A$ leads the firm to charge a lower price on side $B$. This is because the firm has a greater incentive to subsidize the consumers on side $A$, due to the higher
price on side A. These observations lead to the second part of Lemma 1. The changes in the market shares depend on the direct effect from a change in the type and the indirect effect from a change in pricing. It turns out that the direct effect dominates the indirect effects whenever they conflict, and thus the market shares increase with firm’s type. This is shown in the third part of Lemma 1.

5.1. CS-Neutral Mergers

Consumer surplus on each side is given by

\[ CS^A = \log H^A - \beta \log H^B \quad \text{and} \quad CS^B = \log H^B \]  

Consider a merger between firm f and g. Let M be the merged entity with the type \((T^A_M, T^B_M) = (T^A_f + T^A_g + \Delta^A_M, T^B_f + T^B_g + \Delta^B_M)\). I call \(\Delta^A_M\) and \(\Delta^B_M\) as the technological synergies on side A and side B, respectively.

We say that a merger is CS-neutral if the equilibrium value of \(CS^A\) and \(CS^B\) are the same as before the merger. A merger is CS-neutral if and only if two equations

\[ N^A \left( T^A_M, T^B_M, H^A, H^B \right) = N^A \left( T^A_f, T^B_f, H^A, H^B \right) + N^A \left( T^A_g, T^B_g, H^A, H^B \right), \]

\[ N^B \left( T^A_M, T^B_M, H^A, H^B \right) = N^B \left( T^A_f, T^B_f, H^A, H^B \right) + N^B \left( T^A_g, T^B_g, H^A, H^B \right). \]  

The next lemma shows that there is a unique pair of technological synergies that is CS-neutral.

**Lemma 2.** For any merger between firms f and g, there exists a unique pair of technological synergies \((\hat{\Delta}^A, \hat{\Delta}^B)\) such that the merger is CS-neutral on both sides.

**Proof.** In Appendix A.8

Next, we analyze the interaction between the product value on one side and the required technological synergies on the other side. I put the proposition first and then explain the intuition behind the proposition.

**Proposition 8.** Consider a merger between firm f and firm g. The following statements hold.

1. If \(N^A_f > \hat{N}\), then \(\hat{\Delta}^A\) increases with \(T^B_f\).

2. If \(N^A_f < \hat{N}\), then there exists \(\hat{N}^A_g\) such that \(\hat{\Delta}^A\) decreases with \(T^B_f\) if and only if \(N^A_g < \hat{N}^A_g\).

3. There exists \(\hat{\beta} > 0\) such that for all \(\beta \leq \hat{\beta}\), there exists an increasing function \(\xi(\cdot)\) such that \(\hat{\Delta}^B\) increases with \(T^A_f\) if and only if \(N^A_g < \xi(N^B_g)\).

**Proof.** In Appendix A.8

Whether an increase in the size of merging entity on one side increases the required synergies on the other side depends on its relative magnitude of the change in \(\iota\)-markups of merging and merged entities.

Consider the effects of an increase in \(T^A_f\). This increases the subsidization incentive of both merging and merged entities. Which increase in subsidization incentive is greater depends on the revenue structure represented by \(n^A_f/n^B_f\). When \(\beta_A\) is close to 0, then the growth of \(n^A_f/n^B_f\) as
a result of merger makes it more likely that the increase in the subsidization incentive is greater for merged entity, which implies that the less technological synergy is required on side $B$.

The effects of an increase in $T^B_f$ on the technological synergies required on side $A$ is more complicated. On the one hand, an increase in the size on side $B$ decreases the subsidization incentive and increases the size of both merging and merged entities on side $A$. Holding the technological synergy on side $B$ fixed, such increase in size on side $B$ accompanies the increase in $\iota$-markup on both sides. Such an adverse effect is greater for the merged entity. On the other hand, an increase in the size on side $B$ requires a greater technological synergy on side $B$, which may decrease the relative size of adverse effect on side $A$. When the merged entity is not too large on side $A$ and its share on side $B$ does not increase too much, the former adverse effect is not too large, and the latter effect may be enough to compensate the adverse effects. In such cases, the required synergy on side $A$ can decrease.

6. Conclusion

In this study, I examined the relation between the consumer-surplus effects of merger and the strength of network effects in the multiproduct-firm oligopoly. In the presence of network effects, an acquisition of sufficiently small firm improves consumer surplus. A merger from symmetric oligopoly without technological synergies improves the consumer surplus if the network effect is stronger than certain critical value. When merging party is network-minor, the consumer-surplus oriented merger policy is less stringent as the network effects becomes stronger. Overall, the stronger network effects justify the merger between relatively small firms, but makes the merger between two dominant firms that results in an extreme concentration unlikely to be justified. These implications give several theoretical guidance to the competition authorities involving in the merger policy.

My analysis abstracts several important aspects of the merger policy, which leaves the avenue for future research. First, my analytical framework is statics and do not take dynamics into account, such as R&D, investments, sequential mergers, entry incentives, etc. (e.g., Cunningham et al. (2018)). Incorporating such dynamics may alter some policy prescription in the long run. I leave such dynamic analysis for future research.

A. Appendix

A.1. Derivation of demand system and firm pricing in Section 3

I first present a general framework to which our specific analyses belong to.

$$\log h^J_f(p_i) + \alpha_J \log n^J_f + \beta_J \log n^J_f + \varepsilon_i,$$

where $J, I \in \{A, B\}, J \neq I, \alpha^J \in [0, 1)$, and $\varepsilon$ follows the Type-I extreme-value distribution. As a result, the consumer choice probability of product $i \in \mathcal{N}^J_f$ is given by

$$s^J_i = \frac{h^J_f(p_i)(n^J_f)^{\alpha^J}(n^J_i)^{\beta^J}}{\sum_{j' \in \mathcal{F}} \sum_{j \in \mathcal{N}^J_f} h^J_j(p_j)(n^J_j)^{\alpha^J}(n^J_{j'})^{\beta^J}}$$
network share $n_f^J$ is given by

\[ n_f^J = \frac{H_f^J (n_f^J)^{\alpha_J} (n_f^J)^{\beta_J}}{\sum_{f' \in F} H_{f'}^J (n_{f'}^J)^{\alpha_J} (n_{f'}^J)^{\beta_J}}. \]  

(33)

Then, we have

\[ \frac{n_f^J}{n_{f'}^J} = \frac{H_f^J (n_f^J)^{\alpha_J} (n_f^J)^{\beta_J}}{H_{f'}^J (n_{f'}^J)^{\alpha_J} (n_{f'}^J)^{\beta_J}} \]

which can be rewritten as

\[ \frac{n_f^J}{n_{f'}^J} = \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\alpha_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\beta_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\gamma_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\delta_J} \]

(34)

(35)

Finally, we obtain

\[ \frac{n_f^J}{n_{f'}^J} = \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\alpha_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\beta_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\gamma_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\delta_J} \]

(36)

Plugging this equation into the definition of market share function, I obtain

\[ n_f^J = \frac{H_f^J}{\sum_{f' \in F} H_{f'}^J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\alpha_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\beta_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\gamma_J} \left( \frac{H_f^J}{H_{f'}^J} \right)^{1-\delta_J} \]

\[ = \frac{(H_f^J)^{1-\alpha_J} (H_f^J)^{1-\beta_J} (H_f^J)^{1-\gamma_J} (H_f^J)^{1-\delta_J}}{\sum_{f' \in F} (H_{f'}^J)^{1-\alpha_J} (H_{f'}^J)^{1-\beta_J} (H_{f'}^J)^{1-\gamma_J} (H_{f'}^J)^{1-\delta_J}} \]

(37)

\[ D_i^J(p) = \hat{D}_i^J(p; H_f^J(p_f^J), H_f^J(p_f^J), H_f^J(p_f^J)) \]

\[ = n_f^J(p) \times S_i^J(p) \times \left( \frac{h_{iJ}^J(p_i)}{h_i^J(p_i)} \right) \]

\[ = \frac{(H_f^J(p_f^J))^{1-\alpha_J} (H_f^J(p_f^J))^{1-\beta_J} (H_f^J(p_f^J))^{1-\gamma_J} (H_f^J(p_f^J))^{1-\delta_J}}{H_f^J(p_f^J)} \]

(38)

Given this demand system, the profit function of each firm $f$ is given by

\[ \Pi_f(p_f, H^A(p), H^B(p)) = \Pi_f^A(p_f^J, H_f^A(p_f^J), H_f^B(p_f^J), H^A(p), H^B(p)) \]

\[ + \Pi_f^B(p_f^J, H_f^B(p_f^J), H_f^A(p_f^J), H^B(p), H^A(p)) \]

(39)
where

$$\Pi_f^J (p_i^f, H_f^J(p_i^f), H_f^J(p_i^f)) = \sum_{i \in N_f^J} \hat{D}_i^J (p_i, H_f^J(p_i^f), H_f^J(p_i^f), H^J(p))(p_i - c_i).$$

(40)

The first-order condition for profit-maximization of each firm \( f \) is given by

$$0 = \frac{\partial \Pi_f^J}{\partial p_i} + (h_i^J)^{\prime \prime}(p_i - c_i) \left( \begin{array}{c}
\frac{\partial \Pi_f^J}{\partial H_f^J} + \frac{\partial H^J}{\partial H_f^J} \frac{\partial \Pi_f^J}{\partial H^J} + \frac{\partial H^J}{\partial \Pi_f^J} \\
\frac{\partial \Pi_f^J}{\partial H_f^J} + \frac{\partial H^J}{\partial \Pi_f^J}
\end{array} \right)
$$

$$= \hat{D}_i^J - \hat{D}_i^J \frac{(h_i^J)^{\prime \prime}}{(h_i^J)^{\prime \prime}(p_i)} (p_i - c_i) + (h_i^J)^{\prime \prime} \left( \frac{1}{(1 - \alpha_J)(1 - \alpha_I) - \beta_J \beta_I H_f^J} \right)
$$

$$\left( \frac{1}{1 - \alpha_J(1 - \alpha_I) - \beta_J \beta_I} \Pi_f^J + \frac{\beta_I}{1 - \alpha_I \frac{N_f^J}{N_f^J}} \Pi_f^J \right)
$$

(41)

From this equation, we observe that

$$- \frac{(h_i^J)^{\prime \prime}(p_i)}{(h_i^J)^{\prime \prime}(p_i)} (p_i - c_i) = \mu_f^J
$$

(42)

**A.2. Proof of Proposition 1**

**A.2.1. Proof of Proposition 1**

I prove Proposition 1-1 for CES-type demand and MNL-type demand separately.

Before proceeding to individual proofs, I introduce several notations that are used in both proofs. First, let

$$\Omega_f^J (p_f) := \left( H_f^J(p_f) \right)^{\frac{1-\alpha_J}{1-\alpha_J + \beta_J \beta_I}} \left( H_f^J(p_f) \right)^{\frac{\beta_J}{1-\alpha_J + \beta_J \beta_I}}
$$

and

$$H_{-f}^J = \sum_{f \in F \setminus \{f\}} \Omega_f^J (p_f^f).$$

Then, the profit-maximization problem of firm \( f \) the problem can be rewritten as

$$\max_{p_f \in \mathbb{R}^{N_f^A \cup N_f^B}} G_f(p_f) := \Pi_f \left( p_f, \Omega_f^{AB}(p_f) + H_{-f}^A, \Omega_f^{BA}(p_f) + H_{-f}^B \right)
$$

(43)

I show that the solution to (43) takes unique finite value and given by the first-order condition (10). Here, I list the steps of the proof. For CES-type demand,
1. To whatever extent a subaggregator \( H_f^I \) of firm \( f \) on one side grows, the profit on the other side \( \Pi_f^I \) is bounded above.

2. Setting zero price for some good is never optimal.

3. Setting infinite price for some good is never optimal.

4. The optimal prices should satisfy the first-order condition \( (10) \).

For MNL-type demand,

1. Fixing \( p_f^A, p_f^B \) that maximizes \( G_f(p_f^A, p_f^B) \) is finite and unique. Let \( \tilde{p}_f^B(p_f^A) \) denote such \( p_f^B \).

2. \( p_f^A \) that maximizes \( G_f(p_f^A, \tilde{p}_f^B(p_f^A)) \) is finite and unique.

3. Setting infinite price for some good is never optimal.

4. The optimal prices should satisfy the first-order condition \( (10) \).

**CES-type demand** I first show that whatever the value of subaggregator \( H_f^I \) is, the value \( \Pi_f^I \) is bounded. To see this, note that

\[
\Pi_f^I = \frac{\Omega^I_f}{H_f^I} \sum_{j \in N_f^I} \frac{-(h_f^j)'(p_j) + c_j}{H_f^I(p_f^j)} (p_j - c_j) \leq \sum_{j \in N_f^I} \frac{-(h_f^j)'(p_j)}{H_f^I(p_f^j)} (p_j - c_j) = \frac{\sum_{j \in N_f^I}(\sigma_f-1)\alpha_f p_f^j \tilde{p}_f^j(p_f^j-c_j)}{\sum_{k \in N_f^J} a_k p_k^\sigma},
\]

which has finite maximum with respect to \( p_f^j \).

Next, I show that the price of each product should be in the set \((0, \infty)\), and thus the optimal pricing should be characterized by the equation \( (10) \). To see this, suppose that \( p_i \leq 0 \) for some \( i \in N_f^J \). Then, \( h_f^j(p_j) = \infty \) and thus the \( D_f^j(p) = 0 \) for all \( j \in N_f^J \setminus \{i\} \). Therefore the profit of firm \( f \) is given by

\[
D_f^j(p_i - c_i) + \Pi_f^I.
\]

Since \( \Pi_f^I \) is finite as shown above, \( D_f^j(p) = \infty \), and \( p_i - c_i < 0 \), the profit is negative. Thus, setting non-positive for some product is never optimal for firm \( f \).

The fact that firm never sets infinite price is shown in the same manner as Nocke and Schutz (2018b).

Thus, the optimal profile of prices is interior, which implies that the optimal prices should satisfy the first-order condition \( (10) \).

**MNL-type demand** I first show that all firms’ prices are bounded below. Next, I show that all firm’ prices are bounded above.

Fix \( p_f^A := (p_f^A)_{i \in N_f^A} \). Then, I show that the value of \( (p_f^B) := (p_f^B)_{i \in N_f^B} \) that maximize the profit of firm \( f \) has finite absolute values. To see this, note that

\[
\text{sign} \left( \frac{\partial G(p_f^j)}{\partial p_i} \right) = \text{sign} \left( 1 - \frac{p_i - c_i}{\lambda_j} + \frac{(1 - \alpha_I)\alpha_J + \beta_I \beta_J}{(1 - \alpha_I)(1 - \alpha_J) - \beta_I \beta_J} \Pi_f^I - \frac{\beta_I}{(1 - \alpha_I)(1 - \alpha_J) - \beta_I \beta_J} \right) \frac{(1 - n_f^I)}{n_f^I} \Pi_f^I.
\]

(46)
the last term converges to 0 as \( p_i \to -\infty \) because \( n_i' \to 1 \) as \( p_i \to -\infty \). The sum of the second and the third terms is nonnegative as \( p_i \to -\infty \) because

\[
\frac{p_i - c_i}{\lambda_J} + \frac{(1 - \alpha_I)(1 - \alpha_J)(\alpha_I + \beta_I \beta_J)\frac{1}{n_J'}}{\Pi_J'} \geq \frac{p_i - c_i}{\lambda_J} + \Pi_J'
\]

(47)

as \( p_i \to -\infty \). Thus we have \( \partial G / \partial p_i > 0 \) for sufficiently small \( p_i \).

The fact that firm never sets infinite price is shown in the same manner as Nocke and Schutz (2018b).

As a result, fixed the values of \( p_J' \), the optimal pricing for \( p_J' \) is given by the common \( \iota \)-markup pricing, which is given by (10). Let \( \mu_J'(p_J') \) be the optimal \( \iota \)-markup on side \( J \) given the profile of prices on the other side \( p_J' \). Then, under MNL-type demand, we have \( p_i = c_i + \lambda_J \mu_J'(p_J') \) for \( i \in N_J' \).

Next, I show that the optimal value of \( p_J' \) that maximizes \( G(p) \) where \( p_J' = (c_i + \lambda_J \mu_J'(p_J'))_{i \in N_J'} \) is finite. To see this, it is sufficient to show that

\[
\frac{(1 - n_J')}{n_J'} \Pi_J' \to 0
\]

as \( p_J \to -\infty \) for some \( N_J' \), because if it is, I can use the same proof as in the case where \( p_J' \) is fixed. To verify this, note that \( \lim_{p_J \to -\infty} \mu_J'(p_J') = \infty \), \( \lim_{p_J \to -\infty} n_J' = 1 \), and \( \lim_{p_J \to -\infty} n_J' = 1 \). Thus, we have

\[
\lim_{p_J \to -\infty} \frac{(1 - n_J')}{n_J'} \Pi_J' = \lim_{p_J \to -\infty} (1 - n_J') \mu_J'
\]

\[
= \lim_{p_J \to -\infty} \frac{H_J' \mu_J'}{H_J' + \Omega_J'}
\]

\[
= \frac{\lim_{p_J \to -\infty} \frac{d\mu_J'}{dp_J}}{\lim_{p_J \to -\infty} \left( \frac{d\mu_J'}{dp_J} \frac{d\Omega_J'}{d\mu_J'} + \left( \frac{h_J'}{H_J'} \right) \frac{d\Omega_J'}{dH_J'} \right)}
\]

\[
= \{(1 - \alpha_I)(1 - \alpha_J) - \beta_I \beta_J\} \frac{H_J'}{\left( (1 - \alpha_I) \frac{d\mu_J'}{dp_J} + \beta_J \frac{h_J'}{H_J'} \right) \Omega_J'}
\]

\[
= 0
\]

(48)

Again, the fact that firm never sets infinite price is shown in the same manner as Nocke and Schutz (2018b).

Thus, the optimal profile of prices is interior, which implies that the optimal prices should satisfy the first-order condition (10).
A.2.2. Proof of Proposition 1.2

Suppose that $\beta_A = \beta_B = 0$. Then, two sides of markets are independent, and thus it suffice to focus on one side of the market. The $\iota$-markup is uniquely given by the equation (FOC-MNL) for MNL-type demand and (FOC-CES) for CES-type demand. Finally, since the market share equation $N_0(\gamma(T)/H)$ defined by is monotonically decreasing in $H$, $\lim_{x \to 0} N_0(x) = 0$, and $\lim_{x \to \infty} N_0(x) = 1$, the intermediate value theorem implies that the value of aggregator $H$ that satisfies $\sum f N_0(\gamma(T)/H) = 1$ is unique.

A.2.3. Proof of Proposition 1.3

I show that there is the unique pair of $\iota$-markup that satisfies the system of equations (10), and the unique pair of aggregator that satisfies the system of equation (10) when the demand system is given by MNL-type demand, $\alpha_A = \alpha_B = 0$, and at least one of $\beta_A$ or $\beta_B$ is sufficiently close to 0.

Without loss of generality, suppose that $\beta_B = 0$. After several manipulations, the system of first-order conditions (10) can be rewritten as

$$g_A(\mu_A^f, \mu_B^f) = (1 - n_f^A)\mu_A^A - 1 = 0,$$

$$g_B(\mu_A^f, \mu_B^f) = (1 - n_f^B)\mu_B^B - 1 + \beta_A n_f^A - \frac{n_f^A}{n_f^B} = 0.$$  

Let $g(\mu_A^f, \mu_B^f) := \{g_A(\mu_A^f, \mu_B^f), g_B(\mu_A^f, \mu_B^f)\}$. To show that this system of equations has unique solution, I show that the determinant of $D(g(\mu_A^f, \mu_B^f))$ is positive. A calculation leads to

$$\det G_f = \left(\frac{n_f^A}{1 - n_f^A} + 1 - n_f^A\right) \left(\frac{n_f^B}{1 - n_f^B} + 1 - n_f^B + \beta_A \left(\frac{1}{n_f^B} - \frac{1}{1 - n_f^B}\right)\right)$$

$$+ \beta_A^3 \frac{(n_f^A)^2}{n_f^B(1 - n_f^A)} > 0,$$

where

$$G_f := \left(\begin{array}{cc}
\frac{\partial g_A}{\partial \mu_A^f} & \frac{\partial g_A}{\partial \mu_B^f} \\
\frac{\partial g_B}{\partial \mu_A^f} & \frac{\partial g_B}{\partial \mu_B^f}
\end{array}\right),$$

and the last inequality follows from the fact that

$$\frac{n_f^B}{1 - n_f^B} + 1 - n_f^B + \beta_A \left(\frac{1}{n_f^B} - \frac{1}{1 - n_f^B}\right)$$

$$= \frac{1}{n_f^B(1 - n_f^B)} \left((n_f^B)^3 - (n_f^B)^2 + n_f^B + \beta_A n_f^A(1 - 2n_f^B)\right)$$

is positive. To see this, all the terms in expression (53) is positive when $n_f^B < 1/2$. When $n_f^B \geq 1/2$, the last term in expression (53) is nonpositive. In that case, by the fact that

---

\( \beta_A n_f^A \in (0, 1) \) the following inequality holds:

\[
\nu(n_f^B) := (n_f^B)^3 - (n_f^B)^2 + n_f^B + \beta_A n_f^A (1 - 2n_f^B) \geq (n_f^B)^3 - (n_f^B)^2 - n_f^B + 1, \tag{54}
\]

which is positive for all \( n_f^B \in (0, 1) \). This is because \( \nu(1) = 0 \), and \( \nu'(n_f^B) < 0 \) for all \( n_f^B \in (0, 1) \).

Thus, the pair of \( t \)-markups that satisfies the first-order condition is unique.

Finally, I show that there is unique equilibrium. To do this, I present several comparative statics of \( m^J \), \( J \in \{A, B\} \), with respect to several parameters \( x \). This is given by the Implicit Function Theorem

\[
G_f \left( \frac{\partial m^A}{\partial x}, \frac{\partial m^B}{\partial x} \right) = - \left( \frac{\partial \mu_A}{\partial x}, \frac{\partial \mu_B}{\partial x} \right). \tag{55}
\]

by Cramer’s Rule, I obtain

\[
\frac{\partial \mu_A}{\partial x} = \frac{\det \left( -\frac{\partial \mu_A}{\partial x} \frac{\partial \mu_B}{\partial x} \right)}{\det G_f}, \quad \frac{\partial \mu_B}{\partial x} = \frac{\det \left( -\frac{\partial \mu_A}{\partial x} \frac{\partial \mu_B}{\partial x} \right)}{\det G_f}. \tag{56}
\]

Using this comparative statics in \( t \)-markups, I conduct a comparative statics in market shares \( N^A \) and \( N^B \):

\[
\frac{\partial N^A}{\partial x} = \frac{\partial n^A}{\partial x} - \frac{\partial m^A}{\partial x} n^A - \beta_A \frac{\partial m^B}{\partial x} n^A - \frac{\partial \mu_A}{\partial x} n^A \tag{57}
\]

\[
\frac{\partial N^B}{\partial x} = \frac{\partial n^B}{\partial x} - \frac{\partial m^B}{\partial x} n^B + \frac{\partial \mu_B}{\partial x} n^B \tag{58}
\]

Based on this observation, I derive the effects of \( H^A \) and \( H^B \) on \( N^A \) and \( N^B \).

Fist, for \( H^A \), we have

\[
\frac{\partial m^A}{\partial H^A} = -\frac{n_f^A}{H^A \det(G_f)} \mu_f \left( n_f^B \mu_f^B + 1 - n_f^B + \beta_A \frac{n_f^A}{n_f^B} \right) < 0
\]

\[
\frac{\partial m^B}{\partial H^A} = \frac{n_f^A}{H^A \det(G_f)} \left( \beta_A \frac{n_f^A}{n_f^B} \mu_f^A (1 - \beta_A) + \beta_A \frac{1 - n_f^A}{n_f^B} \right) > 0,
\]

and thus

\[
\frac{\partial N^A}{\partial H^A} = -\frac{n_f^A}{H^A \det(G_f)} (1 - n_f^A) \left( n_f^B \mu_f^B + 1 - n_f^B + \beta_A (1 + \beta_A) \frac{n_f^A}{n_f^B} \right) < 0 \tag{59}
\]

\[
\frac{\partial N^B}{\partial H^A} = -\frac{n_f^A}{H^A \det(G_f)} \beta_A \frac{n_f^A}{n_f^B} \mu_f^A (1 - \beta_A) + \beta_A \frac{1 - n_f^A}{n_f^B} n_f^B < 0. \tag{60}
\]

For \( H^B \), we have

\[
\frac{\partial m^A}{\partial H^B} = \frac{n_f^B}{H^B \det(G_f)} \left( \mu_f^B + \beta_A \frac{n_f^A}{(n_f^B)^2} \right) \beta_A n_f^A \mu_f^A > 0
\]

\[
\frac{\partial m^B}{\partial H^B} = -\frac{n_f^B}{H^B \det(G_f)} \left( \mu_f^B + \beta_A \frac{n_f^A}{(n_f^B)^2} \right) (n_f^A \mu_f^A + 1 - n_f^A) < 0,
\]
and thus
\[
\frac{\partial N^A}{\partial H^B} = \beta_A \frac{n_j^A n_j^B}{n_j^B H^B \det(G_f)} \left( n_f^B \mu_f^B + \beta_A \frac{n_j^A}{n_j^B} \right) (1 - n_j^A) > 0, \tag{61}
\]
\[
\frac{\partial N^B}{\partial H^B} = -\frac{n_j^B}{H^B} \frac{1}{\det(G_f)} \left( (n_f^A \mu_f^A + 1 - n_f^A)(1 - n_f^B) + \beta_A \frac{n_j^A}{n_j^B} n_f^A \mu_f^A \right) < 0. \tag{62}
\]

As a result, we have
\[
\det \left( \sum \frac{\partial N^A}{\partial H} \sum \frac{\partial N^B}{\partial H^B} \right) > 0, \tag{63}
\]
which implies that the pair \((H^A, H^B)\) that satisfies the condition \((15)\) is unique.

A.3. Proof of Proposition 2

Suppose that \(T_g\) is sufficiently close to 0. In both cases with MNL and CES-type demand systems, we have \(N(0, \alpha) = 0\), the LHS minus RHS in \((??)\) can be approximated around \(\tilde{T}_g = 0\) by
\[
\frac{\partial N}{\partial x} \left( \gamma(T_f)/H^*, \alpha \right) \frac{\gamma'(T_f) - \gamma'(0) H^*}{\gamma'(T_g)} > 0. \tag{64}
\]
Thus, the condition \((??)\) holds for sufficiently small value of \(\tilde{T}_g\).

A.4. Proof of Proposition 3

First, prove the proposition in the MNL case. By the equation \(N((\gamma(T)/H^*, \alpha) = 1/|F|\), we obtain
\[
H^* = |F| \gamma(T) \exp \left( -\frac{|F|}{|F| - 1} \right). \tag{65}
\]
Thus, we have
\[
N \left( \frac{\gamma(2T)}{H^*}, \alpha \right) = N \left( 2^{\frac{1}{\alpha}} \exp \left( \frac{|F|}{|F| - 1} \right), \alpha \right), \tag{66}
\]
which is increasing in \(\alpha\). Further, we have \(N(\gamma(2T)/H^*, \alpha) = 2/|F|\) at \(\hat{\alpha}\) such that
\[
2^{\frac{1}{\hat{\alpha}}} = \exp \left( \frac{|F|}{(|F| - 2)(|F| - 1)} \right). \tag{67}
\]
Thus, if \(\alpha > \hat{\alpha}\), the merger between two firms improves the consumer surplus. Since \(\frac{|F|}{(|F| - 2)(|F| - 1)}\) decreases with \(|F|\), \(\hat{\alpha}\) decreases with \(|F|\).

Next, I show the proposition in the case of CES-type demand. First, the equilibrium level of the aggregator is given by
\[
H^* = |F| \gamma(T) \left( \frac{(\sigma - 1)|F| - 1}{\sigma - \alpha} \right)^{\frac{\sigma - 1}{\sigma - \alpha}}. \tag{68}
\]
Thus, we have
\[
N\left(\gamma(T)H^{-\alpha}, \alpha\right) = N\left(2^{1-\alpha} \frac{(\sigma-1)\mathcal{F}^{-1}}{\mathcal{F}^{-1} - (\sigma-1)|\mathcal{F}|}, \alpha\right),
\]
(69)

Let \(\hat{\alpha}\) such that
\[
\Lambda(\hat{\alpha}, |\mathcal{F}|) := 2^{\hat{\alpha}} - \frac{(\sigma-1)\mathcal{F}^{-1}}{\mathcal{F}^{-1} - (\sigma-1)|\mathcal{F}|} = 0.
\]
(70)

Then, a merger between two firms is CS-neutral at \(\alpha = \hat{\alpha}\). Since \(\Lambda(0, |\mathcal{F}|) < 0\) and
\[
\frac{\partial\Lambda}{\partial \alpha} = 2^{\hat{\alpha}} \log 2 + (\sigma-1)2^{\hat{\alpha}} \frac{\sigma-1}{\mathcal{F}^{-1} - (\sigma-1)|\mathcal{F}|} - \frac{\sigma-1}{\mathcal{F}^{-1} - 1\mathcal{F}(\sigma - \hat{\alpha}) - 2(\sigma-1)}
\]
at \(\alpha = \hat{\alpha}\), the merger between two firms is CS-increases if and only if \(\alpha > \hat{\alpha}\). To see that \(\hat{\alpha}\) decreases with \(|\mathcal{F}|\), I apply the implicit function theorem to obtain
\[
\frac{d\hat{\alpha}}{d|\mathcal{F}|} = -\frac{\frac{\partial\Lambda}{\partial |\mathcal{F}|}}{\frac{\partial\Lambda}{\partial \alpha}},
\]
where
\[
\frac{\partial\Lambda}{\partial |\mathcal{F}|} = (\sigma-1)2^{\hat{\alpha}} \frac{(1-\hat{\alpha})\left(\sigma - \hat{\alpha} - \frac{2(\sigma-1)}{|\mathcal{F}|}\right)}{\left(\sigma - \hat{\alpha} - \frac{2(\sigma-1)}{|\mathcal{F}|}\right)(|\mathcal{F}| - 1)^2} > 0.
\]
(71)

A.5. Proof of Proposition 5 and Proposition 6

I first analyze the concavity and the convexity of the function \(\tilde{N}'(x)\) with respect to the network share, and then use this analysis to prove Proposition 5 and Proposition 6.

In MNL demand system, we have
\[
N_0'(x) = \frac{N^3 - 2N^2 + N}{N^2 - N + 1}
\]
at \(N = N_0(x)\). Then, we have
\[
\frac{d}{dN} (N_0'(x)) = \frac{(1 - N)(N^3 - N^2 + 3N - 1)}{(N^2 - N + 1)^2},
\]
(72)

which is nonnegative in \(N \in [0, \hat{N}]\) and negative in \((\hat{N}, 1]\) for some critical value \(\hat{N} \in (0, 1)\) that satisfies
\[
\hat{N}^3 - \hat{N}^2 + 3\hat{N} - 1 = 0.
\]

Further, we have
\[
\frac{d^2}{dN^2} (N_0'(x)) = \frac{2}{(N^2 - N + 1)^2}(-N^3 + 3N^2 - 1),
\]
(73)
which is negative in \( N \in [0, \tilde{N}) \) and nonnegative in \( N[\tilde{N}, 1) \) for \( \tilde{N} \in (1/2, 1) \). In summary, \( N'(x) \) is increasing and concave in \( N \in [0, \tilde{N}) \), nonincreasing and concave in \( N \in [\tilde{N}, N] \), and decreasing and convex in \( N \in (\tilde{N}, 1) \).

Using this result, I prove Proposition \(^5\) and Proposition \(^6\).

The required synergies \( \hat{\Delta} \) decreases with \( \alpha \) if and only if

\[
A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \geq A(T_f)B(T_f) + A(T_g)B(T_g)
\]

(74)

where

\[
A(T) = N_0 \left( \frac{\gamma(T)}{H^*} \right) \frac{\gamma(T)}{H^*}
\]

(75)

and

\[
B(T) = \frac{H^*}{\gamma(T)} \frac{d}{d\alpha} \left( \frac{\gamma(T)}{H^*} \right)
\]

(76)

Suppose that two merging firms \( f \) and \( g \) are network-minor. If the merged entity is network-dominant, the LHS of (74) is positive while the RHS of (74) is negative. Next, suppose that the merged entity is network-minor. Since any firm with network share greater than 1/2 is network-dominant, \( N_f + N_g < 1/2 \) hold. Thus, we have \( A(T_f + T_g + \hat{\Delta}) \leq A(T_f) + A(T_g) \) by the concavity of \( N'_0(x) \) in \( N \in [0, \tilde{N}) \). Finally, we have the following inequality

\[
A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \geq (A(T_f) + A(T_g))B(T_f + T_g + \hat{\Delta})
\]

\[
\geq A(T_f)B(T_f) + A(T_g)B(T_g).
\]

(77)

Thus, \( \hat{\Delta} \) decreases with \( \alpha \).

Next, I show that \( \hat{\Delta} \) for the merger between firm \( f \) and \( g \) decreases with \( \alpha \) if firm \( f \) is network-minor and the firm \( g \) is network-dominant and if \( N_f + N_g < \tilde{N} \). This can be observed by

\[
A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \geq A(T_f)B(T_f)
\]

(78)

and

\[
A(T_g)B(T_g) \leq 0.
\]

(79)

\subsection*{A.6. Proof of Proposition \(^7\)}

When \( N_f + N_g = 1 \), then \( A(T_f + T_g + \hat{\Delta}) = 0 \).

\[
T_f + T_g + \hat{\Delta} = \left[ H^*(N_f + N_g) \exp \left( \frac{1}{1 - N_f - N_g} \right) \right]^{1-\alpha}.
\]

(80)

Thus, we have

\[
\log(T_f + T_g + \hat{\Delta}) = (1 - \alpha) \left[ \log H^* + \log(N_f + N_g) + \frac{1}{1 - N_f - N_g} \right]
\]

As a result,

\[
N'(x) \log(T_f + T_g + \hat{\Delta}) \to 0 \text{ as } N_f + N_g \to 1
\]

(81)

by applying the l’Hôpital’s rule. Thus, \( A(T_f + T_g + \hat{\Delta})B(T_f + T_g + \hat{\Delta}) \to 0 \) as \( N_f + N_g \to 1 \).

Thus, if two merging firms are network-dominant, the LHS of the equation (74) is zero, while the RHS of the equation (74) is positive. Thus, \( \hat{\Delta} \) increases with \( \alpha \).
A.7. Proof of Lemma 1

To see the effects of $T_f^A$ and $T_f^B$ on $n_f^A$ and $n_f^B$ note that

$$\frac{\partial m^A}{\partial T_f^A} = n_f^A \frac{1}{T_f^A \det(G_f)} \left[ \mu_f^A \left( n_f^B \mu_f^B + 1 - n_f^B + \beta_A \frac{n_f^A}{n_f^B} \right) + \mu_f^A \beta_A^2 \frac{n_f^A}{n_f^B} \right] > 0$$

$$\frac{\partial m^B}{\partial T_f^A} = n_f^A \frac{1}{T_f^A \det(G_f)} \left( \beta_A \frac{n_f^A}{n_f^B} \mu_f^A (1 - \beta_A) + \beta_A \frac{1 - n_f^A}{n_f^B} \right) < 0,$$

and thus

$$\frac{\partial N_f^A}{\partial T_f^A} = n_f^A \frac{1}{T_f^A \det(G_f)} (1 - n_f^A) \left( n_f^B \mu_f^B + 1 - n_f^B + \beta_A (1 + \beta_A) \frac{n_f^A}{n_f^B} \right) < 0 \quad (82)$$

$$\frac{\partial N_f^B}{\partial T_f^A} = n_f^A \frac{1}{T_f^A \det(G_f)} \left( \beta_A n_f^A \mu_f^A (1 - \beta_A) + \beta_A \frac{1 - n_f^A}{n_f^B} \right) n_f^B < 0. \quad (83)$$

Further, note that

$$\frac{\partial m^A}{\partial T_f^B} = n_f^A \frac{1}{T_f^B \det(G_f)} \left( \beta_A n_f^A \mu_f^A (1 - n_f^B) + \beta_A^3 \frac{n_f^A}{n_f^B} \mu_f^A \right) > 0$$

$$\frac{\partial m^B}{\partial T_f^B} = n_f^A \frac{1}{T_f^B \det(G_f)} \left( n_f^A \mu_f^A + 1 - n_f^A \right) \left( n_f^B \mu_f^B + \beta_A (1 - \beta_A) \frac{n_f^A}{n_f^B} \right) + \beta_A n_f^A \mu_f^A \frac{n_f^A}{n_f^B} > 0,$$

and thus

$$\frac{\partial N_f^A}{\partial T_f^B} = \beta_A \frac{n_f^A}{T_f^B \det(G_f)} (1 - n_f^A) \left( 1 - n_f^B + \beta_A^2 \frac{n_f^A}{n_f^B} \right) > 0 \quad (84)$$

$$\frac{\partial N_f^B}{\partial T_f^B} = \frac{n_f^A}{T_f^B \det(G_f)} \left( n_f^A \mu_f^A + 1 - n_f^A \right) \left( 1 - n_f^B + \beta_A^2 \frac{n_f^A}{n_f^B} \right) > 0. \quad (85)$$

A.8. Proof of Lemma 2

Let $N_k^J$ be the equilibrium market share of firm $k \in \{ f, g \}$ on side $J \in \{ A, B \}$. Let $N_M^J = N_f^J + N_g^J$ for each $J \in \{ A, B \}$. Consider the merging entity’s type $(T_M^A, T_M^B)$ such that

$$N^A(T_M^A, T_M^B, H^A, H^B) = N_M^A$$

$$N^B(T_M^A, T_M^B, H^A, H^B) = N_M^B \quad (86)$$

By the first-order condition, we obtain

$$\mu_M^A = \frac{1}{1 - N_M^A} \quad (87)$$

$$1 - (1 - N_M^B) \mu_M^B - \frac{N_M^A}{N_M^B} (1 + N_M^A) \mu_M^A = 0. \quad (88)$$
Thus, we obtain the unique markup such that the market shares given the aggregators $H^A$ and $H^B$ is given by the unique pair of $\mu_M^A$ and $\mu_M^B$ respectively. Finally, the pair $(T_M^A, T_M^B)$ must satisfy the following system of equations:

\[
N_M^A = \frac{T_M^A(T_M^B)^\beta \exp(-\mu_M^A - \beta \mu_M^B)}{H^A},
\]

\[
N_M^B = \frac{T_M^B \exp(-\mu_M^B)}{H^B}.
\]  

(89)

By the latter equation, $T_M^B$ is uniquely determined. Further, once $T_M^B$ is given, $T_M^A$ is also uniquely determined. Let $(\hat{T}_M^A, \hat{T}_M^B)$ be such pair of types. As a result, there is a unique pair of technological synergies $\hat{A}^J_M := \hat{T}_M^J - T_f^J - T_g^J$.

**Proof of Proposition 8** First, we derive the formula for $d\hat{A}^J / dT_f^J$ for $J, I \in \{A, B\}$. Using the Implicit Function Theorem, we obtain the following system of equations

\[
\left(\begin{array}{c}
\frac{d\hat{A}^J}{d\theta} \\
\frac{d\hat{A}^I}{d\theta}
\end{array}\right) = - \left( \begin{array}{cc}
\frac{\partial N_M^A}{\partial \theta} & \frac{\partial N_M^A}{\partial \theta} \\
\frac{\partial N_M^B}{\partial \theta} & \frac{\partial N_M^B}{\partial \theta}
\end{array} \right) \left( \begin{array}{c}
\frac{\partial N_M^A}{\partial \theta} \\
\frac{\partial N_M^B}{\partial \theta}
\end{array} \right)
\]

(90)

**Effect of $T_f^B$ on $\hat{A}^A$** By the above derivation, we have

\[
\text{sign} \left( \frac{d\hat{A}^A}{dT_f^B} \right) = \text{sign} \left( \frac{\partial N_f^A}{\partial T_f^B} \frac{\partial N_M^B}{\partial T_f^M} - \frac{\partial N_f^B}{\partial T_f^B} \frac{\partial N_M^A}{\partial T_f^M} \right)
\]

(91)

Using the results appeared in the proof of Lemma 1, the terms inside the bracket in right-hand side can be rewritten as follows:

\[
\frac{\beta_A}{T_f^BT_f^M \det(G_f) \det(G_M)} \left( 1 - n_f^B + \beta_f^A n_f^A \right) \left( 1 - n_M^B + \beta_M^A n_M^A \right) \\
\times \left( n_f^A(1 - n_f^A)n_m^B(n_M^A\mu_M^A + 1 - n_M^A) - n_f^A(1 - n_M^A)n_f^B(n_f^A\mu_f^A + 1 - n_f^A) \right)
\]

(92)

This is positive if and only if

\[
\frac{N_f^A(1 - N_f^A)^2}{1 - N_f^A + (N_f^A)^2} N_M^B > \frac{N_M^A(1 - N_M^A)^2}{1 - N_M^A + (N_M^A)^2} N_f^B.
\]

(93)

Next, as observed in the proof of Proposition 4, the function $f(N) = N(1-N)^2/(1-N+N^2)$ is increasing in $N \in [0, \hat{N}]$ and decreasing in $x \in (\hat{N}, 1)$. Thus, if $N_f^A > \hat{N}$, this inequality holds. On the contrary, if $N_M^A < \hat{N}$ and $N_f^B$ is sufficiently small so that the difference between $N_f^B$ and $N_M^B$ is small, then the reverse of the inequality holds. This proves Proposition 8-1 and Proposition 8-2.
Effect of $T^A_f$ on $\hat{\Delta}^B$

Similarly, we have the following derivation:

$$\text{sign}\left(\frac{d\hat{\Delta}^B}{dT^A_f}\right) = \text{sign}\left(\frac{\partial N^B_f \partial N^A_M}{\partial T^A_f \partial T^A_M} - \frac{\partial N^A_f \partial N^B_M}{\partial T^A_f \partial T^A_M}\right).$$ (94)

Using the results appeared in the proof of Lemma 1, the terms inside the bracket can be rewritten as follows:

$$\frac{\beta^A}{T^A_f T^A_M \det(G_f) \det(G_M)} \left[ \left(n^A_f \mu^A_f (1 - \beta_A) + 1 - n^A_f \right) \left(n^A_M \mu^A_M + 1 - n^A_M + \beta_A \left(1 + \beta_A \frac{n^A_M}{n^B_M}\right)\right) \right.$$

$$- \left(n^A_M \mu^A_M (1 - \beta_A) + 1 - n^A_M \right) \left(n^B_f \mu^B_f + 1 - n^B_f + \beta_A (1 + \beta_A) \frac{n^A_f}{n^B_f}\right) \left(1 - n^A_f\right) \left(n^B_f \mu^B_f + 1 - n^B_f\right) \right]$$

(95)

If $\beta_A$ is sufficiently close to 0, this can be approximated as

$$\frac{\beta^A}{T^A_f T^A_M \det(G_f) \det(G_M)} \left[ \left(n^A_f \mu^A_f (1 - \beta_A) + 1 - n^A_f \right) \left(n^A_M \mu^A_M + 1 - n^A_M\right) \left(n^B_f \mu^B_f + 1 - n^B_f\right) \right.$$

$$- \left(n^A_M \mu^A_M (1 - \beta_A) + 1 - n^A_M \right) \left(n^B_f \mu^B_f + 1 - n^B_f\right) \left(1 - n^A_f\right) \left(n^B_f \mu^B_f + 1 - n^B_f\right) \right]$$

(96)

This is positive if and only if

$$\frac{1 - N^B_f}{1 - N^B_f + (N^B_f)^2} \frac{1 - N^A_f + (N^A_f)^2}{(1 - N^A_f)^2} > \frac{1 - N^B_M}{1 - N^B_M + (N^B_M)^2} \frac{1 - N^A_M + (N^A_M)^2}{(1 - N^A_M)^2}. (97)$$

Note that the function

$$\eta(N) = \frac{1 - N}{1 - N + N^2}$$

is decreasing in $N$ and the function

$$\omega(N) = \frac{(1 - N + N^2)}{(1 - N)^2}$$

is increasing in $N$. Further, since $N^A_M = N^I_f + N^I_g$ at $\Delta^I_f = \hat{\Delta}^I_f$, there exists an increasing function $\xi(\cdot)$ such that the above inequality holds if and only if

$$N^A_g < \xi(N^B_g).$$ (98)
References


Cunningham, Colleen, Florian Ederer, and Song Ma, “Killer Acquisitions,” August 2018.


