Abstract. We study two-stage versions of clearinghouse models. In the first stage, supplying firms advertise list prices; in a second stage they set final retail prices. A firm’s retail price is constrained to be no higher than its list price. In contrast to the mixed-strategy equilibria of single-stage games, a unique profile of distinct prices is supported by the play of pure strategies along the equilibrium path. Therefore, we predict stable price dispersion across firms. Unlike previous work, the equilibrium strategies we derive are robust to ex-post deviations and are in line with the findings of empirical studies of price dispersion. We also find novel results in applications of our model to a variety of settings including sales, prominence, advertising and search.

Introduction

Seemingly identical products are often sold at different prices by several firms. The “clearinghouse” framework, most notably associated with Varian’s (1980) “model of sales” and other established contributions (for example, Shilony, 1977; Rosenthal, 1980; Narasimhan, 1988) explains price dispersion and is now embedded as a workhorse pricing model throughout the industrial organization literature. Clearinghouse models typically generate price dispersion via the mixed-strategy Nash equilibria of single-stage simultaneous-move pricing games. Such equilibria are naturally susceptible to ex post deviations, and (ignoring any repeated-play effects) predict dynamically uncorrelated variation in firms’ pricing positions. This is inconsistent with any ability of retailers to offer rapid discounts in response to the list prices of others, and with the lack of sufficiently frequent temporal variation in otherwise disperse prices.

Empirical studies of price dispersion have attributed a large proportion of the variation to frictions or heterogeneity on the consumer-side of the market, typically leading to a voice of support for clearinghouse models. For example, Kaplan and Menzio (2015) study a panel of 50,000 US households’ purchases at brick and mortar stores, and attribute up to 35-55% of the total variation in prices to such frictions (Kaplan and Menzio, p.1184). Also studying offline

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prices, Aas, Wulfsberg, and Moen (2018) find marked dispersion with prices exhibiting high persistence over time, where spells of the same price typically last 1-3 months but many last for over a year. They also report strong dynamic correlation across the pricing distribution: stores charging prices in a particular quartile of the distribution tend to stay in that quartile with approximately 0.8-0.9 probability month-to-month. Gorodnichenko, Sheremirov, and Talavera (2018) examine daily online pricing data for 50,000 goods over 22 product categories in the US and UK over 2010-2012. They report that online prices exhibit similar dispersion to those offline and that prices are fixed for long spells of the order of 7-20 weeks, and “clearly do not adjust every instant”. They conclude that prices tend to vary cross-sectionally rather than temporally i.e., a specific firms tends to consistently charge high or low prices, rather than flipping between the two.

We suggest a simple alternative to the standard framework that produces an equilibrium that is ex-post Nash and is consistent with stable price dispersion (in the sense that firms can maintain distinct prices over multiple periods of play) across an industry. In order to achieve this while retaining the spirit of the classic paradigm, we seek a model which is able to produce a pure-strategy pricing on the equilibrium path while also being simple and empirically plausible. To this end, we suggest a two-stage model in which firms first advertise list prices and then, in a second stage, are free to lower (but not to raise) their prices. Because second-stage prices can be no higher than the first-stage prices, they represent discounts from those list prices. Under several specifications, we show that there is a unique set of prices that are supported by the equilibrium play of pure strategies. Firms’ prices differ (including when firms are otherwise symmetric), and second-period discounts are not used (but the threat of them determines equilibrium prices). Amongst asymmetric firms we pin down the identities of high-price and low-price suppliers.

There are at least two applied motivations for why there may be no upward deviations from an initially-quoted price. Firstly, legal constraints may force a firm to meet any initial published offer. For example, in many countries if a firm advertised a low price but then charged more at the point of sale for largely strategic reasons, it would fall under many authorities’ definition of misleading advertising or deceptive pricing, which can lead to investigation or prosecution. Secondly, customers may think any attempts to charge prices above those listed or advertised socially unacceptable or may otherwise reduce their demand. For example, if a list price sets consumers’ reference points, loss aversion (Kahneman and Tversky, 1979) implies a higher elasticity of demand for prices above the list price than below (for a recent analysis see, e.g., Ahrens, Pirschel, and Snower, 2017). A related, long-standing stream of research in marketing has documented how “advertised reference prices” can significantly impact factors

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Helpfully, our equilibrium shares several of the more well-known comparative statics of the more standard single-stage specifications. However, we also present many new results, which we are able to investigate owing both to the relative analytic simplicity of pure-strategy prices, and to the uniqueness of our price profile. We first examine the classic “shoppers and captives” paradigm, and then present several extensions, including prominence, advertising, and search. In each case, we report results that are either different, novel, or both, relative to the existing literature.

In the benchmark Varian (1980) model, symmetric price-setting firms sell homogenous products. Customers are either shoppers (“informed” customers in Varian’s terminology) who buy at the lowest price, or instead are captive to one firm (“uninformed” about other suppliers). Starting from any profile of relatively high prices, a firm undercuts the lowest price and captures the shoppers. Further steps down a staircase of prices erode profit margins. At sufficiently low prices firms abandon the hunt for shoppers and instead exploit loyal customers, which lifts prices back up to the monopoly level. This “Edgeworth cycle” logic rules out an equilibrium in pure strategies. Varian (1980) constructed a symmetric mixed-strategy equilibrium in which firms randomize over an interval which extends downward from the reservation price of captive customers; Baye, Kovenock, and de Vries (1992) characterized the full set of (uncountably-many) equilibria.

Applying our two-stage model, one firm can make an early aggressive move by posting a low list price, while others use high list prices to exploit fully their loyal customers. An aggressively low limit price dissuades any competitors from discounting in the second stage, and so all shoppers are captured by the single low-price firm. Of course, ex post there is an incentive to hike that price; this is ruled out by our assumption that a customer’s order must be fulfilled at a price no higher than the quoted list price. It remains to check whether the aggressive firm does indeed wish to charge such a low price; our results confirm this. In summary, we find a unique price profile that is implemented via pure strategy choices along the equilibrium path.

Working with asymmetric firms, we identify the limit-pricing first-mover that captures the shoppers. A firm is more aggressive if it has a smaller captive customer base (and so is more willing to give up the exploitation of them) or a lower marginal cost (and so is more able to post a profitable low list price in the first stage). These parameters determine the firm that can offer the lowest undominated list price, and it is this firm that captures the shoppers.

An advantage of our approach is that it predicts price dispersion in pure strategies across a set of competing suppliers, and yet yields expected profits that equal those that emerge from

More formally: if there is a unique lowest price then the lowest-price firm could profitably raise that price; and if multiple firms offer the lowest price then one of them faces an incentive to undercut the others.
a single-stage model. A disadvantage is that, in a captives-and-shoppers world, the dispersion is only across two prices. This is because there is, in effect, only a single shopper type that performs price comparison, and only one of the competing firms successfully captures (with a sufficiently low initial list price) the business of all such shopper types. Naturally, it is unlikely in reality that there are only captives and shoppers.

To obtain a richer pattern of price dispersion, we extend our approach in various directions informed by various applied settings. In all the settings we consider, we find a unique profile of \( n \)-distinct prices that are played as pure-strategies by \( n \) firms, on the equilibrium path. When firms are asymmetric, we are (generally, uniquely) able to predict which firm charges which price.

The first richer setting we consider is a model of prominence in a triopoly. There, one firm is prominent and has some captive consumers, while its non-prominent rivals only reach consumers who also see the prominent firm’s price. We find that each firm changes a distinct price: the prominent firm is the most expensive, while the non-prominent firm with larger reach is the cheapest. Extending the analysis to consider the incentives of a monopolist “prominence provider”, we find that the firm with the largest reach tends to be made prominent. This increases the asymmetry of the market which reduces consumer welfare.

Second, we consider build upon classic models of informative advertising (Butters, 1977; Grossman and Shapiro, 1984), by allowing potential buyers to be randomly and independently aware of each firm, and where the advertising reach of each firm corresponds to the fraction of buyers who see its price. In equilibrium, every firm chooses a positive advertising spend which generates the full set of buyer consideration sets: captive consumers, those who see any two prices, shoppers who see all prices, etc. Again, this produces a unique profile of \( n \) distinct prices that are robust to ex-post deviations. Firms with larger advertising spends charge higher prices.

We endogenize firms’ costly advertising decisions via a three-stage model: firms choose advertising (the breadth of awareness), followed by our two-stage pricing game; this adds a list price stage and positive advertising costs to the model of Ireland (1993). We find that one distinct largest-spend firm emerges, where its \( n - 1 \) rivals employ “puppy-dog ploy” strategies: advertising to an audience of at most half of buyers. Under the symmetric “random mailbox postings” technology of Butters (1977), we find that the leading firm advertises at exactly twice the intensity of all its rivals.

Finally, we visit models of search. We demonstrate our setup modification and equilibrium refinement in the classic sequential search model of Stahl (1989) where, due to the simplicity of dealing with pure strategies, we obtain an analytic solution for consumers’ reservation price. We then visit the fixed-sample search model of Janssen and Moraga-González (2004). Although the core trade-offs firms face are the same in our model and their’s, we report several substantive departures from their results, driven by the fact that single-stage models effectively force equilibrium to be in mixed-strategies.
A Model of Sales

Here we modify the classic model of sales to allow for two stages of pricing: the advertisement of list prices followed by the opportunity to offer discounts. The main result of this section reports a unique set of prices that are chosen as pure strategies along an equilibrium path.

A Two-Stage Pricing Game. There are \( n \) firms and two stages of play. At the first stage firms simultaneously advertise list prices, where \( \bar{p}_i \) is the list price of firm \( i \in \{1, 2, \ldots, n\} \). These prices are observed by everyone. At the second stage the firms simultaneously set final retail prices. The final price \( p_i \) of each firm \( i \) must satisfy a “no price hike” constraint: \( p_i \leq \bar{p}_i \). A choice \( p_i < \bar{p}_i \) means that firm \( i \) offers a discount. Firm \( i \) faces a constant marginal cost \( c_i \geq 0 \).

Buyers are all willing to pay \( v \), which exceeds the marginal cost of every firm. A mass \( \lambda_i > 0 \) of buyers are “loyal” or “captive” to firm \( i \). A mass \( \lambda_S > 0 \) of “shoppers” buy from the cheapest firm. If two or more firms charge the same price then we assume (for convenience) that the shoppers buy from the firm (or firms) with the lowest marginal cost.\(^4\) Ignoring tied prices for now, the payoff (profit) of firm \( i \) is \([p_i - c_i][\lambda_i + \lambda_S I[p_i < p_j]]\), where “\( I[\cdot] \)” is the usual indicator function.

Our second stage is the setting of Varian (1980, 1981), albeit generalized to allow for asymmetries in firms’ costs and customers’ loyalty. Any equilibrium of such a single-stage symmetric game involves mixing by at least two firms, and generates profits equal to those earned by serving only loyal customers at the maximum price \( v \) (Baye, Kovenock, and de Vries, 1992).

We seek an equilibrium in which firms choose pure strategies. In such an equilibrium the second-stage prices equal (as we show) the first-stage list prices i.e., no discounts are offered.\(^5\)

Definition. A profile of prices generates an equilibrium in pure strategies if there is a subgame perfect equilibrium in which those prices are chosen as pure strategies on the equilibrium path.

We do not require pure strategies to be chosen off the equilibrium path. Our equilibria are supported by mixed equilibria in subgames that are reached following some first-stage deviations.

A Unique Equilibrium Outcome in Pure Strategies. In the classic Varian model, a firm faces an incentive to capture the shoppers by undercutting the lowest competing price. This force walks the firms down a staircase of prices, until prices are so low that firms lift prices upward to focus on fully extracting the maximum revenue from loyal customers.

\(^4\)This tie-break rule allows us easily to establish the existence of Nash equilibria in all possible subgames. See Footnote 26 for details.

\(^5\)We can also work with a weaker solution concept: we do not require equilibria in subgames that are reached following a multi-player deviation or following a single-player deviation that is equilibrium-dominated.
In our model a first-stage advertised price commits a firm to a second-stage price cap. If the cap is set sufficiently low then a firm can ensure that its competitors have no wish to undercut in the second stage. The equilibrium structure, then, involves all but one firm pricing high while the remaining firm posts a low limit price to dissuade any second-stage discounts.

Turning to our formal analysis, we begin by noting firm \(i\) is able to guarantee a profit of at least \(\lambda_i(v - c_i)\) by setting \(p_i = \bar{p}_i = v\) and exploiting loyal customers, even if shoppers go elsewhere. A strictly lower price can at best win the business of all shoppers and so generate a profit of at most \((\lambda_i + \lambda_S)(p_i - c_i)\). It follows that firm \(i\) will never set a list price (and final sales price) where this falls below \(\lambda_i(v - c_i)\): list prices \(\bar{p}_i < p_i^\dagger\) are strictly dominated, where

\[
p_i^\dagger \equiv \frac{\lambda_i v + \lambda_S c_i}{\lambda_i + \lambda_S}.
\]  

Eliminating strictly dominated prices, firm \(i\) chooses \(\bar{p}_i \in [p_i^\dagger, v]\). The minimum undominated price \(p_i^\dagger\) is increasing in \(c_i\) and \(\lambda_i\). If these firm-specific parameters are lower then a firm can be more aggressive, in the sense that it is more willing (lower \(\lambda_i\)) and able (lower \(c_i\)) to compete for shoppers’ business. To reflect this, we label the firms in order of increasing aggressiveness. The statements of our results are simplified if this order is strict: \(p_1^\dagger > \cdots > p_n^\dagger\).

**Definition.** The competing firms are “strictly asymmetric” if their minimum undominated prices are strictly ordered: \(p_1^\dagger > \cdots > p_n^\dagger\). A higher-indexed firm is described as “more aggressive”.

Henceforth we often assume (with very little loss of generality) that the firms are strictly asymmetric. We will also discuss situations in which firms are exactly symmetric, and our results extend appropriately to other situations in which subsets of firms are the same.\(^6\)

Next we note some natural properties that must be satisfied in any pure-strategy equilibrium of a second-stage subgame. Firstly, the lowest-price firm must be unique: if two (or more) firms tie for the lowest price, then one would face an incentive to undercut. Secondly, all firms charge their list prices: given that the lowest-price firm is unique, it has no incentive to discount; a higher-priced firm sells only to loyal customers, and so can profit by raising any discounted price all the way up to its list price.

It follows that any equilibrium with on-path pure strategies involves a unique lowest list price. This cheapest firm subsequently captures all shoppers. Given that higher-price firms sell only to their loyal customers, their list prices must be as high as possible. We conclude that some firm \(j\) charges a list price \(\bar{p}_j < v\), whereas all other firms \(i \neq j\) set \(\bar{p}_i = v\).

Given there are no discounts, in the second stage no firm is able to raise price. The lowest-price firm has no incentive to lower its price. We must check, however, whether one of the high-price firms wishes to offer a second-stage discount and undercut \(\bar{p}_j\). Doing so is unprofitable if and only if \(\bar{p}_j \leq p_i^\dagger\). We conclude that \(\bar{p}_j \leq \min_{i \neq j} p_i^\dagger\). If this inequality holds strictly then firm \(j\)

\(^6\)Accommodating fully those other situations (in which firms are not strictly asymmetric) lengthens the statements of our results without generating any new insights.
could safely raise its advertised price at the first stage, hence $\bar{p}_j = \min_{i \neq j} p^*_i$. For firm $j$, any list price below $p^*_j$ is strictly dominated, and so $p^*_j \leq \bar{p}_j = \min_{i \neq j} p^*_i$. This implies that the lowest-price is offered by the most aggressive firm. Lemma 1 summarizes.7

**Lemma 1 (Necessary Prices).** If a profile of prices generates an equilibrium in pure strategies then the most aggressive firm, $n$, sets $\bar{p}_n = p^*_{n-1}$ and captures shoppers, while less aggressive firms $i < n$ set $\bar{p}_i = v$ and sell only to their loyal customers. No firm offers a discount.

Lemma 1 characterizes the (unique) profile of list prices that can form the equilibrium play of pure strategies. However, it does not establish that such an equilibrium exists.

Consider a strategy profile that satisfies Lemma 1: $\bar{p}_n = p^*_{n-1}$ and $\bar{p}_i = v$ for $i < n$. For the high-price firms, the only possibly-profitable first-stage deviation is a price cut. Given that firm $n$ is guaranteed to price at or below $p^*_{n-1}$, any downward deviation by $i < n$ leaves firm $i$ strictly worse off. Similarly, firm $n$ already serves all shoppers and cannot gain by deviating downward.

The remaining first-stage deviation is an increase in list price by firm $n$. Such a deviation leads to a subgame in which there is no pure-strategy Nash equilibrium. Nevertheless, in any mixed-strategy equilibrium of this subgame firm $n$ mixes over an interval with $p^*_{n-1}$ as its minimum, earning the same profit as on the equilibrium path. The profits of firms $i < n$ are also as on the equilibrium path.

**Lemma 2 (Second-Stage Subgame).** Consider a subgame in which firm $n$ faces a price cap satisfying $v \geq \bar{p}_n > p^*_{n-1}$ and others are unconstrained, so that $\bar{p}_i = v$ for all $i < n$. There is at least one mixed-strategy Nash equilibrium of this subgame. In any such equilibrium, firm $n$ earns expected profit $(\lambda_S + \lambda_n)(p^*_{n-1} - c_n)$ while each firm $i < n$ earns expected profit $\lambda_i(v - c_i)$.

For example, if firm $n$ deviates to a list price satisfying $p^*_{n-2} > \bar{p}_n > p^*_{n-1}$ then in the subgame there is a unique Nash equilibrium: firms $i < n - 1$ maintain their list prices ($p_i = \bar{p}_i = v$), while firms $n - 1$ and $n$ mix continuously over the interval $[p^*_{n-1}, \bar{p}_n)$ with distribution functions

$$F_{n-1}(p) = 1 - \frac{\lambda_S(p^*_{n-1} - c_n) - \lambda_n(p - p^*_{n-1})}{\lambda_S(p - c_n)}$$

and

$$F_n(p) = 1 - \frac{\lambda_{n-1}}{\lambda_S} \frac{v - p}{p - c_{n-1}},$$

(2)

and place remaining mass at $\bar{p}_{n-1}$ and $\bar{p}_n = v$, respectively. In fact, under some relatively weak parameter restrictions this is an equilibrium for any $\bar{p}_n > p^*_{n-1}$.8

The profits reported in Lemma 2 are the same as those earned when firms use the list prices described in Lemma 1. This means that an upward deviation in list price by firm $n$ is not profitable. Finally, within (off-path) subgames following any other choices of list prices, any equilibrium may be played. Together, this leads us to our first main result.

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7Versions of this lemma hold when firms are not strictly asymmetric. For example, if firms are symmetric then any one of them can play the role of the lowest-list-price competitor that captures all of the shoppers.

8For example, if firms share a common marginal cost then this is an equilibrium for any $\bar{p}_n > p^*_{n-1}$. 
Proposition 1 (Pure Strategies on the Equilibrium Path). There is a unique profile of prices that generates an equilibrium in pure strategies. There are no second-stage discounts. The most aggressive firm advertises a low price $\bar{p}_n = p_{n-1}^\dagger$ which is just low enough to dissuade others from undercutting, and wins the business of all the shoppers. All other firms sell only to loyal customers at the maximum possible price.

Recall that firms are strictly asymmetric. If firms are symmetric there are $n$ price profiles that generate an equilibrium with the play of pure strategies. These differ only via the identity of the aggressive firm that advertises the low price and captures the shoppers.

Profits, Performance, and Pricing Patterns. The list-price strategies reported in Proposition 1 yield a profit of $\lambda_i(v - c_i)$ to each firm $i < n$ (the monopoly profit from serving only loyal customers) and a profit of $(\lambda_S + \lambda_n)(p_{n-1}^\dagger - c_n)$ for the most aggressive firm $n$. From Lemma 2, we see that these profits are identical to those earned in the mixed-strategy equilibria of a pricing game when firm $n$ chooses a higher list price $\bar{p}_n > p_{n-1}^\dagger$. Of course, one such deviation is for firm $n$ to choose $\bar{p}_n = v$. Following such a deviation, no firm is constrained in the second stage, and so such a subgame is equivalent to the single-stage pricing game studied (for symmetric firms) by Varian (1980) and Baye, Kovenock, and de Vries (1992).

Corollary 1 (Profits). The unique profile of prices that generates an equilibrium in pure strategies yields profits equal to those earned in a standard single-stage model of sales.

This further implies that the user of any model that incorporates a Varian (1980) pricing stage as a subgame can replace that stage with our two-stage list-price-then-discount version without upsetting any earlier-stage behaviour of profit-seeking firms.

The replacement of a single-stage with a two-stage pricing model does not influence (expected) profits. It can, however, matter for welfare. This is because output is allocated differently in the two different model forms. For both, any output supplied to a loyal customer is supplied by the associated firm; there is no change here. In our model, however, shoppers are served by a single firm; in contrast, the mixed-strategy equilibrium of a model of sales allocates shoppers randomly across the firms. If costs are heterogeneous, and if the most aggressive firm has the lowest marginal cost (recalling that a reduction in marginal cost $c_i$ lowers the corresponding firm’s minimum undominated price $p_i^\dagger$) then the two-stage model’s equilibrium is more efficient. Of course, if the most aggressive firm achieves that status because of a relatively small mass of loyal customers, then it is possible that the list-prices-and-discounts scenario results in lower welfare.

Corollary 2 (Welfare and Consumer Surplus). If the asymmetry of firms is driven by heterogeneity of costs, so that the most aggressive firm has the lowest marginal cost, then an equilibrium with on-path pure strategies in a two-stage model generates higher welfare and higher consumer surplus than in any equilibrium of a single-stage pricing game.
Turning to pricing patterns, the symmetric mixed-strategy equilibrium characterized by Varian (1980) involves randomization by all firms, and so the profile of realized prices involves generic distinct prices charged by everyone. Baye, Kovenock, and de Vries (1992) showed that any equilibrium requires at least “two to tango” in the sense that two of the firms randomize over the range of undominated prices, while the other $n - 2$ charge the maximum price. In such equilibria we see three distinct prices. Here, however, we report an equilibrium in which “one firm dances alone” in the sense that shoppers used only one supplier, and so the extent of price dispersion consists of only two distinct prices. As we demonstrate in later sections, that we predict two distinct prices is an artefact of the captives-and-shoppers setup where the only consumers to compare prices see the same prices.

**Comparative Statics and Symmetry.** The effect of the composition of consumers, $\lambda_i, \lambda_S$, and the number of firms, $n$, on prices is a major comparative static of interest in related models. For examples under assumptions of symmetry see e.g., Janssen and Moraga-González (2004); Morgan, Orzen, and Sefton (2006); Stahl (1989); Rosenthal (1980). Also worth noting is that such models typically make one of two assumptions about how the number of loyal consumers depends on $n$. In Rosenthal (1980), entrants each bring their own loyal customer base and do not reduce the loyal base of incumbents. However, in Varian (1980), there is a fixed total mass of loyal consumers who are divided evenly between all firms. Under an assumption of symmetry, we obtain the following results in our two-stage model of sales.

**Corollary 3 (Consumer Composition and Symmetry).** Under the symmetric modelling assumption $\lambda_i = \lambda_j$ for all $i, j$, the unique profile of equilibrium prices has the property that expected price rises with $\lambda_i$ and falls with $\lambda_S$.

**Corollary 4 (Entry and Symmetry).** Under the symmetric modelling assumption $\lambda_i = \lambda_j$ for all $i, j$, the unique profile of equilibrium prices has the property that expected price rises with $n$. The result also holds if $\lambda_i = \lambda_L/n$ for some fixed $\lambda_L > 0$.

The same relationships are found in the canonical single-stage models of sales, and so, under symmetric modelling assumptions, we add support for these comparative statics. However, we now discuss two important matters. First, we explain that the results reflect the same tensions in both models, but emerge more cleanly from ours. Second, we show how our model’s unique profile of equilibrium prices makes predictions under assumptions of asymmetry where single-stage models cannot. To aid the discussion, we label the core incentives found in models of sales:

i) The *business-stealing* incentive is that firms set a low price in order to sell to shoppers.

ii) The *surplus-appropriation* incentive is that firms set a high price in order to exploit their loyal consumers.$^{10}$

$^9$From here on, we maintain the assumption of the previous literature and set $c_i = 0$ for all $i$ (note that zero is just a normalization).

$^{10}$The italicized terms are taken from Janssen and Moraga-González (2004).
In both single- and two-stage models, more shoppers means a higher business-stealing incentive, and hence lower prices, while more loyals means a higher surplus-appropriation incentive, and hence higher prices. In this sense, both setups generate “search externalities” (in the spirit of Armstrong, 2015) because all consumers benefit from an increase in the number of shoppers. However, in the unique mixed equilibrium of symmetric single-stage models, an increase in the mass of shoppers (loyals) causes a first-order stochastic shift left (right) in the pricing distribution of every firm. In our two-stage model, more shoppers (loyals) means the lowest price falls (rises), while the other $n-1$ high prices remain at the monopoly price.

To illustrate the difference with single-stage models regarding the effect of entry we follow Rosenthal (1980) and assume that each firm’s mass of loyal consumers is independent of $n$, denoting $\lambda_i = \lambda_L > 0$ for $i = 1 \ldots n$. As before, denote the mass of shoppers $\lambda_S > 0$. In the classic single-stage models, the unique symmetric Nash equilibrium is in mixed strategies and described by a unique, continuous CDF. In Rosenthal’s version of the model, the business-stealing and surplus-appropriation effects are showcased in their clearest forms. Force (i) pushes equilibrium prices down whereas (ii) pushes them up. In the mixed-strategies equilibrium of the classic models, force (i) operates via the lowest order statistic: holding the distribution fixed, if one takes more draws, the first order statistic falls. Force (ii) arises because when there are more firms, the chance that any one firm is cheapest falls, and in equilibrium firms shift their focus to setting high prices to exploit their loyal customer base. Rosenthal’s main result is that as $n$ increases, force (ii) increases by more than (i), so that expected price rises. In the limit of his model, as $n \to \infty$, the expected price converges to the monopoly price while the expected lowest price converges to the minimum of the support of prices. In our two-stage model, the unique profile of pure prices has one firm pricing at $p^\dagger = v \frac{\lambda_L}{\lambda_L + \lambda_S}$ and $n-1$ firms pricing at the monopoly price, $v$. Remarkably, both these price points are independent of $n$. This is because forces (i) and (ii) are already fully incorporated in the calculation of $p^\dagger$: the forces are balanced in this equilibrium price such that there is neither any profitable business-stealing, nor any surplus appropriation opportunity. In the one-stage game, the fact that strategies are mixed prevents this from occurring. Instead, the lower bound of the support of the equilibrium CDF, which the distribution of the lowest price collapses towards as $n$ grows large, is $p^\dagger$. In our model, the low and the high price are independent of $n$, and expected price rises with $n$ because prices are asymmetric: exactly one firm charges the low price while $n-1$ charge the high. In sum, the same underlying forces drive the same result in both the classic single-stage and our two-stage paradigm. However, our model’s profile of pure-prices balances these forces considerably more cleanly, which among other benefits, allows us to progress more easily to richer modelling assumptions.

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11 The other main assumption, made e.g., by Varian (1980), that $\lambda_i = \lambda_L / n$ for all $i$ does not affect our discussion substantively. For comparability, we also assume symmetric marginal costs, normalized to zero.

12 When loyal bases are independent of $n$, the support is independent of $n$. 

This brings us to a discussion of symmetry. In classic models of sales there are at least three reasons why symmetric equilibria are sought. Firstly, authors typically make symmetric modelling assumptions such as those in Corollary 4, so searching for symmetric equilibria is natural. Second, under symmetric modelling assumptions, there is a unique symmetric equilibrium. And, as Baye, Kovenock, and de Vries (1992) showed, models of sales typically have infinitely-many asymmetric equilibria. However, our game yields a unique asymmetric profile of on-path pure strategies with symmetric or asymmetric modelling assumptions. This allows us to examine the robustness of the comparative statics of single-stage set-ups to a loosening of symmetry assumptions.

First consider the effect of consumer composition under the assumption that each firm has a distinct number of loyal consumers. In our model, increasing the mass of shoppers decreases the lowest price, as it did under symmetry. Regarding captive consumers, only the second-smallest firm’s mass of loyals affects price, and it only affects the smallest firm’s price. Single-stage models are unable to give this prediction because at least two firms must tango i.e., play mixed strategies (Baye, Kovenock, and de Vries, 1992). More generally, the multiple equilibria of single-stage models inhibits precise predictions. However, our model predicts that it is changes in the market power of the smaller firms that affects the price which serves shoppers, and hence generates externalities. In contrast, changes (except for large falls) in the market power of larger firms do not affect equilibrium prices because these firms charge the monopoly price.

Regarding the effect of entry under an asymmetric distribution of loyals, we report the following result.

**Corollary 5 (Entry and Asymmetry).** Under the asymmetric modelling assumption \( \lambda_i > \lambda_j \iff i < j \), if firm \( n \) has a sufficiently small number of loyal customers relative to the other incumbents, \( \lambda_n < (n\lambda_{n-1} - \lambda_S)/(n+1) \), the unique profile of equilibrium prices has the property that the expected price with \( n + 1 \) firms is lower than with \( n \) firms.

In our profile of equilibrium prices, the firm with the fewest loyal consumers, \( n \), is the firm which charges the low price, which is an increasing function of \( \lambda_{n-1} \). Therefore, when the new and smallest firm \( (n + 1) \) arrives, they become the low-price firm, setting an even lower price than \( n \) did when they were the low-price firm. Therefore, if \( \lambda_n \) is small enough relative to \( \lambda_{n-1} \), \( n + 1 \)'s entry lowers the low price by enough to offset the effect of more firms charging the high price, resulting in a lower expected price. This is the opposite of the prediction made by single-stage models.

**A Model of Prominence**

Suppose that there are three firms, each with a base of \( \phi_i > 0 \) captive consumers. In addition, let all consumers be informed of one, prominent, firm’s price. Without loss of generality, let the prominent firm be firm 1 so that consideration sets are \( \{1\} \), \( \{1, 2\} \), \( \{1, 3\} \), held by \( \phi_1, \phi_2, \phi_3 \) consumers, respectively. Also without loss, let \( \phi_2 \geq \phi_3 \).
We now apply our two-stage game. We first derive the candidate price profiles to be played in pure-strategies on the equilibrium path. In any such profile, firm 1’s retail price must be positive because it can always guarantee itself at least $v\phi_1$ by charging $v$. Firm 1 cannot tie with another firm because it would undercut them. Firm 1 must charge the unique-highest price; if a non-prominent firm charged the unique-highest price, that firm would prefer to undercut 1’s price. Because it charges the highest price firm 1 charges $v$. The non-prominent firms charge prices as high as possible such that firm 1 does not wish to undercut them. The cheapest such firm therefore charges

\[ v\phi_1 = p(\phi_1 + \phi_2 + \phi_3) \iff p = v\phi_1/(\phi_1 + \phi_2 + \phi_3). \]  

(3)

Only one of the non-prominent firms can charge this lowest price. If both did, either firm could then profitably raise their price slightly because it would not provoke 1 to undercut them. Similarly, the second-cheapest firm, $i$, charges

\[ v\phi_1 = p(\phi_1 + \phi_i) \iff p = v\phi_1/(\phi_1 + \phi_i). \]  

(4)

These retail prices cannot be lower than firms’ list prices or firms could profitably raise their retail price. Under the symmetric assumption $\phi_2 = \phi_3$, $\bar{p}_1 = v$ along with (3) and (4) define a unique profile of prices.

When $\phi_2 > \phi_3$, there are two candidate profiles. Both entail the prominent firm being the most expensive with $\bar{p}_1 = v$, but differ regarding the non-prominent firms’ prices because (4) depends on whether $i = 2$ or $i = 3$. We show the larger non-prominent firm must be the cheapest i.e., $i = 3$ in (4). If instead $i = 2$ such that $\bar{p}_2 > \bar{p}_3$, there is an interval of prices $(v\phi_1/(\phi_1 + \phi_2), v\phi_1/(\phi_1 + \phi_3))$ which firm 1 never prices within (because such a price yields profit lower than $v\phi_1$). As a result, we show (for details see the web appendix) that following a deviation of firm 3 to a list price $\hat{p}_3 \in (v\phi_1/(\phi_1 + \phi_2), v\phi_1/(\phi_1 + \phi_3))$, 3 makes strictly higher profit in any Nash equilibrium of the ensuing subgame. However, if $\bar{p}_3 > \bar{p}_2$ the analogous deviation is not available because $\phi_2 > \phi_3$. We are left with one candidate profile of prices.

The following lemma summarizes.

**Lemma 3 (Necessary Prices).** If a profile of prices generates an equilibrium in pure strategies, the prominent firm charges $\bar{p}_1 = v$ and sells only to their captive consumers, while the smaller non-prominent firm plays $p_3 = v\phi_1/(\phi_1 + \phi_3)$ and the larger plays $p_2 = v\phi_1/(\phi_1 + \phi_2 + \phi_3)$. No firm offers a discount.

The price profile in Lemma 3 is unique. When $\phi_2 > \phi_3$, the assignment of firms to prices is also unique. When $\phi_2 = \phi_3$ there are (trivially) two such assignments.

Next, we establish that an equilibrium featuring the profile of prices of Lemma 3 exists. No firm can profitably deviate down: the prominent firm does not want to undercut others by construction, while the non-prominent firms would only reduce their profit. It remains to check whether any non-prominent firm has a profitable deviation to a higher list price. Such deviations lead to subgames with no pure-strategy Nash equilibrium. However, because the deviation violates firm 1’s “no undercutting” constraint, these subgames feature mixed equilibrium strategies.
In the equilibrium we derive, the prominent firm and the deviating firm \( i \) mix over supports with the common minimum \( \bar{p}_i \). As such, the deviating firm makes no higher profit than when it charged \( \bar{p}_i \), which is the lowest price the prominent firm is willing to drop to in order to undercut \( i \) (details are in the appendix).

**Lemma 4 (Second-Stage Following a Deviation).** Consider a subgame in which: (i) all firms other than the non-prominent firm \( i > 1 \) charge \( \bar{p}_i \) as described in Lemma 3 and (ii) firm \( i \) deviates upward to a higher list price. There is a mixed-strategy Nash equilibrium of this subgame in which firms earn the same profit as they did with the original list prices.

With the addition that we can allow for any equilibria within other subgames following other profiles of list prices, Lemmas 3 and 4 allow us to report a subgame-perfect equilibrium that supports the play of pure strategies on the equilibrium path.

**Proposition 2 (Pure Strategies on Path: Prominence).** When non-prominent firms are asymmetric, i.e., \( \phi_2 > \phi_3 \), a unique profile of distinct prices generates an equilibrium in pure strategies. There are no second stage discounts. The prominent firm sets the monopoly price. The larger non-prominent firm sets the lowest price. Prices are

\[
\bar{p}_1 = v, \quad \bar{p}_3 = \frac{v\phi_1}{(\phi_1 + \phi_3)}, \quad \bar{p}_2 = \frac{v\phi_1}{(\phi_1 + \phi_2 + \phi_3)}.
\]

(5)

Profits are \( \pi_i = \bar{p}_i \phi_i \).

Unlike the sales and advertising models, firms with larger reach can be cheaper. Because all consumers see the prominent firm’s price, non-prominent firms have no captive consumers. A larger reach makes a non-prominent firm’s price more attractive to undercut for the prominent firm, suppressing the non-prominent firm’s price. However, each firm’s profit is increasing in \( \phi_i \): the increase in sales offsets the reduction in price.

As firm 1’s market power i.e., \( \phi_1 \), increases, price cuts hurt it more. As a result, its non-prominent rivals can sustain higher prices without being undercut by 1. It follows that non-prominent firms’ prices are increasing in \( \phi_1 \). Firm 1’s price is constant in \( \phi_1 \). Together this says consumers are worse off the larger the prominent firm is. It also follows that while the prominent firm’s profit does not depend on its rivals’ reach, non-prominent firms’ profits are increasing in \( \phi_1 \).

Now suppose there is a monopolist, \( M \), who makes exactly one firm prominent. For example, \( M \) may be department store which chooses which firm’s product to display in the window, or a website that decides which firm’s product to put on its home page or at the top of a list of search results.

We add more stages to the game prior to the two pricing stages. \( M \) makes a take it or leave it offer, \( t \) to one firm. Once offered \( t \), the firm can either: (i) accept, pay \( t \) to \( M \) and become the prominent firm; or (ii) reject. If the firm rejects, \( M \) chooses which firm becomes prominent, but does not receive any payment for it. Let firms be ordered such that \( \phi_1 > \phi_2 > \phi_3 \).
The prices of Proposition 2 show that being prominent is strictly more profitable than being non-prominent for any firm. However, how profitable prominence is for \( i \) depends on which firm is prominent when \( i \) is non-prominent. For example, when firm 1 is prominent (denoted \( P = 1 \)), it makes \( \pi_{i}^{P=1} = v\phi_{1} \). But when firm 3 is prominent, firm 1 makes \( \pi_{i}^{P=3} = v\phi_{3}/(\phi_{1} + \phi_{2} + \phi_{3}) \). Therefore, were \( M \) to make firm 3 prominent if its offer were turned down, firm 1 would be willing to pay up to
\[
\pi_{1}^{P=1} - \pi_{1}^{P=3} = \frac{v\phi_{1}(\phi_{1} + \phi_{2})}{\phi_{1} + \phi_{2} + \phi_{3}}
\]
to become prominent. Therefore, this is highest amount \( M \) can charge firm 1 if it makes firm 3 prominent when 1 rejects. Suppose \( M \) makes an offer to \( i \) and makes \( j \) prominent if \( i \) rejects and that firm \( i \) accepts any offer up to its willingness to pay for prominence. It follows that \( M \) maximizes profit by maximizing \( i \)'s willingness to pay:
\[
t^{\ast} = \max_{i,j} \left\{ \pi_{i}^{P=i} - \pi_{1}^{P=j} \right\}.
\]
The choices \( i = 1 \) and \( j = 3 \) solve (7), which gives Proposition 3.

**Proposition 3 (Prominence Provision).** In any equilibrium, the prominence provider offers
\[
t^{\ast} = \frac{v\phi_{1}(\phi_{1} + \phi_{2})}{\phi_{1} + \phi_{2} + \phi_{3}}
\]
to firm 1 who accepts and becomes the prominent firm.\(^{13}\) In the pricing stages, a unique profile of prices generates an equilibrium in pure strategies. These are the prices of Proposition 2. Profits are \( \pi_{M} = t^{\ast}, \pi_{1} = v\phi_{1}\phi_{3}/(\phi_{1} + \phi_{2} + \phi_{3}), \pi_{2} = v\phi_{1}\phi_{2}/(\phi_{1} + \phi_{2} + \phi_{3}), \pi_{3} = v\phi_{1}\phi_{3}/(\phi_{1} + \phi_{3}) \).

Inspection of the prominence provider’s profits shows
\[
\frac{\partial \pi_{M}}{\partial \phi_{1}} > \frac{\partial \pi_{M}}{\partial \phi_{2}} > 0 > \frac{\partial \pi_{M}}{\partial \phi_{3}}.
\]
Therefore, \( M \)'s profit is generally increasing in the asymmetry of the market. This is because non-prominent firms tend to do worse when the prominent firm is smaller. For example, take \( \phi = (\phi_{1}, \phi_{2}, \phi_{3}) > 0 \) and \( \phi' = (\phi_{1} + \Delta, \phi_{2}, \phi_{3} - \Delta) > 0 \) for some \( \Delta > 0 \). By (9), \( M \) makes higher profit under \( \phi' \) than \( \phi \). In other words, the incentive to provide prominence is increasing in the inequality between sellers in the market. Furthermore, \( M \) increases this inequality by choosing to make the largest firm prominent. Because consumer surplus is decreasing in the size of the prominent firm, \( M \)'s incentives are orthogonal to consumers’.

### A Model of Advertising

Our equilibrium in pure strategies predicts stable price dispersion in the model of sales, but only across two prices. In essence, there is only a single type of price-comparing shopper and so (in pure strategies) a single price captures all shoppers. A richer ladder of prices requires multiple types of shoppers.

\(^{13}\)If one were to allow for ties in \( \phi_{i} \), \( t^{\ast} \) is unique up to firm indexing.
We do this here using an advertising specification that builds upon Butters (1977), Chioveanu (2008), Eaton, MacDonald, and Meriluoto (2010), Grossman and Shapiro (1984), and Ireland (1993): each price is exposed to an independent fraction of potential buyers. We find that a unique profile of distinct prices is supported by the equilibrium play of pure strategies.

We also evaluate the endogenous choice of advertising strategies. We find that a single firm chooses relatively high exposure for a high list price; other firms limit their exposures in order to prevent the advertisement of revenue-eroding lower list prices.

**A Two-Stage Pricing Game with Independent Advertising.** The game played by \( n \) firms is as before: firms advertise list prices, and then follow by setting weakly lower final sale prices. To simplify exposition we assume symmetry of costs, and then simplify again (without further loss of generality) by setting marginal cost to zero for all members of the industry.

On the demand side, an independent fraction \( \alpha_i \) of buyers is aware of firm \( i \). Hence the probability that a buyer is aware of firms \( i \) and \( j \) but no others is \( \lambda_{(i,j)} = \alpha_i \alpha_j \prod_{k \in \{i,j\}} (1 - \alpha_k) \), where the notation “\( \lambda_{(i,j)} \)” should be clear. Recycling earlier notation, the mass of shoppers is \( \lambda_S = \prod_{i=1}^{n} \alpha_i \), and the mass of buyers who are loyal to firm \( i \) is \( \lambda_i = \alpha_i \prod_{j \neq i} (1 - \alpha_j) \).

**Definition.** The competing firms are strictly asymmetric if \( 1 \geq \alpha_1 > \alpha_2 > \cdots > \alpha_n > 0 \). A lower-indexed firm, with pricing awareness or advertising reach, is described as a larger firm.

Just as before, we can accommodate ties (so that subsets of firms share the same type) but our results can be stated more cleanly when firms are distinct.

**A Unique Equilibrium Outcome in Pure Strategies.** We again seek a unique outcome in pure strategies. Necessarily, this involves the play of a pure-strategy Nash equilibrium in the pricing subgame. In the loyals-and-shoppers model we noted that the lowest-price firm must be unique. Here we can say something stronger: all final retail prices must be unique. If any subset of firms post the same price then there is positive probability that a buyer sees precisely this “consideration set” of suppliers and so all of them have an incentive to undercut.

Given that prices are distinct, any otherwise-unconstrained firm could locally raise price without sales. This implies that all firms must be constrained: in the second stage all firms charge their list prices, and so no discounts are offered on the equilibrium path.

We now characterize the necessary properties of first-stage list prices. For now, suppose that \( \bar{p}_1 > \cdots > \bar{p}_n \) so that larger firms list higher prices. (We will confirm that this is the case.) The largest firm only sells to its own captive customers, and so sets the maximum list price: \( \bar{p}_1 = v \). The next price must be low enough to dissuade the larger firm from undercutting in the second stage. That largest firm sells to a fraction \( \alpha_1 \prod_{i>1} (1 - \alpha_i) \) of buyers: those who are aware of the price \( p_1 \) but unaware of the \( n - 1 \) cheaper firms. In contrast, charging slightly below \( p_2 \) means that only the \( n - 2 \) cheapest firms provide competition; the undercut sells to
a fraction \( \alpha_1 \prod_{i>2} (1 - \alpha_i) \) of buyers. There is no incentive to undercut the next list price if

\[
\bar{p}_1 \alpha_1 \prod_{i>1} (1 - \alpha_i) \geq \bar{p}_2 \alpha_1 \prod_{i>2} (1 - \alpha_i) \quad \Leftrightarrow \quad \bar{p}_2 \leq (1 - \alpha_2) \bar{p}_1. \tag{10}
\]

If this “no undercutting” inequality holds strictly then the second firm can locally raise \( \bar{p}_2 \) without any loss of sales, and so we conclude that \( \bar{p}_2 = (1 - \alpha_2) \bar{p}_1 \). It remains, however, to check whether the second firm wishes to raise \( \bar{p}_2 \) still further. One possibility would be to match \( \bar{p}_1 \). The worst-case scenario is that this higher price sells only to captives of this second firm. Assuming this worst-case scenario, such an upward deviation must be unprofitable, and so

\[
\bar{p}_2 \alpha_2 \prod_{i>2} (1 - \alpha_i) \geq \bar{p}_1 \alpha_2 (1 - \alpha_1) \prod_{i>2} (1 - \alpha_i) \quad \Leftrightarrow \quad \bar{p}_2 \geq \bar{p}_1 (1 - \alpha_1) \quad \Leftrightarrow \quad 1 - \alpha_2 \geq 1 - \alpha_1, \tag{11}
\]

which holds if \( \alpha_1 \geq \alpha_2 \). This shows that the higher price must be charged by the larger firm.

We have dealt here with the two highest list prices. The same logic can be applied as we move down the sequence, which allows us to characterize fully the list prices that must be charged if pure strategies are played along the equilibrium path. Lemma 5 summarizes.

**Lemma 5 (Necessary Prices).** If a profile of advertised list prices generates an equilibrium in pure strategies then they satisfy \( \bar{p}_1 = v \) and \( \bar{p}_i = (1 - \alpha_i) \bar{p}_{i-1} \) for \( i > 1 \).

We proceed as before. Consider any strategy profile in which the firms charge list prices which satisfy Lemma 5. By construction, and on the equilibrium path, no firm offers a discount in the second stage. In the first stage, no firm wishes to deviate with a lower list price: such a firm could (in any case) choose such a lower list price in the second stage, while lowering the first-stage list price can only serve to push down the prices of competing firms. It follows that the only possible profitable first-stage deviation is for a firm to raise its list price.

The largest firm already charges the maximum price \( \bar{p}_1 = v \). For other firms, an increased list price necessarily violates the “no undercutting” constraint in the relevant subgame, which is resolved by the play of a mixed-strategy Nash equilibrium in that subgame. Such an equilibrium, however, generates payoffs that offer no improvement over the equilibrium path.

Consider, for example, an upward deviation by the second-largest firm to \( \bar{p}_2 > (1 - \alpha_2)v \). In the second-stage subgame, there is an equilibrium in which all other prices are maintained (so that \( p_i = \bar{p}_i \) for \( i > 2 \)) while the top two firms play continuous mixed strategies over the interval \( [(1 - \alpha_2)v, \bar{p}_2) \). The distribution functions satisfy

\[
F_1(p) = \frac{1}{\alpha_1} \left[ 1 - \frac{(1 - \alpha_2)v}{p} \right] \quad \text{and} \quad F_2(p) = \frac{1}{\alpha_2} \left[ 1 - \frac{(1 - \alpha_2)v}{p} \right] \tag{12}
\]

within that interval, with all remaining mass placed at \( v \) and \( \bar{p}_2 \) respectively. For price-setting purposes these firms ignore the existence of other competitors and play the mixed-strategy equilibrium of a standard loyals-and-shoppers game. They enjoy expected profits equal to \( v \alpha_i \prod_{j>2} (1 - \alpha_j) \) for each \( i \in \{1, 2\} \). These are equal to the payoffs received on the equilibrium path. Examining upward list-price deviations for other firms yields Lemma 6.
Lemma 6 (Second-Stage Following a Price Deviation). Consider a subgame in which: (i) all firms other than \( i \) maintain list prices which satisfy Lemma 5 and (ii) firm \( i \) deviates upward to a higher list price. There is a mixed-strategy Nash equilibrium of this subgame in which firms earn the same profit as they did with the original list prices.

Lemma 6 establishes that the remaining deviations in the first stage are not profitable. For any other subgames we can pick any Nash equilibrium. Doing so we can construct a subgame perfect equilibrium that supports the play of pure strategies on path.

Proposition 4 (Pure Strategies on the Equilibrium Path). If firms are strictly asymmetric, then a unique profile of prices generates an equilibrium in pure strategies. Distinct list prices are higher for the larger firms. These prices are

\[
\bar{p}_1 = v \quad \text{and} \quad \bar{p}_i = v\prod_{j=2}^{i}(1 - \alpha_j) \quad \text{for } i > 1. \tag{13}
\]

Moving from larger to smaller firms, prices become relatively closer: \( \bar{p}_i/\bar{p}_{i-1} \) is decreasing in \( i \).

Across the industry cross-section, profits are proportional to firms’ sizes. They satisfy

\[
\pi_i = v\alpha_i\prod_{j=2}^{n}(1 - \alpha_j). \tag{14}
\]

This proposition considers the case with completely asymmetric firms. With symmetric firms (so that \( \alpha_i = \alpha \) for all \( i \) and for some \( \alpha \)) we can characterize a sequence of list prices that are unique apart from the labelling of the firms. For this symmetric case,

\[
\bar{p}_i = v(1 - \alpha)^{i-1} \quad \text{and} \quad \pi_i = v\alpha(1 - \alpha)^{n-1}. \tag{15}
\]

These outcomes are also obtained if we take the unique outcome for a fully asymmetric specification and allow the asymmetries to disappear.

Firms’ (expected) profits have the feature that the awareness parameters \( \alpha_i \) enter symmetrically for firms \( i > 1 \) but not for the largest firm \( i = 1 \). This suggests that the endogenous choice of advertising may be asymmetric (as found in related models e.g., Ireland, 1993). We investigate this next.

Endogenous and Costly Advertising. The \( n \) firms now participate in three stages of play. At the first stage they simultaneously choose their advertising policies: firm \( i \) chooses the (independent) probability \( \alpha_i \in [0, 1] \) that a buyer is aware of its final price. The second and third stages of play follow our list-and-discount specification. Firm \( i \)’s cost of advertising \( C_i(\alpha_i) \) is smoothly and strictly increasing, convex, satisfying \( C_i(0) = 0 \) and \( C_i'(0) < v \). When firms are asymmetric we order them such that \( C_i'(\alpha) < \cdots < C_n'(\alpha) \) for all \( \alpha \in (0, 1) \).

An illustrative cost function is provided by the classic “random mailbox postings” technology described by Butters (1977). Suppose that the unit mass of potential buyers is divided into \( 1/\Delta \) segments each of size \( \Delta \). Each segment here corresponds to mailbox. A single advertisement costs \( \gamma_i\Delta \) for firm \( i \), and randomly hits one of the segments. Hence, with a total spend of
\( C_i(\alpha_i), \) a firm is able to distribute \( C_i(\alpha_i)/(\gamma_i \Delta) \) advertisements. It follows that
\[
\alpha_i = 1 - (1 - \Delta)C_i(\alpha_i)/(\gamma_i \Delta).
\] (16)

Taking the limit as \( \Delta \to 0, \) we observe that \( (1 - \Delta)C_i(\alpha_i)/(\gamma_i \Delta) \to \exp(-C_i(\alpha_i)/\gamma_i). \) Solving for \( C_i(\alpha_i) \) suggests a cost specification which satisfies
\[
C_i(\alpha_i) = \gamma_i \log \left( \frac{1}{1 - \alpha_i} \right),
\] (17)
where (for asymmetric firms) we assume that \( 0 < \gamma_1 < \cdots < \gamma_n. \)

Our solution concept is as before. We seek a profile of pure strategies (for both advertising choices and subsequent list and retail prices) along the equilibrium path, and we also look for the play of pure strategies following any first-stage deviations of advertising choices.

**Definition.** A profile of advertising strategies generates an equilibrium in pure strategies if there is a subgame perfect equilibrium in which pure strategies are played both on the equilibrium path, and along the equilibrium path starting from any second-stage subgame.

Proposition 4 characterizes firms’ expected profits in any list-and-discount subgame when firms choose different advertising strategies. The same profit expressions continue to apply when some firms are the same. Given that firms are not yet ordered by their (now endogenous) choice of advertising exposure, we can write these expected profits as
\[
\pi_i = \begin{cases} 
v \alpha_i \prod_{j \neq i} (1 - \alpha_j) & \alpha_i > \max_{j \neq i} \{\alpha_j\} \text{ and } \\
v \alpha_i (1 - \alpha_i) \prod_{j \notin \{i,k\}} (1 - \alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j \neq i} \{\alpha_j\}, \end{cases}
\] (18)
and where both expressions apply when firm \( i \) ties to be the largest firm.

An equilibrium (in the sense of our definition above) is generated by a pure-strategy Nash equilibrium of the simultaneous-move game in which each firm \( i \) maximizes \( \pi_i - C_i(\alpha_i). \)

The sales revenue earned by a firm reacts differently to a marginal increase in its advertising reach depending on whether that firm is the largest. The largest firm sets the highest price and therefore does not worry about another firm undercutting them; it sets the monopoly price. Therefore for the largest firm, an increase in \( \alpha_1 \) increases its expected revenue linearly. In contrast, smaller firms’ equilibrium prices are set to deter undercutting of larger firms. Therefore for smaller firms, there are two competing effects: fixing second-period prices, an increase in \( \alpha_i \) scales up sales; however, it also forces its second-period price down (and that of any smaller firms because of the recursive nature of prices). In fact,
\[
\frac{\partial \pi_i}{\partial \alpha_i} = \begin{cases} 
v \prod_{j \neq i} (1 - \alpha_j) & \alpha_i > \max_{j \neq i} \{\alpha_j\} \text{ and } \\
v(1 - 2\alpha_i) \prod_{j \notin \{i,k\}} (1 - \alpha_j) & \alpha_i < \alpha_k \text{ where } \alpha_k = \max_{j \neq i} \{\alpha_j\}. \end{cases}
\] (19)
For a smaller firm, revenue is decreasing in advertising exposure when a firm reaches a majority of buyers. If not, then this revenue kinks upward as \( \alpha_i \) passes through the maximum
advertising exposure of competing firms. Specifically,

\[
\lim_{\alpha_i \mapsto \alpha_{\max_i}, \alpha_i \neq j} \frac{\partial \pi_i}{\partial \alpha_i} = \frac{1 - \max_{j \neq i} \alpha_j}{1 - 2 \max_{j \neq i} \alpha_j} > 1,
\]

where the strict inequality holds because (once dominated strategies have been eliminated) every firm chooses positive advertising exposure. This immediately implies that no firm will ever choose its advertising reach to be exactly equal to the maximum of others, and so there is (in any pure-strategy equilibrium) a unique largest firm.

For smaller firms, the fact advertising increases sales revenue only if \(\alpha_i < \frac{1}{2}\) implies that in equilibrium all firms other than the largest restrict awareness to a minority of potential buyers. These observations are recorded in Lemma 7.

Lemma 7 (Properties of Advertising Choices). In any pure-strategy equilibrium of the advertising game, there is a unique largest firm. Other firms advertise to a minority of buyers.

On the revenue side, the largest firm always faces an incentive to increase its exposure. Labelling this firm as \(k\), it is straightforward to confirm that, in equilibrium, \(\partial \pi_k / \partial \alpha_k \geq 1/2^{n-1}\). Hence, if \(C'(1) < 2^{n-1}\) then firm \(k\) chooses \(\alpha_k = 1\) and advertises to everyone. More generally, as costs fall (e.g., allowing the cost coefficients \(\gamma_i\) to fall to zero in the Butters (1977) specification) the audience of the largest firm expands to give it complete market coverage.

An advertising equilibrium is characterized by the specification of a leading (and largest) firm \(k\) and \(n\) advertising choices which satisfy the \(n-1\) first-order conditions

\[
\frac{C'_k(\alpha_k)}{v} = \prod_{j \neq k} (1 - \alpha_j) \quad \text{and} \quad \frac{C'_i(\alpha_i)}{v} = (1 - 2\alpha_i) \prod_{j \neq (i,k)} (1 - \alpha_j) \quad \forall i \neq k.
\]

To fully characterize an equilibrium we also need to check for any non-local deviations. For example, one of the smaller firms \(i \neq k\) has the option to deviate and choose \(\alpha_i > \alpha_k\) and become the largest firm. The proof of Proposition 5 checks such remaining details.

Proposition 5 (Pure-Strategies on Path: Endogenous Advertising). There is at least one equilibrium of the advertising-then-pricing game in which firms choose pure strategies along the equilibrium path, and also do so following any first-stage deviation.

In any such equilibrium, one firm chooses a strictly higher advertising level than all the others, sets a list price equal to the maximum willingness to pay of consumers, and only sells to buyers who are uniquely aware of its product. Other firms advertise to at most half of potential buyers.

In equilibrium a leading firm advertises distinctly more than others. Proposition 5 does not identify this firm. If the advertising cost functions are not too different then any firm can play this role. If they are different then the leading firm enjoys relatively lower advertising costs.\(^{14}\) The other minority-audience firms can, however, be ordered given the structure of the advertising cost functions. For example, if \(k = 1\) then advertising choices satisfy \(\alpha_1 > \cdots > \alpha_n\).

\(^{14}\)Formally: there is some \(k^*\) such that there is an equilibrium in which any \(k \in \{1, \ldots, k^*\}\) leads the industry.
If firms are symmetric \((C_i(\alpha_i) = C(\alpha_i) \text{ for all } i)\) then the first-order conditions simplify appreciably. Writing \(\alpha\) for the common advertising choice of the smaller firms,

\[
\frac{C'(\alpha_k)}{v} = (1 - \alpha)^{n-1} \quad \text{and} \quad \frac{C''(\alpha)}{v} = (1 - 2\alpha)(1 - \alpha)^{n-2}.
\]  

A special case is when advertising is free. There is, of course, a pathological equilibrium in which multiple firms choose \(\alpha_i = 1\) and subsequent prices are competed down to marginal cost. Putting this aside (or by allowing costs to be close to free) the “free advertising” case yields \(\alpha = \frac{1}{2}\) for \(n - 1\) firms, and complete coverage for one firm.

**Proposition 6 (Equilibrium with Free Advertising).** If advertising is free then, in an equilibrium in which firms earn positive profits, the largest firm chooses maximum advertising exposure, while others advertise to half of potential buyers. Labelling firms appropriately, final retail prices satisfy \(\bar{p}_i = \frac{v}{2^i} - 1\). The largest firm earns twice the profit of each smaller firm.

As a final case, we consider the cost specification derived from random mailbox postings suggested by Butters (1977). For the cost function of \((17)\), the marginal cost of increased advertising satisfies \(C'(\alpha) = \gamma/(1 - \alpha)\). Setting \(\gamma = 1\) without loss of generality (for this cost coefficient only matters relative to the valuation \(v\) of buyers for the product) and requiring \(v > 1\) (otherwise all firms choose zero advertising) the relevant first-order conditions become

\[
\frac{1}{v(1 - \alpha_k)} = (1 - \alpha)^{n-1} \quad \text{and} \quad \frac{1}{v} = (1 - 2\alpha)(1 - \alpha)^{n-1}.
\]

These equations solve recursively. Substituting the second into the first, we find that \(\alpha_k = 2\alpha\): no matter what the level of cost, the large firm has twice the advertising reach of all smaller firms. The solution for \(\alpha\) satisfies the natural comparative-static property that \(\alpha\) is increasing in the product valuation \(v\), and so is decreasing in the advertising cost parameter \(\gamma\).

**Proposition 7 (Equilibrium with Random Mailbox Postings).** If the cost of advertising reach is determined by a random mailbox postings technology, so that \(C'(\alpha) = -\gamma \log(1 - \alpha)\), and firms are symmetric, then the largest firm chooses advertising awareness equal to double that of the competing small firms. Advertising is increasing in buyers’ willingness to pay.

**Related Models.** The “independent awareness” advertising technology that we use here is, of course, not new to this paper. A closely related and under-appreciated extension of the model of sales was developed by Ireland (1993). He studied a two-stage model in which first-stage (costless) advertising choices are followed by a single stage of pricing. His pricing stage (very naturally) generates an equilibrium in mixed strategies with exactly the same payoffs as those reported here. This naturally implies that the first-stage equilibrium advertising choices are asymmetric, as described in Proposition 6. Other authors study versions of the classic model of sales but with a pre-pricing stage in which firms determine their loyal shares, also finding asymmetric equilibrium advertising outlays (see e.g., Chioveanu, 2008; Ronayne and Taylor, 2018). In contrast to these papers, our result maintains the prediction of asymmetric

\[15\text{An explicit solution is easily obtained when } n = 2: \alpha = \frac{3}{4} - \frac{1}{4} \sqrt{1 + \frac{8}{v}}.\]
advertising intensities while allowing for the on-path play of pure strategies. Moreover, we also show that the two-to-one advertising ratio (in which the lead firm advertises to twice the audience of other firms) does not require free advertising; it emerges readily via the random-mailbox-postings advertising technology.

A Model of Buyer Search

For both models considered so far, the access of buyers to firms’ prices has been either exogenous or influenced by firms’ marketing choices. Here we allow for endogenous consumer search.

To do this, we modify the models of Stahl (1989) and Janssen and Moraga-González (2004). Stahl studied a framework where firms set prices, while some consumers collect prices sequentially incurring a search cost each time. Janssen and Moraga-González studied an oligopoly version of the classic Burdett and Judd (1983) model: firms set prices, while some consumers use a costly fixed-sample search technology to obtain quotations. Just as in the previous two sections, our firms set their prices over two stages of play: they first set list prices, and then have the opportunity to discount (but not to raise) those prices. In both models, to preserve comparability we assume that such firms are symmetric, with constant marginal cost normalized to zero. On the demand side, both models assume a unit mass of consumers (all of whom are willing to pay at most $v$) of which fraction $\lambda_S \in [0,1]$ are (exogenously) shoppers who see all $n$ final price offers. The difference between the two models is in how the remaining $1 - \lambda_S$ “searchers” suffer costly search to reveal prices. Unlike the original models, one consequence of our two-stage framework is that we are not forced to justify the existence of shoppers by assuming that the distribution of search costs has exactly two elements in its support, $\kappa$ and 0. Alternatively, we can assume the purely informational asymmetry that all consumers are subject to costly search but that proportion $\lambda_S$ (shoppers) observe firms’ first-stage pricing (which directs them to the cheapest firm), while searchers are completely uninformed until they search.

A Pricing Game with Costly Sequential Search. Here we replace the single-stage pricing game studied by Stahl (1989) with our two-stage version. Accordingly, we assume that the $1 - \lambda_S$ searchers receive their first search for free, but then incur cost $\kappa$ for each subsequent search. Firms are ex-ante symmetric from the perspective of searchers, so that searchers are willing to sample firms at random. In the unique equilibrium of Stahl, firms set prices below

---

16 We have in mind a situation in which buyers choose their search strategies at the time same as firms engage in their two stages of pricing. Of course, such a game has no proper subgames and so we cannot use subgame perfection as our solution concept. Formally, then, we consider a game in which buyers move first and simultaneously choose their search policies. These search policies are observed by firms, before they begin their two stages of play. Thus, the first pricing stage occurs at the start of a proper subgame. With a finite number of buyers, an individual buyer would recognize the influence that the search choice has on the firms’ future prices. Here, however, we assume that relevant player set includes a continuum of potential buyers. An individual buyer has no measurable influence on future play, and so such a buyer acts as though taking an action simultaneously. This enables us to use subgame perfection as our solution concept.

17 To argue that shoppers have zero search costs Stahl appeals to the fact that some people enjoy shopping whereas Janssen and Moraga-González suppose that some people have a negligible opportunity cost of time.
searchers’ “reservation price” \( r \), such that these consumers receive their first (free) price quote and purchase it immediately i.e., search no further. If firms were to deviate to some non-equilibrium price, we follow the standard assumption of passive beliefs: upon seeing an out-of-equilibrium price, consumers’ beliefs about other firms’ prices do not change. Below we report the analogous equilibrium to Stahl’s but with our two-stage modification.

**Proposition 8 (Sequential Search).** There is a subgame perfect equilibrium in which firms choose pure strategies along the equilibrium path where \( n - 1 \) firms charge the high price

\[
\min \{ r^*, v \},
\]

one firm charges the low price

\[
\min \left\{ \frac{r^* - 1}{1 - \lambda_S (n - 1)}, \frac{1 - \lambda_S}{1 + \lambda_S (n - 1)} \right\},
\]

where

\[
r^* = \frac{n - 1}{n} \frac{1 + \lambda_S (n - 1)}{\lambda_S},
\]

and there are no second-stage discounts.

The equilibrium is the symmetric version of our Proposition 1 modified by the inclusion of a reservation price. The high price charged by \( n - 1 \) firms is equal to the equilibrium reservation price, \( r^* \) if \( r^* \leq v \). That is, the price \( n - 1 \) firms charge is as high as possible such that the searchers never search a second store. In equilibrium, the high-price firms do not gain from unilaterally choosing a higher price (but still less than \( v \)) because if they did so, searchers would continue their search, buying for sure from the next store.

The relative simplicity of our model’s pure-strategy pricing enables us to obtain simple analytic expressions for variables of interest, including \( r^* \), allowing us to carry out comparative statics exercises easily. It is immediate from Proposition 8 that, as in Stahl (1989), the distribution of market prices converges continuously to marginal cost (zero) à la Bertrand as \( \kappa \to 0 \), and to the monopoly price \( (v) \) à la Diamond (1971) as \( \kappa \to \infty \), \( \lambda_S \to 0 \), or \( n \to \infty \). However, unlike Stahl (1989) the distribution of market prices in our model does not continuously converge to marginal cost as \( \lambda_S \to 1 \). In our asymmetric pure-price profile, the high-price firms’ price converges to a strictly positive value, while the low-price firm’s price, which serves shoppers in equilibrium, tends to zero. This is in contrast to models with symmetric mixed strategies such as Stahl’s. By forcing a single stage (and to a lesser extent symmetry), these models effectively require that multiple firms’ equilibrium strategies exhibit some interior balance of the trade-off between the business-stealing and surplus-appropriation incentives. As \( \lambda_S \to 1 \), the business-stealing incentive dominates, and firms’ equilibrium distribution places more and more weight on low prices. Because strategies are mixed, there is still some positive probability that prices

---

18 We note that in our model, \( r \), does not share the interpretation as in Stahl (see his discussion on p702) because in our model, firms’ (degenerate) price distributions are asymmetric. Nonetheless, \( r^* \) acts as a reservation price in equilibrium because consumers would prefer to buy immediately at \( p \) rather than continue their search iff \( p \leq r^* \).

19 For a discussion of the result that the market converges to the monopoly outcome as \( n \to \infty \), the reader is referred to our earlier discussion of “entry and symmetry”.
will be “high”, even as $\lambda_S \to 1$, and more precisely, there is a profitable ex-post deviation for every firm with probability one for any $\lambda_S < 1$. Our two-stage model relieves equilibrium of this constraint. Our profile of prices has the feature that in equilibrium one low-price firm completely exploits the business-stealing incentive, selling to shoppers with probability one, while the others completely exploit surplus-appropriation, selling only to their loyal customers. This structure of our equilibrium strategies holds for any $\lambda_S < 1$: because shoppers are globally informed, equilibrium only requires one low price to serve them, but single-stage models preclude this possibility. Perhaps a more reasonable criterion for market efficiency is the average price paid. Then, in both our equilibrium and Stahl’s, the average price paid converges continuously to the competitive price (zero) as $\lambda_S \to 1$ for the straightforward reason that shoppers are flooding the market.

**A Pricing Game with Costly Fixed-Sample Search.** Here we replace the single-stage pricing game studied by Janssen and Moraga-González (2004) (henceforth JMG) with our two-stage version. In contrast to sequential search, searchers first apply for quotations from firms, then upon receipt, choose the cheapest. Specifically, searchers choose how many quotes to gather, $q \in \{0, 1, \ldots, n\}$ at cost $\kappa q$ where $0 < \kappa < v$, which randomly reveals (without replacement) $q$ out of the $n$ price offers. We follow JMG by writing $\mu_q$ for the proportion of searchers who pay for $q$ price quotations. Using natural notation, it follows that the fractions of potential buyers who see only a price quotation from firm $i$ (and so are loyal to that firm) or compare the prices of firms $i$ and $j$ are

$$\lambda_i = \frac{\mu_1(1 - \lambda_S)}{n} \quad \text{and} \quad \lambda_{ij} = \frac{2\mu_2(1 - \lambda_S)}{n(n - 1)}.$$  \hspace{1cm} (24)

Once again, we seek equilibria with pure strategies along the equilibrium path.

**Buyer Search Strategies in Equilibrium.** A familiar property of costly search is that buyers obtain at most two price quotations, and some obtain a single quotation. The rough logic is that there are (at least weakly) decreasing returns to increasing the sample size of a search, and so for each (homogeneous) buyer there is either a single optimal sample size $q$ (so that $\mu_q = 1$ for this $q$) or neighboring sample sizes $q$ and $q + 1$ are both optimal (so that $\mu_q + \mu_{q+1} = 1$).\hspace{1cm}20 If all buyers obtain at least two price quotations ($q \geq 2$) then every firm is sure to face head-to-head competition with at least one other firm. This forces prices (also in our two-stage game) down to zero. However, if prices are zero then consumers optimally obtain only a single price quotation; a contradiction. This means that $q \in \{0, 1\}$, and so Lemma 1 of JMG holds here too.

**Lemma 8 (Number of Quotations in Equilibrium).** In equilibrium, searchers gather at most two quotations. At least some searchers gather exactly one quotation, so that $\mu_1 > 0$.

\hspace{1cm}20When $n = 2$ we can also construct situations in which buyers are indifferent between three options $q \in \{0, 1, 2\}$.  

Based on this their version of this lemma, JMG considered three possibilities: (i) a low search intensity equilibrium in which \( \mu_0, \mu_1 > 0 \) and \( \mu_0 + \mu_1 = 1 \); (ii) a moderate search intensity equilibrium in which \( \mu_1 = 1 \); and (iii) a high search intensity equilibrium in which \( \mu_1, \mu_2 > 0 \) and \( \mu_1 + \mu_2 = 1 \). We will show that the low-intensity equilibrium exists only for a set of parameters that is vanishing in \( n \), and that the moderate-intensity equilibrium does not exist (for generic parameter choices) with two-stage pricing. In the high-intensity equilibrium we find different comparative statics with respect to the number of firms, \( n \). Specifically, when searchers’ decisions are treated as exogenous, we show that expected price is not increasing in \( n \), contrary to JMG’s Proposition 6. When searchers’ decisions are endogenous, we show that the relations which JMG rely on to show that equilibrium search intensity is non-monotonic in \( n \), no longer hold.

**Low and Moderate Search Intensity.** We combine possibilities (i) and (ii) above by setting \( \mu_2 = 0 \) and consider scenarios in which searchers either pay to find one quotation or do not search. This places the firms squarely within a world of loyals and shoppers, where the mass of buyers loyal to \( i \) is \( \lambda_i = \mu_i(1 - \lambda_S)/n \). We know (from Proposition 1, with symmetric firms) that \( n - 1 \) firms advertise a list price equal to \( v \) and fully exploit loyal customers. The remaining firm advertises a list price low enough to dissuade discounts from others. From equation (1), this is

\[
p^\dagger = \frac{\lambda_i v}{\lambda_i + \lambda_S} = \frac{\mu_1(1 - \lambda_S)v}{\mu_1(1 - \lambda_S) + \lambda_S n} \quad \Leftrightarrow \quad \frac{v - p^\dagger}{n} = \frac{\lambda_S v}{\mu_1(1 - \lambda_S) + \lambda_S n}.
\]

A single search earns a consumer surplus \( v - p^\dagger \) if the “limit price” firm \( n \) supplies the quotation, which happens with probability \( 1/n \); this generates the expression on the right. Search is optimal if this exceeds the quotation cost \( \kappa \). The fact that a buyer is searching for a single lowest price means that the gains from search (just like the costs) are linearly increasing in the number of quotations \( q \); the probability that \( q \) searches without replacement find the cheapest firm is \( q/n \). This means that if it is strictly preferred to search for one quotation rather than none, then (necessarily) it is strictly optimal to search for quotations from all \( n \) suppliers. This implies that for generic parameter values there can be no “moderate search intensity” equilibrium in which all buyers search for exactly one price quotation.

**Lemma 9 (No Moderate Search Intensity).** For generic parameters, \( \lambda_S v \neq \kappa (1 + \lambda_S (n - 1)) \), there is no equilibrium in which (i) all searchers gather exactly one price quotation, so that \( \mu_1 = 1 \); and (ii) firms choose pure strategies along the equilibrium path.

For our list-prices-and-discounts game (with pure-strategy play) we rule out many of the cases reported in Propositions 1–3 of JMG (Section 3, pp. 1094–1098). In their world, the use of single-stage pricing à la Varian (1980) means that firms choose mixed strategies, and so there are many different possible prices. This implies that there are decreasing returns to increased consumer search, which contrasts with the linearity that we obtain here.\(^{22}\)

\(^{21}\)Notice that the low-intensity equilibrium, where \( \mu_0 > 0 \), would not exist if the first search were free, as in e.g., Stahl (1989).

\(^{22}\)We note that one way to recover existence for a non-degenerate set of parameters would be to assume that search costs are strictly increasing, rather than constant.
It follows that an equilibrium in which searchers obtain at most one price quotation must involve low search intensity, in the sense that a strictly positive mass of them do not search at all. Searchers must be indifferent to paying the cost $\kappa$ of that single search, and so

$$\frac{\lambda_S v}{\mu_1(1 - \lambda_S) + \lambda_S n} = \kappa \iff \mu_1 = \frac{\lambda_S}{1 - \lambda_S} \left( \frac{v}{\kappa} - n \right).$$

We obtain an equilibrium so long as this solution satisfies $\mu_1 \in (0, 1)$. Necessarily (so that $\mu_1 < 1$) this means that there cannot be too few firms. In our model, this is because too few firms can make the probability with which a searcher finds a low price so high that all searchers want to search. However, unlike JMG, we also require there not to be too many firms (so that $\mu_1 > 0$). This is because only one firm charges a low price, and so a single search at cost $\kappa$ at best generates a benefit of $v$ (if that low price is zero) with probability $1/n$. Rearranging these constraints to be in terms of $\kappa/v$ gives the result below.

**Lemma 10 (No Low Search Intensity for Large $n$).** If consumers’ cost-to-value ratio satisfies

$$\frac{\kappa}{v} \in \left( \frac{\lambda_S}{\lambda_S(n-1)+1}, \frac{1}{n} \right),$$

which collapses as $n \to \infty$, then there is a subgame perfect equilibrium with low search intensity in which a fraction

$$\mu_1 = \frac{\lambda_S}{1 - \lambda_S} \left( \frac{v}{\kappa} - n \right)$$

of searchers obtain a single price quotation, $\mu_0 = 1 - \mu_1$, and firms play pure strategies on the equilibrium path. One firm sets a low list price $p^\dagger = v - n\kappa$, while all other firms charge a high list price equal to $v$. There are no second-stage discounts.

We note a contrast to Propositions 4–5 of JMG. They showed the existence of a low-search-intensity equilibrium when $n$ is large. In contrast, we find that the set of parameters for which there exists an equilibrium is vanishing in $n$. With single price quotations, the equilibrium property (with pure strategy play) that only one firm chooses a price lower than $v$ means that buyers are searching for the proverbial needle in the haystack when $n$ is large. Hence, even for low cost-to-value ratios, searching consumers can find it not worthwhile to search at all.

More generally the fraction of searchers who search (and so, ultimately, welfare) is decreasing in $n$. An increase in $n$ also pushes down the lowest price while making the highest prices (equal to $v$) more frequent, and so makes the distribution of prices (looking across the industry) riskier in the usual second-order sense, while the average price remains constant. These results reinforce those reported in Proposition 5 of JMG.

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23The fact that the lowest price of our pure-strategy profile is decreasing in $n$, tempers, but does not undo this effect.

24More specifically, JMG’s expression for expected consumer surplus normalized by $v$, $\phi(\mu_1)$ had the feature that $\phi(\mu_1) \to 0$ as $\mu_1 \to 0$ because their symmetric equilibrium in mixed strategies continuously converges to the Bertrand outcome as $\mu_1 \to 0$ (see their Figure 2). However, as discussed in our treatment of sequential search, this is not true under our pure-price profile: there is only one low price, and hence it is possible for the cost-to-value ratio, $\kappa/v$, to be too high for any searchers to search in equilibrium, even if $\mu_1$ is very small. In fact, for us $\phi(\mu_1) \to 1/n$ as $\mu_1 \to 0$, and so as $n$ increases, finding the low price becomes impossible, and any cost-to-value ratio too high.
Our final observation is that if the number of pure shoppers (who exogenously see all prices) is negligible then trade in a low-search-intensity equilibrium collapses: $\mu_1 \downarrow 0$ as $\lambda_S \downarrow 0$.

**High Search Intensity.** The more interesting case is when some searchers ask for two price quotations. Each firm faces a positive probability of going head-to-head with any other firm. This (as it did in the advertising model) ensures that any equilibrium with the on-path play of pure strategies must involve distinct prices. We order the firms such that $\bar{p}_1 > \cdots > \bar{p}_n$.

Fixing buyers’ search strategies, we now characterize the staircase of prices. Looking for prices supported by the on-path play of pure strategies, we seek list prices that satisfy “no undercutting” constraints: firm $i < n$ with list price $\bar{p}_i$ does not wish to undercut (at the second discounting stage) the list price $\bar{p}_{i+1}$ of firm $i+1$ and so capture those additional searchers who see price quotations from both firms $i$ and $i+1$. For $i < n - 1$, this constraint is

$$
\left( \lambda_i + \sum_{j<i} \lambda_{ij} \right) \bar{p}_i \geq \left( \lambda_i + \lambda_{(i+1)} + \sum_{j<i} \lambda_{ij} \right) \bar{p}_{i+1}.
$$

On the left-hand side, maintaining list price $\bar{p}_i$ wins sales from the mass $\lambda_i$ of effectively-loyal customers who see only this price, and also wins sales $\sum_{j<i} \lambda_{ij}$ from comparisons with all higher-priced rivals. On the right hand side, undercutting $\bar{p}_{i+1}$ grabs additional sales of $\lambda_{(i+1)}$ (the searchers who compare $i$’s price against the next cheapest firm on the pricing staircase). Of course, if this constraint held as a strict inequality then firm $i+1$ could safely raise $\lambda_{(i+1)}$, and so it must holds as an equality. (We check all of the details in the proof of Lemma 11.) For firm $n-1$, there is an additional incentive to undercut firm $n$: by charging below $\bar{p}_n$ this firm can also capture the shoppers (who see all prices). Hence, the relevant equation is

$$
\left( \lambda_{n-1} + \sum_{j<n-1} \lambda_{j(n-1)} \right) \bar{p}_{n-1} = \left( \lambda_{n-1} + \lambda_{n-1(n)} + \sum_{j<n-1} \lambda_{j(n-1)} + \lambda_S \right) \bar{p}_n.
$$

Finally, the highest list price must satisfy $\bar{p}_1 = v$ as such a firm sells only to loyal customers. These three equations together pin down the entire profile of prices that must form part of any equilibrium with the on-path play of pure strategies. The terms $\lambda_i$ and $\lambda_{ij}$ are, in turn, determined endogenously by the search strategies of potential buyers via equation (24).

These equations yield a unique candidate for a profile of prices that can be supported by the play of pure strategies. Just as before, there is no incentive (by construction) for any firm to undercut with a discount at the second stage of pricing, and there is no incentive for any firm to offer a lower list price (for such a firm could always cut price at the second stage). It remains to check for any profitable deviation upwards at the list-price stage. The proof of Lemma 11 does just that, and so pins down prices for any choice of buyer search strategy.

**Lemma 11 (Equilibrium Prices with Search for Two Quotations).** Suppose that $\mu_1 > 0$, $\mu_2 > 0$ and $\mu_1 + \mu_2 = 1$. For the two-stage pricing game played by the $n$ firms, a unique set of prices is supported by a subgame perfect equilibrium in which pure strategies are played on the
equilibrium path. List prices are all distinct and are not discounted in the second stage:

\[
\bar{p}_i = \begin{cases} 
  v \left[ 1 + 2 \frac{i-1}{n-1} \frac{\mu_1}{\mu_1 - \mu_i} \right]^{-1} & \text{for } i < n \\
  v \left[ 1 + 2 \frac{i-1}{n-1} \frac{1-\mu_1}{\mu_1 - \mu_i} + n \frac{\lambda_S}{\mu_1 (1-\lambda_S)} \right]^{-1} & \text{for } i = n.
\end{cases}
\]  
(31)

Allowing the number of firms to grow large, the distribution of prices set across the industry converges to a continuous distribution \( F(\cdot) \) where a single firm sets the low price \( \bar{p}_\infty = 0 \), while all other firms compete for searchers over \([v \mu_1/(2-\mu_1), v]\), where

\[
F(p) = \begin{cases} 
  0 & \text{for } p \leq \frac{v \mu_1}{2-\mu_1} \\
  1 - \frac{\mu_1}{1-\mu_1} \frac{v - p}{2p} & \text{for } p \in \left[\frac{v \mu_1}{2-\mu_1}, v\right) \\
  1 & \text{for } p \geq 1.
\end{cases}
\]  
(32)

With these equilibrium prices to hand, we denote expected price for some given number of firms as \( \mathbb{E}[p; n] \) and make the following remark.

**Corollary 6 (Exogenous Search and Entry).** Under the prices of Lemma 11, expected price does not increase with \( n \). Specifically, \( \mathbb{E}[p; 2] \geq \mathbb{E}[p; n] \) for all \( n \), and \( \mathbb{E}[p; 2] > \mathbb{E}[p; 3] > \mathbb{E}[p; 4] \).\(^{25}\)

This shows that in our two-stage pricing model, JMG’s Proposition 6 does not hold. As discussed earlier in the context of loyals and shoppers, under the symmetric mixed strategies of the single-stage game, it is uncertain whether a particular firm will win shoppers (i.e., be the cheapest). And as the number of firms grows, the probability of being the cheapest falls exponentially. Firms react to this exponential fall in equilibrium by shifting mass to higher prices. This force is not present under our equilibrium price profile: the lowest price \( \bar{p}_n \) internalises this force completely.

To see the difference between the one- and two-stage models more clearly in the present model of search, first notice that the minimum and maximum of the support of equilibrium prices in the one-stage model coincides with the lowest and highest equilibrium price in our two-stage model (\( \bar{p}_n \) and \( v \), respectively). In the symmetric mixed-strategy equilibrium of single-stage models, all firms are effectively forced to mix over the whole support \([\bar{p}_n, v]\), for any \( n \). However, our profile of prices exhibits a clean separation of the forces exerted on firms: one firm charges \( \bar{p}_n \), which serves shoppers, while firms \( i < n \) charge prices that are independent of \( \lambda_S \), set solely to prevent other firms from undercutting them and stealing the searchers for whom they are cheapest. This difference in the equilibrium of the two models can be seen most clearly when \( n \to \infty \). Then, the two-stage model’s equilibrium features a single low price (\( \bar{p}_n = 0 \)) which serves shoppers, and a distribution of prices over \([v \mu_1/(2-\mu_1), v]\) which reflects the (imperfect) competition for the \( 1 - \mu_1 \) searchers who check two prices. When instead \( n < \infty \) increases, our model’s pure prices tend to reduce the expected price, as reported in Corollary 6: \( \bar{p}_n \) falls while the other \( n - 1 \) prices form a distribution (which becomes (32) as \( n \) grows large). Firms

\(^{25}\)Expected price does not generally have a clean analytic solution for \( n > 4 \), but simulations suggest that expected price decreases with \( n \).
in JMG are trading-off the same forces, but their assumptions force the equilibrium into mixed strategies, ultimately causing the opposite comparative static.

Here we make the observation that although both the one- and two-stage models predict the same relationship between expected price and $n$ in the “loyals and shoppers” setups of earlier sections, they make different predictions here. This is because the present model of search has more than one distinct group of active (non-loyal) consumers in equilibrium. Specifically, there are two: $(1 - \lambda_S)\mu_2$ more-active searchers and $\lambda_S$ shoppers. Indeed, if one sets $\lambda_S = 0$ in JMG, then prices are independent of $n$ because then there are only the more-active searchers who check exactly two prices (hence firms behave as if they only had one rival, regardless of $n$). Similarly, for $\lambda_S = 0$ in our model, the lowest and highest equilibrium prices are independent of $n$. As $n$ increases, the intermediate prices shuffle around in order to accommodate the increasing number of distinct prices, but in the limit the distribution of price tends to exactly that of JMG.

Now we consider endogenous search. Lemma 11 generates a (unique) price profile for any $\mu_1$. From this we can work out the expected gain to acquiring one or two price quotations. An equilibrium is obtained when the gain from the second quotation (this is weakly lower than from the first quotation) equals $\kappa$. Denote the equilibrium value of $\mu_1$ for a given $n$, $\mu^*_1(n)$.

To investigate this, we first consider the case where $n = 2$. For this case, the high-price firm charges $v$ (yielding no consumer surplus) while the low-price firm charges $\bar{p}_n$. A single quotation yields consumer surplus $v - \bar{p}_n$ with probability $1/2$, whereas two quotations generate this with certainty; hence for the special case of $n = 2$ the marginal benefit of the first and second quotation are equalized. An equilibrium is obtained if

$$\kappa = \frac{v - \bar{p}_n}{2} = \frac{1 - \mu_1(1 - \lambda_S)}{2 - \mu_1(1 - \lambda_S)}.$$  \hspace{1cm} (33)

Solving yields $\mu^*_1(2) = (v - 2\kappa)(1 + \lambda_S)/(v - \kappa)$. Hence we obtain a high-search-intensity equilibrium where the fraction of more-active searchers increases as search becomes more costly. (This is necessary in order to lower prices sufficiently to induce buyers to search at all.) Similar workings yield $\mu^*_1(3)$ and we report that $\mu^*_1(2) > \mu^*_1(3)$, as in JMG’s Lemma 6.

We can also make progress when the number of firms is very large. Taking $n \to \infty$, Lemma 11 reports the distribution of prices charged. Taking expectations,

$$\lim_{n \to \infty} \mathbb{E}[p | \mu_1] = \frac{v}{2} \mu_1 \log \left( \frac{2 - \mu_1}{\mu_1} \right).$$ \hspace{1cm} (34)

Extending this, we write $F_{\min}(\cdot)$ for the distribution of the minimum of $q = 2$ prices taken from such a large industry. This distribution satisfies

$$F_{\min}(p) \equiv \lim_{n \to \infty} \Pr[\min\{p', p''\} \leq p] = 1 - (1 - F(p))^2 = 1 - \frac{1}{4} \left( \frac{\mu_1}{1 - \mu_1} \right)^2 \left( \frac{v - p}{p} \right)^2.$$ \hspace{1cm} (35)
where \( p' \) and \( p'' \) are prices quoted by randomly chosen (without replacement) firms. Hence,

\[
\lim_{n \to \infty} E[\min\{p', p''\} | \mu_1] = \frac{v}{2} \left( \frac{\mu_1}{1 - \mu_1} \right)^2 \left( \frac{2(1 - \mu_1)}{\mu_1} - \log \left( \frac{2 - \mu_1}{\mu_1} \right) \right).
\] (36)

The difference between the expectations in equations (34) and (36) is the incentive for a buyer to acquire a second price quotation. Equating this to the search cost \( \kappa \) characterizes an equilibrium in an industry with a large number of firms. Specifically,

\[
\lim_{n \to \infty} (E[p | \mu_1] - E[\min\{p', p''\} | \mu_1]) = \frac{v}{2} \frac{\mu_1}{1 - \mu_1} \left( \frac{1}{1 - \mu_1} \log \left( \frac{2 - \mu_1}{\mu_1} \right) - 2 \right) = \kappa.
\] (37)

The left-hand side of this equation is single-peaked, rising from zero at \( \mu_1 = 0 \) and falling back to zero at \( \mu_1 = 1 \). This implies that if \( \kappa \) is sufficiently large there is no solution; but if \( \kappa \) is sufficiently small there are exactly two equilibrium values for \( \mu_1 \). The proof of our next proposition also establishes that the incentive to acquire a second price quotation for \( n \to \infty \) lies everywhere below the incentive when \( n = 2 \). For means that equilibria in the two cases must be distinctly different, and involve greater search (and so lower prices) when the industry is very large rather than very small.

**Proposition 9 (Equilibrium Buyer Search).** Consider subgame perfect equilibria in which (i) firms charge pure strategies along the equilibrium path; and (ii) buyers search for either one or two price quotations. If \( n = 2 \) and \( \kappa/v \in (\lambda_S/(1 + \lambda_S), 1/2) \), then there is a unique equilibrium in which

\[
\mu_1^*(2) = \frac{v - 2\kappa}{(v - \kappa)(1 - \lambda_S)}.
\] (38)

There is some \( \kappa \) such that if \( \kappa < \bar{\kappa} \) then, for \( n \) sufficiently large, there are two equilibria, both of which involve greater buyer search for two quotations than for the case where \( n = 2 \).

In progress...
This appendix contains the proofs of any result that is not given already through arguments in the main text.

**Proof of Lemma 2.** The conditions of Theorem 5 of Dasgupta and Maskin (1986, p. 14) are met, and guarantee the existence of a mixed-strategy Nash equilibrium. Suppose that \( p_i \sim F_i(\cdot) \), and write \( \bar{s}_i \) for the upper bound to the support of this mixed strategy.

Claim (i). No firm places an atom strictly below its list price. This absence of atoms implies that the expected profit of a firm is continuous in its price except at others’ list prices.

An atom at \( p_i < \bar{p}_i \) is justified only if it has a chance of capturing the shoppers. No other firm \( j \neq i \) prices just above \( p_i \); it would be better for \( j \) to undercut and capture the atom. Similarly, any less efficient \( (c_j \geq c_i) \) firm would never set \( p_j = p_i \). The only firms that might do so are strictly more efficient \( (c_j < c_i) \) because they (by our tie-break rule) can win at tied prices; but that means that firm \( i \) loses against them anyway. We conclude that firm \( i \) can raise \( p_i \) locally without losing sales, and yet increasing the profit from loyal customers; a contradiction.

Claim (ii). Firm \( n \) performs strictly better than it does by exploiting only loyal customers.

For \( i < n \) prices strictly below \( p_{n-1}^i \) are strictly dominated. No firm places an atom at \( p_{n-1}^i \). This means that firm \( n \) can capture all shoppers by setting \( p_n = p_{n-1}^i \), and so \( n \)'s expected profit weakly exceeds \((p_{n-1}^i - c_n)(\lambda_0 + \lambda_n) > (p_n^i - c_n)(\lambda_0 + \lambda_n) = (v - c_n)\lambda_n \geq (\bar{p}_n - c_n)\lambda_n \).

Claim (iii). A firm’s pricing mixture includes its list price: \( \bar{s}_i = \bar{p}_i \) for all \( i \).

If \( \bar{p}_j < \bar{s}_i < \bar{p}_i \) then firm \( i \) would never choose \( p_i \in (\bar{p}_j, \bar{p}_i) \): sales are only to loyal customers (firm \( j \) is guaranteed to be cheaper) and firm \( i \) would do better to raise price further. Suppose instead that \( \bar{p}_j > \bar{s}_i \) for all \( j \). For the same reason, firm \( j \) never chooses \( p_j \in (\bar{s}_i, \bar{p}_j) \). From claim (i) there is no atom at \( \bar{s}_i \). Firm \( i \) could again raise price from \( \bar{s}_i \) without losing sales. We conclude that \( \bar{s}_i < \bar{p}_i \) can be optimal only if some \( j \) places an atom at \( \bar{p}_j = \bar{s}_i \). This must be firm \( n \) (all others satisfy \( \bar{p}_j = v > \bar{s}_i \)). Firm \( i \) does not place an atom at \( \bar{s}_i \), from claim (i). Hence firm \( n \) sells only to loyal customers with an atom at \( \bar{p}_n \). This contradicts claim (ii).

Claim (iv). Firm \( i < n \) earns an expected payoff of \( \lambda_i(v - c_i) \).

Suppose that \( \bar{p}_n < v \). From claim (iii), the mixed strategy of \( i < n \) includes its maximum price at which no shoppers buy and so firm \( i \) earns \( \lambda_i(v - c_i) \) from sales to loyal customers.

---

26 A condition is that the sum of payoffs (here, aggregate profit) is upper semi-continuous in actions (here, prices). Industry revenue is continuous. Consider costs when the allocation of output changes discontinuously as prices change. For upper semi-continuity we require the allocation to maximize industry profit, and so minimize aggregate cost, at any tied prices. This is achieved by breaking ties in favor of efficient suppliers.
Suppose instead that \( \tilde{p}_n = v \). If every firm were to place an atom at \( v \) then at least one firm would undercut the others. Hence some firm \( j \) places no atom at \( v \). Other firms \( i \neq j \) are willing to price at or close to \( v \), and are always beaten on price by firm \( j \). These firms earn \( \lambda_i(v - c_i) \). If \( n \neq j \), then firm \( n \) would earn \( \lambda_n(v - c_n) \), which contradicts claim (ii). Hence \( j = n \), and all firms \( i < n \) earn \( \lambda_i(v - c_i) \) as claimed. Note that all firms \( i < n \) must place an atom at \( v \). If some \( i < n \) were to join \( n \) by omitting the atom, then once again firm \( n \)'s expected profit would fall to \( \lambda_n(v - c_n) \), in contradiction to claim (ii).)

Firm \( i \)'s profit can exceed \( \lambda_i v \) only if all other more efficient firms have an atom at \( v \).

Claim (v). Firm \( n \) earns an expected payoff of \((\lambda_0 + \lambda_n)(p_{n-1}^\dagger - c_n)\).

Suppose that firm \( n \) never prices below some \( p > p_{n-1}^\dagger \). Firm \( n - 1 \) could price within the interval \((p_{n-1}^\dagger, \min\{p, p_{n-2}^\dagger\})\) and sell to all shoppers, at a price yielding a profit that strictly exceeds \((\lambda_s + \lambda_{n-1})p_{n-1}^\dagger = \lambda_{n-1}v \). This contradicts claim (iv). Hence the mixed strategy of firm \( n \) extends down to \( p_{n-1}^\dagger \), which captures all shoppers and yields the claimed profit. \( \square \)

**Proof of Lemma 4.** We write \( p_i^\star \) for the list prices of Lemma 3, which generate profits \( \pi_i^\star = p_i^\star \phi_i \). Consider an unilateral upward deviation in list price to \( \tilde{p}_i \) by firm \( i > 1 \). Below we supply a Nash equilibrium in the ensuing subgame.

Suppose \( i = 3 \) and \( \tilde{p}_3 \in (p_3^\dagger, p_3^\star] \). The following strategies constitute a Nash equilibrium. Firm 2 charges \( p_2 = p_2^\star \). Firms 1 and 3 mix continuously over the interval \([p_3^\star, \tilde{p}_3] \) via

\[
F_1(p) = 1 - \frac{p_3^\star}{p}, \quad F_3 = 1 - \frac{(v - p)\phi_1}{p\phi_3},
\]

(39)

placing any residual mass at \( p_1^\star \) and \( \tilde{p}_3 \), respectively. Firm 3’s profit is \( \pi_3^\star \).

Suppose \( i = 2 \) and \( \tilde{p}_2 \in (p_2^\dagger, p_2^\star] \). The following strategies constitute a Nash equilibrium. Firm 3 charges \( p_3 = p_3^\star \). Firms 1 and 2 mix continuously over the interval \([p_2^\star, \tilde{p}_2] \) via

\[
F_1(p) = 1 - \frac{p_2^\star}{p}, \quad F_2 = 1 - \frac{(v - p)\phi_1}{p(\phi_2 + \phi_3)},
\]

(40)

placing any residual mass at \( p_1^\star \) and \( \tilde{p}_2 \), respectively. Firm 2’s profit is \( \pi_2^\star \).

Suppose \( i = 2 \) and \( \tilde{p}_2 \in (p_3^\star, p_3^\dagger] \). The following strategies constitute a Nash equilibrium. Firm 1 mixes via

\[
F_1(p) = 1 - \frac{p_2^\star}{p} \quad \text{for } p \in [p_2^\star, \tilde{p}_2]
\]

(41)

and places residual mass at \( p_1^\star \). Firm 2 mixes via

\[
F_2(p) = \begin{cases} 
1 - \frac{(v - p)\phi_1}{p(\phi_2 + \phi_3)} & \text{for } p \in [p_2^\star, \tilde{p}_2] \\
1 - \frac{(v - p)\phi_1}{p\phi_2} & \text{for } p \in [p_3^\star, \tilde{p}_2],
\end{cases}
\]

(42)
and places residual mass at $\tilde{p}_2$. Firm 3 mixes via
\begin{equation}
F_3(p) = \begin{cases} 
1 - \frac{(v - p)\phi_1}{p(\phi_2 + \phi_3)} & \text{for } p \in [p_2^*, \tilde{p}] \\
2 - \frac{(v - p)\phi_1}{p\phi_3} & \text{for } p \in [\tilde{p}, p_3^*], 
\end{cases}
\end{equation}
where $\tilde{p} = \nu_1\phi_2/(\phi_1\phi_2 + \phi_3(\phi_2 + \phi_3))$. Firm 2’s profit is $\pi_2^*$.

Proof of Proposition 3. Here we show that $M$’s optimal choice is $i = 1, j = 3$. Recall $\pi_M = \pi_i^{P=i} - \pi_i^{P=j}$ for $i \neq j$. Once one firm is prominent, the equilibrium of Proposition 2 follows. Then for $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$:
\begin{equation}
\pi_i^P = \begin{cases} 
\nu P & \text{if } P = i \\
\nu P \phi_1/(\phi_P + \phi_1) & \text{if } P = j, i > k \\
\nu P \phi_1/(\phi_P + \phi_1 + \phi_k) & \text{if } P = j, i < k.
\end{cases}
\end{equation}
Using these expressions it can be easily checked that $i = 1, j = 3$ maximizes $\pi_i^{P=i} - \pi_i^{P=j}$. □

Proof of Lemma 5. For now, label firms so that $\tilde{p}_1 > \cdots > \tilde{p}_n$. (The argument in the text confirms that prices are distinct.) If firms maintain list prices on the equilibrium path then firm $i$ earns a profit $\tilde{p}_i \alpha_i \prod_{k>i}(1 - \alpha_k)$. Undercutting a cheaper firm $j > i$ earns (arbitrarily close to) $\tilde{p}_j \alpha_i \prod_{k>j}(1 - \alpha_k)$. Comparing these terms yields the inequality $\tilde{p}_j \geq \tilde{p}_i \prod_{j > k > i}(1 - \alpha_k)$. This inequality is satisfied if and only if $\tilde{p}_i \leq (1 - \alpha_i)\tilde{p}_{i-1}$ for every $i > 1$.

This holds as an equality. If not then we find the lowest $i$ such that the inequality is strict. Such a firm $i$ can locally raise its list price (and final retail price) without violating any “no undercutting” inequalities in the second stage, and so without losing any sales; a contradiction. We conclude that $\tilde{p}_i = (1 - \alpha_i)\tilde{p}_{i-1}$ for $i > 1$. The argument in the text shows that $\tilde{p}_i = v$. Repeated substitution of the relevant equality yields $\tilde{p}_i = v \prod_{j=2}^i(1 - \alpha_j)$ for $i > 1$.

Finally, we confirm the ordering of the firms. Firm $i > 1$ earns profit $v \prod_{j=2}^i(1 - \alpha_j)$ with probability $\alpha_i \prod_{j>i}(1 - \alpha_j)$, yielding an expected profit of $\nu \alpha_i \prod_{j=2}^i(1 - \alpha_j)$. Pricing at $v$ would (at worst) yield a profit of $\nu \alpha_i \prod_{j \neq i}(1 - \alpha_j)$. Comparing this terms, necessarily $1 - \alpha_i \geq 1 - \alpha_1$, or equivalently $\alpha_1 \geq \alpha_i$. We conclude that the largest firm must charge the highest price. □

Proof of Lemma 6. We write $p_i^*$ for the list prices that satisfy Lemma 5 and generate profits $\pi_i = \nu \alpha_i \prod_{j=2}^i(1 - \alpha_j)$. Consider an upward deviation in list price by firm $i > 1$. Suppose that this deviant list price satisfies $p_i^* < \tilde{p}_i \leq p_{i-1}^*$. Consider the following strategy profile: all firms $j < i - 1$ and $j > i$ charge their list prices, so that $p_j = \tilde{p}_j = p_j^*$; firms $j = i - 1, i$ mix continuously over the interval $[p_i^*, \tilde{p}_i)$ with distribution satisfying
\begin{equation}
F_j(p) = \frac{1}{\alpha_j} \left[ 1 - \frac{p_i^*}{p} \right],
\end{equation}
and \( i - 1 \) places residual mass at \( \bar{p}_{i-1} \). (It is readily verified that \( F_j(p) \) satisfies \( F_j(p^*_i) = 0 \), is increasing, satisfies \( F_j(p) \leq 1 \) for \( p \leq \bar{p}_j \), and that there are no profitable deviations.) These strategies constitute a Nash equilibrium.

Deviations by \( i \) to higher list prices are handled as follows. For example, if \( i \) deviates to a list price satisfying \( p^*_{i-1} < \bar{p}_i \leq p^*_{i-2}(1 - \alpha_i)^{1/2} \) then players \( j = i - 1, i \) mix by (45) and \( i - 1 \) places residual mass at \( \bar{p}_{i-1} \). If however, \( p^*_{i-2}(1 - \alpha_i)^{1/2} < \bar{p}_i \leq p^*_{i-2}, j = i - 2, i - 1, i \) mix in the interval \([p^*_i, p^*_{i-1}]\) according to

\[
F_j(p) = \frac{1}{\alpha_j} \left[ 1 - \left( \frac{p^*_i}{p} \right)^{1/2} \right],
\]

while firms \( j = i - 2, i \) also mix in the interval \([p^*_{i-2}(1 - \alpha_i)^{1/2}, \bar{p}_i)\) according to

\[
F_j(p) = \frac{1}{\alpha_j} \left[ 1 - \frac{p^*_i}{p(1 - \alpha_{i-1})} \right],
\]

and place remaining mass at \( \bar{p}_j \). (Again it is readily checked that these strategies yield the payoffs \( \pi_j \) and there are no profitable deviations.) These strategies constitute a Nash equilibrium. One can then consider all higher list-price deviations by firm \( i \) iteratively, concluding in all cases that \( i \)'s profit is \( \pi_i \), and hence that \( i \) has no profitable upward deviation in its list price \( \hat{p}_i \).

Proof of Proposition 5. Here we show there is an equilibrium where firm 1 chooses the outright highest advertising exposure \( \alpha^*_1 = \max_{i \neq 1} \{ \alpha^*_i \} \). Equilibrium of the pricing subgames are those of Proposition 4. Advertising choices satisfy the first-order conditions (21) so there are no local deviations for any firm. The remaining deviation checks are non-local:

(i) \( i \) deviates to \( \hat{\alpha}_1 \leq \alpha^*_j \) where \( j: \alpha^*_j = \max_{i > 1} \{ \alpha^*_i \} \). Firm \( j \) satisfies their first-order condition at \( \alpha^*_j \). Therefore, the best such deviation for 1 is to \( \hat{\alpha}_1 = \alpha^*_j \) (1’s revenue (cost) curve is the same (flatter) for \( \hat{\alpha}_1 \in [0, \alpha^*_j] \) than \( j \)'s over the same interval when \( \alpha_i = \alpha^*_i \) for \( i \neq j \)). But by continuity and 1’s first-order condition, 1’s profit at any \( \hat{\alpha}_1 \geq \alpha^*_j \) is less than at \( \alpha^*_1 \). Hence no such non-local deviation is profitable.

(ii) \( i > 1 \) deviates to \( \hat{\alpha}_i \geq \alpha^*_i \). Firm 1 satisfies their first-order condition at \( \alpha^*_1 \). Therefore, the best such deviation for \( i > 1 \) is to \( \hat{\alpha}_i = \alpha^*_i \) (\( i \)'s revenue (cost) curve is flatter (steeper) for \( \hat{\alpha}_i \in [\alpha^*_1, 1] \) than 1’s over the same interval when \( \alpha_i = \alpha^*_i \) for \( i > 1 \)). But by continuity and \( i \)'s first-order condition, \( i \)'s profit at any \( \hat{\alpha}_i \leq \alpha^*_i \) is less than at \( \alpha^*_i \). Hence no such non-local deviation is profitable.

\[ \Box \]

Proof of Proposition 6. We look for equilibria with pure-strategy advertising choices where following any profile of advertising intensities (\( \alpha^*_i \) for \( i = 1, \ldots, n \)), the equilibrium of the pricing subgame is that of Proposition 4. We put aside trivial equilibria where more than one firm chooses \( \alpha_i = 1 \) which lead to marginal cost pricing. All firms have zero costs and are therefore symmetric. It follows that although the profile of equilibrium advertising choices we report is unique, the assignment of firms to advertising choices is not. Subject to this disclaimer, the
main text explains that one firm will advertise with the outright highest intensity, and we label this firm 1.

By (18) the profit of firm 1 is strictly (and linearly) increasing in \( \alpha_1 \) for any \( \alpha_i < 1 \) for \( i > 1 \), hence \( \alpha_1^* = 1 \). Given \( \alpha_1^* = 1 \), (18) shows that the profit of the non-largest firms is maximized at \( \alpha_i = 1/2 \) for any \( \alpha_j < 1 \) where \( j \neq 1, i \), hence \( \alpha_i^* = 1/2 \) for \( i > 1 \).

\[ \square \]

**Proof of Proposition 7.** We look for equilibria with pure-strategy advertising choices where following any profile of advertising intensities \( (\alpha_i^* \text{ for } i = 1, \ldots, n) \), the equilibrium of the pricing subgame is that of Proposition 4. All firms have the same cost function and are therefore symmetric. It follows that although the profile of equilibrium advertising choices we report is unique, the assignment of firms to advertising choices is not. Subject to this disclaimer, the main text explains that one firm will advertise with the outright highest intensity, and we label this firm 1.

Firms \( i, j > 1 \) must satisfy their first-order conditions given in (21) but with \( C_i = C \). Taking the ratio of \( i \)'s and \( j \)'s condition yields

\[
\frac{C'(\alpha_i)}{C'(\alpha_j)} = \frac{(1 - 2\alpha_i)(1 - \alpha_j)}{(1 - 2\alpha_j)(1 - \alpha_i)}.
\]

If \( \alpha_i > (\alpha_j \text{ the LHS } > 1 (\text{but the RHS } < 1(> 1). However, if \( \alpha_i = \alpha_j \), (48) is satisfied.

Hence \( \alpha_1^* = \alpha_i^* \). Letting \( C(\alpha) = -\log(1-\alpha) \) gives (23), the solution to which gives the values of \( \alpha_i^* \) and \( \alpha_i^* \text{ for } i > 1, \text{ and that } \alpha_1^* = 2\alpha_i^* \). Similar reasoning to that in the proof of Proposition 5 rules out profitable non-local deviations.

\[ \square \]

**Proof of Lemma 11.** Abusing notation a little, we write \( \lambda_i \equiv \lambda \) for the mass of buyers who see a single price quotation from a firm \( i \), and \( \lambda_{ij} \equiv \lambda_{x2} \) for the mass who see quotations from distinct firms \( i \) and \( j \). Equation 29, expressed as an equality, becomes

\[
\bar{p}_{i+1} = \frac{\lambda + (i - 1)\lambda_{x2}}{\lambda + i\lambda_{x2}} \bar{p}_i.
\]

Beginning from \( \bar{p}_1 = \nu \) and substituting iteratively yields the general solution

\[
\bar{p}_i = \begin{cases} 
\frac{\lambda\nu}{\lambda + (i-1)\lambda_{x2}} & i \leq n - 1 \\
\frac{\lambda\nu}{\lambda + \lambda_{S} + (n-1)\lambda_{x2}} & i = n.
\end{cases}
\]

Next we note that \( \lambda = (1 - \lambda_{S})\mu_1/n = (1 - \lambda_{S})(1 - \mu_2)/n \) and \( \lambda_{x2} = 2(1 - \lambda_{S})\mu_2/[n(n - 1)] \). Substituting these expressions yields the solution stated in the lemma.

\[ \star \text{ To be completed } \ldots \star \]

\[ \square \]


Suppose \( n = 3 \) and \( \phi_2 > \phi_3 \). This appendix shows that the prices \( \bar{p}_1 = v, \bar{p}_2 = v\phi_1/(\phi_1 + \phi_2), \bar{p}_3 = v\phi_1/(\phi_1 + \phi_2 + \phi_3) \) cannot be played as pure strategies on the equilibrium path. If they were, \( \pi_3 = v\phi_1/(\phi_1 + \phi_2 + \phi_3) \). However, below we show that in any equilibrium of the subgame following first-stage deviation of firm 3 to \( \hat{p}_3 \in (v\phi_1/(\phi_1 + \phi_2), v\phi_1/(\phi_1 + \phi_3)) \), firm 3’s profit is strictly higher (Lemma W17).

Denote \( s_i, \bar{s}_i > 0 \) as the minimum and maximum of the support of prices for firm \( i, S_i \).

**Lemma W1.** Some firms employ mixed strategies.

**Proof.** If not, firms charge their list prices. But then firm 1 profits from undercutting \( \bar{p}_2 \). □

**Lemma W2.** There is zero probability of a tie in price.

**Proof.** If not, then either firm with probability mass on the same price can move their mass to a slightly lower price for a discrete gain in sales. □

**Lemma W3.** \( \bar{s}_1 \geq \bar{s}_2, \bar{s}_3 \).

**Proof.** Suppose \( \bar{s}_i < \bar{s}_1 \). Then firm \( i \) profits by moving all probability mass on prices above \( \bar{s}_1 \), to prices below \( \bar{s}_1 \). □

**Lemma W4.** \( \bar{s}_1 > \bar{s}_2, \bar{s}_3 \).

**Proof.** Lemma W3 gives \( \bar{s}_1 \geq \bar{s}_2, \bar{s}_3 \). Now suppose \( \bar{s}_i < \bar{s}_1 \). If 1 has a mass point at \( \bar{s}_1 \), \( \pi_1 < v\phi_1 \) (but 1 is guaranteed at least \( v\phi_1 \)). If 1 has no mass point at \( \bar{s}_1 \), then \( \pi_1 = 0 \) (but there is always some price small enough to guarantee \( \pi_1 > 0 \)). □

**Lemma W5.** \( \bar{s}_1 = v \) and \( \pi_1 = v\phi_1 \).

**Proof.** Immediate from Lemma W4. □

**Lemma W6.** \( \bar{s}_1 \leq \min\{\bar{s}_2, \bar{s}_3\} \).

**Proof.** Suppose \( \bar{s}_1 > \bar{s}_i \). Then firm \( i \) charges \( \bar{p}_i \) (else firm \( i \) profits by moving all probability mass to \( \bar{p}_i \)). But then firm \( P \) profits from undercutting \( \bar{p}_u \). □

**Lemma W7.** \( \bar{s}_1 = \min\{\bar{s}_2, \bar{s}_3\} \).

**Proof.** Lemma W6 gives \( \bar{s}_1 \leq \min\{\bar{s}_2, \bar{s}_3\} \). Suppose \( \bar{s}_1 < \min\{\bar{s}_2, \bar{s}_3\} \). Then firm 1 profits by moving all probability mass on prices below \( \min\{\bar{s}_2, \bar{s}_3\} \) to slightly higher prices. □

**Lemma W8.** No firm places mass at any point in \( [\bar{s}_1, \bar{p}_2] \).

**Proof.** Suppose there were, at some \( p \in [\bar{s}_1, \bar{p}_2] \). By Lemma W2 only one firm places mass at \( p \). Then there must be some interval \( (p, p + \epsilon) \) in no firm’s support (if there were, such a firm could profitably move this mass to slightly below \( \bar{s}_1 \)). But then, the firm with the mass point at \( p \) could profitably shift this mass to a slightly higher price. □
Lemma W9. \( s_1 = v \phi_1 / (\phi_1 + \phi_2 + \phi_3) \).

Proof. By Lemmas W5 and W8, \( \pi_1 = v \phi_1 = s_1 (\phi_1 + \phi_2 + \phi_3) \).

Lemma W10. \( [s_1, \bar{p}_2] \subset S_i \).

Proof. If not, \( \exists (x, y) \subset (s_1, \bar{p}_2), (x, y) \notin S_i \) and where \( x \in S_i \setminus \bar{1} \) (by Lemma W7 there is such an \( i \)). Firm \( i \) can profitably move the mass it assigns to prices slightly below \( x \) to any price in \( (x, y) \).

Lemma W11. \( \not\exists (x, y) \in [s_1, \bar{p}_2] \) such that \( (x, y) \notin S_2, S_3 \).

Proof. If not, \( (x, y) \notin S_1 \), contradicting Lemma W10.

Lemma W12. \( F_1 = 1 - s_1/p \) for \( p \in [s_1, \bar{p}_2] \).

Proof. Take \( i \neq 1 : s_i = s_1 \). Suppose \( [s_1, \bar{p}_2] \subset S_i \). Then for \( p \in [s_1, \bar{p}_2] \cap S_i \),

\[
\pi_i = s_1 \phi_i = p \phi_i (1 - F_i(p)) \iff F_i(p) = 1 - s_1/p.
\] (51)

Now suppose \( \exists (x, y) \in [s_1, \bar{p}_2], (x, y) \notin S_i \), where \( [s_1, x] \in S_i \). Take \( j : (x, x + \epsilon) \in S_j \) (by Lemma W11, \( j \) exists). Firm \( j \) must be indifferent over prices charging \( p \downarrow x \) and \( p \in (x, x + \epsilon) \),

\[
\pi_j = x \phi_j (1 - F_1(x)) = p \phi_j (1 - F_1(p)) \iff F_1(p) = 1 - s_1/p.
\] (52)

Lemma W13. If \( s_3 < \bar{p}_2 \), \( \pi_3 = s_1 \phi_3 \).

Proof. For \( p \in [s_3, \bar{p}_2] \cap S_3 \), by Lemma W12, \( \pi_3 = p \phi_3 (1 - F_1(p)) = s_1 \phi_3 \).

Lemma W14. \( (\bar{p}_2, v \phi_1 / (\phi_1 + \phi_3)) \notin S_i \).

Proof. The best 1 does with \( p \in (\bar{p}_2, v \phi_1 / (\phi_1 + \phi_3)) \) is to sell \( \phi_1 + \phi_3 \) units, making \( \pi_1 = p (\phi_1 + \phi_3) < v \phi_1 \), which is made by charging \( v \).

Lemma W15. \( s_3 \geq \bar{p}_2 \).

Proof. If not, \( \pi_3 = s_1 \phi_3 \) from Lemma W13. However, by Lemma W14 \( p \in (\bar{p}_2, v \phi_1 / (\phi_1 + \phi_3)) \) yields

\[
p \phi_3 (1 - F_1(\bar{p}_2)) = p \phi_3 s_1 / \bar{p}_2 > s_1 \phi_3 \iff p > \bar{p}_2.
\] (53)

Lemma W16. \( s_3 = \bar{p}_3 \).

Proof. By Lemma W15, \( s_3 \geq \bar{p}_2 \). By Lemma W14, prices \( [\bar{p}_2, \bar{p}_3] \) are strictly less profitable for 3 than \( \bar{p}_3 \).

Lemma W17. \( s_3 = \bar{p}_3 \) and \( \pi_3 = s_1 \phi_3 \bar{p}_3 / \bar{p}_2 > s_1 \phi_3 \).

Proof. By Lemma W16, \( s_3 = \bar{p}_3 = \bar{p}_3 \). At this price, and recalling \( \bar{p}_3 > \bar{p}_2 \),

\[
\pi_3 = \bar{p}_3 \phi_3 (1 - F_1(\bar{p}_2)) = s_1 \phi_3 \bar{p}_3 / \bar{p}_2 > s_1 \phi_3.
\] (54)