Information, Bertrand-Edgeworth Competition
and the Law of One Price

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Abstract

It is a widespread empirical observation that homogeneous goods often trade at
different prices within the same market. This work proposes a theoretical foundation
for this phenomenon in the context of a capacity-constrained price game. The sellers
have asymmetric information about the state of the market modelled by a partition
of the state space, and evaluate uncertain profits in a way consistent with ex ante
ambiguity aversion. It is demonstrated that if the market demand is uniformly elastic
in each state then a pure strategy price equilibrium exists. In the equilibrium sellers
may post different prices in the market violating the law of one price. Moreover,
the market demand may be rationed between the sellers, resulting in positive trade
taking place at different prices.

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1 Introduction

A significant body of empirical research has demonstrated that homogeneous goods are often sold at different prices within the same market. This observation is usually termed “price dispersion”.¹ The classical models of price competition; Bertrand competition (in which sellers meet all demand forthcoming to them) and Bertrand-Edgeworth competition (in which sellers may have capacity constraints), struggle to explain this observation because in these models trade usually takes place, in a pure strategy equilibrium, at the minimum price posted in the market. The way in which price dispersion has tended to be explained is by sellers, in various contexts, resorting to the use of atomless mixed strategies (which often has the added advantage of resolving non-existence problems regarding pure strategy equilibrium).² However, the use of mixed strategies by sellers posting prices in the marketplace has always remained contentious and has been refuted in experiments.³ As Friedman (1988, p.608) remarked “it is doubtful that the decision-makers in firms shoot dice as an aid to selecting output or price”.

In this work, we are able to remedy this problem and explain price dispersion as the result of a pure strategy price equilibrium which results from the sellers having asymmetric information about the state of the market. The sellers are endowed with fixed quantities of a perfectly homogeneous good, and in equilibrium, the market demand may be rationed between the sellers. This work also contributes to the literature on the

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¹A large empirical study of this phenomenon, looking at internet prices, is Baye et al. (2004).
existence of pure strategy equilibria in Bertrand-Edgeworth competition when sellers hold incomplete and asymmetric information about the market. Whilst it is well-known that equilibria will only generally exist in mixed strategies even with complete information, a result commonly referred to as the Edgeworth paradox, we provide conditions that guarantee the existence of a pure strategy equilibrium that are straightforward to characterize, implement and interpret.

Starting with the classical Bertrand-Edgeworth duopoly, where capacity constrained sellers compete directly in prices, we introduce asymmetric information of the type usually studied in the context of general equilibrium models. The uncertainty that sellers face is modelled by an information partition over the possible states of the world. Sellers cannot distinguish between states contained in the same element of their information partition. The prices that sellers choose must be measurable with respect to their private information. Demand is distributed in proportion to the sellers’ endowments of the good when prices are tied, and efficient rationing occurs if different prices are posted in the market and the cheapest seller is unable to satisfy all of their forthcoming demand.

With complete information, which is a special case when all information partitions contain only singleton elements, pure strategy equilibria will generally not exist because it is profitable for a seller to deviate to a higher price and sell only to the residual demand that their competitor cannot satisfy. This non-existence problem is compounded when sellers have asymmetric information about the state of the market as we must specify how each seller evaluates ex ante uncertain profits. To address this, we consider ambiguity averse sellers with maximin expected utilities (MEU), following Gilboa & Schmeidler (1989). Using this ex ante decision rule, the sellers assign probabilities to the lowest possible ex post profits they know could be realized.

Under incomplete information, we show that a pure strategy equilibrium exists at the minimum competitive price in each element of the sellers’ information partitions, providing that market demand is sufficiently elastic across all market states. The intuition

\footnote{Glycopantis and Yannelis (2005) contains many papers which analyze asymmetric information of the type which we introduce in the Bertrand-Edgeworth game.}
follows that with complete information, each seller would choose the competitive price in each state and supply their full capacity. Upward price deviations reduce profit due to the sensitivity of demand and downward price deviations reduce profit because the firm already sells their full capacity. Sales are therefore independent of the market state and only price varies. With incomplete information, sellers post the lowest competitive price from all possible future states. If a firm deviated above the lowest competitive price and the worst state prevails, profit would be lower than at the competitive price.

As noted earlier, equilibria in Bertrand-Edgeworth games can often be difficult to find and/or characterize. Our model has the advantage that the equilibrium is easy to construct and analyze. One simply has to find the competitive equilibrium prices in each state of the world, and then construct the sellers price strategies based upon the information partitions (see the example in section 2.4). This work substantially generalises previous results based on complete information. In particular, a modified version of the well-known observation that a pure strategy equilibrium can only exist where all firms charge the same competitive price, continues to pass through (Shubik, 1958). The main difference is that with multiple possible states and maximin utilities, the only pure strategy equilibrium that can exist involves each firm charging the lowest competitive price for any possible realisation of the market demand.

1.1 Related Literature

The literature on capacity-constrained price competition has primarily focused on an environment with complete information, where it is well-known that equilibria generally only exist in mixed strategies (Dasgupta & Maskin, 1986; Maskin, 1986). We outline existing remedies to the problem of equilibrium existence in pure strategies under complete information, before considering the small number of papers that study related, but distinct, types of incomplete information.

A similar work to ours is Tasnádi (1999), who provides conditions on the demand function that restore pure strategy equilibrium under complete information. Specifically,
the competitive price associated with each seller supplying their capacity is the unique equilibrium, providing that demand is elastic at this price. This ensures that upward price deviations, which generally destabilize the equilibrium, decrease revenue. Our model nests Tasnádi’s (1999) restrictions as a special case when firms hold complete information. However, we go further by permitting incomplete information and show that the resulting unique pure strategy equilibrium can involve price dispersion, rationing of demand and excess demand. Related to this, Madden (1996) also imposes uniform elasticity to show outcome equivalence between Cournot competition and Kreps and Scheinkman’s (1983) model where firms choose capacities prior to Bertrand competition, for a wide range of rationing rules.

Another remedy to the non-existence problem has been to impose further requirements on the properties of any potential deviation from the proposed equilibrium. Bade (2005) considers sellers with multiple objectives, such as profit and sales maximisation, and sellers have incomplete preferences that prevent evaluation of the trade-off between each objective. Incomplete preferences restore the competitive price as the unique equilibrium because upward price deviations, which would otherwise overturn the equilibrium, reduce sales and sellers cannot quantify the aggregate impact. In our framework, incomplete information amongst sellers complicates the existence of pure strategy equilibrium, rather than restoring it. Iskakov et al (2018) consider sellers that are cautious, in the sense that any profitable deviation must also not induce a counter-deviation by another player that would leave the initial deviator worse off than their original position. Their solution of equilibrium in stable strategies is similar to the Von-Neumann-Morgenstern stable strategies used in cooperative game theory.

Pure strategy equilibrium can also be restored by allowing firms to post prices sequentially (Deneckere & Kovenock, 1992; Shubik & Levitan, 1980) or by considering a dynamic game where each consumer visits the market in a different period and firms may price discriminate between periods (Dudey, 1992; Deneckere & Peck, 2012). Pure strategy equilibrium also returns if sellers simultaneously choose a list price before si-
multaneously deciding whether to offer a discounted price to all consumers (García Díaz et al, 2009). The seller with the lowest list price signals to the rival that it can behave as a monopolist for the demand that he cannot meet. Other remedies include imposing a cost on firms that cannot satisfy all of their demand (Dixon, 1992), confining sellers to integer pricing (Dixon, 1993; Chowdhury, 2008), permitting firms to choose both price and output where actual output is the minimum of their chosen output and their demand (Dixon, 1992) and introducing a public social-surplus maximising seller (Balogh & Tasnádi, 2012; 2016).

Finally, several authors study the asymptotic properties of mixed strategy equilibria as the number of sellers increases. If demand is rationed proportionally, the mixed strategy converges, in a probabilistic sense, to the competitive price but the monopoly price remains in the support of the distribution (Allen & Hellwig, 1986). If demand is rationed efficiently, the highest price in the distribution also decreases with the number of sellers (Vives, 1986). In contrast with each of these approaches, we restore equilibrium existence without modifying the timing structure, imposing additional costs, requiring integer pricing or revising the firms’ objectives.

A small literature analyses equilibrium existence with demand uncertainty. Dana (1999) considers a competitive market with fixed capacities, and sellers only know the probability of each demand state, with symmetric information. The unique pure strategy equilibrium involves non-random intra-firm price dispersion, where sellers specify the amount of output available at each price. In particular, the price of each specific unit is given by the marginal cost divided by the probability that specific unit sells.

In our model, however, we do not require that firms can attach probabilities to demand states and we allow asymmetric information across sellers. This leads to inter-firm price dispersion, where consumers buy from different firms that charge different prices, instead of intra-firm dispersion identified by Dana (1999), where consumers purchase from the same firm at different prices. Therefore, Dana’s (1999) model explains why a firm assigns different prices to different units of the same product but our model
explains why prices differ between firms for the same product. The two frameworks provide complementary but distinct explanations for non-random price dispersion.

It is also interesting that our equilibrium under incomplete information features excess demand whenever the lowest demand state in a firm’s information partition does not occur. This is consistent with the idea that incomplete information amongst firms can result in unsatisfied demand in the market. This outcome also differs from Dana (1999) where excess demand is eliminated by the price-quantity schedules. In both Dana (1999) and our model, price dispersion also disappears when the lowest demand state arises, but the explanation is different. In our case, all firms supply their capacity at the lowest price but in Dana (1999), all firms only sell their lowest priced units. Furthermore, price dispersion is guaranteed if a higher demand state arises in Dana’s (1998) model but the existence of price dispersion depends on the information structures of sellers in our framework.

Other papers consider demand uncertainty when firms choose capacities before competing in prices, following Kreps and Scheinkman (1983). Hviid (1990) considers a linear demand function where the intercept is a random variable with uniform distribution, which is only realised once capacities and prices are set. There exists no pure strategy capacity choice that induces a single-price equilibrium unless capacity and pricing stages are fully sequential. 5 With simultaneous capacity choices, no pure strategy equilibrium exists even if demand is realised before the pricing stage (Hviid, 1991). Reynolds and Wilson (2000) and Lepore (2012) also consider demand uncertainty only at the capacity stage. Gabszewicz and Poddar (1997) replace capacity-price competition with capacity-output competition, where output is bounded by the preceding capacity investments.

de Frutos and Fabra (2011) consider a capacity-price duopoly where the amount of consumers, each with a common reservation price, is a random variable until capacities are chosen. There exists no pure strategy price equilibrium unless demand exceeds

5The exogenously appointed first-mover chooses high capacity and forces the second-mover to choose low capacity. In return, the larger seller chooses his price first and permits the smaller seller to sell their capacity first, at the same price.
total capacity or the capacity of the smallest firm exceeds demand. Non-existence also persists when demand remains uncertain at the pricing stage, unless the capacity of the smallest firm can satisfy the highest possible level of demand, which will never arise from the capacity stage. Sun (2017) also shows that the non-existence of pure strategy price equilibrium returns in Dudey’s (1992) dynamic Bertrand-Edgeworth game, if the number of buyers in each period becomes uncertain. We depart from this literature by fixing the choice of (possibly asymmetric) capacities and focusing on the requirements for a pure strategy price equilibrium under a more general type of demand uncertainty.

2 The Bertrand-Edgeworth Game

The model consists of a finite set of sellers \( N = \{1, 2\} \), who are producing a single perfectly homogeneous good. The uncertainty will be modelled by a finite set \( \Omega = \{\omega_1, ..., \omega_m\} \) which is the set of possible states of the world. There is a probability distribution, \( \mu \), over the set \( \Omega \) which describes the probability of each state occurring. It shall be assumed that \( \mu(\omega) > 0 \) for every \( \omega \in \Omega \) so no state of the world is redundant. Each seller is endowed with a fixed quantity \( q_i > 0 \) of the good.\(^6\) The total quantity of the good which can be traded in the market is \( q_1 + q_2 \). There is a state-contingent market demand function for the homogeneous good given by \( D : \mathbb{R}_{++} \times \Omega \to \mathbb{R}_+ \). The following conditions are imposed upon the demand function.

Assumption 2.1 For every \( \omega \in \Omega \) and every \( x \in (0, \infty) \), \( D(x, \omega) > 0 \). The function \( D(\cdot, \omega) \) is \( C^1 \) and \( D'(x, \omega) < 0 \) for every \( x \in (0, \infty) \).

The private information of seller \( i \) is modelled by a partition, \( P_i \), of the set \( \Omega \). Whenever two states of the world are in the same element of the partition \( P_i \), it means that seller \( i \) is unable to distinguish between those two states. A function \( f : \Omega \to \mathbb{R}_+ \) will be called \( P_i \)-measurable if, whenever \( \omega_p \in E \) and \( \omega_q \in E \) for some \( E \in P_i \), then

\(^6\)We are assuming that each seller has zero marginal cost to supply the good. However, one could easily add a constant marginal cost of production and this would make no difference to the results.
\[ f(\omega_p) = f(\omega_q). \] Facing these information restrictions, the strategy set of seller \( i \) in the game is

\[ L_i = \{ f : \Omega \to \mathbb{R}^+ : f \text{ is } P_i \text{ - measurable} \}. \]

Let \( L = \times_{i \in N} L_i \) be the joint strategy set. The primitives of a **Bertrand-Edgeworth game with asymmetric information** game can be summarized as \( G = \{ N, \Omega, (P_i, q_i)_{i \in N}, D, \mu \} \). The price elasticity of the market demand in state \( \omega \in \Omega \) is:

\[ \epsilon(x, \omega) = D'(x, \omega) \frac{x}{D(x, \omega)}. \]

The market demand will be called **uniformly elastic** if \( \epsilon(x, \omega) \leq -1 \) for every \( x \in (0, \infty) \) and every \( \omega \in \Omega \). Let \( R(x, \omega) = xD(x, \omega) \) so \( R(x, \omega) \) is the total revenue available in the market at price \( x \) in state \( \omega \in \Omega \).

### 2.1 The Ex Post Payoffs

After fixing a set of strategies \( f \in L \), to specify the payoffs which a seller receives ex post, a rationing rule is required because a seller may not set the lowest price, but the other seller may not be able to serve all the market demand. We consider the most widely used rationing rule in the literature: efficient, or “surplus-maximizing”, rationing which is consistent with those buyers with the highest valuation of the good being served first.\(^7\) Under this rule, the demand which the higher-priced seller faces is a horizontal displacement of the market demand. If the sellers tie at the same price, we shall make the standard assumption that they split the market demand in proportion to the quantities of the good they are endowed with.

Given a set of strategies \( f \in L \), let \( D_j = \min\{ D(f_j(\omega), \omega), q_j \} \). The demand which

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\(^7\)An alternative interpretation of the efficient rationing rule is that buyers are served randomly at first, but are then able to retrade the good amongst themselves, which then results in the same allocation of the good. See, amongst others, Baye and Kovenock (2008) and Vives (1999, pp.124-5).
seller $i$ faces under efficient rationing is:

$$D_i^E(f, \omega) = \begin{cases} \max\{0, D(f_i(\omega), \omega) - D_j(\omega)\}, & \text{if } f_i(\omega) > f_j(\omega); \\
\frac{q_i}{q_i + q_2} D(f_i(\omega), \omega), & \text{if } f_i(\omega) = f_j(\omega); \\
D(f_i(\omega), \omega) & \text{if } f_i(\omega) < f_j(\omega). \end{cases}$$

As seller $i$ is only endowed with $q_i$ units of the good, the demand which seller $i$ actually meets in state $\omega \in \Omega$ is given by $D_i^A(f, \omega) = \min\{q_i, D_i^E(f, \omega)\}$. Therefore, the ex post payoff of seller $i$ in state $\omega \in \Omega$ is:

$$u_i(f, \omega) = f_i(\omega) D_i^A(f, \omega).$$

### 2.2 The Ex Ante Payoffs

Before each seller has received the information regarding which element in $P_i$ the state of the world is in, how should the sellers evaluate their expected payoff? Given we are assuming that sellers cannot distinguish between different states of the world contained in the same element in $P_i$, it is not unreasonable to assume that sellers cannot assign probabilities to those states. In this context, it is not possible for sellers to calculate standard Bayesian expected utilities. We consider a well-known alternative to Bayesian expected utilities: maximin expected utilities (MEU). If a seller knows that the state of the world is contained in $E \in P_i$, we consider the case where the seller is pessimistic and assigns all the probability associated with event $E$, which is $\mu(E)$, to the minimum ex post payoff in $E$. Let $X$ be the set of probability distributions over $\Omega$:

$$X = \{x \in \mathbb{R}^\Omega : x(\omega) \geq 0 \text{ for every } \omega \in \Omega \text{ and } \sum_{\omega \in \Omega} x(\omega) = 1\}$$

and let $M_i$ be the set of probability distributions which agree with seller $i$’s private information:

$$M_i = \{x \in X : x(E) = \mu(E) \text{ for every } E \in P_i\}.$$

Given a set of strategies $f \in L$ the ex ante payoff of seller $i$ is:

$$U_i(f) = \min_{x \in M_i} \left[ \sum_{\omega \in \Omega} x(\omega) u_i(f, \omega) \right].$$
An alternative, but equivalent expression, is:

\[ U_i(f) = \sum_{E \in P_i} \mu(E) [\min_{\omega \in E} u_i(f, \omega)]. \]

**Remark 2.1** The most prominent early application of maximin expected utilities was in Gilboa and Schmeidler (1989) who characterized this type of decision rule and noted that it can explain the Ellsberg (1961) violations of subjective expected utility theory. Recently, maximin expected utilities have been used in a wide range of papers, including Correia-da-Silva and Hervés-Beloso (2009), He and Yannelis (2015) and de Castro and Yannelis (2018).

**Remark 2.2** This model of a Bertrand-Edgeworth game with asymmetric information contains, as a special case, the standard complete information game. If one specifies the information partitions of the sellers to be \( P_i = \{\{\omega_1\}, \{\omega_2\}, ..., \{\omega_m\}\} \) for every \( i \in N \) then each seller can distinguish every state of the world and the model is a complete information game. Moreover, the calculation of maximin expected utilities then coincides with standard Bayesian utilities.

### 2.3 Existence of Pure Strategy Equilibrium and the Law of One Price

Now that the ex ante and ex post payoffs have been defined, we can introduce the equilibrium concept. A set of strategies \( f \in L \) is a **pure strategy price equilibrium** if, for every \( i \in N \),

\[ U_i(f) \geq U_i(f'_i, f_{-i}) \text{ for every } f'_i \in L_i. \]

We shall say that a pure strategy price equilibrium, \( f \in L \), **violates the law of one price** if \( f_1(\omega) \neq f_2(\omega) \) for some \( \omega \in \Omega \). That is to say, a pure strategy price equilibrium violates the law of one price if there is at least one state of the world when the sellers post different prices in the market. The first result gives some useful properties of the market demand function.

**Proposition 2.1** Fix a Bertrand-Edgeworth game with asymmetric information \( G = \{N, \Omega, (P_i, q_i)_{i \in N}, D, \mu\} \). If \( D(x, \omega) \) is uniformly elastic then the following are true:

(i) \( R'(x, \omega) \leq 0 \) for every \( x \in (0, \infty) \).
(ii) \( \lim_{x \to 0} D(x, \omega) = \infty \) and \( \lim_{x \to \infty} D(x, \omega) = 0 \).

(iii) For each \( \omega \in \Omega \) there exist unique prices \( p^*(\omega) \) such that \( D(p^*(\omega), \omega) = q_1 + q_2 \).

**Proof.** (i) From the definition \( R(x, \omega) = xD(x, \omega) \), therefore:

\[
R'(x, \omega) = D(x, \omega) + xD'(x, \omega) = D(x, \omega)(1 + \epsilon(x, \omega)).
\]

As \( \epsilon(x, \omega) \leq -1 \) for every \( x \in (0, \infty) \), \( R'(x, \omega) \leq 0 \) for every \( x \in (0, \infty) \).

(ii) Suppose a contradiction: that \( \lim_{x \to 0} D(x, \omega) = y > 0 \). Then \( \lim_{x \to 0} xD(x, \omega) = 0 \). As \( R'(x, \omega) \leq 0 \), this implies \( R(x, \omega) \leq 0 \) for every \( x \in (0, \infty) \) and contradicts \( xD(x, \omega) > 0 \) for every \( x \in (0, \infty) \). Hence, \( \lim_{x \to 0} D(x, \omega) = 0 \). Suppose a contradiction:

\( \lim_{x \to \infty} D(x, \omega) = y > 0 \). Then \( \lim_{x \to \infty} xD(x, \omega) = \infty \) and contradicts \( R'(x, \omega) \leq 0 \) for every \( x \in (0, \infty) \). Hence, \( \lim_{x \to \infty} D(x, \omega) = 0 \).

(iii) It follows from (ii) that the range of \( D(\cdot, \omega) \) is \((0, \infty)\). Therefore, for each \( \omega \in \Omega \) there exists a \( p^*(\omega) \) such that \( D(p^*(\omega), \omega) = q_1 + q_2 \). The uniqueness of such prices follows from \( D(\cdot, \omega) \) being decreasing on \((0, \infty)\). \( \blacksquare \)

Using the \( p^*(\omega) \) prices defined in part (iii) of the previous result, define the strategies of the sellers to be as follows. For each \( E \in P_i \) let:

\[
f_i^*(E) = \min_{\omega \in E} p^*(\omega).
\]

By construction these strategies are \( P_i \)-measurable, so \( f^* \in L \). The following result gives some of the properties of these strategies.

**Proposition 2.2** Fix a Bertrand-Edgeworth game with asymmetric information

\( G = \{N, \Omega, (P_i, q_i)_{i \in N}, D, \mu\} \). Suppose the demand \( D(x, \omega) \) is uniformly elastic and the sellers play the strategies \( f^* \in L \). Then:

(i) \( D_i^A(f^*, \omega) = q_i \) for every \( \omega \in \Omega \).

(ii) \( U_i(f^*) = \sum_{E \in P_i} \mu(E)f_i^*(E)q_i \).

**Proof.** (i) If the sellers use strategies \( f^* \in L \) then \( f_i^*(\omega) \leq p^*(\omega) \) for every \( \omega \in \Omega \). Therefore \( D(f_i^*(\omega), \omega) \geq q_1 + q_2 \), \( D_i^E(f^*, \omega) \geq q_i \), and consequently, \( D_i^A(f^*, \omega) = q_i \) for every \( \omega \in \Omega \).
The next result demonstrates that the strategies $f^* \in L$ are a pure strategy price equilibrium of the Bertrand-Edgeworth game.

**Proposition 2.3** Fix a Bertrand-Edgeworth game with asymmetric information $G = \{N, \Omega, (P_i, q_i)_{i \in N}, D, \mu\}$. If $D(x, \omega)$ is uniformly elastic, then the strategies $f^* \in L$ are a pure strategy price equilibrium.

**Proof.** Suppose the sellers play the strategies $f^* \in L$. It follows from part (i) of Proposition 2.2 that using these strategies each seller is able to sell all their quantity of the good they are endowed with.

If for some $E \in P_i$ seller $i$ were to deviate and play $f_i(E) < f_i^*(E)$, then $u_i(f_i, f_j^*, \omega) = f_i(E)q_i < f_i^*(E)q_i = u_i(f^*, \omega)$ for every $\omega \in E$. This is not a profitable deviation.

Suppose for some $E \in P_i$ seller $i$ were to deviate and play $f_i(E) > f_i^*(E)$. From part (i) of Proposition 2.2 we know that using strategies $f^* \in L$ seller $i$ obtains the same payoff $f^*(E)q_i$ across all states in $E$. To show that deviating to $f_i(E) > f_i^*(E)$ is not a profitable deviation, given the maximin ex ante utilities, we only have to find one state in $E$ where the payoff does not increase above $f^*(E)q_i$.

Consider the state $\omega_E = \{\omega \in E : p^*(\omega) \leq p^*(\omega') \quad \forall \quad \omega' \in E\}$. In state $\omega_E$, using strategies $f^*$, seller $i$ either ties at price $p^*(\omega_E)$ with seller $j$, or seller $j$ posts a strictly lower price. Hence:

$$u_i(f^*, \omega_E) = p^*(\omega_E)q_i = p^*(\omega_E)(D(p^*(\omega_E), \omega_E) - q_j).$$

where the second equality follows from $D(p^*(\omega_E), \omega_E) = q_1 + q_2$. To show that deviating to $f_i(E) > f_i^*(E)$ is not a profitable deviation, we need to demonstrate that the function $g(x) = x(D(x, \omega_E) - q_j)$ is decreasing in $x$. The derivative is:

$$g'(x) = D(x, \omega_E) - q_j +xD'(x, \omega_E) = D(x, \omega_E)(1 - q_j/D(x, \omega_E) + \epsilon(x, \omega_E)).$$

As $\epsilon(x, \omega_E) \leq -1$ for every $x \in (0, \infty)$ it follows that $g'(x) \leq 0$. Therefore, deviating to $f_i(E) > f_i^*(E)$ is not a profitable deviation in state $\omega_E$. ■
Remark 2.3 One might reasonably ask whether the uniform elasticity of the market demand can be relaxed and still guarantee the existence of a pure strategy price equilibrium. As is well-known, Nash equilibria in Bertrand-Edgeworth games often only exist in mixed strategies. However, in the current model, we can be more precise. Suppose there is a market demand $D^*(x, \omega)$ and $\epsilon(x, \omega) \in (0, -1)$ for every $x \in (0, \infty)$ so demand is always inelastic in one state of the world. Then, it is possible to find a Bertrand-Edgeworth game with asymmetric information $G = \{N, \Omega, (P_i, q_i)_{i \in N}, D, \mu\}$ with $D = D^*$, such that a pure strategy price equilibrium fails to exist. This results follows directly from Remark 2.2, that the current model includes the complete information game as a special case, and Proposition 2.3 of Tasnádi (1999). Therefore, it does not seem possible to significantly weaken the condition of uniform elasticity and still guarantee the existence of a pure strategy equilibrium.

We are now able to present our main result which gives precise conditions under which a Bertrand-Edgeworth game with asymmetric information possesses a pure strategy price equilibrium which violates the law of one price.

Proposition 2.4 Fix a Bertrand-Edgeworth game with asymmetric information $G = \{N, \Omega, (P_i, q_i)_{i \in N}, D, \mu\}$. Suppose the following three conditions are satisfied:

(i) The demand $D(x, \omega)$ is uniformly elastic.

(ii) $p^*(\omega) \neq p^*(\omega')$ whenever $\omega \neq \omega'$.

(iii) $P_1 \neq P_2$.

Then the game possesses a pure strategy price equilibrium which violates the law of one price.

Proof. Let the sellers play the strategies $f^* = (f^*_1, f^*_2) \in L$. It follows from Proposition 2.3 that these are a pure strategy price equilibrium of the game. Suppose a contradiction: $f^*_1(\omega) = f^*_2(\omega)$ for every $\omega \in \Omega$. As $f^*_1 \in L_1$ and $f^*_2 \in L_2$, $f^*_1(\omega) = f^*_2(\omega)$ for every $\omega \in \Omega$, together with $p^*(\omega) \neq p^*(\omega')$ whenever $\omega \neq \omega'$ imply $P_1 = P_2$. This contradicts $P_1 \neq P_2$. Hence, there must be at least one $\omega \in \Omega$ such that $f^*_1(\omega) \neq f^*_2(\omega)$.

The conditions (i)-(iii) in Proposition 2.4 are tight in the following sense. If one were to dispense with (i), but retain (ii) and (iii), it follows from Remark 2.3 that a game can be found which fails to possess a pure strategy price equilibrium. If one retains condition
(i), and dispenses with either (ii) or (iii), then a game can be found in which the pure strategy price equilibrium defined by the \( p^*(\omega) \) prices does not violate the law of one price. Therefore, all three conditions in Proposition 2.4 are required to ensure that a pure strategy price equilibrium exists, defined by the \( p^*(\omega) \) prices, which violates the law of one price.

2.4 An Illustrative Example

Consider a market in which there are three states of the world, \( \Omega = \{\omega_1, \omega_2, \omega_3\} \), and the prior is \( \mu(\omega_1) = \mu(\omega_2) = \mu(\omega_3) = 1/3 \). The market demands in the three states are \( D(x, \omega_1) = x^{-1} \), \( D(x, \omega_2) = x^{-1/2} \) and \( D(x, \omega_3) = x^{-1/3} \). The quantities of the good the sellers are endowed with are \( q_1 = 1 \) and \( q_2 = 2 \). The information partitions of the two sellers are \( P_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\} \) and \( P_2 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\} \). Given these market primitives, the \( p^* \) prices are \( p^*(\omega_1) = \frac{1}{3} \), \( p^*(\omega_2) = \frac{1}{9} \) and \( p^*(\omega_3) = \frac{1}{27} \). It follows from Proposition 2.3 that the strategies:

\[
\begin{align*}
    f_1^*(\{\omega_1, \omega_2\}) &= \frac{1}{9} \quad \text{and} \quad f_1^*(\omega_3) = \frac{1}{27} \\
    f_2^*(\omega_1) &= \frac{1}{3} \quad \text{and} \quad f_2^*(\{\omega_2, \omega_3\}) = \frac{1}{27}.
\end{align*}
\]

are a pure strategy price equilibrium. The ex ante expected utilities of the sellers at the equilibrium are \( U_1(f^*) = \frac{7}{81} \) and \( U_2(f^*) = \frac{22}{81} \).

This example violates the law of one price because \( f_1^*(\omega_1) \neq f_2^*(\omega_1) \) and \( f_1^*(\omega_2) \neq f_2^*(\omega_2) \). As \( \mu(\{\omega_1, \omega_2\}) = 2/3 \) the sellers post different prices in the market with ex ante probability 2/3.

3 Concluding Remarks

The purpose of this paper has been to provide a theoretical foundation for the commonly observed phenomenon of perfectly homogeneous goods trading at different prices within the same market, without resorting to the usual, but contentious, device of sellers using
mixed strategies. To this end, we have introduced a capacity-constrained price game which permits asymmetries of information of the type usually studied in the context of general equilibrium models. The information of the sellers has been modelled by a partition of a state space, and consequently, prices posted in the market must be measurable with respect to the private information of each seller. The main result has been to demonstrate that if the market demand is uniformly elastic, the competitive equilibrium prices differ in each state of the world, and if the sellers have different information partitions, then a pure strategy price equilibrium exists which violates the law of one price. Furthermore, given the rationing rule, even the seller posting a higher price may make positive sales in equilibrium. To conclude, the following two extensions of the model are interesting areas for further research.

- Throughout the paper we have assumed that the uncertainty only affected the market demand. A richer model could permit the possibility that the endowments of the good the sellers bring to the market are also state dependent. In this case, each seller’s endowment would be a function $q_i : \Omega \to \mathbb{R}_{++}$. One could impose the condition that the function $q_i$ is measurable with respect to each seller’s private information so that no seller could infer more about the state of the market by observing their endowment. In this richer model, it is an open question whether the strategies defined by the $p^*(\omega)$ prices still constitute a pure strategy price equilibrium, and whether conditions, such as those in Proposition 2.4, which determine when the law of one price is violated could be found.

- The advantage of the models of Bertrand competition and Bertrand-Edgeworth competition is that they provide a direct foundation for prices in the marketplace without resorting to the fiction of the Walrasian auctioneer. The model we have analyzed assumed constant zero marginal costs. Most of the literature indicates that with more general convex costs an equilibrium only exists in mixed strategies, and is often difficult to characterize. However, Dixon (1992) noted that if sellers are permitted to specify both price and quantity pairs, provided all but one seller could supply the whole market
demand subject to a no bankruptcy constraint, then the competitive equilibrium of the market could be sustained as a pure strategy equilibrium. It would be interesting to see whether the framework of sellers posting both price and quantity pairs in the market could be extended to permit asymmetries of information of the type studied in this paper, and what the implications are for the law of one price.
References


