Ethical Poverty Lines: Existence and Implications*

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ABSTRACT

This paper provides a theoretical foundation for poverty lines. Social preference conditions are provided which are both necessary and sufficient for a poverty line to arise endogenously. Characterizations are provided for the existence of both relative and absolute poverty lines. In each case, one of the conditions is a familiar weak monotonicity property. The other conditions are simple consistency requirements.

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1 Introduction

Many of the important debates on poverty in recent times have revolved around estimating the appropriate poverty line, which is crucial for identifying the poor (Anand et al. 2010, Chen and Ravallion 2010, Reddy and Pogge 2009, Deaton 2004). The calibration of the poverty line, starting with the work of Rowntree (1901) has mainly been seen as a statistical exercise of estimating the income required for a given standard of living (Ravallion 1998, 2012). Despite the importance and long history of the identification method, research on the conceptual underpinnings of the poverty line has, remarkably, been very limited, particularly compared to that on aggregation methods for poverty measurement (Foster et al. 2010, Zheng 1997). The purpose of this paper is to provide welfare-theoretic foundations for the existence of poverty lines.

Following a seminal paper by Sen (1976), 'identification' of who is poor has been regarded as a prerequisite for estimating poverty. The axiomatic literature on poverty measurement thus treats the poverty line as a given parameter. Indeed, most of the standard axioms which characterize poverty measures presume the existence of a poverty line, calibrated via an empirical exercise. The sensitivity of poverty estimates to the level of the poverty line has long been recognised in the literature on poverty orderings (Foster and Shorrocks 1988a and 1988b, Atkinson 1987). Recently, there has been a growing recognition of a need for a better understanding of the role of poverty lines in poverty measurement (Decerf 2013, Foster et al. 2010).[1]

Our paper contributes to this line of inquiry by providing a systematic theoretical foundation which embeds the concept of a poverty line within a broader social welfare framework. We assume a (social) preference ordering over all possible income distributions and impose a set of intuitive ethical principles on the preference ordering. This gives rise to a poverty line which embodies the essential intuition of the 'focus' axiom, i.e. it is that income cut-off above which any changes to income do not affect social welfare.[2] The exact value of the poverty line will be very much dependent on the prevailing social preferences; it is in this sense that we describe the poverty line discussed here as being 'ethical.'

Our framework is perhaps most closely related to the literature on ethical poverty indices, which discusses the notion of poverty within a social welfare setting (e.g. Hagenaars 1987, Chakravarty 1983, Clark et al. 1981, Blackorby and Donaldson 1980).[3] In these studies, the poverty line is assumed to be exogenously determined. This paper is, to the best of our knowledge, the first to extend such a framework by providing the necessary and sufficient social preference conditions for the existence of

[1] The identification methodology has also been the focus on the recent literature on multidimensional deprivation (see Alkire and Foster, 2011).

[2] The 'focus' axiom in poverty measurement states that poverty is unaffected by changes in incomes of those above the poverty line (Foster 1984).

[3] The emphasis in this strand of literature is on the link between poverty measures and social welfare functions.

a poverty line itself.

The rest of the paper is organized as follows. Section 2 introduces our framework for studying social preferences with regard to poverty and shows how certain preferences can imply the existence of a poverty line. Using the terms in the sense of Foster (1998), we discuss both ‘absolute’ and ‘relative’ poverty lines. Thus, an absolute poverty line is a fixed income cut-off level that is constant across all potential income distributions; a relative poverty line is dependent on the specific distribution. Under concluding remarks in Section 3, we discuss the implications of such poverty lines. All proofs are deferred to the Appendix.

2 Deriving Poverty Lines from Social Preferences

2.1 Notation and Basic Framework

We consider a society of $n \geq 2$ individuals. A profile $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$ represents the distribution of incomes within the society. Although our results below apply, with minor modifications, to profiles within $\mathbb{R}^n$, we restrict incomes to be non-negative, as is usually done in the literature.

A social preference relation, denoted by the symbol $\succcurlyeq$, on $\mathbb{R}^n_+$ is assumed. As usual, $\succcurlyeq$ means weak preference, $\succ$ is strict preference and $\sim$ is indifference. Sometimes the reversed preference symbols are used: $\preccurlyeq$ and $\prec$. The preference $\succcurlyeq$ on $\mathbb{R}^n_+$ is *complete* (i.e., $\mathbf{x} \succcurlyeq \mathbf{y}$ or $\mathbf{x} \preccurlyeq \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n_+$).[4]

For a permutation ρ of the indices in $\{1, \ldots, n\}$, such that $\rho_i := \rho(i)$, we write $\rho(\mathbf{x})$ for the profile $\mathbf{z}$ with $z_i = x_{\rho_i}$ for $i = 1, \ldots, n$. We assume *symmetry* (or *anonymity*): $\mathbf{x} \sim \rho(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n_+$. For simplicity, henceforth, whenever we write $\mathbf{x} \in \mathbb{R}^n_+$ we implicitly assume that $x_1 \leq \cdots \leq x_n$. By symmetry this is not a restriction. Further, we write $y_i\mathbf{x}$ for the profile $\mathbf{x}$ with x_i replaced by y; whenever this notation is used we implicitly assume that the ordering of incomes in a profile remains from lowest to highest, i.e., $x_{i-1} \leq y \leq x_{i+1}$ if $i \in \{2, \ldots, n-1\}$, $x_{i-1} \leq y$ if $i = n$, and $y \leq x_{i+1}$ if $i = 1$. For example, for $\varepsilon \geq 0$ we write $(x_i + \varepsilon)_i\mathbf{x}$, thereby implicitly requiring constraints on the admissible values of ε, such that $x_i + \varepsilon \leq x_{i+1}$ whenever $i \in \{1, \ldots, n-1\}$; in particular, $\varepsilon = 0$ if $x_i = x_{i+1}$. Whenever we use $(x_i + \varepsilon)_i\mathbf{x}$ with $\varepsilon > 0$ it is implicitly assumed that $x_i < x_{i+1}$ if $i \neq n$.

As we are concerned with the existence of poverty lines, we formally define the notion of a poverty line based on Foster (1998).

Our first definition is of an absolute poverty line, which remains constant and thus is the same for all possible income distributions.

Definition 1 *Income $z \in \mathbb{R}_+ \cup \{\infty\}$ is an absolute poverty line if for all $\mathbf{x} \in \mathbb{R}^n_+$ we have (i) for all $x_i < z$ and all $\delta > 0$, $(x_i + \delta)_i\mathbf{x} \succ \mathbf{x}$; (ii) for all $x_i \geq z$ and all $\delta \geq 0$, $(x_i + \delta)_i\mathbf{x} \sim \mathbf{x}$.*

[4] In our interpretation, completeness includes *reflexivity*, that is, $\mathbf{x} \sim \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n_+$.

In the definition of an absolute poverty line we allow for the extreme case in which the poverty line is not finite. Such cases may occur if one sees every possible income level as worth improving ($z = \infty$).

A poverty line might instead be relative, where it depends on the specific income distribution. We define a relative poverty line as follows.

Definition 2 *Given any profile* $\mathbf{x} \in \mathbb{R}_+^n$*, income* $z_{\mathbf{x}} \in \mathbb{R}_+$ *is a relative poverty line if, (i) for all* $x_i < z_{\mathbf{x}}$ *and all* $\delta > 0$*,* $(x_i + \delta)_i\mathbf{x} \succ \mathbf{x}$ *and (ii) for all* $x_i \geq z_{\mathbf{x}}$ *and all* $\delta \geq 0$*,* $(x_i + \delta)_i\mathbf{x} \sim \mathbf{x}$*.*

Our poverty line definitions are consistent with the normatively appealing 'focus' axiom – increases in incomes above the poverty line have no impact and do not lead to a socially preferred outcome.

2.2 Foundations for Poverty Lines

The following axioms impose constraints on the social preference relation $\succcurlyeq$. The first axiom imposes the mild requirement that, *ceteris paribus*, an increase in an individual's income cannot lead to a new distribution of income which is strictly less socially desirable than the original one. Or, put differently, additional income should not be harmful. Formally, in terms of the preference, this requirement comes down to the following axiom.

Axiom 1 *(Weak Monotonicity) The preference relation* $\succcurlyeq$ *on* $\mathbb{R}_+^n$ *satisfies weak monotonicity if for all* $v, w \in \mathbb{R}_+$*, all* $\mathbf{x} \in \mathbb{R}_+^n$ *and all* $i \in \{1, \ldots, n\}$*,* $v \geq w$ *implies* $v_i\mathbf{x} \succcurlyeq w_i\mathbf{x}$*.*

The second axiom is a simple consistency requirement. Consider a situation where an individual's income, say x_i, increases by a certain amount. Suppose that this increase leads to no improvement in the social ranking of the resulting income profile compared to the original one. The axiom says that if this is the case, then no social progress would have been made by increasing this individual's income by some other amount. Moreover, in any income distribution in which an individual has an income at least as high as x_i, an increase in that individual's income also brings no social improvement to that distribution. Conversely, suppose that some increase in an individual's income does lead to an improvement in the social ranking of the resulting income profile compared to the original one. Then increasing that individual's income by some different amount should also lead to some social improvement. Further, in any income distribution, increasing the income of a less well-off individual must also be socially beneficial.

Axiom 2 *(Global Consistency) The preference* $\succcurlyeq$ *on* $\mathbb{R}_+^n$ *satisfies strong consistency if for all profiles* $\mathbf{x} \in \mathbb{R}_+^n$*,* $\mathbf{y} \in \mathbb{R}_+^n$*, and all individuals* $i, j \in \{1, \ldots, n\}$*, we have:*

(i) $(x_i + \varepsilon)_i \mathbf{x} \sim \mathbf{x}$, *for* $\varepsilon > 0 \Rightarrow (y_j + \delta)_j \mathbf{y} \sim \mathbf{y}$, *for* $y_j \geq x_i$ *and* $\forall \delta \geq 0$*;*

(ii) $(x_i + \varepsilon)_i \mathbf{x} \succ \mathbf{x}$, *for* $\varepsilon > 0 \Rightarrow (y_j + \delta)_j \mathbf{y} \succ \mathbf{y}$, *for* $y_j \leq x_i$ *and* $\forall \delta > 0$*.*

Third, we propose a weaker version of our global consistency axiom. The main difference is that our comparisons are now restricted to the same income profile. Thus, if increasing an individual's income does not lead to a social improvement, then increasing the income of a more affluent individual in the *same profile* must also bring no social benefit. Similarly, if increasing an individual's income leads to a social improvement, then increasing the income of a less well-off individual in the *same profile* must also register a social improvement.

Axiom 3 *(Local Consistency) The preference* $\succcurlyeq$ *on* $\mathbb{R}^n_+$ *satisfies weak consistency if for all profiles* $\mathbf{x} \in \mathbb{R}^n_+$*, and all individuals* $i, j \in \{1, \ldots, n\}$*, we have:*

(i) $(x_i + \varepsilon)_i \mathbf{x} \sim \mathbf{x}$, *for* $\varepsilon > 0 \Rightarrow (x_j + \delta)_j \mathbf{x} \sim \mathbf{x}$, *for* $x_j \geq x_i$ *and* $\forall \delta \geq 0$*;*

(ii) $(x_i + \varepsilon)_i \mathbf{x} \succ \mathbf{x}$, *for* $\varepsilon > 0 \Rightarrow (x_j + \delta)_j \mathbf{x} \succ \mathbf{x}$, *for* $x_j \leq x_i$ *and* $\forall \delta > 0$*.*

Remark 1 *Global Consistency (Axiom 2)* $\Longrightarrow$ *Local Consistency (Axiom 3).*

We can now state and prove our results regarding the existence of poverty lines. Our first result concerns absolute poverty lines.

Theorem 1 *Suppose that* $\succcurlyeq$ *is a complete and symmetric preference relation on* $\mathbb{R}^n_+$*. Then the following statements are equivalent:*

(a) There exists an absolute poverty line $z \in \mathbb{R}^n_+ \cup \{\infty\}$*.*

(b) The preference relation satisfies weak monotonicity and global consistency.

If z *is finite, it is uniquely determined.*

Proof. See Appendix. ■

An analogous result for relative poverty lines is as follows.

Theorem 2 *Suppose that* $\succcurlyeq$ *is a complete and symmetric preference relation on* $\mathbb{R}^n_+$*. Then the following statements are equivalent:*

(a) *For any given profile* $\mathbf{x} \in \mathbb{R}_+^n$, *there exists a relative poverty line* $z_x \in \mathbb{R}_+$.

(b) *The preference relation satisfies weak monotonicity and local consistency.*

Proof. See Appendix. ■

It is clear that the absolute poverty line z in Theorem 1 has some properties in common with the relative poverty line $z_{\mathbf{x}}$ in Theorem 2. The key difference is that whereas the $z_{\mathbf{x}}$ in Theorem 2 is dependent on $\mathbf{x} \in \mathbb{R}_+^n$, and is in this sense relative, the income level z in Theorem 1 is independent of the distribution. The absolute poverty line requires a stronger (global) consistency condition than is needed for a relative poverty line. This is also the main reason that relative poverty lines are not necessarily unique, while absolute poverty lines, if finite, must always be.

It remains to be shown that the axioms of weak monotonicity and global (and local) consistency are independent. We again need only consider income profiles that are ranked from lowest to highest.

Example 1 *If the preference relation on* $\mathbb{R}_+^n$ *is represented by* $P(\mathbf{x}) = -\sum_{i=1}^n x_i$ *then global consistency is satisfied (this holds trivially, as there is no profile* $\mathbf{y} \in \mathbb{R}_+^n$ *and* $\varepsilon > 0$ *such that* $(y_i + \varepsilon)_i \mathbf{y} \sim \mathbf{y}$ *or* $(y_i + \varepsilon)_i \mathbf{y} \succ \mathbf{y}$*). Obviously, weak monotonicity does not hold.*

It can be inferred from Remark 1 that the above example also holds if global consistency is replaced by local consistency.

Example 2 *If the preference relation on* $\mathbb{R}_+^2$ *with* $x_1 \leq x_2$ *is represented by*

$$P(\mathbf{x}) = \begin{cases} 0 & \text{if } x_1 = 0, \\ x_1 + x_2 & \text{if } x_1, x_2 \in (0,1), \\ x_1 + 1 & \text{if } x_1 \in (0,1) \text{ and } x_2 \geq 1, \\ 2 + x_2 & \text{if } x_1 \geq 1. \end{cases}$$

then the preference satisfies weak monotonicity. For $\mathbf{x} = (1, 2)$ *local consistency does not hold.*

Since, in the above example, local consistency does not hold, from Remark 1 we can infer that global consistency will also be violated.

3 Concluding Remarks

Despite its ubiquity, there has until now been little attempt to embed the concept of the poverty line within a broader social welfare framework. This is surprising, especially given the volume of literature on the social preference axioms which characterize poverty indices. By demonstrating that poverty lines can arise naturally from existing social preferences over income distributions, our results add to the appeal of the popular 'focus' axiom. Moreover, it can easily be inferred that, with the notable exception of the headcount ratio, all the well-known poverty measures from the literature are consistent with our preference axioms.[5]

The existence of a poverty line does not necessarily imply that it lies within the range of actual incomes in society; it doesn't rule out the possibility that all individuals are poor or that all individuals are non-poor. The existence of absolute poverty lines at $+\infty$ and 0 requires that in all possible income profiles, 'all individuals are poor' and 'no individuals are poor,' respectively. If there is a poverty line at $+\infty$, this implies that society always attaches some importance to an increase in any individual's income, regardless of the income profile. Such preferences do not necessarily embody a concern for poverty. If there is a poverty line at 0, society is always indifferent to any increase in any individual's income, regardless of the income profile. This seems a most unrealistic scenario; such a society does not care at all about either the level or the distribution of income. For a finite absolute poverty line to exist, the preference relation $\succcurlyeq$ must lie somewhere between these two extremes.

Unlike the absolute poverty line, the relative poverty line is not unique. It is non-unique in two respects. Firstly, there is a relative poverty line associated with each income distribution. Thus the relative poverty line may be different for different distributions. Secondly, within a given distribution there can be different relative poverty lines. Consider an income distribution, $\mathbf{x}$, whose bottom four incomes are $(5, 10, 15, 20, ...)$. Suppose we find that a small increase in the second lowest income of $\mathbf{x}$, leads to a distribution which is socially preferred to the original income distribution, while a small increase in the third lowest income leads to a distribution which is socially indifferent to the original distribution. In that case, the relative poverty line $z_{\mathbf{x}}$ can be any income such that $10 < z_{\mathbf{x}} \leq 15$. Thus the relative poverty line for a given income distribution is non-unique. It can be inferred from the proof of Theorem 2 that every relative poverty line for a given income distribution will identify the same individuals as poor or non-poor.

In this paper, we propose some fairly mild normative criteria and demonstrate that when social preferences satisfy these conditions, a poverty line must exist. While we provide welfare-theoretic foundations for poverty lines, we do not derive any specific numerical values for them. Thus the salience of the statistical approach to calibrating poverty lines for policy purposes remains. How to best to extend the welfare-theoretic framework presented in this paper to derive specific poverty lines is left for future research.

[5] The headcount measure is not sensitive to the size of the income shortfall of those below the poverty line - only the fact that an income is below the poverty line. It is therefore quite common for the headcount measure to indicate that a small increase in an income would not lead to any improvement, while a larger increase would do. This lack of sensitivity is widely regarded as a serious drawback of the headcount measure and causes it to fail to satisfy our consistency axioms.

4 Appendix

4.1 Proof of Theorem 1

First, we assume (a) and derive (b). Let $\mathbf{x} \in \mathbb{R}^n_+$ be an arbitrary income profile, $i \in \{1, \ldots, n\}$ an arbitrary individual, and $\varepsilon \geq 0$. Suppose $\varepsilon = 0$. Then completeness (or reflexivity) implies $(x_i + \varepsilon)_i\mathbf{x} \succcurlyeq \mathbf{x}$. Suppose now that $\varepsilon > 0$ and that $x_i < x_i + \varepsilon < x_{i+1}$. If $x_i < z$ then $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$ follows from the definition of an absolute poverty line (Definition 1), hence, $(x_i+\varepsilon)_i\mathbf{x} \succcurlyeq \mathbf{x}$ is concluded. If $x_i \geq z$ then $(x_i+\varepsilon)_i\mathbf{x} \sim \mathbf{x}$ follows from the definition of an absolute poverty line, hence, $(x_i+\varepsilon)_i\mathbf{x} \succcurlyeq \mathbf{x}$ is concluded. Together these three cases imply that $(x_i+\varepsilon)_i\mathbf{x} \succcurlyeq \mathbf{x}$. As x, i and ε were chosen arbitrarily, weak monotonicity follows.

Now we show that global consistency holds. Let $\mathbf{x} \in \mathbb{R}^n_+$ be an arbitrary income profile, $i \in \{1, \ldots, n\}$ be an arbitrary individual, and $\varepsilon > 0$. Suppose $(x_i + \varepsilon)_i\mathbf{x} \sim \mathbf{x}$. Then, by the definition of z it must be that $x_i \geq z$ otherwise $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$ would hold. Hence, for all $\mathbf{y} \in \mathbb{R}^n_+$ and $j \in \{1, \ldots, n\}$ with $y_j \geq x_i$ it follows that $y_j \geq z$. By the definition of an absolute poverty line it follows that $(y_j + \delta)_j\mathbf{y} \sim \mathbf{y}$ for all $\delta \geq 0$. Suppose now that $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$. Then, by the definition of z it must be that $x_i < z$ otherwise $(x_i + \varepsilon)_i\mathbf{x} \sim \mathbf{x}$ would hold. Hence, for all $\mathbf{y} \in \mathbb{R}^n_+$ and all $j \in \{1, \ldots, n\}$ with $y_j < x_i$ it follows that $y_j < z$. By the definition of an absolute poverty line it follows that $(y_j + \delta)_j\mathbf{y} \succ \mathbf{y}$ for all $\delta > 0$. Therefore, global consistency holds. Thus, statement (b) has been derived.

Next we derive statement (a) from statement (b). Let $\mathbf{x} \in \mathbb{R}^n_+$ be an arbitrary income profile, $i \in \{1, \ldots, n\}$ be an arbitrary individual, and $\varepsilon > 0$ with $x_i < x_{i+1}$ if $i < n$. By weak monotonicity either $(x_i + \varepsilon)_i\mathbf{x} \sim \mathbf{x}$ or $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$ follows.

Consider the case that $(x_i + \varepsilon)_i\mathbf{x} \sim \mathbf{x}$. By global consistency it follows that $(y_j + \delta)_j\mathbf{y} \sim \mathbf{y}$ whenever $y_j \geq x_i$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta \geq 0$.

Suppose (CASE 1) that there exists no income profile $\hat{\mathbf{x}} \in \mathbb{R}^n_+$, $\hat{\varepsilon} > 0$ and $k \in \{1, \ldots, n\}$ such that $\hat{x}_k < \hat{x}_{k+1}$if $k < n$, and $(\hat{x}_k + \hat{\varepsilon})_k\hat{\mathbf{x}} \succ \hat{\mathbf{x}}$. Then we set $z := 0$. It follows that $(y_j + \delta)_j\mathbf{y} \sim \mathbf{y}$ whenever $y_j \geq z$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta \geq 0$.

Otherwise (CASE 2), if there exists an income profile $\hat{\mathbf{x}} \in \mathbb{R}^n_+$, $\hat{\varepsilon} > 0$ and $k \in \{1, \ldots, n\}$ such that $\hat{x}_k < \hat{x}_{k+1}$ if $k < n$, and $(\hat{x}_k + \hat{\varepsilon})_k\hat{\mathbf{x}} \succ \hat{\mathbf{x}}$, global consistency implies that $(y_j + \delta)_j\mathbf{y} \succ \mathbf{y}$ whenever $y_j \leq \hat{x}_k$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta > 0$. Let $\hat{\mathbf{x}} \in \mathbb{R}^n_+$ be such that k is maximal with $\hat{x}_k < \hat{x}_{k+1}$ (if $k < n$) and $(\hat{x}_k + \hat{\varepsilon})_k\hat{\mathbf{x}} \succ \hat{\mathbf{x}}$ (otherwise consider, without loss of generality, $\hat{\mathbf{x}}' \in \mathbb{R}^n_+$ with $\hat{x}'_j = \hat{x}_j$ for all $j \in \{1, \ldots, n-1\}$ and $\hat{x}'_n = x_i$). In that case $\hat{x}_k < x_i$.

Take $\tilde{z} := \begin{cases} \min\{x_i, \hat{x}_{k+1}\} & if \quad k < n \\ x_i & otherwise \end{cases}$, and consider $\bar{\mathbf{x}} := \begin{cases} \tilde{z}_{k+1}\hat{\mathbf{x}} & if\, k < n \\ \hat{\mathbf{x}} & otherwise \end{cases}$.

By global consistency it follows that $(\hat{x}_k + \delta)_k\bar{\mathbf{x}} \succ \bar{\mathbf{x}}$ for all $\delta > 0$. Note that $0 < \delta < \tilde{z} - \hat{x}_k$.

Next, consider the set $\{w \in \mathbb{R}_+ : (w + \varepsilon')_k\bar{\mathbf{x}} \sim w_k\bar{\mathbf{x}}$ for some $\varepsilon' > 0\}$. This set has an upper bound (e.g., $\tilde{z}$) and a lower bound (e.g., $\hat{x}_k$). By global consistency this set is connected and, hence, is a bounded interval. Further, by global consistency, its infimum belongs to the set. Hence, we define

$$z := \inf\{w \in \mathbb{R}_+ : (w + \varepsilon')_k\bar{\mathbf{x}} \sim w_k\bar{\mathbf{x}} \text{ for some } \varepsilon' > 0\}. \tag{1}$$

Therefore, for all $\hat{x}_k < v < z$ there exists some $\delta' > 0$ such that $(v + \delta')_k\bar{\mathbf{x}} \succ v_k\bar{\mathbf{x}}$ and by global consistency it follows that $(y_j + \delta)_j\mathbf{y} \succ \mathbf{y}$ whenever $y_j \leq v$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta > 0$. Thus, as the supremum of all such v is z it follows that $(y_j + \delta)_j\mathbf{y} \succ \mathbf{y}$ whenever $\mathbf{y} \in \mathbb{R}^n_+$, $y_j < z$ and $\delta > 0$. Further, for all $v \geq z$ we have $(v + \delta)_k\bar{\mathbf{x}} \sim v_k\bar{\mathbf{x}}$ whenever $\delta \geq 0$. Thus, by global consistency it follows that $(y_j + \delta)_j\mathbf{y} \sim \mathbf{y}$ whenever $y_j \geq v$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta \geq 0$. As the minimum of such v is z it follows that $(y_j + \delta)_j\mathbf{y} \sim \mathbf{y}$ whenever $y_j \geq z$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta \geq 0$. Note that, by construction, z is unique.

Consider now the case that $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$. If there exists $\hat{\mathbf{x}} \in \mathbb{R}^n_+$ such that for some $\hat{\varepsilon} > 0$ and $k \in \{1, \ldots, n\}$we have $(\hat{x}_k + \hat{\varepsilon})_k\hat{\mathbf{x}} \sim \hat{\mathbf{x}}$, then we are back at CASE 2 (with the roles of $\hat{\mathbf{x}}$ and $\mathbf{x}$ interchanged) and find z as defined in Eq. (1). Otherwise (CASE 3), if there is no $\hat{\mathbf{x}} \in \mathbb{R}^n_+$, $k \in \{1, \ldots, n\}$ and $\hat{\varepsilon} > 0$ such that $(\hat{x}_k + \hat{\varepsilon})_k\hat{\mathbf{x}} \sim \hat{\mathbf{x}}$, we define $z := +\infty$. It follows that $(y_j + \delta)_j\mathbf{y} \succ \mathbf{y}$ whenever $y_j < z$, $\mathbf{y} \in \mathbb{R}^n_+$ and $\delta > 0$.

CASES 1–3 and the uniqueness result in CASE 2, complete the derivation of statement (a). This completes the proof of Theorem 1. $\square$

4.2 Proof of Theorem 2

First we show that (a) implies (b). Clearly, if statement (a) is satisfied, then following similar steps as in the Proof of Theorem 1 we can show that weak monotonicity holds.

For local consistency, consider any profile $\mathbf{x} \in \mathbb{R}^n_+$. If $i \in \{1, \ldots, n\}$ is such that $(x_i + \varepsilon)_i\mathbf{x} \sim \mathbf{x}$ for some $\varepsilon > 0$, then $x_i \geq z_{\mathbf{x}}$ from the definition of a relative poverty line (Definition 2). Therefore, it follows that $(x_j + \delta)_j\mathbf{x} \sim \mathbf{x}$ for all $j \geq i$ and $\delta > 0$ as $j \geq i \iff x_j \geq x_i$. Similarly, for any profile $\mathbf{x} \in \mathbb{R}^n_+$, if $i \in \{1, \ldots, n\}$ is such that $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$ for some $\varepsilon > 0$, then $x_i < z_{\mathbf{x}}$ from the definition of a relative poverty line. Therefore, it follows that $(x_j + \delta)_j\mathbf{x} \succ \mathbf{x}$ for all $j \leq i$ and $\delta > 0$ as $j \leq i \iff x_j \leq x_i$. Hence, statement (b) holds.

Next we prove that statement (b) implies statement (a). Take an arbitrary profile $\mathbf{x} \in \mathbb{R}^n_+$. By weak monotonicity it follows that for any $\varepsilon > 0$ either $(x_n + \varepsilon)_n\mathbf{x} \succ \mathbf{x}$ or $(x_n + \varepsilon)_n\mathbf{x} \sim \mathbf{x}$.

(CASE 1) First suppose that $(x_n + \varepsilon)_n\mathbf{x} \succ \mathbf{x}$. Then local consistency implies that $(x_j + \delta)_j\mathbf{x} \succ \mathbf{x}$ for all $j \leq n$ and $\delta > 0$. Set $z_{\mathbf{x}} := x_n + \varepsilon$ and statement (a) follows.

(CASE 2) Suppose instead that $(x_n + \varepsilon)_n\mathbf{x} \sim \mathbf{x}$. There are then two further cases to consider.

(CASE 2i) If there exists some $i < n$, such that $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$ for some $\varepsilon > 0$, then take the maximal value $k \in \{1, ..., n-1\}$ such that $x_k < x_n$ and $(x_k + \varepsilon')_k\mathbf{x} \succ \mathbf{x}$ for some $\varepsilon' > 0$. By local consistency, for all $x_j \leq x_k$ and any $\delta > 0$, $(x_j + \delta)_j\mathbf{x} \succ \mathbf{x}$. We next consider all x_j with $k < j \leq n$. By weak monotonicity it follows that, for any $\hat{\varepsilon} > 0$, either $(x_{k+1} + \hat{\varepsilon})_{k+1}\mathbf{x} \succ \mathbf{x}$ or $(x_{k+1} + \hat{\varepsilon})_{k+1}\mathbf{x} \sim \mathbf{x}$. If $(x_{k+1} + \hat{\varepsilon})_{k+1}\mathbf{x} \succ \mathbf{x}$, then this contradicts the definition of k. Therefore it must be the case that $(x_{k+1} + \hat{\varepsilon})_{k+1}\mathbf{x} \sim \mathbf{x}$. By local consistency, for any $j \geq k+1$ and $\delta \geq 0$, $(x_j + \delta)_j\mathbf{x} \sim \mathbf{x}$. Set $z_{\mathbf{x}} = x_{k+1}$ and statement (a) follows.

(CASE 2ii) If, however, there exist no $i < n$ and $\varepsilon > 0$ such that $(x_i + \varepsilon)_i\mathbf{x} \succ \mathbf{x}$, then from weak monotonicity and local consistency it follows that $(x_j + \delta)_j\mathbf{x} \sim \mathbf{x}$ for all $j \in \{1, \ldots, n\}$ and $\delta \geq 0$. Set $z_{\mathbf{x}} = 0$. Then statement (a) follows.

Therefore, and because $\mathbf{x}$ was arbitrary, we can conclude that statement (a) holds. This completes the proof of Theorem 2. $\square$

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