Growth Effects of the Allocation of Government Expenditure in an Endogenous Growth Model with Physical and Human Capital

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The growth effects of government expenditure is a topic that has received much attention in the literature. Of interest is the question, could the allocation of government expenditure matter for economic growth? Using a theoretical approach, this research sets out to answer this questions. Our framework is a combination of the Uzawa-Lucas two sector model of human capital with the Barro (1990) model of government expenditure in an endogenous growth model. The main contribution is that government expenditure is now split between the human capital sector and the production of goods sector. We find that there is an optimal allocation of government expenditure when the shares of government government expenditure between sectors is endogenously determined. When government expenditure is not optimally allocated, then the growth rate in the long run will be lower by the rate of the miss allocation. In this case, the growth rate of the economy always reverts back to the growth rate of human capital. Using data from Kenya, we solve for the model numerically and find that Kenya would benefit from a higher long run growth rate if the government shifted expenditure from the production sector to the human capital sector.

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1. Introduction

About eight years ago, the government of Kenya began the implementation of Vision 2030, an economic blueprint that is meant to get the country to middle income status by the year 2030. In order for the country to achieve middle income status, it needs to grow at a rate of 10% per annum until the year 2030. While the government has identified several areas from where growth could be attained, the question arises, given its limited resources, what is the most efficient allocation of these resources that will help it achieve the targeted growth. Government fiscal policy would therefore play an important role, and therefore so does the allocation of government spending.

This phenomenon is not unique to Kenya, in fact many governments, developing and developed alike, are faced with the question of how best to allocate the given limited resources in order to achieve maximum growth. The allocation of government expenditure is thus not a new topic, and has been the subject of a lot of debate in the literature. The debate has taken two strands, with one strand looking at long-run growth effects of government expenditure on recurrent and non recurrent expenditure such as the study of Devarajan et al. (1996), while the other strand looks at the growth effects of the allocation of government expenditure between infrastructure on the one hand and expenditure on the social sector such as education and health on the other such as the study of Agenor and Neanidis (2011). It has been argued that not only can the right allocation of government expenditure have growth enhancing effects, but that this growth could also be inclusive and poverty reducing, thereby raising incomes for the poor.

This research therefore asks the question; does the allocation of government expenditure matter for economic growth? The research answers this question in a setting with an endogenous growth model, where we distinguish between physical and human capital in a two sector set up.

Government Expenditure is important for the provision of public goods. Public goods are goods that have the properties of non-rivalry and non-excludability, which make it difficult for the private sector to provide these goods due to lack of an efficient pricing mechanism. However, these goods are also still considered necessary for the smooth flow of an economy, and therefore the government's role in the provision of public goods is very important. Consequently, it is very important that the government allocate resources as efficiently as possible so as to maximise growth, thereby increasing income and possibly also reducing poverty levels.

To carry out the analysis of government policy on economic growth, we propose an endogenous model with physical and human capital such as the Uzawa-Lucas model. We then combine this model with the Barro (1990) model of government expenditure in an endogenous growth model. Our point of departure from Barro (1990); Rebelo (1991); Turnovsky and Fisher (1995) is that we then analyse the sectoral optimal allocation of government expenditure much like Devarajan et al. (1996) and Agenor and Neanidis (2011). However, while Devarajan et al. (1996) use a CES production function to analyse the allocation of government expenditure, this research uses a two sector approach.[1] This is similar to Agenor and Neanidis (2011) who also carry out a sectoral analysis, but rather look at human capital as consisting of health and education sectors, where health has an effect on the human capital.[2]

The research finds that there is an optimal allocation of government expenditure. Where the allocation of government expenditure is optimal, then the marginal productivity of government spending is the same for both sectors. At this point, marginal productivity for the shares of human capital in both the human capital sector and the production of goods sector is the same and in addition, the marginal productivity of human capital and physical capital is the same. The marginal productivity of physical and human capital is also the economy's interest rate. Growth rate in the long run is thus the same for human capital, physical capital, consumption and income.

The rest of this research is organised as follows; in the following section, we give a review of the existing literature on government expenditure and economic growth. In section three we look at the text book Uzawa Lucas model of physical and human capital. The set up of this model is in a closed economy with no government. In section four, we expand the model from section three by introducing government policy, thereby combining the Uzawa Lucas and the Barro (1990) model of government spending in an endogenous growth model. The model will incorporate two types of government expenditure, that is spending on infrastructure and spending on human capital. Section

[1] Devarajan et al. (1996) look at two types of government spending, that is, productive and non productive in a one sector model with a CES production function.

[2] In the framework of Agenor and Neanidis (2011) government expenditure is allocated between the production of goods sector, human capital sector and the health sector. Health impacts human capital and human capital is a factor of production in the goods sector. In addition, government spending on infrastructure is important for all three sectors.

five analyses the long-run growth effects of changes in the allocation of government expenditure keeping government revenue constant, and then looks at an application of the model to the Kenyan economy. Section six concludes.

2. A Review of the Literature

In his seminal paper, Aschauer (1989) looks at the growth effects of productive government expenditure in a framework where government expenditure has an effect on productivity in the economy. Testing for this empirically, he finds that non military government expenditure stocks are indeed important for productivity. Following this research on government expenditure and productivity, came the debate on the effect of government expenditure and economic growth. Barro (1990) looks at the growth effects of government expenditure in an endogenous growth model and finds that there is indeed an optimal level of government expenditure.

Since the Barro (1990) research, several research has looked at the effects of government expenditure on economic growth in an Ak (one sector) endogenous growth framework (Rebelo (1991); Futagami et al. (1993); Turnovsky and Fisher (1995); Turnovsky (1997); Gomez (2004, 2008); Irmen and Kuehnel (2009)). Some of this work has extended the Barro (1990) model to include government expenditure as a stock with congestion (Futagami et al. (1993); Turnovsky (1997); Glomm and Ravikumar (1994, 1997); Gomez (2004, 2008); Irmen and Kuehnel (2009)). Gomez (2008) extends the Ak framework to include physical and human capital, in a model where investment on public capital is irreversible and includes congestion.

A second strand of literature that looks at the growth effects of government expenditure, looks specifically at the allocation of government expenditure on economic growth (Devarajan et al. (1996) and Agenor and Neanidis (2011)). This is perhaps a combination of the Aschauer (1989) and Barro (1990) research.

Devarajan et al. (1996), using a one sector endogenous growth framework, introduce two types of government expenditure, that is, productive and non-productive. They then use a CES production function to analyse the growth effects of the allocation of government expenditure between productive and non-productive sectors. They find that allocation of government spending matters and that government spending on the productive sector is only productive up to a certain point after which productivity declines.

Agenor and Neanidis (2011) also analyse the growth effects of the allocation of government spending in a multiple sector framework, which includes the production of goods sector, education and health sectors. They split government spending into spending on infrastructure, education and health. They point out the externalities associated with shifts in government spending between sectors, where revenue is held constant.

This research develops a model with physical and human capital. However, unlike the Agenor and Neanidis (2011) framework, we do not split the effects of human capital. In addition, we follow the Uzawa-Lucas model, where we assume a human capital intensive sector in the production of human Capital. Our similarity with Agenor and Neanidis (2011) is with government expenditure that is split between the two sectors.

3. An Endogenous Growth Model With Physical and Human Capital

We begin with an endogenous growth model. This model follows the Uzawa-Lucas model of human capital. We follow the analysis as is done by Barro and i Martin (2004).[3]

3.1. Household Preferences

The model assumes homogenous households with the same preferences, perfect capital markets and perfect foresight. Households begin with the same level of assets. From these assumptions, we can therefore derive equilibrium from a single representative household. The objective function is thus:

$$U = \int_0^\infty u(c)e^{-\rho t}dt \tag{1}$$

The function above assumes that total utility facing a household at time zero is the weighted sum of all future utility. ρ is the rate of time preference and is assumed to be $\rho > 0$. The instantaneous utility function satisfies the inada conditions and is assumed to be concave, that is, consumption of private goods are assumed to be such as those of normal goods. We take the utility function to be of the isoelastic form as below:

[3]Barro and i Martin (2004) carry out an analysis of the Uzawa Lucas Model. This can be found in pp.251-252, and 274-276. Section 3.2 above therefore follows this analysis as is shown in their textbook "Economic Growth".

$$u(c) = \frac{c^{1-\theta} - 1}{1 - \theta} \tag{2}$$

Households are faced with the following budget constraint:

$$\dot{a} = w + ra - c \tag{3}$$

Where a is the assets per household, w is the wage income per household, and r is the interest rate, or rate of return on assets.

3.2. Production of Goods

Physical capital, K, Human capital H, and technology, A, are used in the production of aggregate output. A fraction of labour z, is used to produce goods in this sector. Following the Uzawa-Lucas model, H is assumed to derive from the total number of workers multiplied by the total human capital per worker, where the two, human capital per worker and total number of workers, are perfect substitutes. Assuming a Cobb-Douglas production technology, we can express the production function as below:

$$Y = AK^{\alpha} (zH)^{1-\alpha} \tag{4}$$

where $0 < \alpha < 1$, and α is the share of physical capital in output.

3.3. The Human Capital Sector

Human capital is produced using technology B, and the share of human capital that is not used up in the production sector, $1 - z$. This sector is human capital intensive and therefore does not use up any form of physical capital.

$$\dot{H} = B[(1 - z) H] - \delta H \tag{5}$$

3.4. The Planner's Solution

The resource accumulation constraint can be expressed as:

$$\dot{K} = AK^{\alpha} (zH)^{1-\alpha} - C - \delta K \tag{6}$$

In solving for the Planner's solution, we begin by inserting equation (2) into equation (1). The Current-value Hamiltonian can be expressed, using equation (5) and equation (6) as:

$$J = u(C) + \nu \left[AK^{\alpha} (zH)^{1-\alpha} - C - \delta K\right] + \mu \left[B\left[(1-z)H\right] - \delta H\right] \tag{7}$$

ν and μ are the co-state variables associated with the constraints in equation (6) and (5) respectively. Given the optimality conditions, following from the first order condition that $\frac{\partial J}{\partial C} = 0$ and the co-state condition $\dot{\nu} = -\frac{\partial J}{\partial K}$, and given the transversality condition $lim_{t\to\infty} \nu K exp - \rho t = 0$, the growth rate of consumption will be (see Appendix for workings of the optimality conditions):

$$\frac{\dot{C}}{C} = \frac{1}{\theta} \left(A\alpha \left(\frac{K}{zH}\right)^{-(1-\alpha)} - \delta - \rho \right) \tag{8}$$

The expression $\left(A\alpha \left(\frac{K}{zH}\right)^{-(1-\alpha)} - \delta \right)$ is also the net marginal product of capital net of depreciation, and is equal to the rate of return r. If the net marginal product exceeds the rate of time preferences, then the growth rate of consumption is positive. The net marginal product of capital depends on capital and the share of human capital in the production sector.

The expression below follows from the first order conditions that $\frac{\partial J}{\partial z}$ are equal to zero (see appendix equation (43)):

$$\frac{\mu}{\nu} = \frac{AK^{\alpha} (zH)^{-\alpha} (1-\alpha)}{B} \tag{9}$$

Equation (9) is an expression of the shadow price of human capital. It captures the idea that the marginal product of human capital is the same for both the goods production sector as well as the human capital sector. Using the the shadow price of human capital and the condition that $\dot{\mu} = -\frac{\partial J}{\partial H}$, we get the growth rate of a change in human capital to be (see workings in the appendix, also follows Barro and i Martin (2004)):

$$\frac{\dot{\mu}}{\mu} = -B + \delta \tag{10}$$

3.5. Reduced Form Equations

From the resource accumulation constraint and the change in capital equations, we can get the growth rates of capital and human capital as:

$$\frac{\dot{K}}{K} = Az^{(1-\alpha)} \left(\frac{K}{H}\right)^{-(1-\alpha)} - \frac{C}{K} - \delta \tag{11}$$

and

$$\frac{\dot{H}}{H} = B(1-z) - \delta \tag{12}$$

Differentiating the first order condition that $\frac{\partial J}{\partial z} = 0$ with respect to time and inserting equations (9), (10), (11) and (12) gives us the growth rate of z as below (See Appendix for workings):

$$\frac{\dot{z}}{z} = \frac{(1-\alpha)B}{\alpha} + Bz - \frac{C}{K} \tag{13}$$

We follow Barro and i Martin (2004), we define the following ratios; $\chi = \frac{C}{K}$ and $\omega = \frac{K}{H}$. χ, ω and $\frac{\dot{z}}{z}$ give us a system of equations, for which the growth rate at steady state will be equal to zero. Solving for this will give the following values at steady state, where the state variable ω will begin at some value $\omega(0)$(complete workings are in Appendix A):

$$\omega = \left(\alpha\frac{A}{B}\right)^{\frac{1}{1-\alpha}} \frac{\theta-1}{\theta} + \frac{\delta(1-\theta)+\rho}{B\theta} \tag{14}$$

$$\chi = B\left(\frac{1}{\alpha} - \frac{1}{\theta}\right) + \frac{\delta(1-\theta)+\rho}{\theta} \tag{15}$$

$$z = \frac{\theta-1}{\theta} + \frac{\delta(1-\theta)+\rho}{B\theta} \tag{16}$$

3.6. Balanced Growth Path

We can now derive the reduced form growth rate of capital and human capital in the long run by inserting equations (14), (15) and (16) above into equation (11) and (12):

$$\frac{\dot{K}}{K} = \frac{\dot{H}}{H} = \frac{1}{\theta}\left(B - \delta - \rho\right) \tag{17}$$

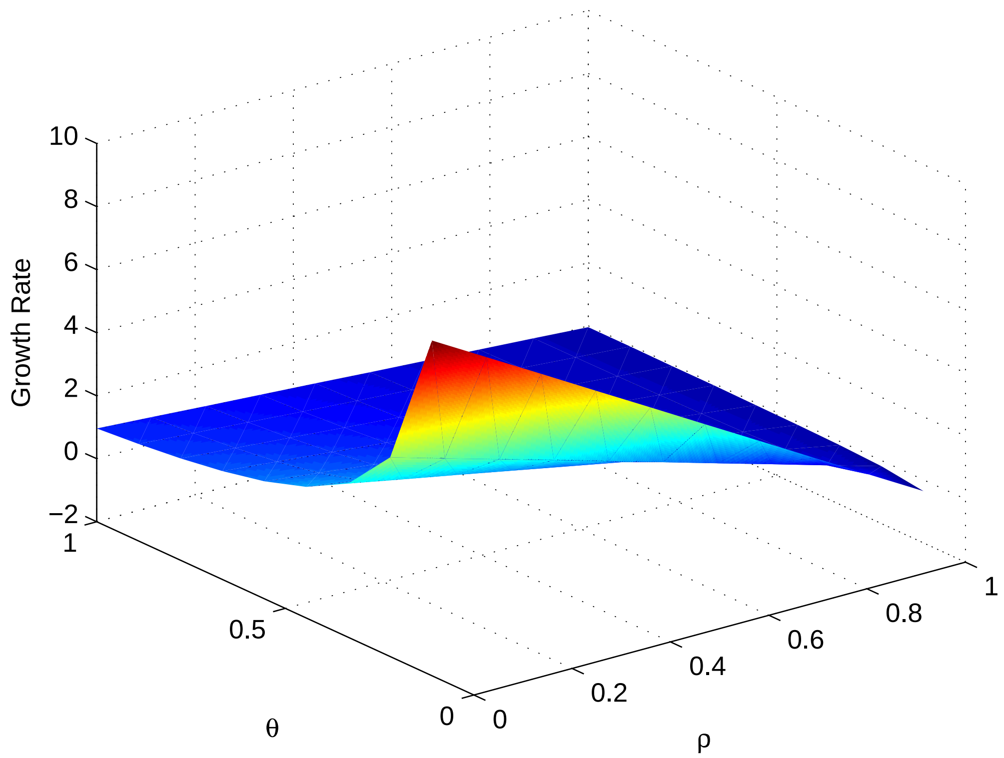
When we solve for the growth rate of output and consumption, we get the growth rates as below:

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{1}{\theta}\left(B - \delta - \rho\right) \tag{18}$$

The expression $(B - \delta)$is the net marginal product of capital at steady state. This is also the rate of interest or the price of investment. This expression is equal to the marginal product of human capital, which means that at steady state, the marginal product of capital and human capital, as well as the interest rate must be the same. In the long run therefore, the growth rate of output, consumption, capital and human capital, as shown in equations (17) and (18) will be the same.

The diagram below represents the growth rate at Balanced Growth Path.

Figure 1: A Graphical Presentation of the Balanced Growth Path



The growth rate depends on θ and ρ, which are the elasticity of substitution and the rate of time preference respectively. A large θ means individuals are less willing to give up consumption today for more consumption tomorrow, and therefore save less, while the opposite is the case for a small θ. A high ρ means that consumption today is preferred to consumption tomorrow, and therefore reduces the consumption growth rate.

From Figure (1) above, we begin with a rate of time preference of 0.01, and at this rate, with a low elasticity of substitution, here also 0.01, then individuals are more willing to save, and give up consumption today for consumption tomorrow. The growth rate is therefore high. However, where the rate of time preference is high and the elasticity of substitution is large, then the growth rate in the Balanced growth path is low, as is shown in the diagram above. On the other hand, where elasticity of substitution is low and the rate of time preference is high, then the growth rate is much lower than if the opposite were the case, that is, if the elasticity of substitution is high and the rate of time preference is low. In this case, preference for consumption today is higher than the preference for consumption tomorrow.

4. Endogenous Growth With Government Policy - Composition of Government Expenditure

The government operates a balanced budget, that is, its expenditure is equal to revenue collected. The revenue is obtained through a flat rate tax on output. The government spends part of its revenue on the education sector and the invests the rest in public capital, which is used in the production sector. Government therefore has the following constraint:

$$G = vG + (1 - v)\,G \tag{19}$$

and

$$G = \tau Y$$

where $\tau \in (0, 1)$

4.1. The Production of Goods

In addition to physical capital, K, human capital H, and technology, A, government public investment G, is now also used in the production of aggregate output. Like in the previous section, a fraction of labour z, is used to produce goods in this sector, where H is assumed to derive from the total number of workers multiplied by the total human capital per worker, where the two, human capital per worker and total number of workers, are perfect substitutes. A fraction of government expenditure v is used for public capital in the production sector. Assuming a Cobb-Douglas production technology, the production function can be expressed as:

$$Y = AK^{\alpha} (vG)^{\beta} (zH)^{1-\alpha-\beta} \tag{20}$$

We assume constant returns to scale to factors of production, where $\alpha, \beta \in (0,1)$

4.2. Human Capital Sector

Technology B, and the share of human capital that is not used up in the production sector, $1-z$, as well as the share of government expenditure not used up in the production sector $1-v$ are used in the production of human capital. Government spending in this sector includes spending on materials that are used in the production of human capital such as books. The human capital production function can be expressed as:

$$\dot{H} = B\left[(1-z)H\right]^{1-\eta}\left[(1-v)G\right]^{\eta} - \delta H \tag{21}$$

Like in the production sector, we assume constant returns to scale to factor inputs, $\eta \in (0,1)$

4.3. Welfare Maximising Equilibrium - A Planner Approach

The resource accumulation constraint can be expressed by equation (21) and the equation below:

$$\dot{K} = AK^{\alpha} (vG)^{\beta} (zH)^{1-\alpha-\beta} - C - G - \delta K \tag{22}$$

Taking into account equation (1), (2), (21) and (22), the current value Hamiltonian can be expressed as:

$$\begin{aligned} J = u(C) + \nu &\left[AK^{\alpha}(vG)^{\beta}(zH)^{1-\alpha-\beta} - C - G - \delta K\right] \\ +\mu &\left[B\left[(1-z)H\right]^{1-\eta}\left[(1-v)G\right]^{\eta} - \delta H\right] \end{aligned} \tag{23}$$

We then assume a Planner who has complete information and therefore maximises the consumer's lifetime utility by selecting C, z and v, with ν and μ being the co-state variables associated with equations (22) and (21) respectively. The first order optimality conditions for z and v are presented in the appendix in equation (54) and (55). From equation (54) we derive the shadow price of human capital as below:

$$\frac{\mu}{\nu} = \frac{(1-\alpha-\beta)(1-\tau)AK^{\alpha}(vG)^{\beta}(Hz)^{-\alpha-\beta}}{(1-\eta)B\left[(1-z)H\right]^{-\eta}\left[(1-v)G\right]^{\eta}} \tag{24}$$

The shadow price of human capital shows that the marginal product of human capital must be the same in both the human capital sector and the goods sector. Inserting the shadow price of human capital into the first order optimality condition for v leads to the following condition:

$$\frac{z}{1-z}\frac{\beta}{1-\alpha-\beta} = \frac{\eta}{1-\eta}\frac{v}{(1-v)} \tag{25}$$

Equation (25) is the condition derived by taking into account the shadow price of human capital in equation (24) which implies that the marginal product of human capital must be the same in both the goods sector and the human capital sector. In addition, the above condition implies that the marginal product of government expenditure must be the same for the two sectors, and that the share of government expenditure, and the share of human capital in both sectors are positively related, so that given β and η, then an increase in output can only be achieved by increasing both z and v (Barro and i Martin (2004)).

This implies therefore, that at the point where marginal productivity for human capital and for government expenditure are the same in both the human capital and the goods production sector, then the ratio of government expenditure to human capital must be equal to 1, since increasing one of these factor inputs without increasing the other has no effect on output.

With this in mind, the growth rates of the co state conditions ν and μ, as well as the growth rates of consumption, C, capital, K, human capital, H, as well as the growth rate of z can now give us a system of equations from which we can solve for the optimal Balanced Growth Path. Note that the value of z in the Balanced Growth Path is proportional to the value of v, which means that we can determine the value of v explicitly in the Balanced Growth Path. The growth rates are as shown in the appendix, equations (59), (60), (61), (62), (63) and (64).

As with the Uzawa-Lucas model, we define the values $\omega = \frac{K}{H}$ and $\chi = \frac{C}{K}$. Where we set the state variable $\omega(0) = \omega_0$, and the steady state growth rates of ω, χ, and z as equal to zero, together with the transversality condition that $lim_{t \to \infty} \nu K exp - \rho t = 0$, then the steady state values of ω, χ and z are as given in the appendix, equations (65), (66) and (67). Inserting these steady state values into the growth rates of income, capital, consumption and human capital give us the growth rates of income, capital, consumption and human capital in the Balanced Growth Path as below:

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = \frac{1}{\theta} (B - \delta - \rho) \tag{26}$$

Equation (26) gives us the optimal Balanced Growth Path growth rate of income, consumption, capital and human capital, in an economy with government policy. At this growth rate, the marginal productivity for human capital is the same for both the human capital and production of goods sector, and the same is the case for the marginal productivity of government expenditure. The Balanced Growth Path depends positively on B, which is the technology of human capital, depends inversely on θ, the elasticity of substitution, and negatively on ρ, the rate of time preference.

As in Barro (1990), we note that in a centrally planned economy such as the one defined above, this growth rate is also Pareto Optimal. This is because in a decentralised economy, price is the main resource allocation mechanism, and therefore individuals who are willing and able to pay will benefit more from the resource allocation.

The value $(B - \delta)$is also the economy's interest rate, since this is the marginal product for physical and human capital. The marginal product for physical and human capital must be the same in the long run.

5. Long Run Effects of Changes in Government Policy

In this section, we proceed by analysing the steady state effects of changes in government expenditure. In this case, government revenue does not change, rather, instead there is a change from one type of spending to another. We will begin by analysing the steady state effects of a change in government expenditure from human capital sector to the production of goods sector, after which we will proceed to look at the steady state effects of a change in government expenditure from production of goods sector to human capital sector.

5.1. An Increase in Government Spending in the Production of Goods Sector

Suppose that the government makes a decision to shift expenditure from the human capital sector to the production of goods sector. Since government revenue remains the same, an increase in the share of government spending on the goods sector can only be achieved by a decrease in the share of government spending on the human capital sector. This is illustrated below:

$$\partial v = -\partial \left(1 - v\right)$$

There are two major implications for this shift:

1. *From the condition in equation (25), the implication is that the ratio of the share of government expenditure to the share of human capital in the human capital sector is now less than one as expressed below*

$$\frac{1-v}{1-z} \leq 1 \tag{27}$$

2. *However, from the same condition, increasing the share of government expenditure in the production of goods sector without increasing other factor inputs, does not increase output. Therefore, the ratio of the share of government expenditure to the share of human capital that is useful in the production sector remains 1, without any growth in production*

$$\frac{v}{z} \equiv 1 \tag{28}$$

Equation (28) is a transversality condition and sets an upper bound on the growth rate of income. Taking the implications from equations (27) and (28) above into account, we can now rewrite the economy resource accumulation constraint, and the change in human capital as below:

$$\dot{K} = Az^{1-\alpha}\left(\frac{K}{H}\right)^{\alpha} - C - G - \delta K \tag{29}$$

and

$$\dot{H} = B\left[(1-z)\,H\right]^{1-\eta}\left[(1-v)\,G\right]^{\eta} - \delta H \tag{30}$$

Note that while the equation for change in human capital (30) appears not to change, the values of $(1-z)$ and $(1-v)$are not optimally chosen since v is now lower than it would be at optimal levels. Thus, the new value v is as represented below:

$$v^{\star} = v + v*$$

From the above two equations, we can now rewrite the Hamiltonians (Appendix, equation (68)).The system of equations for our analysis, that is, the growth rate of the co state conditions for capital and human capital ν and μ, the growth rate of capital, human capital and consumption are as expressed in the appendix equations (69), (70), (71), (72) and (73). From the appendix we see that the system can be reduced into a system of two equations for ω and χ. Solving for the values of ω and χ and inserting these values into the growth rates of capital, human capital, consumption and income give us the Balanced Growth Path of capital, human capital, consumption and income as below:

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = B\,(1-z)^{1-\eta}\,(1-v^{\star})^{\eta} - \delta \tag{31}$$

and

$$\frac{\dot{K}}{K} = \frac{\dot{H}}{H} = B\,(1-z)^{1-\eta}\,(1-v^{\star})^{\eta} - \delta \tag{32}$$

The growth rate of income, consumption and capital is now the same as the growth rate of human capital. The growth rate depends on technology, B, the share of human capital in the human capital sector $(1-z)$, and the share of government expenditure in the human capital sector $(1-v^{\star})$.

Intuitively, production of output depends on capital, human capital and government expenditure, and we assume constant returns to factors of production. Output therefore increases with an increase to all factor inputs. For the case above, where government reduces spending on the human capital sector, and assuming constant returns to factors in the human capital sector as well, then change in human capital declines by the amount of decline in government expenditure, that is, $-\partial v$. The growth rate of human capital is now lower than it was before. The marginal productivity of all factor inputs in the human capital sector increases.

This lower growth rate is now carried to the production of goods sector, since the factor input, human capital, is much lower. Therefore, output and capital also decline by the rate of decline of government spending.

Note that $B\left(1-z\right)^{1-\eta}\left(1-v^{\star}\right)^{\eta}-\delta$ is now the marginal product of physical and human capital in both sectors. In the long run, the marginal product is the same for both physical and human capital and is also equal to the economy interest rate or reward to physical capital.

This results support the finding of Devarajan et al. (1996) who find that increasing government expenditure on what is considered productive spending such as infrastructure, is only beneficial up to a given point, after which spending is better shifted to non-productive sectors such as government spending on consumption.

5.2. An Increase in Government Spending in the Human Capital Sector

Now let us look at a scenario where government increases spending on the human capital sector, holding revenue constant. As with the previous section, increasing spending on the human capital sector means a similar decline in spending on the production of goods sector. This can be illustrated below:

$$\partial\left(1-v\right)=-\partial v \tag{33}$$

Similarly, as with the section above, there will be two major implications:

1. *The condition in equation (25) implies that the ratio of government spending to human capital in the production sector is now less than one as illustrated below:*

$$\frac{v}{z} \leq 1 \tag{34}$$

2. *The share of government spending to human capital in the human capital sector remains equal to one, since an increase in government spending on human capital does not increase the growth rate of human capital unless there is a corresponding increase in the share of human capital. This is illustrated below:*

$$\frac{1-v}{1-z} = 1 \tag{35}$$

Note that v now decreases by the amount $-\partial v$. v is now no longer optimal and can be expressed below:

$$v^{\star} = v - v * \partial v$$

Equation (35) is a transversality condition that bounds the growth rate of human capital, so that even with an increase in the share of government spending in the human capital sector, it is not possible for human capital to grow to infinity. Thus human capital only increases with an increase in both factors of production in the human capital sector.

Based on this, the new resource accumulation constraint and equation for growth rate of human capital will be:

$$\dot{K} = Az^{1-\alpha}\left(\frac{K}{H}\right)^{\alpha}\left(\frac{v^{\star}G}{zH}\right)^{\beta} - C - G - \delta K \tag{36}$$

and

$$\dot{H} = B\left(1-z\right)H - \delta H \tag{37}$$

Equations (36) and (37) will give us the new Hamiltonians (Appendix equation (76)). From the hamiltonians, we can now compute the growth rates which give us the system of equations from which we can solve the optimality conditions. Equations (77), (78), (79), (80) and (81) in the appendix give us the growth rates of capital, human capital, consumption and the growth rates of the co state conditions for the equations for the change in capital and human capital.

Reducing the system into two equations for χ and ω, solving for these and inserting them into the growth rates of income, consumption, capital and human capital give us the following growth rates in the Balanced Growth Path:

$$\frac{\dot{K}}{K} = \frac{\dot{H}}{H} = B\,(1 - z) - \delta \tag{38}$$

and

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = B\,(1 - z) - \delta \tag{39}$$

Equations (38) and (39) give us the growth rates of capital, human capital, income and consumption in the Balanced Growth path. The growth rate depends on B, the technology used in the human capital sector as well as z, the share of human capital used in the human capital sector. A larger z, which is the share of human capital in the production of goods sector, will reduce the growth rate in the Balanced Growth Path. However, the increase is bound by the transversality condition in equation (34) so that z can only increase by the proportion by which government expenditure increased in the human capital sector. Government spending in the human capital sector must therefore decrease, given the condition in equation (34) to achieve optimal growth in the Balanced Growth Path.

$B\,(1 - z) - \delta$ is the marginal productivity of human capital as well as the marginal productivity of physical capital. From equations (38) and (39) above, the marginal productivity for physical and human capital must be the same in the long run. This is also the interest rate for capital in the economy.

5.3. An Application of Government Policy

In this section, we show the effects of changes in the share of government expenditure by setting the shares of human capital and government expenditure in both sectors as exogenous rather than endogenously determined. In doing so, we first solve for the growth rates in the Balanced Growth Path. From the Appendix section (C), the growth rates of capital, human capital, consumption and income are the same and are equal to:

$$\frac{\dot{Y}}{Y} = \frac{\dot{C}}{C} = B\left(1-z\right)^{1-\eta}\left(1-v\right)^{\eta} - \delta \tag{40}$$

and

$$\frac{\dot{K}}{K} = \frac{\dot{H}}{H} = B\left(1-z\right)^{1-\eta}\left(1-v\right)^{\eta} - \delta \tag{41}$$

Equation (40) and (41) above are the equations for growth in the Balanced Growth Path, where the share of human capital and the share of government expenditure are exogenously determined. The model parameters are B, which is the technology parameter in the human capital sector, η, which is the output parameter for government expenditure in the human capital sector, δ, the depreciation rate, z, the share of human capital in the production of goods sector and v, the share of government expenditure in the production of goods sector.

For the parameters we use Kenya country statistics from the Kenya Economic Survey (of Statistics (9 15)). We select $\eta = 0.27$ and $\delta = 0.04$. The choice of η is based on the SPEED (of Public Expenditure for Economic Development (2013)) data, which shows that Kenya has a total government expenditure to GDP ratio of 0.27. Following the Barro model, the optimal government expenditure is that where the share of government spending to income is also the same as the output elasticity of government expenditure. Since $(B - \delta) = r$, which is the interest rate, and the interest rate for Kenya is at 11% as per the current Central Bank interest rate, then we calculate $B = 0.15$.

From the data in the Kenya Economic Survey (of Statistics (9 15)), government spending on development expenditure is 2.1 times that of government spending on the education sector. We thus take a $v^{\star} = 0.67$.

The data on labour in the various sectors for Kenya shows that formal employment in education sector was 1.8 times more than formal employment in the category "professional, scientific and technical". This is in line with the model's assumption that the education sector is human capital intensive.From this, we take the $z^{\star} = 0.35$. $v^{\star}$ and $z^{\star}$indicate that these are exogenously determined and are not necessarily the optimal shares.

Figure 2: Changing Government Policy

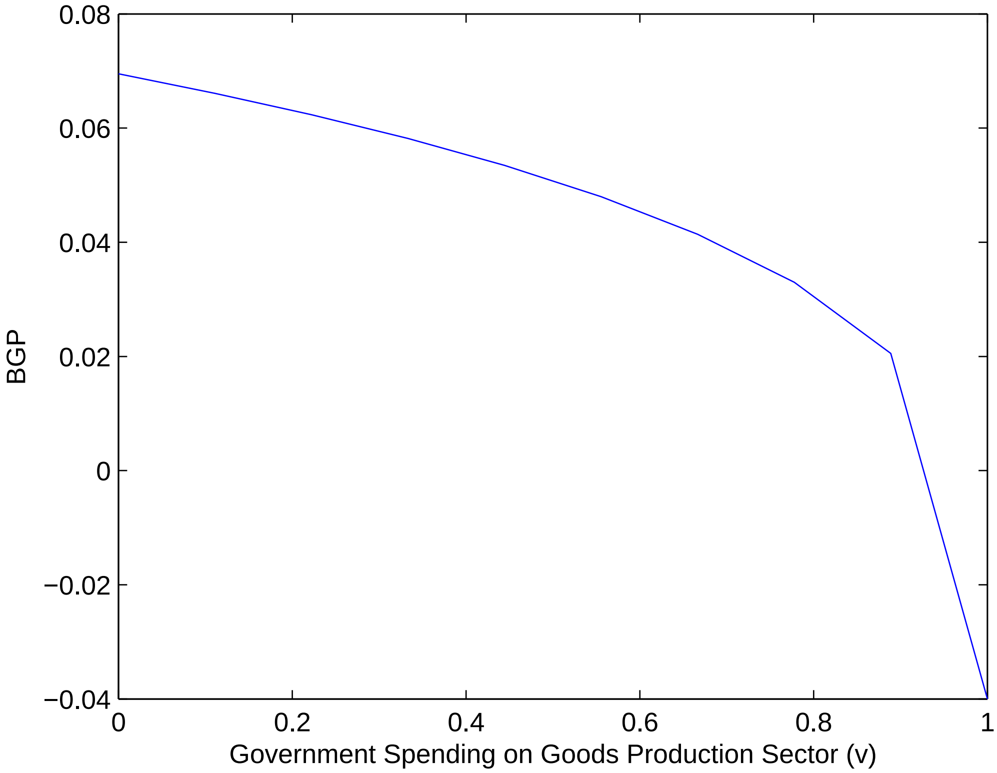


Figure (2) above shows how changes in government policy might affect growth rates in the long run.

At $v = 0.67$, the long run growth rate is 0.042, given a $z = 0.35$ and $\eta = 0.27$. A reduction in v, that is, an increase in government spending on human capital sector, could increase growth in the long run, given the parameter values above. In the case of Kenya, therefore, the government would need to increase spending on the human capital sector in order to achieve higher growth in the long run.

Given the parameters above for the data for Kenya, the graph implies that in order to increase the long run growth for the economy, the government needs to invest more on the human capital sector, as compared to spending on the production of goods sector.

6. Conclusion

Government expenditure is important for economic growth, and the main arguments for government expenditure are the public good case. Public goods normally have two qualities, they are non-rival and non-excludable, so that it would be difficult to price such good since it is difficult to determine exactly how much they are worth to individual consumers. The Government therefore steps in to provide public goods that are much needed in the economy.

An important question arising from government expenditure is whether government spending is optimal such that given a specific level of government expenditure, growth is maximised. Non optimal spending implies that resources are not being used in such a way as to generate maximum value from them. Governments also have the additional problem of a budget constraint, which further limits how they can provide for public goods in an economy.

This research looks at the allocation of government expenditure between two sectors and the effect of this on economic growth. Using the Uzawa Lucas model, and combining this with the Barro (1990) model of government spending and economic growth in an endogenous model, the research first determines the optimal allocation of government spending in an endogenous growth model. We find that optimal allocation of government spending is where the marginal product of government expenditure are the same for both the human capital and the production of goods sector. At this point, the marginal product of human capital is also the same for the two sectors, and human capital and government expenditure have a positive relationship, such that an increase in one unit of human capital will require a unit increase of government expenditure for productivity to increase.

In addition, the growth rate in the long run will be the same for human capital, physical capital, consumption and income. The marginal product for human capital and physical capital will be the same, and will be equal to the interest rate in the economy. The growth rate therefore depends on the savings rate and the technology, as is the case with standard endogenous growth models.

The chapter then proceeds by looking at two different scenarios for changes in government policy in the case where government spending is exogenously determined. We find that there are certain optimality conditions that need to be fulfilled, and where these are not fulfilled, then growth rates in the long run are not optimal. In particular, based on the condition that in optimality, marginal productivity for all factors have to be the same in both sectors, then increasing one factor input without increasing the others does not increase the level of output. Therefore an increase in government expenditure from one sector to another without increasing the revenue levels lowers the long run growth rate by the amount increased. The long run growth rate then depends on the growth rate of human capital.

Finally, we analyse the model based on data from Kenya, a developing country. We find that government spending on the production of goods sector is higher than government spending on human capital. Policy implications for this are that in the long run, the economy will grow at a rate that is lower than the optimal growth rate. This is an important finding, since for a developing economy, a higher growth rate in income has implications for for poverty levels and for the future development path that the country would like to take.

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A. Appendix

A.1. Endogenous Growth Model With Physical and Human Capital - A Planner Approach

From the Hamiltonians in equation (7), we can derive the following First Order Conditions that $\frac{\partial J}{\partial C}$ and $\frac{\partial J}{\partial z}$ are equal to Zero:

$$u'(C) = \nu \tag{42}$$

and

$$A(1-\alpha)H\left(\frac{K}{zH}\right)^{\alpha}\nu - HB\mu = 0 \tag{43}$$

The co-state condition $\dot{\nu} = -\frac{\partial J}{\partial K}$, and $\dot{\mu} = -\frac{\partial J}{\partial H}$ can be expressed as:

$$\dot{\nu} = -\nu\left[A\alpha z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)} - \delta\right] \tag{44}$$

and

$$\dot{\mu} = -\mu\left[B(1-z) - \delta\right] - \nu A(1-\alpha)\left(\frac{K}{H}\right)^{\alpha} z^{(1-\alpha)} \tag{45}$$

We proceed by finding the growth rates of ν, μ, K and H. The growth rates of K and H are derived from the resource accumulation constraint and the equation for change in human capital, (6) and (5) respectively. The growth rates of the co state conditions for capital and human capital are derived from equations (44) and (45) above. In addition, we eliminate the shadow price of human capital for the co state condition of human capital. The growth rates can thus be presented below as:

$$\frac{\dot{K}}{K} = Az^{(1-\alpha)}\left(\frac{K}{H}\right)^{-(1-\alpha)} - \frac{C}{K} - \delta \tag{46}$$

$$\frac{\dot{H}}{H} = B(1-z) - \delta \tag{47}$$

$$\frac{\dot{\nu}}{\nu} = -\left[A\alpha z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)} - \delta\right] \tag{48}$$

$$\frac{\dot{\mu}}{\mu} = -B + \delta \tag{49}$$

To get the growth rate of z, we differentiate equation (43) with respect to time and insert equations (46), (47), (48) and (49):

$$\frac{\dot{z}}{z} = \frac{(1-\alpha)\,B}{\alpha} + Bz - \frac{C}{K}$$

Following Barro and Sala-i-Martin, in order to find the steady state values, we begin by defining the following ratios $\omega = \frac{K}{H}$ and $\chi = \frac{C}{K}$. At steady state, the growth rates of ω, χ and z will be equal to zero, and where we set one of the variables, $\omega(0) = \omega_0$, we can define the growth rates as:

$$\frac{\dot{\omega}}{\omega} = Az^{1-\alpha}\omega^{-(1-\alpha)} - \chi - B\,(1-z) = 0 \tag{50}$$

$$\frac{\dot{\chi}}{\chi} = \frac{1}{\theta}\left(A\alpha z^{1-\alpha}\omega^{-(1-\alpha)} - \delta - \rho\right) - Az^{1-\alpha}\omega^{-(1-\alpha)} + \chi + \delta = 0 \tag{51}$$

$$\frac{\dot{z}}{z} = \frac{(1-\alpha)\,B}{\alpha} + Bz - \chi = 0 \tag{52}$$

Solving for equations (50), (51) and (52) gives us the values of ω, χ and z at steady state.

A.2. Endogenous Growth With Government Policy - A Planner Approach

From the Hamiltonians in equation (23), we can derive the first order conditions that a change in C, z and v are equal to zero as below:

$$u'(C) = \nu \tag{53}$$

$$\begin{aligned}\frac{\partial J}{\partial z} &= \nu\left(1-\alpha-\beta\right)AK^{\alpha}\left(vG\right)^{\beta}\left(zH\right)^{-\alpha-\beta}\\ &-\mu\left(1-\eta\right)B\left[\left(1-z\right)H\right]^{-\eta}\left[\left(1-v\right)G\right]^{\eta}=0\end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial J}{\partial v} &= \nu\beta A\left(vG\right)^{-(1-\beta)}K^{\alpha}\left(zH\right)^{1-\alpha-\beta}\\ &-\mu\eta B\left[\left(1-v\right)G\right]^{-(1-\eta)}\left[\left(1-z\right)H\right]^{1-\eta}=0\end{aligned} \tag{55}$$

The co state conditions for the equations of the resource accumulation constraint and the growth rate of human capital, $\dot{\nu}=-\frac{\partial J}{\partial K}$ and $\dot{\mu}=-\frac{\partial J}{\partial H}$, respectively can be expressed below. In addition, note the government revenue constraint. Since $G=\tau Y$, where τ is the tax rate, or in other words, the rate at which the Planner appropriates resources for government revenue, then G is a portion of income as expressed below:

$$\dot{\nu}=-\nu\left[A\alpha z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)}\left(\frac{vG}{zH}\right)^{\beta}-\delta\right] \tag{56}$$

$$\begin{aligned}\dot{\mu} &= -\mu\left[B\left(1-z\right)\left(\tfrac{(1-v)G}{(1-z)H}\right)^{\eta}-\delta\right]\\ &-\nu A\left(1-\alpha-\beta\right)z^{1-\alpha}\left(\tfrac{K}{H}\right)^{\alpha}\left(\tfrac{vG}{zH}\right)^{\beta}\end{aligned} \tag{57}$$

$$G=\tau AK^{\alpha}\left(vG\right)^{\beta}\left(zH\right)^{1-\alpha-\beta} \tag{58}$$

Since the values of the variables K, H and C do not therefore affect the dynamics of the system, then following Barro and Xala-i-Martin we rather express the system in terms of the ratios $\omega=\frac{K}{H}$ and $\chi=\frac{C}{K}$. Taking into account the condition that the ratio of the share of government expenditure to human capital, $\frac{v}{z}=1$, the growth rates of μ, ν, K, H and C are expressed below. Note that we take into account the government revenue constraint in the resource accumulation constraint, which gives as the growth rate of capital as being less the amount of resources allocated for government expenditure.

$$\frac{\dot{\mu}}{\mu}=-B+\delta \tag{59}$$

$$\frac{\dot{\nu}}{\nu}=-A\alpha z^{1-\alpha}\omega^{-(1-\alpha)}+\delta \tag{60}$$

$$\frac{\dot{K}}{K}=\left(1-\tau\right)Az^{1-\alpha}\omega^{-(1-\alpha)}-\chi-\delta \tag{61}$$

$$\frac{\dot{H}}{H} = B\left(1-z\right) - \delta \tag{62}$$

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left(A\alpha z^{1-\alpha}\omega^{-(1-\alpha)} - \delta - \rho\right) \tag{63}$$

The growth rate of z can be determined by getting the time derivative of equation (54) and inserting equations (59), (60), (61), (62) and (63):

$$\frac{\dot{z}}{z} = \frac{1}{\alpha}B - B\left(1-z\right) - \tau A z^{1-\alpha}\omega^{-(1-\alpha)} - \chi \tag{64}$$

Where we set the state variable $\omega\left(0\right) = \omega_0$, the steady state growth rates of ω, χ, and z are equal to zero, the transversality condition that $lim_{t\to\infty}\nu Kexp - \rho t = 0$, then the steady state values can be defined as:

$$\chi = B\left[\frac{(1-\tau)}{\alpha} - \frac{1}{\theta}\right] + \frac{\delta\left(1-\theta\right)+\rho}{\theta} \tag{65}$$

$$\omega = \left(\frac{\alpha A}{B}\right)^{\frac{1}{1-\alpha}}\left(\frac{\theta-1}{\theta} + \frac{\delta\left(1-\theta\right)+\rho}{B\theta}\right) \tag{66}$$

$$z = \frac{\theta-1}{\theta} + \frac{\delta\left(1-\theta\right)+\rho}{B\theta} \tag{67}$$

B. Changes in Government Policy

B.1. A Shift Towards Spending on the Production Sector

The Hamiltonians can now be expressed as below:

$$\begin{aligned} J = u\left(C\right) + \nu\left[AK^{\alpha}\left(zH\right)^{1-\alpha} - C - G - \delta K\right] \\ +\mu\left[B\left[\left(1-z\right)H\right]^{1-\eta}\left[\left(1-v\right)G\right]^{\eta} - \delta H\right] \end{aligned} \tag{68}$$

The growth rates of K, H, C, $\mu$and ν can be defined as:

$$\frac{\dot{K}}{K} = \left(1-\tau\right)Az^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)} - \frac{C}{K} - \delta \tag{69}$$

$$\frac{\dot{H}}{H} = B(1-z)\left(\frac{1-v^{\star}}{1-z}\right)^{\eta} - \delta \tag{70}$$

$$\frac{\dot{C}}{C} = \frac{1}{\theta}\left[A\alpha z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)} - \delta - \rho\right] \tag{71}$$

$$\frac{\dot{\mu}}{\mu} = -B(1-\eta)\left(\frac{1-v^{\star}}{1-z}\right)^{\eta} + \delta \tag{72}$$

$$\frac{\dot{\nu}}{\nu} = -A\alpha z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)} + \delta \tag{73}$$

Defining the ratios $\omega = \frac{K}{H}$ and $\chi = \frac{C}{K}$, setting the state variable $\omega(0) = \omega_0$, and the steady state growth rates of ω and χ are equal to zero, the transversality condition that $lim_{t\to\infty}\nu K exp - \rho t = 0$, then the steady state values can be defined as:

$$\begin{gathered}\chi = (1-\tau)\left[\tfrac{\delta(1-\theta)+\rho+\theta B(1-z)^{1-\eta}(1-v)^{\eta}}{\alpha}\right] \\ -B(1-z)^{1-\eta}(1-v^{\star})^{\eta}\end{gathered} \tag{74}$$

$$\omega = \left[\frac{\alpha A}{\delta(1-\theta)+\rho} + \frac{\alpha A}{\theta B(1-z)^{1-\eta}(1-v^{\star})^{\eta}}\right]^{\frac{1}{1-\alpha}} z \tag{75}$$

B.2. A Shift Towards Spending on the Human Capital Sector

Taking into account equations (36) and (37), the new Hamiltonians becomes:

$$\begin{gathered}J = u(C) + \nu\left[AK^{\alpha}(vG)^{\beta}(zH)^{1-\alpha-\beta} - C - G - \delta K\right] \\ +\mu\left[B\left[(1-z)H\right] - \delta H\right]\end{gathered} \tag{76}$$

We proceed by defining the growth rates of K, H, C, ν and μ as below:

$$\frac{\dot{K}}{K} = (1-\tau)Az^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)}\left(\frac{v^{\star}}{z}\right)^{\beta} - \frac{C}{K} - \delta \tag{77}$$

$$\frac{\dot{H}}{H} = B(1-z) - \delta \tag{78}$$

$$\frac{\dot{C}}{C}=\frac{1}{\theta}\left(\alpha A z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)}\left(\frac{v^{\star}}{z}\right)^{\beta}-\delta-\rho\right) \tag{79}$$

$$\frac{\dot{\nu}}{\nu}=-\alpha A z^{1-\alpha}\left(\frac{K}{H}\right)^{-(1-\alpha)}\left(\frac{v^{\star}}{z}\right)^{\beta}+\delta \tag{80}$$

$$\frac{\dot{\mu}}{\mu}=-B+\delta \tag{81}$$

Defining the ratios $\omega=\frac{K}{H}$ and $\chi=\frac{C}{K}$, setting the state variable $\omega\left(0\right)=\omega_0$, and the steady state growth rates of ω and χ are equal to zero, the transversality condition that $lim_{t\to\infty}\nu K exp-\rho t=0$, then the steady state values can be defined as:

$$\chi=\left(1-\tau\right)\left[\frac{\delta\left(1-\theta\right)+\rho+\theta B\left(1-z\right)}{\alpha}\right]-B\left(1-z\right) \tag{82}$$

$$\omega=\left[\frac{\alpha A\left(\frac{v}{z}\right)^{\beta}}{\delta\left(1-\theta\right)+\rho+\theta B\left(1-z\right)}\right]^{\frac{1}{1-\alpha}}z \tag{83}$$

C. Changes in Government Policy - An Application

This section solves for the Balanced Growth path growth rates where the share of human capital and the share of government expenditure in both sectors are exogenously determined. We begin with the current value Hamiltonians as below:

$$\begin{aligned}J=u\left(C\right)+\nu\left[AK^{\alpha}\left(vG\right)^{\beta}\left(zH\right)^{1-\alpha-\beta}-C-G-\delta K\right]\\+\mu\left[B\left[\left(1-z\right)H\right]^{1-\eta}\left[\left(1-v\right)G\right]^{\eta}-\delta H\right]\end{aligned} \tag{84}$$

Households then choose consumption and take the shares of factor inputs into the production of goods and human capital sector as given. The First Order Condition that a change in the Hamiltonians due to a change in consumption is equal to zero and the co-state conditions that $\dot{v}=-\frac{\partial J}{\partial K}$ and $\dot{\mu}=\frac{\partial J}{\partial H}$ are as given in equations (53), (56) and (57).

Expressing the system in terms of growth rates of K, H, C, ν and μ will yield the following equations:

$$\frac{\dot{K}}{K} = (1-\tau)\, A z^{1-\alpha} \left(\frac{K}{H}\right)^{-(1-\alpha)} \left(\frac{v}{z}\right)^{\beta} - \frac{C}{K} - \delta \tag{85}$$

$$\frac{\dot{H}}{H} = B\,(1-z) \left(\frac{1-v}{1-z}\right)^{\eta} - \delta \tag{86}$$

$$\frac{\dot{C}}{C} = \frac{1}{\theta} \left(\alpha A z^{1-\alpha} \left(\frac{K}{H}\right)^{-(1-\alpha)} \left(\frac{v}{z}\right)^{\beta} - \delta - \rho\right) \tag{87}$$

$$\frac{\dot{\nu}}{\nu} = -\alpha A z^{1-\alpha} \left(\frac{K}{H}\right)^{-(1-\alpha)} \left(\frac{v}{z}\right)^{\beta} + \delta \tag{88}$$

$$\frac{\dot{\mu}}{\mu} = -\,(1-\eta)\, B \left(\frac{1-v}{1-z}\right)^{\eta} + \delta \tag{89}$$

The system can be reduced to two equations in two unknowns. Defining the ratios $\omega = \frac{K}{H}$ and $\chi = \frac{C}{K}$, setting the state variable $\omega(0) = \omega_0$, and the steady state growth rates of ω and χ are equal to zero, the transversality condition that $lim_{t\to\infty} \nu K exp - \rho t = 0$, then the steady state values of ω and χ can be defined as:

$$\chi = (1-\tau) \left[\frac{\delta\,(1-\theta) + \rho + \theta B\,(1-z)\left(\frac{1-v}{1-z}\right)^{\eta}}{\alpha}\right] - B\,(1-z)\left(\frac{1-v}{1-z}\right)^{\eta} \tag{90}$$

and

$$\omega = \left[\frac{\alpha A \left(\frac{v}{z}\right)^{\beta}}{\delta\,(1-\theta) + \rho + \theta B\,(1-z)\left(\frac{1-v}{1-z}\right)^{\eta}}\right]^{\frac{1}{1-\alpha}} z \tag{91}$$

Inserting these values into the growth rates of capital, human capital, consumption and income give us the growth rates in the Balanced Growth Path.