Heterogeneous Preferences and Asset Prices under Endogenously Incomplete Markets*

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Abstract

In this paper, I study how heterogeneous preferences (heterogeneity in risk aversion and time discount factor) affect asset prices and risk sharing in a two-agent endowment economy, when markets are endogenously incomplete due to the contracting friction of limited enforcement. I find that heterogeneous preferences lead to more conditional variation in the stochastic discount factor (SDF), which results in a higher and more volatile equity premium. In contrast to the standard findings under heterogeneous preferences, the long run distribution of agents’ consumption is stationary and nondegenerate, since limited enforcement entitles the agents to the option of autarky for all times.

Keywords: Heterogeneous Preferences, Equity Premium, Limited Enforcement, Stochastic Discount Factor

JEL : G12, E44, D5

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1 Introduction

Since Mehra and Prescott (1985) document the famous equity premium puzzle, several outstanding explanations for the puzzle have been proposed, such as Campbell and Cochrane (1999) external habit, Bansal and Yaron (2004) long run risk, and Barro (2006) rare disaster. Meanwhile, the strand of literature in heterogeneous agents with incomplete markets and portfolio constraints only partially resolve the puzzle with a modest equity premium, see for example Heaton and Lucas (1996) and Alvarez and Jermann (2001). This paper argues that introducing heterogeneous preferences helps this literature towards the fully resolution of the equity premium puzzle.

In this paper, I introduce heterogeneity in risk aversion and time discount factor into Alvarez and Jermann (2001), which feature a two-agent endowment economy with idiosyncratic income risk, aggregate risk, and endogenously incomplete markets due to the contracting friction of limited enforcement. I study the corresponding implications for asset pricing and risk sharing. I find that preference heterogeneity boosts the mean and volatility of equity premium quantitatively. I also find that the long run distribution of agents’ consumption is stationary, in contrast to the conventional wisdom under heterogeneous preferences. The reason is that limited enforcement entitles the agents to the option of autarky for all times.

Limited enforcement (Kehoe and Levine (1993); Alvarez and Jermann (2000)) means that a risk-sharing contract between agents cannot be fully enforced in the form of constraints for the agents that staying in the contract should be not worse than defaulting on the contract, whereas defaulting means being excluded from financial markets permanently and staying in autarky forever. Alvarez and Jermann (2001) show that endogenously incomplete markets generated by this contracting friction lead to limited risk sharing and a volatile SDF. The two preference parameters—risk aversion and time discount factor—are the key determinants of the amount of risk shared and the properties of the SDF. As the recent papers, such as

\footnote{See Cochrane (2017) for an overview of this macro asset pricing literature.}
Xiouros and Zapatero (2010), Bhamra and Uppal (2014), and Gărleanu and Panageas (2015), show that preference heterogeneity has substantial impact on asset prices, it seems natural to explore how preference heterogeneity may affect risk sharing and asset prices when limited enforcement is present and ensures a nondegenerate stationary equilibrium.

First, I derive the risk sharing implications of embedding preference heterogeneity into the economy with limited enforcement by studying the evolution of the relative Pareto weight (RPW) of the two agents over time. I show that the RPW is affected by two factors related to preference heterogeneity, i.e., the relative ratio of time discount factors of the two agents, and the interaction between heterogeneous risk aversion and aggregate risk. All else equal, the more patient agent tends to have a higher RPW, and the less risk-averse agent tends to have a higher RPW in booms and a lower RPW in recessions. This is absent in Ligon, Thomas, and Worrall (2002), because in their economy there is no aggregate risk and agents have the same time preference.

Ligon et al. prove that there exists an interval for the RPW to fall into for each state, and the RPW takes only boundary values of these intervals in the long run if agents cannot fully share their income risk. When heterogeneous preferences and aggregate risk are present, I show that the RPW takes not only boundary values of those intervals as in Ligon et al., but also certain values inside those intervals. In addition, the RPW does not go to zero or infinity—i.e., no agents die or dominate in the long run, since enforcement constraints entitle the agents to the option of autarky for all times, and agents can consume their non-zero endowment.

Next, I demonstrate that preference heterogeneity combined with limited enforcement generates a positive equity premium (or term premium here) in a two-state example with only idiosyncratic income risk (i.e., no aggregate risk), while agents’ endowment shares are symmetrically distributed. Enforcement constraints induce discount rate shocks as the marginal pricer changes over time, depending on which agent is not constrained. When agents have symmetric endowment and the same preference parameters, the SDF is also
symmetric across agents. Although the marginal pricing agent is changing, the conditional SDF does not vary. Thus there is no time variation in the price-dividend ratio, and the equity premium is zero.

Introducing heterogeneous preferences breaks down the symmetry of the SDF across agents. The conditional SDF varies, resulting in a time-varying price-dividend ratio and a positive equity premium. In contrast to homogeneous risk aversion, the amount of risk sharing increases little for the low-risk-aversion agent, but decreases dramatically for the high-risk-aversion agent, leading to a high consumption volatility for the latter. When the more risk-averse agent is unconstrained and becomes the marginal pricer, the SDF is large and volatile, resulting in a sizable equity premium. As for the case of heterogeneous time preference, the more patient agent has a greater chance to be the marginal pricer due to his higher patience level, even though the two agents have symmetric endowment distributions. As a result, the SDF is more affected by the more patient agent. When he cannot trade away most of his income risk with the less patient agent, the equity premium is high.

Last, I use the recursive Lagrangian method of Marct and Marimon (2016) to solve a calibrated model with both idiosyncratic and aggregate risk. As the intuition of the two-state example carries through, I show that heterogeneous preferences generate asymmetric risk sharing across agents and lead to a higher and more volatile equity premium than homogeneous preferences. Undesirably, the risk-free rate could be too volatile. In addition, heterogeneous time preference shows more promise than heterogeneous risk aversion for better matching asset pricing moments. In particular, the former could produce 7.05% mean equity premium and 27.87% equity volatility, with a moderate risk aversion around 3 and reasonable heterogeneous time discount factors 0.85 and 0.75 for the two agents.

This paper’s contribution to asset pricing literature is to show that idiosyncratic income

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2I use CRRA (constant relative risk aversion) utility. Heterogeneous risk aversion implies heterogeneous EIS (elasticity of intertemporal substitution). The problem might be solved by Epstein-Zin preference, which separates EIS and relative risk aversion so that agents can have heterogeneous risk aversion, but the same EIS at the same time. I do not pursue it here, however, because of the associated computational difficulties for the combination of Epstein-Zin and enforcement constraints.
risk and incomplete markets become more relevant for asset prices when agents have heterogeneous preferences. Alvarez and Jermann (2001) and Krueger and Lustig (2010) show that idiosyncratic income risk does not affect (multiplicative) equity premium if its distribution is independent of aggregate risk and aggregate risk is i.i.d. over time. I show that preference heterogeneity renders the discount rate shocks induced by enforcement constraints to become asymmetric across agents, leading to time variation in the conditional SDF and price-dividend ratio, and thus a non-zero equity premium even without any aggregate risk present. Quantitatively, heterogeneous time preference performs better than heterogeneous risk aversion at matching asset pricing moments when aggregate risk is present. I show that the SDF under heterogeneous time preference is equivalent to one in a representative agent economy with a time-varying time discount factor. This relates to the recent Albuquerque, Eichenbaum, Luo, and Rebelo (2015), which show that a representative agent model with exogenous time preference shocks accounts for key asset pricing moments. My model gives an endogenous interpretation of the time preference shocks as being driven by enforcement constraints and the time-varying marginal pricing agent.

As for the paper’s contribution to risk sharing literature, I generalize the theoretical results of Ligon, Thomas, and Worrall (2002) on the evolution of the RPW over time under limited enforcement to include preference heterogeneity and aggregate risk. In particular, I show that the relative time discount factor and the interaction between heterogeneous risk aversion and aggregate risk affect the evolution of the RPW.

This paper is related to several strands of literature. First, it is related to the literature on heterogeneous preferences and asset prices. For early contributions, see Dumas (1989) and Wang (1996). The two papers, along with Basak and Cuoco (1998), draw an undesirable implication: with positive growth, less risk-averse agents will dominate the economy in the long run. To ensure stationarity, the literature has introduced habit into preferences (Chan and Kogan (2002), Xiouros and Zapatero (2010), and Bhamra and Uppal (2014)) or overlapping generations with agents vanishing each period (Gomes and Michaelides (2008) and
In this paper, enforcement constraints naturally emerge as a device to ensure stationary long-run distribution, as agents always have the option to choose autarky and would not end up with zero consumption. In addition, the papers cited above examine how heterogeneous preferences affect asset prices with only aggregate risk. In contrast, I include both aggregate risk and idiosyncratic risk and show explicitly how the two risks and preference heterogeneity are intertwined through the evolution of the RPW to determine consumption allocations, risk sharing, and asset prices.

Second, this paper relates to the literature on limited enforcement. For early contributions, see Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000). Alvarez and Jermann (2001) study its implications for asset pricing. Krueger, Lustig, and Perri (2008) test the SDF generated from limited enforcement and find support in U.S. consumption data. Beker and Espino (2015) incorporate heterogeneous beliefs into Alvarez and Jermann (2001) to explain return momentum and reversals. I also build on Alvarez and Jermann (2001) and examine how preference heterogeneity affects risk sharing and asset prices. Chien and Lustig (2010) show that less risk sharing can be sustained by allowing agents to file for bankruptcy instead of excluding them from financial markets forever, and this improves the asset pricing predictions. Ai and Bhandari (2016) study the asset pricing implications of uninsurable tail risk in labor productivities when markets are endogenously incomplete due to principal-side limited commitment. Cao (2014) shows that agents with incorrect beliefs survive by holding on to their nonfinancial wealth under limited commitment.


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4There is also a literature on heterogeneous beliefs and asset prices. Borovika (2016) shows that when agents have heterogeneous beliefs, Duffie-Epstein-Zin preferences lead to long-run outcomes in which both agents survive or more incorrect agents dominate. See that paper for more references on heterogeneous beliefs.

5Gomes and Michaelides (2008) is an exception.
heterogeneous preferences using structural estimation with Indian village consumption data. There is no aggregate risk in Laczó’s model. And her focus is risk sharing; my focus is asset pricing. Krueger and Perri (2006) study the implications of limited enforcement for consumption inequality. Rampini and Viswanathan (2016) show that limited enforcement can explain household’s insurance pattern. In addition, see Kehoe and Perri (2002) for an international application.

This paper is also related to the literature on how incomplete markets and portfolio constraints affect asset prices. Mankiw (1986) and Constantinides and Duffie (1996) show that equity premium will increase if the cross-sectional volatility of non-tradable idiosyncratic risk is higher in recessions. The main difference is that in my model the markets are endogenous incomplete due to enforcement constraints, while the literature usually assumes exogenous incomplete markets due to limited securities to trade, trade frictions, or exogenous borrowing constraints, etc. Storesletten, Telmer, and Yaron (2007) extend the endowment economy of Constantinides and Duffie (1996) to overlapping generations, along with capital accumulation. They show that idiosyncratic risk inhibits the intergenerational sharing of aggregate risk, but capital accumulation mitigates it by providing self-insurance. Chabakauri (2013) finds that tighter margins and leverage constraints generate higher risk premia. Rytchkov (2014) finds that state-dependent and time-varying margin constraints reduce risk-free rate, but increase risk premium.

This paper proceeds as follows. Section 2 outlines the model. Section 3 uses a two-state example to gain the intuition of the model. Section 4 presents calibration and quantitative results, and Section 5 concludes.

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2 Model

In this section, I outline the model. I first adopt the promised utility approach to formulate the contracting problem, derive the evolution of the RPW, and show its long-run properties. I then use the recursive Lagrangian method of Marcet and Marimon (2016) (see also Kehoe and Perri (2002)) to set up the planner’s problem and present a computation algorithm to solve the consumption allocations. Last, I decentralize the economy, following Alvarez and Jermann (2000), and pin down the asset prices with the solved allocations.

2.1 Environment

Two agents are endowed with a random stream of income \( e_{it}, \ i = 1, 2 \). Aggregate endowment is \( e_t = e_{1t} + e_{2t} \) and grows over time \( g_{t+1} = e_{t+1}/e_t \). Agents’ income share is \( \hat{e}_{it} = e_{it}/e_t \). I assume that \( z = (g, \hat{e}) \) jointly follows a finite-state Markov process. I denote \( z^t = (z_0, z_1, z_2, ..., z_t) \) as the history up to \( t \). Transition probabilities from \( t - 1 \) to \( t \) are denoted as \( \pi(z^t | z^{t-1}) \).

Agents have CRRA utility, but may differ in the relative risk aversion coefficient and time discount factor. The utility function is

\[
U_i(t) = \frac{c_i^{1-\gamma_i}}{1-\gamma_i},
\]

and lifetime utility is defined as

\[
U_i^t(c_i) \equiv E_t \sum_{j=0}^{\infty} \beta_i^j u_i(c_{it+j}),
\]

where \( c_{it} \) is agent \( i \)'s consumption at time \( t \), \( \gamma_i \) is the relative risk aversion coefficient, and \( \beta_i \) is the time discount factor. Contracts are not fully enforceable. If the agent defaults, he will be excluded from financial markets forever and remain in autarky. Agents’ consumption
choices satisfy the participation constraints,

\[ U^t_i(c_i) \geq U^t_i(e_i), \quad i = 1, 2, \quad t = 0, 1, 2, \ldots, \]

where \( U^t_i(e_i) \) is the utility value of autarky starting from time \( t \). Note that for notational convenience, I write out the states \( z' \) only when it is necessary to avoid confusion.

### 2.2 Promised Utility Formulation

I take the promised utility approach to formulate the contracting problem. Pareto frontiers should satisfy

\[
V(w, z) = \max_{c, w', z'} \{ u(c_1) + \beta_1 \sum_{z' \in Z} \pi(z'|z) V(w'(z'), z') \}
\]

\[
c_1 + c_2 \leq e,
\]

\[
\lambda : \quad u(c_2) + \beta_2 \sum_{z' \in Z} \pi(z'|z) w'(z') \geq w,
\]

\[
\beta_2 \pi(z'|z) \eta^2(z') : \quad w'(z') \geq U^2(z'), \quad z' \in Z,
\]

\[
\beta_1 \pi(z'|z) \eta^1(z') : \quad V(w'(z'), z') \geq U^1(z'), \quad z' \in Z,
\]

where \( V \) is the lifetime utility of agent 1, \( w \) is the promised life-time utility to agent 2, \( \lambda \) is the Lagrangian multiplier on the promise-keeping constraint, and \( \eta^i(z') \) is the state-dependent Lagrangian multiplier on agent \( i \)'s participation constraints.

The first-order conditions are

\[
\frac{u^1(c_1)}{u^2(c_2)} = \lambda, \quad (2.1)
\]

\[
-V'(w'(z'), z') = \frac{\lambda + \eta^2(z') \beta_2}{1 + \eta^1(z') \beta_1},
\]
together with the envelope condition

\[ -V'(w, z) = \lambda. \]

There is aggregate growth in the economy. To obtain a stationary economy, I normalize some variables as follows:

\[
\hat{c}_t \equiv \frac{c_t}{e_t}, \quad \hat{e}_t \equiv \frac{e_t}{e_t}, \\
\hat{w} \equiv \frac{w}{e^{1-\gamma}}, \quad \hat{V}(\hat{w}, z) \equiv \frac{V(w, z)}{e^{1-\gamma}}, \\
\hat{U}_i(z) \equiv \frac{U_i(z)}{e^{1-\gamma}}, \\
\hat{\pi}_i(z'|z) \equiv \frac{\pi(z'|z)}{\sum_{z'} \pi(z'|z)g(z')^{1-\gamma}}, \\
\hat{\beta}_i(z) \equiv \beta_i \sum_{z'} \pi(z'|z)g(z')^{1-\gamma}.
\]

The original recursive problem can then be rewritten as

\[
\hat{V}(\hat{w}, z) = \max_{\hat{c}_1, \hat{w}'(z')} \{ u(\hat{c}_1) + \hat{\beta}_1 \sum_{z' \in Z} \hat{\pi}_1(z'|z)\hat{V}(\hat{w}'(z'), z') \}
\]

\[
\hat{c}_1 + \hat{c}_2 \leq 1, \\
\hat{\lambda} : \quad u(\hat{c}_2) + \hat{\beta}_2 \sum_{z' \in Z} \hat{\pi}_2(z'|z)\hat{w}'(z') \geq \hat{w}, \\
\hat{\beta}_2 \hat{\pi}_2(z'|z) \hat{\eta}_2(z') : \quad \hat{w}'(z') \geq \hat{U}_2(z'), \quad z' \in Z, \\
\hat{\beta}_1 \hat{\pi}_1(z'|z) \hat{\eta}_1(z') : \quad \hat{V}(\hat{w}'(z'), z') \geq \hat{U}_1(z'), \quad z' \in Z,
\]

where \( \hat{\lambda} \) and \( \hat{\eta}_i(z') \) are Lagrangian multipliers on the constraints after normalization.

The first-order conditions are

\[
\frac{u_1(\hat{c}_1)}{u_2(\hat{c}_2)} = \hat{\lambda}, \quad (2.2)
\]
\[-\dot{V}'(\dot{w}'(z'), z') = \frac{\hat{\lambda} + \hat{\eta}^2(z') \beta \hat{\pi}_2}{1 + \hat{\eta}^1(z') \beta \hat{\pi}_1} \]
\[= \frac{\hat{\lambda} + \hat{\eta}^2(z') \beta \hat{\pi}_2}{1 + \hat{\eta}^1(z') \beta \hat{\pi}_1} g(z')^{\gamma_1 - \gamma_2}, \quad (2.3)\]

together with the envelope condition

\[-\dot{V}'(\dot{w}, z) = \hat{\lambda}. \quad (2.4)\]

It is easy to see from equations 2.1 and 2.2 that

\[\hat{\lambda} = \frac{\lambda}{c_i^{\gamma_2 - \gamma_1}}.\]

\(\hat{\lambda}\) equals the marginal utility of consumption share, which is also the (normalized or preference-adjusted) RPW (of agent 2 with respect to agent 1) in the planner’s problem, which will be discussed in subsection 2.4.\(^7\) Combining equations 2.3 and 2.4, I obtain

\[\hat{\lambda}' = \frac{\hat{\lambda} + \hat{\eta}^2(z') \beta \hat{\pi}_2}{1 + \hat{\eta}^1(z') \beta \hat{\pi}_1} g(z')^{\gamma_1 - \gamma_2}. \quad (2.5)\]

The above equation describes how idiosyncratic risk, aggregate risk, and preference heterogeneity are intertwined to influence RPW and, thus, consumption allocations, risk sharing, and asset prices. When agent \(i\) gets a high idiosyncratic income shock tomorrow, his participation constraint binds, \(\hat{\eta}^i > 0\), and his RPW rises. When both agents are not constrained, \(\hat{\eta}^1 = \hat{\eta}^2 = 0\), the RPW of the more patient agent will increase due to the term \(\beta_2 / \beta_1\), and the RPW of the less risk-averse agent will rise in booms and decline in recessions due to the term \(g(z')^{\gamma_1 - \gamma_2}\).

2.3 Risk Sharing

I characterize the evolution of \(\hat{\lambda}\) over time and its long-run properties.

\(^7\)For convenience, I call \(\hat{\lambda}\) RPW hereafter.
Proposition 1. Suppose agents can have heterogeneous preferences. A constrained-efficient contract can be characterized as follows: There exist $S$ state-dependent intervals $[\lambda_s, \bar{\lambda}_s]$, $s = 1, 2, ..., S$ such that given $\lambda_t$ and next period occurring state $s$, $\lambda_{t+1}$ updates as:

$$
\hat{\lambda}_{t+1} = \begin{cases} 
\lambda_s, & \text{if } \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} < \lambda_s \\
\hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}}, & \text{if } \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} \in [\lambda_s, \bar{\lambda}_s] \\
\bar{\lambda}_s, & \text{if } \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} > \bar{\lambda}_s.
\end{cases}
$$

Proof. The proof follows Ligon, Thomas, and Worrall (2002). First, there exist $S$ state-dependent intervals $[V_s, \bar{V}_s]$ and $[w_s, \bar{w}_s]$ for agent 1's and agent 2's possible lifetime utility values, following Thomas and Worrall (1988). Obviously, $V_s = \bar{U}_s^1$ and $w_s = \bar{U}_s^2$. Second, let $\lambda_s = -\hat{V}'(w_s)$ and $\bar{\lambda}_s = -\hat{V}'(\bar{w}_s)$. Then considering the three cases

- If $\hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} < \lambda_s$, $\hat{\lambda}_{t+1} \in [\lambda_s, \bar{\lambda}_s] > \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}}$. Equation 2.5 then implies $\hat{n}_s^2 > 0$. Thus $\hat{w}_s = w_s$. Hence $\hat{\lambda}_{t+1} = -\hat{V}'(w_s) = \lambda_s$.

- If $\hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} > \bar{\lambda}_s$, $\hat{\lambda}_{t+1} \in [\lambda_s, \bar{\lambda}_s] < \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}}$. Equation 2.5 then implies $\hat{n}_s^1 > 0$. Thus $\hat{V}_s = V_s$ and $\hat{w}_s = \bar{w}_s$. Hence $\hat{\lambda}_{t+1} = -\hat{V}'(\bar{w}_s) = \bar{\lambda}_s$.

- If $\hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}} \in [\lambda_s, \bar{\lambda}_s]$, it must have $\hat{n}_s^1 = \hat{n}_s^2 = 0$, and hence $\hat{\lambda}_{t+1} = \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}}$. Suppose the contrary, $\hat{n}_s^2 > 0$ and $\hat{n}_s^1 = 0$. Then $\hat{w}_s = w_s$ and $\hat{\lambda}_{t+1} = \lambda_s$. But equation (2.5) implies $\hat{\lambda}_{t+1} > \hat{\lambda}_t g_s^{\gamma_1-\gamma_2 \frac{\beta_2}{\beta_1}}$. Contradiction. The symmetric argument holds for $\hat{n}_s^2 = 0$ and $\hat{n}_s^1 > 0$. And it is not possible that $\hat{n}_s^2 > 0$ and $\hat{n}_s^1 > 0$.

$\square$

Corollary 1. Suppose agents have the same preferences, i.e., $\beta_1 = \beta_2, \gamma_1 = \gamma_2$. Given $\hat{\lambda}_t$
and next period occurring state $s$, $\hat{\lambda}_{t+1}$ updates as:

\[
\hat{\lambda}_{t+1} = \begin{cases} 
\lambda_s, & \text{if } \hat{\lambda}_t < \lambda_s \\
\hat{\lambda}_t, & \text{if } \hat{\lambda}_t \in [\lambda_s, \bar{\lambda}_s] \\
\bar{\lambda}_s, & \text{if } \hat{\lambda}_t > \bar{\lambda}_s.
\end{cases}
\]

This is the same result as in Ligon, Thomas, and Worrall (2002). When there are no enforcement constraints, the first best is achieved and $\hat{\lambda}$ is constant over time, $\hat{\lambda}_{t+1} = \hat{\lambda}_t$ for all $t$. With participation constraints present, next period $\hat{\lambda}$ remains unchanged if possible, and changes the minimum amount to be inside the interval of the possible values of (preference-adjusted) RPW if this period $\hat{\lambda}$ is outside the interval. When heterogeneous preferences are present, the extra term $g_s^{\gamma_1 - \gamma_2} \frac{\beta_2}{\beta_1}$ shows up in the updating rule of $\hat{\lambda}$.

**Proposition 2.** Assume agents have heterogeneous preferences.

(i) Both agents survive in the long run, i.e., $\lim_{T \to \infty} \hat{\lambda}_T \neq 0$ and $\lim_{T \to \infty} \hat{\lambda}_T \neq \infty$.

(ii) There exists no constrained efficient contract that features constant consumption shares.

(ii) When the constrained efficient contract features limited risk sharing, the long-run ergodic set of the RPW is several certain boundary $\hat{\lambda}$s plus sets of other points inside $\hat{\lambda}$ intervals.

**Proof.**

(i) From Proposition 1, $\lim_{T \to \infty} \hat{\lambda}_T \in [\min_s \lambda_s, \max_s \bar{\lambda}_s]$. Thus, $\lim_{T \to \infty} \hat{\lambda}_T \neq 0$ and $\lim_{T \to \infty} \hat{\lambda}_T \neq \infty$.

(ii) From Corollary 1, if there is an overlapping interval for all $[\lambda_s, \bar{\lambda}_s]_{s=1}^S$, a constrained efficient contract achieves constant consumption under homogeneous preferences. From Proposition 1, even if an overlapping interval exists, there is some positive probability that $\lim_{T \to \infty} \hat{\lambda}_T$ will be outside the interval under heterogeneous preferences, due to the term $g_s^{\gamma_1 - \gamma_2} \frac{\beta_2}{\beta_1}$. Hence there exists no constrained efficient contract that features constant consumption shares.

(ii) From Corollary 1, $\lim_{T \to \infty} \hat{\lambda}_T$ will only take values from $\{\lambda_s, \bar{\lambda}_s\}_{s=1}^S$ when agents have homogeneous preferences. From Proposition 1, $\lim_{T \to \infty} \hat{\lambda}_T$ will take values not only from
\{\lambda_s, \bar{\lambda}_s\}_{s=1}^S \text{ but also inside } [\lambda_s, \bar{\lambda}_s]_{s=1}^S \text{ due to the term } g_s^{\gamma_1 - \gamma_2} \frac{\beta_s}{\beta_1}, \text{ when agents have heterogeneous preferences.} \\

2.4 Recursive Lagrangian Formulation

One difficulty inherent to computation associated with the promised utility approach is finding the maximum promised utility \(\bar{w}_s\) for each state \(s\), which is endogenous determined. Thus I resort to the relatively manageable recursive Lagrangian approach—introducing the RPW as the co-state variable—to compute the model solutions. The section on the promised utility approach is retained to facilitate proofs for the evolution of the RPW.

Given initial Pareto weight \(\phi_i\), the planner’s problem is

\[
\max_{\{c_{1t},c_{2t}\}} \left[ E_0 \sum_{t=0}^{\infty} \sum_{i=1,2} \beta_t^i \phi_i u^i(c_{it}) \right] \\
U_t^i(c_i) \geq U_t^i(e_i), \quad i = 1, 2, \quad t = 0, 1, 2, ... \\
c_{1t} + c_{2t} = e_t, \quad t = 0, 1, 2, ...
\]

I follow Marcet and Marimon (2016) (see also Kehoe and Perri (2002)) and use a recursive Lagrangian formulation to solve the problem. Let the Lagrange multiplier on the enforcement constraints be \(\{\mu_{it}\}_{t=0}^{\infty}\). The Lagrangian is

\[
\mathcal{L} = \sum_{i=1,2} E_0 \sum_{t=0}^{\infty} \beta_t^i \left[ \phi_i u^i(c_{it}) + \mu_{it} \left[ E_t \sum_{j=0}^{\infty} \beta_t^j u^i(c_{it+j}) - U_t^i(e_i) \right] \right].
\]

By Abel’s summation, \(\mathcal{L}\) can be simplified as

\[
\mathcal{L} = \sum_{i=1,2} E_0 \sum_{t=0}^{\infty} \beta_t^i \left[ H_{it} u^i(c_{it}) - \mu_{it} U_t^i(e_i) \right] \\
H_{it} = H_{it-1} + \mu_{it} \\
H_{i,1} = \phi_i,
\]
where $H_{it}$ is the time-varying Pareto weight for agent $i$ at $t$ and it equals the sum of initial Pareto weight $\phi_i$ and all historical multipliers up to $t$, $\{\mu_{i,s}\}_{s=0}^t$. First-order conditions wrt. consumption imply

$$\frac{u^1_i(c_{1t})}{u^2_i(c_{2t})} = \frac{H_{2t}}{H_{1t}} \left( \frac{\beta_2}{\beta_1} \right)^t,$$

where $u^i(c_{it})$ denotes the marginal utility of consumption for agent $i$. Redefine the multipliers,

$$\lambda_t = \frac{H_{2t}}{H_{1t}} \left( \frac{\beta_2}{\beta_1} \right)^t, \quad \nu_{it} = \frac{\mu_{it}}{H_{it}},$$

where $\lambda_t$ is the RPW of agent 2 to agent 1. The law of motion for this can then be derived as

$$\lambda_t = \frac{1 - \nu_{1t}}{1 - \nu_{2t}} \lambda_{t-1} \frac{\beta_2}{\beta_1}.$$

When agent $i$’s enforcement constraint binds, $\nu_{it} > 0$, his RPW goes up. When both agents are not constrained, $\nu_{it} = 0 \ i = 1, 2$, the more patient (higher $\beta$) agent has an increase in his RPW.

**Definition 1.** Given initial Pareto weights $\phi_i$, constrained efficient allocations for the growth economy are $\{c_1, c_2, \nu_1, \nu_2, \lambda\}$, which satisfy equations (2.6), (2.7), complementary slackness condition (2.8), and resource constraint (2.9):

$$\frac{u^1_i(c_{1t})}{u^2_i(c_{2t})} = \lambda_t, \quad (2.6)$$

$$\lambda_t = \frac{1 - \nu_{1t}}{1 - \nu_{2t}} \lambda_{t-1} \frac{\beta_2}{\beta_1}, \quad (2.7)$$

$$\lambda_{-1} = \frac{\phi_2}{\phi_1},$$

$$\nu_{it} \geq 0, \quad \nu_{it} \left[ U_t^i(c_t) - U_t^i(e_t) \right] = 0, \quad (2.8)$$

$$c_{1t} + c_{2t} = e_t. \quad (2.9)$$
To obtain a stationary economy, I normalize some variables as follows:

\[
\hat{c}_{it} \equiv \frac{c_{it}}{e_t}, \quad \hat{e}_{it} \equiv \frac{e_{it}}{e_t}, \\
\hat{\lambda}_t \equiv \frac{\lambda_t}{\gamma_2 - \gamma_1}, \quad \hat{\lambda}_{-1} \equiv \lambda_{-1}, (e_{-1} \equiv 1) \\
\hat{\lambda}_{-1} \equiv \phi_2 \phi_1, \\
\hat{U}_{it}(\hat{c}_i) \equiv \frac{U_i(c_i)}{e_t^{1 - \gamma_i}}, \quad \hat{U}_{it}(\hat{e}_i) \equiv \frac{U_i(e_i)}{e_t^{1 - \gamma_i}}, \\
\hat{\pi}_t(z_{t+1}|z_t) \equiv \frac{\pi(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}}{\sum_{z_{t+1}} \pi(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}}, \\
\hat{\beta}_t(z_t) \equiv \beta_t \sum_{z_{t+1}} \pi(z_{t+1}|z_t)g_{t+1}^{1-\gamma_i}.
\]

**Definition 2.** Given the initial Pareto weights \(\phi_i\), constrained efficient allocations for the stationary economy are \(\{\hat{c}_1, \hat{c}_2, \nu_1, \nu_2, \hat{\lambda}\}\), which satisfy equations (2.10), (2.11), complementary slackness condition (2.12), and resource constraint (2.13):

\[
\frac{u_1^i(\hat{c}_t)}{u_2^i(\hat{c}_t)} = \hat{\lambda}_t, \\
\hat{\lambda}_t = \frac{1 - \nu_{1t} \hat{\lambda}_{-1} g_{t+1}^{\gamma_2 - \gamma_1} \beta_2}{1 - \nu_{2t}}, \\
\hat{\lambda}_{-1} = \phi_2 \phi_1, \\
\nu_{it} \geq 0, \quad \nu_{it} \left[ \hat{U}_i^i(\hat{c}_i) - \hat{U}_i^i(\hat{e}_i) \right] = 0, \\
\hat{c}_{1t} + \hat{c}_{2t} = 1.
\]

It is easy to derive the equivalence between the growth economy and stationary economy using the redefined variables. The recursive Lagrangian formulation is consistent with the promised utility formulation. In particular, \(\hat{\lambda}\), the normalized RPW in (2.11), is exactly the marginal utility ratio of consumption share in (2.5). One difference is that \(\hat{\lambda}\) is implied by the envelope condition in the promised utility formulation, while it is a state variable in the recursive Lagrangian formulation.
2.5 Computation

Denote state variables \( x = \{ \hat{\lambda}, z \} \), where \( z \) is the joint Markov process of idiosyncratic uncertainty and aggregate uncertainty, and the added co-state variable \( \hat{\lambda} \) is the RPW of agent 2 wrt. agent 1. Denote the set of policy functions and value functions as \( F(x) = \{ \hat{c}_i(x), \hat{\lambda}'(x), \nu_i(x), W_i(x) \} \), \( i = 1, 2 \), where

\[
W_i(x) = u(\hat{c}_i(x)) + \beta_i(z) \sum_{z'} \pi_i(z'|z) W_i(x').
\]

The computation algorithm follows several steps:

1. Set up a grid \( X \) over the state space.
2. Set the full risk-sharing solution as the initial guess \( F^0(x) \).
3. For each \( x \in X \), guess that neither enforcement constraint binds.
   
   3a. If satisfied, set the new policies and value functions \( F^1(x) \) to be \( F^0(x) \)
   
   3b. If agent 1’s or agent 2’s constraint is not satisfied, impose the binding constraint and recalculate the solution as \( F^1(x) \).
4. Iterate until \( |F^n(x) - F^{n-1}(x)| < \epsilon \).

Linear interpolation is used for states not on the grid points. See Appendix A.1 for more details.

2.6 Decentralization and Asset Prices

I follow Alvarez and Jermann (2000) to decentralize the economy.\(^8\)

**Definition 3.** A competitive equilibrium with solvency constraints \( \{ B_i \} \) that is not too tight for initial conditions \( \{ a_{i,0} \} \) has quantities \( \{ a_i \} \) and prices \( \{ q \} \) s.t.

\(^8\)Preference heterogeneity does not affect proofs for the welfare theorems.
1. For each $i$, given $\{a_{i,0}\}$ and prices $\{q\}$, $\{c_i, a_i'\}$ solves

$$J_i (a_i(z^t)) = \max \left\{ u^i(c_i(z^t)) + \beta_i E \left[ J_i (a_i(z^{t+1})) \right] \right\}$$

$$c_i(z^t) + \sum_{z^{t+1}|z^t} a_i(z^{t+1})q(z^{t+1}|z^t) = e_i(z^t) + a_i(z^t),$$

$$a_i(z^{t+1}) \geq B_i(z^{t+1}), \quad (2.14)$$

where $a_i$ is agent $i$’s Arrow security holdings, $q$ is the price of Arrow security, $B_i$ is agent $i$’s endogenous determined borrowing constraint, and $J_i$ is agent $i$’s value function.

2. Good market and asset markets clear,

$$\sum_i c_i(z^t) = e(z^t)$$

$$\sum_i a_i(z^{t+1}) = 0$$

3. Solvency constraint is not-too-tight,

$$J_i (B_i(z^{t+1})) = U^i_{t+1}(e_i).$$

The Euler equation can be derived as

$$q_t(z^{t+1}|z^t) = \beta_t \pi(z^{t+1}|z^t) \frac{u^i_t(c_{it+1}(z^{t+1}))}{u^i_t(c_{it}(z^t))} + \frac{\zeta_{it+1}(z^{t+1})}{u^i_t(c_{it}(z^t))},$$

where $\zeta_{it+1}(z^{t+1})$ is the Lagrangian multiplier on the solvency constraint (2.14). Since the two agents cannot have binding constraints at the same time, it follows that $\zeta_{1t+1} = 0$ or $\zeta_{2t+1} = 0$ or both. Thus

$$q_t(z^{t+1}|z^t) = \pi(z^{t+1}|z^t) \max_{i=1,2} \left( \beta_t \frac{u^i_t(c_{it+1}(z^{t+1}))}{u^i_t(c_{it}(z^t))} \right). \quad (2.15)$$

The agent whose solvency constraint is not binding has the highest intertemporal marginal
rate of substitution (IMRS) and prices the Arrow security at that state.

The asset pricing equation is

\[ E_t[M_{t+1} R_{t+1}] = 1, \]

where the SDF \( M_{t+1} \) is

\[ M_{t+1} = \max_{i=1,2} \left( \beta_i \frac{u_i(c_{i,t+1})}{u_i(c_{i,t})} \right) = \max_{i=1,2} \left( \beta_i \left( \frac{\hat{c}_{i,t+1}}{c_{i,t}} \right)^{-\gamma_i} \left( \frac{e_{t+1}}{e_t} \right)^{-\gamma_i} \right). \tag{2.16} \]

The risk-free rate is

\[ R_{f,t+1} = 1/E_t[M_{t+1}] \]

and the return of aggregate consumption claim is

\[ R_{s,t+1} = \frac{P_{t+1} + e_{t+1}}{P_t} = \frac{1 + \frac{P_{t+1}}{e_{t+1}} e_{t+1}}{P_t} \frac{P_{t+1}}{e_{t+1}} \]

where

\[ P_t = E_t \left[ M_{t+1} \left( 1 + \frac{P_{t+1}}{e_{t+1}} \right) \frac{e_{t+1}}{e_t} \right]. \]

\( P_t \) is the price of aggregate consumption claim. And the equity premium is defined as

\[ R_{e,t+1} = R_{s,t+1} - R_{f,t+1}. \]

I use the solution of consumption allocations from the planner’s problem to pin down the SDF, then the price-consumption ratio and asset returns.
3 Two-state Example

I introduce preference heterogeneity into the two-state example from Alvarez and Jermann (2001). I show that it induces discount rate shocks, which—combined with the limited risk sharing generated by enforcement constraints—generate a positive and volatile equity premium with the absence of aggregate risk.

Figure 1 compares the long-run consumption allocation under homogeneous risk aversion with that under heterogeneous risk aversion, where agent 2 has a higher risk aversion (2.7) than agent 1 (1.5). Agents have the same time discount factor, which is fixed at $\beta = 0.65$. With the same risk aversion, the two agents have symmetric consumption shares because transition probability and endowment shares are symmetric. The upper solid line shows the consumption share when the agent has a higher income share, while the lower dashed line shows the consumption share for a lower income share. As risk aversion increases, agents change from autarky to limited risk sharing to full risk sharing. The squares (for agent 1) and diamonds (for agent 2) present a particular case in which agents have different risk aversion parameters, i.e., $\gamma_1 = 1.5$ and $\gamma_2 = 2.7$. The more risk-averse agent 2 pays a premium to the less risk-averse agent 1 for insurance so that agent 2 consumes less than agent 1 in both states ($c_{1L} > c_{2L}$ and $c_{1H} > c_{2H}$). In addition, agent 1’s consumption allocation deviates little from autarky if they have had the same low risk aversion of 1.5, but much less risk is shared for agent 2 than when they have the same high risk aversion of 2.7.

[Insert Figure 1 about here]

[Insert Figure 2 about here]

Figure 2 compares the long-run consumption allocation under homogeneous time discount

---

9The transition probability and endowment shares are

$$
\begin{bmatrix}
0.75 & 0.25 \\
0.25 & 0.75
\end{bmatrix},
\begin{bmatrix}
\hat{e}_1(s_1) \\
\hat{e}_1(s_2)
\end{bmatrix} =
\begin{bmatrix}
0.641 \\
0.359
\end{bmatrix},
\begin{bmatrix}
\hat{e}_2(s_1) \\
\hat{e}_2(s_2)
\end{bmatrix} =
\begin{bmatrix}
0.359 \\
0.641
\end{bmatrix}.
$$

10Without aggregate risk, the equity premium is essentially the term premium of a perpetual consol bond.
factor with that under heterogeneous time discount factor where agent 1 has a higher patience level (0.7) than agent 1 (0.5). Agents have the same time risk aversion coefficient, which is fixed at $\gamma = 3.0$. Similarly, the two agents have symmetric consumption shares when they have the same patience level $\beta$. The upper solid (lower dashed) line shows the consumption share when the agent has a higher (lower) income share. As $\beta$ rises, agents change from autarky to limited risk sharing to full risk sharing. The squares (for agent 2) and diamonds (for agent 1) present a particular heterogeneous $\beta$ case $\beta_1 = 0.7$ and $\beta_2 = 0.5$. The more patient agent 1 pays a premium to the less patient agent 2 for insurance, so that agent 1 consumes less than agent 2 in both states ($c_{1L} < c_{2L}$ and $c_{1H} < c_{2H}$).

When agents have the same preference, there will be no risk premium because agents are symmetric and there is no aggregate risk. Although the agent who prices the asset is time-varying, the conditional SDF, and thus price-(aggregate) consumption ratio, do not change due to the symmetry assumption. But when agents have heterogeneous preferences, a positive risk premium instead results. Preference heterogeneity breaks the symmetry of SDF and induces conditional variation in SDF, and thus in price-consumption ratio. Therefore, preference heterogeneity generates asymmetric discount rate shocks and leads to positive equity premium, even without aggregate risk.

[Insert Table 1 about here]

Table 1 dissects how heterogeneous preference produces a positive equity premium. Panel A contrasts the case of heterogeneous $\gamma$ with homogeneous $\gamma$ while keeping $\beta$ fixed at 0.65. When agents have the same $\gamma$ (1.5 or 2.7), there is no variation in conditional SDF due to the symmetric transition probability matrix and symmetric endowment shares. Thus the stock return equals the risk-free rate ($R_s = R_f$) state by state, and there is no equity premium. When agents have different $\gamma$, however, the symmetry in SDF breaks down. Especially for the transition from $s_t = s_2$ to $s_{t+1} = s_1$, which is priced by the more risk-averse agent 2, SDF is much higher because he has high consumption volatility and higher risk aversion. This boosts the mean and volatility of SDF. Despite no aggregate risk, conditional variation
in SDF generates variation in the price-consumption ratio, resulting in a positive equity premium. Panel B contrasts the case of heterogeneous $\beta$ with homogeneous $\beta$ while keeping $\gamma$ fixed at 3.0. The results are similar to those in Panel A. When agents have the same $\beta$ (0.7 or 0.5), there is no variation in conditional SDF. Thus $R_s = R_f$ state by state, and there is no equity premium. But when agents have different $\beta$, the symmetry in SDF breaks down. Especially for the transition from $s_t = s_1$ to $s_{t+1} = s_2$, which is priced by the more patient agent 1, SDF is much higher due to high consumption volatility and higher patience. In addition, agent 1 is also the marginal pricer for the transition from $s_t = s_2$ to $s_{t+1} = s_2$, as he has a larger IMRS (larger $\beta$ and no consumption change for either agent). Although aggregate risk is not present, conditional variation in SDF generates variation in the price-consumption ratio and positive equity premium as a result.

4 Quantitative Results

4.1 Calibration

I assume that there are four states and agents are symmetric in their endowment processes. I follow the high $\beta$ annual calibration of Alvarez and Jermann (2001) with 10 moments to pin down 10 parameters, including 2 aggregate growth rates, 2 idiosyncratic income shares, 2 aggregate transition probabilities, and 4 idiosyncratic transition probabilities. See Appendix A.2 for details. For preference parameters $\beta$ and $\gamma$, I pick values around the numbers used in Alvarez and Jermann (2001) ($\beta = 0.78$ and $\gamma = 3.5$). I simulate the model 1,000 times and 5,000 periods each time, with the first 500 periods burned out to obtain the long-run distribution.

4.2 Heterogeneous Risk Aversion

Figure 3 shows how the degree of risk sharing measured by consumption volatility and SDF change when agents have heterogeneous risk aversion $\gamma$, but the same time preference $\beta$. 
I compare the case in which both agents have the same risk aversion of 2.5 or 4 with the case in which one agent has risk aversion $\gamma_1 = 2.5$ and the other has $\gamma_2 = 4$ for a range of $\beta$. When $\beta$ is small, risk sharing incentive is low. Agents stay in autarky and consumption volatility is high. As $\beta$ increases, the degree of risk sharing rises. When $\beta$ becomes large enough, agents achieve full risk sharing. The degree of risk sharing for the heterogeneous risk aversion case lies between the case of both low risk aversion and the case of both high risk aversion for most $\beta$. In particular, the amount of risk sharing does not increase much for the less risk-averse agent 1, but decreases a fair amount for the more risk-averse agent 2, compared with the case of homogeneous risk aversion. In addition, agent 2’s consumption profile is more volatile than agent 1’s. Note that when $\beta > 0.96$, heterogeneous risk aversion generates less risk sharing for both agents than homogeneous risk aversion. Even when $\beta$ is large enough, constant consumption shares (i.e., zero standard deviation for consumption shares) cannot be achieved, as proved by Proposition 2.

[Insert Figure 3 about here]

The magnitude of $\beta$ affects SDF through two opposing channels. On the one hand, higher $\beta$ leads to higher and more volatile SDF directly given individual consumption growth. On the other hand, higher $\beta$ induces more risk sharing, and thus lower consumption volatility, reducing the size and volatility of SDF. When $\beta$ is small, agents stay in autarky and only the first effect is present. Therefore, the mean and volatility of SDF increase with $\beta$. As $\beta$ becomes larger, the second effect dominates and the mean and volatility of SDF decline with $\beta$. For a range of moderate $\beta$, heterogeneous risk aversion generates higher and more volatile SDF than homogeneous risk aversion. Specifically, when $\beta$ is between 0.78 and 0.85, heterogeneous agents share limited risk, and the more risk-averse agent with volatile consumption pushes up the mean and volatility of SDF in the states in which he is not constrained and is the marginal pricer.

[Insert Figure 4 about here]
Figure 4 shows how the mean and volatility of asset returns change when agents have heterogeneous risk aversion but the same time preference. The two effects of $\beta$ on SDF are present inversely on the risk-free rate. The mean of $R_f$ ($E(R_f)$) first declines and then increases with $\beta$. But the volatility of $R_f$ ($\sigma(R_f)$) declines all the way down. This is because the direct effect of $\beta$ on $\sigma(R_f)$ always dominates the indirect effect. When $\beta$ lies in between 0.78 and 0.85, limited risk is shared and heterogeneous risk aversion produces lower $E(R_f)$ than homogeneous risk aversion. For most ranges of $\beta$, heterogeneity in risk aversion generates higher equity premium and higher equity volatility, because the more risk-averse agent 2 feels unsafe holding stocks in recessions when his income volatility is higher and he cannot trade away most of his income risk with the less risk-averse agent 1. In addition, $\sigma(R_f)$ is much higher when agents have different risk aversion. This results in an overly volatile risk-free rate.

### 4.3 Heterogeneous Time Preference

Figure 5 shows how the degree of risk sharing and SDF change when agents have heterogeneous time preference but the same risk aversion. I compare the case in which both agents have the same time preference $\beta$ of 0.75 or 0.85 with the case in which agent 1 has $\beta$ of 0.85, while agent 2 has 0.75 for each $\gamma$. When $\gamma$ is small, risk sharing incentive is low. Agents stay in autarky and consumption volatility is high. As $\gamma$ increases, the degree of risk sharing rises. When $\gamma$ becomes large enough, agents achieve full risk sharing. The degree of risk sharing for the heterogeneous $\beta$ case lies between the case of both low $\beta$ and the case of both high $\beta$ for $\gamma \in [3, 4.2]$. Specifically, the amount of risk sharing does not increase much for the less patient agent 2, but decreases a fair amount for the more patient agent 1, in contrast to the corresponding case of homogeneous $\beta$. In addition, agent 2’s consumption profile is more volatile than agent 1’s. Note that when $\gamma > 4.7$, the heterogeneous time discount factor generates less risk sharing for both agents than homogeneous time discount factors do. Even when $\gamma$ is large enough, constant consumption shares cannot be achieved,
as proved by Proposition 2.

[Insert Figure 5 about here]

The magnitude of $\gamma$ affects SDF through two opposing channels. On the one hand, higher $\gamma$ leads to higher and more volatile SDF directly given individual consumption growth. On the other hand, higher $\gamma$ induces more risk sharing and thus smaller individual consumption growth, reducing the size and volatility of SDF. When $\gamma$ is small, agents stay in autarky and only the first effect is present. Therefore, the mean and vol of SDF increase with $\gamma$. As $\gamma$ becomes larger, the second effect dominates and the mean and volatility of SDF decline with $\gamma$. For most $\gamma$, heterogeneous $\beta$ generates a higher mean of SDF, because the more patient agent will price more states including all transitions with no state change $s_{t+1} = s_t$, and thus no consumption change. Yet the volatility of SDF for heterogeneous $\beta$ is smaller than for same low $\beta$ when $\gamma \in [2.9, 4.7]$, because agents have higher consumption volatility in the latter case.

[Insert Figure 6 about here]

Despite the low SDF volatility, heterogeneous $\beta$ produces a high and volatile equity premium, as shown in Figure 6. There is a spike for mean equity premium $E(R_e)$ and equity return volatility $\sigma(R_s)$ at $\gamma$ values where agents change from autarky to little risk sharing. This is because when the more patient agent 1 receives a higher income share (and thus is not constrained) and prices the assets, he requires a high compensation for bearing risk, as little of his income risk can be traded away with the less patient agent 2. The pattern of the mean risk-free rate $E(R_f)$ corresponds inversely with the mean of SDF $E(M)$. Heterogeneous $\beta$ generates variation in conditional SDF, leading to an excessively volatile risk-free rate when $\gamma$ is large. As demonstrated by the two-state example, preference heterogeneity in $\gamma$ or $\beta$ renders discount rate shocks asymmetric, generates conditional SDF variation, and results in volatile asset returns.
4.4 Asset Pricing Moments

Table 2 presents the simulation moments for asset prices and consumption allocation. Panels A and B show the effect of heterogeneous risk aversion and heterogeneous time preference, respectively. For Panel A, $\beta$ is chosen to match the mean of risk-free rate $E(R_f)$ as closely as possible after $\gamma$ is chosen around the value 3.5 from Alvarez and Jermann (2001). For Panel B, $\gamma$ is chosen to match $E(R_f)$ as closely as possible after $\beta$ is chosen around the value 0.78 from Alvarez and Jermann (2001). Panel A shows that heterogeneity in $\gamma$ ($\gamma_1 = 2.5, \gamma_2 = 4.0$) increases the mean equity premium and equity volatility, 4.11% and 12.70%, respectively, in comparison with homogeneous low risk aversion ($\gamma_1 = \gamma_2 = 2.5$), 1.59% and 6.64%. But heterogeneous risk aversion does not necessarily improve over homogeneous high risk aversion ($\gamma_1 = \gamma_2 = 4.0$), where equity premium is slightly higher (4.52%) and equity volatility is slightly lower (10.74%) in the latter. In addition, the risk-free rate is too volatile (10.93%). Panel B shows that heterogeneity in $\beta$ ($\beta_1 = 0.85, \beta_2 = 0.75$) does not necessarily increase SDF volatility, but produces a much higher equity premium (7.05%) and more volatile equity returns (27.87%) than homogeneous $\beta$ (3.99% and 9.94% for $\beta_1 = \beta_2 = 0.75$ and 2.09% and 7.42% for $\beta_1 = \beta_2 = 0.85$). Moreover, risk-free rate volatility rises little (6.96% vs. 6.14% and 4.08%), although it is already more volatile than in the data (4.01%). Alvarez and Jermann (2001) matches the volatility of risk-free rate 5.67% from Mehra and Prescott (1985). Given that I follow their calibration, the heterogeneous $\beta$ case shows much promise for better matching the mean and volatility of the equity premium. That the risk-free rate is too volatile is because the main mechanism of the model originates from the discount rate shocks induced by enforcement constraints, which become asymmetric with the presence of preference heterogeneity. I mention a potential remedy for this in footnote 1.

[Insert Table 2 about here]

[Insert Table 3 about here]

As proved by Alvarez and Jermann (2001) and Krueger and Lustig (2010), when the dis-
tribution of idiosyncratic shocks is independent of aggregate shocks and aggregate shocks are i.i.d. (in short, as independent risk), the consumption share \( \hat{c} \) does not depend on aggregate uncertainty. Hence the term premium is zero and the multiplicative equity premium is the same as in a representative agent economy. Table 3 shows that heterogeneous preferences generate a positive term premium and higher equity premium than homogeneous preferences when risk is independent.\(^{11}\) Panel A shows the results for heterogeneous risk aversion when \( \beta \) is fixed at 0.5. When agents have the same \( \gamma = 2.5 \) or 3.5, the term premium is zero and the multiplicative equity premium is small (less than 0.5%). But when agents have different risk aversion (\( \gamma_1 = 2.5 \) and \( \gamma_2 = 3.5 \)), the term premium becomes 0.72% and the multiplicative equity premium rises to 2.44%. But all returns become quite volatile. Panel B shows the results for the case of heterogeneous time discount factor when \( \gamma \) is fixed at 3.0. Similarly, when agents have the same \( \beta = 0.45 \) or 0.55, the term premium is zero and the equity premium is small (less than 0.5%). But when agents have heterogeneous time preference (\( \beta_1 = 0.55 \) and \( \beta_2 = 0.45 \)), a small difference in \( \beta \) (0.1) causes a 4.37% term premium and a 7.72% multiplicative equity premium. In addition, the returns of equity and perpetual bond are much more volatile than the risk-free rate. In contrast to the case of heterogeneous \( \gamma \), heterogeneous \( \beta \) can generate a higher and more volatile equity premium without inducing too much volatility in the risk-free rate.

[Insert Table 4 about here]

In the benchmark model, all agents’ income is labor income; agents are not endowed with any assets at the beginning. I relax this assumption by letting the agents be endowed with both labor income and half of a Lucas tree initially. The Lucas tree bears fruits as dividend income, which is a constant fraction of total income, \( \omega = D_t/e_t \).\(^{12}\) The rest \( 1 - \omega \) fraction of \( e_t \) is labor income, and the agents’ shares of labor income are subject to idiosyncratic

\(^{11}\)The risk aversion and time discount factor parameters are chosen for qualitative illustration, not to match moments quantitatively.

\(^{12}\)Other parameter values are not changed. The only difference, therefore, is that \( \omega = 0 \) in the benchmark model while it is positive here.
shocks, as in the benchmark model. In default, the Lucas tree will be seized and the agents will consume only labor income in autarky. Table 4 shows how asset prices change as $\omega$ varies from zero to 10%.\textsuperscript{13} Asset prices are very sensitive to the magnitude of $\omega$. As it increases, autarky becomes less attractive and agents share more risk. The volatility of consumption share and SDF decrease. The risk-free rate becomes much higher, while the equity premium declines a lot, from 3.45% to 0.63%, 4.11% to 0.72%, and 7.05% to 2.59%, for homogeneous preferences, heterogeneous risk aversion, and heterogeneous time discount factor, respectively. In order for limited enforcement to matter more for the equity premium, I could alleviate the punishment for default, such as allowing agents to trade a risk-free bond as in Krueger and Perri (2006), or allowing agents to come back to financial markets several years after defaulting. These less severe punishment mechanisms, instead of permanent exclusion from financial markets, can reduce optimal risk sharing and increase the equity premium.\textsuperscript{14}

5 Conclusion

In this paper, I show that heterogeneous preferences have substantial impact on risk sharing and asset prices when markets are endogenously incomplete. I introduce heterogeneity in risk aversion and time discount factor into a two-agent endowment economy with limited enforcement, idiosyncratic income risk, and aggregate risk. I show that in contrast to the conventional findings under heterogeneous preferences, both agents survive in the long run due to enforcement constraints. I find that preference heterogeneity boosts the mean and volatility of the equity premium quantitatively, because it generates asymmetric risk sharing across agents and more conditional variation in the SDF. In particular, heterogeneous time preference holds great promise for helping the literature in heterogeneous agents with incomplete markets towards the fully resolution of the equity premium puzzles.

\textsuperscript{13}Chien, Cole, and Lustig (2012) calibrate the fraction of collateralizable income $\omega$ to be 10%.

\textsuperscript{14}Another way to boost the equity premium is to view equity as a levered aggregate consumption claim.
References


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Table 1: SDF and Returns Across States: Heterogeneous v.s. Homogeneous Preferences

This table shows the asset prices in the two-state example for each transition from state $t$ to state $t+1$. Panel A fixes $\beta = 0.65$ and compares the case of homogeneous $\gamma$ with heterogeneous $\gamma$. Panel B fixes $\gamma = 3.00$ and compares the case of homogeneous $\beta$ with heterogeneous $\beta$. $\gamma_i$ is agent $i$’s risk aversion parameter and $\beta_i$ is agent $i$’s time discount factor. SDF stands for stochastic discount factor, $R_s$ is the (gross) aggregate market return, $R_f$ is the (gross) risk-free rate, and $R_e$ is the equity premium.

<table>
<thead>
<tr>
<th>Panel A</th>
<th>$\gamma_1 \neq \gamma_2$ vs. $\gamma_1 = \gamma_2$</th>
<th>$\gamma_1 \neq \gamma_2$ vs. $\gamma_1 = \gamma_2$</th>
<th>$\gamma_1 \neq \gamma_2$ vs. $\gamma_1 = \gamma_2$</th>
</tr>
</thead>
<tbody>
<tr>
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<td>state $t+1$</td>
<td>$\gamma_1$</td>
<td>$\gamma_2$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1.50</td>
<td>2.70</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.40</td>
<td>1.41</td>
</tr>
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<th>$\beta_1 \neq \beta_2$ vs. $\beta_1 = \beta_2$</th>
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Table 2: Moments for Asset Pricing and Risk Sharing

This table presents the unconditional moments from the model simulations. The moments are averaged across 1,000 simulations, each with 5,000 periods and the first 500 periods burned. $E(R_f)$ ($\sigma(R_f)$) is the mean (volatility) of risk-free rate, $E(R_s - R_f)$ is the mean of the equity premium, and $\sigma(R_s)$ is the equity return volatility. Sharpe stands for the Sharpe ratio, $E(M)$ ($\sigma(M)$) is the mean (volatility) of SDF, and $\sigma(ln(\hat{c}_i))$ is the volatility of agent $i$’s consumption share. U.S. data sample moments of market excess return, market return volatility, and Sharpe ratio are from Bansal and Yaron (2004). Real risk-free rate sample moments are from Chien and Lustig (2010) using a long sample (1928-2007). AJ denotes the calibration and results from Alvarez and Jermann (2001). Panels A and B show the effects of heterogeneous $\gamma$ and heterogeneous $\beta$, respectively. The preference parameters are chosen around the values from Alvarez and Jermann (2001). For the heterogeneous $\gamma$ ($\beta$) case, risk aversion coefficients are first chosen, and then time discount factor is chosen to match $E(R_f)$ as closely as possible.

<table>
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<th>$E(R_s - R_f)$</th>
<th>$\sigma(R_f)$</th>
<th>$\sigma(R_s)$</th>
<th>Sharpe</th>
<th>$E(M)$</th>
<th>$\sigma(M)$</th>
<th>$\sigma(ln(\hat{c}_1))$</th>
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<td>0.82</td>
<td>4.11</td>
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<td>0.99</td>
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35
Table 3: Moments under Independent Risk

This table shows the unconditional moments under independent risk, where aggregate risk is i.i.d. and the distribution of the idiosyncratic risk is independent of aggregate risk (M1=0, M2,8,9,10=1 in the calibration in Table A1). Moments are averaged across 1,000 simulations, each with 5,000 periods and the first 500 periods burned. \(E(R_f)\) (\(\sigma(R_f)\)) is the mean (volatility) of the risk-free rate, \(E(R_s - R_f)\) (\(E(R_b - R_f)\)) is the mean of the equity premium (the term premium of a perpetual bond), and \(\sigma(R_s)\) (\(\sigma(R_b)\)) is the volatility of the equity return (perpetual bond return). Sharpe stands for the Sharpe ratio, \(E(M)\) (\(\sigma(M)\)) is the mean (volatility) of SDF, and \(\sigma(ln(\hat{c}_i))\) is the volatility of agent \(i\)'s consumption share. \(E(R_s)/E(R_f) - 1\) is the multiplicative equity premium. Panel A fixes \(\beta = 0.5\) and contrasts homogeneous risk aversion of \(\gamma = 2.5\) and \(\gamma = 3.5\) with heterogeneous risk aversion of \(\gamma_1 = 2.5\) and \(\gamma_2 = 3.5\). Panel B fixes \(\gamma = 3.0\) and contrasts homogeneous time discount factor of \(\beta = 0.45\) and \(\beta = 0.55\) with heterogeneous time discount factor of \(\beta_1 = 0.55\) and \(\beta_2 = 0.45\).

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<th>(E(R_f))</th>
<th>(E(R_b - R_f))</th>
<th>(E(R_s - R_f))</th>
<th>(\sigma(R_f))</th>
<th>(\sigma(R_b))</th>
<th>(\sigma(R_s))</th>
<th>Sharpe</th>
<th>(\sigma(ln\hat{c}_1))</th>
<th>(\sigma(ln\hat{c}_2))</th>
<th>(E(R_s)/E(R_f) - 1)</th>
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<tr>
<td>Panel A</td>
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<td>0.00</td>
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<td>0.09</td>
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<td>0.296</td>
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<tr>
<td>(\gamma = 3.5)</td>
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<td>0.279</td>
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<td>(\beta_1 = \beta_2)</td>
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<tr>
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<td>0.43</td>
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<td>0.222</td>
<td>0.37</td>
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<td>4.37</td>
<td>6.77</td>
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<td>25.82</td>
<td>33.02</td>
<td>0.21</td>
<td>0.273</td>
<td>0.277</td>
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</table>
Table 4: Moments under Positive Assets

This table shows the unconditional moments under positive collateral income. Moments are averaged across 1,000 simulations, each with 5,000 periods and the first 500 periods burned. \( \omega \) is the share of dividend income as of total income, which is seizable in default. \( E(R_f) \) (\( \sigma(R_f) \)) is the mean (volatility) of the risk-free rate, \( E(R_s - R_f) \) is the mean of equity premium, and \( \sigma(R_s) \) is the equity return volatility. Sharpe stands for the Sharpe ratio, and \( \sigma(ln(\hat{c}_i)) \) is the volatility of agent \( i \)'s consumption share. Panel A shows the results under homogeneous preferences \( \beta = 0.78 \) and \( \gamma = 3.5 \), as in Alvarez and Jermann (2001). Panel B shows the results under heterogeneous risk aversion \( \beta = 0.81, \gamma_1 = 2.5 \) and \( \gamma_2 = 4 \). Panel C shows the results under heterogeneous time discount factor \( \gamma = 3.2, \beta_1 = 0.85 \), and \( \beta_2 = 0.75 \).

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<th>( E(R_s - R_f) )</th>
<th>( \sigma(R_f) )</th>
<th>( \sigma(R_s) )</th>
<th>Sharpe</th>
<th>( \sigma(ln(\hat{c}_1)) )</th>
<th>( \sigma(ln(\hat{c}_2)) )</th>
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<td>Panel A:</td>
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<td>3.45</td>
<td>5.56</td>
<td>9.31</td>
<td>0.37</td>
<td>0.28</td>
<td>0.28</td>
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<tr>
<td>( \beta_1 = \beta_2, \gamma_1 = \gamma_2 )</td>
<td>0.01</td>
<td>13.93</td>
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<td>4.70</td>
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<td>6.28</td>
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<td>4.11</td>
<td>10.93</td>
<td>12.70</td>
<td>0.32</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>( \gamma_1 \neq \gamma_2 )</td>
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<td>11.01</td>
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<td>6.83</td>
<td>9.03</td>
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<td>( \beta_1 \neq \beta_2 )</td>
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<td>8.43</td>
<td>4.65</td>
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<td>12.32</td>
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<td>21.42</td>
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<td>10.83</td>
<td>0.24</td>
<td>0.14</td>
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Figure 1: Consumption Share in the Two-state Example: Homogeneous $\gamma$ vs. Heterogeneous $\gamma$

Note: This figure shows the long-run stationary consumption shares of agents 1 and 2 in the two-state example. The horizontal axis denotes risk aversion and the vertical axis consumption shares. The agents’ time discount factors are the same, and fixed at $\beta = 0.65$. The upper solid (lower dashed) line depicts how the agent’s consumption share changes with risk aversion when he receives a high (low) endowment realization under homogeneous risk aversion ($\gamma_1 = \gamma_2$). The squares and diamonds present a particular case in which agents have different risk aversion coefficients, i.e., $\gamma_1 = 1.5$ and $\gamma_2 = 2.7$. Squares denote the consumption share of the less risk-averse agent 1 ($\gamma_1 = 1.5$) when he receives a high (upper square) or low (lower square) endowment realization. Diamonds are for the more risk-averse agent 2 ($\gamma_2 = 2.7$). $c_{1L} = 0.377$, $c_{1H} = 0.628$, $c_{2L} = 0.372$, $c_{2H} = 0.623$. 
Figure 2: Consumption Share in the Two-state Example: Homogeneous $\beta$ vs. Heterogeneous $\beta$

Note: This figure shows the long-run stationary consumption shares of agents 1 and 2 in the two-state example. The horizontal axis denotes time discount factor and the vertical axis consumption shares. The agents’ risk aversion coefficients are the same, and fixed at $\gamma = 3.0$. The upper solid (lower dashed) line depicts how the agent’s consumption share changes with the time discount factor when he receives a high (low) endowment realization under homogeneous time discount factor ($\beta_1 = \beta_2$). The squares and diamonds present a particular case in which agents have different time discount factors, i.e., $\beta_1 = 0.7$ and $\beta_2 = 0.5$. Squares denote the consumption share of the less patient agent 2 ($\beta_2 = 0.5$) when he receives a high (upper square) or low (lower square) endowment realization. Diamonds are for the more patient agent 1 ($\beta_1 = 0.7$). $c_{1L} = 0.422$, $c_{1H} = 0.531$, $c_{2L} = 0.469$, $c_{2H} = 0.578$. 
Figure 3: Risk Sharing: Heterogeneous vs. Homogeneous Risk Aversion

Note: This figure shows how agents’ consumption shares and the resulting SDF change with time discount factor $\beta$. Agents have the same $\beta$, but may have different risk aversion coefficients $\gamma$. The upper part of the figure presents the standard deviation of consumption shares, which measures the degree of risk sharing. The diamond-marked line indicates that both agents have the same risk aversion $\gamma = 2.5$, while the square-marked line the same $\gamma = 4.0$. The other two lines present the case of heterogeneous risk aversion, $\gamma_1 = 2.5$ (the dashed line for agent 1) and $\gamma_2 = 4.0$ (the solid line for agent 2). The more risk-averse agent 2 has less volatile consumption than the less risk-averse agent 1 when there is non-zero risk sharing. The lower part of the figure presents the mean and standard deviation of SDF. The diamond-marked (square-marked) line stands for homogeneous risk aversion $\gamma = 2.5$ ($\gamma = 4.0$), while the solid line stands for heterogeneous risk aversion $\gamma_1 = 2.5$ and $\gamma_2 = 4.0$. 
Figure 4: Asset Prices: Heterogeneous vs. Homogeneous Risk Aversion

Note: This figure shows how the unconditional moments of the risk-free rate and equity returns change with time discount factor $\beta$. Agents have the same $\beta$, but may have different risk aversion coefficient $\gamma$. The diamond-marked (square-marked) line stands for homogeneous risk aversion $\gamma = 2.5$ ($\gamma = 4.0$), while the solid line stands for heterogeneous risk aversion $\gamma_1 = 2.5$ and $\gamma_2 = 4.0$. 
Figure 5: Risk Sharing: Heterogeneous vs. Homogeneous Time Discount Factor

Note: This figure shows how agents’ consumption shares and the resulting SDF change with risk aversion $\gamma$. Agents have the same $\gamma$, but may have different time discount factor $\beta$. The upper part of the figure presents the standard deviation of consumption shares, which measures the degree of risk sharing. The diamond-marked line indicates that both agents have the same time discount factor $\beta = 0.75$, while the square-marked line the same $\beta = 0.85$. The other two lines present the case of heterogeneous time discount factor $\beta_1 = 0.85$ (the dashed line for agent 1) and $\beta_2 = 0.75$ (the solid line for agent 2). The more patient agent 1 has less volatile consumption than the less patient agent 2 when there is non-zero risk sharing. The lower part of the figure presents the mean and standard deviation of SDF. The diamond-marked (square-marked) line stands for homogeneous time discount factor $\beta = 0.75$ ($\beta = 0.85$), while the solid line stands for heterogeneous time discount factor $\beta_1 = 0.85$ and $\beta_2 = 0.75$. 
Figure 6: Asset Prices: Heterogeneous vs. Homogeneous Time Discount Factor

Note: This figure shows how the unconditional moments of the risk-free rate and equity returns change with risk aversion $\gamma$. Agents have the same $\gamma$, but may have different time discount factor $\beta$. The diamond-marked (square-marked) line stands for homogeneous time discount factor $\beta_1 = 0.75$ ($\beta = 0.85$), while the solid line stands for heterogeneous time discount factor $\beta_1 = 0.75$ and $\beta_2 = 0.85$. 

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A APPENDICES

A.1 Computation

Let state variables be $x = (\lambda, z)$. The policy and value functions are $\hat{c}_i(x), \lambda'(x), \nu_i(x), W_i(x)$, where the value function

$$W_i(x) = u(\hat{c}_i(x)) + \beta_i(z) \sum_{z'} \pi_i(z'|z)W_i(x').$$  \hspace{1cm} (A.1)

The computation algorithm follows these steps:

1. Set up a grid $X$ over the state space.
2. Set the initial guess to be the solution to the planner’s problem without enforcement constraints.
   $$\nu^0_i(x) = 0, z^0(x) = z, \hat{c}^0_i(x),$$ and $W^0_i(x)$ satisfy (2.10), (2.13) and (A.1).
3. Consider three possible binding patterns of enforcement constraints:
   
   - Neither constraint binds
   - Agent 1’s constraint binds
   - Agent 2’s constraint binds

3.1 For each $x \in X$, compute allocations that assume neither constraint binds.
3.2 Then check

$$u(\hat{c}^0_i(x)) + \beta_i(z) \sum_{z'} \pi(z'|z)W^0_i(x') \geq U^i(\dot{e}_i(z)) \text{ for } i = 1, 2.$$ \hspace{1cm} (A.2)

3.2.1 If (A.2) is satisfied for $i = 1, 2$, then set new policies

$$\nu^1_i(x) = \nu^0_i(x), \lambda'^1(x) = \lambda'^0(x), \hat{c}^1_i(x) = \hat{c}^0_i(x), W^1_i(x) = W^0_i(x).$$
3.2.2 If (A.2) is satisfied for $i = 2$ but not $i = 1$, then set $\nu_2^1(x) = 0$, solve $\nu_1^1(x), \lambda^1(x), \hat{c}_1^1(x)$, and $\hat{c}_2^1(x)$ from (2.10), (2.11), (2.13) and

$$u^1(\hat{c}_1^1(x)) + \beta(z) \sum_{z'} \pi(z'|z)W_1^1(x') = U^1(\hat{e}_1(z)). \quad (A.3)$$

Set $W_1^1(x)$ as the LHS (left-hand-side) of (A.3) and $W_2^1(x)$ as the LHS of (A.2).

3.2.3 If (A.2) is satisfied for $i = 1$ but not $i = 2$, then set $\nu_1^1(x) = 0$, solve $\nu_2^1(x), \lambda^1(x), \hat{c}_1^1(x)$, and $\hat{c}_2^1(x)$ from (2.10), (2.11), (2.13) and

$$u^2(\hat{c}_2^1(x)) + \beta(z) \sum_{z'} \pi(z'|z)W_2^1(x') = U^2(\hat{e}_2(z)). \quad (A.4)$$

Set $W_1^1(x)$ as the LHS of (A.2) and $W_2^1(x)$ as the LHS of (A.4).

4.1 If the difference between $(\nu_0^i(x), \lambda^0(x), \hat{c}_0^i(x), W_0^i(x))$ and $(\nu_1^i(x), \lambda^1(x), \hat{c}_1^i(x), W_1^i(x))$ is small enough for each $x \in X$, then stop.

4.2 If not, then set the initial guess equal to the new set of policy, multiplier, and value functions. Keep iterating until the value functions and policy functions converge.

### A.2 Calibration

The endowment process is

<table>
<thead>
<tr>
<th>State</th>
<th>$g$</th>
<th>$\hat{e}_2$</th>
<th>$\hat{e}_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$g_L$</td>
<td>$\frac{1}{2} - \theta$</td>
<td>$\frac{1}{2} + \theta$</td>
</tr>
<tr>
<td>2</td>
<td>$g_H$</td>
<td>$\frac{1}{2} - \eta$</td>
<td>$\frac{1}{2} + \eta$</td>
</tr>
<tr>
<td>3</td>
<td>$g_L$</td>
<td>$\frac{1}{2} + \theta$</td>
<td>$\frac{1}{2} - \theta$</td>
</tr>
<tr>
<td>4</td>
<td>$g_H$</td>
<td>$\frac{1}{2} + \eta$</td>
<td>$\frac{1}{2} - \eta$</td>
</tr>
</tbody>
</table>
The transition matrix is

\[
\begin{bmatrix}
p_{LL} \pi_{LL} & p_{LH}(1 - \pi_{LL}) & (1 - p_{LL})\pi_{LL} & (1 - p_{LH})(1 - \pi_{LL}) \\
p_{HL}(1 - \pi_{HH}) & p_{HH} \pi_{HH} & (1 - p_{HL})(1 - \pi_{HH}) & (1 - p_{HH})\pi_{HH} \\
(1 - p_{LL})\pi_{LL} & (1 - p_{LH})(1 - \pi_{LL}) & p_{LL} \pi_{LL} & p_{LH}(1 - \pi_{LL}) \\
(1 - p_{HL})(1 - \pi_{HH}) & (1 - p_{HH})\pi_{HH} & p_{HL}(1 - \pi_{HH}) & p_{HH} \pi_{HH}
\end{bmatrix}
\]

Aggregate growth rate follows a Markov process, and the idiosyncratic income shares follow a Markov process conditional on the aggregate transition. \(\pi_{LL} \ (\pi_{HH})\) denotes the aggregate transition probability from recession (boom) to recession (boom). \(p_{ij}\) denotes the idiosyncratic transition probability of agents having the same relative status (higher or lower than the other agent) of income share conditional on the aggregate state transition from \(i\) to \(j\).

There are 10 parameters to be calibrated in addition to the preference parameters:

\[g_L, \ g_H, \ \theta, \ \eta, \ \pi_{LL}, \ \pi_{HH}, \ p_{LL}, \ p_{LH}, \ p_{HL}, \ p_{HH}.\]

The calibrated parameters, following Alvarez and Jermann (2001), are in Table A1.

The resulting endowment process is

<table>
<thead>
<tr>
<th>State</th>
<th>(g)</th>
<th>(\hat{e}_2)</th>
<th>(\hat{e}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9602</td>
<td>0.3562</td>
<td>0.6438</td>
</tr>
<tr>
<td>2</td>
<td>1.0402</td>
<td>0.3562</td>
<td>0.6438</td>
</tr>
<tr>
<td>3</td>
<td>0.9602</td>
<td>0.6438</td>
<td>0.3562</td>
</tr>
<tr>
<td>4</td>
<td>1.0402</td>
<td>0.6438</td>
<td>0.3562</td>
</tr>
</tbody>
</table>

And the resulting transition matrix is

\[
\begin{bmatrix}
0.1414 & 0.8200 & 0.0309 & 0.0077 \\
0.2637 & 0.6820 & 0.0486 & 0.0057 \\
0.0309 & 0.0077 & 0.1414 & 0.8200 \\
0.0486 & 0.0057 & 0.2637 & 0.6820
\end{bmatrix}
\]
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aggregate growth rate in recessions (L)</td>
<td>$g_L$</td>
<td>0.9602</td>
<td>Mehra-Prescott (1985): $\rho(g) = -0.14$</td>
</tr>
<tr>
<td>Aggregate growth rate in booms (H)</td>
<td>$g_H$</td>
<td>1.0402</td>
<td>M2: $E(g) = 1.83%$</td>
</tr>
<tr>
<td>Transition probability from L to L</td>
<td>$\pi_{LL}$</td>
<td>0.1723</td>
<td>M3: Std($g$) = 3.57%</td>
</tr>
<tr>
<td>Transition probability from H to H</td>
<td>$\pi_{HH}$</td>
<td>0.6877</td>
<td>NBER 1889-1991: M4: Pr(boom)/Pr(recession) = 2.65</td>
</tr>
<tr>
<td>High income share in recessions</td>
<td>$1/2 + \theta$</td>
<td>0.6438</td>
<td>Alvarez-Jermann (2001):</td>
</tr>
<tr>
<td>High income share in booms</td>
<td>$1/2 + \eta$</td>
<td>0.6438</td>
<td></td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on LL</td>
<td>$p_{LL}$</td>
<td>0.8206</td>
<td>M5: Std($\hat{e}_i(z)$) = 0.296</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on HH</td>
<td>$p_{HH}$</td>
<td>0.9907</td>
<td>M6: $\rho(\ln \hat{e}_i(z)) = 0.9$</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on LH</td>
<td>$p_{LH}$</td>
<td>0.8444</td>
<td>M7: $\sum_{i=1}^{L} [\hat{e}<em>i(L) - \frac{1}{2}]^2 = \sum</em>{i=1}^{H} [\hat{e}_i(H) - \frac{1}{2}]^2$</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on LH</td>
<td>$p_{HL}$</td>
<td>0.9917</td>
<td>M8: $\sigma_{HH} = \frac{\text{std}(\ln \hat{e}_i(z+1)</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on HH</td>
<td>$p_{HH}$</td>
<td>0.9907</td>
<td>M9: $\sigma_{LL} = \frac{\text{std}(\ln \hat{e}_i(z+1)</td>
</tr>
<tr>
<td>Idiosyncratic transition probability of agent 2 low-low (high-high) income shares conditional on HL</td>
<td>$p_{HL}$</td>
<td>0.9917</td>
<td>M10: $\frac{\sigma_{HH}}{\sigma_{HH}} = \frac{\text{std}(\ln \hat{e}_i(z+1)</td>
</tr>
</tbody>
</table>