Dynamic Market Making and Asset Pricing *

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Abstract

We develop a dynamic model of market making with asymmetric information and imperfectly competitive market makers. Our model captures key features of market making in many financial markets: market makers optimally make offsetting trades in bid and ask markets by adjusting bid and ask prices and they tend to avoid the risk of long or short positions. We solve for equilibrium endogenous bid and ask prices and trading volume analytically. We find that when market makers have significant market power, other traders optimally smooth out their trading even though they are not strategic, consistent with empirical findings. The market power of market makers dampens and spreads out trading spikes due to new information arrivals or liquidity shocks. Traders tend to trade fast on their private information and postpone their trading due to hedging needs until later periods. As a result, both trading volume and bid-ask spread may exhibit U-shaped patterns. In addition, when trading frequency increases, our model suggests that market makers’ profit may monotonically decrease if information arrives only at certain periods while it first increases and then decreases if information arrives gradually over time or liquidity shock arises at certain periods. Other market participants are always better off with more trading rounds. Our model thus may help explain why trading frequency is low in markets where market makers have significant market power while it is high when competition among market makers is intensive.

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Keywords: market making, liquidity, asset pricing, imperfect competition, bid-ask spread, trading volume

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1 Introduction

Market makers in many financial markets tend to avoid the risk of long or short positions and have significant market power.\(^1\) Especially after new banking regulations and the Volcker Rule following the financial crisis of 2008, market makers tend to offset trades instead of carrying inventory due to decreased risk-bearing capacity of market makers.\(^2\) One of the most important features distinguished market making from proprietary trading is that market making should have rapid inventory turnover with the vast majority of profits from bid-ask spreads instead of from price appreciation.\(^3\) How does offset trading of market makers who have market power and mainly care about the round-trip profit from bid-ask spread affect the dynamics of bid-ask spreads and trading volume in the presence of asymmetric information? Does market regulation such as increasing trading frequency of markets where market makers have significant power benefit market participants? In this paper, we develop a dynamic market making model where market makers have market power and mainly gain from bid-ask spread by offsetting trades to address these questions.

We find that the dynamics of trading volume and bid-ask spread is largely determined by the structure of information flow and liquidity shocks. When both information and liquidity shocks arrive over time or they mainly arrives at the opening and closing of the market, both trading volume and bid-ask spread exhibit U-shaped patterns—high trading volume and bid-ask spread at the beginning and the end of the trading day and low trading volume and bid-ask spread in the middle of the day.\(^4\) New information arrivals or liquidity shocks generate trading spikes. When market makers have significant market power, other traders optimally smooth out their trading even though they are not strategic. Trading might persist after information is incorporated in the price, consistent with the empirical finding that most prices adjustment occurs in the first trade.\(^5\) In addition, we find that the market power of

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\(^2\) Schultz (2017) and Bao, O’Hara, and Zhou (2018) show that after the Volcker Rule, dealers are less willing to take bonds into inventory and they have relied more on prearranged trades that are quickly offset by trades in the opposite direction. Goldstein and Hotchkiss (2018) show that dealers have a high propensity to offset trades for riskier and less actively traded bonds. Choi and Huh (2018) also find that the fraction of offsetting trades increases after the new banking regulations were adopted following the 2008 financial crisis.

\(^3\)See Financial Stability Oversight Council (2011).

\(^4\)See, for example, Jain and Joh (1988) and Foster and Viswanathan (1993) on trading volume, Mcinish and Wood (1992) and Brock and Kleidon (1992) on bid-ask spread. Jain and Joh (1988) find that the average trading volume is highest during the first hour of the day for each day of the week. The average first hour volume is at least 50 percent more than the average hourly volume during the day. The average trading volume declines monotonically until the fourth hour but increases again in the fifth and sixth hours. Mcinish and Wood (1992) find that spreads are higher at the beginning and end of the day relative to the interior period.

\(^5\)See, for example, Holthausen, Leftwich, and Mayers (1990).
market makers tends to dampen trading spikes due to either new information arrivals or new liquidity shocks. Moreover, when trading frequency increases, our model suggests that market makers’ profit may monotonically decrease if information arrives only at market openings while it first increases and then decreases if information arrives gradually over time or liquidity shock arises at certain periods. Both informed and uninformed traders’ welfare increases monotonically when the number of trading rounds increases regardless of the structure of information flow and liquidity shocks.

Specifically, we consider a multi-period model of trading in a market where all trades go through designated market makers. There are three types of risk averse investors: informed investors, uninformed investors, and designated market makers who are also uninformed. Both informed and uninformed traders are perfectly competitive and market makers are imperfectly competitive. Traders can trade one risk-free asset and one risky security at periods $0, 1, \ldots, T$ to maximize their expected constant absolute risk averse (CARA) utility from the terminal wealth at period $T + 1$. All may be endowed with some shares of the risky security whose payoff becomes public at period $T + 1$. At each trading period $t = 0, 1, \ldots, T$, informed investors observe a private signal about the terminal payoff of the security at period $T + 1$ and thus have trading demand motivated by private information. Informed investors also have non-information-based incentives to trade, which we term as a liquidity shock and model as a random endowment of a non-tradable asset. The non-tradable risky asset whose payoff is perfectly correlated with stock payoff can only be liquidated at period $T + 1$. It follows that informed investors also have trading demand motivated by the liquidity needs for hedging. At each trading period $t$, informed and uninformed investors sell to the designated market makers at the bid price or buy from them at the ask price or do not trade at all. We model the oligopolistic competition among the market makers as a Cournot competition. Specifically, we assume that market makers simultaneously choose how much to buy at bid given the inverse supply function (a function of the market makers’ purchasing quantity) of all other participants and how much to sell at ask given the inverse demand function (a function of the market makers’ selling quantity) of all other participants. The posted bid and ask prices are the required prices to achieve the optimal amount market makers choose to trade.

Solving for the equilibrium endogenous bid and ask prices and depth analytically in a dynamic model is very challenging. The difficulty lies in the fact that the wealth of each trader is path dependent; it depends on the whole history of the traders’ demand and the endogenously determined bid and ask prices. To make the model tractable and to capture the realistic feature of market making in many financial markets—market makers tend to offset trades instead of carrying inventory, we focus on the limiting case where market makers
do not carry inventory over time.\footnote{We proved formally that market makers optimally do not carry any inventory when they are extremely risk averse.} Therefore, in our model, market makers trade in both the “ask” market and the “bid” market and optimally make offsetting trades with other traders to maximize the expected profit from each round of trading conditional on the history of order flows submitted by traders. This not only allows us to solve for the equilibrium endogenous bid and ask prices and depth analytically in the dynamic model but also captures realistic features of market making in most financial markets.

We find that the optimal ask (bid) price is the weighted average of the reservation price of the uninformed and that of the informed investors.\footnote{The reservation price is the critical price such that an investor buys (sells) the risky security if and only if the ask (bid) is lower (higher) than this critical price.} The weights depend on informed and uninformed investors’ risk aversion and conditional variance of the fundamental value at period $t$, as well as the number of each type of traders. Facing the demand and supply functions of other investors, an oligopolistic market maker optimally trades off the prices and quantities. Similar to the static model of \textit{Liu and Wang (2016)}, in our dynamic model, the bid and ask spread at period $t$ is equal to the absolute value of the reservation price difference at period $t$ divided by the number of market makers plus one. The difference between buyers’ (sellers’) reservation prices and the ask (bid) price is also proportional to the absolute value of the reservation price difference at period $t$. Therefore the trading amount of both informed and uninformed investors and thus the aggregate trading volume at period $t$ all increase with the reservation price difference at period $t$.

If both information and liquidity shocks arrive over time or they mainly arrive at the opening and closing of the market, the average trading volume exhibits a U-shaped pattern—the average trading volume at the beginning and the end of the trading day is high while the average trading volume in the middle of the day is low. The higher trading volume at the beginning and at the end of the day results from greater trading incentives between informed and uninformed traders due to new information arrivals or hedging demand. The U-Shape tends to be more substantial when the volatility of hedging demand is higher or when informed traders’ private information is noisier. Intuitively, uninformed traders are more willing to trade with informed traders when it is more likely that informed traders trade to hedge the risk from the non-tradable asset instead of speculating on their private information. Interestingly, traders trade more smoothly over a longer period if the competition among market makers becomes less intensive. This is because the monopoly power of market makers increases the cost of trading too aggressively. When competition among market makers increases, the average of bid-ask spread decreases and it becomes less costly for traders to trade more aggressively at the beginning when information arrives or at the end to re-
balance their portfolios. Therefore, for some trading periods during the middle of the day, the expected trading volume can be higher in a market with oligopolistic market makers than that in a market with competitive market makers.

To further illustrate the impact of information arrivals on the dynamics of trading volume and bid-ask spread, we examine both the case when information or liquidity shock only arrives at certain time and the case when information and liquidity shock arrives gradually at a certain rate. We find that the arrivals of new information or liquidity shock generate spikes in trading volume. In some sense, the arrival of information or liquidity shock has a ripple effect on the average trading volume and this effect tends to be more substantial when the competition among market makers becomes more intensive. When there are fewer market makers, traders tend to smooth out their trading throughout the whole trading period since the market power of market makers makes trading aggressively more costly. Therefore, the monopoly power of market makers tends to dampen trading spikes due to information arrivals or liquidity shocks. If both information and liquidity shocks arrive gradually at a certain rate, uninformed traders’ estimate of the fundamental value of the asset and the amount of hedging demand becomes more precise over time. The trading volume exhibits a U-shaped pattern. This is because informed traders tend to act fast on their private information since others might trade on the same information and they tend to postpone their trading due to hedging needs until later dates to reduce trading costs because of smaller adverse selection effect. As a result, trading volume tends to be concentrated at the market opening and closure.

Since both the bid-ask spread and trading volume at period $t$ are proportional to the absolute value of the reservation price difference at period $t$, when information arrives at certain points during the trading periods, similar to the dynamics of trading volume, the arrivals of new information also generate spikes in bid-ask spread. In addition, the expected bid-ask spread tends to exhibit larger spikes than those of the expected trading volume. This is because expected trading volume is proportional to the expected bid-ask spread and the proportionality coefficient decreases over time with new information arrivals. If both information and liquidity shocks arrive gradually at a certain rate, since market makers can charge wider bid-ask spreads when the trading incentives are greater, it follows that bid-ask spreads are wider at market opening and closure.

We also study how market makers’ profit changes with the number of trading rounds while fixing the information environment. When information only arrives at certain periods, surprisingly, we find that a faster market with more trading rounds makes market makers less profitable. This is because, with more trading rounds, informed and uninformed traders tend to spread out their orders over time in oligopolistic markets to reduce total trading
costs. In some sense, adding more trading rounds undermines market makers’ market power. When information arrives at a constant rate over time or liquidity shock arises at certain periods, as the number of trading rounds increases, market makers’ profit first increases and then decreases. Therefore, when information arrives at a constant rate, there exists an optimal trading frequency for market makers. Intuitively, adding more trading rounds has two opposite effects on market makers’ expected profit. On one hand, it decreases the expected bid-ask spread at each period since market makers’ market power is diluted. On the other hand, it increases trading opportunities for market makers to charge bid-ask spreads for more rounds even though the magnitude of bid-ask spread from each round decreases.

When the total precision of information becomes higher, then the benefit from being able to trade more rounds increases, and thus the optimal number of trading rounds for market makers is larger. In addition, the optimal number of trading rounds is smaller when there are fewer market makers. This implies that, oligopolistic market makers might not necessarily prefer to make the market more frequently. Our model might help explain why in some OTC markets (e.g., municipal bond markets) where market makers have significant market power, yet the trading frequency is relatively low. It also helps explain why market makers prefer higher trading frequency when the competition among market makers is more intensive. Regardless of the information structure, both informed and uninformed traders always benefit from more trading rounds due to narrower bid-ask spreads.

Our model generates several new testable empirical predictions. For example, our model predicts that, during periods of news events, the variation of trading volume tends to be smaller for securities with fewer imperfectly competitive market makers. The U-shaped pattern of the bid-ask spread tends to be more pronounced for stocks with fewer market makers while the U-shaped pattern of the trading volume tends to be more pronounced for stocks with less amount of private information, more hedgers, or more market makers. Our model explains the empirical findings that liquidity demanders pay most of the bid-ask spread while non-dealer liquidity suppliers get much more favorable prices from market makers. Our model can also help shed some light on the impact of trading platforms regulation on traders’ welfare. Increasing trading frequency tends to hurt market makers, but it benefits both informed and uninformed traders.

The remainder of the paper proceeds as follows. In Section 1.1, we briefly describe some applicable markets and discuss additional related literature. In Section 2 we present the model. In Section 3 we derive the equilibrium bid and ask prices, bid and ask depth, and trading volume. In Section 4 we study the dynamics of bid-ask spreads and trading volume. We study the impact of trading frequency on market makers’ profit as well as informed and uninformed investors’ welfare in Section 5. Section 6 discusses several practical implications
of our model. We conclude in Section 7. All proofs are provided in the Appendix.

1.1 Applicable Markets and Related Literature

In many financial markets, such as NYSE, NASDAQ and Paris Bourse, designated market-makers are employed to facilitate trading. These market-makers are required to maintain two-sided markets during exchange hours and are obligated to buy and sell at their displayed bids and offers. In a quote-driven, electronic market such as the London Stock Exchange (LSE), public investors generally cannot trade directly among themselves and need to trade through market makers (e.g., Charitou and Panayides (2009)). This suggests that our model applies to some of these financial markets where designated market makers facilitate trades.

Competition among market-makers is imperfect in many financial markets. For example, Christie and Schultz (1994) suggest that Nasdaq dealers may implicitly collude to maintain wide spreads. Biais, Bisèire, and Spatt (2010) analyze trades and order placement on Nasdaq and a competing electronic order book, Island. They conclude that competition among market-makers in these markets is still imperfect even after the introduction of electronic markets. In addition, the opaqueness and illiquidity of many dealers’ markets make these markets even less competitive (e.g., Ang et al. (2013)).

Investors in most of the OTC markets almost always trade with designated dealers (market makers) who typically quote a pair of bid and ask prices that are contingent on order sizes. Market makers in these markets also tend to frequently engage in offsetting trades within a short period of time with other customers or with other dealers (Shachar (2012)). This suggests that our model applies well to some OTC markets where neither bilateral negotiation with a market maker nor searching for a market maker is critical. For example, in bond markets after the introduction of the Transaction Reporting and Compliance Engine (TRACE) and OTCQX and OTCQB stock markets, investors can observe both bid and ask prices and do not need to conduct much search for or significant negotiation with market makers. In addition, market makers in many financial markets tend to withdraw from trading as their inventory or short positions reach preset parameters (O’Hara (2015)). They tend to avoid the risk of long or short positions in favor of earning bid/ask spreads on large trading volumes. Especially after the Volcker Rule and new banking regulations...
following the financial crisis of 2008, dealers tend to offset trades instead of taking bonds into inventory (Schultz (2017), Bao et al. (2018), Goldstein and Hotchkiss (2018), and Choi and Huh (2018)). Bessembinder, Jacobsen, Maxwell, and Venkataraman (2018) document that less capital commitment from dealers is required in the part of the corporate bond market where electronic communications have reduced search costs and trading is concentrated. Therefore, our model captures critical features that market makers optimally make offsetting trades in bid and ask markets and they care about the round-trip profit rather than assuming zero expected profit on each trade (e.g., Garman (1976), Lyons (1995), and Virtu Financial (2014)).

Admati and Pfleiderer (1988) generate a U-shaped pattern of trading volume when liquidity traders concentrate in particular time periods. However, their model implies that trading cost is low at time of the day with higher trading volume. This is inconsistent with the empirical finding of Foster and Viswanathan (1993) that high trading costs are found at times of the day with higher trading volume, which is consistent with our model. In our model, a U-shaped pattern of both trading volume and bid-ask spread can arise when information arrives at beginning and the end of trading period or when both information and liquidity shocks arrive over time. Brock and Kleidon (1992) show that transactions demand at open and close is greater and less elastic than at other times of the trading day due to periodic market closure and this periodic demand generates high volume and concurrent wide spreads. In their model, buy and sell orders are Poisson distributed in time. In our model, both buyers and sellers optimally do dynamic programming and choose optimal trading quantities each period. Both trading volume and bid/ask prices are solved endogenously in the equilibrium. Our model setting allows us to generate several empirically testable implications on the magnitude of the U-shaped patterns across different stocks. Liu and Wang (2016) study market-making in the presence of asymmetric information and inventory risk. However, they only focus on a static model and are silent on the dynamics of trading volume and bid-ask spread. Budish, Cramton, and Shim (2015) argue that financial exchanges should use frequent batch auctions and orders should be processed in a batch auction instead of serially because frequent batch auctions eliminate the mechanical arbitrage rents and stop the high-frequency trading arms race. Our result that market makers might be worse off while other market participants are always better off when the number of trading rounds times of extreme market volatility, we believe the reduction in risk is an appropriate trade-off that is in keeping with our aim of generating consistently strong revenue from trading.”

10Existing inventory-based market making literature ignores information asymmetry (e.g., Garman (1976), Stoll (1978), and Ho and Stoll (1981)). In contrast to most information-based (rational expectations) models (e.g., Grossman and Stiglitz (1980), Kyle (1985), Glosten and Milgrom (1985)), in our model a market maker faces discretionary uninformed investors, has significant market power, and profits from bid-ask spreads.
increases does not contradict with Budish et al. (2015). In our model, orders are submitted and executed at discrete time and traders do not engage in high-frequency trading arms race. Our model implies that market makers can be worse off while other market participants are always better off when trading frequency increases in the absence of sniping. Traders in our model trade more aggressively to exploit their private information, however, they respond to symmetrically observed public information simultaneously. Studying how the arms race among high-frequency traders affects investors’ welfare is beyond the scope of this paper.

2 The Model

We consider a multi-period model of trading in a market where all trades go through oligopolistic market makers. There are $N_I$ identical informed investors, $N_U$ identical uninformed investors, and $N_M$ designated market makers who are also uninformed. We assume that both $N_I$ and $N_U$ are much greater than $N_M$, and thus informed and uninformed traders are perfectly competitive and market makers are imperfectly competitive. Traders can trade one risk-free asset and one risky security at dates $0, 1, ..., T$. There is zero supply of the risk-free asset, which also serves as the numeraire and thus the risk-free interest rate is normalized to 0. The total supply of the risky asset is $\theta$ and each share of the stock pays a liquidation value of $V \sim \mathcal{N}(\tilde{V}, \tau_V^{-1})$ at the final date $T + 1$, where $\tilde{V}$ is a constant, $\tau_V^{-1} > 0$, and $\mathcal{N}$ denotes the normal distribution. No investor is endowed with any risk-free asset. Each type $i \in \{I, U, M\}$ investor is endowed with $\theta_i$ shares of risky assets before date 0. Therefore, the total supply of the risky security $\theta$ is equal to the total endowment $N_I \theta^I + N_U \theta^U + N_M \theta^M$.

On each date $t = 0, 1, ..., T$, informed investors may observe a private signal

$$v_t = V + \varepsilon_t,$$

about stock’s liquidation value $V$, where $\varepsilon_t \sim \mathcal{N}(0, \tau_{\varepsilon_t}^{-1})$ is independent noise. To prevent informed investors’ private information from being fully revealed in equilibrium, on each date $t = 0, 1, ..., T$, we assume that each informed investor is also endowed with $X_t$ total number of shares of a non-tradable risky asset on date $t$,

$$X_t = X_{t-1} + \eta_t,$$

where $X_{-1} = 0$, $\eta_t \sim \mathcal{N}(0, \tau_{\eta_t}^{-1})$, $\eta_t$ is the number of shares received by each informed trader on date $t$ ($t = 0, 1, ..., T$) and independent of $V$, $\varepsilon_t$, and $\eta_t$ where $t \neq t'$. The non-tradable risky asset can only be liquidated on date $T + 1$. Without loss of generality, we assume that each unit of the non-tradable risky asset pays a liquidation value of $V - \bar{V}$ at the final date
\( T + 1. \)

All trades must go through the designated market makers at dates \( t = 0, 1, ..., T \). Specifically, \( I \) and \( U \) investors sell to the designated market makers at the bid \( B_t \) or buy from them at the ask \( A_t \) or do not trade at all. We model the oligopolistic competition among market makers as a Cournot competition. Specifically, we assume that market makers simultaneously choose how much to buy at bid given the inverse supply function (a function of the market makers’ purchasing quantity) of all other participants and how much to sell at ask given the inverse demand function (a function of the market makers’ selling quantity) of all other participants. The posted bid and ask prices are the required prices to achieve the optimal amount market makers choose to trade.

Define \( Z_t \equiv (Z_0, Z_1, ..., Z_t) \) for any stochastic process \( \{Z_t\} \), i.e., \( Z_t \) represents the history of \( Z_t \) up to and including \( t \). On each date \( t \), after observing private signals \( v_t \), each informed investor chooses a demand schedule \( \Theta^I_t (v_t, X_t; \cdot) \). Each uninformed trader chooses a demand schedule \( \Theta^U_t (\cdot) \). The schedules \( \Theta^I_t \) and \( \Theta^U_t \) are traders’ strategies. Given bid price \( B_t \) and ask price \( A_t \), the quantities demanded by informed and uninformed investors can be written

\[
\theta^I_t = \Theta^I_t (v_t, X_t, A_t, B_t) \quad \text{and} \quad \theta^U_t = \Theta^U_t (A_t, B_t).
\]

Let \( F^i_t \) denote a type \( i \) investor’s information set on date \( t \) for \( i \in \{I, U, M\} \). Investors’ information sets can be written as

\[
F^I_t = \{v_t, X_t, A_t, B_t\}, \quad F^U_t = F^M_t = \{A_t, B_t\}. \tag{3}
\]

Given \( A_t, B_t \), for type \( i \in \{I, U\} \), a type \( i \) investor’s problem is to solve

\[
\max_{\theta^i_t} \mathbb{E}[-e^{-\lambda^i W^i_{T+1}} | F^i_t], \tag{4}
\]

where \( W^i_{T+1} \) is a type \( i \) investor’s final wealth, \( \lambda^i \) is the risk aversion coefficient of type \( i \) traders and

\[
W^I_{T+1} = \sum_{t=0}^{T} \left[(\theta^I_t - \theta^I_{t-1})^+ B_t - (\theta^I_t - \theta^I_{t-1})^+ A_t\right] + \theta^I_T V + X_T (V - \bar{V}) \tag{5}
\]

\[
W^U_{T+1} = \sum_{t=0}^{T} \left[(\theta^U_t - \theta^U_{t-1})^+ B_t - (\theta^U_t - \theta^U_{t-1})^+ A_t\right] + \theta^U_T V
\]

where \( x^+ := \max(0, x) \), and \( x^- := \max(0, -x) \).

\(^{11}\)In general, we can assume that the non-tradable asset has a payoff that has a covariance with \( V \) and is realized and becomes public on date \( T + 1 \). The correlation between the non-tradable asset and the security results in a non-information based, liquidity demand for the risky asset to hedge the non-tradable asset payoff. This generalization would not yield qualitatively different results.
Let $\alpha_j^t$ and $\beta_j^t$ be the number of shares that market maker $j \in \{1, 2, ..., N_M\}$ sells at ask (i.e., ask depth) and buys at bid (i.e., bid depth) respectively. Given the demand schedules of the informed and the uninformed ($\Theta^I_t(A_t, B_t)$ and $\Theta^U_t(A_t, B_t)$), the bid price $B_t(\alpha_j^t, \beta_j^t)$ (i.e., the inverse supply function) and the ask price $A_t(\alpha_j^t, \beta_j^t)$ (i.e., the inverse demand function) can be determined by the following stock market clearing conditions at the bid and ask prices.

$$\sum_{j=1}^{N_M} \alpha_j^t = \sum_{i=I, U} N_i(\Theta^I_t(A_t, B_t) - \Theta^I_{t-1}(A_{t-1}, B_{t-1}))^+,$$

$$\sum_{j=1}^{N_M} \beta_j^t = \sum_{i=I, U} N_i(\Theta^I_t(A_t, B_t) - \Theta^I_{t-1}(A_{t-1}, B_{t-1}))^-,$$

(6)

where the left-hand sides represent the total sales and purchases by market makers respectively and the right-hand sides represent the total purchases and sales by other investors respectively.

For expositional simplicity, we assume that each market maker’s initial inventory before date 0, $\bar{\theta}_M = 0$. In general, the wealth of each trader is path dependent; it depends on the whole history of the trader’s demand and the endogenously determined bid and ask prices.\(^\text{12}\) To make the model tractable and to capture the realistic feature of market making in many financial markets, we assume that market makers do not carry inventory over time and they focus on the round-trip profit from bid-ask spread by offsetting trades. Then for $j = 1, 2, ..., N_M$, the designated market maker $M_j$’s problem is equivalent to solve

$$\max_{\alpha_j^t, \beta_j^t} E[\alpha_j^tA_t - \beta_j^tB_t | F^M_t],$$

(7)

this implies that market makers who trade in both the “ask” market and the “bid” market optimally make offsetting trades with other traders to maximize the expected profit from each round of trading conditional on market makers’ information set at period $t$, $F^M_t$.\(^\text{13}\)

This leads to the definition of a subgame perfect Nash equilibrium as follows.

\(^\text{12}\)More specifically, at each period, there are potentially eight cases in equilibrium: both $I$ and $U$ investors buy, both $I$ and $U$ investors sell, $I$ investors buy and $U$ investors sell, $I$ investors sell and $U$ investors buy, $I$ investors buy and $U$ investors do not trade, $I$ investors sell and $U$ investors do not trade, $I$ investors do not trade and $U$ investors buy, $I$ investors do not trade and $U$ investors sell. Therefore, in our dynamic model, at period $t$, there are potentially $8^t$ cases. The dimension grows exponentially and thus the model becomes intractable.

\(^\text{13}\)In general, for $j = 1, 2, ..., N_M$, the designated market maker $M_j$’s problem is

$$\max_{\alpha_j^t, \beta_j^t} E[-e^{-\lambda M^{W^M_{t+1}} | F^M_t}],$$

where $W^M_{t+1} = \sum_{t=0}^{T} [\alpha_j^tA_t(\alpha_j^t, \beta_j^t) - \beta_j^tB_t(\alpha_j^t, \beta_j^t)] + \theta^{M,j}_T V.$

(8)
Definition 1. An equilibrium \( (\Theta^I_t(A_t, B_t), \Theta^U_t(A_t, B_t), A_t, B_t, \alpha^I_t, \beta^I_t) \) is such that

1. given any \( A_t \) and \( B_t \), \( \Theta^I_t(A_t, B_t) \) solves a type \( i \) investor’s Problem (4) for \( i \in \{I, U\} \), where the information set of the informed is \( \mathcal{F}^I_t = \{v_t, X_t, A_t, B_t\} \) and the information set of the uninformed is \( \mathcal{F}^U_t = \{A_t, B_t\} \);

2. given \( \Theta^I_t(A_t, B_t) \) and \( \Theta^U_t(A_t, B_t) \), \( \alpha^I_t \) and \( \beta^I_t \) solve the market makers’ Problem (7), where the information set of the market makers is \( \mathcal{F}^M_t = \{\Theta^I_t(A_t, B_t), \Theta^U_t(A_t, B_t)\} \);

3. \( A_t := A_t(\alpha^I_t, \beta^I_t) \) and \( B_t := B_t(\alpha^I_t, \beta^I_t) \) clear both the risky security and the risk-free asset markets, where \( A_t(\alpha^I_t, \beta^I_t) \) and \( B_t(\alpha^I_t, \beta^I_t) \) solve Equation (6); and

4. for every realization of the signals \( v_t \) and endowment shocks \( X_t \), the beliefs of all investors are consistent with the joint conditional probability distribution in equilibrium.

2.1 Discussions on the assumptions of the model

In this subsection, we discuss our main assumptions and whether these assumptions are important for our main results.

In our model, we assume oligopolistic market makers compete for orders using Cournot competition. In contrast to Bertrand competition, Cournot competition allows market makers to keep some market power, which is a key feature of our model.\textsuperscript{14} In addition, to capture the fact that market makers in reality tend to avoid the risk of long or short positions in favor of earning bid/ask spreads, market makers’ objective function is assumed to maximizing the expected profit from each round of trading conditional on the information set. This captures the limiting case when market makers are risk averse and have limited risk-bearing capacity and thus they optimally do not carry inventories over time. Our framework would not be tractable to study the dynamics of market makers’ optimal inventory level, instead we focus on the dynamics of endogenous trading volume and bid-ask spread when market makers mainly care about the round-trip profit and do not carry much inventory over time due to the pre-set risk limits (e.g., Virtu Financial (2014)).

Similar to Vayanos and Wang (2012) and Goldstein, Li, and Yang (2014), to keep information from being fully revealed in equilibrium, we assume that informed investors have

\textsuperscript{14}As is well-known, it takes only two Bertrand competitors to reach a perfect competition equilibrium (and thus no market maker has any market power). However, market prices can be far from the perfect competition ones (e.g., Christie and Schultz (1994), Biais et al. (2010)).
two trading motives. They have liquidity shocks in addition to private information. One can interpret this assumption as there are some pure liquidity traders who trade in the same direction as the informed. The assumption that all informed traders have the same information and the same liquidity shock is only for simplicity so that there are only two groups of non-market-makers in the model. Our main results still hold when they have different information and different liquidity shocks. Intuitively, no matter how many heterogenous investor groups there are, the equilibrium bid and ask prices would divide these groups into a Buy group, a Sell group, and a No trade group. Therefore, as long as the characteristics of the Buy and Sell group investors are similar to those in our model, our main results still hold.

In some OTC markets and for relatively illiquid stocks in some centralized markets, it can be costly for non-market-makers to find and directly trade with each other. Therefore, most trades are through market makers, as we assume in the model. The market makers in our model can buy at the bid from some investors and sell at the ask to other investors at time $t$ as long as the reservation prices of informed traders and uninformed traders at time $t$ are different and thus both groups have incentives to trade. This assumption captures the fact that in many OTC markets, when a dealer receives an inquiry from a client, she commonly contacts other clients (or dealers) to see at which price and by how much she can unload the inquired trade before she trades with the initial client. In centralized markets, orders on opposite sides come in relatively frequently. Even for markets where there is a delay between offsetting trades and thus market makers may need to wait a period of time for the offsetting trades, as long as they have reasonable estimates of the next order, they will choose qualitatively similar trading strategy.

3 The Equilibrium

In this section, we solve for the dynamic equilibrium bid and ask prices and trading volume in closed form. We focus on a piecewise linear equilibrium where the equilibrium trading prices can be expressed as a piecewise linear function of the state variables in the model. At period $t$, informed investors’ private information $v_t$ and liquidity shock $\eta_t$ both affect the informed investor’s demand and thus the equilibrium prices, from market prices, other investors can only infer the value of the composite signal

$$s_t := v_t - h_t \eta_t = V + \varepsilon_t - h_t \eta_t,$$

(9)
where constant $h_t$ will be determined below in equation (13). Thus both ask prices $A_t$ and bid prices $B_t$ are informational equivalent to $s_t$. Given past prices, observing the current price is equivalent to observe $s_t$, which is a linear combination of informed investors’ private signal and amount of the non-tradable risky asset. As a result, the information set at time $t$ for uninformed investors $\mathcal{F}_t^U = \{\mathcal{F}_t^U\}$.

In order to derive each investor’s optimal stock holding, we need to compute the conditional expectations, given his information set. In the linear equilibrium, calculating the conditional expectations of the state variables is a linear filtering problem. For type $i \in \{I, U, M\}$, we define the expectation and variance of any stochastic process $Z_t$ as well as the covariance of $Z_t$ and $Z'_t$ conditional on $\mathcal{F}_t$ as $\hat{Z}_t := E[Z_t \mid \mathcal{F}_t], o_{Z,t} := \text{Var}[Z_t \mid \mathcal{F}_t]$, and $o_{Z'Z,t} := \text{Cov}(Z_t, Z'_t \mid \mathcal{F}_t)$. We obtain the following lemma on the dynamic Kalman filtration.

**Lemma 1.** Informed traders’ expected value of $V$ at time $t$, $\hat{V}_t^I$ and uninformed traders’ expected value of $V$ and $X_t$ at time $t$, $(\hat{V}_t^U, \hat{X}_t^U)$ are determined by the following stochastic differential equations:

$$\dot{\hat{V}}_t^I = \hat{V}_{t-1}^I + K_{V,t}(v_t - \hat{V}_{t-1}^I),$$

(10)

$$\dot{\hat{V}}_t^U = \hat{V}_{t-1}^U + K_{V,t}(s_t - \hat{V}_{t-1}^U), \quad \hat{X}_t^U = \hat{X}_{t-1}^U + K_{X,t}(s_t - \hat{V}_{t-1}^U)$$

(11)

and $\hat{X}_{U,1} = 0, \hat{V}_{I,1} = \hat{V}_{U,1} = \bar{V}$. The conditional variance of $V$ at time $t$ for informed traders $\sigma_{V,t}^I = K_{V,t}^{-1}$. The conditional variance $o_{V,t}^I, o_{X,t}^I$ and covariance $o_{V,X,t}$ for uninformed traders are defined in equation (A-16) in the Appendix. $K_{V,t}^I$ is defined in (A-11), $K_{V,t}^U$ and $K_{X,t}^U$ are defined in (A-15) in the Appendix A.1.

For uninformed traders, $(\hat{X}_t^U, \hat{V}_t^U)$ follows a Gaussian Markov process under uninformed traders’ information set $\mathcal{F}_t^U$ since $(\hat{V}_t^U, \hat{X}_t^U)$ can be expressed as a recursive equation of $(\hat{V}_{t-1}^U, \hat{X}_{t-1}^U)$ with the noisy terms in $s_t$ as innovations. Similarly, for informed traders, $(\hat{V}_t^I)$ follows a Gaussian Markov process under informed traders’ information set $\mathcal{F}_t^I$.

It can be shown that uninformed traders can infer from prices $\hat{V}_t^I - \mu_t X_t$, which is a composite signal of informed traders’ estimation about stock value $\hat{V}_t^I$ and informed traders’ liquidity shock $X_t$ at time $t$. Thus the composite signal $s_t$ is informationally equivalent to

$$S_t := \hat{V}_t^I - \mu_t X_t = \hat{V}_t^U - \mu_t \hat{X}_t^U,$$

(12)

where $\mu_t$ is the noise-signal ratio for uninformed traders. As shown in Appendix A.1, the
noise-signal ratio $\mu_t$ and constant $h_t$ are

$$\mu_t = \lambda^I \omega_{V,t} = \frac{\lambda^I}{\tau_V + \sum_{s=0}^t \tau_{\epsilon,s}}, \quad h_t = \frac{\lambda^I}{\tau_{\epsilon,t}}.$$  \hspace{1cm} (13)

Equation (13) implies that the noise-signal ratio for uninformed traders $\mu_t$ decreases over time since prices become more and more informative about informed traders’ estimation of the stock value.

At time $t$, given $A_t$ and $B_t$, the optimal demand schedule of a type $i$ investor ($i \in \{I, U\}$) is

$$\Theta^I_t(A_t, B_t) - \Theta^{i-1}_t(A_{t-1}, B_{t-1}) = \begin{cases} 
\frac{P^{iR}_t - A_t}{\gamma^I_t}, & A_t < P^{iR}_t, \\
0, & B_t \leq P^{iR}_t \leq A_t, \\
\frac{B_t - P^{iR}_t}{\gamma^U_t}, & B_t > P^{iR}_t,
\end{cases} \hspace{1cm} (14)$$

where $P^{iR}_t$ is the reservation price of type $i$ traders at time $t$ (i.e. type $i$ traders don’t trade at $P^{iR}_t$, a non-market-maker buys (sells, respectively) the security if and only if the ask price is lower (the bid price is higher, respectively) than this critical price). $\gamma^I_t$ is a constant coefficient which will be determined below ($\gamma^U_t$ and $\gamma^I_t$ can be computed by equations (A-60) and (A-72) respectively). Define the reservation price difference in the reservation prices of $I$ and $U$ investors at time $t$ as

$$\Delta_t = P^{IR}_t - P^{UR}_t.$$  \hspace{1cm} (15)

Define $P^i_t$ as the price at which type $i \in \{I, U\}$ traders are trading at time $t$. For $i \in \{I, U\}$, $P^i_t = A_t$ if a type $i$ trader buys at $t$, and $P^i_t = B_t$ if a type $i$ trader sells at $t$. For both informed and uninformed traders, the return on each share of the stock depends on whether they buy or sell since they buy from market makers at ask prices and sell to market makers for bid prices. For general case, the return depends on the signs of both $\Delta_{t+1}$ and $\Delta_t$. For example, if $\Delta_t > 0$ and $\Delta_{t+1} < 0$, then it can be shown that informed traders buy at time $t$ and sell at time $t + 1$, therefore $P^I_t = A_t$ and $P^I_{t+1} = B_{t+1}$. It follows that the return on each share of the stock $P^I_t - P^I_{t+1} = B_{t+1} - A_t$ in this example. Under the assumption that market makers do not carry inventory over time, it turns out that the trading price for type $i$ trader $P^i_t$ has the same expression regardless of the sign of $\Delta_t$. This resolves the path-dependent problems and makes the model tractable. The following theorem provides the equilibrium bid and ask prices and equilibrium quantities in closed-form.
Theorem 1. 1. The equilibrium ask and bid prices for $t = 0, 1, 2, 3, \ldots, T$ are

\[
A_t = P_{t}^{UR} + \frac{N_M}{N_M + 1} \frac{\gamma_I^U/N_U}{\gamma_I^I/N_I + \gamma_I^U/N_U} \Delta_t + \frac{\Delta^+_t}{N_M + 1},
\]

\[
B_t = P_{t}^{UR} + \frac{N_M}{N_M + 1} \frac{\gamma_U^U/N_U}{\gamma_U^I/N_I + \gamma_U^U/N_U} \Delta_t - \frac{\Delta^-_t}{N_M + 1}.
\]

(16)

where coefficients $\gamma_I^I$ and $\gamma_U^U$ can be computed recursively by equations (A-60) and (A-72).

Theorem 1 implies that the optimal ask (bid) price is the weighted average of the reservation price of the uninformed and that of the informed investors. The weights on $P_{t}^{IR}$ and $P_{t}^{UR}$ depend on informed and uninformed investors’ risk aversion and conditional variance of the fundamental value at time $t$, as well as the number of each type of traders.

2. The bid-ask spread is

\[
A_t - B_t = \frac{|\Delta_t|}{N_M + 1}.
\]

(17)

3. The equilibrium quantities demanded are

\[
N_I(\theta_I^I - \theta_{I-1}^I) = -N_U(\theta_U^U - \theta_{U-1}^U) = \frac{N_M}{N_M + 1} \frac{\Delta_t}{\gamma_I^I/N_I + \gamma_U^U/N_U}.
\]

(18)

4. The equilibrium ask and bid depths are

\[
\alpha_t = \beta_t = \frac{1}{N_M + 1} \frac{|\Delta_t|}{\gamma_I^I/N_I + \gamma_U^U/N_U}.
\]

(19)

We plot the above demand and supply functions and equilibrium spreads in Figure 1. Figure 1 shows that the higher the bid price $B_t$, the more a market maker can buy from other investors, and the lower the ask price $A_t$, the more a market maker can sell to other investors. Facing the demand and supply functions of other investors, an oligopolistic market maker optimally trades off the prices and quantities. Similar to the results of classical models on oligopolistic firms who set a market price to maximize profit, the bid and ask spread is equal
Figure 1. Demand/Supply Functions and Bid/Ask Spreads

to the absolute value of the reservation price difference $|\Delta_t|$ divided by $N_M + 1$. Different from these oligopolistic firms, however, the market makers are sellers in one market and buyers in another, and make profit from the bid-ask spread. In addition, as implied by Theorem 1, the difference between $P_t^{IR}$ ($P_t^{UR}$) and the ask (bid) price is also proportional to the absolute value of the reservation price difference $|\Delta_t|$. Therefore the trading amount of both $I$ and $U$ investors and thus the aggregate trading volume all increase with $|\Delta_t|$.

Therefore Theorem 1 implies that bid-ask spreads and trading volume can move in the same direction over time, because both trading volume and bid-ask spread increase with $|\Delta_t|$. Lin, Sanger, and Booth (1995) find that trading volume and effective spreads are positively correlated at the beginning and the end of the day. Chordia, Roll, and Subrahmanyam (2001) find that the effective bid-ask spread is positively correlated with trading volume. Our model suggests that these positive correlations may be caused by changes in the valuation difference of investors.

**Proposition 1.** The reservation prices for traders $I$ and $U$ are given by

\[
P_t^{IR} = \delta_t^I (\hat{V}_t^I - \mu_t X_t) + (1 - \delta_t^I)\hat{V} + g_t^I \hat{X}_t^I + f_t^I \bar{\theta} - \gamma_t^I \theta_{t-1}^I,
\]
\[
P_t^{UR} = \delta_t^U \hat{V}_t^U + (1 - \delta_t^U)\hat{V} + g_t^U \hat{X}_t^U + f_t^U \bar{\theta} - \gamma_t^U \theta_{t-1}^U.
\]

The reservation price difference at time $t$ is

\[
\Delta_t = (\delta_t^I - \delta_t^U)(\hat{V}_t^U - \hat{V}) + (g_t^I - g_t^U - \delta_t^I \mu_t)\hat{X}_t^U + (f_t^I - f_t^U)\bar{\theta} - \gamma_t^I \theta_{t-1}^I + \gamma_t^U \theta_{t-1}^U,
\]

where $\delta_t^U$ and $\delta_t^I$ can be computed by equations (A-61) and (A-73) respectively, $\gamma_t^U$ and $\gamma_t^I$
can be computed by equations (A-60) and (A-72), $f_t^I$ and $f_t^U$ can be computed by equations (A-63) and (A-75) respectively, and $g_t^I$ and $g_t^U$ can be computed by equations (A-62) and (A-74) respectively. Their values at $t = T$ are given by equations (A-95) and (A-96).

Proposition 1 implies that the reservation price for informed traders is a linear function of state variables $S_t = \hat{V}_t^I - \mu_t X_t$, $\hat{X}_t^U$, and informed investors’ previous inventory level. The reservation price for uninformed investors is a linear function of $\hat{V}_t^U$, $\hat{X}_t^U$, and uninformed investors’ previous inventory level. Interestingly, informed investors’ reservation price at trading period $t$ depends on uninformed investors’ estimate about the amount of informed investors’ non-tradable asset $\hat{X}_t^U$. This is because $\hat{X}_t^U$ affects the expected trading price at period $t + 1$ which in turn affects informed traders’ reservation price at time $t$.

Proposition 1 also implies that the difference in reservation prices $\Delta_t$ depends on $\hat{X}_t^U$, total supply, traders’ previous inventory, as well as $\hat{V}_t^I - \hat{V}$. If liquidity shocks only arrived at the beginning, then it can be proved that the difference in reservation prices $\Delta_t$ would not depend on $\hat{V}_t^I - \hat{V}$. Intuitively, if the uninformed investors had the same liquidity shock at the beginning, then the uninformed investors would just trade the same amount as the informed investors (which can be inferred from the equilibrium price) and thus would have the same reservation price as the informed investors; thus the reservation price difference comes from the difference in the estimation of the liquidity shock. If both private information and endowment shocks might arrive at each period, then the difference in reservation prices depends on both $\hat{X}_t^U$ and $\hat{V}_t^U$.

4 Dynamics of Trading Volume and Bid-Ask Spread

In this section, we study the dynamics of trading volume and bid-ask spread under different specifications of the information flow and endowment shocks. To make our intuition more clear, in Sections 4.1 and 4.2, we mainly focus on the case where traders only receive endowment shocks at the beginning and they rebalance portfolios whenever they observe new information about the fundamental value of the asset. To illustrate key heterogenous effects of speculation and hedging on the dynamics of trading volume and bid-ask spread, in Section 4.3, we analyze more general cases where traders rebalance portfolios due to new information arrival or additional endowment shocks after market opening.
4.1 Dynamics of Trading Volume

Theorem 1 implies that the trading volume at time \( t \), denoted by \( \text{Vol}_t \), is

\[
\text{Vol}_t = \frac{N_M}{N_M + 1} \frac{|\Delta_t|}{\gamma_t^I/N_I + \gamma_t^U/N_U}.
\]

(24)

It can be shown that \( \Delta_t/(\gamma_t^I/N_I + \gamma_t^U/N_U) \) is normally distributed with mean \( \mu_t^\Delta \) and variance \( (\sigma_t^\Delta)^2 \) and thus the average trading volume at time \( t \) is

\[
E[\text{Vol}_t] = \frac{N_M}{N_M + 1} \left[ \sqrt{\frac{2}{\pi}} \sigma_t^\Delta e^{-\frac{1}{2} \frac{(\mu_t^\Delta)^2}{(\sigma_t^\Delta)^2}} + \mu_t^\Delta \left( 1 - 2 \Phi \left( -\frac{\mu_t^\Delta}{\sigma_t^\Delta} \right) \right) \right],
\]

(25)

where \( \Phi \) is normal cumulative distribution function, \( \mu_t^\Delta \) is given by Equation (A-107) and \( \sigma_t^\Delta \) is given by Equation (A-111) in Appendix A.4. Using equation (25), we can compute the average trading volume at each period \( t \).

Figure 2 illustrates the dynamics of the expected trading volume against trading duration for different \( N_M \). Parameter values are \( T = 20, N_I = 10, N_U = 100, \lambda^I = 0.8, \lambda^U = 1, \tilde{\theta} = 1.1, \tau_V = 0.9, \tau_{\eta,t} = 10^6 \) except for \( \tau_{\eta,0} = 1.1, \) and \( \tau_{\varepsilon,t} = 10^{-6} \) except for \( \tau_{\varepsilon,0} = 1 \) and \( \tau_{\varepsilon,T} = 0.1 \).

Figure 2 illustrates the dynamics of the expected trading volume for the cases when \( N_M = 1 \) (solid curve) and \( N_M = 10 \) (dashed curve) against the trading periods \( t/T \) when information arrives mainly at the beginning and the end of the trading day.\(^{15}\) As we can

\(^{15}\)The dynamics of trading volume and bid-ask spread depend on the structure of information flow and liquidity shocks, however, under each specification of information flow and endowment shocks, the general pattern of all figures in this section stays qualitatively the same under a wide range of parameter values. Note that the new endowment shock \( \eta_t \sim \mathcal{N}(0, \tau_{\eta,t}^{-1}) \). In all figures thoughtout this section, when we set
see from Figure 2, the expected trading volume exhibits a U-shaped pattern against the trading duration—high trading volume at the beginning and the end of the trading day and low trading volume in the middle of the day (e.g., Jain and Joh (1988)). The higher trading volume at the beginning and at the end of the day results from greater trading incentives between informed and uninformed traders due to new information arrivals and hedging demand.

Even though both informed and uninformed traders are not strategic, they optimally choose which period to trade as well as the optimal trading quantity at that period. Interestingly, Figure 2 illustrates that traders tend to trade more patiently over a longer period when market makers are less competitive (e.g., $N_M$ is smaller). In this example, after information arrives at the beginning, most of the trades occur in the first two rounds when $N_M = 10$ while most of the trades are done in approximately eight rounds when $N_M = 1$. This is because the monopoly power of market makers makes it too costly for other traders to trade aggressively. When competition among market makers increases, the average of bid-ask spread decreases and it becomes less costly for traders to trade more aggressively at the beginning when information arrives or at the end to re-balance their portfolio to optimally hedge the risk from the non-tradable asset. Our model implies that when market makers have significant market power, other traders trade gradually over time even though they are not strategic and prices in the trades after the first round reveal the same information as revealed by the price in the first trade. This is consistent with the empirical findings of Holthausen et al. (1990). They examine how quickly prices attain new equilibrium levels after large-block transactions and find that prices adjust within at most three trades, with most of the adjustment occurring in the first trade.

Figure 3 depicts how the uncertainty about the hedging demand and the precision of informed traders’ private information at market opening affect the magnitude of the U-shaped pattern of the trading volume. The left subfigure illustrates the dynamics of the expected trading volume for the case when $\tau_{\eta,0} = 0.5$ (dashed curve) and $\tau_{\eta,0} = 1.1$ (solid curve) against the trading periods $t/T$. The right subfigure illustrates the dynamics of the expected trading volume when $\tau_{\epsilon,0} = 1$ (solid curve) and $\tau_{\epsilon,0} = 2$ (dashed curve). As illustrated in Figure 3, the U-Shape tends to be more substantial when the volatility of hedging demand is higher (e.g., $\tau_{\eta,0}$ is lower) or when informed traders’ private information is noisier (e.g., $\tau_{\epsilon,0}$ is lower). This is because uninformed traders are more willing to trade with informed traders when it is more likely that informed traders trade to hedge the risk from

$\tau_{\eta,t} = 10^6$, then $\tau_{\eta,t}^{-1} \approx 0$. This implies $\eta_t \approx 0$ and there is no endowment shock at date $t$. Information noise $\varepsilon_t \sim N(0, \tau_{\epsilon,t}^{-1})$, when we use $\tau_{\epsilon,t} = 10^{-6}$, then it means the information is purely noise, i.e., at time $t$, there is no information arrival.
The non-tradable asset instead of speculating on their private information. An empirically testable implication of Figures 2 and 3 is that the U-shaped pattern of the trading volume tends to be more pronounced for stocks with less amount of private information, more hedgers, or more market makers.

To further illustrate the impact of information arrivals on the dynamics of trading volume, we examine both the case when information only arrives at certain time and the case when information arrives gradually at a constant rate while informed traders receive no new endowment shocks after $t = 0$. Panel (a) in Figure 4 illustrates the dynamics of trading volume for the case when information arrives only at certain points of time during the trading period for $N_M = 1$ (solid curve) and $N_M = 10$ (dashed curve). As we can see from this figure, the arrivals of new information generate spikes in trading volume. In some sense, the arrival of information has a ripple effect on the average trading volume. More interestingly, this information ripple effect tends to be more substantial when the competition among market makers becomes more intensive. When there are fewer market makers, the market power of market makers makes trading aggressively more costly. Therefore, traders tend to smooth out their trading over a longer period. This implies that the monopoly power of market makers tends to dampen trading spikes due to information arrivals. An empirically testable implication is that the variation of trading volume tends to be smaller for stocks with less competitive market makers during periods of news events.

Panel (b) in Figure 4 illustrates the dynamics of trading volume for the case when infor-
Figure 4. The Dynamics of Expected Trading Volume Against Trading Duration for Different Information Structure. Parameter values are \( T = 20, N_I = 10, N_U = 100, \lambda_I = 0.8, \lambda_U = 1, \bar{\theta} = 1.1, \tau_V = 0.9, \tau_{\eta,t} = 10^6 \) except for \( \tau_{\eta,0} = 1.1 \). In Panel (a), \( \tau_{\varepsilon,t} = 10^{-6} \) except for \( \tau_{\varepsilon,0} = \tau_{\varepsilon,7} = \tau_{\varepsilon,15} = 1 \). In Panel (b), \( \tau_{\varepsilon,t} = 3/T \).

4.2 Dynamics of Bid-Ask Spread

Theorem 1 implies that the bid and ask spread is equal to the absolute value of the reservation price difference \( |\Delta_t| \) divided by \( N_M + 1 \) and thus the expected bid-ask spread at period \( t \) is given by

\[
E[A_t - B_t] = \frac{1}{N_M + 1} E|\Delta_t|.
\]  

Since both trading volume and bid-ask spread increase with \( |\Delta_t| \), the expected bid-ask spread tends to be proportional to the expected trading volume. When information arrives mainly at the opening and closing of the market, the average bid-ask spread exhibits a U-shaped pattern against the trading duration—wider bid-ask spreads at the beginning and the end of the trading day and narrower bid-ask spread in the middle of the day, as illustrated.
in Figure 5.

![Graph](image)

**Figure 5.** The Dynamics of Expected Bid-Ask Spread Against Trading Duration for Different $N_M$ and $T$. Parameters are $N_I = 10$, $N_U = 100$, $\lambda^I = 0.8$, $\lambda^U = 1$, $\theta = 1.1$, $\tilde{V} = 1$, $\tau_V = 0.9$, $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = 1.1$, $\tau_{\varepsilon,t} = 10^{-6}$ except for $\tau_{\varepsilon,0} = 1$ and $\tau_{\varepsilon,T} = 0.1$.

Figure 5 depicts how the number of market makers and the number of trading rounds affect the magnitude of the U-shaped pattern of the bid-ask spread. The left subfigure plots the dynamics of the expected bid-ask spread for the case when $N_M = 1$ and $T = 10$ (dot-dashed curve) and $N_M = 1$ and $T = 20$ (dashed curve) against the trading periods $t/T$. The right subfigure illustrates the dynamics of the expected bid-ask spread when $N_M = 1$ and $T = 10$ (dot-dashed curve) and $N_M = 5$ and $T = 10$ (dashed curve). As illustrated in Figure 5, the U-Shape tends to be more substantial when there are fewer market makers or fewer trading rounds since market makers tend to have greater market power and can optimally set wider bid-ask spreads. An empirical testable implication of Figures 5 is that the U-shaped pattern of the bid-ask spread tends to be more pronounced for stocks with fewer market makers.

From equations (24) and (26), we can see that, different from the average bid-ask spread, average trading volume not only depends on the absolute difference of reservation prices $|\Delta_t|$, but also depends on $1/ (\gamma^I_t / N_I + \gamma^U_t / N_U)$ which serves as a multiplier. The multiplier $1/ (\gamma^I_t / N_I + \gamma^U_t / N_U)$ depends on how much weight traders put on the next period’s information which tends to decrease over time. As a result, different from the dynamics of trading volume which tends to decrease over time and might exhibit a L-shape, when information arrives at a constant rate over time, bid-ask spread exhibits a U-shape, as illustrated in Figure 6, similar to the case when information only arrives at market opening and closing. The wider bid-ask spread at the end of the trading period is due to a larger reservation price difference toward market closure since traders will not be able to trade with each other anymore after period $T$. Figure 6 illustrates the dynamics of expected bid-ask spread for the case $T = 20$.
(dot-dashed curve) and the case $T = 50$ (dashed curve). As we can see from Figure 6, the expected bid-ask spread decreases when traders can trade for more rounds (e.g., a larger $T$). Figure 6 also implies that the U-Shape tends to be more substantial when there are fewer trading rounds, consistent with Figure 5.

![Figure 6](image_url)

**Figure 6.** The Dynamics of Expected Bid-Ask Spread Against Trading Duration with Information Arriving at A Constant Rate. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 100$, $\lambda^I = 0.8$, $\lambda^U = 1$, $\bar{\theta} = 1.1$, $\bar{V} = 1$, $\tau_V = 0.9$, $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = 1.1$, and $\tau_{\epsilon,t} = 3/T$.

We further examine the impact of increasing trading rounds on both bid and ask prices. As depicted in Figure 7, the expected bid price is higher and the expected ask price is lower when traders have more rounds of trading. This implies that traders pay lower purchase price ($A_t$ is lower) to market makers and get higher sale price ($B_t$ is higher) from market makers for each round of trading when there are more rounds of trading.

### 4.3 Heterogenous Effects of Speculation and Hedging on the Dynamics of Bid-Ask Spread and Trading Volume

In this subsection, we examine more general cases where traders update their portfolio for either speculation or hedging motives. We first illustrate how arrivals of new private information and endowment shocks affect the dynamics of trading volume. Figure 8 illustrates the case when informed traders receive new endowment shocks at dates $t = 0, 7, 14$. In addition to the endowment shocks, informed traders observe private information at date 0 in Panel (a) and they observe private information at dates $t = 0, 7, 14$ in Panel (b). As illustrated in Panel (a) of Figure 4, the arrivals of new information generate spikes in trading volume. The arrival of information has a ripple effect on the average trading volume and the
Figure 7. The Dynamics of Expected Ask and Bid Prices Against Trading Duration for Different $T$. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 100$, $\lambda^I = 0.8$, $\lambda^U = 1$, $\bar{\theta} = 1.1$, $\bar{V} = 1$, $\tau_V = 0.9$, $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = 1$, and $\tau_{\eta,T} = 3$, $\tau_{\varepsilon,t} = 3/T$.

The magnitude of the effect tends to get smaller over time. Figure 8 shows that the arrivals of new endowment shocks also generate spikes in trading volume, however, different from the ripple effects from information arrivals, the ripple effects from arrivals of endowment shocks tend to become larger over time when the number of market makers is not too small. Intuitively, both uninformed investors and market makers become less uncertain about the fundamental value of the asset in later periods with more information revealed by trading. The adverse selection effect tends to be smaller over time and thus informed traders optimally postpone trading due to hedging demands until a later time when the market power of market makers is not significant. If market makers’ market power is significant (e.g., $N_M = 1$), traders optimally hedge some of the risk from endowment shocks in earlier periods because they would have to pay wider bid-ask spread if they hedge all the risk near the market closure.
In general, informed traders tend to trade aggressively to speculate on their private information since others trade on the same information. However, they tend to wait until later periods to rebalance their portfolio with new arrivals of endowment shocks to reduce the adverse selection cost. Due to market makers’ market power, they smooth out trading due to either private information or endowment shocks to reduce the total trading cost. In Panel (a) of Figure 8, at period $t = 7$, even though the trading volume spike due to new endowment shocks can be more significant with $N_M = 1$ than that with $N_M = 5$, we can see that the market power of market makers still tend to dampen the overall trading volume spikes—the solid curve ($N_M = 1$) is smoother than the dashed curve ($N_M = 5$) over the whole trading period. Panel (b) of Figure 8 depicts the case when informed traders observe new information and receive new endowment shocks at dates $t = 0, 7, 14$. Similar to Panel (a) of Figure 4 and Panel (a) of Figure 8, the market power of market markers dampen overall trading volume spikes. Different from Panel (a) of Figure 4, because of new arrivals of endowment shocks, the trading volume spikes become more significant in a later period.

Figure 9 illustrates the dynamics of bid-ask spread and trading volume when informed traders observe private information and receive endowment shocks at market opening and informed traders only receive endowment shocks at a constant rate after market opening. Different from the case where only information arrives at a constant rate after market opening, bid-ask spread increases over time, as shown in the right panel of Figure 9. This is
Figure 9. The Dynamics of Expected Trading Volume and Bid-Ask Spread Against Trading Duration with Endowment Shocks Arriving at A Constant Rate. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 100$, $T = 20$, $\lambda^I = \lambda^U = 1$, $\bar{\theta} = 1.1$, $\bar{V} = 1$, $\tau_V = 0.9$, $\tau_{\epsilon,t} = 10^{-6}$ except for $\tau_{\epsilon,0} = 1$, and $\tau_{\eta,t} = 10$.

because the reservation price difference increases over time with new arrivals of endowment shocks. Anticipating wider bid-ask spread toward market closure, to reduce total trading costs, other traders optimally shift some trading to earlier periods instead of waiting until market closure, as illustrated in the left panel of Figure 9.

We next study the dynamics of bid-ask spread and trading volume when informed traders observe private information and receive endowment shocks at each period. As we can see from Figure 10, both trading volume and bid-ask spread exhibit U-shaped patterns. The intuition is as follows. Even though both informed and uninformed traders tend to smooth out their trading due to market makers’ market power, informed traders tend to act fast on their private information since others might trade on the same information and they tend to postpone their trading due to hedging needs until later dates to reduce trading costs because of smaller adverse selection effect. In addition, informed traders optimally hedge some of the risk from endowment shocks at earlier periods prior to market closure when market makers have significant market power ($N_M = 1$ in Figure 10). As a result, trading volume tends to be concentrated at the market opening and closure and the peak of trading volume near market closure might be reached at a period prior to market closure when market makers’ market power is significant. Since market makers can charge wider bid-ask spreads when the trading incentives are greater, it follows that bid-ask spreads are wider at market opening and closure.
Figure 10. The Dynamics of Expected Trading Volume and Bid-Ask Spread Against Trading Duration with Both Information and Endowment Shocks Arriving at Constant Rates. Parameter values are $T = 14$, $N_M = 1$, $N_I = 10$, $N_U = 100$, $\lambda^I = \lambda^U = 1$, $\bar{\theta} = 1.1$, $\tau_V = 0.9$, $\tau_{\eta,t} = 10$ and $\tau_{\epsilon,t} = 0.1$.

5 Slow Versus Fast Markets

In this section, we study how trading frequency affects the profits of market makers from making the market and the welfare of other traders.

5.1 Market Makers’ Profits

In our model, market makers trade in both the “ask” market and the “bid” market and optimally make offsetting trades with other traders to maximize the expected profit from each round of trading. The total profit for each market maker in a market with $N_M$ oligopolistic market makers and $T$ trading rounds is given by

$$\text{Profit}(N_M, T) = \frac{1}{(N_M + 1)^2} \sum_{t=0}^{T} \left| \gamma_{it}^I/N_I + \gamma_{it}^U/N_U \right|.$$

Therefore, the expected profit for each market maker is

$$\mathbb{E}[\text{Profit}(N_M, T)] = \frac{1}{N_M^2} \sum_{t=0}^{T} (\gamma_{it}^I/N_I + \gamma_{it}^U/N_U) \left[ (\mu_{it}^\Delta)^2 + (\sigma_{it}^\Delta)^2 \right],$$

where $\mu_{it}^\Delta$ is as defined in equation (A-107) and $\sigma_{it}^\Delta$ is as defined in equation (A-111) in the Appendix A.4.
Figure 11 illustrates how market makers’ profit changes with the number of trading rounds when information and endowment shocks only arrive at market openings, i.e., $\tau_{i,t} = 0$ except for $t = 0$. Surprisingly, as we can see from Panels (a) and (b) of Figure 11, a faster market with more trading rounds makes market makers less profitable. This is because, with more trading rounds, informed and uninformed traders tend to spread out their orders over time in oligopolistic markets to reduce total trading costs due to bid-ask spread. Both the average bid-ask spread and trading volume at each trading round decrease with the number of trading periods, thus market makers’ total profit tends to be reduced.

Panels (a) and (b) of Figure 12 illustrate how market makers’ profit changes with the number of trading rounds when endowment shocks only arrive at beginning and information arrives at a constant rate over time for two different total precisions of information flows, i.e., $\sum_{t=1}^{T-1} \tau_{i,t} = 2$ (dashed curve) or 3 (solid curve). In this case, as the number of trading
rounds increases, market makers’ profit first increases and then decreases. Therefore, when information arrives at a constant rate, there exists an optimal trading frequency for market makers. Intuitively, adding more trading rounds has two opposite effects on market makers’ expected profit. On one hand, it decreases both the trading volume and the expected bid-ask spread as illustrated in Figure 6 since traders tend to spread out their trading over more trading periods to reduce the average bid-ask spreads. On the other hand, it increases the trading opportunities for market makers to charge the bid-ask spread for more rounds even though the magnitude of the bid-ask spread for each round decreases.

Figure 12. Market Makers’ Expected Total Profit Against the Number of Trading Rounds with Information Arriving at A Constant Rate. Parameters are $N_I = 10$, $N_U = 100$, $\lambda^I = 0.8$, $\lambda^U = 1$, $\bar{\theta} = 1.1$, $\bar{V} = 1$, $\tau_V = 0.9$, $\bar{\eta}, t = 10^6$ except for $\tau_{\eta, 0} = 1.1$, and $\tau_{\epsilon, t}$ as indicated.

As we can see from Figure 12, when the total precision of information becomes higher (solid curve), then the benefit from being able to trade more rounds increases, and thus the optimal number of trading rounds for market makers is larger. Figure 12 also implies that, oligopolistic market makers might not necessarily prefer to make the market more frequently. In addition, comparing the left subfigure ($N_M = 1$) to the right subfigure ($N_M = 5$), we can see that the optimal number of trading rounds is smaller when there are fewer market makers. Our model might help explain why in some OTC markets (e.g., municipal bond markets) where market makers have significant market power, yet the trading frequency is relatively low.

Figures 13 and 14 illustrate how market makers’ profit changes with the number of trading rounds. Panel (a) of Figure 13 depicts the case when a new piece of private information arrives at date $T/2$ in addition to the private information and endowment shocks at 0. Panel (a) of Figure 14 depicts the case when informed traders observe private information
Figure 13. Market Makers’ Expected Total Profit Against the Number of Trading Rounds with only Information (Panel (a)) or Endowment Shocks (Panel (b)) Arriving After Market Opening. Parameter values are $N_M = 1, N_I = 10, N_U = 100, \lambda^I = \lambda^U = 1, \hat{\theta} = 1.1, \tau_V = 0.9$. In Panel (a), $\tau_{\varepsilon,t} = 10^{-6}$ expect for $\tau_{\varepsilon,0} = \tau_{\varepsilon,T/2} = 1$, and $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = 3$. In Panel (b), $\tau_{\varepsilon,t} = 10^{-6}$ expect for $\tau_{\varepsilon,0} = 1$, and $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = \tau_{\eta,T/2} = 3$.

and receive endowment shocks at both dates 0 and $T/2$ while the trading due to private information tends to dominate the trading due to hedging (by setting relatively large value for $\tau_{\varepsilon,T/2}$). Consistent with Panel (a) of Figure 11, when information only arrives at certain times, a faster market with more trading rounds makes market makers less profitable. The intuition is similar. With more trading rounds, informed and uninformed traders spread out their orders over time. Both the average bid-ask spread and trading volume at each trading round decrease with the number of trading periods, thus market makers’ total profit tends to be reduced.

Panel (b) of Figure 13 depicts the case when new endowment shock is received at date $T/2$ in addition to the private information and endowment shocks at 0. Panel (b) of Figure 14 depicts the case when informed traders observe private information and receive endowment shocks at both dates 0 and $T/2$ while the trading due to hedging tends to dominate the trading due to private information (by setting relatively small value for $\tau_{\varepsilon,T/2}$). In this case, as the number of trading rounds increases, market makers’ profit first increases and then decreases. Intuitively, since informed traders tend to postpone trading due to endowment shocks to later dates because the adverse selection effects tend to be smaller over time. If traders can only trade fewer rounds, they tend to optimally trade less due to larger adverse selection effects. This in turn might decrease market makers’ trading profit. Therefore, adding more trading rounds initially increases market makers’ profit. If adding too many
trading rounds, then both the trading volume and the expected bid-ask spread are reduced since traders tend to spread out their trading over more trading periods. As a result, market makers’ profit starts to decrease again, as illustrated in Panel (b) of Figures 13 and 14.

Figure 14. Market Makers’ Expected Total Profit Against the Number of Trading Rounds with Both New Information and Endowment Shocks Arriving After Market Opening. Parameter values are $N_M = 1, N_I = 10, N_U = 100, \lambda^I = \lambda^U = 1, \bar{\theta} = 1.1, \tau_V = 0.9$. In Panel (a), $\tau_{\epsilon,t} = 5$ and $\tau_{\eta,t} = 10^2$. In Panel (b), $\tau_{\epsilon,t} = 1$ and $\tau_{\eta,t} = 10$.

5.2 Welfare of Informed and Uninformed Traders

We now study how trading frequency affects informed and uninformed traders’ welfare. Trader $i$’s ($i \in \{I, U\}$) welfare is defined as

$$W^i = E[ J^i_t ] ,$$

where $J^i_t$ is defined in the proof of Proposition 1 in the Appendix. Informed and uninformed traders’ welfare is computed in Appendix A.5.

If information and endowment shocks only arrive at market openings, as illustrated in Figure 15, informed and uninformed traders’ welfare increases monotonically when the number of trading rounds increases.

We also examine the case when information may arrive gradually over time or endowment shocks arrive at certain time after market openings or gradually over time and the case when both information and endowment shock arrive over time, we find that both informed and uninformed traders still benefit from higher trading frequency. This suggests that, regardless of the information structure, more trading rounds tend to reduce market makers’ power and
Figure 15. Informed and Uninformed Traders’ Welfare Against the Number of Trading Rounds. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 100$, $\lambda^I = 0.8$, $\lambda^U = 1$, $\tilde{\theta} = 1.1$, $\tilde{V} = 1$, $\tau_V = 0.9$, $\tau_{\eta,t} = 10^6$ except for $\tau_{\eta,0} = 1.1$, and $\tau_{\epsilon,t} = 0$ except for $\tau_{\epsilon,0} = 3$.

benefit other market participants. Since market makers might be worse off with more trading rounds (Figures 13 and 14) while both informed and uninformed traders are better off with more trading rounds, the total certainty equivalent wealth of all market participants can be better off or worse off with more trading rounds depending on the population weights of market makers and investors.

6 Practical Implications

Our model captures main features of many financial markets where market makers play significant roles and have market power. However, despite their market power, many market makers have become very risk averse after financial crisis and try to avoid the risk of long or short positions. We now discuss some practical implications of our model.

6.1 Customer Liquidity Provision

After the Volcker Rule and new banking regulations following the financial crisis of 2008, traditional liquidity providers (dealers) in bond markets become more risk averse. Non-dealers (customers) increasingly provide liquidity to other customers who are demanding liquidity. Our model captures this feature. Uninformed traders in our model are providing liquidity since they trade in the opposite direction to that of informed traders who are demanding liquidity. Intuitively, liquidity demanders should pay most of the trading cost while non-dealer liquidity suppliers should be able to trade at a price comparable to competitive
Figure 16. Informed traders’ expected trading price (dashed curve), uninformed traders’ trading price (dot-dashed curve), and expected competitive price (solid curve) against trading duration. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 20$, $\lambda^I = \lambda^U = 1$, $T = 10$, $V = 1$, $\tau_V = 0.9$, $\tau_{\varepsilon,t} = 0.1$, $\tau_{\eta,t} = 10$, and total supply $\bar{\theta} = 1.1$. Each Investor’s initial endowment is $\bar{\theta}^I = \bar{\theta}^U = 0.037$ in Panel (a) and $\bar{\theta}^I = 0.11$, $\bar{\theta}^U = 0$ in Panel (b).

Figure A.8 plots informed traders’ expected trading price $E[P^I_t]$ (dashed curve), uninformed traders’ trading price $E[P^U_t]$ (dot-dashed curve), and expected competitive price (solid curve) against trading duration. Figure A.8 presents both the case where traders do not change their trading directions (Panel (a)) and the case where traders might optimally change their trading directions at some point of time (Panel (b)) due to new arrivals of information or endowment shocks. As we can see from Figure A.8, consistent with our intuition, informed traders always pay most of the bid-ask spread regardless of their trading directions while uninformed traders’ expected trading price does not deviate much from the expected competitive price. As liquidity suppliers, uninformed traders in our model get much more favorable prices from market makers.

Our result is consistent with the empirical findings of Choi and Huh (2018) who document that customers increasingly provide liquidity particularly after the new banking regulations were adopted following the 2008 financial crisis. They argue that conventional bid-ask spread measures underestimate the cost of dealers’ liquidity provision to customers. They also find that customers who are demanding liquidity from dealers tend to pay higher spreads, this is consistent with our model prediction.

Even though a dynamic setting where market makers might carry inventory is not tractable in our paper, the one-period version of our model can be applied to study the impact of inventory shocks of a risk-averse market maker on other market participants’ trad-
Figure 17. Expected ask price (solid curve), expected bid price (dashed curve), and expected competitive price (dot-dashed curve) against $\sigma_X$. Parameters are $N_M = 1$, $N_I = 10$, $N_U = 100$, $\lambda^I = \lambda^U = 1$, $\bar{V} = 3$, $\tau_V = 0.4$, $\tau_{\varepsilon,0} = 0.8$, $\bar{\theta}^I = \bar{\theta}^U = 1$, $\bar{\theta}^M = 200$ in Panel (a) and $\bar{\theta}^M = -200$ in Panel (b).

6.2 Order Splitting in Corporate Bond Markets

The number of US primary dealers has declined to 22 in 2013 from 46 in 1988.\footnote{Please see detailed analysis in Appendix A.8.} When the number of market makers decreases, the competition among market makers becomes less

\footnote{See, for example, BlackRock’s Report (2014).}
intensive and thus market maker’s market power becomes more significant. As predicted by our model, traders trade more smoothly over a longer period because the monopoly power of market makers increases the cost of trading too aggressively. This implies that traders tend to slice large orders when market makers become less competitive. As illustrated in Figure 18, after financial crisis of 2008, the average number of trades has increased but the size of these trades has declined. Bessembinder et al. (2018) also find that block trade frequency and the average trade size decreased during the bank regulatory period and the Volcker period. This is consistent with our model prediction.

7 Conclusion

In this paper, we develop a multi-period model of trading in which market makers have market power and set bid and ask prices in the presence of information asymmetry and hedging demands. We find that the dynamics of trading volume and bid-ask spread are largely determined by the structure of the information flow and hedging needs. Both trading volume and bid-ask spread exhibit U-shape patterns against the trading duration when both information and liquidity shocks arrive over time or they mainly arrive at the opening and closing of the market. The arrival of information or liquidity shocks has a ripple effect on trading volume. Traders smooth out their trading over time even though information is incorporated into prices in the first trade. This is consistent with empirical findings that most of the price adjustment occurs in the first trade and traders tend to split large orders when the competition among market makers becomes less intensive.
The market power of market makers tends to dampen trading spikes due to either new information arrivals or liquidity shocks. Moreover, when trading frequency increases, our model suggests that market makers’ profit monotonically decreases if information arrives only at certain times while it first increases and then decreases if information arrives gradually over time. Both informed and uninformed traders’ welfare increases monotonically when the number of trading rounds increases.

Our model generates several new testable empirical predictions. For example, our model predicts that the variation of trading volume tends to be smaller for stocks with less competitive market makers during periods of news events. The U-shaped pattern of the bid-ask spread tends to be more pronounced for stocks with fewer market makers while the U-shaped pattern of the trading volume tends to be more pronounced for stocks with less amount of private information, more hedgers, or more market makers. Our model explains the empirical findings that liquidity demanders pay most of the bid-ask spread while non-dealer liquidity suppliers get much more favorable prices from market makers. Moreover, our model can also help shed some light on the impact of trading platforms regulation on traders’ welfare. Increasing trading frequency tends to hurt market makers, but it benefits both informed and uninformed traders.
Appendix

A.1 Proof of Lemma 1

To get Lemma 1, we use the results in the following lemma.

Lemma 2. The system of interest is a $n \times 1$ vector of underlying state variables: $z_t$, which follows the process $z_t = a_{z,t}z_{t-1} + b_{z,t}\zeta_t$, where $\zeta_t$ is a $k \times 1$ the process noise vector. Moreover, assume $\zeta_t$ is white noise distributed as $\mathcal{N}(0, \Sigma_{\zeta,t})$. Thus, $a_{z,t}$ is a $n \times n$ matrix and $b_{z,t}$ is a $n \times k$ matrix. The signals are given by an $m \times 1$ vector, $s_t = a_{s,t}z_t + b_{s,t}\zeta_t$, where the signal noise is perfectly correlated with process noise, $a_{s,t}$ is an $m \times n$ matrix, and $b_{s,t}$ is an $m \times k$ matrix.

The first and second moment estimations $\hat{z}_t = \mathbb{E}_t[z_t], o_t = \mathbb{E}[(z_t - \hat{z}_t)(z_t - \hat{z}_t)^\top], \text{ where } \mathbb{E}_t[.] = \mathbb{E}_.|\mathcal{S}_t]$ and $\mathcal{S}_t$ contains the information up to time $t$, $\mathcal{S}_t = \{s_0, s_1, s_2, ..., s_t\}$. Then Kalman filtering is

\begin{align}
\hat{z}_t &= a_{z,t}\hat{z}_{t-1} + k_t[s_t - \mathbb{E}_{t-1}[s_t]], \\
o_t &= (I_{n\times n} - k_t a_{s,t})(a_{z,t}o_{t-1}a_{z,t}^\top + b_{z,t}\Sigma_{\zeta,t}b_{z,t}^\top) - k_t b_{s,t}\Sigma_{\zeta,t}b_{z,t}^\top,
\end{align}

(A-1)

where $k_t$ is defined in equation (A-4).

Proof of Lemma 2: Kalman filtering contains two parts: predict and update. The predicted (a priori) state estimate is $\hat{z}_{t|t-1} = \mathbb{E}_{t-1}[z_t] = a_{z,t}\hat{z}_{t-1}$, and the innovation (measurement pre-fit residual) is

\begin{equation}
s_t - \mathbb{E}_{t-1}[s_t] = a_{s,t}z_t + b_{s,t}\zeta_t - a_{s,t}\hat{z}_{t|t-1} = a_{s,t}a_{z,t}(z_{t-1} - \hat{z}_{t-1}) + (a_{s,t}b_{z,t} + b_{s,t})\zeta_t.
\end{equation}

(A-2)

The innovation (pre-fit residual) variance and covariance matrix are

\begin{align}
\text{Var}_{t-1}[s_t] &= a_{s,t}a_{z,t}o_{t-1}a_{z,t}^\top + (a_{s,t}b_{z,t} + b_{s,t})\Sigma_{\zeta,t}(a_{s,t}b_{z,t} + b_{s,t})^\top, \\
\text{Cov}_{t-1}(z_t, s_t) &= a_{z,t}o_{t-1}a_{z,t}^\top + b_{z,t}\Sigma_{\zeta,t}(a_{s,t}b_{z,t} + b_{s,t})^\top,
\end{align}

(A-3)

and the optimal Kalman gain $k_t$ is

\begin{equation}
k_t = \text{Cov}_{t-1}(z_t, s_t)(\text{Var}_{t-1}[s_t])^{-1},
\end{equation}

(A-4)

where $\text{Cov}_{t-1}(z_t, s_t)$ and $\text{Var}_{t-1}[s_t]$ are as defined in equation (A-3). Therefore, the updated (a posteriori) state estimate $\hat{z}_t$ and the updated (a posteriori) estimate covariance matrix $o_t$ are

\begin{align}
\hat{z}_t &= a_{z,t}\hat{z}_{t-1} + k_t[s_t - \mathbb{E}_{t-1}[s_t]], \\
o_t &= \mathbb{E}[(z_t - \hat{z}_t)(z_t - \hat{z}_t)^\top],
\end{align}

(A-5)
where

\[ z_t - \hat{z}_t = (I_{n \times n} - k_t a_s,t) a_{z,t}(z_{t-1} - \hat{z}_{t-1}) + (b_{z,t} - k_t (a_s,t b_{z,t} + b_{s,t}))\zeta_t. \]  

(A-6)

From equation (A-4), we have \( k_t \text{Var}_{t-1}[s_t] k_t^\top = \text{Cov}_{t-1}(z_t, s_t) k_t^\top. \) Therefore, the updated estimate covariance matrix can be simplified as

\[
\begin{align*}
\alpha_t &= (I_{n \times n} - k_t a_s,t) a_{z,t}\alpha_{t-1} a_{z,t}^\top (I_{n \times n} - k_t a_s,t)^\top + (b_{z,t} - k_t (a_s,t b_{z,t} + b_{s,t})) \Sigma_{\zeta,t} (b_{z,t} - k_t (a_s,t b_{z,t} + b_{s,t}))^\top \\
&= (I_{n \times n} - k_t a_s,t) (a_{z,t}\alpha_{t-1} a_{z,t}^\top + b_{z,t} \Sigma_{\zeta,t} b_{z,t}^\top) - k_t b_{s,t} \Sigma_{\zeta,t} b_{z,t}^\top.
\end{align*}
\]  

(A-7)

Q.E.D.

We now prove Lemma 1 using Lemma 2, the process \( z_t \) and the noise \( \zeta_t \sim \mathcal{N}(0, \Sigma_{\zeta,t}) \) are

\[
\begin{align*}
z_t &= \begin{pmatrix} V \\ X_t \end{pmatrix}, \quad \zeta_t = \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix}, \quad \Sigma_{\zeta,t} = \begin{pmatrix} \tau_{\varepsilon,t}^{-1} & 0 \\ 0 & \tau_{\eta,t}^{-1} \end{pmatrix}.
\end{align*}
\]  

(A-8)

In our model,

\[
\begin{align*}
z_t &= a_{z,t} z_{t-1} + b_{z,t} \zeta_t, \quad a_{z,t} = I_{2 \times 2}, \quad b_{z,t} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\end{align*}
\]  

(A-9)

\( X_t, \hat{V}_t^I \) and \( \hat{X}_t^U \) are the state variables that informed traders keep tracking in their value function, and are Gaussian Markov processes under \( \mathcal{F}_t^I \).

\[
\hat{V}_t^I = \hat{V}_{t-1}^I + K_{V,t}^I e_t^I,
\]  

(A-10)

where

\[
\begin{align*}
e_t^I &= v_t - \hat{V}_{t-1}^I = V - \hat{V}_{t-1}^I + \varepsilon_t, \quad e_t^I \mid \mathcal{F}_{t-1}^I \sim \mathcal{N}(0, \sigma_t^I), \\
\sigma_t^I &= \text{Var}_{t-1}(e_t^I) = o_{V,t-1}^I + \tau_{\varepsilon,t}^{-1}, \quad K_{V,t}^I = o_{V,t}^I \tau_{\varepsilon,t},
\end{align*}
\]  

(A-11)

and the second moment estimation is

\[
\begin{align*}
o_{V,t}^I &= \left( \tau_V + \sum_{s=0}^t \tau_{\varepsilon,s} \right)^{-1}.
\end{align*}
\]  

(A-12)

Uninformed traders’ signal process is

\[
\begin{align*}
s_t &= a_{s,t} z_t + b_{s,t} \zeta_t, \quad a_{s,t} = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad b_{s,t} = \begin{pmatrix} 1 & -h_t \end{pmatrix}.
\end{align*}
\]  

(A-13)
Thus, we have \( a_{s,t}b_{z,t} = 0_{1 \times 2} \). \( \hat{V}^U_t \) and \( \hat{X}^U_t \) are uninformed traders’ state variables, and are Gaussian Markov processes under \( \mathcal{F}^U_t \):

\[
\begin{align*}
\hat{V}^U_t &= \hat{V}^U_{t-1} + K^{U,t}_{\hat{V},t} e^U_t, \\
\hat{X}^U_t &= \hat{X}^U_{t-1} + K^{U,t}_{X,t} e^U_t,
\end{align*}
\]

where

\[
\begin{align*}
\alpha^U_t & = v_t - h_t \eta_t - \hat{V}^U_{t-1} = V - \hat{V}^U_{t-1} - h_t \eta_t + \varepsilon_t, \\
\Sigma^U_t &= \text{Var}_t^U(e^U_t) = o^U_{V,t-1} + \tau_{\varepsilon,t}^{-1} + h_t^2 \tau_{\eta,t}^{-1}, \\
K^{U,t}_{\hat{V},t} &= o^U_{V,t-1}/\Sigma^U_t, \\
K^{U,t}_{X,t} &= (o^U_{V,X,t-1} - h_t \tau_{\eta,t}^{-1})/\Sigma^U_t,
\end{align*}
\]

and the second moment estimations and the initial variance are

\[
\begin{align*}
o^U_{V,t} &= (\tau_{\varepsilon,t}^{-1} + h_t^2 \tau_{\eta,t}^{-1}) o^U_{V,t-1}/\Sigma^U_t, \\
o^U_{X,t} &= o^U_{X,t-1} + \tau_{\eta,t}^{-1} - (o^U_{V,X,t-1} - h_t \tau_{\eta,t}^{-1})^2/\Sigma^U_t, \\
o^U_{V,0} &= \tau_{\varepsilon,0}^{-1}, \\
o^U_{X,0} &= 0, \quad o^U_{V,0} = 0.
\end{align*}
\]

Note that the Kalman gains are not independent since \( \hat{V}^U_t - \mu t X_t = \hat{V}^U_t - \mu t \hat{X}^U_t \). From equations (A-10) and (A-14), we have

\[
\begin{align*}
[K^{I,t}_V - (K^{U,t}_V - \mu_t K^{U,t}_X)] (v_t - \hat{V}^U_{t-1}) + [(K^{U,t}_V - \mu_t K^{U,t}_X) h_t - \mu_t] \eta_t \\
+ [(1 - K^{I}_V) \mu_{t-1} - \mu_t] (X_{t-1} - \hat{X}^U_{t-1}) &= 0. \tag{A-17}
\end{align*}
\]

Thus, the coefficients in front of \( v_t - \hat{V}^U_{t-1}, \eta_t, \) and \( X_{t-1} - \hat{X}^U_{t-1} \) need to be zero and this yields equation (13). Using \( \hat{V}^I_{t-1} - \mu_{t-1} X_{t-1} = \hat{V}^U_t - \mu_{t-1} \hat{X}^U_t \), we can rewrite \( \hat{X}^U_t \) as

\[
\begin{align*}
\hat{X}^U_t &= \hat{X}^U_{t-1} + K^{U,t}_{X,t} \mu_{t-1} (X_{t-1} - \hat{X}^U_{t-1}) + K^{U,t}_{X,t} e^I_t - K^{U,t}_{X,t} h_t \eta_t, \\
&= (1 - K^{U,t}_{X,t} \mu_{t-1}) \hat{X}^U_{t-1} + K^{U,t}_{X,t} \mu_{t-1} X_{t-1} + K^{U,t}_{X,t} e^I_t - K^{U,t}_{X,t} h_t \eta_t. \tag{A-18}
\end{align*}
\]

### A.2 Proof of Theorem 1

Let \( \lambda^M \) denote the risk aversion of market makers. The wealth of market maker \( j \) is

\[
W_{T+1}^{M,j} = W_T^{M,j} + \theta_T^{M,j} V, \quad \text{and} \quad W_t^{M,j} = W_{t-1}^{M,j} + \alpha_t^j A_t - \beta_t^j B_t, \quad \text{for} \ t = 0, 1, 2, \ldots, T.
\]

We solve the problem backward starting from \( t = T \),

\[
E_T^U[-e^{-\lambda^M W_{T+1}^{M,j}}] = -e^{-\lambda^M W_{T+1}^{M,j} - \lambda^M \alpha_t^j A_T + \lambda^M \beta_t^j B_T - \lambda^M \theta_T^{M,j} \hat{V}^U_T + \frac{1}{2} (\lambda^M \theta_T^{M,j})^2 o^U_{V,T}}. \tag{A-19}
\]
We prove the case when $\Delta_T < 0$. The other case when $\Delta_T > 0$ can be proved similarly.

If $\Delta_T < 0$, $I$ investors sell at the bid price and $U$ investors buy at the ask price at time $t = T$, i.e., $P_T^{UR} > A_T > B_T > P_T^{IR}$.

$$A_T = P_T^{UR} - \frac{\gamma_T^U}{N_U} \sum_{j=1}^{N_M} \alpha_T^j, \quad B_T = P_T^{IR} + \frac{\gamma_T^I}{N_I} \sum_{j=1}^{N_M} \beta_T^j.$$ (A-20)

Substituting (A-20) into the objective function and taking the first order conditions with respect to $\alpha_T^j$ and $\beta_T^j$ yield

$$\frac{\gamma_T^U}{N_U} \alpha_T^j + \frac{\gamma_T^I}{N_I} \beta_T^j = -\frac{\Delta_T}{N_M + 1}, \quad \alpha_T^j - \beta_T^j = \frac{\lambda^M \alpha_T^U \theta_{T-1}^{M,j} + \frac{P_T^{UR} \gamma_T^U/\gamma_T^I}{\gamma_T^U/\gamma_T^I + \gamma_T^I/\gamma_T^U} - \hat{V}_T}{\lambda^M \alpha_T^U} - \frac{\gamma_T^I/\gamma_T^U}{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U} \frac{\gamma_T^I/\gamma_T^I}{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U} (N_M + 1).$$

Taking limit $\lambda^M \to \infty$ yields $\alpha_T^j - \beta_T^j = \theta_T^{M,j}$, which implies that market makers hold zero inventory at $t = T$, i.e., $\theta_T^{M,j} = 0$. Therefore,

$$\alpha_T^j = -\frac{1}{{\frac{N_M + 1}{{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}}} \Delta_T}, \quad \beta_T^j = \alpha_T^j - \theta_T^{M,j},$$ (A-21)

where $\theta_T^{M,j} = \theta_T^{M,j-2} + \beta_T^{j-1} - \alpha_T^{j-1}$. The bid and ask prices are

$$A_T = P_T^{UR} + \frac{N_M}{{\frac{N_M + 1}{{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}}} \Delta_T}, \quad B_T = -\frac{\Delta_T}{N_M + 1}.$$ (A-22)

Plugging the equilibrium ask and bid prices and trading quantities at $T$ into $W_T^{M,j}$,

$$W_T^{M,j} = W_{T-1}^{M,j} + \frac{1}{{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}} \left( \frac{\Delta_T}{{N_M + 1}} + \frac{N_M + 1}{{2 \gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}} \theta_T^{M,j} \right)^2 - \left( \frac{N_M + 1}{{2 \gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}} \theta_T^{M,j} \right)^2$$

$$-\theta_T^{M,j} \sum_{j=1}^{N_M} \theta_T^{M,j} + P_T^{UR} \theta_T^{M,j}. \quad \text{(A-23)}$$

We will prove in Section A.2.1 that market makers optimally hold zero inventory, therefore, $\theta_T^{M,j} = 0$, which implies that $\alpha_T^j = \beta_T^j$. Market maker $j$’s value function at $T$ is

$$J_T^{M,j} = -e^{-\lambda^M} \left( W_{T-1}^{M,j} + \frac{1}{{\gamma_T^I/\gamma_T^I + \gamma_T^I/\gamma_T^U}} \left( \frac{\Delta_T}{{N_M + 1}} \right)^2 \right). \quad \text{(A-24)}$$
Now we compute backward one more step. Market maker $j$’s objective function at $T - 1$ is

$$\lim_{\lambda^M \to \infty} \max_{\alpha^j_{T-1}, \theta^M_{T-1}} E_{T-1}^U[J^{M,j}_T].$$

(A-25)

Note that $\theta^M_{T} = \theta^M_{T-1} = 0$, we rewrite market maker $j$’s last period value function as

$$J^{M,j}_T = -e^{-\lambda^M} \left[ W_{T-2}^{M,j} + \alpha^j_{T-1} A_{T-1} - \beta^j_{T-1} B_{T-1} + \frac{\Delta^T}{(N_M + 1)} \left( \frac{1}{\gamma_U/N_I + \gamma_U/N_U} \right) \right].$$

(A-26)

The only uncertainty at time $T - 1$ is from $\Delta_T$. We assume that $P^{UR}_{T-1} > P^{IR}_{T-1}$, i.e., informed traders sell and uninformed traders buy at $T - 1$. The case when $P^{UR}_{T-1} < P^{IR}_{T-1}$ can be proved similarly. $\Delta_T$ can be written as

$$\Delta_T = \Delta_{T-1}^T + b_T^2 - \frac{\gamma_I}{N_I} \sum_{j=1}^{N_M} \beta^j_{T-1} - \frac{\gamma_U}{N_U} \sum_{j=1}^{N_M} \alpha^j_{T-1},$$

(A-27)

where

$$\Delta_{T-1}^T = -\mu_T \Delta^T_U - \gamma_I \theta_{T-2} + \gamma_U \theta_{T-2}, \quad b_T^2 = -\mu_T K^U_{X,T}.$$  

$A_{T-1}$ and $B_{T-1}$ can be written as

$$A_{T-1} = P^{UR}_{T-1} - \gamma_U/N_U \sum_{j=1}^{N_M} \alpha^j_{T-1}, \quad B_{T-1} = P^{IR}_{T-1} + \frac{\gamma_U}{N_U} \sum_{j=1}^{N_M} \beta^j_{T-1}.$$  

Therefore, the expectation can be expressed as

$$E_{T-1}^U[J^{M,j}_T] = -e^{-\lambda^M} \left[ W_{T-2}^{M,j} - \frac{\gamma_I}{N_I} \sum_{j=1}^{N_M} \alpha^j_{T-1} \right] \left( \Delta_{T-1}^T + \frac{\gamma_I}{N_I} \sum_{j=1}^{N_M} (\alpha^j_{T-1} - \theta^M_{T-1}) - \frac{\gamma_U}{N_U} \sum_{j=1}^{N_M} \beta^j_{T-1} \right)^2 \times e^{-\lambda^M} \left[ \gamma_I/N_I \sum_{j=1}^{N_M} \alpha^j_{T-1} \right] \left( P^{UR}_{T-1} - \frac{\gamma_U}{N_U} \sum_{j=1}^{N_M} \alpha^j_{T-1} \right) \left( P^{IR}_{T-1} + \frac{\gamma_U}{N_U} \sum_{j=1}^{N_M} \beta^j_{T-1} \right)^2,$$

(A-28)

where

$$\chi_T = 1 - \frac{2\lambda^M/(\gamma_I/N_I + \gamma_U/N_U)(\mu_T K^U_{X,T}/(N_M + 1))^2}{(\Sigma_{T}^U)^{-1} + 2\lambda^M/(\gamma_I/N_I + \gamma_U/N_U)(\mu_T K^U_{X,T}/(N_M + 1))^2}, \quad \text{and} \quad \lim_{\lambda^M \to \infty} \chi_T = 0.$$
We then repeat the above steps to solve for $\alpha_{T-1}$. Substituting the optimal trading quantity into (14), it is straightforward to show that the ask and

\[ \alpha_{T-1} = \lim_{\lambda^M \to \infty} \frac{-\Delta_{T-1} + \frac{\gamma^I_{T-1}}{N_I}(N_M + 1) \theta^M_{T-2} + \frac{(\gamma^I_{T-1}/N_I - \gamma^U_{T-1}/N_U)\chi_T}{(N_M + 1)(\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U)} \left( \Delta_{T-1} - \sum_{j=1}^{N_M} \theta^M_{T-2,j} \right)}{(\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U)(N_M + 1) - \chi_T N_M (\gamma^I_{T-1}/N_I - \gamma^U_{T-1}/N_U)^2 / (N_M + 1)^2 (\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U)} = -\frac{1}{\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U} \frac{\Delta_{T-1}}{N_M + 1} + \frac{\gamma^I_{T-1}/N_I}{N_M + 1} \theta^M_{T-2,j}. \]

Thus,

\[ \beta_{T-1} = -\frac{1}{\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U} \frac{\Delta_{T-1}}{N_M + 1} - \frac{\gamma^I_{T-1}/N_I}{N_M + 1} \theta^M_{T-2}. \]

Hence, market maker $j$’s wealth at $t = T - 1$ can be written as

\[ W^M_{T-1,j} = W^M_{T-2,j} + \frac{1}{\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U} \left( \frac{\Delta_{T-1}}{N_M + 1} + \frac{N_M + 1}{2} \frac{\gamma^I_{T-1}/N_I}{N_M + 1} \theta^M_{T-2,j} \right)^2 - \left( \frac{N_M + 1}{2} \frac{\gamma^I_{T-1}/N_I}{N_M + 1} \theta^M_{T-2,j} \right)^2. \]

Substituting $\theta^M_{T-2,j} = 0$, market maker $j$’s value function at $T - 1$ can be expressed as

\[ J^M_{T-1} = -e^{-\lambda^M_{W^M_{T-2,j}}} \frac{\lambda^M_{X_T}}{(N_M + 1)^2 (\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U)} \left( \frac{\Delta_{T-1}}{N_M + 1} + \frac{N_M + 1}{2} \frac{\gamma^I_{T-1}/N_I}{N_M + 1} \theta^M_{T-2,j} \right)^2 - \frac{\lambda^M_{\theta^M_{T-2,j} \theta^M_{T-2,j}}}{\gamma^I_{T-1}/N_I + \gamma^U_{T-1}/N_U} \left( \frac{\Delta_{T-1}}{N_M + 1} \right)^2. \]

We then repeat the above steps to solve for $\alpha^M_{T-2}$ and take the infinity limit of $\lambda^M$ to obtain

\[ \alpha^M_{T-2} = -\frac{1}{\gamma^I_{T-2}/N_I + \gamma^U_{T-2}/N_U} \frac{\Delta_{T-2}}{N_M + 1} + \frac{\gamma^I_{T-2}/N_I}{\gamma^I_{T-2}/N_I + \gamma^U_{T-2}/N_U} \theta^M_{T-3,j}. \]

substituting $\theta^M_{T-3,j} = 0$, the objective function of market maker $j$ is equivalent to $\max_{\alpha^M_t} \alpha^M_t (A_t - B_t)$. The optimal quantities each market maker chooses to buy and sell at time $t$ are

\[ \alpha^M_t = \beta^M_t = \frac{1}{N_M + 1} \frac{\Delta_t}{\gamma^I_t/N_I + \gamma^U_t/N_U}. \]

Substituting the optimal trading quantity into (14), it is straightforward to show that the ask and bid prices are as in Theorem 1.
A.2.1 Proof of the Result that Extremely Risk Averse Market Makers Do not Hold Inventory

For simplicity, in this proof we set $N_M = 1$. We assume that the market maker has zero initial endowment, i.e., $\hat{\theta}_0 = 0$. The market maker’s final wealth at $T + 1$ is $W_{T+1} = W_T + \theta_T^M V$. At period $t (t = 0, 1, \ldots, T)$,

$$W_t^M = W_{t-1}^M + \alpha_t A_t - \beta_t B_t, \quad \theta_t^M = \theta_{t-1}^M + \beta_t - \alpha_t.$$  \hfill (A-33)

Thus, the objective function of the market maker at $t = T$ is

$$\max_{\alpha_T, \beta_T} E_T^M [-e^{-\lambda M W_{T+1}^M}].$$  \hfill (A-34)

The wealth at period $T + 1$ depends on the market maker’s previous inventory $\theta_{T-1}^M$ as studied in Liu and Wang (2016). As $\lambda^M \to \infty$, the eight cases described in Liu and Wang (2016) become: (1) informed traders buy and uninformed traders sell if $-\frac{N_M}{2\gamma_T} \Delta_T < \theta_{T-1}^M < \frac{N_M}{2\gamma_T} \Delta_T$, (2) informed traders buy and uninformed traders do not trade if $\frac{N_M}{2\gamma_T} \Delta_T \leq \theta_{T-1}^M < \frac{N_M}{\gamma_T} \Delta_T$, (3) both informed and uninformed buy if $\theta_{T-1}^M \geq \max\{-\frac{N_M}{\gamma_T} \Delta_T, \frac{N_M}{\gamma_T} \Delta_T\}$, (4) informed investors do not trade and uninformed traders buy if $-\frac{N_M}{2\gamma_T} \Delta_T \leq \theta_{T-1}^M \leq \frac{N_M}{\gamma_T} \Delta_T$, (5) informed traders sell and uninformed traders buy if $\frac{N_M}{\gamma_T} \Delta_T < \theta_{T-1}^M < -\frac{N_M}{2\gamma_T} \Delta_T$, (6) informed traders sell and uninformed traders do not trade if $\theta_{T-1}^M \leq \frac{N_M}{\gamma_T} \Delta_T$, (7) both informed and uninformed traders sell if $\theta_{T-1}^M \leq \min\{-\frac{N_M}{\gamma_T} \Delta_T, \frac{N_M}{\gamma_T} \Delta_T\}$, and (8) informed traders do not trade and uninformed traders sell if $-\frac{N_M}{\gamma_T} \Delta_T \leq \theta_{T-1}^M \leq -\frac{N_M}{2\gamma_T} \Delta_T$. Since $\Delta_T$ is normally distributed, the probability of each case is cumulative density function of normal variable depending on $\theta_{T-1}^M$. We denote the probability of these eight cases as $\text{Prob}_j^T (\theta_{T-1}^M)$ respectively, where $j = 1, 2, \ldots, 8$. If $\theta_{T-1}^M > 0$, then the probabilities of cases (1), (2), (3), (4), and (5) are not zero. If $\theta_{T-1}^M < 0$, then the probabilities of cases (1), (5), (6), (7), and (8) are not zero. If $\theta_{T-1}^M = 0$, then the probabilities of case (2), (3), (4), (6), (7), and (8) are all zero. We will now show that $\theta_{T-1}^M = 0$, otherwise infinite risk averse market makers’ utility will explode. If $\theta_{T-1}^M \neq 0$, we prove the case when $\theta_{T-1}^M > 0$, the case when $\theta_{T-1}^M < 0$ can be proved similarly. Note that if $\theta_{T-1}^M > 0$, then the probabilities of cases (1), (2), (3), (4), and (5) are not zero. In case (2), informed traders buy and uninformed traders do not trade at $t = T$, i.e., in this case, $\beta_T = 0$ and the expected utility is

$$E_T^M [-e^{-\lambda M W_{T+1}^M}] = -e^{-\lambda M [W_{T-1}^M + \alpha_T A_T + (\theta_{T-1}^M - \alpha_T) V_T^U - \frac{1}{2} \lambda M (\theta_{T-1}^M - \alpha_T)^2 V_T^U]},$$  \hfill (A-35)

where $A_T = P_{IR}^T - \frac{\gamma_T}{N_I} \alpha_T$. Thus, the first order condition with respect to $\alpha_T$ gives

$$\alpha_T = \frac{P_{IR}^T - V_T^U + \lambda M V_T^U \theta_{T-1}^M}{\lambda M V_T^U + 2 \gamma_T / N_I}.$$  \hfill (A-36)
We express the above equations in terms of $\frac{1}{\lambda^M}$,

$$\alpha_T = \theta^M_{T-1} + o(\frac{1}{\lambda^M}), \quad \Delta_T = P^T_{IR} - \frac{\gamma^T}{N_t^M} \theta^M_{T-1} + o(\frac{1}{\lambda^M}), \quad (A-37)$$

$$J^M_t = -e^{-\lambda^M \left[ W^M_{T-1} + \theta^M_{T-1} (P^T_{IR} - \frac{\gamma^T}{N_t^M} \theta^M_{T-1}) + o(\frac{1}{\lambda^M}) \right]}, \quad (A-38)$$

where $o(\frac{1}{\lambda^M})$ means higher orders of $\frac{1}{\lambda^M}$. Thus, as $\lambda^M \to \infty$, $o(\frac{1}{\lambda^M}) \to 0$. In the previous period, the market maker maximizes $E^M_{T-1} [J^M_t]$ by choosing the optimal $\alpha_{T-1}$ and $\beta_{T-1}$. The market maker makes the market and $\beta_{T-1} = \theta^M_{T-1} + \alpha_{T-1} - \theta^M_{T-2}$,

$$\Psi^2_{T-1}(\theta^M_{T-1}) = E^M_{T-1} [J^M_t \mid \frac{N_t^I}{2 N_T^T} \Delta_T \leq \theta^M_{T-1} \leq \frac{N_t^I}{\gamma^T} \Delta_T] = -\exp \left\{ -\lambda^M \left[ \alpha(\frac{1}{\lambda^M}) + W^M_{T-2} + A_{T-1} \alpha_{T-1} - B_{T-1} (\theta^M_{T-1} + \alpha_{T-1} - \theta^M_{T-2}) \right] \right\} \times \exp \left\{ -\lambda^M \left[ \theta^M_{T-1} \left( E^M_{T-1} [P^T_{IR} | \text{case}(2)] - \frac{\gamma^T}{N_t^M} \theta^M_{T-1} \right) \right] \right\} \times \exp \left\{ -\lambda^M \left[ -\frac{1}{2} \lambda^M \text{Var}^M_{T-1} [P^T_{IR} | \text{case}(2)] (\theta^M_{T-1})^2 \right] \right\}. \quad (A-39)$$

Similarly, we can also compute the conditional expected utility for cases (1), (3), (4), (5), and denote them as $\Psi^k_{T-1}(\theta^M_{T-1})$, where $k = 1, 2, ..., 5$. Therefore, the expected utility at $t = T - 1$ is $E^M_{T-1} [J^M_t] = \sum_{k=1}^{5} \text{Prob}^k_{T-1}(\theta^M_{T-1}) \Psi^k_{T-1}(\theta^M_{T-1})$. If the probability of case (2) is not zero, then the expected utility contains the conditional expected utility of the second case which contains the term of $-\exp \left\{ -\lambda^M \left[ \theta^M_{T-1} \left( E^M_{T-1} [P^T_{IR} | \text{case}(2)] - \frac{\gamma^T}{N_t^M} \theta^M_{T-1} \right) \right] \right\} \times \exp \left\{ -\lambda^M \left[ -\frac{1}{2} \lambda^M \text{Var}^M_{T-1} [P^T_{IR} | \text{case}(2)] (\theta^M_{T-1})^2 \right] \right\}$. This implies that when $\lambda^M \to \infty$, the expected utility will go to negative infinity. Since $\Delta_t \in F_t$ is a normal random variable at $t - 1$, the probability of case (2) is positive except when $\theta^M_{T-1} = 0$. The above argument can be applied to every period and we can conclude that when market makers are extremely risk averse, they optimally hold zero inventory every period, $\theta^M_t \to 0$.

A.3 Proof of Proposition 1

Conjecture that the value functions of informed and uninformed traders have the following form

$$J^I_t = -\rho^I_t e^{-\lambda^I \left[ W^I_{t+\delta} + \frac{1}{2} (\Phi^I_t) \Phi^I_t + (\Phi^I_t)^\top \Phi^I_t \theta^I_{t-1} + \frac{1}{2} \lambda^I \theta^I_{t-1} \right]},$$

$$J^U_t = -\rho^U_t e^{-\lambda^U \left[ W^U_{t+\delta} + \frac{1}{2} (\Phi^U_t) \Phi^U_t + (\Phi^U_t)^\top \Phi^U_t \theta^U_{t-1} + \frac{1}{2} \lambda^U \theta^U_{t-1} \right]},$$

(A-40) (A-41)
and the reservation prices are

\[
P^I_{t}^{R} = \delta^I_t (\bar{V}^I_t - \mu_t X_t) + (1 - \delta^I_t) \bar{V} + g^I_t \bar{X}^U_t + f^I_t \theta - \frac{\gamma^I_t}{N_I} \sum_{i=1}^{N_I} \theta^I_{t-1}, \quad (A-42)
\]

\[
P^U_{t}^{R} = \delta^U_t \bar{V}^U_t + (1 - \delta^U_t) \bar{V} + g^U_t \bar{X}^U_t + f^U_t \theta - \frac{\gamma^U_t}{N_U} \sum_{j=1}^{N_U} \theta^U_{t-1}. \quad (A-43)
\]

Define

\[
\theta^I_{t-1} = \frac{1}{N_I} \left( (1 - w^I_t) \sum_{i=1}^{N_I} \theta^I_{t-1} + w^U_t \sum_{j=1}^{N_U} \theta^U_{t-1} \right), \quad \theta^U_{t-1} = \frac{1}{N_U} \left( w^I_t \sum_{i=1}^{N_I} \theta^I_{t-1} + (1 - w^U_t) \sum_{j=1}^{N_U} \theta^U_{t-1} \right)
\]

\[
w^I_t = \frac{N_M}{N_M + 1} \frac{\gamma^I_t / N_I}{\gamma^I_t / N_I + \gamma^U_t / N_U}, \quad w^U_t = \frac{N_M}{N_M + 1} \frac{\gamma^U_t / N_U}{\gamma^I_t / N_I + \gamma^U_t / N_U}. \quad (A-44)
\]

Note that by definition \(N_I \theta^I_{t-1} + N_U \theta^U_{t-1} = \bar{\theta} \) in equilibrium. Thus, traders’ trading prices are

\[
P^I_t = (1 - w^I_t) P^I_{t}^{R} + w^I_t P^U_{t}^{R} = \left[ (1 - w^I_t) \delta^I_t + w^I_t \delta^U_t \right] \bar{V} + \left[ (1 - w^I_t) g^I_t + w^I_t (g^U_t + \delta^U_t \mu_t) \right] \bar{X}^U_t + \left[ (1 - w^I_t) f^I_t + w^I_t f^U_t \right] \bar{\theta} - \gamma^I_t \theta^I_{t-1}, \quad (A-45)
\]

\[
P^U_t = w^U_t P^I_{t}^{R} + (1 - w^U_t) P^U_{t}^{R} = \left[ w^U_t \delta^I_t + (1 - w^U_t) \delta^U_t \right] \bar{V}^U_t + \left[ w^U_t (g^I_t - \delta^I_t \mu_t) + (1 - w^U_t) g^U_t \right] \bar{X}^U_t + \left[ w^U_t f^I_t + (1 - w^U_t) f^U_t \right] \bar{\theta} + \left[ 1 - w^U_t \delta^I_t - (1 - w^U_t) \delta^U_t \right] \bar{V} - \gamma^U_t \theta^U_{t-1}. \quad (A-46)
\]

where

\[
H^I_{P,t} = \begin{pmatrix}
(1 - w^I_t) \delta^I_t + w^I_t \delta^U_t \\
-\mu_t [(1 - w^I_t) \delta^I_t + w^I_t \delta^U_t] \\
(1 - w^I_t) \delta^I_t + w^I_t \delta^U_t + \delta^I_t \mu_t \\
(1 - w^I_t) \delta^I_t + w^I_t \delta^U_t \\
1 - [(1 - w^I_t) \delta^I_t + w^I_t \delta^U_t] \\
\end{pmatrix}, \quad H^U_{P,t} = \begin{pmatrix}
w^I_t \delta^I_t + (1 - w^I_t) \delta^U_t \\
w^U_t \delta^I_t + (1 - w^I_t) \delta^U_t + (1 - w^U_t) g^I_t \\
w^U_t \delta^U_t + (1 - w^U_t) f^U_t \\
w^I_t \delta^I_t + (1 - w^U_t) f^I_t \\
w^U_t \delta^I_t + (1 - w^U_t) f^U_t \\
\end{pmatrix}. \quad (A-47)
\]
Therefore, traders’ equilibrium positions are

\[ \theta_{t,i} = -\frac{w_i}{\gamma_i} \left[ (\delta_i^U - \delta_i^I)(\hat{V}_t^I - \mu_i X_t) - (\delta_i^U - \delta_i^I) \hat{V} + (\mu_i \delta_i^U + g_i^I - g_i^I) \bar{X}_t^I + (f_i^U - f_i^I) \hat{\theta} \right] + \theta_{t-1}^I = \left( H_{\theta,t}^U \right)^\top \Phi_t^I + \theta_{t-1}^I, \]  

\[ \theta_{t,j} = \frac{w_j}{\gamma_j} \left[ (\delta_j^U - \delta_j^I)(\hat{V}_t^I - \mu_j X_t) - (\delta_j^U - \delta_j^I) \hat{V} + (\mu_j \delta_j^U + g_j^I - g_j^I) \bar{X}_t^I + (f_j^U - f_j^I) \hat{\theta} \right] + \theta_{t-1}^U = \left( H_{\theta,t}^U \right)^\top \Phi_t^U + \theta_{t-1}^U, \]  

where

\[ H_{\theta,t}^U = -\frac{w_i}{\gamma_i} \begin{pmatrix} \delta_i^U - \delta_i^I \\ - (\delta_i^U - \delta_i^I) \mu_i \\ \mu_i \delta_i^U + g_i^I - g_i^I \\ f_i^U - f_i^I \\ - (\delta_i^U - \delta_i^I) \end{pmatrix}, \quad H_{\theta,t}^U = \frac{w_i}{\gamma_i} \begin{pmatrix} \delta_i^U - \delta_i^I \\ \mu_i \delta_i^U + g_i^I - g_i^I \\ f_i^U - f_i^I \\ - (\delta_i^U - \delta_i^I) \end{pmatrix}, \] 

and in equilibrium

\[ \theta_t^I = \frac{\theta_{t,i}}{N_M + 1} + \frac{w_{t+1}^I \hat{\theta}}{N_I}, \quad \theta_t^U = \frac{\theta_{t,j}}{N_M + 1} + \frac{w_{t+1}^U \hat{\theta}}{N_U}. \]  

The coefficients in trading prices can be expressed recursively. \( \gamma_i^U \) and \( \gamma_j^I \) can be computed by Eq. (A-60) and Eq. (A-72) respectively, \( \delta_i^U \) and \( \delta_j^I \) can be computed by Eq. (A-61) and Eq. (A-73) respectively, \( f_i^U \) and \( f_j^I \) can be computed by Eq. (A-63) and Eq. (A-75) respectively, and \( g_i^U \) and \( g_j^I \) can be computed by Eq. (A-62) and Eq. (A-74) respectively. Their values at \( t = T \) are given by Eq. (A-95) and Eq. (A-96).

The coefficients in uninformed traders’ value function \( \rho_i^U, M_i^U, C_i^U, m_i^U \) are given by Eq. (A-87) - Eq. (A-90). The coefficients in informed traders’ value function \( \rho_i^I, M_i^I, C_i^I, m_i^I \) are given by Eq. (A-83) - Eq. (A-86). Their values at \( t = T \) are given by Eq. (A-102), Eq. (A-103) and Eq. (A-104).

We first solve the uninformed traders’ problem. Rewrite uninformed traders’ value function at \( t \) as

\[ J_t^{U,j} = -\rho_t^U e^{-\lambda_t^U [w_i^{U,i} - \theta_{t-1}^{U,i} + \theta_{t-1}^{U,j} + f_t^U + \frac{1}{2}(\Phi_t^U)^\top M_t^U \Phi_t^U + (C_t^U)^\top \Phi_t^U + \frac{1}{2} m_t^U (\theta_{t-1}^{U,i})^2]}, \]  

where, \( \Phi_t^U := (\hat{V}_t^U, \bar{X}_t^I, \hat{\theta}, \bar{V})' \), and from the proof of Lemma 1,

\[ \Phi_t^U = \Phi_{t-1} + F_t^U e_t^U, \quad F_t^U = (K_t^U, K_{X,t}^U, 0, 0)', \] 

In addition,

\[ P_t^U = \left( H_{\theta,t}^U \right)^\top \Phi_t^U - \gamma_t^U \theta_t^{U,i} - \gamma_t^U \theta_t^{U,j} + \left( H_{\theta,t}^U \right)^\top F_t^U e_t^U. \]
Thus, we can rewrite the value function as

\[
J_t^{U,j} = -\rho_t^U e^{-\lambda_t^U \left[ W_t | U_t \theta_t^{U,j} | F_t \theta_t^{U,j} + (H_{P_t}^{U})^\top \phi_{t}^{U,j} \theta_t^{U,j} - \gamma_t^{U,j} \theta_t^{U,j} + (C_t^{U})^\top \phi_{t}^{U,j} \theta_t^{U,j} \right]}
\]

\[
	imes e^{\frac{1}{2} \left( \frac{1}{2} \left( \phi_{t-1}^{U,j} \right) M_t^{U} \phi_{t}^{U,j} + \frac{1}{2} m_t^{U} \left( \theta_t^{U,j} \right)^2 + \frac{1}{2} (F_t^{U})^\top M_t^{U} F_t^{U} \left( e_t^{U,j} \right)^2 \right)}
\]

\[
	imes e^{-\lambda_t^U \left[ \left( H_{P_t}^{U} \right)^\top F_t^{U} \theta_t^{U,j} + (C_t^{U})^\top F_t^{U} \theta_t^{U,j} + (\phi_{t}^{U,j})^\top M_t^{U} F_t^{U} \right]}
\].

(A-55)

Hence, the expectation at \( t - 1 \) is

\[
E_{t-1} [J_t^{U,j}] = -\rho_t^U e^{-\lambda_t^U \left[ W_t | U_t \theta_t^{U,j} | F_t \theta_t^{U,j} + (H_{P_t}^{U})^\top \phi_{t}^{U,j} \theta_t^{U,j} - \gamma_t^{U,j} \theta_t^{U,j} + (C_t^{U})^\top \phi_{t}^{U,j} \theta_t^{U,j} \right]}
\]

\[
	imes \sqrt{\sum_{t} \Omega_t^U} e^{\frac{1}{2} \left( \frac{1}{2} \left( \phi_{t-1}^{U,j} \right) M_t^{U} \phi_{t}^{U,j} + \frac{1}{2} m_t^{U} \left( \theta_t^{U,j} \right)^2 + \frac{1}{2} (F_t^{U})^\top M_t^{U} F_t^{U} \right)^2},
\]

(A-56)

where \( \Xi_t^U = \left[ \left( \Sigma_t^U \right)^{-1} + \lambda_t^U (F_t^U)^\top M_t^U F_t^U \right]^{-1} \). Taking the first order condition with respect to \( \theta_t^{U,j} \) yields

\[
-P_t^{U,j} + (H_{P_t}^{U})^\top \phi_{t}^{U,j} - \gamma_t^{U,j} \theta_{t-1}^{U} = \lambda_t^U \Xi_t^U (H_{P_t}^{U})^\top F_t^{U} (F_t^{U})^\top \left[ H_{P_t}^{U} \theta_t^{U,j} + C_t^{U} \theta_t^{U,j} + M_t^{U} \phi_{t}^{U,j} \right],
\]

(A-57)

and the second order condition requires \( \lambda_t^U \Xi_t^U \left[ \left( H_{P_t}^{U} \right)^\top F_t^{U} \right]^2 > 0 \), which is satisfied automatically. Note that in equilibrium market clearing condition gives \( \theta_t^{U,j} = \theta_{t-1}^{U,j} / (N_M + 1) + \hat{w}_t / N_U \). Thus, in equilibrium, by the first order condition, uninformed trader’s position is

\[
\theta_{t-1}^{U,j} - \theta_{t-2}^{U,j} = \frac{P_{t-1}^{UR} - P_{t-1}^{U}}{\gamma_{t-1}},
\]

(A-58)

where

\[
P_{t-1}^{UR} = \delta_{t-1}^U \hat{V}_t^U + (1 - \delta_{t-1}^U) \tilde{V} + g_{t-1}^U \tilde{X}_t^U + f_{t-1}^U \tilde{\theta} - \gamma_{t-1}^U \theta_{t-2}^U,
\]

(A-59)

\[
\gamma_{t-1}^U = \frac{\gamma_t^U}{N_M + 1} + \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top \left( H_{P,t}^{U} + \frac{C_t^U}{N_M + 1} \right),
\]

(A-60)

\[
\delta_{t-1}^U = \left[ (H_{P,t}^{U})^\top - \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top M_t^U \right]_{1,1},
\]

(A-61)

\[
g_{t-1}^U = \left[ (H_{P,t}^{U})^\top - \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top M_t^U \right]_{1,2},
\]

(A-62)

\[
f_{t-1}^U = \left[ (H_{P,t}^{U})^\top - \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top M_t^U \right]_{1,3}
\]

\[
- \left[ \gamma_t^U + \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top C_t^U \right] \frac{w_t^U}{N_U}.
\]

(A-63)

It can be verified that

\[
1 - \delta_{t-1}^U = \left[ (H_{P,t}^{U})^\top - \lambda_t^U \Xi_t^U (H_{P,t}^{U})^\top F_t^{U} (F_t^{U})^\top M_t^U \right]_{1,4}.
\]

(A-64)
We next solve the informed traders’ problem. Rewrite informed traders’ value function at $t$ as

$$J_{t,i}^I = -p_t^I e^{-\lambda^I \left[ W_{t-1}^{I,i} - \theta_{t-1}^I P_t^I + \frac{1}{2} (\Phi_t^I - H_{t-1}^I \sigma_{t-1}^I)^2 \right]}, \quad (A-65)$$

where $\Phi_t^I := H_t^I \Phi_{t-1}^I + F_t^I u_t^I$, and from the proof of Lemma 1,

$$\Phi_t^I = \left( \hat{V}_t^I, X_t, \hat{X}_t^U, \bar{\theta}, \hat{V}_t^I \right)', \quad u_t^I = \left( e_t^I \right)'$$

$$\Sigma_t^I = \begin{pmatrix} \sigma_t^I & 0 \\ 0 & \tau_{0,t} \end{pmatrix}, \quad F_t^I = \begin{pmatrix} K_{t,t}^I \\ K_{X_t,t}^I \end{pmatrix}.$$  \quad (A-66)

In addition,

$$P_t^I = (H_{P_t}^I)'^T \Phi_t^I - \gamma_t^I \theta_{t-1}^I = (H_{P_t}^I)'^T \Phi_{t-1}^I - \gamma_t^I \theta_{t-1}^I + (H_{P_t}^I)' \gamma_t^I u_t^I.$$  \quad (A-67)

The value function can be rewritten as

$$J_{t,i}^I = -p_t^I e^{-\lambda^I \left[ W_{t-1}^{I,i} - \theta_{t-1}^I P_t^I + \theta_{t-1}^I ((H_{P_t}^I)'^T H_t^I \Phi_{t-1}^I - \gamma_t^I \theta_{t-1}^I) + (C_t^I)'^T H_t^I \Phi_{t-1}^I \theta_{t-1}^I \right]} \times e^{-\lambda^I \left[ (H_t^I \Phi_{t-1}^I)'^T M_t^I H_t^I \Phi_{t-1}^I + \frac{1}{2} (\theta_{t-1}^I)^2 + \frac{1}{2} (F_t^I u_t^I)'^T M_t^I F_t^I u_t^I \right]} \times e^{-\lambda^I \left[ (\theta_{t-1}^I (H_t^I)'^T F_t^I + \theta_{t-1}^I (C_t^I)'^T F_t^I + (H_t^I \Phi_{t-1}^I)'^T M_t^I F_t^I) u_t^I \right]}.$$  \quad (A-68)

Hence, the expectation at $t - 1$ is

$$E_{t-1}[J_{t,i}^I] = -p_t^I e^{-\lambda^I \left[ W_{t-1}^{I,i} - \theta_{t-1}^I P_t^I + \theta_{t-1}^I ((H_{P_t}^I)'^T H_t^I \Phi_{t-1}^I - \gamma_t^I \theta_{t-1}^I) + (C_t^I)'^T H_t^I \Phi_{t-1}^I \theta_{t-1}^I + \frac{1}{2} (H_t^I \Phi_{t-1}^I)'^T M_t^I H_t^I \Phi_{t-1}^I + \frac{1}{2} \theta_{t-1}^I)^2 \right]} \times \sqrt{\frac{\Xi_t^I}{\Sigma_t^I}} e^{(\lambda^I \Sigma_t^I)'^T F_t^I + \theta_{t-1}^I (C_t^I)'^T F_t^I + (H_t^I \Phi_{t-1}^I)'^T M_t^I F_t^I} \Xi_t^I [\theta_{t-1}^I (H_t^I)'^T F_t^I + \theta_{t-1}^I (C_t^I)'^T F_t^I + (H_t^I \Phi_{t-1}^I)'^T M_t^I F_t^I]^T, \quad (A-69)$$

where $\Xi_t^I = \left[ (\Sigma_t^I)^{-1} + \lambda^I (F_t^I)'^T M_t^I F_t^I \right]^{-1}$. Taking the first order condition with respect to $\theta_{t-1}^I$ yields

$$-P_t^I + (H_{P_t}^I)' H_t^I \Phi_{t-1}^I - \gamma_t^I \theta_{t-1}^I = \lambda^I \left[ (H_{P_t}^I)'^T F_t^I \Xi_t^I (F_t^I)'^F_t^I \right] [H_{P_t}^I \theta_{t-1}^I + C_t^I \theta_{t-1}^I + M_t^I H_t^I \Phi_{t-1}^I], \quad \text{and the second order condition requires } \lambda^I (H_{P_t}^I)'^F_t^I \Xi_t^I (F_t^I)'^H_{P_t}^I > 0,$$

which is satisfied under certain conditions. Note that in equilibrium market clearing condition gives $\theta_{t-1}^I = \theta_{t-1}^I/(N_M + \text{certain conditions. Note that in equilibrium market clearing condition gives } \theta_{t-1}^I = \theta_{t-1}^I/(N_M +$
1) + \frac{w_t}{N_I}. Thus, in equilibrium, from the first order condition, uninformed trader’s position is

$$\theta_{t-1}^{I,i} - \theta_{t-2}^{I,i} = \frac{P_{t-1}^{IR} - P_{t-1}^{I}}{\gamma_{t-1}^{I}}$$

(A-70)

where

$$P_{t-1}^{IR} = \delta_{t-1}^{I}(\check{V}_t - \mu_{t-1}X_t) + (1 - \delta_{t-1}^{I})\check{V} + g_{t-1}^{I}X_{t-1} + f_{t-1}^{I}\check{\theta} - \gamma_{t-1}^{I}\theta_{t-2}^{I}$$

(A-71)

and

$$\gamma_{t-1}^{I} = \frac{\gamma_{t}^{I}}{N_M + 1} + \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T \left( H_{P,t}^I + \frac{C_t^I}{N_M + 1} \right),$$

(A-72)

$$\delta_{t-1}^{I} = \left[ (H_{P,t}^I)^T H_t^I - \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T M_t^I H_t^I \right]_{1,1},$$

(A-73)

$$g_{t-1}^{I} = \left[ (H_{P,t}^I)^T H_t^I - \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T M_t^I H_t^I \right]_{1,3},$$

(A-74)

$$f_{t-1}^{I} = \left[ (H_{P,t}^I)^T H_t^I - \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T M_t^I H_t^I \right]_{1,4}$$

$$- \left[ \gamma_{t}^{I} + \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T C_t^I \right] \frac{w_t^I}{N_I}.$$ (A-75)

It can be verified that

$$-\mu_{t-1} \delta_{t-1}^{I} = \left[ (H_{P,t}^I)^T H_t^I - \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T M_t^I H_t^I \right]_{1,2},$$

(A-76)

$$1 - \delta_{t-1}^{I} = \left[ (H_{P,t}^I)^T H_t^I - \lambda^T (H_{P,t}^I)^T F_t^I \Xi_t^I (F_t^I)^T M_t^I H_t^I \right]_{1,5}. $$ (A-77)

Knowing \(\gamma_{t-1}^{I}, \gamma_{t-1}^{U}, \delta_{t-1}^{I}, \delta_{t-1}^{U}, g_{t-1}^{I}, g_{t-1}^{U}, f_{t-1}^{I}, f_{t-1}^{U},\) and \(\mu_{t-1}\), we can construct \(w_{t-1}^{I}, w_{t-1}^{U}, H_{P,t-1}^{I}, H_{P,t}^{U}, H_{\theta,t-1}^{I,},\) and \(H_{\theta,t}^{U}\) by equations (A-44), (A-47), and (A-50). Thus, we can express the equilibrium positions as

$$\theta_{t-1}^{I,i} = (H_{\theta,t-1}^{I})^T \Phi_{t-1}^{I,i} + \theta_{t-2}^{I,i}, \quad \theta_{t-1}^{U,i} = (H_{\theta,t-1}^{U})^T \Phi_{t-1}^{U,i} + \theta_{t-2}^{U,i}.$$ (A-78)

By construction, the market clearing condition will automatically hold. Moreover,

$$\theta_{t-1}^{I} = \frac{(H_{\theta,t-1}^{I})^T \Phi_{t-1}^{I} + \theta_{t-2}^{I}}{N_M + 1}, \quad \theta_{t-1}^{U} = \frac{(H_{\theta,t-1}^{U})^T \Phi_{t-1}^{U} + \theta_{t-2}^{U}}{N_M + 1}.$$ (A-79)

where

$$\hat{H}_{\theta,t-1}^{I} = \frac{H_{\theta,t-1}^{I}}{N_M + 1} + (0, 0, \frac{w_t^U}{N_I}, 0)' \quad \text{and} \quad \hat{H}_{\theta,t-1}^{U} = \frac{H_{\theta,t-1}^{U}}{N_M + 1} + (0, 0, \frac{w_t^I}{N_U}, 0)'.$$ (A-80)
Lastly, we need to verify the value function at time $t - 1$:

$$J_{t-1}^l = -\rho_{t-1}^l e^{-\lambda t} \left[ W_{t-1}^l + \frac{1}{2} \Phi_{t-1}^l \right] \Phi_{t-1}^l + (C_{t-1}^l)^\top \Phi_{t-1}^l \left( t_{t-1} + \frac{1}{2} m_{t-1}^l (\theta_{t-1}^l)^2 \right],$$  

(A-81)

$$J_{t-1}^U = -\rho_{t-1}^U e^{-\lambda U} \left[ W_{t-1}^U + \frac{1}{2} \Phi_{t-1}^U \right] \Phi_{t-1}^U + (C_{t-1}^U)^\top \Phi_{t-1}^U \left( t_{t-1} + \frac{1}{2} m_{t-1}^U (\theta_{t-1}^U)^2 \right],$$  

(A-82)

where

$$\rho_{t-1}^l = \rho_t^l \sqrt{|\Xi_t^l|/|\Sigma_t^l|},$$  

(A-83)

$$M_{t-1}^l = (H_{t}^l)^\top M_{t}^l H_{t}^l + m_{t}^l \hat{H}_{\theta,t-1}^l (\hat{H}_{\theta,t-1}^l)^\top + 2 \hat{H}_{\theta,t-1}^l (C_{t}^l)^\top H_{t}^l$$

$$+ \lambda^l (H_{P,t}^l)^\top F_t^l \Xi_l^l (F_t^l)^\top H_{P,t}^l \hat{H}_{\theta,t-1}^l (H_{\theta,t-1}^l)^\top$$

$$- \lambda^l \left[ (C_{t}^l)^\top \hat{H}_{\theta,t-1}^l + M_{t}^l H_{t}^l \right]^\top F_t^l \Xi_l^l (F_t^l)^\top \left[ (C_{t}^l)^\top \hat{H}_{\theta,t-1}^l + M_{t}^l H_{t}^l \right],$$  

(A-84)

$$C_{t-1}^l = \frac{(H_{t}^l)^\top C_{t}^l}{N_M + 1} + \frac{m_{t}^l}{N_M + 1} \hat{H}_{\theta,t-1}^l + \lambda^l H_{P,t}^l$$

$$+ \lambda^l \left[ (C_{t}^l)^\top \hat{H}_{\theta,t-1}^l + M_{t}^l H_{t}^l \right]^\top F_t^l \Xi_l^l (F_t^l)^\top C_{t}^l,$$  

(A-85)

$$m_{t-1}^l = \frac{m_{t}^l}{(N_M + 1)^2} + \lambda^l (H_{P,t}^l)^\top F_t^l \Xi_l^l (F_t^l)^\top H_{P,t}^l - \frac{\lambda^l}{(N_M + 1)^2} (C_{t}^l)^\top F_t^l \Xi_l^l (F_t^l)^\top C_{t}^l,$$  

(A-86)

and

$$\rho_{t-1}^U = \rho_t^U \sqrt{|\Xi_t^U|/|\Sigma_t^U|},$$  

(A-87)

$$M_{t-1}^U = M_{t}^U + m_{t}^U \hat{H}_{\theta,t-1}^U (\hat{H}_{\theta,t-1}^U)^\top + 2 \hat{H}_{\theta,t-1}^U (C_{t}^U)^\top + \lambda^U \Xi_l^U \left( (H_{P,t}^U)^\top F_t^U \right)^2 H_{\theta,t-1}^U (H_{\theta,t-1}^U)^\top$$

$$- \lambda^U \Xi_t^U \left[ (C_{t}^U)^\top F_t^U (\hat{H}_{\theta,t-1}^U)^\top + (F_t^U)^\top M_{t}^U \right]^\top \left[ (C_{t}^U)^\top F_t^U (\hat{H}_{\theta,t-1}^U)^\top + (F_t^U)^\top M_{t}^U \right],$$  

(A-88)

$$C_{t-1}^U = \frac{C_{t}^U}{N_M + 1} + \frac{m_{t}^U}{N_M + 1} \hat{H}_{\theta,t-1}^U + \lambda^U \Xi_l^U \left( (H_{P,t}^U)^\top F_t^U \right)^2 H_{\theta,t-1}^U$$

$$- \lambda^U \Xi_t^U \left[ (C_{t}^U)^\top F_t^U (\hat{H}_{\theta,t-1}^U)^\top + (F_t^U)^\top M_{t}^U \right]^\top,$$  

(A-89)

$$m_{t-1}^U = \frac{m_{t}^U}{(N_M + 1)^2} + \lambda^U \Xi_t^U \left( (H_{P,t}^U)^\top F_t^U \right)^2 - \lambda^U \Xi_t^U \left( \frac{(C_{t}^U)^\top F_t^U}{N_M + 1} \right)^2.$$  

(A-90)
Now we solve informed and uninformed traders’ last period problem. Their expected utility at time $t = T$ are

$$E^I_T \left[ -e^{-\lambda^I W_{T+1}^I} \right] = -e^{-\lambda^I [W_{T+1}^I + \theta_{T-1}^I (\hat{V}^I_T - P^I_T) + X_T \hat{V}^I_T - \frac{1}{2} \lambda^I o^I_{VT} (\theta_{T-1}^I + X_T)^2]},$$  

(A-91)

$$E^U_T \left[ -e^{-\lambda^U W_{T+1}^U} \right] = -e^{-\lambda^U [W_{T+1}^U + \theta_{T-1}^U (\hat{V}^U_T - P^U_T) - \frac{1}{2} \lambda^U o^U_{VT} (\theta_{T-1}^U)^2]}.$$  

(A-92)

Thus, the first order condition with respect to $\theta_{T-1}^I$ and $\theta_{T-1}^U$ yields

$$\theta_{T-1}^I - \theta_{T-1}^I = \frac{P_{T-1}^{IR,i} - P^I_T}{\gamma_{T}^I}, \quad \theta_{T-1}^U - \theta_{T-1}^U = \frac{P_{T-1}^{UR,j} - P^U_T}{\gamma_{T}^U}. \quad \text{(A-93)}$$

where

$$P_{T-1}^{IR,i} = \hat{V}^I_T - \mu_T X_T - \gamma_{T-1}^I \theta_{T-1}^I, \quad P_{T-1}^{UR,j} = \hat{V}^U_T - \gamma_{T-1}^U \theta_{T-1}^U. \quad \text{(A-94)}$$

Thus,

$$\mu_T = \lambda^I o^I_{VT}, \quad \gamma_{T}^I = \lambda^I o^I_{VT}, \quad \gamma_{T}^U = \lambda^U o^U_{VT}, \quad \delta_{T}^I = \delta_{T}^U = 1, \quad g_{T}^I = g_{T}^U = 0, \quad f_{T}^I = f_{T}^U = 0. \quad \text{(A-95)}$$

The second order condition requires that $\gamma_{T}^I > 0, \quad \gamma_{T}^U > 0$, which are satisfied automatically. And the average reservation prices are

$$P_{T}^{IR} = \hat{V}^I_T - \mu_T X_T - \gamma_{T}^I \theta_{T-1}^I, \quad P_{T}^{UR} = \hat{V}^U_T - \gamma_{T}^U \theta_{T-1}^U. \quad \text{(A-96)}$$

Hence, by market clearing condition the trading prices are

$$P_{T}^I = \hat{V}^I_T - \mu_T X_T - \frac{1}{N_I} \sum_{i=1}^{N_I} \theta_{T-1}^I, \quad P_{T}^U = \hat{V}^U_T - \frac{1}{N_U} \sum_{j=1}^{N_U} \theta_{T-1}^U. \quad \text{(A-97)}$$

The equilibrium positions of informed and uninformed trader are

$$\theta_{T-1}^I = -\frac{\gamma_{T}^I \mu_T}{\gamma_{T}^I} \hat{X}^I_T + \theta_{T-1}, \quad \theta_{T-1}^U = \frac{\gamma_{T}^U \mu_T}{\gamma_{T}^U} \hat{X}^U_T + \theta_{T-1}. \quad \text{(A-98)}$$

The equilibrium positions are

$$\theta_{T-1}^I = -\frac{w_{T}^I \mu_T}{\gamma_{T}^I} \hat{X}^I_T + \theta_{T-1}, \quad \theta_{T-1}^U = \frac{w_{T}^U \mu_T}{\gamma_{T}^U} \hat{X}^U_T + \theta_{T-1}. \quad \text{(A-99)}$$

Plugging the equilibrium trading prices and positions back to the expected utility, we obtain the
value functions at $T$

$$J_{T}^{L,j} = -\rho_{T}^{L}e^{-\lambda_{T}^j \left[ W_{T}^{L,j} + X_{T}(\dot{\theta}_{T}^{j} - \bar{\theta}^{j}) + \frac{1}{2} \gamma_{T}^{j} \left( -\frac{w_{T}^{L,j} \mu_{T}^{j}}{\gamma_{T}^{j}} \dot{X}_{T}^{j} + \theta_{T}^{j-1} \right)^{2} \right]}$$, \quad (A-100)

$$J_{T}^{U,j} = -\rho_{T}^{U}e^{-\lambda_{T}^{U} \left[ W_{T}^{U,j} + \frac{1}{2} \gamma_{T}^{U} \left( \frac{w_{T}^{U,j} \mu_{T}^{U}}{\gamma_{T}^{U}} \dot{X}_{T}^{U} + \theta_{T}^{U-1} \right)^{2} \right]}$$, \quad (A-101)

where

$$\rho_{T}^{L} = 1, \quad \rho_{T}^{U} = 1.$$ \quad (A-102)

The value functions can be rewritten as

$$J_{T}^{L,j} = -\rho_{T}^{L}e^{-\lambda_{T}^{L} \left[ W_{T}^{L,j} + \frac{1}{2} (\Phi_{T}^{L})^{T} M_{T}^{L} \Phi_{T}^{L} + (C_{T}^{L})^{T} \Phi_{T}^{L} \dot{X}_{T}^{L} + \frac{1}{2} m_{T}^{L} (\theta_{T}^{L-1})^{2} \right]}$$,

$$J_{T}^{U,j} = -\rho_{T}^{U}e^{-\lambda_{T}^{U} \left[ W_{T}^{U,j} + \frac{1}{2} (\Phi_{T}^{U})^{T} M_{T}^{U} \Phi_{T}^{U} + (C_{T}^{U})^{T} \Phi_{T}^{U} \dot{X}_{T}^{U} + \frac{1}{2} m_{T}^{U} (\theta_{T}^{U-1})^{2} \right]}$$,

where

$$M_{T}^{L} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -\mu_{T} & 0 & 0 & 0 \\ 0 & 0 & (w_{T}^{L} \mu_{T})^{2} / \gamma_{T}^{L} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}, \quad C_{T}^{L} = \begin{pmatrix} 0 \\ -w_{T}^{L} \mu_{T} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad m_{T}^{L} = \gamma_{T}^{L}, \quad (A-103)$$

and

$$M_{T}^{U} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & (w_{T}^{U} \mu_{T})^{2} / \gamma_{T}^{U} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{T}^{U} = \begin{pmatrix} 0 \\ w_{T}^{U} \mu_{T} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad m_{T}^{U} = \gamma_{T}^{U}. \quad (A-104)$$

### A.4 The Expected Trading Volume

It can be shown that

$$\frac{N_{M}}{N_{M} + 1} \frac{\Delta_{t}}{\gamma_{T}^{L} / N_{I} + \gamma_{T}^{U} / N_{U}} = -\frac{w_{s}^{U}}{\gamma_{s}^{U} / N_{U}} \left[ (\delta_{s}^{U} - \delta_{t}^{U}) V_{s}^{U} - (\delta_{s}^{U} - \delta_{t}^{U}) V + (\mu_{s}^{U} + g_{s}^{U} - g_{t}^{U}) X_{s}^{U} + (f_{s}^{U} - f_{t}^{U}) \hat{\theta} \right]$$

$$+ \sum_{s=1}^{t-1} \frac{N_{M}}{(N_{M} + 1)^{t-s}} \left\{ -\frac{w_{s}^{U}}{\gamma_{s}^{U} / N_{U}} \left[ (\delta_{s}^{U} - \delta_{s}^{L}) V_{s}^{U} - (\delta_{s}^{U} - \delta_{s}^{L}) V + (\mu_{s}^{U} + g_{s}^{U} - g_{s}^{L}) X_{s}^{U} + (f_{s}^{U} - f_{s}^{L}) \hat{\theta} \right] + w_{s}^{L} \hat{\theta} \right\}$$

$$-w_{t}^{L} \hat{\theta} + \frac{N_{M}}{(N_{M} + 1)^{t} N_{U} \bar{\theta}^{U}} \sim N(\mu_{t}^{L}, (\sigma_{t}^{L})^{2}). \quad (A-105)$$
Therefore, the average trading volume at time $t$ is

$$E[\text{Vol}]=\sqrt{\frac{2}{\pi}}\sigma_\Delta^\Delta e^{-\frac{1}{2}\left(\frac{\mu_\Delta^\Delta}{\sigma_\Delta^\Delta}\right)^2} + \mu_\Delta^\Delta \left(1 - 2\Phi\left(-\frac{\mu_\Delta^\Delta}{\sigma_\Delta^\Delta}\right)\right), \quad \text{(A-106)}$$

where $\Phi$ is normal cumulative distribution function, $\mu_\Delta^\Delta$ is given by equation (A-107) and $\sigma_\Delta^\Delta$ is given by equation (A-111) below.

$$\mu_\Delta^\Delta = \frac{N_M}{(N_M + 1)t} N_U \bar{\theta} - \left[\frac{w^U}{\gamma^U/N_U} (f^U - f^I) + w^I\right] \bar{\theta} + \sum_{s=1}^{t-1} \frac{N_M}{(N_M + 1)^{t-s}} \left[\frac{w^U}{\gamma^U/N_U} (f^U_s - f^I_s) + w^I_s\right] \bar{\theta}. \quad \text{(A-107)}$$

$\sigma_\Delta^\Delta$ depends on the variance of $\hat{X}^U$. Since $\hat{X}^U = \mu_\Delta^{-1}(\hat{V}^I_t - \hat{V}^I_t) + X_t$, and

$$\hat{V}^I_t = (\tau_V + \sum_{n=1}^t \tau_{\varepsilon,n})^{-1} \left(\tau_V \bar{V} + V \sum_{n=1}^t \tau_{\varepsilon,n} + \sum_{n=1}^t \tau_{\varepsilon,n} \varepsilon_n\right),$$

$$\hat{V}^U_t = (\tau_V + \sum_{n=1}^t \tau_n)^{-1} \left(\tau_V \bar{V} + V \sum_{n=1}^t \tau_n + \sum_{n=1}^t \tau_n \varepsilon_n - \sum_{n=1}^t \tau_n h_n \eta_n\right),$$

where $\tau_t = \left(\frac{1}{\tau_{\varepsilon,t}} + \frac{\eta^2}{\tau_X}\right)^{-1}$. We can rewrite

$$\frac{w^U}{\gamma^U/N_U} \left[\left(\delta^U_t - \delta^I_t\right) \hat{V}^U_t + (\mu_\Delta \delta^U_t + g^U_t - g^I_t) \hat{X}^U_t\right] = d_t \bar{V} + d^\varepsilon_t V + \sum_{n=1}^t d_{t,n} \varepsilon_n + \sum_{n=1}^t d_{t,n} \eta_n, \quad \text{(A-108)}$$

where

$$d_t = \frac{w^U}{\gamma^U/N_U} \left[\left(\delta^U_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^U_t - \left(\delta^I_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^I_t\right] \tau_V,$$

$$d^\varepsilon_t = \frac{w^U}{\gamma^U/N_U} \left[\left(\delta^U_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^U_t \sum_{n=1}^t \tau_n - \left(\delta^I_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^I_{t,n} \sum_{n=1}^t \tau_{\varepsilon,n}\right],$$

$$d_{t,n}^\varepsilon = \frac{w^U}{\gamma^U/N_U} \left[\left(\delta^U_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^U_{t,n} \tau_n - \left(\delta^I_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^I_{t,n} \tau_{\varepsilon,n}\right],$$

$$d_{t,n}^\eta = \frac{w^U}{\gamma^U/N_U} \left[\left(\delta^U_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \mu_t - \left(\delta^I_t + \frac{g^U_t - g^I_t}{\mu_t}\right) \sigma^I_{t,n} \tau_{\varepsilon,n}\right]. \quad \text{(A-109)}$$
Since \( V, \{ \varepsilon_t \} \) and \( \{ \eta_t \} \) are independent, \( \sigma_t^\lambda \) comes from the variance of these three sources. Define

\[
- \frac{w_t^U}{\gamma_t^U / NU} \left[ (\delta_t^U - \delta_t^I) \dot{V}_t + (\mu_t^I + \rho_t^U - g_t^I) \dot{X}_t \right] \\
+ \sum_{s=1}^{t-1} \frac{N_M}{(N_M + 1) t-s} \frac{w_s^U}{\gamma_s^U / NU} \left[ (\delta_s^U - \delta_s^I) \dot{V}_s + (\mu_s^I + \rho_s^U - g_s^I) \dot{X}_s \right] \\
= D_t \ddot{V} + D_t^V \dot{V} + \sum_{n=1}^{t} D_{t,n}^\varepsilon \varepsilon_n + \sum_{n=1}^{t} D_{t,n}^\eta \eta_n,
\]

where

\[
D_t = -d_t + \sum_{s=1}^{t-1} \frac{N_M}{(N_M + 1) t-s} d_s, \quad D_t^V = -d_t^V + \sum_{s=1}^{t-1} \frac{N_M}{(N_M + 1) t-s} d_s^V, \\
D_{t,n}^\varepsilon = -d_{t,n}^\varepsilon + \sum_{s=n}^{t-1} \frac{N_M}{(N_M + 1) t-s} e_s, \quad D_{t,n}^\eta = -d_{t,n}^\eta + \sum_{s=n}^{t-1} \frac{N_M}{(N_M + 1) t-s} \eta_s.
\]

Therefore,

\[
(\sigma_t^\lambda)^2 = \frac{(D_t^V)^2}{\tau_V} + \sum_{n=1}^{t} \frac{(D_{t,n}^\varepsilon)^2}{\tau_{\varepsilon,n}} + \sum_{n=1}^{t} \frac{(D_{t,n}^\eta)^2}{\tau_{\eta,n}}.
\]

### A.5 Welfare Computation

Before trading, each informed trader has an initial holding \( \theta_{-1}^I = \bar{\theta}^I \) and each uninformed trader has an initial holding \( \theta_{-1}^U = \bar{\theta}^U \), so that

\[
N_I \ddot{\theta}^I + N_U \ddot{\theta}^U = \ddot{\theta}, \quad \theta_{-1}^I = \frac{\bar{\theta}^I}{N_M + 1} + \frac{w_0^U}{N_I} \bar{\theta}, \quad \theta_{-1}^U = \frac{\bar{\theta}^U}{N_M + 1} + \frac{w_0^I}{N_U} \bar{\theta}.
\]

Also note that \( E[V] = \bar{V} \) and \( X_{-1} = 0 \). We can rewrite the state vector at time \( t = 0 \) as

\[
\Phi_0^I = \Phi_{-1}^I + F_0^I u_0^I, \quad \Phi_0^U = \Phi_{-1}^U + F_0^U u_0^U,
\]

where

\[
\Phi_{-1}^I = (\bar{V}, X_{-1}, X_{-1}, \bar{\theta}, \bar{V}), \quad u_0^I = V - \bar{\varepsilon}_0, \quad \Phi_{-1}^U = (\bar{V}, X_0, \bar{\theta}, \bar{V}'), \quad u_0^U = u_0^I - h_0 \eta_0,
\]

\[
F_0^I = \begin{pmatrix} K_{V,0}^I & 0 & K_{U,0}^I & 0 & 0 \\ 0 & 1 & -K_{U,0}^I h_0 & 0 & 0 \end{pmatrix}^T, \quad F_0^U = \begin{pmatrix} K_{V,0}^U & K_{U,0}^U & 0 & 0 \end{pmatrix}^T.
\]
Uninformed trader's welfare is the unconditional expectation of $J_{1}^{U,i}$. We can write $J_{1}^{U,i}$ as

$$J_{1}^{U,i} = -\rho_{1}e^{-\lambda_{1}t\sigma_{t}} \Phi_{1}^{+} (H_{P_{1}}^{U} + C_{1}^{U} - \frac{1}{2}(\Phi_{0}^{U} - \frac{1}{2}(\Phi_{0}^{U})^2) \gamma_{1}^{U} \sigma_{t}^{-2}) \epsilon_{1}^{U}$$

(A-116)

Thus, the welfare of uninformed traders is

$$\mathcal{W}^{U} = \mathbb{E}[J_{1}^{U,i}] = -\sqrt{\frac{\pi}{\sigma_{t}^{2}}} \rho_{1}e^{-\lambda_{1}t\sigma_{t}} \Phi_{1}^{+} (H_{P_{1}}^{U} + C_{1}^{U} - \frac{1}{2}(\Phi_{0}^{U} - \frac{1}{2}(\Phi_{0}^{U})^2) \gamma_{1}^{U} \sigma_{t}^{-2}) \epsilon_{1}^{U}$$

Similarly, we can write $J_{1}^{I,i}$ as

$$J_{1}^{I,i} = -\rho_{1}e^{-\lambda_{1}t\sigma_{t}} \Phi_{1}^{+} (H_{P_{1}}^{U} + C_{1}^{U} - \frac{1}{2}(\Phi_{0}^{U} - \frac{1}{2}(\Phi_{0}^{U})^2) \gamma_{1}^{U} \sigma_{t}^{-2}) \epsilon_{1}^{U}$$

Thus, the welfare of informed traders is

$$\mathcal{W}^{I} = \mathbb{E}[J_{1}^{I,i}] = -\sqrt{\frac{\pi}{\sigma_{t}^{2}}} \rho_{1}e^{-\lambda_{1}t\sigma_{t}} \Phi_{1}^{+} (H_{P_{1}}^{U} + C_{1}^{U} - \frac{1}{2}(\Phi_{0}^{U} - \frac{1}{2}(\Phi_{0}^{U})^2) \gamma_{1}^{U} \sigma_{t}^{-2}) \epsilon_{1}^{U}$$

### A.6 The Average Price and Bid-Ask Spread

In equilibrium, the average bid-ask spread is

$$\mathbb{E}[A_{t} - B_{t}] = \frac{1}{N_{M} + 1} \mathbb{E}[\Delta_{t}] = \left( \gamma_{I}^{I} N_{I} + \gamma_{U}^{I} N_{U} \right) \mathbb{E}[\text{Vol}_{t}] \frac{1}{N_{M}}.$$  

(A-117)

Thus, by Eq. (A-106) we have

$$\mathbb{E}[A_{t} - B_{t}] = \frac{1}{N_{M}} \left( \gamma_{I}^{I} N_{I} + \gamma_{U}^{I} N_{U} \right) \left( \sqrt{\frac{2}{\pi}} \sigma_{t}^{\Delta_{t}} e^{-\frac{1}{2} \sigma_{t}^{2}} + \mu_{t}^{\Delta} \left( 1 - 2 \Phi \left( -\frac{\mu_{t}^{\Delta}}{\sigma_{t}^{\Delta}} \right) \right) \right),$$

(A-118)

where $\mu_{t}^{\Delta}$ is given by equation (A-107) and $\sigma_{t}^{\Delta}$ is given by equation (A-111).

### A.7 Market Makers’ Expected Profits

Each market maker’s profit depends on the number of market makers and the trading frequency:

$$\text{Profit}(N_{M}, T) = \frac{1}{(N_{M} + 1)^{2}} \sum_{t=0}^{T} \frac{\Delta_{t}^{2}}{\gamma_{I}^{I} N_{I} + \gamma_{U}^{I} N_{U}} = \frac{1}{N_{M}^{2}} \sum_{t=0}^{T} \left( \gamma_{I}^{I} N_{I} + \gamma_{U}^{I} N_{U} \right) \text{Vol}_{t}^{2}.$$  

(A-119)
Therefore, the expected profit of each market maker is

\[
E[\text{Profit}(N_M, T)] = \frac{1}{N^2} \sum_{t=0}^{T} \left( \frac{\gamma_I}{N_I} + \frac{\gamma_U}{N_U} \right) \left[ (\mu^A)^2 + (\sigma^A)^2 \right],
\]

where \( \mu^A \) is given by equation \((A-107)\) and \( \sigma^A \) is given by equation \((A-111)\).

### A.8 Equilibrium Bid and Ask Prices When Risk-Averse Market Makers Have Large Positions

For simplicity, we assume \( \delta_I = \delta_U = \delta \) and focus on the limiting case where \( \lambda^M \to \infty \). Denote \( \nu := \text{Var}^U[V]/\text{Var}^I[V] \). If \( \bar{\theta}^M > 0 \), market makers need to offload their large position and there are five cases: (1) informed traders buy and uninformed traders sell if \( \Delta > \frac{2\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), and \( A = ((2N_I \nu + N_U)P^{IR} + N_U P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), B = (N_I \nu P^{IR} + (N_I \nu + 2N_U)P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), \) (2) informed traders sell and uninformed traders buy if \( \Delta < \frac{-2\delta \text{Var}^U[V]}{N_U} \bar{\theta}^M \), \( A = (N_I \nu P^{IR} + (N_I \nu + 2N_U)P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), B = ((2N_I \nu + N_U)P^{IR} + N_U P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), (3) informed traders buy and uninformed traders do not trade if \( -\frac{2\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M < \Delta \leq \frac{-2\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), \( A = P^{IR} - \delta \text{Var}^U[V] \bar{\theta}^M/N_U, B \leq A, \) and (5) informed traders do not trade and uninformed traders buy if \( -\frac{2\delta \text{Var}^U[V]}{N_U} \bar{\theta}^M < \Delta < -\frac{\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), \( A = P^{IR} - \delta \text{Var}^U[V] \bar{\theta}^M/N_U, B \leq P^{IR} \).

If \( \bar{\theta}^M < 0 \), market makers need to buy to close existing short position and there are five cases: (1) informed traders buy and uninformed traders sell if \( \Delta > \frac{-2\delta \text{Var}^I[V]}{N_U} \bar{\theta}^M \), and \( A = ((2N_I \nu + N_U)P^{IR} + N_U P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), B = (N_I \nu P^{IR} + (N_I \nu + 2N_U)P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), \) (2) informed traders sell and uninformed traders buy if \( \Delta < \frac{2\delta \text{Var}^U[V]}{N_I} \bar{\theta}^M \), \( A = (N_I \nu P^{IR} + (N_I \nu + 2N_U)P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), B = ((2N_I \nu + N_U)P^{IR} + N_U P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), (3) informed traders sell and uninformed traders do not trade if \( \frac{2\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M < \Delta \leq \frac{\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), \( B = P^{UR} - \delta \text{Var}^U[V] \bar{\theta}^M/(N_I \nu), A \geq P^{UR}, \) (4) both informed and uninformed investors buy if \( \frac{\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \leq \Delta \leq \frac{\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), \( B = (N_I \nu P^{IR} + (N_I \nu + 2N_U)P^{UR} - 2\delta \text{Var}^U[V] \bar{\theta}^M)/(2(N_I \nu + N_U)), A \geq B, \) and (5) informed traders do not trade and uninformed traders sell if \( \frac{2\delta \text{Var}^U[V]}{N_U} \bar{\theta}^M \leq \Delta < \frac{\delta \text{Var}^I[V]}{N_I} \bar{\theta}^M \), \( B = P^{UR} - \delta \text{Var}^U[V] \bar{\theta}^M/N_U, A \geq P^{IR} \). The competitive price where nobody has market power is \( P = (N_I \nu P^{IR} + N_U P^{UR} - \delta \text{Var}^U[V] \bar{\theta}^M)/(N_I \nu + N_U) \).
References


