Consumption Ratcheting and Loss Aversion∗

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Abstract

This paper investigates the optimal consumption, portfolio selection, and risk attitude of an economic agent who faces partial irreversibility of consumption decisions, formalizing the theory proposed by Duesenberry (1949). The irreversibility generates consumption ratcheting and dynamic loss aversion. We derive optimal policies and a measure of risk aversion implied by the optimal portfolio in closed form. The optimal consumption policy involves an inaction interval for the consumption-wealth ratio; when the ratio is inside the interval it is optimal not to adjust consumption, and when the ratio is outside the interval, it is optimal to adjust consumption immediately to restore the ratio to the nearest boundary of the interval. In particular, we disentangle the effects of loss aversion from those of risk aversion, and show that loss aversion determines the frequency of consumption adjustments and the shape of the risky share inside the interval, which provides various novel implications for consumption decisions and risk attitudes. We also provide an extension of our model to that with durable or multiple goods.

JEL Classification Codes: C61, D11, D15, E21, G11

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1 Introduction

This critique is based on a demonstration that two fundamental assumptions of aggregate demand theory are invalid. These assumptions are (1) that every individual’s consumption behaviour is independent of that of every other individual, and (2) that consumption relations are reversible in time (Duesenberry, 1949).

We study a model with partially irreversible decisions on consumption expenditures and analyze the optimal portfolio selection and attitudes toward risk. Motivated by the second part of the quote from Duesenberry (1949), we aim to provide the microfoundations for the irreversibility of consumption expenditure decisions. The irreversibility means if an agent increases or decreases consumption expenditure now, she cannot costlessly reverse the decision in the future, similar to a firm’s investment decision involving sunk costs. We model the irreversibility by introducing a proportional utility cost for each consumption adjustment. Since the decisions can be reversed by incurring costs, irreversibility in our model is partial, similar to that studied by Abel and Eberly (1996) and Dixit and Pindyck (1998) for a firm’s investment decisions. We find that a consumption irreversible decision generates consumption ratcheting and dynamic loss aversion. Moreover, we clearly disentangle the effects of risk aversion and loss aversion on households’ consumption and risky investment decisions. The model is tractable and admits closed-from expressions for optimal policies.

A university or foundation may find it difficult to reverse its decision to increase expenditures, since expenditures typically involve implicit or explicit long-term commitments to faculty or to donors (Dybvig (1995)). Increases in consumption expenditures for a household often come with purchases of durable goods with imperfect resale markets or other long-term commitments, e.g., getting married or having children (Chetty and Szeidl (2007); Grossman and Laroque (1990); Hindy and Huang (1992); Postlewaite et al. (2008)). An agent facing such costly adjustments would regret in some contingencies of the future if she increased expenditures immediately. Hence, she has an incentive to wait and to learn more about the uncertain environment in order to avoid the possibility of such a regret, similar to a firm when making an irreversible investment decision (Dixit and Pindyck (1998)). For a decrease in consumption expenditure Duesenberry (1949) explained a similar learning and delayed adjustment process; habit and learning lead to delayed and incremental adjustments (called ratcheting) to consumption expenditures.1

We propose a formal model which features partial irreversibility, learning and ratcheting

1“At any moment a consumer already has a well-established set of consumption habits... Suppose a man suffers a 50 per cent reduction in his income and expects this reduction to be permanent. Immediately after the change he will tend to act in the same way as before... In retrospect he will regret some of his expenditures. In the ensuing periods the same stimuli as before will arise, but eventually he will learn to reject some expenditures and respond by buying cheap substitutes for the goods formerly purchased (Duesenberry 1949, p. 24).”
The utility model we propose is also closely related to loss aversion in the prospect theory (Kahneman and Tversky (1979)). Taking the current level of consumption (or consumption-wealth ratio) as a reference point, loss aversion in our model is defined as the ratio of the utility gain from increasing consumption to the utility loss from decreasing consumption. We find that loss aversion depends only on the subjective discount rate and the cost adjustment parameters, independent of risk aversion. We also show that loss aversion summarizes the combined effects of upward and downward adjustment costs, and hence, the subjective discount rate, risk aversion and loss aversion are the three preference parameters which determines consumption and investment (risk taking) decisions. By using this observation, we investigate the effects of loss aversion on optimal polices that are clearly distinguished from those of risk aversion. In this sense, our paper provides not only a theory of consumption ratcheting but also a theory of dynamic loss aversion, arising from the irreversibility of consumption decisions with a minimal departure from the standard von Neumann-Morgenstern utility setup.

![Diagram of Consumption adjustment in (X, c)-domain](image)

**Figure 1: Consumption adjustment in (X, c)-domain**

The optimal consumption policy is characterized by two numbers, say c and $\bar{c}$ (with

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2We note that the partial irreversibility of a consumption decision can arise if there exists a cost only for one directional change, i.e., there exists a cost only for a downward adjustment with an upward adjustment being costless, or vice versa. This is a feature different from other models of costly consumption adjustment, e.g., that of Grossman and Laroque (1990), which assume positive transaction costs both for the increase of consumption and for its decrease.

3Empirical estimates of loss aversion are greater than 1. In our model, loss aversion is 1 only if there is no costs of adjustment (i.e., the Merton case). Otherwise, loss aversion is greater than 1.
When the ratio \( c_t / X_t \) of the previous consumption level to wealth, which we call the *consumption-wealth ratio*, is within the interval, consumption is not adjusted. When the ratio is outside \([\bar{c}, \check{c}]\), called the *inaction* interval, consumption is adjusted immediately so that the ratio is restored to the nearest boundary of the interval: Figure 1 gives a graphical illustration of the consumption adjustment.\(^4\) We show that this inaction interval becomes wider as loss aversion increases, i.e., the agent adjusts consumption less frequently as loss aversion increases. This result implies that loss aversion is important for matching several moments to the consumption data. Moreover, due to the nature of the optimal policy the consumption-wealth ratio is not constant, but time-varying, whereas it would be constant in the absence of the partial irreversibility. We show, however, that there exists a stationary distribution for the ratio and derive its density in closed form.

The consumption policy in our model is consistent with two well-documented empirical regularities: the excess smoothness of consumption (Deaton, 1987) and its excess sensitivity (Flavin, 1981). First, consumption does not respond to a permanent shock as long as the shock does not push up or down the consumption-wealth ratio to a boundary of the inaction interval, implying excess smoothness. Shocks to permanent income accumulate while consumption is not adjusted, and an excessively sensitive adjustment is made when it is changed. Both the excess smoothness and the excess sensitivity disappear, however, for a large shock or a sequence of shocks that pushes the consumption-wealth ratio to a boundary of the inaction interval. In other words, the partial irreversibility of consumption implies that consumption tends to respond immediately to a large shock or consecutive shocks in the same direction, exhibiting a behavior consistent with the permanent income hypothesis (Jappelli and Pistaferri (2010)).

We also discuss asset pricing implications of the model.\(^5\) Our optimal consumption policy implies that consumption is adjusted only infrequently, so the consumption-based capital asset pricing model and the Hansen-Jagannathan bound do not hold. We provide simulation results similar to those in Marshall and Parekh (1999) and Jeon et al. (2018) and show our model generates moments of aggregate consumption that match well with those of the US data. In particular, the auto-correlation of consumption matches well with the data when loss aversion is in the neighborhood of 2, empirically plausible values

\[^4\]The optimal policy in our model is similar to that in the model of Grossman and Laroque (1990) in the sense that both policies involve inaction intervals. Their optimal policy, however, requires a third number to which the ratio \( c_t / X_t \) is adjusted. The optimal policy in our model does not require the third number; the consumption-wealth ratio is adjusted to the nearest boundary of the inaction interval. Similar policies arise in models of portfolio selection with proportional transaction costs (Constantinides (1986); Davis and Norman (1990); Jang et al. (2007)).

\[^5\]In this paper we regard investment opportunities as fixed. This can be justified by regarding them as constant to returns physical technologies, as in Constantinides (1990). In our companion paper (Choi et al. (2019)) we study a competitive equilibrium with heterogeneous agents in which one class of agents has the same preferences as in this paper. We derive asset prices endogenously, given an aggregate endowment process, and show that the model matches well with the data.
of loss aversion (Kahneman et al. (1990), Tversky and Kahneman (1991), Benartzi and Thaler (1995), and Barberis et al. (2001)).

The optimal portfolio policy implies a U-shaped relationship between financial wealth and the share of risky assets in wealth, called the risky share. If loss aversion is greater than 1, the U-shaped risky share takes the maximum value at the boundaries of the inaction region and the minimum value inside the inaction region.\(^6\) The minimum value decreases with loss aversion while the maximum value is determined by risk aversion and unchanged by loss aversion. The U-shape relationship has an interesting implication for the effect of a wealth change on the risky share. The empirical literature is inconclusive about the effect; Calvet et al. (2009) and Calvet and Sodini (2014) show that the risky share tends to increase with an increase in wealth, evidences consistent with the habit or commitment models or households having decreasing relative risk aversion (DRRA), and Chiappori and Piaiella (2011) and Brunnermeier and Nagel (2008) show a neutral or slightly negative relationship between the risky share and wealth change, providing evidence consistent with the standard CRRA model. We show that indeed both a upward adjustment and a downward adjustment of the risky share can happen in response to an increase in wealth in our model depending on a sample path period used for an empirical analysis since the relationship is U-shaped (See Section 6 in more detail).

Attitudes toward risk are characterized by the revealed coefficient of relative risk aversion (RCRRA), i.e., the coefficient of relative risk aversion (CRRA) inferred by an outside observer who observes the portfolio of the agent but assumes the agent does not face any irreversibility of consumption expenditure decisions. The RCRRA shows an inverted U-shape relationship with the consumption-wealth ratio and wealth. The RCRRA is higher than the CRRA of the felicity function inside the inaction interval and is equal to the CRRA at its boundary points. More precisely, the RCRRA increases with loss aversion while its minimum value is fixed as the level of risk aversion. The fact that the RCRRA varies according to changes in the consumption-wealth ratio can help reconcile the different views of risk aversion in the decision theory, the structural estimation or behavioral literature, and the asset pricing literature. Most asset pricing models use the relative risk aversion coefficient of a 10 or higher to match asset pricing moments.\(^7\) The estimated relative risk aversion of individuals, however, takes values between 0.7 and 2 in the decision theory, structural estimation or behavioral literature.\(^8\) The RCRRA can be low near a boundary of the inaction interval and can be substantially high at points far from the boundary points, and hence both the high estimates and the low estimates are consistent with our model (see Section 6.3 and simulation results in Figures 12 and 13). This result

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\(^6\)If loss aversion is equal to 1, the problem collapses to the Merton case, in which the optimal risky share is always constant.

\(^7\)See, e.g., Bansal et al. (2012) Cocco et al. (2005), and references therein. There is research that suggests a very high value for the CRRA. For example, Kandel and Stambaugh (1991) argues $\gamma = 29$.

is a consequence of the separate effects of risk aversion and loss aversion: the lowest value is equal to the agent’s actual risk aversion and the highest value (or the difference between the highest and lowest values) is the contribution of loss aversion.

The optimization problem is highly complicated, involving both the partially irreversable consumption decision and portfolio choice. We simplify the problem by transforming it into a dual problem, which involves only choice of consumption, by using the linearization of the dynamic budget constraint developed by Cox and Huang (1989) and Karatzas et al. (1987). The dual problem is a control problem choosing two non-decreasing processes, called singular control.\textsuperscript{9} We obtain a closed form solution to the dual problem by using the super-contact conditions (Dumas (1991)). Then, we establish a duality relationship, which allows us to recover the value function of the original problem from that of the dual problem.\textsuperscript{10} We also provide several extensions such as a model with a durable good and a model with multiple goods.

Dynamic consumption and portfolio selection models can have important normative implications. Specifically, according to Dybvig (1995) “these models can be used to prescribe policies for managing portfolios” (p. 287). There exists a vast array of continuous-time models of consumption and portfolio selection with implementable closed-form portfolio policies (see e.g., Merton (1969, 1971, 1973), Kim and Omberg (1996)). One observes, however, that the prescribed policies have not been used for practical asset management.\textsuperscript{11} The lack of applicability might be because the models do not capture constraints and needs of households and institutions. We propose an alternative model with fairly simple closed-form prescriptions which might cater to the asset management needs of agents who face irreversibility in their consumption expenditure decisions.

After reviewing the literature in Section 1.1, the rest of the paper is organized as follows. Section 2 describes the model. Section 3 explains the solution to the model and derives optimal policies in closed form. It also provides a stationary distribution of the consumption-wealth ratio and discusses risk attitude. Section 4 theoretically summarizes the effects of loss aversion on optimal policies that are disentangled from the effects of risk aversion. Section 5 studies the asset pricing implications of the model. Section 6 explains investment in the risky assets. Section 7 discusses extensions of the model. Section 8 concludes the paper. Appendix contains proofs and technical details. It also has a section showing the results of a back test that uses historical data on the S&P 500 index when the model is applied to long-term asset management.

\textsuperscript{9}Portfolio choice with proportional transaction costs also involves a singular control problem (see Constantinides (1986), Davis and Norman (1990), Jang et al. (2007)).

\textsuperscript{10}A similar method is used by Miao and Zhang (2015) in a different context.

\textsuperscript{11}Cochrane (2014) observes, “Even highly sophisticated hedge funds typically form portfolios with one-period mean-variance optimizers—despite the fact that mean-variance optimization for a long-run investor assumes i.i.d. returns, while the funds’ strategies are based on complex models of time-varying expected returns, variances, and correlations. Beyond formal portfolio construction, their informal thinking and marketing is almost universally based on one-period mean-variance analysis, ignoring Mertonian state variables.”
1.1 Related Literature

Our paper is motivated by the classical work of Duesenberry (1949). The first part of his critique quoted in the preamble to this paper is closely related to external habit formation and thus has significantly contributed to developing the modern habit models (Abel (1990), Constantinides (1990), and Campbell and Cochrane (1999)). Part of our results resembles those from the habit models. For example, both our model and habit models generate time-varying risk aversion. However, ours are fundamentally different from the habit models in the sense that the agent in the habit models becomes more conservative when the current consumption level gets closer to the habit stock, while the RCRRA in our model is the highest when the consumption-wealth ratio is far from either of the boundary points inside the inaction interval. We show that the RCRRA tends to be high when the agent has no gains or losses from risky investments, whereas the effective risk aversion increases in a habit model when the agent has experienced losses from investments. There are also a handful of studies from the previous literature that are based on the second part of the critique (Dybvig (1995) and Jeon et al. (2018)). Dybvig (1995) and Jeon et al. (2018) assume that the consumption decision is fully irreversible (other than allowing a predetermined depreciation). Therefore, their models are an extreme special case of our model; their model is a case where the cost of the downward adjustment of consumption expenditure is infinite and the upward adjustment cost is zero, and hence loss aversion is infinite.\footnote{More precisely, Dybvig (1995) and Jeon et al. (2018) assume that consumption is not allowed to decrease.}

We formulate the consumption ratchet effect with partial irreversibility by allowing costly downward adjustments, which result in the U-shaped relationship between wealth and the risky share and its novel implications that the previous literature did not obtain.

Our model is obviously related to the loss aversion literature (Kahneman and Tversky (1979), Kahneman et al. (1990), Tversky and Kahneman (1991), Benartzi and Thaler (1995), and Barberis et al. (2001)). Our paper provides a theory of dynamic loss aversion in the sense that the reference point is changing over time, i.e., the previous level of consumption, $c_{t-1}$, is a reference point at each instant $t$. While most of the loss aversion literature does not consider the separate and independent effects of risk aversion, our model includes both risk aversion and loss aversion with a minimal departure from the standard von Neumann-Morgenstern utility setup. Moreover, our approach clearly disentangles the two effects on consumption decisions and the agent’s risk attitude over time.

Our model is also related to dynamic consumption and investment models with durable goods or consumption commitments (Grossman and Laroque (1990), Hindy and Huang (1992, 1993), Hindy et al. (1997), Flavin and Nakagawa (2008), and Chetty and Szeidl (2007, 2016)). Our model generalizes that of Hindy and Huang (1993) to the case where there exists a resale market for the durable good.\footnote{By using an isomorphism developed by Schroder and Skiadas (2002) we can show that our model generalizes} Since resale markets are prevalent

\footnote{More precisely, Dybvig (1995) and Jeon et al. (2018) assume that consumption is not allowed to decrease.}

\footnote{By using an isomorphism developed by Schroder and Skiadas (2002) we can show that our model generalizes}
in the real world, our model is more realistic than theirs. Our model and those of consumption commitments (Grossman and Laroque (1990), Flavin and Nakagawa (2008), and Chetty and Szeidl (2007, 2016)) also share common features. Consumption adjustments are infrequent, consumption responses to moderate wealth shocks exhibit excess smoothness and excess sensitivity, and the excess smoothness and the excess sensitivity disappear when consumption responds to large wealth shocks, both in these models and in our model. There exists, however, a significant difference between our model and the consumption commitment models. In our model consumption adjustment is incremental and hence exhibits *ratcheting*, which Duesenberry (1949) proposed as a realistic description of individuals’ consumption behavior, whereas consumption adjustment in commitment models is bulky and does not exhibit ratcheting. The difference comes from different mechanisms behind costly consumption adjustments in the two classes of models. In our model the costs arise because of learning and adjustment to new environments, whereas they are mainly transaction costs for durable goods, e.g., houses, in consumption commitment models.\footnote{As a result, the optimal policies in the two classes of models are also technically very different. See footnote 4 for more detail.}

Risk aversion and responses of the optimal portfolio to wealth shocks are similar qualitatively across the two classes of models, e.g., exhibiting high risk aversion for a small shock and low risk aversion for a large shock or for consecutive shocks in the same direction. They can, however, be substantially different quantitatively.\footnote{For example, the relationship between the risky share and the consumption-wealth ratio is U-shaped but tilted; the smallest risky share and the highest RCRRA occur at a point closer to the left-end point of the inaction interval with realistic parameter values. Such an asymmetry in the risky share and in the risk attitude has not been documented in the consumption commitment literature.} Furthermore, we have a closed-form expression for the optimal portfolio, but no closed-from expression is available for the existing commitment models. In these respects our model can be viewed as complementing the consumption commitment models.

Finally we mention potential reinterpretation and generalizations of the model. We can reinterpret our model as that of a durable good. A generalization is to introduce another consumption good which can be adjusted without incurring costs. The generalization is similar to the extension of the model of Grossman and Laroque (1990) by Flavin and Nakagawa (2008) and Chetty and Szeidl (2007, 2016). Similar to Flavin and Nakagawa (2008), we show that the elasticity of intertemporal substitution in the presence of the partial irreversibility is generally lower than that implied by the curvature of the felicity function. Flavin and Nakagawa (2008) use this fact to explain the low EIS of non-durable consumption. Our model can provide the same explanation. Finally, another extension is to study a model of competitive equilibrium with two agents, where an agent faces the partial irreversibility and the other doesn’t, and to analyze asset pricing implications. We provide such analysis in our companion paper (Choi et al. (2019)).

the model of durable goods proposed by Hindy and Huang (1993) to the case where there exist resale markets for durable goods.
2 Model

2.1 Preference, Financial Market, and Optimization Problem

Preference: An agent lives in an infinite-horizon one-good economy. For an exogenously given consumption level $c_{0-}$, we consider a preference represented by the following ex-pected utility function:

$$U = \mathbb{E} \left[ \sum_{t=0}^{\infty} \delta^t \left( u(c_t) - \alpha(\Delta u(c_t))^+ - \beta(\Delta u(c_t))^\_ \right) \right].$$

(1)

Here $\mathbb{E}$ denotes expectation, $\delta$ the subjective discount factor ($0 < \delta < 1$), $c_t$ consumption at time $t$, $u$ a strictly increasing real-valued function, called the felicity function, $\alpha, \beta$ are nonnegative constants, $\Delta u(c_t) \equiv u(c_t) - u(c_{t-1})$ for $t > 0$ and $\Delta u(c_0) \equiv u(c_0) - u(c_{0-})$ and superscript $^+$ ($^-$) means the positive (negative) part of a real-valued function, i.e., $f^+ = \max(f, 0)$ ($f^- = \max(-f, 0)$). The first part of the utility function is the ordinary expected utility with felicity function $u$, the second part $\alpha(\Delta u(c_t))^+$ is the utility cost of adjusting consumption upward, and the third part $\beta(\Delta u(c_t))^\_$ is the utility cost for of adjusting consumption downward. Constant $\alpha$ ($\beta$) is a proportional cost of an upward (downward) consumption adjustment such that $\alpha \geq 0, \beta \geq 0$, and $(\alpha, \beta) \neq (0, 0)$.

To gain understanding of the costs let us consider the following approximation for small changes in consumption

$$(\Delta u(c_t))^+ \approx u'(c_t) \Delta c_t^+, \quad (\Delta u(c_t))^\_ \approx u'(c_t) \Delta c_t^\_,$$

(2)

where $\Delta c_t^+$ denotes a positive change and $\Delta c_t^\_$ a negative change in consumption. Hence, the utility cost terms are equal to reductions in the agent’s utility when there exist pecuniary costs of adjustment proportional to changes. The terms can also be interpreted as learning and psychological costs if they are proportional to the magnitude of changes and the marginal utility. Such costs can occur in the presence of explicit or implicit commitments and breaking the commitments requires either psychological or pecuniary costs.

We can provide another interpretation of the utility costs from the standpoint of the prospect theory (see e.g., Kahneman and Tversky (1979)). The utility change due to a change in consumption is given by

$$\Delta U = \begin{cases} (1 - \alpha)(\Delta u(c_t))^+ & \text{if } \Delta c_t > 0, \\ -(1 + \beta)(\Delta u(c_t))^\_ & \text{if } \Delta c_t < 0. \end{cases}$$

(3)

The slope of change is larger for a decrease than for an increasing, similar to that due to loss aversion (Tversky and Kahneman (1991), Benartzi and Thaler (1995), Barberis et al. (2001)) in the prospect theory. Similar changes in utility can arise when there is a need for learning to re-optimize consumption to adjust to new income levels (Brunnermeier (2004)).
When consumption is adjusted, it will have a permanent effect on future consumption, since the agent will be able to reverse the decision only by paying costs. Hence, the utility change in in (3) does not capture the effect of the adjustment on future consumption. We will consider the total effect below.

We now consider a continuous-time limit of the utility function. Heuristically, the limit takes the following form, as the length of the consecutive time points tends to zero,

$$U = E \left[ \int_0^\infty e^{-\delta t} \left( u(c) dt - \alpha (du(c))^+ - \beta (du(c))^-(t) \right) \right],$$

where $\delta > 0$ is the subjective discount rate, and $c$ is the rate of consumption expenditure, which we will simply call consumption, and $\alpha (du(c))^+$ means the utility cost of an upward consumption adjustment and $\beta (du(c))^-$ is the utility cost of a downward consumption adjustment. The terms $(du(c))^+$ and $(du(c))^-$ are, however, not precisely defined. We provide a rigorous formal definition as follows: consider the following utility function

$$U = E \left[ \int_0^\infty e^{-\delta t} \left( u(c) dt - \alpha ||u(c)||^+ - \beta ||u(c)||^- \right) \right],$$

where the terms $||u(c)||^+$ and $||u(c)||^-$ denote positive and negative variations of $u(c)$ over $[0, t]$, which we will call positive and negative variation processes (here we denote a process $X = (X_t)_{t=0}^\infty$ simply by $X_t$). Since there is no possibility of confusion, we will use the following simpler notation:

$$u^+_t := ||u(c)||^+ \quad \text{and} \quad u^-_t := ||u(c)||^-.$$  \hspace{1cm} (6)

The positive and negative variation processes can become infinite at a finite time. If the processes become infinite, then the corresponding utility costs are infinite. Indeed it is not optimal to adjust consumption so that the positive variation process becomes infinite in a finite time interval if $\alpha \neq 0$ and the negative variation process becomes infinite in a finite time interval if $\beta \neq 0$. The positive variation, negative variation and total variation are, however, all infinite if any one of them is infinite (Theorem 2.6, Wheeden (2015)). Accordingly, if the consumption process $c_t$ has an infinite total variation over a finite time interval $[0, t]$, then the process $u(c_t)$ also has an infinite total variation and infinite positive and negative variations, and the utility costs are infinite since we have assumed $(\alpha, \beta) \neq (0, 0)$. Thus, we can preclude any consumption policy which results in a process $c_t$ with an infinite total variation over a finite time interval as sub-optimal. Consequently, without loss of generality, we confine our attention to a consumption process with a finite total variation over any finite time interval.

The existence of the adjustment costs make a decision to change consumption partially irreversible, meaning if she changes consumption now, she cannot costlessly reverse the

\[16\] The positive (negative) variation over $[T_1, T_2]$ of a process $X_t$ is defined as $\sup_{[t_0, t_1, \ldots, t_N]} \sum (X_{t_{i+1}} - X_{t_i})^+$ (sup $[t_0, t_1, \ldots, t_N]$ $\sum (X_{t_{i+1}} - X_{t_i})^-$) where $[t_0, t_1, \ldots, t_N]$ is an arbitrary partition of $[T_1, T_2]$. The total variation is defined as the sum of the positive and negative variations.

\[17\] More precisely, we carefully show in the proof that the optimal consumption is a finite total variation.
decision in the future. The irreversibility will induce her not to act immediately but to
wait and learn about the evolution of the market environment, similar to a firm when
making investments incurring sunk costs, as we will see in the next section (Dixit and
Pindyck (1996, 1998), Abel and Eberly (1996)).

When the agent instantaneously increases her consumption by a small amount, from
c to c + dc at t, it will have a permanent effect on the agent’s future consumption. The
present value of the utility change is equal to \( \frac{1}{\delta} \Delta u(c) \). Taking the adjustment cost into
consideration, the marginal utility gain of increasing consumption is \( \left( \frac{1}{\delta} - \alpha \right) (\Delta u(c))^+ \), and
the marginal utility loss of decreasing consumption is \( \left( \frac{1}{\delta} + \beta \right) (\Delta u(c))^− \). Consequently,
the marginal utility gain for an increase in consumption from the current reference point
is smaller than the marginal utility loss for a decrease, consistent with a utility function
exhibiting loss aversion.

Note that the agent would not increase consumption unless \( \frac{1}{\delta} - \alpha > 0 \), since the
total utility gain would not be positive otherwise. This observation leads to the following
assumption.

**Assumption 1.**

\[ 0 \leq \alpha < \frac{1}{\delta}. \]

There is no parameter restriction for \( \beta \in [0, \infty] \).

Following the definition in the loss aversion literature (e.g., Tversky and Kahneman
(1991) and Benartzi and Thaler (1995)), we define loss aversion \( L \) to be the ratio of the
marginal utility loss to the marginal utility gain, i.e.,

\[ L := \frac{1}{\Delta u(c)} \frac{(\frac{1}{\delta} + \beta)(\Delta u(c))^+}{(\frac{1}{\delta} - \alpha)(\Delta u(c))^−}. \]

Remarkably, we can rewrite the utility function (5) by using loss aversion \( L \), as the
following proposition shows.

**Proposition 2.1.** Suppose that \( c_t \) is an admissible consumption process such that its posi-
tive and negative variation processes discounted by the subjective discount rate \( \delta \), approach
0 as \( t \) tends to infinity. Then, utility function (5) can be rewritten as follows

\[ U = \frac{u(c_{0−})}{\delta} + (1 - \delta \alpha)E \left[ \int_0^\infty e^{-\delta t} \left( u_t^+ - Lu_t^- \right) dt \right]. \]

**Proof.** The proof is given in Appendix A.

According to Proposition 2.1 the agent’s utility is equal to the sum of three components
if the consumption process satisfies the growth condition in it: (1) the present value (PV)
of the utility value of the previously given level of consumption, (2) the PV of the increases
in the utility value due to increases in consumption, and (3) the negative of the decreases
in the utility value due to decreases in consumption multiplied by the loss aversion \( L \). If
\( L = 1 \), i.e., \( \alpha = 0, \beta = 0 \), then the agent is not loss averse, the utility value is equal to
the ordinary one, the sum of the PV of the utility value of the given level of consumption
and the difference between the PV of utility increases and the PV of utility decreases. If \( L > 1 \), i.e., \( \alpha \neq 0 \) or \( \beta \neq 0 \), then agent exhibits loss aversion and the reduction in the utility value due to decreases in consumption gets a higher weight equal to \( L \).

Noticeably, the proportions \( \alpha, \beta \) of consumption adjustment costs have an influence on the ordinal preference of the agent only through loss aversion. The proportion \( \alpha \) of the upward adjustment cost has a direct effect on the utility value by reducing it by a fraction \( \delta \alpha \) but does not influence the ordinal preference except indirectly through its effect on loss aversion. Consequently, different combinations of \( (\alpha, \beta) \) with the same \( L \) results in the same preferences when restricted to consumption processes satisfying the natural growth condition in the proposition. This has an important implication for the agent’s optimal policies, as we will see in Section 4.

Dybvig (1995) and Jeon et al. (2018) consider the limiting case \( \beta = \infty \), i.e., \( L = \infty \). Hence downward adjustment of consumption is never optimal in their model. Schroder and Skiadatas (2002) have shown that the model of Dybvig (1995) is isomorphic to that of a durable good and local substitution by Hindy and Huang (1993). Hence, our model extends Hindy and Huang’s model of durable good adjustments via the isomorphism, which we will discuss in Section 7.1.

Finally we assume that the felicity function takes the following form

\[
\begin{align*}
  u(c) = \begin{cases} 
  c^{1-\gamma} & \text{if } \gamma > 0, \gamma \neq 1, \\
  \log c & \text{if } \gamma = 1.
  \end{cases}
\end{align*}
\]

That is, the agent has constant relative risk aversion with the coefficient of relative risk aversion (CRRA) \( \gamma \). In fact, the CRRA assumption is not required for the solution analysis in the paper. More precisely, we do not need it when we derive optimal policies and characterize the duality relationship in Section 3. We use the assumption from Section 4 regarding disentangling the effects of risk aversion and the effects of loss aversion.

**Financial Market:** We consider a standard continuous-time financial market as in Grossman and Laroque (1990) and Flavin and Nakagawa (2008). Namely, there exist a riskless asset with a constant interest rate \( r \) and \( n \)-risky assets, where the value of the \( i \)-th risky asset (including accumulated dividends) \( \hat{z}_{i,t} \) follows

\[
d\hat{z}_{i,t} = \hat{z}_{i,t}(\mu_i dt + dw_{i,t}),
\]

where \( w_{1,t}, \ldots, w_{n,t} \) are arithmetic Brownian motions without drift and have positive definite instantaneous covariance matrix \( \Sigma \). Assuming \( \mu_1, \ldots, \mu_n \) and \( \Sigma \) are constant, a general equilibrium consideration leads naturally to the capital asset pricing model (CAPM) and a two-fund separation theorem. We give their proofs in Appendix N. Due to the two-fund separation theorem, the agent’s portfolio selection can be described as an allocation between the riskless asset and the market portfolio, which consists of all risky assets in the economy, at each instant. We can easily show that the value \( S_t \) of the market portfolio, which we will call the *risky asset* satisfies

\[
dS_t/S_t = \mu dt + \sigma dB_t,
\]
for some constants $\mu, \sigma, \mu > r$, and $B_t$ is a 1-dimensional standard Brownian constructed from $w_{1,t}, \ldots, w_{n,t}$. We will consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F}$ generated by $B_t$.\footnote{The probability space is endowed with an augmented filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by the Brownian motion $B_t$. See Karatzas and Shreve (1998) for details.}

As in Grossman and Laroque (1990) and in Flavin and Nakagawa (2008) we abstract from labor income and assume that the agent’s wealth is kept in the form of financial assets. The assumption is equivalent to an alternative assumption that the agent receives a stream of labor income whose full capitalized value can be utilized for investments in financial assets without any restriction.\footnote{See Koo (1998) for calculation of capitalized value of a stream of income in the presence or absence of frictions.}

In the alternative assumption the agent’s total wealth is the sum of financial wealth and human wealth (the present value of labor income) and a fraction of human wealth can be regarded as her permanent income. We will stick to the assumption that the agent’s wealth is kept in the form of financial assets, and resort occasionally to the alternative assumption when we discuss effects of changes of the permanent income on optimal policies.

**Optimization Problem:** The agent’s wealth process $(X_t)_{t=0}^\infty$ evolves according to the following dynamics:

$$dX_t = [rX_t + \pi_t(\mu - r) - c_t]dt + \sigma \pi_t dB_t, \quad X_0 = X > 0,$$

(10)

where $\pi_t$ is the dollar amount invested in the risky asset at time $t$. We define admissible set $\Pi$ of $(c_t, \pi_t)$: (i) $c_t$ is $\mathcal{F}_t$-adapted, non-negative, right-continuous with left limits (RCLL), has bounded variation, and integrable over any finite time interval, i.e., $\int_0^t c_t dt < \infty$ for all $t \geq 0$ almost surely, (ii) $\pi_t$ is $\mathcal{F}_t$-measurable adapted and square integrable, i.e., $\int_0^t ||\pi_t||^2 dt < \infty$ for all $t \geq 0$ almost surely, and (iii) $c_t$ satisfies

$$\mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( \max(0, -u(c_t)) dt + \alpha du_t^+ + \beta du_t^- \right) \right] < \infty,$$

(11)

where $u_t^+$ and $u_t^-$ are defined in (6).

We now state the problem as follows.

**Problem 1** (Primal Problem (Dynamic Version)).

Given $c_0 = c > 0$ and $X_0 = X > 0$, we consider the following optimization problem of the agent:

$$V(X, c) = \max_{(c_t, \pi_t) \in \Pi} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right],$$

(12)

subject to the wealth process (10).

Here we use the utility function in (1) instead of its rewritten form (8) in the statement of the problem, since it does not impose the growth condition which the latter requires.
It is also easier to deal with utility function (1) than the other when deriving the solution. We will, however, resort to the form in (8) involving loss aversion when interpreting optimal policies.

We make the following assumption which guarantees the existence of a solution to the classical consumption and portfolio selection problem without adjustment costs (Merton (1969, 1971)).

Assumption 2. 

$$K \equiv r + \frac{\delta - r}{\gamma} + \frac{\gamma - 1}{2\gamma^2} \theta^2 > 0 \quad \text{with} \quad \theta \equiv \frac{\mu - r}{\sigma}.$$ 

3 Optimal Consumption and Portfolio Policies

We will derive optimal consumption and portfolio policies. Problem 1 is dynamic in character, since it is subject dynamic constraint (10). We transform the problem to a static problem by the well-known method of linearizing the budget constraint developed by Karatzas et al. (1987) and Cox and Huang (1989) (Subsection 3.1). We then transform the problem into a singular control problem (Problem 3, called the dual problem). Theorem 3.1 establishes a duality relationship between the value function of the dual problem and Problem 1.

3.1 Linearization of the Dynamic Budget Constraint

We first transform the dynamic budget constraint (10) into a static constraint by using the linearization technique developed by Cox and Huang (1989) and Karatzas et al. (1987). For this purpose we define, for $t \geq 0$,

$$\xi_t \equiv e^{-rt}Z_t, \quad \text{and} \quad Z_t \equiv e^{-\frac{1}{2}\theta^2 t - \theta B_t}.$$ 

Here $\xi_t$ is called the stochastic discount factor or state price density. It is used to calculate the present value of a risky cash flow perfectly correlated with the value of the market portfolio. The present value can be calculated by using the stochastic discount factor, taking an expectation (see Cox and Huang (1989), Karatzas et al. (1987)). Since an admissible consumption policy $(c_t)_{t=0}^{\infty}$ is financed by the agent’s wealth which changes according to investment returns on the market portfolio, we can discount the consumption flow by the stochastic discount factor and obtain its present value (PV):

$$\text{PV} = \mathbb{E} \left[ \int_0^{\infty} \xi_t c_t dt \right].$$ 

Then, we can rewrite the constraint faced by the agent as a budget constraint in the static form; the present value is less than or equal to the agent’s initial wealth $X_0 = X$

$$\mathbb{E} \left[ \int_0^{\infty} \xi_t c_t dt \right] \leq X.$$ (13)

By using the above constraint, we can restate Problem 1.
Problem 2 (Primal Problem (Static Version)).

Given \( c_0 = c > 0 \) and \( X_0 = X > 0 \), we consider the following optimization problem of the agent:

\[
\max_{(c_t, \pi_t) \in \Pi} E \left[ \int_0^\infty e^{-\delta t} (u(c_t)dt - \alpha d\pi_t^+ - \beta d\pi_t^-) \right] \tag{14}
\]
subject to budget constraint (13).

We will show that Problem 1 is equivalent to Problem 2 and call both of them the primal problem. In other words, the optimized value in Problem 2 is the same as \( V(X, c) \).

We now consider the Lagrangian of Problem 2:

\[
L = E \left[ \int_0^\infty e^{-\delta t} (u(c_t)dt - \alpha d\pi_t^+ - \beta d\pi_t^-) \right] + y \left( X - E \left[ \int_0^\infty \xi_t c_t dt \right] \right) \tag{15}
\]

where \( y > 0 \) is the Lagrange multiplier for the budget constraint (13). We define the process

\[
y_t = ye^{\delta t} \xi_t, \quad t \geq 0
\]

which will play the role of the Lagrange multiplier for the problem starting at time \( t \geq 0 \), and thus \( (y_t)_{t=0}^\infty \) represents the process of the marginal utility (shadow price) of wealth.

We will describe how the marginal utility of wealth is related to the agent’s optimal consumption policy.

Maximization of the Lagrangian (15) involves the following dual problem.

Problem 3 (Dual problem).

\[
J(y, c) = \max_{c_t \in \Pi(c)} E \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - y_t c_t)dt - \alpha d\pi_t^+ - \beta d\pi_t^- \right) \right] \tag{16}
\]

where

\[
h(y, c) \equiv u(c) - yc.
\]

and \( \Pi(c) \) is the class of all admissible consumption policies \( c_t \in \Pi \) satisfying (11).

Note that the state variables for the dual problem (Problem 3) are the marginal value of wealth \( y \) and the given (previous) level of consumption \( c = c_{0-} \), so that the dual value function can be written as a function of \( (y, c) \). The dual problem does not involve choice of portfolio and thus depends only on the choice of consumption.\(^20\)

We will show that the

\[
c_t = c_{0-} + c_t^+ - c_t^-.
\]

By using the decomposition in (17), we will derive a singular control problem (or Hamilton-Jacobi-Bellman equation with singular controls) for \( J(y, c) \).
dual value function $J(y,c)$ and the value function $V(X,c)$ satisfy a duality relationship and hence the value function can be obtained from the dual value function. We will provide a solution to the dual problem by applying a standard method to solve singular control problems developed by Davis and Norman (1990) and Fleming and Soner (2006).

### 3.2 Solution to the Dual Problem and Duality Relationship

The dual problem involves selecting an optimal consumption process. In a formal sense the problem is similar to that of a firm’s investment with costly reversibility (see e.g., Abel and Eberly (1996)). Similar to the investment problem, the optimal policy involves inaction; the agent decides whether to adjust or not to adjust consumption. We provide an intuitive explanation here and will provide a rigorous proof in Appendix B.

Suppose that the agent adjusts consumption by a small amount $dc$ over an infinitesimal time period $[t, t + dc)$, then the benefit can be calculated in utility terms:

$$J(y,t,c_t + dc) - J(y,t,c_t - dc) \approx J_c(y,t,c_t - dc)dc,$$

where $J$ is the dual value function, i.e., the optimized value of the dual problem, and we follow the convention to denote the partial derivative with a subscript, i.e., $J_c \equiv \partial J/\partial c$.\(^{21}\)

The cost of adjustment is given by

$$\text{Adjustment Cost} = \begin{cases} \alpha du^+ = \alpha u'(c_t - dc) & \text{if } dc > 0, \\ \beta du^- = -\beta u'(c_t - dc) & \text{if } dc < 0. \end{cases}$$

Thus, the adjustment is optimal only when the benefit is greater than or equal to the cost, i.e., $J_c(y,t,c_t - dc) \geq \alpha u'(c_t - dc)$ if $dc > 0$ and $-J_c(y,t,c_t - dc) \geq \beta u'(c_t - dc)$ if $dc < 0$. Note that the quantity $J_c(y,t,c_t - dc)$, the marginal valuation of consumption implied by the dual value function, is different from the marginal utility of consumption $u'(c_t - dc)$; as indicated in equation (18) the former measures the benefit of adjusting consumption, taking into consideration the total effects of the decision on the future utility values, and the latter measures the benefit of marginal increase in consumption over an infinitesimal time period $[t, t + dt)$ and is proportional to the cost of adjustment according to our assumption about the agent’s utility in equation (12). The quantity $J_c(y,t,c_t - dc)$ can become negative; it is negative when the previous consumption level has become relatively high due to infrequent adjustments. Hence, inaction is optimal, i.e., the agent does not adjust consumption if $-\beta u'(c_t - dc) < J_c(y,t,c_t - dc) < \alpha u'(c_t - dc)$. When no action is optimal, the dual value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\delta J(y,c) = \frac{\theta^2 y^2}{2} \frac{\partial^2 J(y,c)}{\partial y^2} + (\delta - r)y \frac{\partial J(y,c)}{\partial y} + u(c) - yc.$$  \(^{19}\)

\(^{21}\)Stokey (2009) provides an excellent explanation of optimal polices involving inaction. See, in particular, Chapter 11.

\(^{22}\)Here we have assumed that the dual value function is continuously differentiable with respect to $c$; this property follows from the usual economic consideration for optimality that leads to smooth pasting and super-contact conditions (see Dumas (1991)).
The HJB equation tells us the balance in the optimization problem; the left-hand side of equation (19) is the rate of return in utility terms required by the agent and the right-hand side is the expected rate of return as the sum of the expected change in the utility value $J$ and the instantaneous utility flow $u(c) - yc$ for the dual problem.

When the agent increases consumption, the smooth pasting condition implies (see e.g., Dumas (1991))

$$J_c(y_t, c_t) = \alpha u'(c_t),$$  \hspace{1cm} (20)

and when the agent reduces consumption.

$$J_c(y_t, c_t) = -\beta u'(c_t).$$  \hspace{1cm} (21)

The agent adjusts consumption upward only when her marginal utility of consumption reaches the high value $\alpha u'(c_\ell)$ and adjusts downward when her marginal utility of consumption reaches the low value $-\beta u'(c_\ell)$. Hence, the agent’s optimal decision can be described by three regions in the state space as explained in the following proposition.

**Proposition 3.1.** The optimal consumption policy of Problem 3 can be described by three regions, the inaction region (NR), the increasing region (IR), and the decreasing region (DR) of the state space $\mathcal{D} = \{(y, c) | y > 0, c > 0\}$.

$$\text{IR} = \{(y, c) \in \mathcal{D} | J_c(y, c) = \alpha u'(c)\},$$

$$\text{NR} = \{(y, c) \in \mathcal{D} | -\beta u'(c) < J_c(y, c) < \alpha u'(c)\},$$

$$\text{DR} = \{(y, c) \in \mathcal{D} | J_c(y, c) = -\beta u'(c)\}.$$

Consumption is not adjusted in the inaction region, but is adjusted upward in the increasing region, and is adjusted downward in the decreasing region.

According to Proposition 3.1 optimal consumption policy is determined by the ratio of the marginal valuation of consumption implied by the dual value function, $J_c(y, c)$, to the marginal utility of consumption, $u'(c)$. We define the ratio as a new function $H$:

$$H \equiv \frac{J_c(y, c)}{u'(c)},$$  \hspace{1cm} (23)

which can be interpreted as the normalized marginal valuation of consumption.

Differentiating the HJB equation (19) with respect to $c$, we obtain

$$\frac{\theta^2 y^2}{2} \frac{\partial^2 J_c(y, c)}{\partial y^2} + (\delta - r)y \frac{\partial J_c(y, c)}{\partial y} + u'(c) - y - \delta J_c(y, c) = 0.$$  \hspace{1cm} (24)

Dividing the left-hand side of equation (24) by $u'(c)$, we arrive at

$$\frac{\theta^2 y^2}{2} \frac{d^2 H}{dz^2} + (\delta - r)z \frac{dH}{dz} + 1 - z - \delta H = 0 \text{ where } z \equiv \frac{y}{u'(c)}.$$  \hspace{1cm} (25)

Hence, $H$ satisfies an ordinary differential equation with respect to $z$, which is the ratio of the marginal utility of wealth, $y$, and the marginal utility of consumption, $u'(c)$ and will be called the marginal utility ratio. In the absence of adjustment costs, the two
marginal utilities are the same, but they are not necessarily equalized in their presence, and hence the marginal utility ratio \( z_t = y_t/(c_t)^{-\gamma} \) can vary over time. Proposition 3.1 implies that consumption is not adjusted when \(-\beta < H(z_t) < \alpha\), and is adjusted upward when \( H(z_t) = \alpha\), and downward when \( H(z_t) = -\beta\). We know consumption is adjusted upward when the marginal utility of wealth is low and adjusted downward if the opposite is true. Hence, there exists an interval of the marginal utility ratio, \((z_\alpha, z_\beta)\), such that consumption is adjusted upward if \( z_t < z_\alpha \) and adjusted downward if \( z_t > z_\beta \) and inaction is optimal if \( z_\alpha < z < z_\beta \); when an adjustment is made, it is made until the marginal utility ratio reaches the nearest boundary of \((z_\alpha, z_\beta)\). See Figure 2 for the graphical illustration of the regions and adjustment of consumption.

\[
H(z) = \begin{cases} 
\alpha & \text{if } z \leq z_\alpha \\
-\beta & \text{if } z \geq z_\beta.
\end{cases} \tag{26}
\]

and the super-contact condition implies the following (Dumas (1991)):

\[
H'(z) = 0, \quad \text{if } z \leq z_\alpha \text{ or if } z \geq z_\beta. \tag{27}
\]

We provide a summary of how to solve equation (25) with boundary conditions (26) and (27). The solution to the equation involves the roots of a quadratic equation\(^\text{23}\): let \( m_1 \) and \( m_2 \) be the positive and negative roots of the following quadratic equation

\[
\frac{\theta^2}{2}m^2 + (\delta - r - \frac{\theta^2}{2})m - \delta = 0. \tag{28}
\]

\(^\text{23}\)See Dixit and Pindyck (1996) for a solution to a second-order differential equation involving roots of a quadratic equation.

Figure 2: DR-region, NR-region, and IR-region and consumption adjustment

In particular, we have

\[
H(z) = \begin{cases} 
\alpha & \text{if } z \leq z_\alpha \\
-\beta & \text{if } z \geq z_\beta.
\end{cases} \tag{26}
\]

and the super-contact condition implies the following (Dumas (1991)):

\[
H'(z) = 0, \quad \text{if } z \leq z_\alpha \text{ or if } z \geq z_\beta. \tag{27}
\]
Then, a general solution to (25) takes the following form
\[ H(z) = D_1 \left( \frac{z}{z_\alpha} \right)^{m_1} + D_2 \left( \frac{z}{z_\alpha} \right)^{m_2} + \frac{1}{\delta} - \frac{z}{r}, \]
for \( z_{\alpha} < z < z_{\beta} \),
where \( D_1, D_2 \) are constants to be determined later. Conditions (26) and (27) imply
\[ H(z_{\alpha}) = \alpha, \ H'(z_{\alpha}) = 0, \ H(z_{\beta}) = \beta, \ H'(z_{\beta}) = 0. \quad (29) \]
Note (29) is a system of four equations in four unknowns, \( D_1, D_2, z_\alpha, \) and \( z_\beta \). A simple calculation shows that the solutions are given by
\[ D_1 = \frac{(\alpha - \frac{1}{\delta})m_2 + (m_2 - 1)\frac{z_{\alpha}}{z_\alpha}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{\delta})m_1 + (m_1 - 1)\frac{z_{\alpha}}{z_\alpha}}{m_1 - m_2}, \]
\[ z_\alpha = (1 - \delta \alpha)m_1 - 1 \frac{L w^{m_1}}{w^{m_1} - 1}, \quad z_\beta = (1 + \delta \beta)m_1 - 1 \frac{w^{m_1} - 1}{w^{m_1} - w}, \]
where \( L \) is the loss aversion defined in (7) and \( w \in (0, \frac{1}{\delta}) \) is a unique root to the equation
\[ (m_1 - 1)m_2(1 - w^{m_1} - m_2)(L w^{m_1} - 1) - m_1(m_2 - 1)(w^{m_1} - w)(L - w^{m_2}) = 0. \quad (30) \]
Moreover, we can show that the normalized marginal valuation of consumption is decreasing in the marginal utility ratio, i.e.
\[ H'(z) < 0 \text{ for } z \in (z_{\alpha}, z_{\beta}). \quad (31) \]

The dual value function \( J \) can be constructed from \( H \) by taking an appropriate integration of \( J_\epsilon(y,c) = u'(c)H\left(\frac{y}{u(c)}\right) \). The detail is given in Appendix B. We provide the dual value function in the following proposition.

**Proposition 3.2.** The dual value function \( J(y,c) \) of Problem 3 is a convex function of \( y \) and given by
\[
J(y,c) = \begin{cases} 
\frac{D_1yc}{(1 - \gamma + \gamma m_1)z_\alpha \gamma z_\alpha} \left( \frac{yc}{c^{\gamma z_\alpha}} \right)^{m_1-1} & \text{for } (y,c) \in \mathbb{NR}, \\
\frac{1}{\delta} u(c) - \frac{yc}{r} & \text{for } (y,c) \in \mathbb{NR}, \\
J \left( y, \frac{y}{z_\alpha} \right) + \alpha \left( u(c) - u(I \left( \frac{y}{z_\alpha} \right)) \right) & \text{for } (y,c) \in \mathbb{IR}, \\
J \left( y, \frac{y}{z_\beta} \right) - \beta \left( u(c) - u(I \left( \frac{y}{z_\beta} \right)) \right) & \text{for } (y,c) \in \mathbb{DR},
\end{cases}
\]
and the regions \( \mathbb{IR}, \mathbb{NR}, \) and \( \mathbb{DR} \) are rewritten by
\[
\mathbb{IR} = \{(y,c) \in \mathbb{D} \mid y \leq u'(c)z_\alpha\}, \\
\mathbb{NR} = \{(y,c) \in \mathbb{D} \mid u'(c)z_\alpha \leq y \leq u'(c)z_\beta\}, \\
\mathbb{DR} = \{(y,c) \in \mathbb{D} \mid u'(c)z_\beta \leq y\},
\]
where \( I(z) \equiv (u')^{-1}(z) = z^{1-\delta}, \mathbb{NR} \) is the closure of \( \mathbb{NR} \) in \( \mathbb{R} \). Constants \( m_1, m_2, D_1, D_2, z_\alpha, \) and \( z_\beta \) are given in the proof.
Proof. The proof is given in Appendix B.

We now proceed to find the value function of the primal problem (Problem 2) by
using the dual value function obtained in Proposition 3.2. Let us define, for an admissible
consumption process \( C = (c_t)_{t=0}^{\infty} \) satisfying the budget constraint (13),
\[
V(X, C) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right], \\
J(y, C) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} (u(c_t) - ye^{\delta t} \xi c_t) dt \right].
\]

Since the consumption plan satisfies the budget constraint, we can easily obtain the fol-
lowing inequality: for every \( y > 0 \),
\[
V(X, C) \leq V(X, C) + y \left( x - \mathbb{E} \left[ \int_0^\infty \xi c_t dt \right] \right) \\
= J(y, C) + yx.
\]

This implies that
\[
V(X, C) \leq \inf_{y > 0} [J(y, C) + yx].
\]
Hence, the value function \( V(X, C) \) for Problem 2 satisfies the following:
\[
V(X, c) = \max_{(c_t) \in \Pi^b(c)} V(X, C) \leq \max_{(c_t) \in \Pi^b(c)} \min_{y > 0} \left[ J(y, C) + yx \right] \leq \min_{y > 0} \left[ \max_{(c_t) \in \Pi^b(c)} J(y, C) + yx \right]
\]
(36)

where \( \Pi^b(c) \) denotes the set of admissible consumption plans satisfying the budget con-
straint. The last inequality is valid because of the usual inequality that the maximum of
minima of a function is smaller than or equal to the minimum of its maxima. Indeed the
inequalities in (36) are valid as equalities, and we will provide the proof in Appendix C.
This is the content of the following theorem.

Theorem 3.1. The value function \( V(X, c) \) of Problem 1 is the concave conjugate of the
dual value function \( J(y, c) \), i.e.,
\[
V(X, c) = \min_{y > 0} [J(y, c) + yX].
\]
(37)

In addition, there exists a unique solution \( y^* \) for the minimization problem (37).

Proof. The proof is given in Appendix C.

Theorem 3.1 establishes a duality relationship between the value function \( J \) and the
dual value function \( J \) and allows us to obtain the value function from the dual value
function. Namely, the value function is the concave conjugate of the dual value function
if we use a standard term from convex analysis. A well-known result from the convex
analysis (e.g., Proposition 1 and Theorem 1 in Section 8.6 of Luenberger (1969)) implies
\[
J(y, c) = \max_{X > 0} [V(X, c) - yX],
\]
i.e., the dual value function is the convex conjugate of the value function.
3.3 Optimal Consumption to Wealth Ratio

We now explain optimal consumption policies. If the given initial level of consumption $c_{0-}$ and the marginal utility of wealth $y_0$ is such that the initial marginal utility ratio $y_0/u'(c_{0-})$ lies in the increasing region $\text{IR}$ or in the decreasing region $\text{DR}$, then consumption is immediately adjusted to the nearest boundary of the inaction region $\text{NR}$. If the initial marginal utility ratio $y_0/u'(c_{0-})$ is inside the $\text{NR}$-region, then consumption is not adjusted, i.e., consumption is set equal to $c_{0-}$, until the marginal utility ratio $y_t/u'(c_{0-})$ goes outside the region; consumption is adjusted downward if and only if the marginal utility of wealth $y_t$ goes above $u'(c_0)z_{\alpha}$ and adjusted downward if and only if it goes below $u'(c_0)z_{\alpha}$. Proposition 3.3 provides an explicit characterization of the optimal consumption process. Figure 3 shows simulated paths of optimal consumption $c_t$, the marginal utility ratio $y_t/u'(c_t-)$ and the cumulated adjustment processes $c_t^+, c_t^-$. 

**Proposition 3.3.** For given $c_{0-}$, the optimal consumption $c_t^*$ for $t \geq 0$ is given by

$$c_t^* = c_{0-} + c_t^{*+} - c_t^{*-} \quad \text{with} \quad c_{0-}^{*+} = c_{0-}^{*-} = 0,$$

where $y_t^* = y^*e^{\delta t}\xi_t$ and $y^*$ is the unique solution to the minimization problem (37) and

$$c_t^{*+} = \max \left\{ 0, -c_{0-} + \sup_{s \in [0,t]} \left( c_s^{*-} + I(\frac{y_s^*}{z_{\alpha}}) \right) \right\},$$

$$c_t^{*-} = \max \left\{ 0, -c_{0-} + \sup_{s \in [0,t]} \left( c_s^{*+} - I(\frac{y_s^*}{z_{\alpha}}) \right) \right\}.$$  \hspace{1cm} (39)

**Proof.** The proof is given in Appendix D. \hfill \Box

Proposition 3.3 describes the optimal consumption path in terms of the marginal utility ratio. Alternatively, we can describe optimal policies by using the previous level of consumption $c_{t-}$ and current wealth $X_t$. The duality relationship in Theorem 3.1 implies

$$X_t = -J_p(y_t, c_{t-}).$$  \hspace{1cm} (40)

Equation (10) and the dual value function in Proposition 3.2 allow us to derive relationship between wealth and the marginal utility of wealth $y_t$ in the dual problem. The following theorem provides the relationship.

**Theorem 3.2.** For $t > 0$, wealth process $X_t$ and the marginal utility of wealth $y_t$ satisfies

$$X_t = \frac{c_t - c_{t-}}{r} + \left( \frac{D_1 m_1}{(1 - \gamma + \gamma m_1)z_{\alpha}} (\frac{y_t}{(c_{t-})^{\gamma}z_{\alpha}}) - m_{1-1} + \frac{D_2 m_2}{(1 - \gamma + \gamma m_2)z_{\alpha}} (\frac{y_t}{(c_{t-})^{\gamma}z_{\alpha}}) - m_{2-1} \right).$$  \hspace{1cm} (41)

In addition, there exist two positive numbers $\underline{c}$ and $\bar{c}$ (and $\underline{x} = \frac{1}{\underline{c}}$, $\bar{x} = \frac{1}{\bar{c}}$) such that $c_t^* = c_{t-}$, where $c_t^*$ is the optimal consumption at $s$, for $s \geq t$ if and only if

$$\underline{c} < \frac{c_t}{X_s} < \bar{c} \quad \text{or} \quad \underline{x}c_{t-} < X_s < \bar{x}c_{t-}.$$  \hspace{1cm} (42)

The explicit forms of $\underline{c}$ and $\bar{c}$ are given in Appendix E.
Figure 3: Simulation of optimal consumption, $y/u'(c^*)$, $c^*\pm$ and $c^*-\pm$ with $\alpha = 0$, $\beta = 66$. The other parameter values are as follows: $\rho = 0.015$, $\mu = 0.0784$, $r = 0.0086$, $\sigma = 0.2016$, $\gamma = 3.5$, $X = 50$, $c = 1$ and $T = 80$.

Proof. The proof is given in Appendix E.

Theorem 3.2 provides the optimal consumption policy in terms of the consumption-wealth ratio $c_t^-/X_t$. When the ratio stays inside the interval $(c_t, \bar{c})$, consumption is not adjusted, when it goes outside of the interval, consumption is adjusted so that the ratio becomes equal to the nearest boundary of the interval. We will call the interval the inaction interval for the consumption-wealth, since there is no possibility of confusion with the previously named inaction interval for the marginal utility ratio. The optimal consumption policy implies that there exists a stationary distribution of the consumption-wealth ratio (See Section 3.4). Figure 1 shows the regions in the state space consisting of consumption $c_t-$ and wealth $X_t$ and illustrates adjustment of consumption.

3.4 Stationary Distribution

Here we derive the stationary distribution of the consumption-wealth ratio $c_t/X_t$ when the agent adopts the optimal policies for every $t \geq 0$. The stationary distribution can be regarded as a long run distribution of an agent’s consumption-wealth ratio (see. e.g,
Bertola and Caballero (1990)), or a cross-sectional distribution of a population with the same preference parameters which are invariant with the change of time. The stationary distribution can also be used for estimation of parameter values under appropriate assumptions (see, e.g., Ait-Sahalia (1996)).

By Theorem 3.2,

$$\frac{X_t}{c_t} = X\left(\frac{y_t}{c_t - \gamma}\right),$$  \hspace{1cm} (43)

where $$X(\cdot)$$ is defined by

$$X(z) = \frac{1}{r - \theta} \left(\frac{D_1 m_1}{1 - \gamma + \gamma m_1} (\frac{z}{z_\alpha})^{m_1 - 1} + \frac{D_2 m_2}{1 - \gamma + \gamma m_2} (\frac{z}{z_\alpha})^{m_2 - 1}\right).$$  \hspace{1cm} (44)

Note that $$X(z)$$ is a decreasing function in $$z$$ (see Appendix C).

We know that $$\log \frac{y_t}{c_t - \gamma}$$ is a regulated Brownian motion with drift $$(\delta - r)$$ and volatility $$-\theta$$ on $$[\log z_\alpha, \log z_\beta]$$. By Proposition 5.5 in Harrison (1985) or Proposition 10.8 in Stokey (2009), $$\log \frac{y_t}{c_t}$$ has a stationary distribution with the density function

$$p(x) = \frac{\zeta e^{\zeta x}}{z_\beta - z_\alpha},$$  \hspace{1cm} (45)

where $$\zeta = 2(\delta - r)/\theta^2$$.

Figure 4: The density function $$q(x)$$ for stationary distribution of $$c/X$$ with $$\alpha = 0, \beta = 66$$. Other parameter values are as follows: $$\delta = 0.015, r = 0.0086, \mu = 0.0784, \sigma = 0.2016$$ and $$\gamma = 3.5$. 

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By equation (43), for \( x \in [\underline{c}, \bar{c}] \),
\[
\lim_{t \to \infty} \mathbb{P} \left( \frac{c_t}{X_t} \leq x \right) = \lim_{t \to \infty} \mathbb{P} \left( X \left( \frac{y_t}{c_t^{-\gamma}} \right) \geq \frac{1}{x} \right)
\]
\[
= \lim_{t \to \infty} \mathbb{P} \left( \frac{y_t}{c_t^{-\gamma}} \leq \frac{1}{x} \right)
\]
\[
= \int_{\log \frac{x}{y}}^{\log \frac{x}{c_t^{-\gamma}}} p(u)du
\]
\[
= \int_{\underline{c}}^{\bar{c}} \left[ -\frac{p(\log X^{-1}(\frac{1}{x}))}{x^2 X''(X^{-1}(\frac{1}{x}))(X^{-1}(\frac{1}{x}))} \right] du.
\]
This implies that \( \frac{c_t}{X_t} \) has a stationary distribution with the density function
\[
q(x) = -\frac{p(\log X^{-1}(\frac{1}{x}))}{x^2 X''(X^{-1}(\frac{1}{x}))(X^{-1}(\frac{1}{x}))}, \quad x \in [\underline{c}, \bar{c}].
\]

### 3.5 Risky Share and Risk-Attitude (RCRRA)

**The optimal portfolio and risky share:** We first derive the optimal portfolio \( \pi_t \).
Suppose that the consumption-wealth ratio stays in the inaction interval. Applying Itô’s lemma to equation (10), we get
\[
dX_t = -\frac{1}{2} J_{ypp}(y_t, c_{t-}) < dy_t, dy_t > - J_{ypp}(y_t, c_{t-})dy_t
\]
\[
= - [(\delta - r)y_t J_{yyy}(y_t, c_{t-}) + \frac{1}{2} \sigma^2 y_t^2 J_{yyy}(y_t, c_{t-})] dt + \theta y_t J_{yy}(y_t, c_{t-})dB_t,
\]
where \( < dy_t, dy_t > \) denotes the quadratic variation of \( y_t \) and the second equality follows from the fact \( dy_t = (\delta - r)y_t dt - \theta y_t dB_t \). By comparing of equation (48) with (10), we derive
\[
\pi_t = \frac{\theta}{\sigma} y_t J_{yy}(y_t, c_{t-}) = \frac{\mu - r}{\sigma^2} y_t J_{yy}(y_t, c_{t-}).
\]
Since we have obtained the dual value function \( J \) in Proposition 3.2, we can derive the optimal portfolio in terms of the marginal utility of wealth \( y_t \), given \( c_{t-} \). Proposition 3.4 provides the result.

**Proposition 3.4.** Suppose that \( c_{t-} \) and \( X_t \) at time \( t \) are given such that \( \frac{c_t}{X_t} \in [\underline{c}, \bar{c}] \).
Then, the optimal portfolio \( \pi_t^* \) is as follows:
\[
\pi_t^* = \frac{\theta}{\sigma} c_{t-} \left( D_1 m_1 (m_1 - 1) \left( \frac{y_t^*}{c_{t-}^{-\gamma} z_{t-}} \right)^{m_1 - 1} + D_2 m_2 (m_2 - 1) \left( \frac{y_t^*}{c_{t-}^{-\gamma} z_{t-}} \right)^{m_2 - 1} \right),
\]
where \( y_t^* \) is a unique solution to equation (40).

**Proof.** The proof is given in Appendix F. \( \square \)

Based on the classical portfolio selection results, for given \( c_{t-} \) we can rewrite the optimal portfolio and the risky share in terms of \( X_t \) by using the duality. In other words,
if $X_t \in [x_{c_{t-}}, \bar{x}_{c_{t-}}]$, i.e., when the wealth level is inside the inaction region, we have

$$\pi^*_t = \frac{\mu - r}{\gamma \sigma^2} X_t - F(X_t, c_{t-})$$

and

$$\frac{\pi^*_t}{X_t} = \frac{\mu - r}{\gamma \sigma^2} - \frac{F(X_t, c_{t-})}{X_t},$$

(51)

where $F(\cdot, \cdot)$ is a certain function. In this case, the first term is a myopic term and the remaining term is a hedging term. Note the minus sign in front of the second term. Intuitively, the second term in (51) is the demand that crowds out the myopic demand, in order to maintain the current consumption level since frequent changes of the consumption level incur high utility costs. In this sense, the second term itself is positive. As seen in Panel (b) of Figure 5, the risky share is U-shaped and the ratio of the hedging demand $\frac{F(X_t, c_{t-})}{X_t}$ takes the largest value at the certain point inside the inaction region at which the total risky share takes the minimum value, which generates interesting implications. We will discuss the implications of the U-shaped risky share over time in Section 6 in detail.

Figure 6 shows simulated paths of wealth, optimal consumption and optimal portfolio.

Figure 5: Parameter value are as follows : $\delta = 0.015, r = 0.0085, \mu = 0.0784, \sigma = 0.2016, \gamma = 3, X = 50, c = 1, \alpha = 0, \beta = 66$.

**RCRRA**: In the absence of adjustment costs (the case $\alpha = 0, \beta = 0$) the coefficient $\gamma$ of relative risk aversion (CRRA) of the felicity function and the optimal portfolio $\pi^*_t$ is related in the following way (Merton (1969)):

$$\gamma = \frac{\mu - r}{\sigma^2} \frac{X_t}{\pi^*_t}.$$  

(52)

Motivated by the relationship, we define the *revealed coefficient of relative risk aversion (RCRRA)* as follows:

$$\text{RCRRA}(t) \equiv \frac{\mu - r}{\sigma^2} \frac{X_t}{\pi^*_t},$$

(53)

The RCRRA is the coefficient of relative risk aversion inferred from the agent’s portfolio allocation at time $t$ by an outsider who regards the agent as a plain Mertonian agent,
Figure 6: Simulation of wealth, optimal consumption, and optimal portfolio with $\alpha = 0$, $\beta = 66$. The other parameter values are as follows: $\rho = 0.015$, $\mu = 0.0784$, $r = 0.0086$, $\sigma = 0.2016$, $\gamma = 3.5$, $X_0 = 50$, $C_0 = 1$ and $T = 80$.

i.e., the outside observer assumes that $(\alpha, \beta) = (0, 0)$ for the agent. In the presence of the adjustment costs, partial irreversibility of consumption decisions induces the agent to behave more conservatively than a Mertonian agent, and the RCRRA is generally larger than the CRRA $\gamma$. Proposition 3.4 implies that the RCRRA is determined as a function of the marginal utility ratio $y_t^* / u'(c_t -)$ and hence can be regarded as a function of the consumption-wealth ratio $c_t - / X_t$. Hence the RCRRA changes over time according to changes of the consumption-wealth ratio. The following theorem provides properties of the RCRRA.

**Theorem 3.3.** The RCRRA is a function of the consumption-wealth ratio $c_t - / X_t$ over the interval $[c, \bar{c}]$.

(a) $\text{RCRRA} \geq \gamma$ for every value of the consumption-wealth ratio in $[c, \bar{c}]$.

(b) $\text{RCRRA} = \gamma$ if $c_t - / X_t = c$ or $c_t - / X_t = \bar{c}$.

(c) There exists $\hat{c} \in (c, \bar{c})$ such that RCRRA attains its maximum value. The RCRRA is increasing over $[c, \hat{c}]$ and deceasing over $[\hat{c}, \bar{c}]$.

**Proof.** The proof is given in Appendix G. \qed

According to Theorem 3.3(a) RCRRA is generally greater than the CRRA $\gamma$. Theorem 3.3(b) says the RCRRA is equal to $\gamma$ at the boundaries of the inaction interval, an envelope theorem type result; the agent acts as if not facing the costs when she adjusts consumption. Theorem 3.3(c) implies that the RCRRA shows a hump (inverted U-) shape relationship with the consumption-wealth ratio. Figure 7 provides an illustration of the hump shape relationship. We will discuss the implications of the hump-shape RCRRA in detail in Section 6.
Figure 7: RCRRA with respect to consumption-wealth ration $c_t / X$. Parameter value are as follows: $\delta = 0.015$, $r = 0.0086$, $\mu = 0.0784$, $\sigma = 0.2016$, $\gamma = 3.5$, $\alpha = 0$, $\beta = 66$. In this case, $\bar{c} = 0.0139$ and $\bar{c} = 0.0290$. In particular, $\hat{c} = 0.0192$. The maximum RCRRA value is 5.1586.

4 Loss Aversion

In this section, we discuss the relationship between the partial irreversibility of consumption decision and loss aversion.

4.1 Disentangling Loss Aversion from Risk Aversion

Proposition 2.1 implies that the ordinal preference of the agent is dependent only on loss aversion when it is restricted to consumption processes satisfying an appropriate growth condition. Indeed we can show that the optimal policies depend on $\alpha$ and $\beta$ only through loss aversion $L$.

Theorem 4.1. Given $x$ and $c_0$, $\gamma$, and $\delta$, the optimal consumption policy and the optimal portfolio policy are the same for every pair of $(\alpha, \beta)$ if loss aversion $L$ is the same.

Proof. The proof is given in Appendix H.

The theorem implies that the boundaries $x$ an $\bar{x}$ of the inaction interval are the same and the RCRRA for a given consumption-wealth ratio is the same regardless of $\alpha$ and $\beta$ if loss aversion is the same. Intuitively, if there is an increase in the upward adjustment cost $\alpha$, the agent will be less willing to adjust consumption upward. If there is a simultaneous decrease in the downward adjustment cost $\beta$, then the agent will be less worried about the future costs of decreasing consumption, so she has a greater willingness to increase consumption. It turns out that if the changes in $\alpha$ and $\beta$ are such that loss aversion is not changed, then the two opposing effects are exactly balanced and there is no change in the optimal policy.
Theorem 4.1 also means that the subjective discount factor $\delta$, the risk aversion coefficient $\gamma$ and loss aversion $L$ are three important preference parameters that determine optimal consumption and investment decisions. Note that the definition of loss aversion in (7) does not involve the risk aversion coefficient $\gamma$. Accordingly, we can disentangle the effects of risk aversion and loss aversion. As we have seen the agent’s RCRRA near the boundaries of the inaction interval is close to risk aversion $\gamma$. Thus, risk aversion determines the agent’s behavior when the agent is about to adjust consumption. Utility costs $\alpha$ and $\beta$, in combination, determine loss aversion, and consequently, the inaction interval and increase in RCRRA inside the interval can be regarded as effects of loss aversion.

Definition (7) shows that loss aversion is increasing in the subjective discount rate $\delta$. Accordingly, a more impatient agent exhibits higher loss aversion. Intuitively, the more impatient the agent, the higher the effects of the utility costs, and consequently, the higher the loss aversion.

In the following subsections we will investigate how a change in loss aversion has effects on optimal consumption and investment policies.

### 4.2 Consumption

The following proposition provides the technical impact of loss aversion on the frequency of consumption adjustment.

**Proposition 4.1 (Loss Aversion: Consumption).** Let $\bar{x}$ and $\underline{x}$ be the two boundaries defined in Theorem 3.2. Then, $\bar{x}$ decreases with $L$ and $\underline{x}$ increases with $L$.

**Proof.** The proof is given in Appendix I

![Figure 8: $\bar{x}$ and $\underline{x}$ with respect to loss aversion. Parameter value are as follows: $\delta = 0.015, r = 0.0086, \mu = 0.0784, \sigma = 0.2016, \gamma = 3.5$.](image)

Proposition 4.1 tells us that the level of loss aversion determines the length of each inaction interval. More precisely, the inaction interval increases with loss aversion. In other words, a high loss aversion delays consumption adjustments over time. This property
generally has a significant impact on the pattern of consumption including its time series properties such as mean, volatility, and autocorrelation (see Section 5.2).

### 4.3 Risky Investment

![Graph](image-url)

Figure 9: Panels (a) and (b) plot the risky share and the RCRRA curves for different values of loss aversion. Other parameter values are as follows: $\delta = 0.015$, $\mu = 0.0784$, $\sigma = 0.2016$, $r = 0.0015$ and $\gamma = 3.5$.

Next what is the impact of loss aversion on risky investment? First, from (7) we have $L = 1$ only if $\alpha = \beta = 0$ (the Merton case). In other words, if loss aversion is 1, the inaction region disappears and the risky share is always constant. If loss aversion is greater than 1, the inaction region exists and the risky share has a U-shape in the inaction region. In this case, the risky share has the minimum value inside the inaction region and the maximum value at the boundaries (Theorem 3.3). Moreover, the inaction region becomes wider and the risky share decreases with loss aversion (Figure 9(a)). Similarly, the figure shows that the RCRRA increases with loss aversion (9(b)) except at the boundary points.

**Proposition 4.2** (Loss Aversion: Risky Investment). *Inside the inaction interval (except at the boundary points), the risky share decreases with loss aversion and the RCRRA inside the inaction interval increases with loss aversion. Moreover, as loss aversion goes to infinity, the minimum value of the risky share goes to zero and the maximum value of RCRRA goes to infinity.*

Proof. The proof is given in Appendix J.

We will investigate the implications of the different effects of risk aversion and loss aversion in Section 6.

### 4.4 Value of Loss Aversion and Baseline Parameter Values

We have shown that the loss aversion $L$ summarizes the effects of proportional utility costs $\alpha, \beta$ on the preference and optimal policies. Then, what are plausible values of $L$?
The empirical literature of loss aversion provides a potential answer to the question. Tversky and Kahneman (1991) and Benartzi and Thaler (1995) argue that the disutility of giving up something is about twice as great as the utility of acquiring it. Barberis et al. (2001) use 2.25 as the value of loss aversion for their asset pricing model. Thus, the value close to 2 can be regarded as a reasonable one. In accordance with the literature we will choose $L = 2$ as a benchmark value for calibration/numerical exercises later, or equivalently, we will let

$$\beta \approx \frac{1}{\delta} - 2\alpha.$$  

(54)

In Section 5 and 6 we will use the same set of the market parameter values used by Bansal et al. (2012): $\mu = 0.0784, \sigma = 0.2016$, and $r = 0.0086$. If $\delta = 0.015$ (also from Bansal et al. (2012)) and the agent has no utility loss when he/she increases the level of consumption, we will use $\alpha$ and $\beta$ from (54). In addition, we will use $\gamma = 3.5$.\footnote{The reason for choosing $\gamma = 3.5$ is explained in Section 5.2.} Namely, we will use the following baseline parameter values:

$$(\mu = 0.0784, \sigma = 0.2016, r = 0.0086) \text{ and } (\gamma = 3.5, \delta = 0.015, L = 2).$$  

(55)

5 Consumption and Asset Pricing

5.1 Excess Sensitivity and Excess Smoothness of Consumption

Empirical analysis shows that consumption is too smooth (Deaton, 1987) and exhibits excess sensitivity to shocks (Flavin, 1981). These are called the excess smoothness and the excess sensitivity puzzles.

The optimal consumption policy in our model is consistent with the smoothness of the consumption data. Consumption stays constant inside the inaction interval and is adjusted only when the consumption-wealth ratio moves outside the interval. Thus, households with positive upward or downward adjustment costs are less likely to change consumption in response to small unexpected permanent shocks to income.

The excess sensitivity of consumption can also arise in our model. To understand the mechanism, suppose there is a small or medium good shock. After the shock, consumption may not immediately respond. However, it is more likely to increase later since the probability of increasing consumption in the next period becomes higher. In this sense, the consumption adjustment reflects the accumulated effects of all the income changes, and thus appears to be excessive in our model.

However, both the excess sensitivity and the excess smoothness disappear if the size of the shock is large enough. For a large good shock, the wealth level immediately reaches the upper threshold, which makes consumption increase immediately. By the same token, the consumption level decreases immediately after a large bad shock. So, the consumption moves in the large shock events, following the permanent income hypothesis.
5.2 Implications for Asset Pricing

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Table 1: The first ($E[\Delta c]$), second $\sigma(\Delta c)$, and fifth (AC1($\Delta c$)) columns are the mean, standard deviation, and auto-correlation of the consumption growth rates, respectively. The third (EP) and fourth (Std of IMRS) columns are the equity premium and standard deviation of the IMRS generated by each model, respectively. We use the benchmark parameter set in (55). The data in the first row is from Bansal et al. (2012), sampled on an annual frequency during the period from 1930 to 2008.

Here we investigate the role of each preference parameter for matching the consumption time-series data. To do so, we follow Marshall and Parekh (1999) and Jeon et al. (2018) by assuming the return processes of the investment opportunity and by computing the average and standard deviation of simulated sample paths of the aggregate consumption process.

Similar to Constantinides (1990), we regard the assets as constant returns to scale technologies endowed to the economy and solve for optimal consumption. We simulate the optimal decisions of 100 individuals and aggregate the individual consumption cross-sectionally and temporally to obtain a monthly series. Specifically, we calibrate the model to match the following three consumption moments: (i) mean, (ii) standard deviation, and (iii) auto-correlation of consumption to the data. We repeat the experiment 1000 times and compute means and standard errors by using the empirical distributions of the statistics. We explain the simulation method in Appendix K.

Furthermore, notice that the adjustment costs generate a wedge between the marginal utility of consumption and that of wealth, and thus the intertemporal marginal rate of substitution (IMRS) in consumption

$$e^{-\delta \Delta t} u'(c_{t+\Delta t}^a)/u'(c_t^a) = e^{-\delta \Delta t} (c_{t+\Delta t}^a/c_t^a)^{-\gamma},$$

where $c_t^a$ is the aggregate consumption over $[t, t + \Delta t)$, is not equal to the stochastic discount factor (pricing kernel) $H_{t+\Delta t}/H_t$. We compute the theoretical equity premium $EP$ by using the theoretical IMRS computed from consumption:

$$EP = \frac{-\text{cov}\left(e^{-\delta \Delta t} (c_{t+\Delta t}^a/c_t^a)^{-\gamma}, (\tilde{r}_{t+\Delta t}/\tilde{r}_t)\right)}{\mathbb{E}\left(e^{-\delta \Delta t} (c_{t+\Delta t}^a/c_t^a)^{-\gamma}\right)}.$$  \hspace{1cm} (56)
Table 2: The first (E[∆c]), second σ(∆c), and fifth (AC1(∆c)) columns are the mean, standard deviation, and auto-correlation of the consumption growth rates, respectively for each triplet (δ, γ, L). The third (EP) and fourth (Std of IMRS) columns are the equity premium and standard deviation of the IMRS generated by the model, respectively. The market parameters are μ = 0.0784, σ = 0.2016, and r = 0.0086.

where ˜r_t is the simulated return on the risky asset over [t, t + ∆t]. We also compute the standard deviation of the IMRS in consumption to compare it with the theoretical Hansen-Jagannathan bound (Hansen and Jagannathan (1991) and Otrok et al. (2002)).

Table 1 provides the result by using the benchmark parameter set in (55). It shows that our consumption irreversibility model can well match these consumption moments. The actual equity premium (from Bansal et al. (2012)) is about 7% while the equity premium computed by (56) in our simulation is 0.46%. Note also that both the equity premium and the standard deviation of IMRS generated by our model are fairly smaller than those generated from the Merton model with α = 0 and β = 0 (i.e., L = 1). The reason these two values from our model are much smaller than those from the actual data as well as the Merton model is that the set of time that the consumption adjustment happens has a zero measure. In other words, the Euler equation is violated almost surely

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over time except when the consumption-wealth ratio hits the boundaries of the inaction interval.

Table 2 shows several interesting results by using other parameter sets. First, note from Rows A, B, and C that any values around \( L = 2 \) matches the three moments fairly well if \( \delta = 0.015 \) and \( \gamma = 3.5 \). Second, it is more important to notice that for any pairs of \((\alpha, \beta)\)'s the results are the same if \( \delta, L, \alpha, \) and \( \beta \) satisfy (7). This result is consistent with the one in Theorem 4.1 and Proposition 4.1 and confirms that the level of loss aversion is important for consumption decisions rather than the single values of \( \alpha \) and \( \beta \). Third, the mean growth rate slightly increases with loss aversion and the growth rate volatility slightly decreases with loss aversion. The auto-correlation of consumption is not monotone with loss aversion, but we find the autocorrelation very stable around 48% as long as \( L \) is close to 2.

The results in Rows C, D, and E show that if \( L = 2 \) or \( L \) is in the neighborhood of 2, the model matches the consumption auto-correlation well, while the mean and standard deviation of the consumption growth rate are very sensitive to a choice of \( \delta \) and \( \gamma \). The mean and standard deviation decrease as the agent becomes more impatient (i.e., \( \delta \) increases) or more conservative (i.e., \( \gamma \) increases). We find \((\delta, \gamma) = (0.015, 3.5)\) fits the two moments the best, which is one reason why we pick \( \gamma = 3.5 \) for our benchmark risk-aversion parameter.

In summary, our model fits the consumption data well with reasonable values of the market and preference parameters. In particular, it is surprising that the empirically plausible value of loss aversion \( L = 2 \) can generate an auto-correlation close to that of the historical data.

### 6 Investment in Risky Assets

#### 6.1 Effects of Risk Aversion and Loss Aversion

Our model can also disentangle the effects of risk aversion and loss aversion on the optimal risky share. More precisely, as a result of Theorem 3.3 and Proposition 4.2, risk aversion determines the maximum value of the risky share while loss aversion determines the minimum value of the risky share. In addition, the maximum value decreases with risk aversion and the minimum value decreases with loss aversion (see Table 3). In particular, given fixed risk aversion, an increase in loss aversion deepens the curvature of the U-shape in the risky share (see Figure 9). See also Section 4.3 for a summary of these properties.

The above property helps us to understand the well-known puzzle of why the fraction of risky asset holdings of households are smaller than what is predicted theoretically. Households hold \( 6 - 20\% \) of their assets in equity (conditional on participation, up to 40%).\(^{25}\) For example, theoretically the classical Merton model predicts that the risky share should be 191% and 114% when \( \gamma = 0.9 \) and 1.5, respectively, with the benchmark

---

\(^{25}\)See, for example, Guiso and Sodini (2013)
\[ \gamma = 0.9 \quad \gamma = 1.5 \quad \gamma = 3.5 \quad \gamma = 6 \quad \gamma = 10 \]

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<th>( \alpha = \beta = 0 )</th>
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<td>191% 114% 49% 28% 17%</td>
<td>144% 191% 69% 114% 33% 49% 20% 28% 13% 17%</td>
<td>123% 191% 59% 114% 29% 49% 18% 28% 12% 17%</td>
<td>95% 191% 47% 114% 33% 49% 20% 28% 10% 17%</td>
<td>79% 191% 40% 114% 22% 49% 14% 28% 9% 17%</td>
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Table 3: The risky share for different values of \( L \)'s: the first row corresponds to the standard Merton case. The other parameters values are \( \delta = 0.015, r = 0.0086, \mu = 0.0784, \) and \( \sigma = 0.2016. \)

market parameter values: \( \mu = 7.84\%, \sigma = 20.16\%, r = 0.86\%. \) Table 3 shows the risky share for various values of \( L \) and \( \gamma \). The optimal risky share decreases with both risk aversion and loss aversion. It falls into a range consistent with empirically plausible household equity holdings in either one of the following two cases: (i) risk aversion close to 10 and no loss aversion or (ii) a smaller risk aversion with higher loss aversion.

Note that there is an extensive literature on measurement of the coefficient of relative risk aversion. Recent studies based on labor supply or on self reports of personal well-being tend to favor estimates of the coefficient smaller than or close to 1 (see e.g., Chetty (2006), Layard, Mayraz, and Nickel (2008), Bombardini and Trebbi (2012), and Gandelman and Hernández-Murillo (2015)). In contrast, estimates based on asset markets are much higher, typically 10 or higher (see e.g., Bansal et al. (2012)). Our model can reconcile the gap in these two lines of literature. While actual risk aversion of households is low, the additional effect from loss aversion can make households more conservative in risk-taking.

6.2 Implications of the U-Shape Risky Share

![Figure 10: Four different cases of stock price evolution: the parameter values are from the benchmark set, i.e. \((\mu = 0.0784, r = 0.0086, \sigma = 0.2016)\).](image)

We next investigate the question of how the risky share changes in response to increases in wealth. Models with habit, commitment, or DRRA (decreasing relative risk
aversion) often predict that the risky share increases with wealth while standard models with CRRA preferences predict no change in the risky share. The empirical literature is also inconclusive. For example, Calvet et al. (2009) and Calvet and Sodini (2014) provide a result consistent with the habit or commitment or DRRA models, and Chiappori and Piaiella (2011) and Brunnermeier and Nagel (2008) provide evidence consistent with the standard CRRA model and even show a slightly negative relationship between the risky share and wealth. We shed light on on the debate by showing that choices of sample paths generated from one underlying data-generating process (e.g., time-series of the stock market data) can produce the positive, neutral and negative relationships between the risky share and wealth.

Figure 10 shows four types of sample paths we consider. Panel (a) shows a path, corresponding to a typical long-term bull market, and Panel (b) shows a path corresponding to a typical bear market. Panels (c) and (d) have both bull and bear markets during the sample periods and hence we call them mixed markets. We will show that the impact of wealth change on the risky share is different for each sample path. To do so, we generate a population of agents having the joint lognormal density of \((\alpha, \beta)\), in which the mean values of \(\alpha\) and \(\beta\) satisfies (7) with \(\delta = 0.015\) and \(L = 2\) or \(L = 2.33\), i.e., the mean value of loss aversion of the population is 2 or 2.33 with different variances (see the description in Table 4). We simulate their wealth and risky shares over time and conduct regression analysis similar to that by Brunnermeier and Nagel (2008). Consider the following equation

\[
\Delta_k \log \frac{\pi_t}{X_t} = \rho \Delta_k \log X_t, \tag{57}
\]

where \(\Delta_k\) denotes a \(k\)-period(year) first-difference operator, \(\Delta_k y_t \equiv y_t - y_{t-k}\). We give the details of simulation and regression analysis in Appendix M.

The results are summarized in Table 4. The regression results show a positive effect of wealth increase on the risky share when the market is in an upward trend (Panel (a)) and a negative effect when the market is an downward trend (Panel (b)). There is no wealth effect on the risky share or a very small effect (if any) when the market has both mixed or no trends in the sample periods (Panels (c) and (d)).

The intuition behind these results originates from the ratio of risky asset holding dynamics and wealth process. The risky share shows a U-shape relationship with the consumption-wealth ratio within the inaction interval. Hence, there exist two regions inside the interval: the left region (\(L\)-region), in which the risky share decreases with wealth, and the right region (\(R\)-region), in which the risky share increases with wealth (see Figure 6.2). If the market is in an upward trend as in Panel (a), the wealth process is more likely to stay in \(R\)-regions in which the risky share increases with wealth. However, if the market is in a downward trend as in Panel (b), the wealth is more likely to stay in \(L\)-regions, in which the risky share decreases with wealth.

For a discrete time illustration, refer to Appendix L and Figure 15. If there are consecutive good shocks, the wealth process tends to move in the following way: \(X_0 \rightarrow B \rightarrow B_u \rightarrow B_{uu}\). The wealth process will stay longer in the increasing region of each
Table 4: The regression coefficients for markets (a), (b), (c), and (d) in Figure 10: We generate the distribution of households with $\alpha$ and $\beta$ using the log-normal distributions with mean $m_\alpha$, $m_\beta$ and variance $v_\alpha$, $v_\beta$ respectively. The values in the parentheses are p-values greater than 1%. *** means a p-value smaller than 1%. The parameters values are from the benchmark set, i.e., $(\mu = 0.0784, r = 0.0086, \sigma = 0.2016)$ and $(\gamma = 3.5, \delta = 0.015)$.

inaction region since there are more good shocks than bad shocks in size and amount. However, if there are consecutive bad shocks, the wealth process tends to move in the other way: $X_0 \rightarrow A \rightarrow A_d \rightarrow A_{dd}$. During the times of this journey, the wealth process will remain longer in the decreasing region of each inaction region since there are more bad shocks than good shocks. In a scenario such as mixed or no trends (Panels (c) and (d) of Figure 10), it would matter whether the wealth process go through the L-region more often than the R-region or vice versa.

6.3 Time-varying Risk Aversion (RCRRA)

Table 5: The maximum values of RCRRA when the actual value of risk aversion is $\gamma = 0.9$. The other parameters values are $\delta = 0.015, r = 0.0086$, and $\mu = 0.0784$.

The minimum value of RCRRA is the same as the agent’s actual risk aversion. The maximum value of RCRRA increases with loss aversion (see Table 5). RCRRA is changing
over time between these two values. Then, a natural question is when RCRRA is high and low in the time-series sense?

Figures 12 and 13 show simulated sample paths of RCRRA in time. While the actual risk aversion level is $\gamma = 3.5$ in these figures, the RCRRA can take values more than double that. Notice that the upper and lower dotted lines are the boundaries of the inaction region over time in Panel (a) of both figures. The RCRRA tends to be low during the times when there are consecutive large shocks and wealth fluctuates by large magnitudes. For example, the RCRRA is low as the changes in wealth are large in the times between $t = 20$ and $t = 35$ in Figure 12 and between $t = 5$ and $t = 27$ in Figure 13. However, the RCRRA tends to be high during the times when there are small or modest fluctuations in wealth due to moderately alternating good and bad shocks. For example, the RCRRA is high and the changes in wealth are small in the times before $t = 10$ or after $t = 40$ in Figure 12 and in the times after $t = 40$ in Figure 13. The agent in our model exhibits substantial risk aversion when the market is neither bullish nor bearish.
Figure 12: Parameter values are as follows: \( \delta = 0.015, r = 0.0086, \mu = 0.0784, \sigma = 0.2016, T = 80, X = 61.81, c = 1, \) and \( L = 10. \) The maximum RCRRA value is 7.7329.

while the agent looks more aggressive (showing his/her actual risk aversion) when there are large consecutive shocks.

Note that traditional habit models also produce time-varying risk aversion. The mechanism behind changes in risk aversion in our model is, however, different from that in habit models. In our model risk aversion depends on whether the consumption-wealth ratio is close to a boundary of the inaction interval and in habit models it depends on whether consumption is close to the current level of habit stock. Thus, risk aversion tends to increase with a decline in wealth in habit models, whereas it can increase with an increase in wealth in our model.

Grossman and Laroque (1990) and Chetty and Szeidl (2007) have shown that an agent with durable consumption or with consumption commitment exhibits high risk aversion to large shocks to wealth but low risk aversion to small shocks. Our model can produce such attitudes. The RCRRA has the highest value near the center of the inaction interval and the smallest value at its end points. A large shock induces the agent
to adjust consumption and the RCRRA is the lowest at this moment, whereas a small shock doesn’t change the RCRRA much. The models, however, show the relationship by using numerical solutions. We have derived an exact theoretical relationship between the RCRRA and the consumption-wealth ratio in closed form.

7 Re-interpretation and Extensions

In this section we provide a re-interpretation of our model as that of durable goods and other extensions.

7.1 Interpretation as a Model of Durable Goods

We can reinterpret our model as that of a durable good. For this purpose we consider an extension of the model of durability and local substitution by Hindy and Huang (1993) by permitting partial reversibility of purchases of durable goods,26

We consider an agent whose objective is to maximize the following expected utility:

\[ U \equiv \mathbb{E} \left[ \int_{0}^{\infty} \left( e^{-\delta t} u(Z_t) dt - \alpha u'(Z_t) dC_t^+ - \beta u'(Z_t) dC_t^- + t - \beta u'(Z_t) dC_t^- - t \right) \right]. \] (58)

where

\[ Z_t = Z_0 - e^{-qt} q \int_{0}^{t} e^{-q(t-s)} dC_s. \] (59)

Here \( Z_t \) represents a flow of service provided by a stock of durable goods to the agent; her utility is derived from the service. The stock depreciates at a rate \( q \geq 0 \). The process \( C_t \) is the cumulative net purchase of durable goods, and can be decomposed into the cumulative purchase process \( C_t^+ \) and the cumulative sale process \( C_t^- \), i.e.,

\[ C_t = C_t^+ - C_t^- . \] (60)

Both \( C_t^+ \) and \( C_t^- \) are right-continuous with left limits (RCLL) and non-decreasing processes with \( C_0^+ = 0, C_0^- = 0 \). Hindy and Huang (1993) have studied the limiting case \( \alpha = 0, \beta = \infty \), i.e., the case where there is no resale market for durable goods. Note that the utility costs for adjustment in (58) are proportional to the marginal utility and the magnitudes of change, similar to the costs in (12).

As in Hindy and Huang (1993) we assume \( \delta + (1 - \gamma)q > 0 \) that the problem is well-posed. Then, the optimization problem of the agent is given as follows.

**Problem 4.** The agent chooses \( \{ (C_t^+ , C_t^- ) \}_{t=0}^{\infty} \) and \( \{ \pi_t \}_{t=0}^{\infty} \) to maximize

\[ \mathbb{E} \left[ \int_{0}^{\infty} \left( e^{-\delta t} u(Z_t) dt - \alpha u'(Z_t) dC_t^+ - \beta u'(Z_t) dC_t^- + t - \beta u'(Z_t) dC_t^- - t \right) \right], \] (61)

\[ \frac{26}{26} \text{See Cuoco and Liu (2000) for another extension of their model.} \]
where \((X_t)_{t=0}^\infty\) solves

\[
dX_t = [rX_t + \pi_t(\mu - r)]dt - dC_t + \sigma\pi_t dB_t,
\]

subject to initial condition \(X_0 = X\).

Schroder and Skiadas (2002) have shown that the model of Hindy and Huang (1993) is isomorphic to that of Dybvig (1995). The same isomorphism can be applied to show the equivalence of the model in this section and the model we studied in the previous sections. Namely, we make the following definitions:

\[
\tilde{c}_t \equiv e^{qt}Z_t, \quad \tilde{\pi}_t \equiv \frac{e^{qt}\pi_t}{r + q}, \quad \tilde{X}_t \equiv \frac{(X_t + Z_t)e^{qt}}{r + q}, \quad \tilde{\mu} \equiv \mu + q, \quad \tilde{\sigma} \equiv \sigma, \quad \tilde{\gamma} \equiv \gamma, \quad \tilde{\delta} \equiv \delta + (1 - \gamma)q, \quad \tilde{\alpha} \equiv \frac{\alpha}{q}, \quad \tilde{\beta} \equiv \frac{\beta}{q}.
\]

From (59) we know

\[
d(e^{qt}Z_t) = qe^{qt}dC_t = qe^{qt}dC_t^+ - qe^{qt}dC_t^- = d\tilde{c}_t^+ - d\tilde{c}_t^-,
\]

where \(\tilde{c}_t^+ \equiv \int_0^t qe^{qs}dC_t^+\), \(\tilde{c}_t^- \equiv \int_0^t qe^{qs}dC_t^-\) with \(d\tilde{c}_t^+ - d\tilde{c}_t^- = d\tilde{c}_t\). Clearly, \((\tilde{c}_t^+)_{t=0}^\infty\) and \((\tilde{c}_t^-)_{t=0}^\infty\) are non-decreasing processes.

Similarly to Schroder and Skiadas (2002) it can be easily shown that Problem 4 is equivalent to Problem 1 if the variables and parameters are changed to tilted ones. From (63), we deduce

\[
\tilde{c}_t \tilde{X}_t \tilde{\pi}_t = \tilde{\mu} \tilde{X}_t \tilde{\pi}_t = \tilde{\mu} \tilde{X}_t \tilde{\pi}_t (1 - \frac{1}{\tilde{\mu}} \tilde{c}_t \tilde{X}_t) \quad (68)
\]

By Theorem 3.3, we deduce

\[
\text{RCRRA}(t) \geq \gamma \left(1 - \frac{\tilde{c}_u}{\tilde{\mu}}\right).
\]

Figure 14 shows the RCRRA as a function of \(Z/X\).
Figure 14: Parameter value are as follows: $\delta = 0.015, r = 0.0086, \mu = 0.0784, \sigma = 0.2016, \gamma = 1, q = 0.6, \alpha = 0, \beta = 10.$

7.2 Extensions

We discuss extensions of our model. We first consider a multi-good extension of the model and then a model of competitive equilibrium.

A Multi-Good Model: We consider an extension where there exist multiple goods. Damgard et al. (2003) and Flavin and Nakagawa (2008) have provided a similar extension to that of Grossman and Laroque (1990). For simplicity, we consider the case with two goods; consumption of one good can be adjusted without costs and consumption of the other good can be adjusted by incurring costs. Denoting consumption at $t$ the first good by $c_t = (c_{1,t}, \ldots, c_{n,t})$ and that of the second good with adjustment cost by $Z_t$, the objective function takes the following form:

$$U = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( u(c_t, Z_t) dt - \alpha u_Z(c_t, Z_t) dZ_t^+ - \beta u_Z(c_t, Z_t) dZ_t^- \right) \right],$$

where $u$ is a concave, strictly increasing and continuously differentiable function, and $Z_t^+, Z_t^-$ are non-decreasing processes with $Z_0^- = 0$ satisfying

$$Z_t = Z_t^+ - Z_t^-.$$

With an assumption on the price $P_t, P_{Z,t}$ of the two goods which guarantee the completeness of the financial market we can transform the problem into a dual problem, following a procedure similar to that in Subsection 3.1. The dual problem is a singular control problem, involving choices of $Z_t^+, Z_t^-$. Then, arguments in Section 3 (see e.g., Chen and Dai (2013), Liu (2004)) imply that there exists in general an inaction region in the state space $(X_t, Z_{t-}, P_t)$, where $P_t = (P_t, P_{Z,t})$ is a vector of the prices of the goods.

Suppose that the two goods are complementary, i.e., $u_{cZ} > 0$. Then, the same argument as in Flavin and Nakagawa (2008) implies the elasticity of intertemporal substitution (EIS)
of the first good (i.e., the freely adjustable good) is lower than the value implied by the curvature of the felicity function inside the inaction interval.

A Competitive Equilibrium: We next consider a competitive equilibrium with two agents. The first agent (agent 1) has the utility function (5) facing adjustment costs. The second agent (agent 2) has an ordinary utility function:

$$U = E \left[ \int_{0}^{\infty} e^{-\delta t} u(c_t) dt \right].$$

There exists a firm which generates aggregate output $Y_t$ at $t$, whose evolution satisfies the dynamics

$$dY_t = \mu_Y Y_t dt + \sigma_Y Y_t dB_t,$$

with $\mu_Y, \sigma_Y > 0$. The output is perishable and cannot be stored. The agents are endowed with shares of the firm.

In our companion paper Choi et al. (2019) we show that there exists a competitive equilibrium and derive the risk-free interest rate, prices of the zero coupon bond maturing at $T > 0$ and the price of the stock of the firm in closed form.

8 Concluding Remarks

We have studied a model with the partial irreversibility of consumption decisions by introducing the utility costs of adjustment. The irreversibility generates consumption ratcheting and dynamic loss aversion. Moreover, our model disentangles the effects of loss aversion from those of risk aversion on the agent’s consumption decisions and risk-taking behaviors. We have derived optimal consumption and portfolio policies in closed form. We have shown that the optimal policies have the following properties:

- Given the market parameters, initially given level of consumption, the subjective discount rate and the coefficient of relative risk aversion, the optimal consumption and portfolio policies are determined by loss aversion.

- There exists an inaction interval for the consumption-wealth ratio. If the ratio is inside the inaction interval, then consumption is not adjusted. If the ratio is outside the inaction interval, consumption is immediately adjusted so that the consumption-wealth ratio is restored to the nearest boundary of the inaction interval. The inaction interval becomes wider as loss aversion increases, which implies that the frequency of consumption adjustments decreases with loss aversion.

- The above property can explain several consumption puzzles such as the excess smoothness, the excess sensitivity, and the magnitude hypothesis. We also calibrate the model to match the consumption auto-correlation data. We find that the consumption moments are well matched by using empirically plausible loss aversion.
The risky share shows a U-shaped relationship with financial wealth. The U-shape becomes more pronounced as loss aversion increases, i.e., the maximum value of the risky share is determined by risk aversion and unchanged by loss aversion, while the minimum value decreases with loss aversion. This property can reconcile the empirical rebate on the relationship between financial wealth and the risky share. Moreover, the time-varying risk aversion derived by the optimal investment features high risk-aversion to small and moderate shocks and low risk-aversion to large shocks.

We have also given the interpretation of our model as that of a durable good and discussed potential extensions. Pursuing the interpretation, extensions and their generalizations more extensively and thoroughly can be interesting future research.

References


Appendix

A The proof of Proposition 2.1

Since \( u^+_t, u^-_t \) are positive and negative variation processes of \( u(c_t) \) with \( u^+_0 = u^-_0 = 0 \), we can use the integration by parts formula to derive

\[
-\alpha \int_0^T e^{-\alpha t} du^+_t - \beta \int_0^T e^{-\alpha t} du^-_t = -\delta \int_0^T e^{-\alpha t} (\alpha u^+_t + \beta u^-_t) dt - \left[ e^{-\alpha t} (\alpha u^+_t + \beta u^-_t) \right]_{t=0}^T
\]

(71)

with any \( T > 0 \).

By the assumption of the proposition,

\[
\lim_{T \to \infty} e^{-\alpha T} E [u^+_T] = \lim_{T \to \infty} e^{-\alpha T} E [u^-_T] = 0.
\]

(72)

Since \( u(c_t) = u(c_{0-}) + u^+ - u^- \), we have

\[
U = E \left[ \int_0^\infty e^{-\alpha t} u(c_t) dt - \alpha \int_0^\infty e^{-\alpha t} du^+_t - \beta \int_0^\infty e^{-\alpha t} du^-_t \right]
\]

\[
= E \left[ \int_0^\infty e^{-\alpha t} (u^+_t - u^-_t - \delta \alpha u^+_t - \delta \beta u^-_t) dt \right] + E \left[ \int_0^\infty e^{-\alpha t} u(c_{0-}) dt \right]
\]

\[
= (1 - \delta \alpha E) \left[ \int_0^\infty e^{-\alpha t} (u^+_t - Lu^-_t) dt \right] + \frac{u(c_{0-})}{\delta}.
\]

(73)

B Derivation of the dual value function \( J(y, c) \)

In this section, we derive a solution to Problem 3 by solving the associated HJB equation. Since the process \( u^+_t \) and \( u^-_t \) are difficult to deal with directly, we derive a different version of the dual problem.

By the deterministic Itô’s formula (Theorem 14.26, Pascucci (2011)), we can obtain that

\[
du(c_t) = u'(c_{t-}) dc_t + u(c_t) - u(c_{t-}) - u'(c_{t-}) \Delta c_t,
\]

(74)

and

\[
d||u(c_t)|| = u'(c_{t-}) d||c_t|| + ||u(c_t) - u(c_{t-})|| - u'(c_{t-}) \Delta ||c_t||,
\]

(75)

where \(||c_t||\) and \(||u(c_t)||\) are the total-variation process of \( c_t \) and \( u(c_t) \), respectively.

Since \( (c_t) \) is a finite variation process, \( (c_t) \) can be decomposed as

\[
c_t = c_{0-} + c^+_t - c^-_t,
\]

(76)

where \( c^+_t \) and \( c^-_t \) are the positive and negative variation process of a consumption \( (c_t) \) over \([0-, t] \). That is,

\[
dc_t = dc^+_t - dc^-_t \quad \text{and} \quad d||c_t|| = dc^+_t + dc^-_t.
\]

Hence, we can deduce that

\[
du^+_t = u'(c_{t-}) dc^+_t + \frac{(u(c_t) - u(c_{t-})) + u(c_t) - u(c_{t-}))}{2} - u'(c_{t-}) \Delta c^+_t,
\]

\[
du^-_t = u'(c_{t-}) dc^-_t + \frac{(u(c_t) - u(c_{t-})) + u(c_{t-}) - u(c_t))}{2} - u'(c_{t-}) \Delta c^-_t,
\]

(77)

where

\[
\Delta c^+_t = c^+_t - c^-_t \quad \text{and} \quad \Delta c^-_t = c^-_t - c^+_t.
\]

Moreover, the mean-value theorem implies

\[
du^+_t = u'(c_{t-}) dc^+_t + \frac{(u(c_t) - u(c_{t-})) + u(c_t) - u(c_{t-}))}{2} - u'(c_{t-}) \Delta c^+_t
\]

\[
= u'(c_{t-}) dc^+_t + (u(c_t) - u(c_{t-}) - u'(c_{t-}) \Delta c^+_t) \mathbb{1}_{\{\Delta c^+_t < 0\}}
\]

\[
\geq u'(c_{t-}) dc^+_t.
\]

(78)
Similarly, we obtain \( du^- \geq u'(c_{l-}) dc^- \). For admissible consumption \((c)\), the following inequality holds:

\[
\mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - y c_t) dt - \alpha du_t^+ - \beta du_t^- \right) \right] \\
\leq \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - y c_t) dt - \alpha u'(c_{l-}) dc_t^+ - \beta u'(c_{l-}) dc_t^- \right) \right].
\]  
(79)

From this observation, we can consider the following problem instead of the dual problem.

**Problem A.**

\[
\tilde{J}(y,c) = \max_{(c_t) \in \mathcal{H}(c)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t) - y c_t) dt - \alpha u'(c_{l-}) dc_t^+ - \beta u'(c_{l-}) dc_t^- \right) \right].
\]  
(80)

Notice that (79) implies that \( J(y,c) \leq \tilde{J}(y,c) \). Problem A involves the choice of two non-decreasing process \((c_t^+)_{t=0}^\infty \) and \((c_t^-)_{t=0}^\infty \) which is called a singular control. Then, our proof has the following steps.

(S1) Find the value function \( \tilde{J}(y,c) \) defined in Problem A (including the verification).

(S2) Show that there exists an optimal consumption strategy \((c^+,c^-)\) for Problem A.

(S3) The optimal strategy found in (S2) is also the optimal strategy for the dual problem. Therefore, we have \( J(y,c) = \tilde{J}(y,c) \).

Now, we will analyze the value function \( \tilde{J}(y,c) \) defined in Problem A. By a standard technique for a singular control problem (see e.g., Davis and Norman (1990), Fleming and Soner (2006)), the value function \( \tilde{J}(y,c) \) satisfies the following HJB equation:

\[
\max\{\mathcal{L}\tilde{J}(y,c) + u(c) - yc, \tilde{J}_y(y,c) - \alpha u'(c_{l-}), -\tilde{J}_c - \beta u'(c_{l-})\} = 0, \quad (y,c) \in \mathcal{D}
\]  
(81)

where \( \mathcal{D} = \mathbb{R}_+ \times \mathbb{R}_+ \) and the differential operator \( \mathcal{L} \) is given by

\[
\mathcal{L} = \frac{\theta^2}{2} y^2 \frac{\partial^2}{\partial y^2} + (\delta - r)y \frac{\partial}{\partial y} - \delta.
\]

The following theorem guarantees that the solution to the HJB equation (81) is the solution to Problem A.

**Theorem B.1** (Verification Theorem).

1. Suppose that the HJB equation (81) has a twice continuously differentiable solution \( \tilde{J}(y,c) : \mathcal{D} \rightarrow \mathbb{R} \) satisfying the following conditions:

   (1) For any admissible consumption strategy \((c^+,c^-)\), the process defined by

   \[
   \int_0^t e^{-\delta s} (-\theta) y_s \tilde{J}_y(y_s,c_s) dB_s, \quad t \geq 0,
   \]

   is a martingale.

   (2) For any admissible consumption strategy \((c^+,c^-)\),

   \[
   \lim_{t \to \infty} \mathbb{E}[e^{-\delta t} \tilde{J}(y_t,c_t)] \geq 0.
   \]

Then, for initial condition \((y_0,c_0) \in \mathcal{D}\) and any admissible consumption strategy \((c^+,c^-)\),

\[
\tilde{J}(y_0,c_0) \geq \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( h(y_t,c_t) dt - \alpha u'(c_{l-}) dc_t^+ - \beta u'(c_{l-}) dc_t^- \right) \right].
\]

2. Given any initial condition \((y,c) \in \mathcal{D}\), suppose that there exist an admissible and continuous consumption strategy \((c^{*+},c^{*-})\) such that, if \( c^{*} \) is the associated consumption process, then

\[
(y,c) \in \left\{ (y,c) \in \mathcal{D} : \mathcal{L}\tilde{J}(y,c) + h(y,c) = 0 \right\},
\]

where \(\mathcal{L}\) is the differential operator given by

\[
\mathcal{L} = \frac{\theta^2}{2} y^2 \frac{\partial^2}{\partial y^2} + (\delta - r)y \frac{\partial}{\partial y} - \delta.
\]
Lebesgue-a.e., \( \mathbb{P} \)-a.s.,

\[
\int_0^t e^{-\delta s} \left( \hat{J}_e(y_s, c^*_s) - \alpha u'(c^*_s) \right) dc^*_s = 0, \quad \text{for all } t \geq 0, \mathbb{P} \text{-a.s.,}
\]

\[
\int_0^t e^{-\delta s} \left( -\hat{J}_e(y_s, c^*_s) - \beta u'(c^*_s) \right) dc^-_s = 0, \quad \text{for all } t \geq 0, \mathbb{P} \text{-a.s.,}
\]

and

\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{-\delta t} \hat{J}(y_t, c^*_t) \right] = 0 \quad \text{(Transversality condition).}
\]

Then, \( \hat{J}(y, c) \) and \((c^+, c^-)\) are the value function and the optimal consumption strategy for Problem A, respectively.

**Proof. (Proof of 1.)**

For given consumption process \( \{c_t\}_{t=0}^\infty \), define a process

\[
M^c_t = \int_0^t e^{-\delta s} \left( (u(c_s) - y_s c_s)ds - \alpha u'(c^-_s)dc^+_s - \beta u'(c^-_s)dc^-_s \right) + e^{-\delta t} \hat{J}(y_t, c_t).
\]

By the generalized Itô’s lemma (See Harrison (1985)),

\[
dM^c_t = e^{-\delta t}(u(c_t) - y_t c_t)dt - \alpha u'(c^-_t)dc^+_t - \beta u'(c^-_t)dc^-_t + \left( e^{-\delta t}d\hat{J}(y_t, c_t) - e^{-\delta t}\delta \hat{J}(y_t, c_t)dt \right)
\]

\[
= e^{-\delta t} \left( \frac{\theta^2}{2} \hat{J}_{pp}(y_t, c_t) + (\delta - r)y_t \hat{J}_p(y_t, c_t) - \delta \hat{J}(y_t, c_t) + u(c_t) - y_t c_t \right) dt
\]

\[
+ e^{-\delta t} (\hat{J}(y_t, c_t) - \alpha u'(c^-_t))dc^+_t + e^{-\delta t} (-\hat{J}(y_t, c^-_t) - \beta u'(c^-_t))dc^-_t
\]

\[
+ e^{-\delta t} (\hat{J}(y_t, c_t) - \hat{J}(y_t, c^-_t) - \alpha u'(c^-_t)\Delta c_t) 1_{\{\Delta c_t > 0\}} + e^{-\delta t} (\hat{J}(y_t, c^-_t) - \hat{J}(y_t, c^-_t) + \beta u'(c^-_t)\Delta c_t) 1_{\{\Delta c_t < 0\}} - \theta e^{-\delta t} y_t \hat{J}_p(y_t, c_t) dB_t
\]

where \((c^+)^c\) and \((c^-)^c\) are the continuous parts of \(c^+\) and \(c^-\), respectively.

Hence, for any fixed \( T > 0 \),

\[
M^c_T - M^c_0 = \int_0^T e^{-\delta s} \left( \frac{\theta^2}{2} \hat{J}_{pp}(y_s, c_s) + (\delta - r)y_s \hat{J}_p(y_s, c_s) - \delta \hat{J} + u(c_s) - y_s c_s \right) ds
\]

\[
+ \int_0^T \left( \hat{J}_e(y_s, c^-_s) - \alpha u'(c^-_s) \right) d(c^+_s) + \int_0^T \left( -\hat{J}_e(y_s, c^-_s) - \beta u'(c^-_s) \right) d(c^-_s)
\]

\[
+ \sum_{1 \leq s \leq T} e^{-\delta s} \left( \hat{J}(y_s, c^-_s) - \hat{J}(y_s, c^-_s) - \alpha u'(c^-_s)\Delta c_s \right) 1_{\{\Delta c_s > 0\}}
\]

\[
+ \sum_{1 \leq s \leq T} e^{-\delta s} \left( \hat{J}(y_s, c^-_s) - \hat{J}(y_s, c^-_s) + \beta u'(c^-_s)\Delta c_s \right) 1_{\{\Delta c_s < 0\}}
\]

\[
+ \int_0^T (-\theta)e^{-\delta s} y_s \hat{J}_p(y_s, c_s) dB_s.
\]

Since

\[
\max \{ \mathcal{L} \hat{J} + u(c) - yc, \hat{J}_e(y, c) - \alpha u'(c), -\hat{J}_e(y, c) - \beta u'(c) \} = 0,
\]

we deduce that

\( (A) \leq 0 \) and \( (B) \leq 0 \).

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Moreover,

\[ (C) = \sum_{t \leq s \leq T} e^{-\delta s} \int_{c_s - \Delta c_s}^{c_s} (\hat{J}(y_s, c_s) - \alpha u'(c)) \, dc \cdot 1_{(\Delta c_s > 0)} \leq 0, \]

\[ (D) = \sum_{t \leq s \leq T} e^{-\delta s} \int_{|\Delta c_s|}^{c_s} (-\hat{J}(y_s, c_s - |\Delta c_s| + e) - \beta u'(c)) \, dc \cdot 1_{(\Delta c_s < 0)} \leq 0, \]  \hspace{1cm} (86)

and by assumption, \( \mathbb{E}[E] = 0 \).

Thus, we can conclude that

\[ E_t[M^*_T - M^*_T] \leq 0, \]

and \( \{M^*_t\}_{t \geq 0} \) is a super-martingale.

This implies that \( \mathbb{E}[M^*_T] \leq J(y_0, c_0-) \) and

\[ \hat{J}(y_0, c_0-) \geq \mathbb{E} \left[ \int_0^T e^{-\delta s} \left( (u(c_s) - y_s c_s) \, ds - \alpha u'(c_{s-}) dc_s^+ - \beta u'(c_{s-}) dc_s^- \right) \right] + e^{-\delta T} \hat{J}(y_T, c_T). \]  \hspace{1cm} (87)

By assumption

\[ \liminf_{T \to \infty} \mathbb{E}[e^{-\delta T} \hat{J}(y_T, c_T)] \geq 0, \]

and Fatou’s lemma, we deduce that

\[ \hat{J}(y_0, c_0-) \geq \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t^+) - y_t c_t^+) \, dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^- \right) \right]. \]  \hspace{1cm} (88)

The relation (88) holds for any admissible consumption strategy \((c^+, c^-)\), we obtain

\[ \hat{J}(y_0, c_0-) \geq \sup_{(c^+, c^-) \in \Pi(c)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( h(y_t, c_t^+) \, dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^- \right) \right]. \]

(Proof of 2.)

By assumption, we can show that in (85)

\[ \mathbb{E}[(A)] = \mathbb{E}[(B)] = \mathbb{E}[(C)] = \mathbb{E}[(D)] = \mathbb{E}[(E)] = 0 \]  \hspace{1cm} for the process \( M^*_t \).

This implies that \( \{M^*_t\}_{t \geq 0} \) is a martingale and

\[ \hat{J}(y_0, c_0-) = \mathbb{E} \left[ \int_0^T e^{-\delta t} \left( (u(c_t^+) - y_t c_t^+) \, dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^- \right) \right] + e^{-\delta T} \hat{J}(y_T, c_T). \]  \hspace{1cm} (89)

The transversality condition leads to

\[ \hat{J}(y_0, c_0-) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t^+) - y_t c_t^+) \, dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^- \right) \right]. \]  \hspace{1cm} (90)

Thus,

\[ \hat{J}(y_0, c_0) = \sup_{(c^+, c^-) \in \Pi(c)} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \left( (u(c_t^+) - y_t c_t^+) \, dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^- \right) \right] \]

and the consumption strategy \((c^{*, +}, c^{*, -})\) attains the maximum. Hence \((c^{*, +}, c^{*, -})\) is the optimal.

\[ \square \]

Now, we will obtain the analytic characterization of the dual value function by using the the variational inequality (81).

As Dai and Yi (2009), we consider the double obstacle problem arising from variational inequality (81) as follows:

\[
\begin{cases}
\mathcal{L}w(y, c) + u'(c) - y \geq 0, & \text{ for } w(y, c) = \alpha u'(c), \\
\mathcal{L}w(y, c) + u'(c) - y \leq 0, & \text{ for } w(y, c) = -\beta u'(c), \\
\mathcal{L}w(y, c) + u'(c) - y = 0, & \text{ for } -\beta u'(c) < w(y, c) < \alpha u'(c),
\end{cases}
\]

\[ \text{for } c \in [0, 1]. \]  \hspace{1cm} (91)
Consider the following substitution:

\[ w(y, c) = u'(c)H(z) \quad \text{and} \quad z = \frac{y}{u'(c)}. \]

Then, the double obstacle problem (91) can be changed by

\[
\begin{align*}
\mathcal{L}H(z) + 1 - z & \geq 0, \quad \text{for } H(z) = \alpha, \\
\mathcal{L}H(z) + 1 - z & \leq 0, \quad \text{for } H(z) = -\beta, \\
\mathcal{L}H(z) + 1 - z &= 0, \quad \text{for } -\beta < H(z) < \alpha,
\end{align*}
\]

(92)

The following proposition provides the exact solution of the double obstacle problem (92).

**Proposition B.1.** The variational inequality (92) has a unique \( C^1 \)-solution, which is

\[
H(z) = \begin{cases} 
\alpha, & \text{for } z \leq z_\alpha, \\
D_1 \left( \frac{z}{z_\alpha} \right)^{m_1} + D_2 \left( \frac{z}{z_\alpha} \right)^{m_2} + \frac{1}{\delta} - \frac{z}{\psi}, & \text{for } z_\alpha < z < z_\beta, \\
-\beta, & \text{for } z \geq z_\beta,
\end{cases}
\]

(93)

where

\[
D_1 = \frac{(\alpha - \frac{1}{2})m_2 + (m_2 - 1)\frac{w^{m_2}}{m_2 - m_1}}{m_2 - m_1}, \\
D_2 = \frac{(\alpha - \frac{1}{2})m_1 + (m_1 - 1)\frac{w^{m_1}}{m_1 - m_2}}{m_1 - m_2},
\]

and \( m_1, m_2 \) are positive and negative roots of following quadratic equation:

\[
\frac{\theta^2}{2}m^2 + (\delta - r - \frac{\theta^2}{2})m - \delta = 0.
\]

and \( z_\alpha, z_\beta \) are defined as

\[
\begin{align*}
z_\alpha &= (1 - \delta\alpha) \frac{m_1 - 1}{m_1} \frac{Lw^{m_1} - 1}{w^{m_1 - 1} - 1}, \\
\end{align*}
\]

\[
\begin{align*}
z_\beta &= (1 + \delta\beta) \frac{m_1 - 1}{m_1} \frac{Lw^{m_1} - 1}{w^{m_1 - 1} - 1}, \\
\end{align*}
\]

with \( L = \frac{1 + \delta\beta}{1 - \delta\alpha}. \)

Here, \( w \) is the unique root to the equation \( f(w) = 0 \) in \((0, 1/L)\), where

\[
f(w) = (m_1 - 1)m_2(1 - w^{1 - m_2})(Lw^{m_1} - 1) - m_1(m_2 - 1)(w^{m_1} - w)(L - w^{m_2}).
\]

Also,

\[
H'(z_\alpha) = H'(z_\beta) = 0, \quad H'(z) < 0 \quad \text{for } z \in (z_\alpha, z_\beta)
\]

and \( H'(z) \) attains minimum at \( z_m \in (z_\alpha, z_\beta) \) defined by

\[
z_m = z_\alpha \cdot \left( \frac{-D_2m_2(m_2 - 1)}{D_1m_1(m_1 - 1)} \right)^{\frac{1}{m_1 - m_2}}.
\]

**Proof.** The uniqueness of the solution is guaranteed due to the maximum principle of the partial differential equation (PDE) theory (see Lieberman (1996)).

Now, we will prove the remain part of proposition in the following steps.

(Step 1) We first consider the following free boundary problem:

\[
\begin{align*}
\mathcal{L}H + 1 - z &= 0, \quad z_\alpha < z < z_\beta, \\
H(z_\alpha) &= \alpha, \quad H'(z_\alpha) = 0, \\
H(z_\beta) &= -\beta, \quad H'(z_\beta) = 0.
\end{align*}
\]

(95)

Then we can extend the solution \( H \) onto \( \mathbb{R}_+ \) by

\[
H(z) = \alpha \text{ if } z \in (0, z_\alpha) \quad \text{and} \quad H(z) = -\beta \text{ if } z \in (z_\beta, \infty).
\]

(96)
Next, we show that \( H(z) \) is the solution to variational inequality (92). We can let the general solution for (95) in the form of

\[
H(z) = D_1 \left( \frac{z}{z_\alpha} \right)^{m_1} + D_2 \left( \frac{z}{z_\beta} \right)^{m_2} + \frac{1}{\delta} - \frac{z}{r}.
\]

From the smooth-pasting condition \( H(z_\alpha) = \alpha \) and \( H'(z_\alpha) = 0 \),

\[
H(z_\alpha) = D_1 + D_2 + \frac{1}{\delta} - \frac{z_\alpha}{r} = \alpha,
\]

\[
H'(z_\alpha) = m_1 D_1 + m_2 D_2 - \frac{1}{r} = 0.
\]

Therefore, \( D_1 \) and \( D_2 \) are given by

\[
D_1 = \frac{(\alpha - \frac{1}{\delta}) m_2 + (m_2 - 1) \frac{z_\alpha}{r}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{\delta}) m_1 + (m_1 - 1) \frac{z_\alpha}{r}}{m_1 - m_2}.
\]

Similarly,

\[
H(z_\beta) = D_1 \left( \frac{z_\beta}{z_\alpha} \right)^{m_1} + D_2 \left( \frac{z_\alpha}{z_\beta} \right)^{m_2} + \frac{1}{\delta} - \frac{z_\beta}{r} = -\beta,
\]

\[
H'(z_\beta) = \frac{m_1 D_1}{z_\alpha} \left( \frac{z_\beta}{z_\alpha} \right)^{m_1 - 1} + \frac{m_2 D_2}{z_\alpha} \left( \frac{z_\beta}{z_\alpha} \right)^{m_2 - 1} - \frac{1}{r} = 0,
\]

and

\[
D_1 = \frac{-(\beta + \frac{1}{\delta}) m_2 + (m_2 - 1) \frac{z_\alpha}{r}}{m_2 - m_1} \left( \frac{z_\alpha}{z_\beta} \right)^{m_1}, \quad D_2 = \frac{-(\beta + \frac{1}{\delta}) m_1 + (m_1 - 1) \frac{z_\alpha}{r}}{m_1 - m_2} \left( \frac{z_\alpha}{z_\beta} \right)^{m_2}.
\]

From (98) and (100),

\[
(\alpha - \frac{1}{\delta}) m_2 + (m_2 - 1) \frac{z_\alpha}{r} = \left( -\beta + \frac{1}{\delta} \right) m_2 + (m_2 - 1) \frac{z_\alpha}{r}, \quad \left( \frac{z_\alpha}{z_\beta} \right)^{m_1},
\]

\[
(\alpha - \frac{1}{\delta}) m_1 + (m_1 - 1) \frac{z_\alpha}{r} = \left( -\beta + \frac{1}{\delta} \right) m_1 + (m_1 - 1) \frac{z_\alpha}{r}, \quad \left( \frac{z_\alpha}{z_\beta} \right)^{m_2}.
\]

Since \( \frac{m_1 m_2}{(m_1 - 1)(m_2 - 1)} = \frac{\delta}{r} \), we deduce that \( z_\alpha \) and \( z_\beta \) satisfy the following coupled-algebraic equations:

\[
J_1(z_\alpha, z_\beta) = 0, \quad J_2(z_\alpha, z_\beta) = 0,
\]

where,

\[
J_1(v_1, v_2) = \left( 1 - \alpha \delta - \frac{m_1}{m_1 - 1} v_1 \right) v_1^{m_1} - \left( 1 + \beta \delta - \frac{m_1}{m_1 - 1} v_2 \right) v_2^{-m_1},
\]

\[
J_2(v_1, v_2) = \left( 1 - \alpha \delta - \frac{m_2}{m_2 - 1} v_1 \right) v_1^{m_1} - \left( 1 + \beta \delta - \frac{m_2}{m_2 - 1} v_2 \right) v_2^{-m_1}.
\]

**Step 2** There exist a unique solution \((z_\alpha, z_\beta)\) of the coupled-algebraic equations (102) with \(0 < z_\alpha < (1 - \alpha \delta)\) and \((1 + \beta \delta) < z_\beta < \infty\).

It is easily confirmed that

\[
J_1(v_1, v_2) = \left( 1 - \alpha \delta - \frac{m_1}{m_1 - 1} v_1 \right) v_1^{m_1} - \left( 1 - \alpha \delta - \frac{m_1}{m_1 - 1} v_2 \right) v_2^{-m_1} - \delta (\alpha + \beta) v_2^{-m_1} - \delta (\alpha + \beta) v_1^{m_1}.
\]

\[
m_1 \int_{z_\alpha}^{z_\beta} \left( 1 - \alpha \delta - \xi^{m_1} \right) \xi^{m_1 - 1} d\xi - \delta (\alpha + \beta) z_\beta^{-m_1},
\]

\[
= m_1 \int_{z_\alpha}^{z_\beta} \left( 1 - \alpha \delta - \xi^{m_1} \right) \xi^{m_1 - 1} d\xi - \delta (\alpha + \beta) z_\beta^{-m_1},
\]
and
\[ J_2(v_1, v_2) = \left(1 + \beta \delta - \frac{m_2}{m_2 - 1} v_1\right) v_1^{-m_2} - \left(1 + \beta \delta - \frac{m_2}{m_2 - 1} v_2\right) v_2^{-m_2} - \delta(\alpha + \beta)v_1^{-m_2} \]
\[ = m_2 \int_{z_{\alpha}}^{z_\beta} (1 + \beta \delta - \xi) \xi^{-m_2 - 1} d\xi - \delta(\alpha + \beta)z_\alpha^{-m_2}. \]

If \( z_\alpha \geq (1 - \delta \alpha) \) or \( z_\beta \leq (1 - \beta \delta) \), then
\[ J_1(z_\alpha, z_\beta) = m_1 \int_{z_{\alpha}}^{z_\beta} ((1 - \delta \alpha) - \xi) \xi^{-m_1 - 1} d\xi - \delta(\alpha + \beta)z_\alpha^{-m_1} < 0, \]
\[ J_2(z_\alpha, z_\beta) = m_2 \int_{z_{\alpha}}^{z_\beta} ((1 + \beta \delta) - \xi) \xi^{-m_2 - 1} d\xi - \delta(\alpha + \beta)z_\alpha^{-m_2} < 0, \]

Thus, it suffice that we only consider the case
\[ 0 < z_\alpha < (1 - \delta \alpha), \quad (1 + \beta \delta) < z_\beta < \infty. \]

Since
\[ \lim_{v_1 \to 0^+} J_1(v_1, v_2) = +\infty \quad \text{and} \quad \lim_{v_1 \to v_2} J_1(v_1, v_2) = -\delta(\alpha + \beta) < 0 \]
and
\[ \frac{\partial J_1}{\partial v_1} = -v_1^{-m_1 - 1}m_1((1 - \delta \alpha) - v_1) < 0 \quad \text{for} \ v_1 \in (0, (1 - \delta \alpha)), \]
we deduce that for given \( (1 + \beta \delta) < v_2 \) there exists a unique \( z_\alpha(v_2) \in (0, (1 - \delta \alpha)) \) such that
\[ J_1(z_\alpha(v_2), v_2) = 0. \]

Since
\[ \frac{\partial J_1}{\partial v_1} = -m_1 v_1^{-m_1 - 1}((1 - \delta \alpha) - v_1), \quad \frac{\partial J_1}{\partial v_2} = m_1 v_2^{-m_1 - 1}((1 + \beta \delta) - v_2), \]
we can obtain that
\[ \frac{\partial}{\partial v_2} J_1(z_\alpha(v_2), v_2) = \frac{\partial J_1}{\partial v_1} \frac{\partial z_\alpha}{\partial v_2} + \frac{\partial J_2}{\partial v_2} = 0 \]
and thus
\[ \frac{\partial z_\alpha}{\partial v_2} = \frac{v_2^{-m_1 - 1}}{z_\alpha(v_2)^{-m_1 - 1}((1 - \delta \alpha) - z_\alpha(v_2))} > 0. \]

Similarly,
\[ \frac{\partial}{\partial v_2} J_2(z_\alpha(v_2), v_2) \]
\[ = \frac{v_2^{-m_1 - 1}((1 + \beta \delta) - v_2)}{z_\alpha(v_2)^{-m_1 - 1}((1 - \delta \alpha) - z_\alpha(v_2))} \cdot (-m_2)(z_\alpha(v_2))^{-m_1 - 1}((1 - \delta \alpha) - z_\alpha(v_2)) + m_2 v_2^{-m_2 - 1}((1 + \beta \delta) - v_2) \]
\[ = -m_2((1 + \beta \delta) - v_2) v_2^{-m_1 - 1}(z_\alpha(v_2))^{-m_1 - 2} \left(1 - \left(\frac{v_2}{z_\alpha(v_2)}\right)^{-m_2}\right) > 0. \]

Moreover,
\[ \lim_{v_2 \to (1 + \beta \delta)} J_2(z_\alpha(v_2), v_2) = m_2 \int_{z_{\alpha}}^{(1 + \beta \delta)} ((1 + \beta \delta) - \xi) \xi^{-m_2 - 1} d\xi - \delta(\alpha + \beta)z_\alpha^{-m_2} < 0, \]
\[ \lim_{v_2 \to \infty} J_2(z_\alpha(v_2), v_2) = +\infty. \]

This implies that there exists a unique \( z_\beta \in (1 + \beta \delta, \infty) \) such that
\[ J_1(z_\alpha(z_\beta), z_\beta) = 0 \quad \text{and} \quad J_2(z_\alpha(z_\beta), z_\beta) = 0. \]
Hence, we conclude that there exist a unique solution \((z_\alpha, z_\beta)\) of the coupled-algebraic equations (102) with \(0 < z_\alpha < (1 - \alpha \delta)\) and \((1 + \beta \delta) < z_\beta < \infty\).

Let us define

\[
w := \frac{z_\alpha}{z_\beta} \in (0, \frac{1}{L}).
\]

From (101),

\[
\frac{m_2}{m_1} \frac{(\alpha - \frac{1}{2}) + (\beta + \frac{1}{2})w^{m_1}}{(\alpha - \frac{1}{2}) + (\beta + \frac{1}{2})w^{m_2}} = \frac{m_2 - 1}{m_1 - 1} \frac{w^{m_1} - w}{w^{m_2} - w},
\]

and we deduce that

\[
f(w) = (m_1 - 1)m_2(1 - w^{1-m_2})(Lw^{m_1} - 1) - m_1(m_2 - 1)(w^{m_1} - w)(L - w^{-m_2}) = 0.
\]

Since \(z_\alpha \in (0, (1 - \delta \alpha))\) and \(z_\beta \in ((1 + \delta \beta), \infty)\) are uniquely determined, it is obvious that \(f(w) = 0\) has a unique solution \(w \in (0, \frac{1}{L})\).

From (101),

\[
z_\alpha = (1 - \delta \alpha) \frac{m_1 - 1}{m_1} \frac{Lw^{m_1} - 1}{w^{m_1-1} - 1} \quad \text{and} \quad z_\beta = (1 + \delta \beta) \frac{m_1 - 1}{m_1} \frac{w^m - 1}{w^{m_1} - w}.
\]

**Step 2** In \(z \in (z_\alpha, z_\beta)\), \(H'(z) < 0\) and attains minimum at \(z_m\).

First, we will show that \(D_1 > 0 \quad \text{and} \quad D_2 < 0\).

Since

\[
D_1 = \frac{(\alpha - \frac{1}{2})m_2 + (m_2 - 1)\frac{z_\alpha}{m_1}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{2})m_1 + (m_1 - 1)\frac{z_\alpha}{m_1}}{m_1 - m_2},
\]

\(D_1 > 0 \iff (\alpha - \frac{1}{\delta})m_2 + (m_2 - 1)\frac{z_\alpha}{r} < 0\).

\(\iff z_\alpha > -\frac{m_2}{m_2 - 1} \frac{r}{\delta} (1 - \delta \alpha)\).

\(\iff (1 - \delta \alpha) \frac{m_1 - 1}{m_1} \frac{Lw^{m_1} - 1}{w^{m_1-1} - 1} > -\frac{m_2}{m_2 - 1} \frac{r}{\delta}(1 - \delta \alpha)\).

\(\iff Lw^{m_1} - 1 > \frac{m_1 - 1}{w^{m_1-1} - 1} > 1\).

\(\iff w^{m_1-1} > Lw^{m_1}\).

Similarly, we can deduce that \(D_2 < 0\).

We know that \(H'(z_m) = H'(z_\beta) = 0\) and

\[
H''(z) = \frac{D_1 m_1 (m_1 - 1)}{z_\alpha^2} \left( \frac{z}{z_\alpha} \right)^{m_1 - 2} + \frac{D_2 m_2}{z_\alpha} \left( \frac{z}{z_\alpha} \right)^{m_2 - 2}.
\]

Since \(H''(z_m) = 0\), it is enough to show that

\[
z_\alpha < z_m < z_\beta.
\]

By the definition of \(b_m\),

\[
z_\alpha < z_m < z_\beta \iff 1 < \left( \frac{-D_2 m_2 (m_2 - 1) \frac{1}{D_1 m_1 (m_1 - 1)}}{m_1 - m_2} \right) \frac{1}{z_\alpha^2} < 1.
\]

Since

\[
D_1 = \frac{(\alpha - \frac{1}{2})m_2 + (m_2 - 1)\frac{z_\alpha}{m_1}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{2})m_1 + (m_1 - 1)\frac{z_\alpha}{m_1}}{m_1 - m_2},
\]

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we can easily check that
\[ 1 < \left( \frac{-D_2m_2(m_2 - 1)}{D_1m_1(m_1 - 1)} \right)^{\frac{1}{m_1-m_2}} \if\implies z_\alpha < (1 - \delta \alpha). \]

Also, we know that
\[ D_1 = \frac{-((\beta + \frac{1}{2})m_2 + (m_2 - 1)\frac{z_\alpha}{z_\beta})^{m_1}}{m_2 - m_1}, \quad D_2 = \frac{-((\beta + \frac{1}{2})m_1 + (m_1 - 1)\frac{z_\alpha}{z_\beta})^{m_2}}{m_1 - m_2}. \]

This implies that
\[ \left( \frac{-D_2m_2(m_2 - 1)}{D_1m_1(m_1 - 1)} \right)^{\frac{1}{m_1-m_2}} < \frac{1}{x} \iff z_\beta > (1 + \delta \beta). \]

Thus, we deduce that
\[ z_\alpha < z_m < z_\beta. \]

and
\[ H''(z) < 0, \text{ on } (z_\alpha, z_m) \quad \text{and} \quad H''(z) > 0, \text{ on } (z_m, z_\beta). \]

Hence, \( H'(z) \) attains minimum at \( z = z_m \) and \( H'(z) < 0 \) on \( (z_\alpha, z_\beta) \).

\textbf{(Step 3)} \( H(z) \) satisfies the variational inequality (92).

- For \( z \in (z_\alpha, z_\beta) \), it is clear that
  \[ \mathcal{L}H + 1 - z = 0. \] (109)

  Since \( H(z_\alpha) = \alpha, H(z_\beta) = -\beta \) and \( H'(z) \) is strictly decreasing function on \( (z_\alpha, z_\beta) \),
  \[ -\beta < H(z) < \alpha \text{ on } (z_\alpha, z_\beta). \]

- For \( z \leq z_\alpha, H(z) = \alpha \) and
  \[ \mathcal{L}H + 1 - z = 1 - \delta \alpha - z \geq 0. \]

- For \( z \geq z_\beta, H(z) = -\beta \) and
  \[ \mathcal{L}H + 1 - z = 1 + \delta \beta - z \leq 0. \]

From \textbf{(Step 1)} ~ \textbf{(Step 3)}, we have proved the desired result. \( \square \)

By Proposition B.1, \( w(y, c) \) given by
\[
w(y, c) = \begin{cases} \alpha u'(c), \\ D_1 u'(c) \left( \frac{y}{u'(c)z_\alpha} \right)^{m_1} + D_2 u'(c) \left( \frac{y}{u'(c)z_\alpha} \right)^{m_2} + \frac{u'(c)}{\delta} - \frac{y}{r}, & \text{for } \frac{y}{u'(c)} \leq z_\alpha, \\ -\beta u'(c), & \text{for } z_\alpha < \frac{y}{u'(c)} < z_\beta, \quad \text{(110)} \\ \frac{y}{u'(c)} \geq z_\beta, & \text{for } \frac{y}{u'(c)} \geq z_\beta, \end{cases}
\]
is a solution of the double obstacle problem (91).

Using the \( w(y, c) \) in the equation (110), we construct the value function \( \tilde{J}(y, c) \) as follows:

(i) For \( z_\alpha u'(c) < y < z_\beta u'(c) \),
\[
\tilde{J}(y, c) = \int_0^y u'(x) D_1 \left( \frac{y}{u'(x)z_\alpha} \right)^{m_1} dx - \int_c^y u'(x) D_2 \left( \frac{y}{u'(x)z_\alpha} \right)^{m_2} dx + \frac{u(c)}{\delta} - \frac{yc}{r} \quad \text{(111)}
\]

(ii) For \( z_\alpha u'(c) \geq y \),
\[
\tilde{J}(y, c) = \tilde{J} \left( y, I \left( \frac{y}{z_\alpha} \right) \right) + \alpha \left( u(c) - u(I \left( \frac{y}{z_\alpha} \right)) \right). \quad \text{(112)}
\]
(iii) For \( z_{\beta} u'(c) \leq y \),
\[
\hat{J}(y, c) = \hat{J} \left( y, I \left( \frac{y}{z_{\beta}} \right) \right) - \beta \left( u(c) - u \left( I \left( \frac{y}{z_{\beta}} \right) \right) \right).
\]  \hspace{1cm} (113)

where the function \( I(c) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is defined by
\[
I(y) \equiv (u')^{-1}(y) = y^{-\frac{1}{\gamma}}.
\]

Remark B.1. It is easy to check that
\[
m_1 > 1 \quad \text{and} \quad m_2 < 0.
\]
and for \( u(c) = \frac{1}{1-\gamma} \) for \( \gamma > 0, \gamma \neq 1 \) and \( u(c) = \log c \) for \( \gamma = 1 \),
\[
m_2 < -\frac{1-\gamma}{\gamma} < m_1.
\]
Thus, the two integrals in (111) are well-defined and \( J(y, c) \) is
\[
J(y, c) = D_1 \frac{yc}{(1 - \gamma + \gamma m_1)z_{\alpha}} \left( \frac{y}{c^{-\gamma}z_{\alpha}} \right)^{m_1-1} + D_2 \frac{yc}{(1 - \gamma + \gamma m_2)z_{\alpha}} \left( \frac{y}{c^{-\gamma}z_{\alpha}} \right)^{m_2-1} + \frac{1}{\delta} u(c) - \frac{yc}{r}.
\]

Remark B.2. For \( \hat{J}(y, c) \) defined in (111), (112) and (113), we can easily confirm that
\[
\hat{J}_c(y, c) = w(y, c).
\]

Proposition B.2. For the value function \( \hat{J}(y, c) \) defined in (111), (112) and (113), the following statements are true:

1. \( \hat{J}(y, c) \) is a twice continuously differentiable and satisfies the HJB equation (81). Moreover, the regions \( IR, NR \) and \( DR \) are represented by
\[
IR = \{ (y, c) \in D \mid y \leq u'(c)z_{\alpha} \},
\]
\[
NR = \{ (y, c) \in D \mid u'(c)z_{\alpha} < y < u'(c)z_{\beta} \},
\]
\[
NR = \{ (y, c) \in D \mid u'(c)z_{\beta} \leq y \},
\]
respectively.

2. For an admissible consumption strategy \( (c_t) \),
\[
\int_0^t (-\theta) y \hat{J}_c(y_s, c_s) ds, \quad \forall t \geq 0
\]
is a martingale.

3. For an admissible consumption strategy \( (c_t) \),
\[
\lim_{t \rightarrow \infty} e^{-\delta t} E \left[ \hat{J}(y_t, c_t) \right] = 0.
\]

Proof.
(Poof of 1.)
First, with reference to the construction of the value function \( \hat{J}(y, c) \), we will show that \( \hat{J} \) is a continuously differentiable if we prove that \( \hat{J}_y, \hat{J}_y, \) and \( \hat{J}_{cc} \) are continuous along the free boundaries \( c = I \left( \frac{y}{z_{\alpha}} \right) \) and \( c = I \left( \frac{y}{z_{\beta}} \right) \).

Then, we can compute
\[
\hat{J}_y(y, c) = \hat{J}_y(y, I \left( \frac{y}{z_{\alpha}} \right)) + \left( \hat{J}_c(y, I \left( \frac{y}{z_{\alpha}} \right)) - \alpha u'(I \left( \frac{y}{z_{\alpha}} \right)) \right) \frac{d}{dy} \left( I \left( \frac{y}{z_{\alpha}} \right) \right) \quad \text{for } y \leq u'(c)z_{\alpha}
\]
\[
= \hat{J}_y(y, I \left( \frac{y}{z_{\alpha}} \right)),
\]
\hspace{1cm} (114)
and
\[
\hat{J}_y(y, c) = \hat{J}_y(y, I(\frac{y}{z_\beta})) - \left( -\hat{J}_c(y, I(\frac{y}{z_\beta}) - \beta u'(I(\frac{y}{z_\beta})) \right) \frac{d}{dy} \left( I(\frac{y}{z_\beta}) \right) \quad \text{for } y \geq u'(c)z_\beta
\]
\[
= \hat{J}_y(y, I(\frac{y}{z_\beta})).
\]

Thus, \( \hat{J}_y(y, c) \) is continuous along the free boundaries.

Similarly, we can obtain
\[
\hat{J}_{yy}(y, c) = \hat{J}_{yy}(y, I(\frac{y}{z_\alpha})), \quad \text{for } y \leq u'(c)z_\alpha,
\]
\[
\hat{J}_{yy}(y, c) = \hat{J}_{yy}(y, I(\frac{y}{z_\beta})), \quad \text{for } y \leq u'(c)z_\beta,
\]

and hence \( \hat{J}_{yy} \) is continuous along the free boundaries.

We know that \( \hat{J}_c(y, c) = u'(c)H(y/u'(c)) \) and \( H(z) \) is \( C^1 \)-function. Thus, it is clear that \( \hat{J}_c(y, c) \) is continuous function and we conclude that \( \hat{J}(y, c) \) is \( C^2 \)-function.

Next, we will show that the value function \( \hat{J}(y, c) \) satisfies the HJB-equation (81).

- **The region NR:**
  Since \( \hat{J}_c(y, c) = u'(c)H(y/u'(c)) \),
  \[
  \text{NR} = \{(y, c) \in \mathcal{D} \mid -\beta u'(c) < \hat{J}_c(y, c) < \alpha u'(c)\}
  = \{(y, c) \in \mathcal{D} \mid -\beta < H(\frac{y}{u'(c)}) < \alpha \}
  = \{(y, c) \in \mathcal{D} \mid z_\alpha < \frac{y}{u'(c)} < z_\beta \}.
  \]

  Also, we can easily confirm that
  \[
  \mathcal{L}\hat{J} + u(c) - yc = 0.
  \]

- **The region IR:**
  We deduce that
  \[
  \text{IR} = \{(y, c) \in \mathcal{D} \mid \hat{J}_c(y, c) = \alpha u'(c)\}
  = \{(y, c) \in \mathcal{D} \mid \frac{y}{u'(c)} \leq z_\alpha \}.
  \]

  Clearly,
  \[
  -\hat{J}_c(y, c) - \beta u'(c) = -(\alpha + \beta)u'(c) < 0.
  \]

  Since \( \hat{J}_y(y, c) = \hat{J}_y(y, I(\frac{y}{z_\alpha})) \) and \( \hat{J}_{yy}(y, c) = \hat{J}_{yy}(y, I(\frac{y}{z_\beta})) \) on IR,
  \[
  \mathcal{L}\hat{J}(y, c) + u(c) - yc
  = \left( \mathcal{L}\hat{J}(y, I(\frac{y}{z_\alpha})) + u(I(\frac{y}{z_\alpha})) - yI(\frac{y}{z_\alpha}) \right) + \delta\hat{J}(y, I(\frac{y}{z_\alpha})) - \delta\hat{J}(y, c) + u(c) - yc
  = \int_c^{I(\frac{y}{z_\alpha})} \left( \delta\hat{J}_c(y, \eta) - (u'(\eta) - y) \right) d\eta
  = \int_c^{I(\frac{y}{z_\alpha})} u'(\eta) \left( \frac{y}{u'(\eta)} - 1 - \delta\alpha \right) d\eta \leq 0 \quad \left( \text{since } \frac{y}{u'(\eta)} < z_\alpha < 1 - \delta\alpha \right. \text{ on IR}. \]

- **The region DR:**
  Similarly,
  \[
  \text{DR} = \{(y, c) \in \mathcal{D} \mid \hat{J}_c(y, c) = \beta u'(c)\}
  = \{(y, c) \in \mathcal{D} \mid \frac{y}{u'(c)} \geq z_\beta \}.
  \]
Thus, \( \hat{J}(y, c) \) satisfies the HJB-equation

\[
\max\{\mathcal{L}\hat{J} + u(c) - yc, \hat{J} \alpha u'(c), -\hat{J} - \beta u'(c)\} = 0.
\]

(Problem of 2.)

Let

\[
N_t = \int_0^t e^{-\delta s} (\theta y_s) \hat{J}(y_s, c_s) dB_s.
\]

To show the process \( N_t \) is a martingale, it is suffice to prove that

\[
\mathbb{E} \left[ \int_0^t \left( e^{-\delta s} (\theta y_s) \hat{J}(y_s, c_s) \right)^2 ds \right] < \infty, \quad \text{for } \forall t \geq 0.
\]

(see Chapter 3 in Oksendal (2005))

First, we consider the case when \((y_t, c_t) \in \text{NR.}\) Then,

\[
I\left(\frac{y_t}{z_\alpha}\right) < c_t < I\left(\frac{y_t}{z_\alpha}\right) \quad \text{or} \quad z_\alpha < \left(\frac{y_t}{u'(c_t)}\right) < z_\beta.
\]

Since

\[
y \hat{J}_y(y, c) = \frac{D_1 m_1 y c}{(1 - \gamma + \gamma m_1) z_\alpha} \left(\frac{y}{u'(c) z_\alpha}\right)^{m_1 - 1} + \frac{D_2 m_2 y c}{(1 - \gamma + \gamma m_2) z_\alpha} \left(\frac{y}{u'(c) z_\alpha}\right)^{m_2 - 1} - \frac{yc}{r},
\]

there exist constants \( K_{11}, K_{12} > 0 \) such that

\[
\left| y \hat{J}_y(y, c) \right| < K_{11} y c_t \leq K_{11} y (-\frac{\gamma}{r}).
\]

When \((y_t, c_t) \in \text{IR, we know that}\)

\[
\hat{J}_y(y, c), = \hat{J}_y(y, I\left(\frac{y_t}{z_\alpha}\right)), \quad \text{if } (y_t, c_t) \in \text{IR.}
\]

In this case,

\[
y \hat{J}_y(y, I\left(\frac{y_t}{z_\alpha}\right)) = \frac{D_1 m_1 y I\left(\frac{y_t}{z_\alpha}\right)}{(1 - \gamma + \gamma m_1) z_\alpha} + \frac{D_2 m_2 y I\left(\frac{y_t}{z_\alpha}\right)}{(1 - \gamma + \gamma m_2) z_\alpha} - \frac{y I\left(\frac{y_t}{z_\alpha}\right)}{r},
\]

Thus, there exist constants \( K_{21}, K_{22} > 0 \) such that

\[
\left| y \hat{J}_y(y, c) \right| < K_{21} y t \leq K_{22} (y) (-\frac{\gamma}{r}).
\]

Similarly, when \((y_t, c_t) \in \text{DR, there exist constants}\) \( K_{31}, K_{32} > 0 \) such that

\[
\left| y \hat{J}_y(y, c) \right| < K_{31} y t \leq K_{32} (y) (-\frac{\gamma}{r}).
\]
By (119), (120) and (121), for any \((y_t, c_t) \in \mathcal{D}\),
\[
[y_t, \tilde{J}(y_t, c_t)] \sim K_4(y_t)^{\frac{1}{1-\gamma}}.
\]
for some constant \(K_4 > 0\).

Hence
\[
\mathbb{E} \left[ \int_0^t \left( e^{-\delta s} (\theta y_s) \tilde{J}(y_s, c_s) \right)^2 dt \right] \leq K_4 \mathbb{E} \left[ e^{-\delta t} (y_t)^{\frac{1}{1-\gamma}} \right] = K_4 \int_0^t e^{-(K_4 y_t)^{\frac{1}{1-\gamma}} - \frac{1}{\delta} (1 - \gamma - \frac{yc}{r}) s} ds < \infty.
\]
This implies that \(N_t\) is a martingale for \(t \geq 0\).

**(Proof of 3.)**

From the admissibility of \((c_t)\), we know that the process \(u(c_t)\) is a finite variation process over \((0, \infty)\). This implies that the process \((u(c_t))_{t=0}^\infty\) is bounded and thus
\[
\lim_{t \to \infty} \mathbb{E}[e^{-\delta t} u(c_t)] = 0. \tag{122}
\]
If \((y_t, c_t) \in \mathcal{NR}\), then
\[
z_\alpha < \left( \frac{y_t}{u(c_t)} \right) < z_\beta.
\]
Since,
\[
|\tilde{J}(y, c)| = \left| D_1 \frac{yc}{(1 - \gamma + \gamma m_1) z_\alpha} \left( \frac{y}{e^{-\gamma z_\alpha}} \right)^{m_1 - 1} + D_2 \frac{yc}{(1 - \gamma + \gamma m_2) z_\beta} \left( \frac{y}{e^{-\gamma z_\beta}} \right)^{m_2 - 1} + \frac{1}{\delta} \frac{c^\gamma - \gamma}{1 - \gamma - \frac{yc}{r}} \right| \tag{123}
\]
there exists a constant \(K_{51} > 0\) such that
\[
|J(y, c)| \leq K_{51} y^{\frac{1}{1-\gamma}}. \tag{124}
\]
If \((y_t, c_t) \in \mathcal{IR}\), the value function \(\tilde{J}(y_t, c_t)\) is given by
\[
\tilde{J}(y_t, c_t) = \tilde{J} \left( y_t, I \left( \frac{y_t}{z_\alpha} \right) \right) + \alpha \left( u(c_t) - u(I(\frac{y_t}{z_\alpha})) \right). \tag{125}
\]
If \((y_t, c_t) \in \mathcal{DR}\), the value function \(\tilde{J}(y_t, c_t)\) is given by
\[
\tilde{J}(y_t, c_t) = \tilde{J} \left( y_t, I \left( \frac{y_t}{z_\beta} \right) \right) - \beta \left( u(c_t) - u(I(\frac{y_t}{z_\beta})) \right). \tag{126}
\]
From (124), (125) and (126), we deduce that there exists constants \(K_{52}, K_{53} > 0\) such that
\[
|J(y, c)| \leq K_{52} y^{\frac{1}{1-\gamma}} + K_{53}|u(c)|, \text{ for all } (y_t, c_t) \in \mathcal{R}. \tag{127}
\]
Hence, we can conclude that for any admissible consumption strategy \((c_t)\),
\[
\lim_{t \to \infty} \mathbb{E} \left[ e^{-\delta t} |\tilde{J}(y_t, c_t)| \right] = 0
\]
and thus \(\lim_{t \to \infty} \mathbb{E} \left[ e^{-\delta t} \tilde{J}(y_t, c_t) \right] = 0\). This completes the proof.

The next proposition provides the optimal consumption strategy \((c^+, c^-)\) for Problem A.

**Proposition B.3.** For given \(y > 0\), the optimal consumption \(c^*_t\) at time \(t\) for Problem A is given by
\[
c^*_t = c_{0-} + c^*_t + - c^*_t - \text{ with } c^*_0 + = c^*_0 - = 0,
\]
where
\[
c^*_t + = \max \left\{ 0, -c_{0-} + \sup_{s \in [0, t]} \left( c^*_s - + I(\frac{y_s}{z_\alpha}) \right) \right\}, \tag{128}
\]
\[
c^*_t - = \max \left\{ 0, -c_{0-} + \sup_{s \in [0, t]} \left( c^*_s + - I(\frac{y_s}{z_\beta}) \right) \right\},
\]
and \(y_t = ye^{\delta t} \xi_t\).
Proof. First, we show that the optimal consumption strategy \((c^{*,+}, c^{*,-})\) given in (128) is admissible. For simplicity, we can assume that there is no initial jump on the consumption strategy. That is, \((y, c_{0-}) \in \mathbb{N}^R\). We can see that the optimal consumption strategy \((c^{*,+}, c^{*,-})\) satisfies that
\[
    z_{\alpha} \leq \frac{y_{t}}{(c^{*,+}_{t})^{-\gamma}} \leq z_{\beta} \quad \text{for all } t \geq 0. \tag{129}
\]
Then, there exist constants \(K_{61}, K_{62} > 0\) such that
\[
    |u(c^*_t)| \leq K_6(y_t)^{-\frac{1}{\gamma}}, \tag{130}
\]
for all \(t \geq 0\).

Thus,
\[
    E \left[ \int_0^\infty e^{-\delta t} \max(0, -u(c^*_t)) dt \right] \leq E \left[ \int_0^\infty e^{-\delta t} |u(c^*_t)| dt \right] \leq K_6 E \left[ \int_0^\infty e^{-\delta t} (y_t)^{-\frac{1}{\gamma}} dt \right] < +\infty.
\]
For the optimal strategy \((c^{+, -})\) in (128), let us define the value function \(P(y, c_{0-})\) as
\[
P(y, c_{0-}) = E \left[ \int_0^\infty e^{-\delta t} (\alpha u'(c^*_t)dc^{*,+}_t + \beta u'(c^*_t)dc^{*,-}_t) \right].
\]
Then, the value function \(P(y, c_{0-})\) satisfies the following PDE:
\[
\begin{align*}
    \mathcal{L}P(y, c_{0-}) = 0, & \quad \text{for } z_{\alpha}u'(c_{0-}) < y < z_{\beta}u'(c_{0-}), \\
    \partial_t P(y, c_{0-}) = 0, & \quad \text{for } z_{\alpha}u'(c_{0-}) \geq y, \\
    \partial_t P(y, c_{0-}) = 0, & \quad \text{for } y \leq z_{\alpha}u'(c_{0-}).
\end{align*}
\]
By applying similar method in Appendix B, it is not difficult to derive the closed-form of \(P(y, c_{0-})\) and thus we can show that \(P(y, c_{0-}) < \infty\). That is,
\[
E \left[ \int_0^\infty e^{-\delta t} (\alpha u'(c^*_t)dc^{*,+}_t + \beta u'(c^*_t)dc^{*,-}_t) \right] < \infty.
\]
Above two inequalities imply \((c^{*,+}, c^{*,-})\) is admissible consumption strategy.

Moreover, by (129) and (130), there exists a constant \(K_7 > 0\) such that
\[
|J(y_t, c^*_t)| = \left| D_1 \frac{y_t c^*_t}{(1 + \gamma + \gamma m_1)z_{\alpha}} \left( \frac{y_t}{c^*_t} \right)^{m_1-1} + D_2 \frac{y_t c^*_t}{(1 + \gamma + \gamma m_2)z_{\beta}} \left( \frac{y_t}{c^*_t} \right)^{m_2-1} \right| + \frac{1}{\delta} \frac{(c^*_t)^{1-\gamma}}{1 - \gamma} \left( \frac{y_t}{c^*_t} \right)^{\frac{1-\gamma}{\delta}} \\
\leq K_7(y_t)^{-\frac{1}{\gamma}}. \tag{132}
\]
This implies
\[
\lim_{t \to \infty} E \left[ e^{-\delta t} |J(y_t, c^*_t)| \right] \leq K_7 \lim_{t \to \infty} E \left[ e^{-\delta t} (y_t)^{-\frac{1}{\gamma}} \right] = K_7 \lim_{t \to \infty} e^{-Kt} = 0. \tag{133}
\]

From the construction of the optimal consumption strategy, it is easy to check that the consumption strategy \((c^{+, -})\) given in (128) satisfies the following assumption in Theorem B.1:
\[
(y_t, c^*_t) \in \{ (y, c) \in \mathcal{D} : \mathcal{L}J(y, c) + h(y, c) = 0 \},
\]
Lebesgue-a.e., \(P\)-a.s.,
\[
\int_0^t e^{-\delta s} \left( J_s(y_s, c^*_s) - \alpha u'(c^*_s) \right) dc^{*,+}_s = 0, \quad \text{for all } t \geq 0, \text{ P-a.s.}, \tag{134}
\]
\[
\int_0^t e^{-\delta s} \left( -J_s(y_s, c^*_s) - \beta u'(c^*_s) \right) dc^{*,-}_s = 0, \quad \text{for all } t \geq 0, \text{ P-a.s.}
\]
Finally we verify that the dual value function \( J(y, c) \) is equal to the value function \( \hat{J}(y, c) \) for Problem A in the following theorem.

**Theorem B.2.** For given \( y_0 = y \) and \( c_{0-} = c \), we have \( J(y, c) = \hat{J}(y, c) \). More precisely, strategy \((c^+, c^-)\) defined in Proposition B.3 is the optimal strategy for the dual value function \( J(y, c) \) and

\[
J(y, c) = \mathbb{E} \left[ \sum_{t=0}^{\infty} e^{-\delta t} (u(c_t) - y_t c_t) dt - \alpha u'(c_{t-}) dc^+_t - \beta u'(c_{t-}) dc^-_t \right].
\]

**Proof.** In (78),

\[
du^+_t = u'(c_{t-}) dc^+_t + (u(c_t) - u(c_{t-})) dt - \alpha u'(c_{t-}) \delta t \mathbb{1}_{\{\Delta c_t > 0\}},
\]

\[
du^-_t = u'(c_{t-}) dc^-_t + (u(c_t) - u(c_{t-})) dt - \beta u'(c_{t-}) \delta t \mathbb{1}_{\{\Delta c_t < 0\}}.
\]

From Proposition B.3, except the initial time 0, the optimal consumption strategy \((c^+, c^-)\) has a continuous path. We first consider the case there is no initial jump on the consumption. That is, suppose \((y, c_{0-}) \in \mathbb{NR}\). This implies

\[
du^+_t = u'(c_{t-}) dc^+_t \quad \text{and} \quad du^-_t = u'(c_{t-}) dc^-_t.
\]

By Proposition B.3, we have

\[
\hat{J}(y_0, c_{0-}) = \mathbb{E} \left[ \sum_{t=0}^{\infty} e^{-\delta t} (u(c^*_t) - c^*_t) dt - \alpha u'(c^*_t) dc^+_t - \beta u'(c^*_t) dc^-_t \right]
\]

and thus \( J(y_0, c_{0-}) = \hat{J}(y_0, c_{0-}) \) for \((y, c_{0-}) \in \mathbb{NR}\).

We consider the case where there is an initial jump on the consumption. There are two cases. First, suppose \((y, c_{0-}) \in \mathbb{IR}\). Then, we obtain that

\[
\hat{J}(y_0, c_{0-}) = \hat{J} \left( y, I \left( \frac{y}{z_\alpha} \right) \right) + \alpha \left( u(c_{0-}) - u(I(\frac{y}{z_\alpha})) \right).
\]

By the above argument,

\[
\hat{J} \left( y, I \left( \frac{y}{z_\alpha} \right) \right) = J \left( y, I \left( \frac{y}{z_\alpha} \right) \right).
\]

From the definition of \( u^+ \), we also have

\[
J(y, c_{0-}) = J \left( y, I \left( \frac{y}{z_\alpha} \right) \right) + \alpha \left( u(c_{0-}) - u(I(\frac{y}{z_\alpha})) \right).
\]

Therefore, we have \( J(y_0, c_{0-}) = \hat{J}(y_0, c_{0-}) \) for \((y, c_{0-}) \in \mathbb{IR}\). Second, a similar argument is applied to show that \( J(y_0, c_{0-}) = \hat{J}(y_0, c_{0-}) \) for \((y, c_{0-}) \in \mathbb{DR}\), which completes the proof. \( \square \)

### C Proof of Theorem 3.1

We will show that the duality relationship in the following steps:

**Step 1** First, we will prove that the dual value function \( J(y, c) \) is strictly convex in \( y \):

By direct computation,

\[
\frac{\partial^2 J}{\partial y^2} = c \left( \frac{D_1 m_1 (m_1 - 1)}{(1 - \gamma + \gamma m_1) z_\alpha} \left( \frac{y}{c^{-\gamma} z_\alpha} \right)^{m_1-1} + \frac{D_2 m_2 (m_2 - 1)}{(1 - \gamma + \gamma m_2) z_\alpha} \left( \frac{y}{c^{-\gamma} z_\alpha} \right)^{m_2-1} \right).
\]

Since \( D_1 > 0, D_2 < 0, 1 - \gamma + \gamma m_1 > 0, \) and \( 1 - \gamma + \gamma m_2 < 0, \)

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we deduce that
\[ \frac{\partial^2 J}{\partial y^2} > 0 \text{ for } y > 0. \]

and thus \( J(y, c) \) is strictly convex in \( y \).

Let us denote the Lagrangian \( L \) defined in (15) by \( L(y, c) \) for the Lagrangian multiplier \( y \) and consumption profile \( c \).

**(Step 2)** We will show that there exist a unique solution \( y^* > 0 \) such that \( y^* \) and the optimal consumption \( \{c^*_t\}_{t=0}^\infty \) maximize the Lagrangian \( L(y, c) \).

From Proposition 3.2, we deduce that
\[
\frac{\partial J}{\partial y}(y,c) = \begin{cases} \frac{c}{r} - \frac{D_1 m_ty}{(1 - \gamma + \gamma m_1)z_\alpha} \left( \frac{y}{e^{-\gamma \alpha}} \right)^{m_1-1} + \frac{D_2 m_yc}{(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{y}{e^{-\gamma \alpha}} \right)^{m_2-1}, & \text{for } (y, c) \in \mathbf{NR}, \\
\frac{\partial J}{\partial y} \left( y, I_R \left( \frac{y}{z_\alpha} \right) \right), & \text{for } (y, c) \in \mathbf{IR}, \\
\frac{\partial J}{\partial y} \left( y, I_R \left( \frac{y}{z_\beta} \right) \right), & \text{for } (y, c) \in \mathbf{DR}. \end{cases} \tag{139}
\]

For a sufficiently small \( y > 0 \), \( (y, c) \in \mathbf{IR} \), and for a sufficiently large \( y > 0 \), \( (y, c) \in \mathbf{DR} \).

This implies that
\[
\lim_{y \to 0} \frac{\partial J}{\partial y}(y,c) = \lim_{y \to 0} \frac{\partial J}{\partial y} \left( y, I_R \left( \frac{y}{z_\alpha} \right) \right) = +\infty, \tag{140}
\]
\[
\lim_{y \to \infty} \frac{\partial J}{\partial y}(y,c) = \lim_{y \to \infty} \frac{\partial J}{\partial y} \left( y, I_R \left( \frac{y}{z_\beta} \right) \right) = 0.
\]

Since \( J(y, c) \) is strictly convex in \( y \), for given \( X > 0 \), there exists a unique \( y^* \) such that
\[
X = -\frac{\partial J}{\partial y}(y^*, c). \tag{141}
\]

By Proposition B.3 and Theorem B.2, there exist optimal consumption strategy \((c^{*,+}, c^{*,-})\) such that
\[
J(y^*, c) = E \left[ \int_0^\infty e^{-\delta t} \left( h(y^*_t, c^*_t) dt - \alpha u'(c^*_t) dc^*_t^{*,+} - \beta u'(c^*_t) dc^*_t^{*,-} \right) \right], \tag{142}
\]
where \( y^*_t = y^* e^{\delta t} H_t, \ (c^{*,+}, c^{*,-}) \in \Pi(c) \) and \( c^*_t = c + c^*_t^{*,+} - c^*_t^{*,-} \).

This means that since the Lagrangian is concave, \( y^* \) and \( c^* \) are maximizers of the Lagrangian.

**(Step 3)** \( c^* \) satisfies the budget constraint with equality.

Define \( y^{*,+} = y^* + h \) and \( y^{*,-} = y^* - h \) with \( y^* \geq h > 0 \) (For convenience of notation we drop the time subscript \( t \)). Then,
\[ \mathcal{L}(c^*, y^{*,+}) \geq \mathcal{L}(c^*, y^*). \]

Since \( c^*, y^* \) maximizers of the Lagrangian \( L \), we have
\[
\lim_{h \downarrow 0} \sup \mathcal{L}(c^*, y^{*,h}) - \mathcal{L}(c^*, y^*) \leq 0, \lim_{h \uparrow 0} \inf \mathcal{L}(c^*, y^{*,h}) - \mathcal{L}(c^*, y^*) \leq 0,
\]

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and thus we deduce
\[ \pm \left( X - \mathbb{E} \left[ \int_0^\infty H_t c_t^* dt \right] \right) \leq 0. \]

This leads to
\[ X = \mathbb{E} \left[ \int_0^\infty H_t c_t^* dt \right] \]

This implies that \( c^* \) satisfies the budget constraint with equality.

**Step 4** \( c^* \) is optimal consumption.

Let \( (c^+, c^-) \in \Pi(c) \) be a feasible consumption strategy, i.e., it is admissible and satisfies the budget constraint.

Since \( c \) satisfies the budget constraint,
\[ E \left[ \int_0^\infty e^{-\delta t} (u(c_t) dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^-) \right] \]
\[ \leq E \left[ \int_0^\infty e^{-\delta t} (u(c_t) dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^-) \right] + y^* \left( X - E \left[ \int_0^\infty H_t c_t^* dt \right] \right) \]
where \( y^* \) is defined in \( \text{(Step 2)} \).

Since \( y^* \) and \( c^* \) maximize the Lagrangian \( L \) and \( c^* \) satisfies the budget constraint with equality,
\[ E \left[ \int_0^\infty e^{-\delta t} (u(c_t) dt - \alpha u'(c_t) dc_t^+ - \beta u'(c_t) dc_t^-) \right] \]
\[ \leq E \left[ \int_0^\infty e^{-\delta t} (u(c_t^*) dt - \alpha u'(c_t^*) dc_t^{*,+} - \beta u'(c_t^*) dc_t^{*,-}) \right] + y^* \left( X - E \left[ \int_0^\infty H_t c_t^* dt \right] \right) \]
\[ = E \left[ \int_0^\infty e^{-\delta t} (u(c_t^*) dt - \alpha u'(c_t^*) dc_t^{*,+} - \beta u'(c_t^*) dc_t^{*,-}) \right]. \]

Therefore, \( (c_t^*)_{t=0}^\infty \) is optimal.

**Step 5** Proof of duality-relationship in \( (37) \)

Since \( c^* \) is optimal consumption, for \( y > 0 \), we deduce
\[ V(X, c) = E \left[ \int_0^\infty e^{-\delta t} (u(c_t^*) dt - \alpha u'(c_t^*) dc_t^{*,+} - \beta u'(c_t^*) dc_t^{*,-}) \right] \]
\[ = E \left[ \int_0^\infty e^{-\delta t} (u(c_t') dt - \alpha u'(c_t') dc_t'^{*,+} - \beta u'(c_t') dc_t'^{*,,-}) \right] + y \left( X - E \left[ \int_0^\infty H_t c_t^* dt \right] \right) \]
\[ \leq \sup_{(c^+, c^-) \in \Pi(c)} E \left[ \int_0^\infty e^{-\delta t} u(c_t) dt \right] + y \left( X - E \left[ \int_0^\infty H_t c_t^* dt \right] \right) \]
\[ = J(y, c) + yX, \]
where \( (c')_{t=0}^\infty \) is the optimal consumption process for Problem 3 for \( y > 0 \). This implies that
\[ V(X, c) \leq \inf_{y > 0} \left( J(y, c) + yX \right). \]

However, we know that
\[ V(X, c) = E \left[ \int_0^\infty e^{-\delta t} (u(c_t^*) dt - \alpha u'(c_t^*) dc_t^{*,+} - \beta u'(c_t^*) dc_t^{*,-}) \right] \]
\[ = J(y^*, c) + y^* X. \]
Thus,

\[ V(X, c) = \min_{y \geq 0} \left( J(y, c) + yX \right). \]

This completes the proof.

**D Proof of Proposition 3.3**

By Proposition B.3 in Appendix B and the proof in Appendix C, we can directly verify the proposition.

**E Proof of Theorem 3.2**

From Theorem 3.1, we know that there exists a unique solution \( y^* \) for the minimization problem (37). The first-order condition implies that

\[
X = -\frac{\partial J}{\partial y}(y^*, c) = \frac{c}{r} - \frac{D_1m_1 c}{(1 - \gamma + \gamma m_1)z_\alpha} \left( \frac{y}{e^{-\gamma z_\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2 c}{(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{y}{e^{-\gamma z_\alpha}} \right)^{m_2 - 1}.
\]

(145)

Since Problem 3 is time-consistent, \( y^*_s = y^* e^{\delta t} H_s \) is the minimizer for the duality relationship starting at \( s \geq 0 \). Thus, for wealth \( X_t \) at time \( t \), we have

\[
X_t = \frac{c_t - c_{t-1}}{r} \left( \frac{D_1m_1 c}{(1 - \gamma + \gamma m_1)z_\alpha} \left( \frac{y^*_s}{(c_{t-1})^{-\gamma z_\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2 c}{(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{y^*_s}{(c_{t-1})^{-\gamma z_\alpha}} \right)^{m_2 - 1} \right).
\]

(146)

During the time in which \( y^*_t \) is inside the NR, the optimal consumption \( c^* \) is constant, i.e., \( c^*_s = c_{t-1} \) and thus the wealth \( X_s \) for \( s \geq t \) is given by

\[
X_s = \frac{c_t - c_{t-1}}{r} \left( \frac{D_1m_1 c}{(1 - \gamma + \gamma m_1)z_\alpha} \left( \frac{y^*_s}{(c_{t-1})^{-\gamma z_\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2 c}{(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{y^*_s}{(c_{t-1})^{-\gamma z_\alpha}} \right)^{m_2 - 1} \right).
\]

(147)

Let us define \( \bar{x}, \bar{x}, \bar{\epsilon} \) and \( \bar{c} \) as follows:

\[
\bar{x} = X(z_\beta), \quad \bar{x} = X(z_\alpha), \quad \bar{\epsilon} = \frac{1}{\bar{x}} \quad \text{and} \quad \bar{c} = \frac{1}{\bar{x}}.
\]

(148)

where \( X(y) \) is

\[
X(y) = \frac{1}{r} - \left( \frac{D_1m_1}{(1 - \gamma + \gamma m_1)z_\alpha} \left( \frac{y}{z_\alpha} \right)^{m_1 - 1} + \frac{D_2m_2}{(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{y}{z_\alpha} \right)^{m_2 - 1} \right).
\]

Since

\[
\frac{\partial X}{\partial y}(y) = -\left( \frac{D_1m_1(m_1 - 1)}{(1 - \gamma + \gamma m_1)z_\alpha^2} \left( \frac{y}{z_\alpha} \right)^{m_1 - 2} + \frac{D_2m_2(m_2 - 1)}{(1 - \gamma + \gamma m_2)z_\alpha^2} \left( \frac{y}{z_\alpha} \right)^{m_2 - 2} \right)
\]

and \( D_1 > 0, D_2 < 0, 1 - \gamma + \gamma m_1 > 0, \) and \( 1 - \gamma + \gamma m_2 < 0, \) \( X(y) \) is strictly increasing function of \( y \).

This means that the consumption stays for \( s \geq t \) constant if and only if

\[
\bar{\epsilon} < \bar{x} c_{t-1} \quad \text{or} \quad \bar{\epsilon} c_s < \bar{x} \quad \text{or} \quad \frac{c_s}{X_s} < \bar{c}.
\]

This completes the proof.
F Proof of Proposition 3.4

By applying the generalized Itô’s lemma (see Harrison (1985)) to the wealth process $X_t$, we have

\[
    dX_t = -\frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dy_{\tau}^* - \frac{1}{2} \frac{\partial^3 J}{\partial y^3} (y_t^*, c_{-}) (dy_{\tau}^*)^2 - \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-}^* - \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-}^*. 
\]  

(149)

If $(y_t^*, c_{-}) \in \text{NR}$, the agent does not adjust his/her consumption. This means that $dc_{t-}^* = dc_{t-} = 0$ and thus

\[
    \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-}^* = \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-} = 0.
\]

If $(y_t^*, c_{-}) \in \text{IR}$, the agent should increase his/her consumption. This implies that

\[
    \frac{\partial J}{\partial c} (y_t^*, c_{-}) = \alpha u' (c_{-}), \quad dc_{t-} = 0,
\]

and hence

\[
    \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-}^* = \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-} = 0.
\]

Similarly, we also obtain

\[
    \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-}^* = \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}) dc_{t-} = 0.
\]

when $(y_t^*, c_{-}) \in \text{NR}$.

Therefore, by comparing the equation (149) with the wealth dynamics (10) and using the fact that $dy_t^* = (\delta - r) y_t^* dt - \theta y_t^* dB_t$, we deduce the optimal portfolio policy $\pi_t^*$ as follows:

\[
    \pi_t^* = \frac{\theta}{\bar{y}_t^*} \frac{\partial^2 J}{\partial y \partial c} (y_t^*, c_{-}),
\]

(150)

and

\[
    \pi_t^* = \frac{\theta}{\bar{c}_{t-}} \left( \frac{D_1 m_1 (m_1 - 1)}{1 - \gamma + \gamma m_1} \frac{y_t^*}{(c_{t-})^{-\gamma}} z_{a}^{m_1 - 1} + \frac{D_2 m_2 (m_2 - 1)}{1 - \gamma + \gamma m_2} \frac{y_t^*}{(c_{t-})^{-\gamma}} z_{a}^{m_2 - 1} \right).
\]

(151)

G Proof of Theorem 3.3

(Proof of (a)).

Without loss of generality, it suffice to assume that

\[
    \xi \leq \frac{c_{t-}}{X_t} \leq \bar{c}_t.
\]

(152)

Theorem 3.2 implies that

\[
    z_{a} \leq z_t^* \leq z_\beta \quad \text{with} \quad z_t^* = y_t^* / (c_{t-})^{-\gamma}.
\]

(153)

By Theorem 3.2 and Proposition 3.4, we deduce that

\[
    \frac{X_t}{c_{t-}} - \frac{\gamma \sigma \pi_t^*}{\bar{y}} c_{t-} = -H'(z_t^*),
\]

(154)

where $H(\cdot)$ is defined in Proposition B.1.

Then,

\[
    G(z_t^*) = \frac{\gamma \sigma \pi_t^*}{\bar{y}} X_t = 1 + \frac{c_{t-}}{X_t} H'(z_t^*) = 1 + \frac{H'(z_t^*)}{X(z_t^*)},
\]

(155)
where $X(\cdot)$ is defined in Appendix E.

By Proposition B.1, we know that $H'(z) \leq 0$ in $[z_\alpha, z_\beta]$ and this lead to $G(z_\alpha) \leq 1$. Hence, RCRRA is always greater than or equal to $\gamma$ for every value of the consumption-wealth ratio in $[\xi, \bar{c}]$.

(Proof of (b)).

Since $H'(z_\alpha) = H'(z_\beta) = 0$, 

$$G(z_\alpha) = G(z_\beta) = 1.$$ 

This means that 

$$\text{RCRRA} = \gamma \text{ if } c_{t-}/X_t = \xi \text{ or } c_{t-}/X_t = \bar{c}. $$

(Proof of (c)).

We will show that there exists a unique $\hat{z} \in (z_\alpha, z_\beta)$ such that $G(\cdot)$ is a strictly decreasing function on $(z_\alpha, \hat{z})$ and strictly decreasing function on $(\hat{z}, z_\beta)$.

$$G'(z) = \frac{H''(z)X(z) - H'(y)X'(z)}{(X(z))^2}. \quad (156)$$

(For convenience of notation we drop the time subscript $t$ and the optimal subscript $*$.)

Let us define the numerator of $G'(z)$ as $\tilde{G}(y)$, i.e.,

$$\tilde{G}(y) = H''(z)X(z) - H'(z)X'(z). \quad (157)$$

By the proof in Proposition B.1, we know that 

$$H''(z_\alpha) < 0, \quad H''(z_\beta) > 0.$$ 

Thus, 

$$G(z_\alpha) < 0 \quad \text{and} \quad G(z_\beta) > 0.$$ 

Since

$$X(y) = \frac{1}{r} - \frac{D_1m_1}{(1 - \gamma + \gamma m_1)z_{\alpha}} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2}{(1 - \gamma + \gamma m_2)z_{\alpha}} \left( \frac{z}{z_{\alpha}} \right)^{m_2 - 1},$$

$$H'(z) = \frac{D_1m_1}{z_{\alpha}} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2}{z_{\alpha}} \left( \frac{z}{z_{\alpha}} \right)^{m_2 - 1} - \frac{1}{r},$$

we have

$$X'(z) = -\frac{D_1m_1(m_1 - 1)}{(1 - \gamma + \gamma m_1)z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 2} + \frac{D_2m_2(m_2 - 1)}{(1 - \gamma + \gamma m_2)z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_2 - 2},$$

$$H'(z) = \frac{D_1m_1(m_1 - 1)}{z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 2} + \frac{D_2m_2(m_2 - 1)}{z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_2 - 2}. $$

Hence,

$$\tilde{G}(z) = \gamma \frac{D_1m_1(m_1 - 1)^2}{(1 - \gamma + \gamma m_1)z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 2} + \frac{D_2m_2(m_2 - 1)^2}{(1 - \gamma + \gamma m_2)z_{\alpha}^2} \left( \frac{z}{z_{\alpha}} \right)^{m_2 - 2} - \frac{\gamma D_1D_2m_1m_2(m_1 - m_2)^2}{(1 - \gamma + \gamma m_1)(1 - \gamma + \gamma m_2)z_{\alpha}^3} \left( \frac{z}{z_{\alpha}} \right)^{m_1 + m_2 - 3} \left( \frac{z}{z_{\alpha}} \right)^{m_1 + m_2 - 3}$$

$$= \frac{\gamma z^{m_2 - 2}}{z_{\alpha}^2} \left( \frac{D_1m_1(m_1 - 1)^2}{r(1 - \gamma + \gamma m_1)} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 2} - \frac{D_1D_2m_1m_2(m_1 - m_2)^2}{(1 - \gamma + \gamma m_1)(1 - \gamma + \gamma m_2)z_{\alpha}} \left( \frac{z}{z_{\alpha}} \right)^{m_1 - 1} + \frac{D_2m_2(m_2 - 1)^2}{r(1 - \gamma + \gamma m_2)} \right) \triangleq \frac{\gamma z^{m_2 - 2}}{z_{\alpha}^2} G(z).$$

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We know that
\[ m_1 > 1, \ m_2 < 0, \ D_1 > 0, \ D_2 < 0, \ 1 - \gamma + \gamma m_1 > 0 \ \text{and} \ 1 - \gamma + \gamma m_2 < 0, \]
thus, \( G(y) \) is a strictly increasing function of \( y \).

Since \( G(z_\alpha) < 0, \ G(z_\beta) > 0 \), we deduce that
\[ G(z_\alpha) < 0, \ G(z_\beta) > 0. \]
Thus, there exists a unique \( \hat{z} \in (z_\alpha, z_\beta) \) such that
\[ G(\hat{z}) = 0. \]
This implies that
\[ G'(z) < 0, \ \text{for} \ z \in (z_\alpha, \hat{z}) \ \text{and} \ G'(z) > 0, \ \text{for} \ y \in (\hat{z}, z_\beta). \]
To sum up, we conclude that RCRRA is a strictly increasing for \( c \) and a strictly decreasing for \( \hat{c} \). Moreover, RCRRA approaches \( \gamma \) when \( c/X \) approaches \( c \) or \( \hat{c} \). Here, \( \hat{c} = 1/A(\hat{z}) \) and \( \hat{z} \in (z_\alpha, z_\beta) \) is a unique solution of the following algebraic equation:
\[ G(z) = \frac{D_1 m_1(m_1 - 1)^2}{r(1 - \gamma + \gamma m_1)} \left( \frac{z}{z_\alpha} \right)^{m_1 - 2} - \frac{D_1 D_2 m_1(m_1 - m_2)^2}{(1 - \gamma + \gamma m_1)(1 - \gamma + \gamma m_2)z_\alpha} \left( \frac{z}{z_\alpha} \right)^{m_1 - 1} + \frac{D_2 m_2(m_2 - 1)^2}{r(1 - \gamma + \gamma m_2)}. \] (158)

H The proof of Theorem 4.1

The proof is divided into the following two steps.

(Step 1) Let us use notations \( \bar{z}(L) \) and \( \underline{z}(L) \) for two boundaries \( \bar{z} \) and \( \underline{z} \) defined in Theorem 3.2 for given loss aversion \( L \). Then, we show that the following statements are true: If \( L_1 = L_2 \),
\[ \bar{z}(L_1) = \bar{z}(L_2) \ \text{and} \ \underline{z}(L_1) = \underline{z}(L_2). \]
Proof: Recall that for any given \( \alpha \) and \( \beta \),
\[ z_\alpha = (1 - \delta \alpha) m_1 - 1 L w^{m_1 - 1} - 1 \]
and
\[ z_\beta = (1 + \delta \beta) m_1 - 1 w^{m_1 - 1} - 1, \]
which depend on \( L \) and \( w \). From (175) and (176), we have
\[ \bar{z} = \frac{1}{r} \left( 1 - \frac{m_1}{1 - \gamma + \gamma m_1} \right) m_2 - m_1 - \frac{m_2}{1 - \gamma + \gamma m_2} m_1 - 1 \]
\[ + \frac{\gamma}{\delta} \left( 1 - \gamma + \gamma m_1 \right) \left( 1 - \gamma + \gamma m_2 \right) z_\alpha \] (159)
and
\[ \underline{z} = \frac{1}{r} \left( 1 - \frac{m_1}{1 - \gamma + \gamma m_1} \right) m_2 - m_1 - \frac{m_2}{1 - \gamma + \gamma m_2} m_1 - 1 \]
\[ + \frac{\gamma}{\delta} \left( 1 - \gamma + \gamma m_1 \right) \left( 1 - \gamma + \gamma m_2 \right) z_\alpha \] (160)
Let \( w = w(L) \) be the unique solution of the algebraic equation in (94) for given \( L \), i.e.,
\[
0 = (m_1 - 1)m_2(1 - w^{1-m_2})(Lw^{m_1} - 1) - m_1(m_2 - 1)(w^{m_1} - w)(L - w^{-m_2}).
\] (161)

If \( L_1 = L_2 \), then \( w(L_1) = w(L_2) \) for any pair of \( \alpha \) and \( \beta \). This implies that if \( L_1 = L_2 \), then \( z_\alpha \) and \( z_\beta \) are the same for any any pair of \( \alpha \) and \( \beta \). Note that (159) and (160) are determined by \( L, w, z_\alpha \) and \( z_\beta \). Therefore,
\[
L_1 = L_2 \implies \bar{x}(L_1) = \bar{x}(L_2) \quad \text{and} \quad \bar{z}(L_1) = \bar{z}(L_2).
\]

(Step 2) The RCRRA is the same for every pair of \( (\alpha, \beta) \) if loss aversion \( L \) is the same.

Proof: For given wealth-consumption ratio \( \frac{X_t}{c_t} \) at time \( t \), there exist a unique \( z^* \in (z_\alpha, z_\beta) \) such that
\[
\frac{X_t}{c_t} = X(z^*)
\]
\[
= \frac{1}{r} \left( \frac{D_1 m_1}{1 - \gamma + \gamma m_1} \left( \frac{z^*}{z_\alpha} \right)^{m_1 - 1} + \frac{D_2 m_2}{1 - \gamma + \gamma m_2} \left( \frac{z^*}{z_\alpha} \right)^{m_2 - 1} \right).
\] (162)

From the definitions of \( D_1, D_2 \) and \( z_\alpha \), we have
\[
D_1 = \frac{m_2}{m_2 - m_1} \left( \frac{1 - \delta \alpha}{1 - \gamma + \gamma m_1} \right) \frac{m_2 - 1}{m_2 - m_1} = - \frac{m_2}{m_2 - m_1} \frac{m_1}{m_1 - 1} \frac{w^{m_1 - 1} - 1}{Lw^{m_1} - 1} + \frac{m_2 - 1}{1} \frac{1}{r}
\]
\[
D_2 = \frac{m_1}{m_1 - m_2} \left( \frac{1 - \delta \beta}{1 - \gamma + \gamma m_2} \right) \frac{m_1 - 1}{m_1 - m_2} = - \frac{m_1}{m_1 - m_2} \frac{m_1}{m_1 - 1} \frac{w^{m_2 - 1} - 1}{Lw^{m_2} - 1} + \frac{m_1 - 1}{1} \frac{1}{r}.
\] (163)

Given the fixed market parameter values, suppose the two agents have the same time preference \( (\delta, \gamma, L) \) but with possibly different values of \( (\alpha, \beta) \). For any pair of \( (\alpha, \beta) \), \( w \) is determined by \( (\delta, \gamma, L) \). Thus, \( \frac{D_1}{z_\alpha} \) and \( \frac{D_2}{z_\alpha} \) are also determined by \( (\delta, \gamma, L) \). That is, the coefficients of \( \frac{1}{z_\alpha} \) of the algebraic equation (162) depend on \( (\delta, \gamma, L) \), not \( (\alpha, \beta) \). This implies that a unique solution \( \frac{X_t}{c_t} \) of the algebraic equation (162) is determined by \( (\delta, \gamma, L) \), independent of \( (\alpha, \beta) \).

Moreover, the RCRRA(t) at time \( t \) is given by
\[
\text{RCRRA}(t) = \frac{1}{r} - \frac{m_1}{(1 - \gamma + \gamma m_1) z_\alpha} \left( \frac{z^*}{z_\alpha} \right)^{m_1 - 1} + \frac{m_2}{(1 - \gamma + \gamma m_2) z_\alpha} \left( \frac{z^*}{z_\alpha} \right)^{m_2 - 1}.
\] (164)

From this fact and the expression of the RCRRA in (164), we conclude that the RCRRA is the same for every pair of \( (\alpha, \beta) \) if loss aversion \( L \) is the same.

In sum, (Step 1) and (Step 2) complete the proof.

I The proof of Proposition 4.1

We will prove the theorem in the following two steps.

(Step 1) Let \( \bar{x}(\alpha, \beta) \) and \( \bar{z}(\alpha, \beta) \) be the two boundaries \( \bar{x} \) and \( \bar{z} \) defined in Theorem 3.2 for given \( (\alpha, \beta) \). Then, the following statements are true.
(a) If \( \alpha_1 > \alpha_2 \) and \( \beta_1 = \beta_2 \),
\[
\bar{x}(\alpha_1, \beta_1) \geq \bar{x}(\alpha_2, \beta_2) \quad \text{and} \quad \bar{z}(\alpha_1, \beta_1) \leq \bar{z}(\alpha_2, \beta_2).
\]
(b) If \( \beta_1 > \beta_2 \) and \( \alpha_1 = \alpha_2 \),
\[
\bar{x}(\alpha_1, \beta_1) \geq \bar{x}(\alpha_2, \beta_2) \quad \text{and} \quad \bar{z}(\alpha_1, \beta_1) \leq \bar{z}(\alpha_2, \beta_2).
\]

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We can consider the penalty problems corresponding to the double obstacle problem in (167) as follows:

$$\phi_\epsilon(\xi) < 0 \text{ for } \xi < \epsilon,$$

$$\psi_\epsilon(\xi) > 0 \text{ for } \xi > -\epsilon,$$

$$\phi_\epsilon(0) = -C_1 \quad (C_1 > 0),$$

$$\psi_\epsilon(0) = C_2 \quad (C_2 > 0),$$

and

$$\lim_{\epsilon \to 0} \phi_\epsilon(v) = \begin{cases} 0, & v \geq 0, \\ -\infty, & v < 0 \end{cases} \quad \text{and} \quad \lim_{\epsilon \to 0} \psi_\epsilon(v) = \begin{cases} 0, & v \leq 0, \\ \infty, & v > 0. \end{cases}$$

Under the transformation

$$v = \ln z \quad \text{and} \quad h(v) = H(e^v),$$

the double obstacle problem in (92) becomes

$$\begin{cases} -L_0 h(v) \leq 1 - e^v, & \text{for } h(v) = \alpha, \\ -L_0 h(z) \geq 1 - e^v, & \text{for } h(v) = -\beta, \\ -L_0 h(z) = 1 - e^v, & \text{for } -\beta < h(z) < \alpha, \end{cases}$$

where

$$L_0 = \frac{\theta^2}{2} \frac{\partial^2}{\partial v^2} + (\delta - r - \frac{\theta^2}{2}) \frac{\partial}{\partial v} - \delta.$$ 

We consider the following penalty problem of the double obstacle problem (167):

$$-L_0 h_\epsilon(v) + \phi_\epsilon(h_\epsilon(v) + \beta_\epsilon) + \psi_\epsilon(h_\epsilon(v) - \alpha_\epsilon) = 1 - e^v,$$

in $-\infty < v < \infty$.

By applying Friedman’s argument (see Friedman (1982)), as $\epsilon \to 0$, the solution $h_\epsilon(v)$ converges to $h(v)$ which is the solution of the double obstacle problem (167).

For notational convenience, let us denote tat

$$h_1(v) = H(e^v; \alpha_1, \beta_1) \quad \text{and} \quad h_2(v) = H(e^v; \alpha_2, \beta_2).$$

We can consider the penalty problems corresponding to $h_1(v)$ and $h_2(v)$ as follows:

$$-L_0 h_1(v) + \phi_\epsilon(h_1(v) + \beta_\epsilon) + \psi_\epsilon(h_1(v) - \alpha_\epsilon) = 1 - e^v,$$

$$-L_0 h_2(v) + \phi_\epsilon(h_2(v) + \beta_\epsilon) + \psi_\epsilon(h_2(v) - \alpha_\epsilon) = 1 - e^v.$$ 

(169)

Since $\beta_\epsilon = \beta_2$, $\beta_\epsilon = \beta_2$, $\beta_\epsilon = \beta_2$.

$$0 = -L_0(h_1(v) - h_2(v)) + \phi_\epsilon(h_1(v) + \beta_\epsilon) - \phi_\epsilon(h_2(v) + \beta_\epsilon) + \psi_\epsilon(h_1(v) - \alpha_\epsilon) - \psi_\epsilon(h_2(v) - \alpha_\epsilon)$$

$$ = -L_0(h_1(v) - h_2(v)) + \phi_\epsilon'(\theta_1)(h_1(v) - h_2(v)) + \psi_\epsilon'(\theta_2)(h_1(v) - h_2(v)) - (\alpha_1 - \alpha_2)$$

$$ \leq -L_0(h_1(v) - h_2(v)) + \phi_\epsilon'(\theta_1)(h_1(v) - h_2(v)) + \phi_\epsilon'(\theta_2)(h_1(v) - h_2(v))$$

$$= -\left[\frac{\theta^2}{2} \frac{\partial^2}{\partial v^2} + (\delta - r - \frac{\theta^2}{2}) \frac{\partial}{\partial v} - (\delta + \phi_\epsilon'(\theta_1) + \psi_\epsilon'(\theta_2))\right](h_1(v) - h_2(v)),$$

where

$$\theta_1 = \eta_1(h_1(v) - h_2(v)) + (1 - \eta_1)(h_2(v) + \beta_\epsilon)$$

$$\theta_2 = \eta_2(h_1(v) - h_2(v)) + (1 - \eta_2)(h_2(v) + \beta_\epsilon),$$

$$0 < \eta_1, \eta_2 < 1.$$ 

Since $\delta + \phi_\epsilon'(\theta_1) + \psi_\epsilon'(\theta_2) > 0$, the maximum principle (see Lieberman (1996)) implies that $h_2(v) \leq h_1(v)$. Letting $\epsilon \to 0$, we deduce that $h_2(v) \leq h_1(v)$ and thus $H(z; \alpha_2, \beta_2) \leq H(z; \alpha_1, \beta_1)$.
By applying the above argument to \( h_1(v) - \alpha_1 \) and \( h_2(v) - \alpha_2 \), it is easily confirmed that
\[
h_1(v) - \alpha_1 \leq h_2(v) - \alpha_2 \quad \text{and} \quad H(z; \alpha_1, \beta_1) - H(z; \alpha_2, \beta_2) \leq \alpha_1 - \alpha_2.
\]
Thus, we deduce that for \( \alpha_1 > \alpha_2 \) and \( \beta_1 = \beta_2 \),
\[
0 \leq H(z; \alpha_1, \beta_1) - H(z; \alpha_2, \beta_2) \leq \alpha_1 - \alpha_2.
\]
(171)

Let us denote that \((z_{\alpha_i}, z_{\beta_i}) (i = 1, 2)\) are the free boundaries of \( H(z; \alpha, \beta) \) corresponding to \((\alpha_i, \beta_i)\) defined in Proposition B.1.

From the definition of free boundary \( z_{\alpha_1} \),
\[
0 = H_2(z_{\alpha_2}) - \alpha_2 \geq H_1(z_{\alpha_2}) - \alpha_1.
\]
Since \( H_1(z) \leq \alpha_1 \) for all \( z \in (0, \infty) \), we deduce that
\[
H_1(z_{\alpha_2}) = \alpha_2
\]
and thus \( z_{\alpha_2} \geq z_{\alpha_1} \). Similarly,
\[
0 = H_1(z_{\beta_1}) + \beta_1 = H_1(z_{\beta_1}) + \beta_2 \geq H_2(z_{\beta_1}) + \beta_2.
\]
The fact that \( H_2(z) \geq -\beta_2 \) for all \( z \in (0, \infty) \) implies that
\[
H_2(z_{\beta_1}) + \beta_2 = 0.
\]
Hence, we obtain \( z_{\beta_1} \geq z_{\beta_2} \).

Sum up, the following relationships can be established.
\[
\alpha_1 > \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2 \implies z_{\alpha_1} \leq z_{\alpha_2} \quad \text{and} \quad z_{\beta_1} \geq z_{\beta_2}.
\]
(172)

Recall the two boundaries \( \bar{x} \) and \( \underline{x} \) defined in (148):
\[
\bar{x} = \frac{1}{r} \left( \frac{D_1 m_1}{1 - \gamma + \gamma m_1} + \frac{D_2 m_2}{1 - \gamma + \gamma m_2} \right) \]
\[
\underline{x} = \frac{1}{r} \left( \frac{D_1 m_1}{1 - \gamma + \gamma m_1} \left( \frac{\alpha}{z_\alpha} \right)^{m_1 - 1} + \frac{D_2 m_2}{1 - \gamma + \gamma m_2} \left( \frac{\beta}{z_\beta} \right)^{m_2 - 1} \right).
\]
(173)

Let us denote that \((\bar{x}_i, \underline{x}_i) (i = 1, 2)\) are the two boundaries \((\bar{x}, \underline{x})\) in (173) corresponding to \((\alpha_i, \beta_i)\). Since
\[
D_1 = \frac{(\alpha - \frac{1}{2}) m_2 + (m_2 - 1) \frac{\beta}{z_\beta}}{m_2 - m_1}, \quad D_2 = \frac{(\alpha - \frac{1}{2}) m_1 + (m_1 - 1) \frac{\alpha}{z_\alpha}}{m_1 - m_2},
\]
we deduce that
\[
\bar{x} = \frac{1}{r} \left( \frac{D_1 m_1}{1 - \gamma + \gamma m_1} \right) + \frac{D_2 m_2}{1 - \gamma + \gamma m_2} \]
\[
= \frac{1}{r} \left( \frac{m_1}{1 - \gamma + \gamma m_1} \frac{(\alpha - \frac{1}{2}) m_2 + (m_2 - 1) \frac{\beta}{z_\beta}}{m_2 - m_1} + \frac{m_2}{1 - \gamma + \gamma m_2} \frac{(\alpha - \frac{1}{2}) m_1 + (m_1 - 1) \frac{\alpha}{z_\alpha}}{m_1 - m_2} \right)
\]
\[
= \frac{1}{r} \left( \frac{m_1}{1 - \gamma + \gamma m_1} \frac{m_2 - 1}{m_2 - m_1} + \frac{m_2}{1 - \gamma + \gamma m_2} \right) \frac{1}{z_\alpha}.
\]
(175)

Hence, \( \bar{x} \) increases as \( z_\alpha \) decreases. Since \( \alpha_1 > \alpha_2 \), the argument (172) implies \( z_{\alpha_1} \leq z_{\alpha_2} \) and thus we obtain that \( \bar{x}_1 \geq \bar{x}_2 \).

From (100),
\[
D_1 = -\frac{(\beta + \frac{1}{2}) m_2 + (m_2 - 1) \frac{\alpha}{z_\alpha}}{m_2 - m_1} \left( \frac{\alpha}{z_\alpha} \right)^{m_1}, \quad D_2 = -\frac{(\beta + \frac{1}{2}) m_1 + (m_1 - 1) \frac{\beta}{z_\beta}}{m_1 - m_2} \left( \frac{\beta}{z_\beta} \right)^{m_2}.
\]
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Hence,

\[ x = \frac{1}{r} \left( \frac{D_1 m_1}{(1 - \gamma + \gamma m_1) \gamma} \left( \frac{z_{\beta}}{z_a} \right)^{m_1 - 1} + \frac{D_2 m_2}{(1 - \gamma + \gamma m_2) \gamma} \left( \frac{z_{\beta}}{z_a} \right)^{m_2 - 1} \right) \]

\[ = \frac{1}{r} \left( \frac{m_1}{1 - \gamma + \gamma m_1} - \frac{(\beta + \frac{1}{2} \frac{m_2}{m_2} + m_2 - 1) \frac{\gamma}{r}}{m_2 - m_1} - \frac{m_2}{1 - \gamma + \gamma m_2} \frac{(\beta + \frac{1}{2} \frac{m_1}{m_1} + m_1 - 1) \frac{\gamma}{r}}{m_1 - m_2} \right) \]

\[ + \frac{\gamma}{r} \left( \frac{1 - \gamma + \gamma m_1}{m_1 m_2} \left( \frac{1 + \delta \beta}{z_a} \right) \right) > 0 \quad (176) \]

Hence, \( x \) increases as \( z_{\beta} \) decreases. Since \( \alpha_1 > \alpha_2 \), (172) implies \( z_{\beta_1} \geq z_{\beta_2} \) and thus we obtain that \( x_1 \leq x_2 \).

This completes the proof of the part (a) in the (Step 1). The proof of part (b) is almost similar to that of part (a).

(Step 2) \( x(L) \) decreases with \( L \) and \( \bar{x}(L) \) increases with \( L \).

Proof: For given \( L_1 > L_2 > 1 \), there exist two pairs \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) such that

\[ L_1 = \frac{1 + \delta \beta_1}{1 - \delta \alpha_1} \quad \text{and} \quad L_2 = \frac{1 + \delta \beta_2}{1 - \delta \alpha_2} \]

Let us temporarily denote that

\[ \hat{\beta}_1 = \frac{L_1 - 1}{\delta} \quad \text{and} \quad \hat{\beta}_2 = \frac{L_2 - 1}{\delta} \]

Clearly, \( \hat{\beta}_1 > \hat{\beta}_2 \).

Theorem 4.1 implies that

\[ \bar{x}(L_1) = \bar{x}(\alpha_1, \beta_1) = \bar{x}(0, \hat{\beta}_1) \quad \text{and} \quad \bar{x}(L_2) = \bar{x}(\alpha_2, \beta_2) = \bar{x}(0, \hat{\beta}_2) \]

By (Step 1), we deduce that

\[ \bar{x}(0, \hat{\beta}_1) \geq \bar{x}(0, \hat{\beta}_2) \quad \text{and} \quad \bar{x}(0, \hat{\beta}_1) \leq \bar{x}(0, \hat{\beta}_2) \]

and thus

\[ \bar{x}(L_1) \geq \bar{x}(L_2) \quad \text{and} \quad \bar{x}(L_1) \leq \bar{x}(L_2) \]

From (Step 1) and (Step 2), we have proved the desired result.

J Proof of Proposition 4.2

We will prove the proposition in the following steps.

(Step 1) The RCRRA increases as \( L \) increases.

Proof: For given \( L_1 > L_2 \), there exits two pairs \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\) such that

\[ L_1 = \frac{1 + \delta \beta_1}{1 - \delta \alpha_1} \quad \text{and} \quad L_2 = \frac{1 + \delta \beta_2}{1 - \delta \alpha_2} \]

By Theorem 4.1, we can assume that

\[ \alpha_1 = \alpha_2 = 0 \quad \text{and} \quad \beta_1 > \beta_2. \]

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That is, it suffice to consider only the case where $\alpha$ is zero.

By using the almost similar argument in Appendix I, it is easily confirmed that

$$z_{\alpha_1} \leq z_{\alpha_2} \text{ and } z_{\beta_2} \leq z_{\beta_1},$$

where $(z_{\alpha_i}, z_{\beta_i})$ $(i = 1, 2)$ are the free boundaries of $H(z; \alpha, \beta)$ corresponding to $(\alpha_i, \beta_i)$ defined in Proposition B.1. This fact means that $z_{\alpha}$ decrease as $L$ increases even if $\alpha$ is zero.

Since

$$\frac{D_1}{z_{\alpha_1}} = -\frac{m_2}{m_2 - m_1} \frac{(1 - \delta \alpha)1}{z_{\alpha_1} m_1} \frac{1}{\delta} + \frac{m_2 - 1}{m_2 - m_1} \frac{1}{r z_{\alpha_1} m_1 - 1},$$

$$= -\frac{m_2}{m_2 - m_1} \frac{1}{z_{\alpha_1}} \frac{1}{1} + \frac{m_2 - 1}{m_2 - m_1} \frac{1}{r z_{\alpha_1} m_1 - 1},$$

and

$$\frac{D_2}{z_{\alpha_2}} = -\frac{m_1}{m_1 - m_2} \frac{(1 - \delta \alpha)1}{z_{\alpha_2} m_2} \frac{1}{\delta} + \frac{m_1 - 1}{m_1 - m_2} \frac{1}{r z_{\alpha_2} m_2 - 1},$$

$$= -\frac{m_1}{m_1 - m_2} \frac{1}{z_{\alpha_2}} \frac{1}{1} + \frac{m_1 - 1}{m_1 - m_2} \frac{1}{r z_{\alpha_2} m_2 - 1},$$

we have

$$\frac{\partial}{\partial z_{\alpha}} \left( \frac{D_1}{z_{\alpha_1}} \right) = \frac{m_1 m_2}{m_2 - m_1} \frac{z_{\alpha_1} - m_{1-1}}{\delta} (1 - z_{\alpha}) > 0,$$

$$\frac{\partial}{\partial z_{\alpha}} \left( \frac{D_2}{z_{\alpha_2}} \right) = \frac{m_1 m_2}{m_1 - m_2} \frac{z_{\alpha_2} - m_{2-1}}{\delta} (1 - z_{\alpha}) < 0.$$  

Since $z_{\alpha}$ decreases as $L$ increases, $\frac{D_1}{z_{\alpha_1}} (> 0)$ decreases and $\frac{D_2}{z_{\alpha_2}} (< 0)$ increases as $L$ increases. Thus, $\frac{m_1 D_1}{z_{\alpha_1}} (> 0)$ and $\frac{m_2 D_2}{z_{\alpha_2}} (< 0)$ decrease as $L$ increases.

On the other hand, we know that for given $x$ and $c_{0-}$ there exists a unique $z^* \in (z_{\alpha}, z_{\beta})$ such that

$$\frac{x}{c_{0-}} = \chi(z^*)$$

$$= \frac{1}{r} - \left( \frac{D_1 m_1}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_{1-1}} + \frac{D_2 m_2}{(1 - \gamma + \gamma m_2) z_{\alpha_2} m_2} (z^*)^{m_{2-1}} \right).$$

Moreover,

$$\frac{\sigma}{\vartheta} \frac{\pi^*}{c_{0-}} = \frac{m_2(m_1 - 1) D_1}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_{1-1}} + \frac{m_2(m_2 - 1) D_2}{(1 - \gamma + \gamma m_2) z_{\alpha_2} m_2} (z^*)^{m_{2-1}}.$$  

We consider the following two cases: (i) $z^*$ decreases as $L$ increases, (ii) $z^*$ increases as $L$ increases.

(i) $z^*$ decreases as $L$ increases.

In (181),

$$\frac{\sigma}{\vartheta} \frac{\pi^*}{c_{0-}} = \frac{m_1(m_1 - 1) D_1}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_{1-1}} + \frac{m_2(m_2 - 1) D_2}{(1 - \gamma + \gamma m_2) z_{\alpha_2} m_2} (z^*)^{m_{2-1}}$$

$$= (m_2 - 1) \left( \frac{1}{r} - \frac{x}{c_{0-}} \right) + \frac{(m_1 - m_2) m_1 D_1}{(1 - \gamma + \gamma m_1) z_{\alpha_1} m_1} (z^*)^{m_{1-1}}.$$  

The (term A) in (182) is independent of $L$. Since $\frac{m_1 D_1}{z_{\alpha_1}} (> 0)$ and $z^*$ decrease as $L$ increases, the (term B) in (182) decreases as $L$ increases. This leads to $\frac{\sigma}{\vartheta} \frac{\pi^*}{c_{0-}}$ decreases as $L$ increases.

(ii) $z^*$ increases as $L$ increases.
In this case, \((z^*)^{m_2-1}\) decreases as \(L\) increases. Moreover,

\[
\frac{\sigma \pi^*}{\theta c_{0-}} = \frac{m_1(m_1-1)D_1}{(1 - \gamma + \gamma m_1)z_\alpha z_{\alpha} m_1} \left( z^* \right)^{m_1-1} + \frac{m_2(m_2-1)D_2}{(1 - \gamma + \gamma m_2)z_\alpha z_{\alpha} m_2} \left( z^* \right)^{m_2-1}
\]

\[
= (m_1-1) \left( \frac{1}{r} - \frac{x}{c_{0-}} \right) + \frac{(m_2-m_1)m_2D_2}{(1 - \gamma + \gamma m_2)z_\alpha z_{\alpha} m_2} \left( z^* \right)^{m_2-1}.
\]

(183)

This leads to

\[
(\text{term C}) \quad (\text{term D})
\]

Since \(\frac{m_2D_2}{z_\alpha} (> 0)\) decreases as \(L\) increases and \((m_2-m_1)/(1 - \gamma + \gamma m_2) > 0\), we deduce that the (term D) in (183) decreases as \(L\) increases. The (term C) in (183) is also independent of \(L\) and thus \(\frac{\sigma \pi^*}{\theta c_{0-}}\) decreases as \(L\) increases.

From two cases (i) and (ii), we conclude that the RCRRA increases as \(L\) increases.

**Proof:** Since \(w \in (0, \frac{1}{r})\) is a unique solution of the algebraic equation (161), we deduce that as \(L\) goes to infinity, \(w\) goes to zero. Thus,

\[
z_a^\infty := \lim_{L \to \infty} z_a = \lim_{w \to 0} (1 - \delta \alpha) \frac{m_1 - 1}{m_1} = \frac{1 - \delta \alpha}{m_1},
\]

\[
z_\beta^\infty := \lim_{L \to \infty} z_\beta = \lim_{w \to 0} (1 + \delta \beta) \frac{m_1 - 1}{m_1} = \infty.
\]

This leads to

\[
D_1^\infty := \lim_{L \to \infty} D_1 = 0 \quad \text{and} \quad D_2^\infty := \lim_{L \to \infty} D_2 = \frac{(1 - \delta \alpha)}{\delta} \frac{1}{(m_2 - 1)}.
\]

(185)

From the definition of two boundaries \(\bar{x}\) and \(\bar{x}\) in (148):

\[
\bar{x}^\infty = \lim_{L \to \infty} \bar{x} = \frac{1}{r} \frac{\gamma(m_2 - 1)}{1 - \gamma + \gamma m_2} \quad \text{and} \quad \bar{x}^\infty = \lim_{L \to \infty} \bar{x} = \frac{1}{r}.
\]

(186)

When the wealth consumption ratio goes to wealth boundary \(\bar{x}\), the portfolio consumption ratio is given by

\[
\lim_{L \to \infty} \frac{\theta}{\sigma} \left( \frac{D_1m_1(m_1-1)}{(1 - \gamma + \gamma m_1)z_\alpha} \left( z_\beta \right)^{m_1-1} + \frac{D_2m_2(m_2-1)}{(1 - \gamma + \gamma m_2)z_\alpha} \left( z_\beta \right)^{m_2-1} \right)
\]

\[
= \frac{\theta}{\sigma} \lim_{z_\beta \to \infty} \frac{D_2^\infty m_2(m_2-1)}{(1 - \gamma + \gamma m_2)z_\alpha} \left( z_\beta \right)^{m_2-1} = 0.
\]

(187)

This implies that as the wealth consumption ratio goes to wealth boundary \(\bar{x}\), the RCRRA goes to infinity. This completes the proof.

**K Simulation Method for Section 5**

We explain the details of simulation, whose results are reported in Section 5. **(Step 1)** Simulation of sample paths of consumption and wealth for \(N\) individuals:

- We divide the interval \([0, 79]\) into \(2 \times 12 \times 79\) subintervals with end points \(t_j = 1, \ldots, 2 \times 12 \times 79\).
- Similar to Marshall and Parekh (1999), we set \(c_{0-} = 1\) for each individual and select each individual’s initial wealth \(z_0\) randomly according to a uniform distribution over \((\bar{x}, \bar{x})\).
- Generate a \(2 \times 12 \times 79\) random vector \(\omega\) that follows a standard normal distribution. Using this vector \(\omega\), we generate the process of the risky asset returns \(\Delta S_t/S_t\), and the dual process \(y^*_t\) in Proposition 3.3 for all the \(N\) individuals. We then simulate the optimal consumption processes for \(N\) individuals.
(Step 2) Aggregation of Consumption:

- Let $C^1, C^2, ..., C^N$ be the simulated consumption processes for the $N$ individuals in (Step 1).
- (Cross sectional aggregation) The cross sectionally aggregated consumption process $CA$ of $C^1, C^2, ..., C^N$ is defined as follows:
  
  For $j = 1, 2, ..., 2 \times 12 \times 79$, 
  
  \[ CA(t_j) = \frac{1}{N} \sum_{i=1}^{N} c^i(t_j). \]

- (Temporal aggregation) We temporally aggregate the cross sectionally aggregated consumption process $CA$ to create monthly $CA^*$ (that is, $\Delta t = 1/12$) as follows:
  
  For $j = 1, 2, ..., 12 \times 79$, 
  
  \[ CA^*(i) = \sum_{j=1}^{2} CA(t_{2 \times (i-1)+j}). \]

(Step 3) Computation of the consumption growth rate, IMRS, and theoretical EP:

- We compute and obtain the following time-series by using the simulated series in (Step 2).
  
  $i = 1, 2, ..., 12 \times 79 - 1$,

  (Consumption growth rate) \( CG(i) = \frac{CA^*(i + 1) - CA^*(i)}{CA^*(i)} \).

  (IMRS) \( I(i) = e^{-\delta \Delta t} \left( \frac{CA^*(i + 1)}{CA^*(i)} \right)^{-\gamma}. \)

  (EP) \( EP(i) = -\frac{\text{cov} \left( e^{-\delta \Delta t} (CA^*(i + 1)/CA^*(i))^{-\gamma}, \left( \frac{r(i + 1)}{r(i)} \right) \right)}{\mathbb{E} [e^{-\delta \Delta t} (CA^*(i + 1)/CA^*(i))^{-\gamma}]} \).

Using these time-series, we compute the desired statistics: the mean and the standard derivation of consumption growth, the IMRS, the theoretical EP and the auto-correlation of aggregated consumption $CA^*$.

Since each time-series depends on the random vector $\omega$, repeat (Step 1)–(Step 3) 1000 times.

L Illustration of Optimal Consumption Policy in Discrete Time

Figure 15 describes a discrete-time version of the wealth and consumption movement. Assume that current wealth is $X_0$ and consumption is $c_0$. Initially the wealth process lies in interval $(A, B)$. Suppose $X_t$ hits $c_0 \bar{x}$ (point $B$) after several good shocks. Then, a new consumption level is set as $c_u$ with a new inaction interval $(A_u, B_u)$ is . At this instant, the wealth level is at point $B_u$. If a positive shock arrives again at point $B_u$, a new consumption level is immediately set as $c_{uu}$ with a new band is interval $(A_{uu}, B_{uu})$. If a negative shock arrives at point $B_u$, however, $X_t$ moves inside $(A_u, B_u)$ and the consumption level stays at $c_u$ until $X_t$ hits either $c_u \underline{x}$ (point $A_u$) or $c_u \bar{x}$ (point $B_u$).

Let us now go back to the time when the wealth process lies in interval $(A, B)$. Suppose $X_t$ hits $c_{02} \bar{x}$ (point $A$) after several bad shocks. Then, a new consumption level is set as $c_d$ with a new inaction interval $(A_d, B_d)$. At this instant, the wealth level is at point $A_d$. If a negative shock arrives again at point $A_d$, a new consumption level is immediately set as $c_{dd}$ and a new inaction interval $(A_{dd}, B_{dd})$. If a positive shock arrives at at point $A_d$, however, $X_t$ moves inside $(A_d, B_d)$ and the consumption level stays at $c_u$ until $X_t$ hits either $c_d \underline{x}$ (point $A_d$) or $c_d \bar{x}$ (point $B_d$).
M Simulation and Regression Analysis

We explain how we generate the samples and conduct regression analysis for the results reported in Section 6.

(Step 1) Simulation of Wealth and Portfolio for $N$-individuals:

- We divide the interval $[0, T]$ into $12 \times T$ subintervals with end points $t_j = 1, 2, ..., 12 \times T$. (Here, we assume that $T$ is a positive integer)

- Suppose that the means $m_\alpha, m_\beta$ and variances $v_\alpha, v_\beta$ for the distributions of adjustment costs $\alpha$ and $\beta$ are given. For each $i$, $i = 1, 2, ... N$ we generate log-normally distributed random variables $\alpha_i$ and $\beta_i$, whose the mean and variance are $(m_\alpha, v_\alpha)$ and $(m_\beta, v_\beta)$, respectively.\textsuperscript{27}

- We set $c_0 = 1$ for each individual and generate individual $i$’s initial wealth $x_0$ randomly according to the uniform distribution over $(\underline{X}, \bar{X})$.

- Generate a $12 \times T$ random vector $\omega$ that follows a standard normal distribution. Using this vector $\omega$, we generate the process of the risky asset returns $\Delta S_i/S_i$, and the dual process $y^*_i$ in Proposition 3.3 for all the $N$ individuals. By Proposition 3.3, we can simulate the optimal wealth and portfolio processes of $N$ individuals.

(Step 2) Regression Analysis:

- Let $(X^1, \Pi^1), (X^2, \Pi^2), ..., (X^N, \Pi^N)$ be the simulated wealth/portfolio processes for $N$ individuals obtained in (Step 1). (Note that $X^i$ and $\Pi^i$ are $(12 \times T + 1)$ random vectors for $i = 1, 2, ..., N$).

- For given $T$ and $k$, there are $(T - k + 1)$ numbers of $\Delta k$.

\textsuperscript{27}The log-normal distribution implies that there are households having fairly large values of $\alpha$’s although their density in the population is very small. We drop those household whose $\alpha$ values violates Assumption 1.
For $i = 1, 2, ..., N$ and $j = 1, 2, ..., (T - k + 1)$ let

$$DR(i, j) \equiv \Delta_k \log \frac{\Pi_i^{j+k}}{X_i^{j+k}} = \log \frac{\Pi_i^{(12 \times (j + k - 1) + 1)}}{X_i^{(12 \times (j - 1) + 1)}},$$

$$X(i, j) \equiv \Delta_k \log X_i^{j+k} = \log X_i^{(12 \times (j + k - 1) + 1)} - \log X_i^{(12 \times (j - 1) + 1)}.$$

- We regress Eq. (57) with OLS using the simulated $DR$ and $X$.

## N Proof of The Two-Fund Separation Theorem and The CAPM

We provide the proof of the CAPM. Note that all the previous proofs are valid when we assume the general case with $n$ assets whose cum-dividend price $\tilde{z}_i,t$, $i = 1, \ldots, n$, follows

$$d\tilde{z}_i,t = \tilde{z}_i,t(\tilde{\mu}_i dt + dw_i,t)$$

with $w_i,t$ are driftless Brownian motions with instantaneous covariance matrix $\Sigma$. By the duality relationship in Theorem 3.1 it can be easily shown that the value function satisfies the following Hamilton-Jacobi-Bellman equation

$$\max_{\pi_t} \left[ u(c_t) - \delta V + \frac{1}{2} V_{XX} \pi_t \Sigma \pi_t + (rX_t - c_t + (\tilde{\mu} - r1)'\pi_t)V_X \right] = 0,$$

where $1$ denotes the $n$-vector of $1$'s. The first-order condition implies the optimal portfolio satisfies

$$\pi_t^* = \frac{V_X}{X_{XX}} \Sigma^{-1}(\tilde{\mu} - r1).$$

Here $\frac{V_X}{X_{XX}}$ determines the amounts invested in the risky assets and depends on the agent’s preference. The weight for each risky asset is determined by the term $\Sigma^{-1}(\tilde{\mu} - r1)$, which is common across all the agents in the economy. Thus, the two-fund separation theorem holds. Aggregating investments in the risky assets results in a market portfolio whose weights for risky assets are determined by $\Sigma^{-1}(\tilde{\mu} - r1)$. Then, a standard argument as in Grossman and Laroque (1990) gives the CAPM.

## O Applications to Long-term Asset Management: A Back-Test

Here we consider an application of our model to long-term asset management. Suppose that there exist two funds: (1) a fund (fund 1) which manages according to the prescriptions of the model in this paper, (2) a fund (fund 2) which manages according to prescriptions of the Merton model (i.e., the case $\alpha = 0, \beta = 0$). Let us assume that each fund starts with $X_0 = 50$ and pays dividends continuously at a rate equal to the optimal rate of consumption in each model. We provide results of a back test assuming that the two funds started in 1950 and invested in the S&P500 stocks according to the proportions in the S&P 500 index. We assume that the risk-free rate has been constant and equal to 2%. We estimate the mean and standard deviation on the return on the index by using monthly data from 1950 to 2016 and obtain $\mu = 8.54\%, \sigma = 15.47\%$, which we use for the back test. The other parameter values we use for the back test are as follows: $\delta = 0.04$, $\alpha = 3$, $\beta = 20$.

Figure 16 shows the performance of the two funds. It shows that dividends paid by fund 1 were much more stable than those paid by fund 2. Over the period between 1950 and 2016 fund 1 steadily increased dividends and reduced dividends only once in 2008 in the middle of the recent global financial crisis. The initial level of dividends paid by fund 1 were lower than that of dividends paid by the second fund, but it became generally higher during the recent period.
Figure 16: $\mu = 0.0854, \sigma = 0.1547, \delta = 0.04, r = 0.02, \gamma = 2, \alpha = 3, \beta = 20, X_0 = 50$.

The levels of wealth under management by the two funds were similar in the initial period, but that of fund 1 was higher than that of fund 2 since 1960s.