Rating Under Asymmetric Information\textsuperscript{1}

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Abstract

A firm’s owner signals quality to a rating agency by the decision to inject cash, with the rating feeding back into the firm’s cost of capital. In a dynamic signaling game, the rating agency observes the firm’s true cash flow blurred by a persistent measurement error. Firms observed with higher measurement error choose higher default cut-off strategies, and the rating agency successively eliminates infeasibly high measurement errors via a non-Markovian rating inducing Bayesian directional learning. This provides a novel explanation for rating inflation and delayed default at lower asset values, despite the rating agency aiming for unbiased ratings.

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1 Introduction

A central issue in financial economics is how investors on financial markets can infer private information about firms seeking financing. Inherent in this issue is a feedback loop. The assessment by investors, or by the market at large, affects the firms’ financing conditions and behavior, which in turn requires an adjustment in the original judgment. Moreover, the information is rarely static, but evolves over time and allows market participants to adjust their assessment as information unfolds.

Our key innovation is the characterization and analysis of learning in a dynamic signaling game in presence of a feedback loop. We consider the strategic interaction of a levered firm and a rating agency as a representative of corporate bond market investors. The rating agency aims to estimate the firm’s distance to default as precisely as possible from an imperfectly observed Markovian cash flow process subject to a measurement error. The firm’s manager-owner maximizes the firm value by choosing to inject cash or default on debt. By not defaulting during distress, he signals quality to the rating agency at the cost of injecting further cash into the distressed firm. Based on this signal, the rating agency adjusts the credit rating, which in turn feeds back into the firms capital cost. In a specific example, learning from the firm’s signaling decreases the credit spread by more than 240 basis points.

Our central result is the rating agency’s learning mechanism. We show that the firm injects cash up to a measurement error dependent cut-off threshold at which the firm declares bankruptcy. This allows the rating agency to learn as the firm survives distress: Once the observed cash flow hits a new low, an overestimated firm defaults but an underrated firm appears to have more substance to the rating agency, as its owner keeps it going. The rating agency updates its estimated default threshold via Bayesian updating once the observed cash flow hits a historic minimum, by one-sidedly ruling out the most overrated measurement error. This result is intuitive: The rating agency cannot have grossly overestimated the firm, as it would have defaulted for the observed level of distress. We coin this Bayesian directional learning.

Bayesian directional learning implies a number of economic consequences: First, it explains ex-post rating inflation. The rating agency rules out overestimated measurement errors if the firm is not defaulting in distress. Then its estimated
quality improves over time, and the previously underrated firm becomes overrated. In particular, immediately before bankruptcy, the worst case from the perspective of the rating agency occurs and the firm is maximally overestimated. Ex-post rating inflation occurs despite the rating agency’s aim to rate as precisely as possible. Second, the equilibrium of the rating game is partially separating. The rating agency can only imperfectly infer the measurement error until the firm defaults, when the rating agency finally knows the measurement error perfectly, but then the information becomes irrelevant. Thirdly, although the observed cash flow process is Markovian, history remains relevant and the rating strategy is non-Markovian in the observed cash flow as only survival through distress yields valuable information. Forth, the distribution of the measurement error drives the learning potential of the rating agency. In a situation which features a skewed distribution of measurement errors with many lemons and a few gems, like the new economy and the subsequent bust of the dot.com bubble in the early 2000s, our model predicts a substantial equity value of this uncertainty at the expense of debt. Additionally, once the market recovers after a huge number of failures, capital costs of the surviving firms drop.

We model the firm through its Markovian cash flow process, which the rating agency can only observe with a persistent measurement error as in Fershtman and Pakes (2012). Due to the nature of our cash flow process, we specifically exclude statistical learning of the process parameters as in Pastor and Veronesi (2009), that is, the cash flow process on its own does not provide valuable information on the measurement error. We restrict the learning mechanism to learning from strategic actions, namely the firm’s cash injections. By definition, the cash flow process does not provide valuable information on its own. To capture the feedback of ratings and the firm’s capital costs, we employ rating-dependent performance sensitive debt as in Manso et al. (2010) and Manso (2013). Modeling the debt cost of capital as a function of the firm’s creditworthiness allows us to analyze effects similar to those occurring with the roll-over of short-term or finite-maturity debt. Hence, performance-sensitive debt should be seen as a modeling alternative to other approaches introducing finite maturity into infinite-horizon structural models, such as Leland and Toft (1996), Leland (1998), He and Xiong (2012a,b), and He and Milbradt (2016).

We consider the firm’s default decision as a dynamic real option exercise game.
in spirit of Grenadier et al. (2016), in which the decision to liquidate the firm is irreversible, following Kruse and Strack (2015). The resulting equilibrium is partially separating due to the Bayesian directional learning which prevents the rating agency from fully inferring the measurement error. Moreover, we derive constraints for such a rating policy such that it maintains the incentives for the rated firm not to mimic another firm that has a higher true cash flow.

Our paper contributes to several strands of the literature. First, it is related to Duffie and Lando (2001), who study a structural model of debt valuation featuring imperfect information. What distinguishes our contribution from Duffie and Lando (2001) is that in their model, the debt contract (in particular, the interest payment) is fixed ex ante, and the creditors’ (secondary bond market’s) learning of the asset process feeds back neither into the cost of debt nor the firm’s endogenous default decision. In contrast, our model is more suitable in describing a repeated interaction in which the firm’s observable credit quality today affects its future financing conditions, and thus also its considerations whether to inject new equity or to default already today. Our framework allows not only qualitative statements on the behavior of firm and rating agency. Moreover, we can explain the dynamic evolution of ratings and credit spreads over time, similar to, e.g., Jarrow et al. (1997) and Duffie and Lando (2001). Thus, we also relate to the asset pricing perspective on credit risk. We calibrate our model to actual credit spreads for the respective rating classes, and we can make predictions on how much learning matters in terms of dollar value for the rated entity. In a specific example we show that learning can decrease the credit spread by more than 240 basis points. This is the motivation why we rely on a rather involved mathematical framework in continuous time. Still, we are able to stay within the class of structural credit risk models with endogenous default, in the tradition of Leland (1994) and Goldstein et al. (2001).

Second, our paper contributes to the literature on learning in games. Bar-Isaac (2003) considers a learning game between a monopolistic seller with a persistent type and a set of buyers with limited information, that similar to our yields a history dependence. However, the present paper is fundamentally different in the economic implications: Neither does Bar-Isaac (2003) feature Bayesian directional learning nor does his mixed-strategy equilibrium imply a partially separating equilibrium.
Additionally, he cannot explain ex-post rating inflation, because his model builds on reputation, but through the mixed strategy, no type can be eliminated, which is in stark contrast to our Bayesian directional learning. Further contributions in the learning literature include Acharya and Ortner (2017) as well as Halac and Kremer (2018), who consider principal agent settings in discrete time. With Hörner and Lambert (2018) we share the property that a rating does not have to be a function of a single Markov process such as our cash flow process. Rather, it is augmented by a second dimension, which in our context is the historical minimum of the observed cash flow. Thus, the rating agency accounts for survival of past times of distress in its estimated default threshold despite the Markovian cash flow that itself does not carry valuable information besides its current state. Board and Meyer-ter Vehn (2013), Fulghieri et al. (2014), Frenkel (2015), and Thomas (2018) also model learning with Bayesian updating. However, their frameworks are not suitable for explaining the dynamics of credit spreads.

Third, our paper contributes to the literature on the strategic exercise of real options in financial economics, which has previously been studied in the context of initial public offerings by Bustamante (2012), for corporate investments by Hirth and Uhrig-Homburg (2010), Grenadier and Malenko (2011), Morellec and Schürhoff (2011), and Grenadier et al. (2014), and for dynamic agency problems for real options in Gryglewicz and Hartman-Glaser (2014). While we employ a similar equilibrium concept as Grenadier et al. (2016), we extend the setup substantially by a feedback effect between the cost structure and the decision to default, as well as introducing performance sensitive debt. Our innovation over the literature is that we allow a firm’s exercise policy to affect its own cost of capital, which can have a dynamic effect on firm value before exercise, rather than only upon exercise.

Finally, we contribute to the literature on credit ratings. Rather than focusing on learning, many recent contributions in the credit rating literature assume perfect observation of issuer quality, see, for example, Bolton et al. (2012) and Hirth (2014). Part of our modeling framework is closely related to Manso (2013). He models the rated entity’s cash flow process in continuous time as we do, but he does not consider information asymmetry between the rated entity and either the rating agency or the financial market. Consequently, he does not analyze learning. Consistent with
his paper, we show that the firm employs a cut-off strategy, and in case that “the cash-flow process of the firm follows a geometric Brownian motion, equilibrium of the game is unique” (Manso (2013, p.543)). However, the case of multiple equilibria, which drives a large part of his paper’s results, requires a mean-reverting cash-flow process. In contrast, our paper offers Bayesian directional learning and explains ex-post rating inflation.

The remainder of the paper is organized as follows: In Section 2, we introduce the rating game between the rating agency and the firm. Section 3 introduces the best response strategies of both players. Subsequently, we derive and compute the rating game equilibrium in Section 4. In Section 5, we provide an extensive analysis of the economic implications of our model. Section 6 concludes.

2 Model

We set off by formally describing the interaction between a levered firm and a rating agency. The rating agency’s success is measured by its accuracy, namely its ability to deviate as little as possible from a precise and unbiased rating. As a motivation for this objective function, note that on a competitive debt market, creditors have to price credit risk as accurate as possible to be successful. The rating agency in our model faces the same differential of information towards the firm as the capital market participants in general, specifically the investors on the bond market. Thus, the function of the rating agency in our model is mainly to simplify the exposition of the interaction between outsiders’ perception of the firm’s quality and the firm’s cost of debt financing.

The rated firm’s value to its equity holders depends on its continuous stream of cash flows net of its interest payments on outstanding debt. The firm generates a stream of non-negative cash flow at the rate $X = (X_t)_{t \geq 0}$ and pays interest on its outstanding debt at the rate $C = (C_t)_{t \geq 0}$. The firm is risk-neutral and aims to maximize the net present value of the cash flows net the interest payments. The firm identifies as its strategy the optimal time to default. Implicitly, the default timing decides if and how much cash is injected by the firm owners, who are financially unconstrained.
The firm’s debt claim is modeled as performance-sensitive debt, see Manso et al. (2010). In a more general interpretation, this can be seen as a way to model the roll-over of maturing debt, which will make the firm’s cost of debt capital similarly dependent of on the outsiders’ current perception of the firm’s credit quality. The interest payment rate $C$ depends on the rating agency’s rating of the firm $R = (R_t)_{t \geq 0}$, which is based on the imperfectly observed cash flow $D = (D_t)_{t \geq 0}$ of the firm’s cash flow $X$ given by $D = \tilde{\theta} X$. The persistent measurement error $\tilde{\theta}$ is drawn and learned by the firm at initial date $t = 0$. The rating agency does not know $\tilde{\theta}$. However, the law of $\tilde{\theta}$ is common knowledge, see, e.g., Grenadier et al. (2016) for a related setup. Based on the information from observing $D$, the rating agency estimates the firm’s critical cash flow level where default occurs, i.e., $\hat{D}^\star = (\hat{D}^\star_t)_{t \geq 0}$, and issues as its strategy the rating as distance to default $R_t = D_t / \hat{D}^\star_t$, which we discuss in more detail below.

The rating agency’s objective is to achieve the highest possible rating accuracy. It learns from the firm’s endogenous survival in low cash flow states. Apart from the current cash flow level, an important firm characteristic is therefore the lowest observed cash flow $E = (E_t)_{t \geq 0}$, with $E_t = \inf_{0 \leq s \leq t} D_s$, $t \geq 0$. From the perspective of the rating agency, $E$ is the information generating process, and the rating agency’s estimate of the firm’s default threshold is a function of this variable, i.e. $\hat{D}^\star = g(E)$.

Specifically, the firm’s cash flow rate $X$ satisfies

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \text{ for } t > 0, X_0 \in \mathbb{R}^+, \quad (1)$$

where $\mu$ and $\sigma$ represent the cash flow’s growth rate and volatility, respectively, with $\mu < r$, $r$ being the risk-free interest rate, and $W = (W_t)_{t \geq 0}$ is a Wiener process. The growth rate $\mu$ and volatility $\sigma$ are common knowledge, but $X$ is known only by the firm. The firm’s asset value at time $t$ can readily be stated as $X_t / (r - \mu)$. As we abstract from costs and benefits of debt such as bankruptcy costs or tax benefits, this value is independent of both the firm’s leverage level and also its default timing. The event of default has no effect on the firm’s asset value, but only on its ownership structure. Though, knowledge of the firm’s asset value requires perfect observation of the cash flow $X$. For an interpretation of the model, one could think of the cash flow more...
broadly as some characteristics determining firm value.

The imperfect observation $D$ of the cash flow is

$$D_t = \tilde{\theta} X_t, t \geq 0,$$

(2)

which depends on the measurement error $\tilde{\theta}$, but has identical dynamics as the cash flow $X$, i.e., $dD_t = \mu D_t dt + \sigma D_t dW_t$, for $t > 0$. As a consequence, no information on $\tilde{\theta}$ can be obtained from the imperfect observation $D$. The persistent measurement error $\tilde{\theta}$ is drawn by nature and learned by the firm at initial date $t = 0$. The rating agency does not know $\tilde{\theta}$. Rather, it overestimates the true cash flow for a type $\theta > 1$, while a type $\theta < 1$ leads the rating agency to underestimate the true cash flow. To put it the other way round, given a specific observation $D_t$ of the cash flow, a higher $\theta$ means that the actual cash flow $X_t$ is lower. It is common knowledge that $\tilde{\theta}$ is a random draw from the distribution $\mathbb{P}_{\tilde{\theta}}$ on $\Theta = [\theta, \bar{\theta}]$, with $0 < \theta < \bar{\theta} < \infty$, and is independent of the Wiener process $W$. Therefore, the rating agency does not have to learn the parameters of the cash flow dynamics, but the learning in our model solely concerns the persistent measurement error $\tilde{\theta}$ as in Fershtman and Pakes (2012). Learning in this setting is restricted to learning from strategic behavior, while pure statistical learning in the form of observing the imperfectly observed cash flow process alone can by definition not reveal any information. In contrast, think of the well-known case of a mean reverting process with unknown mean. In such a situation, a mere observation of the process realization over time can be useful to find better estimates of the unknown parameter. We assume that the distribution $\mathbb{P}_{\tilde{\theta}}$ admits a density, which is bounded from above and away from zero, which is our prior $\phi$. We write $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ to designate the filtration generated by $X$ and $D$, respectively.\footnote{Formally, the firm’s information set at $t$ is given by $\sigma(\tilde{\theta}) \vee \mathcal{F}_t$, for $t \geq 0$. Since $\tilde{\theta}$ is known to the firm at $t = 0$ as well as $\tilde{\theta}$ and $X$ are independent, we can condition on $\tilde{\theta} = \theta$ and work with $\mathbb{F}$.}

The firm issues performance sensitive debt as in Manso et al. (2010), which is specified by the interest payment rate $C$ depending on the rating (performance) $R \geq 1$. In particular, $C$ is a non-increasing function $C : [1, \infty] \to \mathbb{R}^+_0$ which we assume bounded away from zero and bounded from above. Hence satisfies $0 < C = C(\infty) \leq$
C = C(1) = C < ∞. The debt contract is perpetual in nature and does not specify a repayment of principal. Thus, apart from the rating-dependence of C, it can be seen as a consol bond as in Black and Cox (1976) and Leland (1994).

The rating R is specified as follows. Given the available information, the rating agency assigns the firm’s rating based on the estimated default level ˆD#. Formally, the default level ˆD# is strictly positive and G-adapted, that is, the rating agency only employs the information contained in the observed cash flow D, but cannot access the firm’s private information of the real cash flow X, or, equivalently ˜θ (with ˜θ = D/X). Thus, the rating agency can enhance its default assessment by continuously learning the measurement error from observing the firm’s costly cash injection signal in low cash flow states. We define the rating R issued by the rating agency as the distance to default, which we identify as

\[ R_t = \frac{D_t}{\hat{D}^*_t}, t \geq 0. \]  

The higher the firm’s observed cash flow D is relative to the predicted default level ˆD#, the higher will the firm’s rating be. The rating R takes values in [1, ∞], where \( R_t = 1 \) implies a firm that is expected to default immediately, and \( R_t = \infty \) corresponds to a default-free firm from the rating agency’s perspective. As the rating agency notices changes of the imperfectly observed cash flow over time, it adjusts the rating accordingly, because the estimated default threshold ˆD# changes.

The firm observes the cash flow process X and knows the realization of the measurement error ˜θ. Based on this information, the firm chooses the time to default, denoted by \( \tau(\theta) \). Because the firm is aware of the measurement error at the start and ˜θ and the Wiener process W are independent, we can write \( \tau = (\tau(\theta))_{\theta \in \Theta} \), where \( \tau(\theta) \) is an \( \mathbb{P} \)-stopping time, for \( \theta \in \Theta \), see also Footnote 1.

Observing the firm’s decision either to default or to signal quality by not defaulting, the rating agency gradually learns the measurement error of the firm’s cash flow over time. Hence, the rating agency forms its belief \( \pi = (\pi_t)_{t \geq 0} \) regarding the measurement error ˆθ, which we interpret as type in the given signaling game. We restrict our analysis to beliefs which are absolutely continuous with respect to the
prior $\mathbb{P}_\theta$, and thus also $\pi$ has a density, say $\phi^\pi$, with
\[
\phi^\pi_t(\theta) = L^\pi_t(\theta) \phi(\theta), \text{ for } \theta \in \Theta, t \geq 0,
\] (4)
where $L^\pi_t = (L^\pi_t(\theta))_{\theta \in \Theta}$ describes the evolution of the probabilities for each measurement error by using the information available in the market.\(^2\) While we formulate the beliefs here in general, we will later use perfect Bayesian Markov equilibrium as the equilibrium concept. Thus, the rating agency will update its belief about the measurement error according to Bayes rule whenever possible. Because Bayesian updating is infeasible if the rating agency observes actions which are not used by a firm with any possible measurement error, we need to specify off-path beliefs.\(^3\) Following Grenadier et al. (2016), we make the standard assumption that in such a case the beliefs remain unchanged:

**Assumption 1.** If at any $t$, the rating agency’s belief $\pi_t$ and the firm’s action are such that no possible measurement error could use this action in equilibrium, then the belief is unchanged.

By this learning mechanism, the strategy of the firm in all states $\tau(\theta)$ affects the rating agency’s belief $\phi^\pi$, hence the rating agency’s strategy in form of the estimated default level $\hat{D}^\pi$ feeds back into the specific strategy $\tau(\theta)$. Thus, the firm’s strategy is a measurement error-dependent stopping time, which we will later assume to be a Markov strategy in a suitably extended state space. Figure 1 displays an example of the rating agency’s Bayesian updating of its belief: Setting off from a prior (solid line), by observing the firm’s default or survival, the rating agency learns and reassesses the probabilities for each measurement error, which in turn sharpens the rating agency’s belief. As the cash flow evolves, the beliefs become more and more precise (dotted and dashed lines).

\(^2\) $L^\pi_t$ is a family of non-negative $\mathbb{G}_t$-measurable random variables with $\int_{\Theta} L^\pi_t(\theta) \phi(\theta) \, d\theta = 1$, $t \geq 0$.

\(^3\) For example, we will show in later sections that for each measurement error, there will be a default threshold for the firm. If the rating agency observes that the firm does not default, even though firms with all measurement errors should have defaulted, this strategy should not occur in equilibrium, so that we need to specify beliefs in this case.
Figure 1: Bayesian updating of beliefs. This figure displays an example of how the rating agency updates the prior (solid line) to the updated belief at times $t_1$ and $t_2$ (dotted and dashed line, respectively). It plots the density for each measurement error $\tilde{\theta}$, which has a support of $\Theta = [\underline{\theta}, \overline{\theta}] = [0.50, 1.50]$. As time evolves, the rating agency rules out overestimated measurement errors.

The strategies for the firm $(\tau(\theta))_{\theta \in \Theta}$ and the rating agency $\hat{D}^*$, respectively, are now specified in a general form. The rating agency’s learning is given by its belief $\pi_t$, which depends not only on the current observed cash flow $D_t$ but also on its running minimum $E_t$. Thus we have to extend the state space. Note that while the default threshold $\hat{D}^*$ estimated by the rating agency depends only on $E$, the distance to default and thus the firm’s current rating additionally depends on the currently observed cash flow $D$. The Markov property of the state processes, $(X,Y)$ with running minimum $Y = (Y_t)_{t \geq 0}$ of $X$, i.e. $Y_t = \inf_{0 \leq s \leq t} X_s$, $t \geq 0$, or $(D,E)$ as observed by the rating agency, suggests that it is sufficient to consider Markov strategies. The set of admissible Markov strategies $\mathcal{A}_f$ for the firm is given by default levels $f(\theta)$ of firm cash flow as observed by the rating agency, for $\theta \in \Theta$. The first time the cash flow as observed by the rating agency $D$ falls below $f(\theta)/\theta$, the firm defaults, i.e. $\tau(\theta) = \inf \{t \geq 0 : D_t \leq f(\theta)/\theta\}$, for $\theta \in \Theta$.

Strictly speaking, the set of Markov strategies is much larger and consists of stopping times that are given by first entry times in a measurable set $B(\theta) \subseteq \mathbb{R}^2$, i.e. $\tau(\theta) = \inf \{t \geq 0 : (D_t,E_t) \in B(\theta)\}$, $\theta \in \Theta$. However, the subsequent Proposition 2 shows that this restriction is innocent.

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4 We are taking the perspective of the rating agency to avoid problems when discussing the matters from the perspective of both parties, firm and rating agency. The critical default level in the firm cash flow is then $f(\theta)/\theta$, i.e. $\tau(\theta) = \inf \{t \geq 0 : X_t \leq f(\theta)/\theta\}$, since $D = \theta X$.

5 Strictly speaking, the set of Markov strategies is much larger and consists of stopping times that are given by first entry times in a measurable set $B(\theta) \subseteq \mathbb{R}^2$, i.e. $\tau(\theta) = \inf \{t \geq 0 : (D_t,E_t) \in B(\theta)\}$, $\theta \in \Theta$. However, the subsequent Proposition 2 shows that this restriction is innocent.
addition, we require that \( g \) is reasonable from a financial economics perspective. In \((D_t, E_t)\), the predicted default at \( g(E_t) \) should be attainable, and thus \( g(E_t) \leq E_t \), or, \( g \leq Id \), where \( Id \) is the identity on \( \mathbb{R}_0^+ \), \( Id(x) = x \), for \( x \in \mathbb{R}_0^+ \). Firm survival in bad times, that is for a decreasing running minimum \( E_t \), potentially signals quality. Then the estimated default threshold \( \hat{D}^* = g(E_t) \) should be adjusted downwards or remain constant, i.e. \( g \) is non-decreasing. However, we demand that a rating \( R = D/\hat{D}^* \) should not improve when the cash flow hits a new all-time low. Formally, a new all-time low occurs at time \( t \) for \( D_t = E_t \). Then the rating is \( R_t = D_t/\hat{D}^* = E_t/g(E_t) \), and has to be non-increasing in \( E_t \) to avoid a better rating in case of a new all-time low. The latter is equivalent to the constraint that \( g/Id \) being non-increasing. Thus a reasonable strategy \( g \in \mathcal{A}_g \) satisfies

\[
g \leq Id, \ g \text{ non-decreasing, and } g/Id \text{ non-increasing.} \tag{5}\]

We now specify the expected payoffs as functions of both the firm’s and the rating agency’s strategies, \((\tau, \hat{D}^*)\), with \( \tau = (\tau(\theta))_{\theta \in \Theta} \) and \( \tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\} \), \( \theta \in \Theta \), and \( \hat{D}^* = g(E) \), where \( f \in \mathcal{A}_f \) and \( g \in \mathcal{A}_g \). The firm has a measurement-dependent expected payoff equal to the discounted stream of real cash flow minus rating-dependent interest \( C \), as the perceived measurement error feeds back into the interest payments. Specifically, noting that \( X = D/\hat{\theta} \) and \( R = D/\hat{D}^* \), we have

\[
U^{(\theta)}_F(\tau, \hat{D}^*) = \mathbb{E}\left[ \int_0^{\tau(\theta)} e^{-rt} \left( D_t/\theta - C(D_t/\hat{D}^*) \right) \, dt \right], \text{ for } \theta \in \Theta. \tag{6}
\]

If the interest payments exceed the true cash flow at a given time, the firm does not have to default, but the owners can inject further cash. The firm chooses the default time \( \tau \) with the objective to maximize \( U^{(\theta)}_F(\tau, \hat{D}^*) \), namely the present value of cash flows after interest payments to creditors (see Equation (14) below for the formal specification of the firm’s optimization problem). Thus, the decision to delay default includes the decision of how much cash to inject into the firm in a time of distress.

The rating agency chooses the estimated default threshold \( \hat{D}^* \) and thus its rating scale with the objective to maximize its accuracy. Conditional on its belief \( \pi \) and

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\[\text{If } g \text{ is non-decreasing and } g/Id \text{ is non-increasing, then } g \text{ is also continuous.}\]
the firm’s measurement error-dependent liquidation strategy \( \tau(\theta) \), the rating agency expects the costs from a rating with an estimated default threshold \( \hat{D}^* \) to be, given the rating agency’s information,

\[
U_{RA}\pi(\tau, \hat{D}^*) = -\mathbb{E} \left[ \int_0^{\tau} e^{-\rho t} k_t^\pi dt \right],
\]

(7)

with the cost rate

\[
k_t^\pi = \int_{\Theta} (\hat{D}_t^* - f(\theta))^2 \phi_t^\pi(\theta) d\theta, \text{ for } t \geq 0,
\]

(8)

and the rating agency’s discount parameter \( \rho \), which measures the time preference over future lack of accuracy. For the rating agency, we define the expected costs as the squared deviation of the estimated default threshold, conditional on the belief, and the true default threshold, dependent on the perceived measurement error.

As a robustness check, we will generalize this cost rate by allowing for an asymmetric effect on the cost rate. This generalized structure captures different attitudes of the rating agency towards its stakeholders: If it is more concerned with protecting investors, the rating agency would be biased towards avoiding overestimations and estimate the default threshold conservatively. In the extreme case, it would always assume the worst possible measurement error from the investors’s perspective. On the other hand, a rating agency aiming for maximizing revenues and its market share, would rate more progressively to please their clients. A detailed discussion of this aspect is postponed to Appendix A.

We focus on equilibria in pure strategies. The equilibrium concept is perfect Bayesian equilibrium in Markov strategies, which requires that the rating agency’s strategies are sequentially optimal, beliefs are updated according to Bayes’ rule whenever possible, and the equilibrium strategies are Markov. In particular, the Markov property requires that the firm’s and the rating agency’s strategies are only functions of the payoff-relevant information at any time \( t \), i.e., measurement error \( \hat{\theta} \) and the current value of the state process \((D_t, E_t)\) (equivalently \((X_t, Y_t)\)) for the firm, and the beliefs \( \phi_t^\pi \) about \( \hat{\theta} \) and the current value of the state process \((D_t, E_t)\) for the rating agency. The formal definition of the perfect Bayesian equilibrium in Markov
strategies is presented in Appendix II and is subsequently referred to simply as equilibrium. Note that our state space processes for firm and rating agency contain both the observed cash flow process and its running minimum. Therefore, our players have a memory beyond the current value of the cash flow process, although the Markov property holds for the extended state space.

We consider an exogenously specified interest payment rate $C$ which depends on the firm’s rating $R$, and we assume $C$ to be sufficiently sensitive to the rating. However, for the subsequent equilibrium analysis the sensitivity of $C$ cannot be overly excessive. In order to control the sensitivity of $C$, we make the following assumption.

**Assumption 2.** Assume that the interest payment rate $C$ satisfies for some $0 < L_C < 1$ that

$$C(z) \leq C(z') \leq (z/z')^{L_C} C(z), \text{ for } 1 \leq z' \leq z. \quad (9)$$

To confirm the validity of Assumption 2, we give an outlook on the parametrization based on market data that we will use in our analysis. In Figure 2, we illustrate the interest rate and rating structure obtained from the perfect-information case. Appendix I shows that this interest rate structure indeed satisfies Assumption 2.

### 3 Best Responses and Learning

Having specified the strategies of both the firm and the rating agency in Section 2, we can now turn to the best responses of both players, and describe how the rating agency learns the firm’s true cash flow. In particular, the rating agency’s best estimate of the firm’s optimal liquidation decision is a continuously updated default barrier, dependent on the observed cash flow trajectory, and especially on its running

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7 In contrast, if the interest payment rate is not sensitive to the firm’s rating, then the firm’s payoff is not rating-dependent, leaving the firm indifferent to the rating. Then, the rating issued by the rating agency has no feedback effect on the firm strategy and becomes irrelevant, and thus we have no game between the two parties. In this degenerated case, the firm’s default level is a constant $f_1$ in the firm’s real cash flow $X$ and hence linear in the rating agency’s imperfectly observed cash flow $D = \theta X$, i.e. $f(\theta) = \theta f_1$. 

---
minimum. For the observed cash flow trajectory, the rating agency infers up to which degree of overestimation the firm would have defaulted, ruling out the most overestimated types in its consistent belief. In its best response, the firm specifies its optimal liquidation decision as a response to a rating strategy, based on the type, or measurement error. The firm’s strategy consists of the classical equity holder trade-off between the cost of injecting additional funds to cover presently negative net cash flows and the benefit of receiving positive net cash flows in the future, when the cash flow generating process recovers, see, e.g., Black and Cox (1976), Leland (1994), and Goldstein et al. (2001). Our new contribution is that the injection of additional funds makes the rating agency and other external stakeholders update their beliefs about the firm’s quality and thus leads to lower financing costs in the future. These lower future financing costs imply that the injection of additional funds remains attractive for lower present cash flows. Thus, liquidation can be triggered at a later time, which closes the feedback loop.

3.1 Best Response and Learning of the Rating Agency

In this section, we characterize the best response of the rating agency, i.e., the estimated default level $\hat{D}^* = g(E)$, and the rating agency’s consistent belief $\pi$ given the firm’s liquidation strategy $\tau$. The firm’s strategy is type-dependent and is given by $f \in \mathcal{A}_f$ with $\tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\}$, for all $\theta \in \Theta$. The rating agency’s consistent belief is driven by the minimum observed cash flow $E$. At time $t$, types $\theta$
with \( f(\theta) \geq E_t \) can be discarded, since default obviously has not occurred yet. The consistent belief is therefore \( \pi_t = \mathbb{P}_{\tilde{\theta}} | f(\tilde{\theta}) < E_t \) with density given in (11) below. For a given belief \( \pi \), the best response \( g \) is the solution to the maximization problem given in (10). The rating agency optimizes its expected payoff given in Equation (7) over admissible estimated default thresholds of the firm, i.e.,

\[
\sup_{\hat{D}^* = g(E), g \in \mathcal{A}_g} U^\pi_{RA}(\tau, \hat{D}^*) = - \inf_{g \in \mathcal{A}_g} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} \int_\Theta (g(E_t) - f(\theta))^2 \phi^\pi_t(\theta) d\theta dt \right].
\]

(10)

For each \( t \), the optimal rating agency strategy \( g(E_t) \) minimizes the mean squared error for estimating the type-dependent default threshold \( f(\theta) \) using the current belief. The latter is given by the respective conditional expectation, see (12).

This allows us to formulate both a consistent belief and the rating agency’s best response in the following proposition.

**Proposition 1** (Rating Agency’s Best Response). *Let a firm strategy \( \tau \) be given by a function \( f \in \mathcal{A}_f \), with \( \tau(\theta) = \inf\{t \geq 0 : D_t \leq f(\theta)\}, \theta \in \Theta \). Then the rating agency’s consistent belief is given by

\[
\phi^\pi_t(\theta) = \frac{1_{f(\theta) < E_t}}{\int_\Theta 1_{f(\theta') < E_t} \phi(\theta') d\theta'}, \text{ for } \theta \in \Theta \text{ and } 0 \leq t \leq \tau.
\]

(11)

The rating agency’s corresponding best response to \( f \) is given by \( \hat{D}^* = g(E; f) \), with

\[
g(e; f) = \begin{cases} 
\mathbb{E} \left[ f(\tilde{\theta}) | f(\tilde{\theta}) < e \right], & \text{for } e > \inf_{\theta \in \Theta} f(\theta), \\
e, & \text{else}.
\end{cases}
\]

(12)

and is bounded by \( \text{Id} \), i.e. \( g(e; f) \leq e \), for \( e \geq 0 \), as well as non-decreasing. Moreover, if \( f \) is strictly increasing, then \( g(\cdot; f) \) is continuous and strictly increasing on \( f(\Theta) \).

**Sketch of Proof of Proposition 1.** For a given \( f \), the form of the consistent belief \( \pi \) in (11) follows directly from Bayes’ rule. Using the consistent belief \( \pi \), the rating agency maximizes the respective utility in (10), where the key term is the quadratic loss \( \int_\Theta (g(E_t) - f(\theta))^2 \phi^\pi_t(\theta) d\theta \). We look for \( g(E_t) \) which minimizes the squared
distance to the random variable \( f(\tilde{\theta}) | f(\tilde{\theta}) < E_t \). Therefore, the optimal \( g(E_t; f) \) is the expected value of \( f(\tilde{\theta}) | f(\tilde{\theta}) < E_t \), which is in essence (12). The detailed proof is given in Appendix II.

Proposition 1 specifies that for the rating agency, the optimal response to a type-dependent firm liquidation strategy is to issue a rating based on an estimated liquidation barrier in accordance with its consistent belief about the type, namely by observing which types should have defaulted for the observed cash flow process. Economically speaking, a rating agency’s belief is consistent if the likelihood for the type is in accordance with firm’s default behavior. Hence, if the rating agency sets the rating scale inducing a specific default barrier and the imperfectly observed cash flow reaches this level, then a firm of a sufficiently overestimated type defaults while a firm of an underestimated type may not. This observed default behavior provides useful information to the rating agency, and allows it to update its strategy, i.e. the rating scale, consistent with its belief about the types.

Comparing two states with identical current observed cash flow but with different historical cash flow paths, the state with lower historical minimum cash flow exhibits a better rating. However, this does not imply that a decreasing cash flow improves the current rating, in contrary the rating is worsening for a decreasing cash flow, see the specification of the set of admissible rating strategies \( \mathcal{A}_g \) as well as the constraint in (5) and the discussion there. However, the survival of low observed cash flow levels in the past improves future ratings spurred by the rating agency’s learning. Note that learning occurs when the observed cash flow falls to a level where a given type, or, measurement error, may default, which are typically very low levels with a short distance to default.

Proposition 1 characterizes the rating agency’s best response to an arbitrary firm strategy \( f \in \mathcal{A}_f \). The best response \( g(\cdot; f) \in \mathcal{A}_g \) does not necessarily satisfy all conditions in (5). Whereas \( g(\cdot; f) \) is non-decreasing and bounded by \( Id \), the important constraint that \( g(\cdot; f)/Id \) is non-decreasing, which prohibits inappropriate incentives resulting from the rating agency’s strategy, cannot be shown to hold in general. More specifically, the rating agency may not provide incentives that prevent the firm from decreasing its cash flow to immediately profit from reduced interest payments. Instead of imposing the constraint by requiring \( g(\cdot; f)/Id \) to be
non-decreasing in (10), we rather modify \( g(\cdot; f) \) such that (5) holds, by the following transformation\(^8\)

\[
\mathcal{R}(g)(e) = \begin{cases} 
  e \inf \{g(z) / z : 0 < z \leq e \}, & \text{for } e > 0, \\
  0, & \text{for } e = 0.
\end{cases}
\] (13)

The transform \( \mathcal{R} \) maps \( \{ g \in \mathcal{A}_g : g \text{ non-decreasing}, g \leq Id \} \) to \( \{ g \in \mathcal{A}_g : g \text{ satisfies (5)} \} \) and is the identity on the latter set, i.e. \( \mathcal{R}(\tilde{g}) = \tilde{g} \), for \( \tilde{g} \in \{ g \in \mathcal{A}_g : g \text{ satisfies (5)} \} \), see Lemma 1 in Appendix II.

However, note that although the constraint that \( g(\cdot; f)/Id \) is non-decreasing cannot be shown to hold in general, our numerical implementation and all cases that we could think of as practically relevant satisfy this constraint. That is, the transform is formally needed but typically will just turn out to be the identity.

### 3.2 Best Response of the Firm

In the following we characterize the best response of the firm, i.e., its default time \( \tau \), for an admissible rating strategy \( g \in \mathcal{A}_g \) that satisfies (5). Since the firm knows its own cash flow and the type \( \theta \), we can specify the firm’s type-dependent best response \( \tau(\theta; g) \).

For a given rating agency strategy \( g \in \mathcal{A}_g \) satisfying (5) and type \( \theta \), the best response \( \tau(\theta; g) \) is the solution to the optimal stopping problem given in (14) below. Recall the firm’s expected payoff given in Eq. (6). Denote by \( v(\cdot, \cdot; \theta, g) \) the value function, which is given by

\[
v(d, e; \theta, g) = \sup_{\tau \in \mathcal{T}_{(d,e)}} \mathbb{E}_{(d,e)} \left[ \int_0^\tau e^{-rt} (D_t/\theta - C(D_t/g(E_t))) \, dr \right], \text{ for } (d, e) \in \mathcal{C},
\] (14)

where \( \mathcal{C} \{ (d, e) \in \mathbb{R}^2 : 0 \leq e \leq d \} \) is the convex cone on which the imperfectly observed cash flow \( D \) and its running minimum \( E \) take values in and \( \mathcal{T}_{(d,e)} \) is the

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\(^8\) Note that the solving the constrained problem is much more evolving as path dependencies arise leading to a departure from the Markovian setup. For the sake of tractability, we limit the exposition to the case where implicitly it is assumed that the constraint holds.
set of all stopping times with respect to the information generated by \((D, E)\) with starting value \((d, e)\).

Optimal stopping problems are connected to free-boundary value problems, see Peskir and Shiryaev (2006). For the case that the running minimum of a diffusion process is included in the state variables, Heinricher and Stockbridge (1991) and Barron (1993) provide a characterization of the solution of optimal control problems in terms of the solution of an associated free boundary value problem. While we do have a partial differential equation system for the firm’s optimal stopping problem in Equation (14), this is a non-standard system of partial differential equations, as we have to incorporate not only the cash flow, but also the running minimum of the imperfectly observed cash flow the rating agency uses. The value function \(v\), we characterize through a viscosity solution to the following free boundary problem.

Define the differential operator \(\mathcal{L}^{(\theta, g)}\) by

\[
\mathcal{L}^{(\theta, g)} h = \mu d \frac{\partial h}{\partial d} + \frac{1}{2} \sigma^2 d^2 \frac{\partial^2 h}{\partial d^2} + k^{(\theta, g)} - rh, \tag{15}
\]

where \(k^{(\theta, g)}(d, e) = d/\theta - C(d/g(e))\), for \((d, e) \in \mathcal{C}\). Then

\[
\mathcal{L}^{(\theta, g)} v \leq 0, \tag{16}
\]

\[
v \geq 0, \tag{17}
\]

\[
v \cdot \mathcal{L}^{(\theta, g)} v = 0, \tag{18}
\]

with boundary conditions

\[
0 = \frac{\partial v}{\partial d} \text{ on } \partial C^{(\theta, g)}, \text{ and } 0 = \frac{\partial v}{\partial e} \text{ on } \mathcal{D}, \tag{19}
\]

where \(C^{(\theta, g)} = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) > 0\}\) is the continuation region and \(\mathcal{D} = \{(d, d) \in \mathcal{C} : d > 0\}\) is the diagonal. The first boundary condition is due to smooth fit at the edge of the continuation region \(\partial C^{(\theta, g)}\) and the second condition is normal reflection on the diagonal \(\mathcal{D}\), addressing the dependence on the running minimum \(E\).

The best response of the firm to a given rating agency rating strategy \(g\) is the collection of optimal stopping times \(\{\tau(\theta; g)\}_{\theta \in \Theta}\). Under the condition \(g \in \mathcal{A}_g\) and
satisfying (5), we show that the optimal stopping rule $\tau(\theta; g)$ is a cut-off rule in the observed cash flow $D$ and does not depend on its running minimum $E$. Specifically, the firm liquidates at the first hitting time of a threshold $f(\theta; g)$, for all $\theta \in \Theta$, as is shown in the following proposition. The cut-off rule balances the firm’s trade-off between continuing in unfavorable conditions now in order to reap lower interest payments once the rating agency updates the imperfectly observed cash flow. If the imperfectly observed cash flow is above the threshold, the prospect of continuing is attractive. Once it hits the cut-off, the firm liquidates.

**Proposition 2** (Firm’s Best Response). For $g \in \mathcal{A}_g$ satisfying (5), $\theta \in \Theta$, and $d > 0$, the optimal stopping time of (14) is given by

$$
\tau_{(d,d)}(\theta; g) = \inf\{t \geq 0 : D(t) \leq f(\theta; g)\},
$$

(20)

where $f(\theta; g)$ is some positive real constant, i.e. $\tau_{(d,d)}(\theta; g)$ is the first hitting time of the imperfectly observed cash flow $D$ with respect to the barrier $f(\theta; g)$.

**Sketch of Proof of Proposition 2.** We obtain a cut-off rule for the firm’s best response by analysing the early exercise region using the conditions in (5). Firstly, $g$ being non-decreasing implies that the firm defaults when the observed cash flow falls below a critical level. Secondly, $g/Id$ is non-increasing ensures that this critical level is unique as a better rating in case of a new all-time low is prohibited. The detailed proof is given in Appendix II.

This result implies that the best response of the firm $\tau(\theta; g)$ for a rating agency’s strategy $g$ is characterized by a specific default barrier $f(\theta; g)$, for each $\theta \in \Theta$.$^9$

For the path of cash flow, the firm is aware of its over- or underestimation by the rating agency based on the observed cash flow. If the imperfectly observed cash flow decreases beyond a type-dependent level, the firm defaults.

How does the firm’s best response to a given rating strategy $\hat{D}^* = g(E)$ look like? So far, Proposition 2 characterizes the firm’s optimal strategy in terms of a default barrier $f(\theta; g)$ for one specific $\theta$. But the firm’s best response is the mapping $\theta \mapsto f(\theta; g)$ for all $\theta \in \Theta$. We provide more structure on how the barrier changes in

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$^9$Solving the associated free boundary value problem determines the point-wise solution.
the type, see Lemma 2 in Appendix II. In particular: (i) The best response barrier \( f(\cdot; g) \) is non-decreasing. It is based on the observed cash flow, as in Equation (14), such that the measurement error \( \theta \) only enters as the scaling factor of the observed cash flow \( D \). Hence, a higher measurement error \( \theta \) implies a lower true cash flow \( X \) (note that \( X = D/\theta \)), and subsequently a lower default threshold. (ii) The default barrier in terms of the true cash flow \( f(\cdot; g)/Id \) is non-increasing and uniformly bounded in \( g \).\(^{10}\) A firm subject to an overestimated type has a lower true default barrier, as it largely underpays its interest. (iii) The best response barrier \( f(\cdot; g) \) is uniformly Lipschitz continuous in \( g \). (iv) Assumption 2 yields that \( f(\cdot; g) \) is strictly increasing with a strictly positive lower bound for the slope uniformly in \( g \). The two latter properties are crucial for the subsequent equilibrium analysis as it clears the way for applying the Schauder fixed point theorem.

### 4 Rating Equilibrium

The results of the previous section are fundamental for the subsequent equilibrium analysis. Both the rating agency and the firm use Markov strategies, which are both characterized by real-valued functions, see Proposition 1 and Proposition 2. We obtain a perfect Bayesian equilibrium in Markov strategies as introduced by Maskin and Tirole (1988) in spirit of Grenadier et al. (2016) by the Schauder fixed point theorem. However, the application of this fixed point theorem requires that the interest payment is well behaved. Specifically, the existence of the rating game requires a growth constraint on the rating-dependent interest payments, see Assumption 2. Given this assumption, we arrive in the present section at our equilibrium on the rating market. First, Proposition 3 below provides the existence of an equilibrium candidate. Then, Proposition 4 characterizes the solution and verifies the existence and uniqueness, provided specific technical conditions hold.

In particular, the latter non-trivially extends the result of Manso (2013) giving the existence and uniqueness of an equilibrium for the cash flow that follows a geometric Brownian motion to the case of information asymmetry. The underlying reason for the uniqueness is the non-stationarity assumption for the cash flow process.

\(^{10}\) Default happens if \( E_t \leq f(\theta; g) \), or, equivalently \( Y_t = E_t/\theta \leq f(\theta; g)/\theta \).
Multiple equilibria would be likely for a mean-reverting specification, see Manso (2013). Furthermore, notice that \( f(\theta) \) is strictly increasing according to Lemma 2 in Appendix II, that is, a best response has a minimal slope of \( l_f > 0 \) for all given strategies \( g \) of the rating agency. This eliminates classical semi-pooling equilibria, in which some observed types default at the same time, i.e. play the same strategy. While we do not have a semi-pooling equilibrium, the rating agency’s learning implies that the information is only revealed fully at the time the firm actually defaults, prior to which the rating agency can only rule out some types, but cannot fully infer the firm’s observed type. The rating agency learns the exact type at default, but the game ends simultaneously, leaving the rating agency without the possibility to react.

To establish our equilibrium, we apply the Schauder fixed point theorem to the mapping \( T : (f, g) \mapsto (f(\cdot; g), \mathcal{R}(g(\cdot; f))) \), where \( f(\cdot; g) \) is the firm’s best response given in Proposition 2, \( g(\cdot; f) \) is the rating agency’s best response given in Proposition 1, and \( \mathcal{R} \) is the respective transformation defined in (13).

**Proposition 3.** Suppose Assumption 2 holds. Then \( T : (f, g) \mapsto (f(\cdot; g), \mathcal{R}(g(\cdot; f))) \) has at least one fixed point in \( \mathcal{A}_f \times \mathcal{A}_g \). Let \( (f^*, g^*) \) be such a fixed point, if \( \mathcal{R} \circ g(\cdot; f^*) = g(\cdot; f^*) \) then \( (f^*, g^*) \) is an equilibrium.

**Sketch of Proof of Proposition 3.** For the Schauder fixed point theorem, we identify a sufficiently rich subset \( \mathcal{K} \subseteq \mathcal{A}_f \times \mathcal{A}_g \) and prove that \( \mathcal{K} \) is a nonempty convex compact subset of a Banach space, here, the space of continuous functions on a compact set endowed with the sup-norm. Using structural properties of \( (f, g) \in \mathcal{K} \), we prove that \( f(\cdot; g) \) and \( \mathcal{R}(g(\cdot; f)) \) are both continuous functionals. Then Schauder gives us the existence of at least one fixed point \( (f^*, g^*) \). If in addition \( \mathcal{R} \circ g(\cdot; f^*) = g(\cdot; f^*) \) then \( (f^*, g^*) = (f(\cdot; g^*), g(\cdot; f^*)) \) establishing an equilibrium. The detailed proof is given in Appendix II.

Proposition 3 provides us with a candidate for an equilibrium in this very general setup with function-valued strategies, which then has to be verified to be an equilibrium. Practically, for the subsequent analysis of special cases of the rating game’s equilibrium, the transform \( \mathcal{R} \) regulating the rating agency’s strategy \( g \) always takes the form of the identity function, so that no transformation is necessary. In turn,
without employing the transformation, we immediately end up with an equilibrium for the rating game without further ado for all relevant cases.

This result is a pure existence result and no particular guidance is given on how to actually compute such an equilibrium candidate. Now, we provide with Proposition 4 below a result that characterizes an equilibrium candidate as the solution to a two-dimensional ordinary differential equation (ODE) under some technical assumption. Further, an inequality condition is stated, and given this condition holds, the candidate is indeed an equilibrium, which is then also unique. This result is the basis for computing an equilibrium strategy in a fast and efficient way, which we rely on heavily in the following analysis.

For \( g \in \mathcal{A}_g \) satisfying (5) and \( \theta \in \Theta \) consider the value function \( v(\cdot, \cdot; \theta, g) \).

Its boundary \( \partial C^{(\theta,g)} \) can be described by a function \( b(\cdot, \theta; g) \) that assigns to each minimum observed cash flow \( e \) the critical \( d \)-value where the firm defaults, i.e.

\[
b(e, \theta; g) = \inf\{d \geq e : v(d, e; \theta, g) > 0\}, \quad \text{for } e \in [0, f(\theta; g)].
\]  

See Lemma 3 in Appendix III for the properties of \( b \). In order to facilitate the subsequent analysis, we require the following properties.

For \( g \in \mathcal{A}_g \) satisfying (5), we call the collection of solutions \( (v(\cdot, \cdot; \theta, g))_{\theta \in \Theta} \) of the boundary value problem (15-19) sufficiently differentiable, if: (i) \( v(\cdot, \cdot; \cdot, g) \) is continuously differentiable in \( \theta \), (ii) \( v(\cdot, \cdot; \cdot, g) \) allows for interchanging the order of differentiation with respect to \( d \) and \( \theta \) on the interior of \( \bigcup_{\theta \in \Theta} C^{(\theta,g)} \times \{\theta\} \), and (iii) the collection of boundary functions \( (b(\cdot, \cdot; g))_{\theta \in \Theta} \) is continuously differentiable with respect to \( e \) and \( \theta \).

The following proposition characterizes an equilibrium candidate as the solution to an implicit two-dimensional ODE and gives a condition for the candidate being indeed an equilibrium. This ODE is the centerpiece to the numerical computation of the equilibrium. Its proof is given in Appendix III.

**Proposition 4.** Given the setting of Proposition 3, denote by \((f^*, g^*)\) a fixed point of \( T \). Suppose \( f^*, g^* \) and \( \phi \) are continuously differentiable, as well as the collection of solutions \( (v(\cdot, \cdot; \theta, g^*))_{\theta \in \Theta} \) of the boundary value problem (15-19) is sufficiently
differentiable. Then \((f, \hat{g}) = (f^*, g^* \circ f^*)\) satisfies
\[
\begin{pmatrix}
  f'(\theta) \\
  \hat{g}'(\theta)
\end{pmatrix} = \begin{pmatrix}
  \frac{(1 + \eta) \sigma^2}{2(r - \mu)} f(\theta)^2/\theta^2 - \frac{f(\theta)^2}{\theta} - \frac{1}{\Phi(\theta)} C(f(\theta)/\hat{g}(\theta)) - f(\theta)/\theta \\
  \frac{\phi'(\theta)}{\Phi(\theta)} (f(\theta) - \hat{g}(\theta)) - \frac{\partial b}{\partial e}(f(\theta), \theta; \hat{g} \circ f^{-1}) 
\end{pmatrix},
\]

on \((\theta, \overline{\theta})\) with initial condition \((f(\theta), \hat{g}(\theta)) = (f_1^*, f_1^*)\), where \((f_1^*, g_1^*)\) denotes the unique equilibrium of the perfect information case, i.e. \(\Theta_1 = \{1\}\), \(\Phi(\theta) = \int_0^\theta \phi(t) \, dt\), and \(\eta = \frac{1}{\sigma^2} (\mu - \frac{1}{2} \sigma^2 + \sqrt{(\mu - \frac{1}{2} \sigma^2)^2 + 2 r \sigma^2}) > 0\). If
\[
\hat{g}' \leq f' \hat{g} / f, \quad \text{on } (\theta, \overline{\theta}),
\]
then the fixed point \((f^*, g^*)\) is an equilibrium.

Equation (22) is an implicit ODE.\(^\text{11}\) If this implicit ODE admits a unique solution \((f, \hat{g})\) satisfying the differentiability assumptions in Proposition 4, then the fixed point is given by \((f^*, g^*) = (f, \hat{g} \circ f^{-1})\) on \(\Theta \times f(\Theta)\). Now, if condition (23) holds, then the fixed point is an equilibrium that is moreover unique within the set of strategies satisfying the differentiability assumptions.\(^\text{12}\)

5 Implications of Rating Equilibrium

In this section, we summarize a number of equilibrium implications of our model and derive empirically testable implications. Following an illustration of the rating agency’s Bayesian directional learning, we turn to the ex-post rating inflation in equilibrium. Subsequently, we expand the interpretation of our model to different

\(^{11}\)This holds since \(\frac{\partial b}{\partial e}(f(\theta), \theta; \hat{g} \circ f^{-1})\) does not depend on the entire function \(\hat{g} \circ f^{-1}\). Instead, at \(\theta\) it depends on \(f(\theta), \hat{g}(\theta), f'(\theta)\) and \(g'(\theta)\), or, more precisely \(\frac{\partial b}{\partial e}(f(\theta), \theta; \hat{g} \circ f^{-1}) = h(f(\theta), \hat{g}(\theta), \hat{g}'(\theta)/f'(\theta), \theta)\) for some function \(h\), see (41) in Proposition 6 in Appendix III.

\(^{12}\)In contrast to explicit ODEs, the existence and uniqueness of solutions of implicit ODEs, also known as differential algebraic equations (DAEs), is more delicate and hence beyond the scope of this paper. Here, we feel it is more appropriate to point at the related specialist literature such as Kunkel and Mehrmann (2006).
distributions of types. First, our base case corresponds to a mature firm that exhibits lower uncertainty and less learning potential. Second, for a start-up firm, the skewed nature of the firm’s prospects provide a richer dynamic evolution of the rating agency’s assessment of credit quality. Finally, we discuss empirically testable hypotheses.

The firm has debt outstanding with a face value scaled to unity. \( C \) then denotes the interest rate in percentage terms payable by the firm to the debt holders depending on the rating \( R \). The firm generates cash flow per unit debt at the rate \( X \) with expected growth rate of \( \mu = 0 \) and a volatility of \( \sigma = 0.30 \). The rating agency faces a firm whose cash flow it observes imperfectly. The corresponding type \( \tilde{\theta} \) is distributed according to a truncated normal distribution with parameters \( \mu_{\theta}, \sigma_{\theta} \), and truncation to \( \Theta = [\underline{\theta}, \overline{\theta}] \). Accordingly, the density of \( \tilde{\theta} \) is given by

\[
\phi(\theta) = \begin{cases} 
0, & \text{for } \theta < \underline{\theta}, \\
\frac{1}{\sqrt{2\pi} \sigma_{\theta} c_{\theta}} \exp \left( -\frac{1}{2} \frac{(\theta - \mu_{\theta})^2}{\sigma_{\theta}^2} \right), & \text{for } \underline{\theta} \leq \theta \leq \overline{\theta}, \\
0, & \text{for } \theta > \overline{\theta}.
\end{cases}
\]  \hspace{1cm} (24)

where \( c_{\theta} = N((\overline{\theta} - \mu_{\theta})/\sigma_{\theta}) - N((\underline{\theta} - \mu_{\theta})/\sigma_{\theta}) \) and \( N \) is the standard normal cumulative distribution function. The base case parameters are \( \mu_{\theta} = 1.0, \sigma_{\theta} = 0.25 \) and \( \Theta = [\underline{\theta}, \overline{\theta}] = [0.50, 1.50] \). For the original distribution (before updating of beliefs), this means that the rating agency is unbiased, as \( E[\tilde{\theta}] = 1 \). We combine this setup with a risk-free rate \( r \) set to 0.0211.\(^{13}\)

### 5.1 Learning Mechanism

This section turns to the evolution of the rating agency’s assessment of a firm over time. Figure 3 illustrates the best responses of the rating agency and firm, respectively, as well as the rating agency’s continuous learning. It shows a sample path of a strongly underestimated cash flow (\( \theta = 0.5 \)). The rating agency observes an initial cash flow of 0.1366, for which the firm pays interests 0.0955. The true cash flow is 0.1366/0.5 = 0.2732. In Panel a), the sold black line represents the

\(^{13}\) Appendix I provides details about the calibration.
imperfectly observed cash flow, and the dashed black line its running minimum. The solid gray line presents the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of the firm, which is unobservable to the rating agency.

Figure 3: Best responses (strongly underestimated cash flow). This graph displays the best response of the rating agency as an estimated default threshold, dependent on the imperfectly observed cash flow. The firm initially has a strongly underestimated cash flow with $\theta = 0.5$. In Panel a), the solid black line represents the imperfectly observed cash flow, and the dashed black line its running minimum. The solid gray line is the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of the firm, which is unobservable to the rating agency. In Panel b), the black line represents the firm’s privately observed true cash flow, and the solid gray line is the firm’s interest payment. The dashed gray line indicates the default threshold.
As the imperfectly observed cash flow decreases, the rating worsens. However, as the running minimum falls, the rating agency continuously adjusts the best response default threshold, as it rules out the most overestimated types, who would have defaulted for the observed path. As the imperfectly observed cash flow increases again, the rating improves and the corresponding interest payments decrease. In Panel b), it can be seen that the intersection of the true cash flow (black line) and the interest payment (solid gray line) occurs at a lower level of true cash flow. Between the two intersections (i.e., between about \( t = 1.5 \) and \( t = 7 \) years), the firm has to raise new equity to cover the interest payments. This means that the firm’s best response to the rating agency’s strategy is not to default over the ten year horizon illustrated in the graph, visualized by the cash flow process staying above the true default threshold in either panel. The firm’s strategy results in a positive dividend income stream for the firm’s owner in the start and the end of the horizon. However, it means that between the two mentioned intersections, the firm’s owner prevents the firm from defaulting by injecting new cash. Finally, when the observed cash flow again reaches the level of 0.1366 as at the start of the observation period, the firm pays interests 0.0714, what is is a reduction of 241 basis points compared to the initial interests of 0.0955. This substantial change is the direct result of the rating agency’s updated beliefs, after observing a temporarily low cash flow and ruling out a wide range of cash flow overestimations (see also Figure 1).

5.2 Ex-post Rating Inflation

In this section, we focus on the impact of learning on the dynamics of the rating which implies ex-post rating inflation despite the rating agency’s aim to rate as precise and unbiased as possible. The ex-post rating inflation is a direct consequence of Bayesian directional learning that, via the channel of observing cash injection decisions over time and the subsequent updating of the prior, implies an above-average assessment of a firm approaching its true default threshold.

Figure 4 illustrates the best responses of the rating agency and firm, respectively, for a sample path of an only mildly underestimated cash flow (\( \theta = 0.9 \)). The true cash flow is now 0.1366/0.9 = 0.1518. Note though that the rating agency observes
Figure 4: Best responses (mildly underestimated cash flow). This graph displays the best response of the rating agency as an estimated default threshold, dependent on the imperfectly observed cash flow. The firm initially has a mildly underestimated cash flow with $\theta = 0.9$. In Panel a), the solid black line represents the imperfectly observed cash flow, and the dashed black line its running minimum. The solid gray line is the default threshold estimated by the rating agency, and the dashed gray line is the true default threshold of the firm, which is unobservable to the rating agency. In Panel b), the black line represents the firm’s privately observed true cash flow, and the solid gray line is the firm’s interest payment. The dashed gray line indicates the default threshold.

again the cash flow of 0.1366, just as in the underestimated cash flow case. Therefore it again assigns an identical rating, although the true cash flow is much lower in this scenario. The firm’s best response to the rating agency’s strategy is not to default for the first four years illustrated in the graph, visualized by the cash flow process
staying above the true default threshold in either panel. As Panel b) shows, the firm’s strategy results in a positive dividend income stream for the firm’s owner only during the first half year. In contrast, after the intersection of the true cash flow (black line) and the interest payment (gray line) and until the default point in the fourth year, it means that the firm’s owner prevents the firm from defaulting by injecting new cash, also implying signaling of quality and the rating agency’s adjustment of default thresholds.

Figure 4 indicates what we coin ex-post rating inflation. As the minimum observed firm cash flow deteriorates, there will be a time from which on the rating agency will unconsciously inflate the firm’s ratings until the end of the rating game. In Panel 4a, this corresponds to the predicted default threshold falling below the true default threshold from around year 3 onwards. The reason is the rating agency’s Bayesian directional learning, which results in a predicted default threshold corresponding to the expected average of the potentially surviving types, see Eq. (12). For a fixed type $\theta$ with default threshold $f(\theta)$, the predicted default threshold $g(E)$ is smaller than or equal to the true threshold if $g(E_t) \leq f(\theta)$. We capture this event by a stopping time $T_f(\theta)$ defined as $T_f(\theta) = \inf\{t \geq 0 : g(E_t) \leq f(\theta)\}$. From this time $T_f(\theta)$ on, the predicted default threshold is lower than or equal to the true one, that is, $\hat{D}_t^* \leq f(\theta)$, for $T_f(\theta) < t < \tau(\theta)$, and $T_f(\theta) < \tau(\theta)$ almost surely. Accordingly, the rating of the firm for type $\theta$ is inflated from $T_f(\theta)$ on, that is in $[T_f(\theta), \tau(\theta)]$.

Note that this inflation persists if the firm recovers. Not only will the firm have to pay lower coupons if it stays in a regime of low current cash flow. In case the cash flow recovers, the firm enjoys the well-deserved relief in debt payments given the higher cash flow, but also an additional relief due to the persistent inflation. We refer to this property as ex-post rating inflation, as the persistent inflation kicks in at the stopping time $T_f(\theta)$ not known by the rating agency, since the true $\theta$ is not known until default. At default, the true measurement error is revealed. In contrast to reduced-form intensity models, in which default occurs by chance, in this paper default arises from non-observable firm characteristics.

We emphasize that the ex-post rating inflation occurs even though the rating agency has an unbiased objective function. In contrast, the rating agency literature and public discussion typically focus on rating inflation as a result of distorted
incentives or conflicts of interest. In Appendix A, we analyze the effect of a biased objective function. There, we show how a more issuer-friendly rating policy leads to lower default thresholds and the expected amplification of rating inflation. Interestingly, we also show that the most conservative rating policy eliminates ex-post rating inflation and induces the firm to default as under perfect information, though at the cost of increasing the firm’s debt service.

5.3 Impact of Asymmetric Information Distribution on Value

This section quantifies the impact of the distribution of the measurement error on firm value. The two cases under consideration are a mature firm featuring low cash flow uncertainty, as introduced earlier in the base case, and a startup firm. The latter does not only display a higher cash flow risk, but also a substantial skewness. With a large probability, the firm will turn out as a lemon, and the startup fails. With a small probability, the firm will show tremendous potential, allowing for a much enhanced learning for the capital lender. In such an economic environment, the firm usually has no track record out of previous financing, and it will typically neither have access to the public bond market nor be rated by a major rating agency. In this case, the rating agency can be also interpreted as a financial intermediary investing in venture capital and assessing the startup firm with an internal rating.

Figure 5 presents the measurement error distributions for the two cases: Panel 5a
features the mature firm introduced earlier in the base case, with \( \Theta = [\theta, \bar{\theta}] = [0.50, 1.50] \). It has \( \mu_\theta = 1.0 \) and \( \sigma_\theta = 0.25 \), that is, a symmetric distribution with moderate risk. Panel 5b shows the distribution for the startup firm. It has a wider range of \( \Theta = [\theta, \bar{\theta}] = [0.15, 1.50] \), with \( \mu_\theta = 1.2 \) and \( \sigma_\theta = 0.5 \), which implies both a much higher risk as well as a substantial skewness. The extreme distribution implies much higher uncertainty regarding the measurement errors. Much of the probability mass implies moderate overestimation, but a small fraction represents massive underestimation of the true cash flow. In either case the rating agency is initially unbiased, i.e. \( \mathbb{E}[\hat{\theta}] = 1 \).

Figure 6 displays the firm value for a fixed true cash flow depending on the realization of the type \( \theta \), for both the mature and the startup firm. Consider first the mature firm. Panel a) corresponds to an intermediate rating in the perfect information case, whereas Panel b) corresponds to a firm operating exactly at the default barrier in the perfect information case. For the startup firm, Panels c) and d) display the same information.

In general, we observe that the expected equity value increases when comparing the firm value under perfect information with the expected firm value under imperfect information, independent if the firm is mature or a startup. In both Panels a) and c), the firm may benefit or be disadvantaged compared to the perfect information case, since the minimum value is smaller and the maximum value is greater than the perfect information firm value, respectively.

In comparison to the mature firm, the gap between the perfect information case and the expected firm value under imperfect information is larger for the startup firm. This value gap exists for both intermediate ratings (Panels a) and c)), and for firms at the point of default (Panels b) and c), respectively). The skewness of the measurement error distribution drives this effect: For the symmetric distribution of the mature firm, the learning potential is limited; the rating agency has either over- or underestimated the firm, but the error size is likely to be moderate. For the startup firm, the learning potential is sizable. Even for a firm at the estimated point of bankruptcy, it is possible that the rating agency realizes by observing non-default that the firm is extremely valuable, because the owner keeps injecting cash.
5.4 Empirically Testable Implications

The strategic interaction between the firm and the debt market naturally leads to hypotheses that are based on the firm’s signaling. Both default probabilities and expected recovery rates and thus credit spreads are affected by asymmetric information and its dynamic mitigation through the feedback effect. Specifically, a firm’s survival by equity injection in distress should mitigate part of the asymmetric information, as bad firms are ruled out. Thus, the first empirical implication of our model is that equity injection in distress reduces subsequent credit spreads for non-defaulted firms. Moreover, the second empirical implication of our model is that equity injection in distress should lead to less pronounced surprise effects upon
default.

Our empirical predictions are broadly consistent with Duffie and Lando (2001), i.e., increasing information asymmetry leads to an increase in the credit spread. This was already studied extensively by Yu (2005) and Lu et al. (2010) among others. The new angle for an empirical test based on our theory is that equity injection in distress reduces information asymmetry, as bad firms are ruled out (first empirical implication). When testing the size of surprise effect upon default (second empirical implication), we complement Jankowitsch et al. (2014), who document a significant downward jump of bond prices on the default day. We predict that the jump size is mitigated for firms that experienced an equity injection before the default event.

Our theory suggests to focus on equity injections such as rights issues and private placements that took place in financial distress. Equity injection in distress allows to signal quality to the market and debt holders in particular. We therefore expect to observe a positive performance after the equity injection, evidenced by lower credit spreads. Empirical evidence for our model will complement research on equity issues and stock price performance. The positive impact of equity issues and private placements on the stock market performance is discussed for the U.S., see Hertzel et al. (2002) for private placements and Ursel (2006) for rights offerings, and Singapore, see Tan et al. (2002).

6 Conclusion

This paper analyzes a continuous-time rating game between a rating agency and a rated firm. The main friction in the model is a measurement error, which implies imperfectly observed cash flows. Both the rating agency and the firm’s decision are linked through a feedback effect: The rating influences the firm’s capital costs, to which the rating responds again. The firm’s non-default in periods of apparent distress signals that the observed cash flow is overestimated only to a certain extent. Hence, the rating agency’s optimal strategy is to issue a higher rating for the same current cash flow, if the historical minimum has been sufficiently low. The firm responds to the rating strategy by maximizing its firm value by defaulting at a type-dependent default threshold.
The central mechanism of the game is what we coin Bayesian directional learning. The rating agency rules out types over time by noting that the firm does not default for observed cash flow levels too low for more overestimated types to survive. This one-sided narrowing of the measurement error implies ex-post rating inflation. As the rating agency rules out more and more types, at some point the true type is above the rating agency’s estimate, leaving the rating agency to unconsciously overestimate the measurement error. Furthermore, the uncertainty over measurement errors delays the firm’s default, and enhances the firm’s equity value at the expense of creditors.

The empirical implications of our theory are that (1) equity injection in distress reduces information asymmetry and thus subsequent credit spreads, as bad firms are ruled out, and that (2) the surprise effect upon default shown by earlier studies is mitigated for firms that have experienced an equity injection before the default event.

The paper provides a rich framework for studying feedback effects in dynamic structural models in potential extensions. Our structural model framework allows for a broader concept of asymmetric information by adding ambiguity on the type, so that the rating agency is not only limited by its imperfect observation of the firm’s cash flow, but also uncertain about the exact distribution of the measurement error, which better captures the nature of this vaguer concept. In another direction, our model framework carries over to the valuation of real options, which allows for embedding feedback effects between a firm’s investment policy and its valuation by the capital market.

References


A Robustness Check: Generalized Reputation Costs

So far, the cost rate $k\pi$ in Eq. (8) has been defined such that reputation losses are symmetric for an over- or underestimation of the true default threshold. In reality, the rating agency could be more concerned about making a mistake in either direction. On the one hand, consider the case that its objective function is not only driven by the goal of building up a reputation for precise ratings, but also its revenue originates
from the rated entities as in the issuer-pays model. Then, the rating agency will be particularly concerned about avoiding an overestimation of the true default threshold \(\hat{D}_t^* > f(\theta)\). Such an overestimation leads to a higher cost of debt, thus hurts the issuers and, if they have a choice (rating shopping) among several rating agencies applying different degrees of overestimation, potentially lead to lower fee income for the more overestimating agency. On the other hand, regulatory authorities will typically be particularly concerned about avoiding an underestimation of the true default threshold \(\hat{D}_t^* < f(\theta)\). An underestimation leads to more defaults happening without the appropriate warning signs and can thus be dangerous especially in economic downturns. Thus, regulators might want to incentivize rating agencies rather to over- than underestimate the true default threshold. The investors’ preferences regarding over- or underestimation are ambiguous. If debt is fairly priced and the investors can obtain adequate compensation, then they should be indifferent regarding the rating agency’s assessment. In our framework, we assume that investors and rating agency have the same information level. Thus, the debt investors can just ask for the appropriate cost of debt, regardless of which scheme the rating agency applies. If debt investors have regulatory disadvantages of holding low-rated debt, they might even prefer an underestimation of the true default threshold and as such collude with the issuers, see Opp et al. (2013). However, in our way of modeling, the cost of debt is determined directly from the rating. When the debt contract is already fixed, then debt investors will prefer an overestimation of the true default threshold, as it yields higher cash flows to debt.

To allow for such alternative incentives for the rating agency, we generalize the cost rate \(k^\pi\) to \(k^\pi(\alpha)\) defined by

\[
k^\pi(\alpha) = \int_{\Theta} Q(\hat{D}_t^*, f(\theta); \alpha) \phi_\pi^t(\theta) d\theta, \quad \text{for } t \geq 0, \tag{25}
\]

with \(Q: \mathbb{R} \times \mathbb{R} \times [0, 1]\) given by \(Q(d, d; \alpha) = 2(1 - \alpha) 1_{d \leq f} (d - f)^2 + 2 \alpha 1_{d > f} (d - f)^2\). For \(\alpha = 0.5\), Eq. (25) reduces to Eq. (8), i.e., the case of symmetric reputation losses. For \(\alpha = 0\), reputation costs are only driven by underestimation of the default threshold \(\hat{D}_t^* \leq f(\theta)\). In that case, the rating agency can avoid any costs by setting the rating \(D_t^*\) such that \(D_t^* > f(\theta)\) for all \(\theta\) with non-zero belief, i.e. for
all $\theta$ with $f(\theta) < E_t$. The rating agency’s best response then changes from (12) in Proposition 2 to $g_0(e; f) = \min(e, \sup_{\theta \in \Theta} f(\theta))$, for $e \geq 0$. The interpretation of the case $\alpha = 0$ is that the rating agency plays safe by overestimating the default threshold and therefore the default risk by assuming the worst case. This most conservative rating leads to higher interest payments for the firm and hence to earlier defaults accelerating the rating agency’s learning. A further consequence of the most conservative rating approach is that the firm’s credit risk at default is assessed accurately. Accordingly, firms default at the perfect information default threshold regardless of the observed type. While firms are still in business, they make higher interest payments than under perfect information, where the interest differential is increasing for decreasing type. Accordingly, the firm has no benefit from the information asymmetry. In the worst case specification, debt holders benefit at the expense of equity holders.

For $\alpha = 1$, reputation costs are only driven by overestimation of the default threshold ($\hat{D}_t^* > f(\theta)$). The rating agency can avoid any costs by setting the rating $D_t^*$ such that $D_t^* \leq f(\theta)$ for all $\theta$ with non-zero belief, i.e. for all $\theta$ with $f(\theta) < E_t$. The rating agency’s best response then changes from (12) in Proposition 2 to $g_1(e; f) = \min(e, \inf_{\theta \in \Theta} f(\theta))$, for $e \geq 0$. The interpretation of $\alpha = 1$ is that the rating agency underestimates the default threshold and therefore the default risk by assuming the best case. As a consequence, the firm has the full benefit from the information asymmetry. The rating is always better than under perfect information. Accordingly, firms pay less interest than under perfect information, where the interest differential is increasing for increasing type, and firms delay default compared to perfect information regardless of the observed type. In the most progressive rating specification, equity holders benefit at the expense of debt holders.

The equilibria for both cases, most progressive and most conservative, respectively, are obtained in a similar fashion as in Proposition 3 and Proposition 4. In fact, the derivation is much simpler, as the best response of the rating agency has a simple and explicit structure and is therefore omitted here.

Figure 7 displays the impact of alternative incentives of the rating agency on the strategies in equilibrium. Panel 7a shows the equilibrium strategy of the rating
agency: The solid line displays the rating agency’s strategy. The dotted and dashed lines represent the most progressive and conservative cases, respectively. The rating agency that maximally overestimates the type learns much earlier compared to the base case. In contrast, the most progressive rating agency always assumes the best case, that is, it maximally underestimates the type. As the firm subject to the most serious underestimation of type defaults last, the rating agency never updates its estimate and hence does not learn.

Panels 7b and 7c show the firm’s strategy for a maximally progressive and conservative rating agency in both the observed cash flow (Panel 7b), and the true cash flow (Panel 7c), respectively. In Panel 7b, we see the firm default thresholds as a function of type, for the base case (solid line) and maximally over- (dotted) and underestimated cash flow (dashed). A firm which faces a progressive rating agency defaults for a lower observed cash flow, because it profits more from the reduced interest payments. A firm facing a conservative rating agency defaults for a higher observed cash flow, as the cash flow underestimation and the subsequent lower rating and adverse interest payments make it more attractive to stop injecting additional liquidity. In Panel 7c, we show the firm’s strategy in terms of the true cash flow. Comparing the firm’s equilibrium strategy in terms of observed and true cash flow, we observe that for a given type, the ordering of the default thresholds is the same, that is, the firm defaults earliest facing a conservative rating agency, followed by the unbiased and the progressive case. This is intuitive, as the more conservative the rating agency, the higher is the cost of debt for a firm of any given type (except for the lowest possible type). However, the slopes of the default thresholds as a function of type are opposite for the observed and true cash flow representation. This is due to the fact that the measurement error θ enters as the scaling factor of the observed cash flow $D$.

When analyzing the unbiased rating strategy in Section 5.2, we have pointed out that it results in lower predicted default thresholds than the true default threshold of a fixed type from some type-dependent time before actual default, which we coin ex-post rating inflation. For the generalized reputation cost function proposed in the current section, Panel 7c illustrates that ex-post rating inflation is only eliminated for the most conservative rating agency ($\alpha = 0$). Such an agency’s policy makes the
firm default at the perfect-information default threshold regardless of the observed type. Otherwise, all types and all reputation cost functions except for the one with $\alpha = 0$ induce ex-post rating inflation to varying degrees.
Figure 7: Panel a displays the equilibrium strategy of the rating agency $g^*$ for the base case (solid black line) and the most progressive and conservative rating agency (dotted and dashed line) depending on the smallest imperfectly observed cash flow $e$. Also, the default threshold under perfect information is included (gray line). Panel b displays the equilibrium strategy of the firm $f^*$ in terms of the default threshold depending on the type $\theta$ for the base case (solid line) and the most progressive and conservative rating agency (dotted and dashed lines). Panel c displays the equilibrium strategy of the firm $f^*$ in terms of the default threshold in the firm's true cash flow depending on the type for the base case (solid line) and the most progressive and conservative rating agency (dashed and dotted lines), and the equilibrium default threshold in the perfect information case (gray).
For Online Publication

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I Interest Rates and Rating Parametrization

We use Lemma 3.1.2 of Bielecki and Rutkowski (2004),

\[ PD(T) = \mathbb{P} \left( \inf_{t \in [0,T]} X_t \leq f_1^* \right) = \mathbb{P} \left( \inf_{t \in [0,T]} X_t \leq X_0/R \right) \]

\[ = R^{-2} \frac{\mu - \sigma^2/2}{\sigma^2} N \left( \frac{-\ln(R) + (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right) + N \left( \frac{-\ln(R) - (\mu - \sigma^2/2)T}{\sigma \sqrt{T}} \right), \]

where \( N \) is the standard normal cumulative distribution function, \( f_1^* \) is the firm’s default threshold in the case in which the rating agency can observe the cash flow perfectly, i.e. \( \Theta = \{1\} \), and \( R = X_0/f_1^* \) is the distance-to-default type rating associated with the given default probability \( PD(T) \), for time horizon \( T > 0 \). When we solve this equation for each \( PD(10) \), with \( i \) denoting the rating class, we obtain the corresponding rating \( R_i \) in our scale. The results are given in Table I.1. The values of \( C \) on \([1, \infty)\) are obtained by linear interpolation and extrapolation on the log-scale.

Figure I.1 illustrates Assumption 2 in light of the above interest rate structure. Panel I.1a displays the average interest payment rate \( C \) as a function of default probability \( PD \). Figure 2 in the main text displays the average interest payment rate \( C \) as a function of rating \( R \). If the rating turns bad, that is, if the rating implies a high default probability, then the firm’s interest payments increase sharply. On the other hand, if the firm is far away from default, a further increase in the rating has only a small effect on the firm’s interest payments. Panel I.1b draws the slope on the log-log scale of the interest payment rate function \( C \) depending on \( R \) based on the values given in Table I.1. In particular, the slope in the log-log scale lies in \([-L_C, 0)\) with \( L_C = 0.8989 \) and thus satisfies Assumption 2.

For this specification, the interest payment rate determining function \( C \) is calculated under the assumption of perfect information, i.e. \( D = X \), or \( \bar{\theta} = 1 \) and \( \Theta_1 = \{1\} \), from available market data. For seven rating classes AAA, AA, A, BBB, BB, B and C/CCC, the interest rate \( C_i \) is determined by the average effective yield for each rating class, i.e., averaging over the time period 01 January 1997 through 31
For each rating class $i$, the distance-to-default type rating $R_i$ is extracted from the 10-year probability of default $PD_i$, which is collected from Table 25 of Standard and Poor’s (2016) based on U.S. data in the period 1981 through 2015. The risk-free rate $r$ is set to 0.0211, which is the average 3-month T-Bill rate (DTB3) over the time period 01 January 1997 through 31 December 2016 provided by Federal Reserve Economic Data.

Consider the perfect information case as benchmark, i.e. $\Theta_1 = \{1\}$. In this case, the equilibrium $(f^*_1, g^*_1)$ is given by $f^*_1 = g^*_1 = 0.0490$. Comparing this number to Table I.1, we see that in the base case a company being rated B or better is having a cash flow $X$ of $f^*_1 R_B = 0.0490 \times 3.9336 = 0.1928$ or greater. This cash flow exceeds the interest payment rate $C$, which amounts to 0.0916 for a B rated firm or is smaller for a firm with a better rating. However, for the C/CCC rated firm, the corresponding cash flow amounts to $f^*_1 R_{C/CCC} = 0.0490 \times 2.2507 = 0.1103$, which is not sufficient.

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14 The respective Federal Reserve Economic Data identifiers for yield data are BAMLC0A1CAAAY, BAMLC0A2CAAAY, BAMLC0A3CAEY, BAMLC0A4CBBBEY, BAMLH0A1HYBBEY, BAMLH0A2HYBEY, and BAMLH0A3HYCEY. Note that the data for calibration stems predominantly from non performance-sensitive debt instruments with a higher default risk than comparable PSD instruments. However, the higher credit risk of non performance-sensitive debt instruments is perhaps accompanied by a higher interest rate rewarding for the additional risk. Thus, the effect of using non performance-sensitive debt instruments for calibration does not seem to be critical.

15 This case is an equilibrium, see Proposition 4.
### Table I.1: Rating-Dependent Interest Rate

<table>
<thead>
<tr>
<th>Rating Class</th>
<th>10-Year Default Probability</th>
<th>Implied Rating $R$</th>
<th>Interest Rate $C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>0.0086</td>
<td>18.2180</td>
<td>0.0439</td>
</tr>
<tr>
<td>AA</td>
<td>0.0109</td>
<td>16.8281</td>
<td>0.0447</td>
</tr>
<tr>
<td>A</td>
<td>0.0195</td>
<td>13.7125</td>
<td>0.0494</td>
</tr>
<tr>
<td>BBB</td>
<td>0.0464</td>
<td>9.7811</td>
<td>0.0573</td>
</tr>
<tr>
<td>BB</td>
<td>0.1527</td>
<td>5.5756</td>
<td>0.0732</td>
</tr>
<tr>
<td>B</td>
<td>0.2746</td>
<td>3.9336</td>
<td>0.0916</td>
</tr>
<tr>
<td>C/CCC</td>
<td>0.5584</td>
<td>2.2507</td>
<td>0.1513</td>
</tr>
</tbody>
</table>

**Notes:** The 10-year default probabilities are obtained from Table 25 in Standard and Poor’s (2016) and are referring to rated US companies in the period 1981 to 2015. The implied rating is determined using (26). The interest rates are the average of effective corporate yields provided by Federal Reserve Economic Data.

to cover the interest of 0.1513. In this case, the equity holders keep the company alive by injecting additional funds. If the cash flow drops further and reaches the default threshold $f_{1}^{\star} = 0.0490$, the firm defaults on its debt and creditors receive the assets.
II Proof of Main Results

Definition 1 (Perfect Bayesian Equilibrium in Markov Strategies). Strategies \((\tau^\star, \hat{D}^{\star\star})\) and beliefs \(\pi^\star\) constitute a perfect Bayesian equilibrium in Markov strategies (PBEM) if:

1. For every \(0 \leq t < \tau^\star\), \(\theta \in \Theta\), and strategy \(\tau(\theta)\)
   \[
   \mathbb{E} \left[ \int_t^{\tau^\star(\theta)} e^{-rs} \left( D_s/\theta - C(D_s/\hat{D}^{\star\star}_s) \right) ds \bigg| \mathcal{F}_t \right] \geq \mathbb{E} \left[ \int_t^{\tau^\star(\theta)} e^{-rs} \left( D_s/\theta - C(D_s/\hat{D}^{\star\star}_s) \right) ds \bigg| \mathcal{F}_t \right].
   \]

2. For every \(0 \leq t < \tau^\star\) and strategy \(\hat{D}^\star\)
   \[
   -\mathbb{E} \left[ \int_t^{\tau^\star} e^{-\rho s} \int_\Theta (\hat{D}^{\star\star}_s - \mathbb{E} \left[ D_{\tau^\star(\theta)} \big| \mathcal{G}_s \right])^2 \phi_s^{\pi^\star}(\theta) d\theta ds \bigg| \mathcal{G}_t \right],
   \]
   \[
   \geq - \mathbb{E} \left[ \int_t^{\tau^\star} e^{-\rho s} \int_\Theta (\hat{D}^{\star\star}_s - \mathbb{E} \left[ D_{\tau^\star(\theta)} \big| \mathcal{G}_s \right])^2 \phi_s^{\pi^\star}(\theta) d\theta ds \bigg| \mathcal{G}_t \right].
   \]

3. Bayes rule is used to update beliefs \(\pi^\star\) with density \((\phi_s^{\pi^\star})_{t \geq 0} = (L_t^{\pi^\star} \phi)_{t \geq 0}\) whenever possible: For every \(t \geq 0\), if there exists \(\theta_0 \in \Theta\) such that \(t < \tau^\star(\theta_0)\), then
   \[
   \phi_s^{\pi^\star}(\theta) = \frac{\phi_i^{\pi^\star}(\theta) 1_{t < \tau^\star(\theta)}}{\int_\Theta \phi_i^{\pi^\star}(\theta') 1_{t < \tau^\star(\theta')} d\theta'}, \text{ for } \theta \in \Theta,
   \]
   where \(\phi_0^{\pi^\star} = \phi\), i.e. \(L_0^{\pi^\star}(\theta) = 1\), for \(\theta \in \Theta\).

4. The strategies are Markov, i.e.:
   \[
   \tau^\star(\theta) = \inf \{ t \geq 0 : (D_t, E_t) \in \mathcal{E}(\theta) \}, \text{ for a Borel set } \mathcal{E}(\theta) \subseteq \mathbb{R}^+, \theta \in \Theta,
   \]
   \[
   \hat{D}^{\star\star}_t = g(D_t, E_t), \text{ for some function } g : \mathcal{C} \rightarrow \mathbb{R}_0^+ \text{ for } 0 \leq t < \tau^\star.
   \]

Proof of Proposition 1. The function \(f \in \mathcal{A}_f\) specifying the firm’s strategy is given, which is then \(\tau = (\tau(\theta))_{\theta \in \Theta}\) with \(\tau(\theta) = \inf \{ t \geq 0 : D_t \leq f(\theta) \}\), for \(\theta \in \Theta\). The
structure of the rating agency’s belief $\pi$ that is consistent with the firm strategy $f$ as given in (11) follows from Bayes’ rule. Using the consistent belief $\pi$, the rating agency maximizes the respective utility, i.e.

$$\sup_{g \in A} U_{RA}^\pi(\tau, g(E)) = - \inf_{g \in A} \mathbb{E} \left[ \int_0^\tau e^{-\rho t} \int_{\Theta} (g(E_t) - f(\theta))^2 \phi_\pi^\tau(\theta) d\theta dt \right].$$

The above expression is minimized in case $g(E_t)$ minimizes for each $0 \leq t < \tau$

$$\int_{\Theta} (g(E_t) - f(\theta))^2 \phi_\pi^\tau(\theta) d\theta = \frac{\int_\theta (g(E_t) - f(\theta))^2 1_{f(\theta) < E_t} \phi(\theta) d\theta}{\int_{\Theta} 1_{f(\theta) < E_t} \phi(\theta) d\theta} = \mathbb{E} \left[ (g(e) - f(\tilde{\theta}))^2 | f(\tilde{\theta}) < e \right] |_{e=E_t}. $$

In fact, we look for $g(E_t)$ which minimizes the squared distance to the random variable $f(\tilde{\theta})|_{f(\tilde{\theta}) < E_t}$, which is a function $f$ of the random variable describing the type $\tilde{\theta}$ based on the consistent belief $\pi_t$. Therefore, the optimal $g(E_t; f)$ has to be the expected value of $f(\tilde{\theta})|_{f(\tilde{\theta}) < E_t}$, which is in essence (12). This result can also be obtained by solving the first order condition and checking the second order condition for a minimum. For the irrelevant case $t \geq \tau$, or, equivalently, $E_t < \inf_{\theta \in \Theta} f(\theta)$, we set the critical default level at $g(e; f) = e$, for $e \leq \inf_{\Theta \in \Theta} f(\theta)$. Thus, default is predicted to happen immediately as it should have been occurred before. Clearly, $g(\cdot; f) \in A_g$.

From (12) we see that $g(\cdot; f)$ is bounded by $Id$, i.e. $g(e; f) \leq e$, for $e \geq 0$, and that $g(\cdot; f)$ is non-decreasing. Assuming that $f$ is strictly increasing, $f^{-1}$ is well
defined on $f(\Theta)$. Take $e > e' > \inf_{\theta \in \Theta} f(\theta)$ and then

$$g(e; f) - g(e'; f) = \frac{\int_{\theta < f^{-1}(e)} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e') \phi(\theta) \, d\theta} - \frac{\int_{\theta < f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}$$

$$= \frac{\int_{\theta < f^{-1}(e)} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta} - \frac{\int_{\theta < f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}$$

$$= g(e'; f) - \frac{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta} + \frac{\int_{\theta < f^{-1}(e')} f(\theta) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}$$

$$= \frac{\int_{f^{-1}(e)} (f(\theta) - g(e'; f)) \phi(\theta) \, d\theta}{\int_{\theta < f^{-1}(e')} \phi(\theta) \, d\theta}.$$

The above quantity is non-negative and further strictly positive on the interior of $f(\Theta)$ and converges to zero for $e \searrow e'$. Accordingly, $g(\cdot; f)$ is continuous and strictly increasing as claimed. \(\square\)

The subset of $\mathcal{A}_g$ where the constraints in (5) hold is denoted by $\mathcal{A}_g^C = \{ g \in \mathcal{A}_g : g \text{satisfies (5)} \}$.

**Lemma 1.** Suppose that $g \in \mathcal{A}_g$ is non-decreasing and is bounded by $\text{Id}$, i.e. $g(e) \leq e$, for $e \geq 0$, then $\mathcal{R}(g) \in \mathcal{A}_g^C$. Moreover, $\mathcal{R}(g) = g$, for $g \in \mathcal{A}_g^C$.

**Proof of Lemma 1.** We see immediately $\mathcal{R}(g) \leq g \leq \text{Id}. \ To \ observe \ that \ \mathcal{R}(g)$ is
non-decreasing take \( e' \geq e > 0 \) and
\[
\mathcal{R}(g)(e') = e' \inf\{t(z)/z : 0 < z \leq e'\} \\
= e' \left( \inf\{t(z)/z : 0 < z \leq e\} \wedge \inf\{g(z)/z : e < z \leq e'\} \right) \\
\geq \left( \frac{e'}{e} \mathcal{R}(g)(e) \right) \wedge \left( e' \inf\{g(e)/z : e < z \leq e'\} \right) \\
= \left( \frac{e'}{e} \mathcal{R}(g)(e) \right) \wedge g(e) \\
= \mathcal{R}(g)(e) + \left( \frac{e' - e}{e} \mathcal{R}(g)(e) \right) \wedge (g(e) - \mathcal{R}(g)(e)) \\
\geq \mathcal{R}(g)(e).
\]

By definition, \( \mathcal{R}(g)/Id \) is non-increasing. These two properties rule out negative jumps as well as positive jumps, and hence \( \mathcal{R}(g) \) is continuous on \( \mathbb{R}^+ \). To check the continuity at 0, we observe that 0 \( \leq \mathcal{R}(g)(e) \leq e \), and thus \( \lim_{e \downarrow 0} \mathcal{R}(g)(e) = 0 = \mathcal{R}(g)(0) \), as defined in (13). To address the other claimed properties of \( \mathcal{R}(g) \) observe that by \( g \) being non-decreasing it holds that
\[
\mathcal{R}(g)(e) = e \inf\{g(z)/z : 0 < z \leq e\} \\
\leq e \inf\{g(e)/z : 0 < z \leq e\} = eg(e)/e = g(e).
\]

\( \square \)

Proof of Proposition 2. For \( g \in \mathcal{A}_g^C \) and \( \theta \in \Theta \), the early exercise region associated with the optimal stopping time of (14) is denoted by \( \mathcal{E}(\theta; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) = 0\} \), see also (56). The optimal strategy can be written as first entry time of the state process \((D, E)\) with starting value \((d, d), d \geq 0\), in the early exercise region \( \mathcal{E}(\theta; g) \),
\[
\tau_{(d,d)}(\theta; g) = \inf\{t \geq 0 : (D(t), E(t)) \in \mathcal{E}(\theta; g)\}.
\]
Now, \( g \in \mathcal{A}_g^C \), and we can apply the results of Lemma 7. Recall the definition in (57)

\[
\mathcal{D}(\theta; g) = \{ d \in \mathbb{R}_0^+ : (d, d) \in \mathcal{E}(\theta; g) \}, \quad \text{and} \quad D(\theta; g) = \sup \mathcal{D}(\theta; g).
\]

According to Lemma 7 we have that \( \mathcal{D}(\theta; g) = [0, D(\theta; g)] \). Set \( f(\theta; g) = D(\theta; g) \). The stopping time \( \tau_{(d,d)}(\theta; g) \) is depending on the starting value \( (d, d) \). Consider first the case \( d \leq D(\theta; g) = f(\theta; g) \). Then \( d \in \mathcal{D}(\theta; g) \) and by the definition of \( \mathcal{D}(\theta; g) \) we conclude that \( (d, d) \in \mathcal{E}(\theta; g) \). Accordingly, we start in the early exercise region and stop immediately at 0, i.e. \( \tau_{(d,d)}(\theta; g) = 0 \). Since we also have \( D_0 = d \leq D(\theta; g) = f(\theta; g) \), (20) holds true. Now, consider \( d > D(\theta; g) = f(\theta; g) \). The starting value \( (d, d) \) of \( (D, E) \) is not in \( \mathcal{E}(\theta; g) \). The process \( (D, E) \) has continuous paths and \( E \) is the running minimum of \( D \). Thus, if \( (D, E) \) decreases in the second component then it has to travel through the diagonal \( \mathcal{D} \). Since \( \{(d, d) : d \in \mathcal{D}(\theta; g)\} \subseteq \mathcal{E}(\theta; g) \), we have that

\[
\tau_{(d,d)}(\theta; g) = \inf\{ t \geq 0 : (D(t), E(t)) \in \mathcal{E}(\theta; g) \} \\
\leq \inf\{ t \geq 0 : (D_t, E_t) \in \{(d, d) : d \in \mathcal{D}(\theta; g)\} \} \\
= \inf\{ t \geq 0 : D_t \leq f(\theta; g) \}.
\]

Suppose now that \( \tau_{(d,d)}(\theta; g) < \inf\{ t \geq 0 : D_t \leq f(\theta; g) \} \) with some non-negative probability. Thus \( (D, E) \) has to hit \( \mathcal{E}(\theta; g) \) with some non-negative probability before it eventually hits \( (f(\theta; g), f(\theta; g)) \). Then there exists an \( (d', e') \in \mathcal{E}(\theta; g) \) with \( e' > f(\theta; g) \). However, this would contradict (59). Therefore, \( \tau_{(d,d)}(\theta; g) = \inf\{ t \geq 0 : D_t \leq f(\theta; g) \} \) almost surely and (20) holds also in this case. \( \square \)

For our specific default barrier, we can provide more structure on how the barrier changes in the type in Lemma 2.

**Lemma 2.** Given the setting of Proposition 2 and let \( \theta, \theta' \in \Theta \) with \( \theta' \leq \theta \), then

\[
f(\theta'; g) \leq f(\theta; g), \quad \text{and} \quad f \leq \frac{f(\theta; g)}{\theta} \leq \frac{f(\theta'; g)}{\theta'} \leq \tilde{f},
\]

(27)
uniformly in g, where \(0 < f \leq \tilde{f} < \infty\). In particular, \(f(\cdot;g)\) is Lipschitz continuous.

\[
|f(\theta; g) - f(\theta'; g)| \leq L_f |\theta - \theta'|, \text{ for } \theta, \theta' \in \Theta,
\]

where \(L_f = \tilde{f} > 0\) is the uniform Lipschitz constant for all \(g\). Moreover, suppose that Assumption 2 holds, then for \(\theta, \theta' \in \Theta\) with \(\theta' \leq \theta\) it holds that

\[
f(\theta; g) - f(\theta'; g) \geq l_f (\theta - \theta'),
\]

where \(l_f = (1 - L_C) \tilde{f} > 0\) is the uniform constant for all \(g\).

Proof of Lemma 2. For \(\theta, \theta' \in \Theta\), with \(\theta' \leq \theta\), and \(g \in \mathcal{A}_g^C\) is non-decreasing and bounded by \(Id\), we have \(v(\cdot, \cdot; \theta', g) \geq v(\cdot, \cdot; \theta, g)\) by part 2. of Lemma 6 and \(v(\cdot, \cdot; \theta', g), v(\cdot, \cdot; \theta, g) \geq 0\) by part 1. of Lemma 6. From this, the corresponding early exercise regions \(\mathcal{E}(\theta; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) = 0\}\) and \(\mathcal{E}(\theta'; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta', g) = 0\}\), see also (56) for definition, satisfy \(\mathcal{E}(\theta'; g) \subseteq \mathcal{E}(\theta; g)\). This holds in particular on the diagonal \(\mathcal{D}\), i.e. \(\mathcal{E}(\theta'; g) \cap \mathcal{D} \subseteq \mathcal{E}(\theta; g) \cap \mathcal{D}\), and \([0, D(\theta'; g)] = \mathcal{D}(\theta'; g) \subseteq \mathcal{D}(\theta; g) = [0, D(\theta; g)]\), thus \(D(\theta'; g) \leq D(\theta; g)\). Identifying \(f(\theta'; g) = D(\theta'; g)\) and \(f(\theta; g) = D(\theta; g)\) as is done in the proof of Proposition 2 gives \(f(\theta'; g) \leq f(\theta; g)\), establishing the first part of (27). For the second part, we rewrite (20) of Proposition 2 in the firm scale \((x, y)\). With \((d, e) = \theta(x, y)\), we obtain \(\tau_{(d,e)}(\theta; g) = \inf\{t \geq 0 : X_t \leq f(\theta; g)/\theta\}\), and \(f(\theta; g)/\theta\) is the default barrier in the firm scale, for \(\theta \in \Theta\). Regardless of the rating strategy \(g\), the interest payment rate function \(C\) is bounded from below by \(C\) and from above by \(\bar{C}\). Denote by \(\tilde{f}\) the default threshold in the firm scale for the constant interest \(C\) (i.e. by setting \(g = 0\)) and by \(\bar{f}\) the default threshold in the firm scale for the constant interest \(\bar{C}\) (i.e. by setting \(\bar{g} = \infty\), which is interpreted as limiting case, and noting that \(C\) is as in Lemma 6 extended to \([0, \infty]\) by setting \(C|_{(0, 1)} = C(1)\)). From \(g \leq \bar{g} \leq \bar{g}\\) we obtain with part 3. of Lemma 6 that \(w(\cdot, \cdot; \theta, g) \geq w(\cdot, \cdot; \theta, \bar{g}) \geq w(\cdot, \cdot; \theta, \bar{g})\), and hence \(\mathcal{E}(\theta; g) \subseteq \mathcal{E}(\theta; \bar{g}) \subseteq \mathcal{E}(\theta; \bar{g})\), where \(\mathcal{E}(\theta; \bar{g}) = \{(x, y) \in \mathcal{C} : w(x, y; \theta, \bar{g}) = 0\}\), for \(g' \in \bar{g}, \bar{g}\). The boundary cases \(g\) and \(\bar{g}\) also admit a critical default level, which is given by \(\tilde{f} = \frac{\eta}{(1 + \eta) r} C\) and \(\bar{f} = \frac{\eta}{(1 + \eta) r} \bar{C}\), respectively, where \(\eta = \frac{b - \frac{1}{2} \sigma^2}{\sigma^2} + \sqrt{\left(\frac{b - \frac{1}{2} \sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 0\), see, e.g.,
Equation (C.5) in Manso (2013). And thus \( f \leq f(\theta; g)/\theta \leq \tilde{f} \), for all \( \theta \in \Theta \) and \( g \in \mathcal{A}_g^{LC} \) that are non-decreasing and bounded by \( Id \). Part 2. of Lemma 6 gives \( w(\cdot, \cdot; \theta', g) \leq w(\cdot, \cdot; \theta, g) \), yielding \( f(\theta; g)/\theta \leq f(\theta'; g)/\theta' \), what completes the second part of (27). Finally, recall that \( \theta' \leq \theta \) and use the just established results given in (27) to see

\[
0 \leq f(\theta; g) - f(\theta'; g) = \theta \frac{f(\theta; g)}{\theta} - f(\theta'; g) \leq \theta \frac{f(\theta'; g)}{\theta'} - f(\theta'; g) = (\theta - \theta') \frac{f(\theta'; g)}{\theta'} \leq (\theta - \theta') \tilde{f},
\]

what proves (28). Now, assume that Assumption 2 holds. For \( \theta, \theta' \in \Theta \), with \( \theta' \leq \theta \), and \( g \in \mathcal{A}_g^{LC} \) is non-decreasing and bounded by \( Id \), and \( (d, e) \in \mathcal{C} \). From part 1. of Lemma 5 we see \( g_{\theta'} \leq \frac{\theta}{\theta'} g_\theta \) and since \( C \) is non-increasing \( C(x/g_{\theta'}(y)) \leq C((\theta'/\theta)x/g_\theta(y)) \), for \( (x, y) \in \mathcal{C} \). Assumption 2 gives

\[
C(z) \leq C(z') \leq (z/z')^{LC} C(z), \text{ for } 1 \leq z' \leq z.
\]

With \( z' = (\theta'/\theta)x/g_\theta(y) \) and \( z = x/g_\theta(y) \), it holds that \( z' \leq z \), and we obtain that \( C((\theta'/\theta)x/g_\theta(y)) \leq (\theta/\theta')^{LC} C(x/g_\theta(y)) \), and thus

\[
C(x/g_{\theta'}(y)) \leq (\theta/\theta')^{LC} C(x/g_\theta(y)), \text{ for } (x, y) \in \mathcal{C}.
\]

Consider the optimal stopping problem in (14), but now scaled by \( \hat{\theta} = (\theta/\theta')^{LC} \geq 1 \),

\[
\bar{v}(d, e; \hat{\theta}, g_\theta) = \sup_{\tau \in \mathcal{T}(d,e)} \mathbb{E}_{(d,e)} \left[ \int_0^\tau e^{-rt} \left( D_t - \hat{\theta} C(D_t/g_{\theta}(E_t)) \right) \, dt \right],
\]

or, noting that \( (E, D) \) has the same distribution as \( (X, Y) \) provided the starting values are identical, we can express this alternatively

\[
\bar{v}(x, y; \hat{\theta}, g_\theta) = \sup_{\tau \in \mathcal{T}(x,y)} \mathbb{E}_{(x,y)} \left[ \int_0^\tau e^{-rt} \left( X_t - \hat{\theta} C(X_t/g_{\theta}(Y_t)) \right) \, dt \right],
\]

\[\xi\]
and as in (50)

\[ w(x, y; \theta', g) = \sup_{\tau \in \mathcal{S}_{x, y}} \mathbb{E}_{(x, y)} \left[ \int_0^\tau e^{-r t} (X_t - C(X_t / g(\theta'(Y_t)))) \, dt \right], \]

By Proposition 2, the latter two optimal stopping problems admit a critical default level for describing the optimal default time. The value function \( \tilde{v}(\cdot, \cdot; \hat{\theta}, g_\theta) \) is in the rating agency-scale \((D, E)\) and the critical level is given by \( f(\hat{\theta}; g_\theta) \). The value function \( w(\cdot, \cdot; \theta', g) \) is in the firm-scale \((X, Y)\) and critical default level is given by \( f(\theta'; g) / \theta' \). Recalling that the interest payments are ordered uniformly, see (30), we can order the critical default levels, as a higher interest payment leads to a higher critical default value, and hence

\[ f(\hat{\theta}; g_\theta) \geq f(\theta'; g) / \theta'. \]

Now, focus on \( f(\hat{\theta}; g_\theta) \) and recall that \( \hat{\theta} = (\theta / \theta')^{L_C} \geq 1 \), and since \( f(\cdot; g) / Id \) is decreasing by (27)

\[ f(\hat{\theta}; g_\theta) = \hat{\theta} \frac{f(\hat{\theta}; g_\theta)}{\hat{\theta}} \leq \hat{\theta} \frac{f(1; g_\theta)}{1} = \hat{\theta} f(1; g_\theta) = \hat{\theta} \frac{f(\theta; g)}{\theta}, \]

where the last step follows from plugging \((1, g_\theta)\) in (14) and comparing this with (50), to see that \( f(1, g_\theta) \) is also the optimal default level in firm-scale for \((\theta, g)\), what is \( f(\theta; g) / \theta \). Using this, we find a lower bound to

\[ f(\theta; g) - f(\theta'; g) \geq f(\theta; g) - \theta' f(\hat{\theta}; g_\theta) \geq f(\theta; g) - \theta' \frac{\hat{\theta}}{\theta} f(\theta; g) \]
\[ = f(\theta; g) \left( 1 - (\theta' / \theta)^{1-L_C} \right) \geq f(\theta; g) \left( 1 - (\theta' / \theta)^{1-L_C} \right). \]

Taking the limit gives

\[ D_+ f(\theta'; g) = \liminf_{\theta \searrow \theta'} \frac{f(\theta; g) - f(\theta'; g)}{\theta - \theta'} \geq \liminf_{\theta \searrow \theta'} \frac{f(\theta; g) \left( 1 - (\theta' / \theta)^{1-L_C} \right)}{\theta - \theta'} = (1 - L_C) f, \]
implying the claimed inequality.  

\[ \square \]

**Proof of Proposition 3.** A fixed point is obtained by the Schauder fixed point theorem: Let \( \mathcal{K} \) be a nonempty convex compact subset of a Banach space \( \mathcal{V} \), if \( \tilde{T} : \mathcal{K} \to \mathcal{K} \) is continuous, then \( \tilde{T} \) has a fixed point. Set \( \mathcal{V} = C(\Theta, \mathbb{R}) \times C(\Xi, \mathbb{R}) \) endowed with the sup-norm, i.e. \( \| (f, g) \|_{\infty} = \max(\| f \|_{\infty}, \| g \|_{\infty}) \), for \( (f, g) \in \mathcal{V} \), where \( \| f \|_{\infty} = \sup_{\theta \in \Theta} |f(\theta)| \), \( \| g \|_{\infty} = \sup_{\xi \in \Xi} |g(\xi)| \), and the set \( \Xi = [\xi, \bar{\xi}] \) is defined by \( \xi = \theta \cdot f \) and \( \bar{\xi} = \frac{\theta^2 \cdot f}{(\theta \cdot f)} \). Since \( \Theta \) and \( \Xi \) are closed intervals, \( \mathcal{V} \) is a Banach space as required. Next, we define the set \( \mathcal{K} = \mathcal{K}_f \times \mathcal{K}_g \). The set \( \mathcal{K}_f \subseteq C(\Theta, \mathbb{R}_0^+) \) should contain the firm strategies that are relevant for fixed points of \( T \). Based on Lemma 2 define

\[
\mathcal{K}_f = \{ f \in C(\Theta, \mathbb{R}_0^+) : f_0(\theta - \theta') \leq f(\theta') - f(\theta) \leq L_f(\theta - \theta'), \quad \theta f \leq f(\theta) \leq \bar{\theta} f, \quad \text{for} \quad \theta, \theta' \in \Theta \quad \text{with} \quad \theta' \leq \theta \},
\]

where \( 0 < L_f = (1 - L_C) f \leq \bar{f} = L_f \), with \( 0 \leq L_C < 1 \). The set \( \mathcal{K}_g \subseteq C(\Xi, \mathbb{R}_0^+) \) should contain the rating agency strategies that are relevant for fixed points of \( T \). Proposition 1 and Lemma 4 suggest

\[
\mathcal{K}_g = \{ g \in C(\Xi, \mathbb{R}_0^+) : \xi \leq g \leq \text{Id}, g \text{ non-decreasing}, g/\text{Id} \text{ non-increasing} \}. \quad (32)
\]

It is sufficient to constrain the domain of the rating agency strategy \( g \) from originally \( \mathbb{R}_0^+ \) to \( \Xi \) for the following reason. By Proposition 1, we see that \( g(\cdot; f) \), for \( f \in \mathcal{K}_f \subseteq \mathcal{A}_f \), is continuous, non-decreasing, bounded by \( \text{Id} \). Moreover, \( g(\cdot; f) \) is determined by \( f \) on \( f(\Theta) \). For \( e \geq \overline{e}_f = \sup_{\theta \in \Theta} f(\theta) \), it holds \( g(e; f) = g(\overline{e}_f; f) \), and \( g(\cdot; f) = \text{Id} \) on \( [0, \xi] \), with \( e_f = \inf_{\theta \in \Theta} f(\theta) \). Further, \( f(\Theta) \subseteq [\theta_f, \bar{\theta} f] \), for \( f \in \mathcal{K}_f \). This property is preserved when considering \( \overline{\mathcal{R}} \circ g(\cdot; f) \) and the set \( [\theta_f, \frac{\theta^2 \cdot f}{(\theta \cdot f)}] = \Xi \) according to Lemma 1, i.e. \( \overline{\mathcal{R}} \circ g(e; f) = g(\overline{e}_f; f) \), for \( e \geq \frac{\theta^2 \cdot f}{(\theta \cdot f)} = \xi \), and \( \overline{\mathcal{R}} \circ g(e; f) = e \), for \( 0 \leq e \leq \theta \cdot f = \xi \), for all \( f \in \mathcal{K}_f \subseteq \mathcal{A}_f \). For this reason we cut the non-relevant part of \( T \) off, i.e. we consider \( \tilde{T} : \mathcal{K} \mapsto (f(\cdot; g|_{\mathbb{R}_0^+}), \overline{\mathcal{R}}(g(\cdot; f)))|_{\Xi} \), where \( g|_{\mathbb{R}_0^+} \) is understood as the obvious continuation of
\( g \in \mathcal{H}_g \) from \( \Xi \) to \( \mathbb{R}_0^+ \) with

\[
g_{\mid \mathbb{R}_0^+}(e) = \begin{cases} 
  e & \text{for } 0 \leq e < \frac{\xi}{\theta}, \\
  g(e) & \text{for } e \in \Xi, \\
  g(\frac{\xi}{\theta}) & \text{for } e > \frac{\xi}{\theta},
\end{cases}
\]

and thus \( \tilde{T} \) is well-defined. To show that \( \tilde{T} \) has a fixed point, we have to prove that \( \mathcal{H} \subseteq \mathcal{V} \) is nonempty, convex and compact, as well as \( \tilde{T} \) is continuous.

The set \( \mathcal{H} = \mathcal{H}_f \times \mathcal{H}_g \) is nonempty, convex and compact, if the factors \( \mathcal{H}_f \) and \( \mathcal{H}_g \) have these properties. By definition \( \mathcal{H}_f \) and \( \mathcal{H}_g \) are both nonempty. The convexity and compactness follows from Lemma 8 and Lemma 9, respectively. It remains to show that \( g \mapsto f(\cdot; g_{\mid \mathbb{R}_0^+}) \) and \( f \mapsto \hat{R}(g(\cdot; f))_{\mid \Xi} \) are both continuous.

Consider the best response of the firm \( f(\cdot; \cdot) : \mathcal{H}_g \to \mathcal{H}_f, \ g \mapsto f(\cdot; g_{\mid \mathbb{R}_0^+}) \), see Proposition 2. Take \( g, g' \in \mathcal{H}_g \) with \( \|g - g'\|_{\infty} \leq \varepsilon \), for some \( \varepsilon > 0 \). Denote the continuations by \( \hat{g} = g_{\mid \mathbb{R}_0^+} \) and \( \hat{g}' = g'_{\mid \mathbb{R}_0^+} \), then \( \|\hat{g} - \hat{g}'\|_{\infty} \leq \varepsilon \). Fix \( \theta \in \Theta \), then

\[
|\hat{g}_\theta(e) - \hat{g}'_\theta(e)| = \frac{1}{\theta} |\hat{g}(\theta e) - \hat{g}'(\theta e)| \leq \frac{\varepsilon}{\theta}, \text{ for all } e \geq 0,
\]

and hence \( \|\hat{g}_\theta - \hat{g}'_\theta\|_{\infty} \leq \varepsilon/\theta \), for \( \theta \in \Theta \). For now, we fix \( \theta \in \Theta \) and estimate \( |f(\theta; \hat{g}) - f(\theta; \hat{g}')| \). For doing so, we focus on \( \hat{g}_\theta \) and \( \hat{g}'_\theta \). For \( 0 \leq e < \frac{\xi}{\theta} \) we have that \( \hat{g}_\theta(e) = \hat{g}(\theta e)/\theta = \theta e/\theta = \hat{g}'(\theta e)/\theta = \hat{g}'_\theta(e) \). For \( e \geq \frac{\xi}{\theta} \), note that \( \hat{g}_\theta(e) = \hat{g}(\theta e)/\theta \geq \frac{\xi}{\theta} \) and

\[
\hat{g}_\theta'(e) \leq \hat{g}_\theta(e) + \varepsilon/\theta = \hat{g}_\theta(e)(1 + \varepsilon/(\theta \hat{g}_\theta(e))) \leq \hat{g}_\theta(e)(1 + \varepsilon/(\theta \hat{g}_\theta(e))) \leq \hat{g}_\theta(e)(1 + \varepsilon/\xi).
\]

Define \( \lambda(\varepsilon) = 1 + \varepsilon/\xi \geq 1 \), and by symmetry we have

\[
\lambda(\varepsilon)^{-1} \hat{g}_\theta \leq \hat{g}'_\theta \leq \lambda(\varepsilon) \hat{g}_\theta.
\]

The interest payment rate function \( C \) is non-increasing, thus

\[
C(\lambda(\varepsilon)x/\hat{g}_\theta(y)) \leq C(x/\hat{g}'_\theta(y)) \leq C(\lambda(\varepsilon)^{-1}x/\hat{g}_\theta), \text{ for } (x, y) \in \mathcal{C},
\]

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and Assumption 2 applied in the same fashion as in the proof of Lemma 2 gives

\[ \lambda(\varepsilon)^{-L_C} C(x/\hat{g}_\theta(y)) \leq C(x/\hat{g}'_\theta(y)) \leq \lambda(\varepsilon)^{L_C} C(x/\hat{g}_\theta(y)). \]

Using the same arguments as in the proof of Lemma 2, we have

\[ f(\lambda(\varepsilon)^{-L_C}; \hat{g}_\theta) \leq f(\lambda(\varepsilon); \hat{g}) \leq f(\lambda(\varepsilon)^{L_C}; \hat{g}_\theta). \]

Recalling that \( f(\cdot; h_g)/Id \) is non-increasing and \( \lambda_g(\varepsilon) \geq 1 \), we proceed as in the proof of Lemma 2, to obtain

\[ \lambda(\varepsilon)^{-L_C} f(\theta; \hat{g}) \leq f(\theta; \hat{g}') \leq \lambda(\varepsilon)^{L_C} f(\theta; \hat{g}). \]

Now, we can estimate

\[ |f(\theta; \hat{g}) - f(\theta; \hat{g}')| \leq (\lambda_g(\varepsilon)^{L_C} - 1) f(\theta; \hat{g}) \leq \left( (1 + \varepsilon/\xi)^{L_C} - 1 \right) \frac{L_C \theta f}{\xi} \varepsilon, \]

where we used (27) of Lemma 2 in the second step and \( 0 < L_C < 1 \), which is given by Assumption 2, in the third step. Hence we obtain a uniform upper bound in \( \theta \in \Theta \), i.e.

\[ \|f(\cdot; g|_{\mathbb{R}^+_0}) - f(\cdot; g'|_{\mathbb{R}^+_0})\|_\infty = \|f(\cdot; \hat{g}) - f(\cdot; \hat{g}')\|_\infty \leq \frac{L_C \theta f}{\xi} \varepsilon. \]

This implies that \( g \mapsto f(\cdot; g|_{\mathbb{R}^+_0}) \) is continuous on \( \mathcal{K}_g \).

Finally, consider the transformed best response of the rating agency, which is given by \( \mathcal{R}(g(\cdot; \cdot))|_\Xi : \mathcal{K}_f \rightarrow \mathcal{K}_g, f \mapsto \mathcal{R}(g(\cdot; f))|_\Xi \), see Proposition 1. We focus on the best response \( f \mapsto g(\cdot; f) \) for now, the transformation \( \mathcal{R} \) is dealt with later using Lemma 4. Take \( f, f' \in \mathcal{K}_f \) with \( \|f - f'\|_\infty \leq \varepsilon \), for some \( \varepsilon > 0 \). Consider \( e \in [0, f(\theta) \land f'(\theta)] \), then \( g(e; f) = g(e; f') = e \) and

\[ |g(e; f) - g(e; f')| = 0, \text{ for } e \in [0, f(g) \land f'(g)]. \]

Now, \( e \in [f(\theta) \land f'(\theta), f'(\theta) \lor f'(\theta)]. \) Without loss of generality assume \( f(\theta) <
and thus \( f'(\theta) \leq e < f'(\theta) \). From this we see by (12) \( g(e; f') - g(e; f) \leq 0 \), and since \( e < f'(\theta) \) as well as we have \( |f(\theta) - f'(\theta)| \leq \|f - f'|_\infty = \epsilon \), we have

\[
|g(e; f) - g(e; f')| \leq \epsilon, \text{ for } e \in [f(\theta) \land f'(\theta), f(\theta) \lor f'(\theta)].
\]

Consider \( e \in [f(\theta) \lor f'(\theta), e(\epsilon, f, f')] \), where \( e(\epsilon, f, f') = f(\theta) \lor f'(\theta) + L_f \epsilon^{1/2} \). Then by the uniform Lipschitz continuity of \( \mathcal{K}_f \) with Lipschitz constant \( L_f \), see Lemma 8, it follows that \( f(\theta) \land f'(\theta) \leq g(e; f), g(e; f') \leq f(\theta) \lor f'(\theta) + L_f \epsilon^{1/2} \), and thus

\[
|g(e; f) - g(e; f')| \leq \epsilon + L_f \epsilon^{1/2}, \text{ for } e \in [f(\theta) \lor f'(\theta), e(\epsilon, f, f')].
\]

Consider \( e \geq e(\epsilon, f, f') \). Without loss of generality assume \( f^{-1}(e) \leq f'^{-1}(e) \), note that by \( f, f' \in \mathcal{K}_f \) both functions are continuous and strictly increasing with minimum slope \( l_f \) and thus their respective inverse functions exist and are well-defined.
Using the definition of $g(\cdot, f)$ in (12) we can write
\[
g(e; f') = \frac{\int_{\theta}^{f^{-1}(e)} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]
\[
= \frac{\int_{\theta}^{f^{-1}(e)} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'} + \frac{\int_{f^{-1}(e)}^{f'(e')} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]
\[
= \frac{\int_{\theta}^{f^{-1}(e)} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'} + \frac{\int_{\theta}^{f^{-1}(e)} (f'(e') - f(e')) \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'} + \frac{\int_{f^{-1}(e)}^{f'(e')} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]
\[
= g(e; f) - g(e; f) \frac{\int_{\theta}^{f^{-1}(e)} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]
\[
+ \frac{\int_{\theta}^{f^{-1}(e)} (f'(e') - f(e')) \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'} + \frac{\int_{f^{-1}(e)}^{f'(e')} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]

And hence
\[
|g(e; f) - g(e; f')| \leq g(e; f) \frac{\int_{\theta}^{f^{-1}(e)} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'} + \frac{\int_{f^{-1}(e)}^{f'(e')} f'(e') \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]
\[
+ \frac{\int_{\theta}^{f^{-1}(e)} |f'(e') - f(e')| \phi(e') \, de'}{\int_{\theta}^{f^{-1}(e)} \phi(e') \, de'}
\]

(33)

All three expressions on the right hand side need to become small for $\varepsilon \searrow 0$. Observe that $0 \leq f'^{-1}(e) - f^{-1}(e) \leq l_f^{-1} \varepsilon$, where the first inequality follows by assumption and the second by the fact that $f, f'$ are strictly increasing with a minimum slope of $l_f > 0$ and $\|f - f'\|_{\infty} \leq \varepsilon$. Also, $f, f'$ are bounded by $\overline{\mathcal{F}}$ by (27), and so are
Adding up the three expressions we obtain as bound.

This implies that $g(\cdot; f)$ is bounded away from zero and bounded from above by assumption, we see that

$$
\int_\theta^{f^{-1}(e)} \phi(e') \, de' \geq \int_\theta^{f^{-1}(f(\theta) + L_f \varepsilon)} \phi \, de' \geq \int_\theta^{f^{-1}(\theta) + L \varepsilon} \phi \, de' = \phi(f^{-1}(f(\theta) + L_f \varepsilon) - \theta) \geq \phi L f \varepsilon^{1/2} / L_f = \phi \varepsilon^{1/2}.
$$

Using this, we can bound the first expression on the right hand side of (33)

$$
g(e; f) \frac{\int_\theta^{f^{-1}(e)} \phi(e') \, de'}{\int_\theta^{f^{-1}(e)} \phi(e') \, de'} \leq \frac{\overline{\theta f} \frac{e L_f}{\phi} \varepsilon}{\phi} e^{1/2}.
$$

For the second expression in (33) the same arguments apply, now aimed at $f$ rather than $g(\cdot; f)$, and

$$
\int_\theta^{f^{-1}(e)} f'(e') \phi(e') \, de' \leq \int_\theta^{f^{-1}(e)} f'(e') \phi(e') \, de' \leq \frac{\overline{\theta f} \phi \varepsilon^{1/2}}{\phi} f L / f.\n$$

The third expression in (33) can be estimated as follows

$$
\int_\theta^{f^{-1}(e)} |f'(e') - f(e')| \phi(e') \, de' \leq \int_\theta^{f^{-1}(e)} \phi \, de' \leq \varepsilon^{1/2} / \phi.
$$

Adding up the three expressions we obtain as bound

$$
|g(e; f) - g(e; f')| \leq \frac{2 \overline{\theta f} \phi + L f}{\phi L f} \varepsilon^{1/2}, \text{ for } e \in [e(f, f'), \infty).
$$

From there the uniform bound can be expressed as follows

$$
\|g(\cdot; f) - g(\cdot; f')\| \leq \varepsilon + \frac{2 \overline{\theta f} \phi + L f + \phi L f L_f}{\phi L f} \varepsilon^{1/2}. \quad (34)
$$

This implies that $f \mapsto g(\cdot; f)$ is continuous. From Lemma 4 and noting that the restriction to $\Sigma$ does no harm, the continuity of $\mathcal{R}(g(\cdot; f))|_{\Sigma}$ follows. 

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III  ODE Characterization of Best Responses and Potential Equilibria

Proposition 4 gives an ODE characterization of a equilibrium in case a specified condition holds. In the following, this result is derived. First, the best responses of both, rating agency and firm, are characterized by solutions to ODEs systems. Based on these, the equilibrium characterization is established.

III.1  Best Response of Rating Agency ODE

Proposition 5. Given a strategy $f \in A_f$ that is continuous and strictly increasing, the best response $g(\cdot; f)$ given in (12) of Proposition 1 and its transformation $\mathcal{R}(g(\cdot; f))$ defined in (13) can be characterized as follows

$$g(\cdot; f) = \hat{g}_f \circ f^{-1} \quad \text{and} \quad \mathcal{R}(g(\cdot; f)) = \tilde{g}_f \circ f^{-1}, \quad \text{on} \quad f(\Theta),$$

where $\hat{g}_f$ and $\tilde{g}_f$ have initial values

$$\hat{g}_f(\theta) = \hat{\theta}_f(\theta) = f(\theta),$$

and derivative $\hat{g}_f'$ and $\tilde{g}_f'$ that satisfy

$$\hat{g}_f' = \frac{\phi}{\Phi} (f - \hat{g}_f) \quad \text{and} \quad \tilde{g}_f' = f' \frac{\tilde{g}_f}{f} 1_{\tilde{g}_f < \hat{g}_f} + \min \left( \tilde{g}_f', f' \frac{\tilde{g}_f}{f} \right) 1_{\tilde{g}_f = \hat{g}_f},$$

Lebesgue almost everywhere on $(\theta, \overline{\theta})$, where $\Phi(g) = \int_{\theta}^{\overline{\theta}} \phi(t) \, dt$, $\theta \in \Theta$.

Corollary 1. Suppose that $f \in \mathcal{A}_f$ and $\phi$ are both continuously differentiable, then the function pair $(\hat{g}_f, \tilde{g}_f)$ given in Proposition 5 is the solution to the ODE

$$(\hat{g}_f', \tilde{g}_f') = \left( \frac{\phi}{\Phi} (f - \hat{g}_f) , f' \frac{\tilde{g}_f}{f} 1_{\tilde{g}_f < \hat{g}_f} + \min \left( \tilde{g}_f', f' \frac{\tilde{g}_f}{f} \right) 1_{\tilde{g}_f = \hat{g}_f} \right),$$
on \((\theta, \overline{\theta})\) with initial conditions

\[
(g_{\theta}(\theta), g_{\theta}')(\theta) = (f(\theta), f'(\theta)) \quad \text{and} \quad (g'_{\theta}(\theta), g'_{\theta})(\theta) = \left( \frac{1}{2} f'(\theta), \frac{1}{2} f'(\theta) \right).
\]

(39)

**Proof of Proposition 5.** First, define \(\hat{g}_f : \Theta \to \mathbb{R}^+\) by \(\hat{g}_f(\theta) = g(f(\theta); f), \) for \(\theta \in \Theta,\)

where \(g(\cdot; f)\) is given in (12) of Proposition 1. Then \(g(\cdot; f) = \hat{g}_f \circ f^{-1}\) as in (35) and furthermore

\[
\hat{g}_f(\theta) = \frac{1}{\Phi(\theta)} \int_\theta^\Theta f(t) \phi(t) \, dt, \quad \text{for} \quad \theta \in (\theta, \overline{\theta}),
\]

and \(\hat{g}_f(\theta) = g(f(\theta); f) = f(\theta),\) where the latter proves the first part of (36). Noting that \(g(\cdot; f)\) is the quotient of two absolutely continuous and strictly positive functions on \((\theta, \overline{\theta}),\) we see that \(g(\cdot; f)\) is Lebesgue almost everywhere differentiable with derivative \(\hat{g}_f'\) that satisfies according to the quotient rule

\[
\hat{g}_f'(\theta) = \frac{1}{\Phi(\theta)^2} \left( f(\theta) \phi(\theta) \Phi(\theta) - \int_\theta^\Theta f(t) \phi(t) \, dt \phi(\theta) \right) = \frac{\phi(\theta)}{\Phi(\theta)} (f(\theta) - \hat{g}_f(\theta)),
\]

for Lebesgue almost every \(\theta \in (\theta, \overline{\theta}),\) proving the first part of (37). Define \(\tilde{g}_f\) by

\[
\tilde{g}_f(\theta) = \mathcal{R}(g(\cdot; f))(f(\theta)), \quad \text{for} \quad \theta \in \Theta,
\]

then \(\mathcal{R}(g(\cdot; f)) = \tilde{g}_f \circ f^{-1}\) on \(\Theta,\) as claimed in (35). Note \(g(z; f)/z = 1,\) for \(0 < z \leq f(\theta),\) \(g(z; f)/z \leq 1,\) for \(z > 0,\) and \(f \in \mathcal{A}_f\) is continuous and strictly increasing, to see that

\[
\tilde{g}_f(\theta) = \mathcal{R}(g(\cdot; f))(f(\theta)) = e^{\inf_{0 < z \leq e} \frac{g(z; f)}{z}} \left|_{e = f(\theta)} \right. = f(\theta) \inf_{f(\theta) \leq z \leq f(\theta)} \frac{g(z; f)}{z} = f(\theta) \inf_{\theta \leq \theta' \leq \theta} \frac{g_f(\theta')}{f(\theta')}, \quad \text{for} \quad \theta \in \Theta.
\]

The initial value is given by \(\tilde{g}_f(\theta) = f(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} = \hat{g}_f(\theta) = f(\theta)\) establishing the second part of (36). Further, from Lemma 1 it follows that \(\tilde{g}_f\) is continuous and
nondecreasing, bounded by $f$, as well as, that $\tilde{g}_f/f$ is nonincreasing. For $\theta \geq \theta' \in \Theta$, write
\[0 \leq \tilde{g}_f(\theta) - \tilde{g}_f(\theta') = f(\theta) \inf_{\theta \leq z \leq \theta} \frac{\hat{g}_f(z)}{f(z)} - f(\theta') \inf_{\theta \leq z \leq \theta'} \frac{\hat{g}_f(z)}{f(z)} \leq (f(\theta) - f(\theta')) \inf_{\theta \leq z \leq \theta} \frac{\hat{g}_f(z)}{f(z)} \leq (f(\theta) - f(\theta')) \leq L_f |\theta - \theta'|,
\]
by Lemma 2, for $L_f = f > 0$. Thus, $\tilde{g}_f$ is Lipschitz continuous and by this has a derivative $\tilde{g}_f'$ Lebesgue almost everywhere on $\Theta$. By the same rational $f$ has a derivative $f'$ Lebesgue almost everywhere on $\Theta$. Since $\tilde{g}_f/f$ is nonincreasing and $f > 0$, we have Lebesgue almost everywhere on $\Theta$
\[0 \geq (\tilde{g}_f/f)' = \frac{\tilde{g}_f f' - \tilde{g}_f f}{f^2} \quad \iff \quad \tilde{g}_f' \leq f' \frac{\tilde{g}_f}{f}.\]

Denote by $E_f \subseteq \Theta$ the set where $\tilde{g}_f$ equals $\tilde{g}_f$, i.e. $E_f = \{\theta \in \Theta : \hat{g}_f(\theta) = \tilde{g}_f(\theta)\}$. Since $\hat{g}_f$ and $\tilde{g}_f$ are both continuous and $\Theta$ is bounded, $E_f$ is compact. On $\hat{E}_f = E_f \setminus \partial E_f$ we have Lebesgue almost everywhere that $\tilde{g}_f' = \hat{g}_f'$. Now, the boundary $\partial E_f$ has Lebesgue measure 0 and $\tilde{g}_f' \leq f' \frac{\tilde{g}_f}{f}$ Lebesgue almost everywhere on $\Theta$, hence we have Lebesgue almost everywhere on $E_f$
\[\tilde{g}_f' = \min\left(\tilde{g}_f', f' \frac{\tilde{g}_f}{f}\right).
\]
Next, consider $(\Theta, \tilde{\theta}) \setminus E_f$, which is open. Take $\theta \in (\Theta, \tilde{\theta}) \setminus E_f$, then we find an $\varepsilon > 0$ such that $B_\varepsilon(\theta) = \{\theta' \in \mathbb{R} : |\theta - \theta'| < \varepsilon\} \in (\Theta, \tilde{\theta}) \setminus E_f$, i.e. $\tilde{g}_f < \hat{g}_f$ on $B_\varepsilon(\theta)$. Then
\[\tilde{g}_f(\theta) = f(\theta) \inf_{\theta \leq \theta' \leq \theta} \frac{\hat{g}_f(\theta')}{f(\theta')} = f(\theta) \frac{\hat{g}_f(\theta^*)}{f(\theta^*)}
\]
for some $\theta^* < \theta$, since $\tilde{g}_f(\theta)/f(\theta) < \hat{g}_f(\theta)/f(\theta)$ and $\theta \mapsto \inf_{\theta \leq \theta' \leq \theta} \hat{g}_f(\theta')/f(\theta)$ is continuous, where the latter follows from the continuity of $\hat{g}_f$ and $f$. Moreover, the above equality extend by the latter mentioned continuity to $[\theta, \theta + \varepsilon^*]$, for some
\( \epsilon^* > 0 \). And thus for \( \bar{\epsilon} = \epsilon \wedge \epsilon^* \), we have

\[
\hat{g}_f'(\theta') = f(\theta') \frac{\hat{g}_f(\theta^*)}{f(\theta^*)}, \text{ for } \theta' \in B_{\hat{\epsilon}}(\theta).
\]

Now, \( f \) and \( \hat{h}_f \) are absolutely continuous and their derivatives satisfy Lebesgue almost surely on \( B_{\hat{\epsilon}}(\theta) \)

\[
\hat{g}_f' = f' \frac{\hat{g}_f(\theta^*)}{f(\theta^*)} = f' \hat{g}_f.
\]

Putting the pieces together gives the second part of (37). \( \Box \)

**Proof of Corollary 1.** Since \( \phi \) and \( \Phi \) are continuous, the results in (37) hold for all \( \theta \in (\bar{\theta}, \hat{\theta}) \), what then also specifies an ODE. The initial values for the derivative of \( \hat{g}_f \) is obtained using L’Hospital rule

\[
\hat{g}_f'(\theta) = \lim_{\epsilon \searrow 0} \hat{g}_f'(\theta) = \lim_{\epsilon \searrow 0} \frac{\phi(\theta)}{\Phi(\theta)} (f(\theta) - \hat{g}_f(\theta)) = \phi(\theta) \lim_{\epsilon \searrow 0} \frac{f(\theta) - \hat{g}_f(\theta)}{\Phi(\theta)}
\]

\[
= \phi(\theta) \lim_{\epsilon \searrow 0} \frac{f'(\theta) - \hat{g}_f'(\theta)}{\phi(\theta)} = \phi(\theta) \frac{f'(\theta) - \hat{g}_f'(\theta)}{\phi(\theta)} = f'(\theta) - \hat{g}_f'(\theta),
\]

and hence \( \hat{g}_f'(\theta) = f'(\theta)/2 \) as claimed. Then

\[
\hat{g}_f'(\theta) = \lim_{\epsilon \searrow 0} f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} \min \left( \hat{g}_f'(\theta), f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \right) \mathbf{1}_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}
\]

\[
= f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \lim_{\epsilon \searrow 0} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} + \min \left( \hat{g}_f'(\theta), f'(\theta) \frac{\hat{g}_f(\theta)}{f(\theta)} \right) \lim_{\epsilon \searrow 0} \mathbf{1}_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}
\]

\[
= f'(\theta) \lim_{\epsilon \searrow 0} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} + \min \left( f'(\theta)/2, f'(\theta) \right) \lim_{\epsilon \searrow 0} \mathbf{1}_{\hat{g}_f(\theta) = \hat{g}_f(\theta)}.
\]

Observe that \( \hat{g}_f(\theta) = f(\theta) \) and that the slope \( \hat{g}_f'(\theta) = f'(\theta)/2 \) is strictly smaller than \( f'(\theta) > 0 \). Therefore, \( \hat{g}_f/f \) is strictly decreasing in a neighborhood of \( \theta \) and thus \( \hat{g}_f = \hat{g}_f \) on \( [\bar{\theta}, \hat{\theta} + \epsilon] \) for some \( \epsilon > 0 \). Accordingly, \( \lim_{\theta \searrow \theta} \mathbf{1}_{\hat{g}_f(\theta) < \hat{g}_f(\theta)} = 0 \) and \( \lim_{\theta \searrow \theta} \mathbf{1}_{\hat{g}_f(\theta) = \hat{g}_f(\theta)} = 1 \). From there we conclude

\[
\hat{g}_f'(\theta) = \min \left( f'(\theta)/2, f'(\theta) \right) = f'(\theta)/2,
\]
finishing the proof. □

### III.2 Best Response of Firm ODE

For a given \( g \in \mathcal{A}_g^C \) and \( \theta \in \Theta \), consider the free boundary value problem given in (15-19) to characterize the value function \( v(\cdot, \cdot; \theta, g) \). The boundary \( \partial C(\theta, g) \) of the continuation region \( C(\theta, g) \) can be described by a boundary function \( b(\cdot, \theta; g) \) with the following properties.

**Lemma 3.** For given \( g \in \mathcal{A}_g^C \) and \( \theta \in \Theta \), the boundary \( \partial C(\theta, g) \) is characterized by a function \( b : [0, f(\theta; g)] \times \Theta \times \mathcal{A}_g^C : \mathbb{R}_+^+ \times (e, \theta; g) \mapsto b(e, \theta; g) \), i.e.

\[
\partial C(\theta, g) = \{ (b(e, \theta; g), e) : 0 \leq e \leq f(\theta; g) \},
\]

that is non-decreasing and continuous with terminal value \( b(f(\theta; g), \theta; g) = f(\theta; g) \). The restriction of \( b(\cdot, \theta; g) \) to \( [\xi, f(\theta; g)] \), with \( \xi = \theta f \), is Lipschitz continuous with constant \( L_b = \bar{f}/\xi \).

**Proof of Lemma 3.** Lemma 6 part 4 implies that \( b(\cdot, \theta; g) \) is non-decreasing. Part 5 implies that the slope in \( e \) is bounded by \( f(\theta; g)/e \), which also implies continuity. The terminal value \( b(f(\theta; g), \theta; g) = f(\theta; g) \) follows from Lemma 7. The continuity of \( b \) is uniformly for all \( \theta \in \Theta \) with maximum slope \( \bar{f}/\xi \), where \( \bar{f} \) is given in Lemma 2 and \( \xi = \theta f \), see discussion around (32).

**Proposition 6.** For a given continuously differentiable strategy \( g \in \mathcal{A}_g^C \) suppose that the collection of solutions \( (v(\cdot, \cdot; \theta, g))_{\theta \in \Theta} \) of the boundary value problem (15-19) sufficiently differentiable Then the firm’s best response \( f(\cdot; g) \) given in Proposition 2 satisfies

\[
f'(\theta; g) = \frac{(1 + \eta) \sigma^2}{2(r - \mu)} \frac{f(\theta; g)^2}{\theta^2} \frac{1}{C(f(\theta; g)/g(f(\theta; g))) - f(\theta; g)/\theta} \frac{1}{1 - \partial b / \partial e (f(\theta; g), \theta; g)},
\]

for \( \theta \in (\tilde{\theta}, \tilde{\theta}) \), where \( \eta = \frac{\mu - \frac{1}{2} \sigma^2}{\sigma^2} + \sqrt{\left( \frac{\mu - \frac{1}{2} \sigma^2}{\sigma^2} \right)^2 + 2r \sigma^2} > 0 \), and the partial deriva-
tive of the boundary describing function $b$ with respect to $e$ in $(f(\theta;g), \theta)$ is a function of $f(\theta), g(f(\theta))$, and $g'(f(\theta))$, i.e.

$$\frac{\partial b}{\partial e}(f(\theta;g), \theta;g) = h(f(\theta), g(f(\theta)), g'(f(\theta)), \theta), \quad (41)$$

for $\theta \in (\theta, \bar{\theta})$, and some function $h(\cdot, \cdot, \cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \Theta \to \mathbb{R}^+$.

Proof of Proposition 6. In the following, the value function $v(\cdot, \cdot; g), g \in \mathcal{A}^C$, is frequently differentiated on the boundary $\partial C(\theta, g)$. Without stating this explicitly, we assume that the differential is calculated in the interior of $C(\theta, g)$ and by continuity of $v(\cdot, \cdot; g)$, the limit to the boundary is taken. In order to show the claim, we proceed in four steps. First, we differentiate the smooth fit condition in (19) with respect to $\theta$ to obtain

$$0 = \frac{d}{d\theta} \frac{\partial v}{\partial d}(b(e, \theta; g), e, \theta; g)$$

$$= \frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) \frac{\partial b}{\partial \theta}(e, \theta; g) + \frac{\partial^2 v}{\partial \theta \partial d}(b(e, \theta; g), e, \theta; g), \quad (42)$$

for $e \leq f(\theta; g)$. In the second step, we identify $\frac{\partial^2 v}{\partial d^2}$ at the boundary using (16-18). In the third step, we characterize $\frac{\partial v}{\partial \theta}$ in order to calculate $\frac{\partial^2 v}{\partial \theta \partial d}$. In the fourth and final step, we use results of the third step and the normal reflection condition in (19) to determine $\frac{\partial b}{\partial \theta}$.

For the second step observe that $v = \frac{\partial v}{\partial d} = 0$ on $\partial C(\theta, g)$. From this and (16-18) it follows that

$$\frac{1}{2} \sigma^2 b(e, \theta; g)^2 \frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) + \frac{b(e, \theta; g)}{\theta} - C(b(e, \theta; g)/g(e)) = 0,$$

or, equivalently,

$$\frac{\partial^2 v}{\partial d^2}(b(e, \theta; g), e, \theta; g) = 2 \frac{C(b(e, \theta; g)/g(e)) - \frac{b(e, \theta; g)}{\theta}}{\sigma^2 b(e, \theta; g)^2}, \quad (43)$$

for $e \leq f(\theta; g)$.

Now, the third step is taken. For $\theta \in (\theta, \bar{\theta})$ denote by $u$ the differential of $v$ with
respect to $\theta$, which by assumption exists is continuous in $\theta$. The assumed continuity of $b$, the fact that the continuation region $C^{(\theta,g)}$ is open in $C$, and from equations (16-18) we obtain

$$
\mu d \frac{\partial u}{\partial d}(d,e;\theta,g) + \frac{1}{2} \sigma^2 d^2 \frac{\partial^2 u}{\partial d^2}(d,e;\theta,g) - \frac{d}{\theta^2} - ru(d,e;\theta,g) = 0,
$$

for $(d,e) \in C^{(\theta,g)}$. Similar reasoning implies for $(d,e)$ in the interior of $C \setminus C^{(\theta,g)}$ that

$$
u(d,e;\theta,g) = 0,$$

which extends to all $(d,e) \in C \setminus C^{(\theta,g)}$, i.e. also to $\partial C^{(\theta,g)}$, since $v = 0$ on the boundary $\partial C^{(\theta,g)}$, and computing the derivative in $\theta$ gives

$$
\frac{\partial v}{\partial d}(b(e,\theta;g),e;\theta,g) \frac{\partial b}{\partial \theta}(e,\theta;g) + \frac{\partial v}{\partial \theta}(b(e,\theta;g),e;\theta,g) = 0,
$$

and recalling that $\frac{\partial v}{\partial d} = 0$ on $\partial C^{(\theta,g)}$ by the smooth fit condition in (19) and $\frac{\partial v}{\partial \theta} = u$ by definition. This second order ODE in $d$ is not depending explicitly on $e$. Therefore, we can interpret $e$ as well as $\theta$ as fixed parameters, and $u$ is in its general form given by

$$
u(d,e;\theta,g) = -\frac{d}{\theta^2(r-\mu)} + d^{-\eta} L(e,\theta,g) + d^{-\tilde{\eta}} \tilde{L}(e,\theta,g),
$$

for $(d,e) \in C^{(\theta,g)}$, or, equivalently, for $d \geq b(e,\theta;g)$ in case $e \leq f(\theta;g)$ and $d \geq e$ in case $e > f(\theta;g)$, where $L(e,\theta,g)$ and $\tilde{L}(e,\theta,g)$ are constants taking values in $\mathbb{R}$, and

$$
\eta = \mu - \frac{1}{2} \sigma^2 + \sqrt{\left(\frac{\mu - \frac{1}{2} \sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \text{ and } \tilde{\eta} = \mu - \frac{1}{2} \sigma^2 - \sqrt{\left(\frac{\mu - \frac{1}{2} \sigma^2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.
$$

From $r > 0$, we see directly that $\eta > 0$ and $\tilde{\eta} < 0$. It can also be shown that $\tilde{\eta} < -1$, since $\mu < r$, and hence $x^{-\eta} \tilde{L}(e,\theta,g)$ becomes the dominating expression of $u$ for large $d$. Observe that the value function $v$ for large $d$ is approximately $\frac{d}{\theta(r-\mu)}$, and
thus its partial derivative in $\theta$ is approximately $-\frac{d}{\theta^2(r-\mu)}$. Accordingly, the weight of the otherwise dominating expression $d^{-\tilde{\eta}}$ has to be zero, i.e. $\tilde{L}(e, \theta, g) = 0$, and hence

$$u(d, e; \theta, g) = -\frac{d}{\theta^2(r-\mu)} + d^{-\eta} L(e, \theta, g), \text{ for } (d, e) \in C^{(\theta, g)}.$$  

Fixing $\theta$ and focusing on $\partial C^{(\theta, g)}$, we obtain $L(e, \theta, g)$ in terms of the boundary defining function $b(\cdot, \theta; g)$. On the boundary, i.e. $d = b(e, \theta; g)$ with $e \leq f(\theta; g)$, we have that $u(b(e, \theta; g), e, \theta; g) = 0$. Thus

$$L(e, \theta, g) = \frac{b(e, \theta; g)^{1+\eta}}{\theta^2(r-\mu)}, \text{ for } e \leq f(\theta; g),$$  

yielding for $e \leq f(\theta; g)$ that

$$u(d, e; \theta, g) = -\frac{d}{\theta^2(r-\mu)} \left( 1 - \left[ \frac{d}{b(e, \theta; g)} \right]^{-(1+\eta)} \right) 1_{d \geq b(e, \theta; g)}. \quad (44)$$  

The partial derivative with respect to $d$ is then

$$\frac{\partial u}{\partial d}(d, e; \theta, g) = -\frac{1}{\theta^2(r-\mu)} \left( 1 + \eta \left[ \frac{d}{b(e, \theta; g)} \right]^{-(1+\eta)} \right) 1_{d \geq b(e, \theta; g)},$$  

for $e \leq f(\theta; g)$. In particular, we obtain for $d = b(e, \theta; g)$ that

$$\frac{\partial^2 v}{\partial \theta \partial d}(b(e, \theta; g), e; \theta, g) = \frac{\partial u}{\partial d}(b(e, \theta; g), e; \theta, g) = -\frac{1+\eta}{\theta^2(r-\mu)}, \quad (45)$$  

where the derivative in $(b(e, \theta; g), e) \in \partial C^{(\theta, g)}$ is understood in the limit from the interior of $C^{(\theta, g)}$ and the interchange of the order of differentiation follows from the initial assumption.

For the fourth step, note that the boundary function satisfies $f(\theta; g) = b(f(\theta; g), \theta; g)$,
and differentiating this expression with respect to $\theta$ gives
\[ f'(\theta; g) = \frac{\partial b}{\partial e} (f(\theta; g), \theta; g) f'(\theta; g) + \frac{\partial b}{\partial \theta} (f(\theta; g), \theta; g), \]
where $f'(\cdot; g)$ denotes $\frac{\partial f}{\partial \theta} (\cdot; g)$. Now, solve for $\frac{\partial b}{\partial \theta}$ to obtain
\[ \frac{\partial b}{\partial \theta} (f(\theta; g), \theta; g) = f'(\theta; g) \left( 1 - \frac{\partial b}{\partial e} (f(\theta; g), \theta; g) \right). \quad (46) \]

Finally, we set $e = f(\theta; g)$, hence $b(f(\theta; g), \theta; g) = f(\theta; g)$, and plug (43), (45) and (46) in (42), to see that
\[ 0 = 2 \frac{C(f(\theta; g)/g(f(\theta; g))) - \frac{f(\theta; g)}{\theta}}{\sigma^2 f(\theta; g)^2} f'(\theta; g) \left( 1 - \frac{\partial b}{\partial e} (f(\theta; g), \theta; g) \right) - \frac{1 + \eta}{\theta^2 (r - \mu)}, \]
and solving for $f'(\theta; g)$ gives (40).

The partial derivative $\frac{\partial b}{\partial e}(f(\theta; g), \theta; g)$ in (40) is now analyzed. For doing so, define the function $\hat{v}: \mathbb{R}^+ \times \Theta \times \mathbb{R}^+ \rightarrow \mathbb{R}$, $(d; \theta, g) \mapsto \hat{v}(d; \theta, g)$ by
\[ \hat{v}(d; \theta, g) = \sup_{\tau \in \mathcal{T}_d} \mathbb{E}_d \left[ \int_0^\tau e^{-rt} (d_t / \theta - C(d_t / g)) dt \right], \]
where $\mathcal{T}_d$ is the set of all stopping times with respect to the information generated by $D$ with starting value $d$ and $\mathbb{E}_d$ is the corresponding expectation. In contrast to the function $v$ defined in (14), the direct dependence on the minimum observed cash flow $E$ is eliminated. Instead, an optimal stopping problem in the observed cash flow $D$ is given parameterized by $g \in \mathbb{R}^+$ and $\theta \in \Theta$. However, for $e \leq f(\theta; g)$, the connection between the coordinates $d$ and $e$ of $v$ is dissolved and we have
\[ v(d, e; \theta, g) = \hat{v}(d; \theta, g(e)), \text{ for } (d, e) \in \mathcal{C}, e \leq f(\theta; g), \theta \in \Theta. \]

Accordingly, we can use $\hat{v}$ instead of $v$ in order to characterize the partial derivative $\frac{\partial b}{\partial e}(f(\theta; g), \theta; g)$. Using the differentiability assumptions on the value function
in the proof of Proposition 6, \( g \) is eliminated. Instead, a conventional optimal stopping problem in the observed function \( v \) defined in \( D \) with starting value \( d \) and \( T \) allows us to obtain this quantity numerically. Define the function

\[
\frac{\partial b}{\partial e} (f(\theta;g), \theta; g) = \lim_{\Delta e \to 0} \frac{1}{\Delta e} \left( b(f(\theta;g), \theta; g) - b(f(\theta;g) - \Delta e, \theta; g) \right)
\]

\[
= \lim_{\Delta e \to 0} \frac{1}{\Delta e} \left( \sup \{d \geq e = f(\theta;g) : v(d,e;\theta, g) = 0\} - \sup \{d \geq e = f(\theta;g) - \Delta e : v(d,e;\theta, g) = 0\} \right)
\]

\[
= \lim_{\Delta e \to 0} \frac{1}{\Delta e} \left( \sup \{d > 0 : \hat{v}(d;\theta, g(f(\theta;g))) = 0\} - \sup \{d > 0 : \hat{v}(d;\theta, g(f(\theta;g) - \Delta e)) = 0\} \right).
\]

Noting that \( f(\theta;g) = b(f(\theta;g), \theta; g) = \sup \{d > 0 : \hat{v}(d;\theta, g(f(\theta;g))) = 0\} \), gives

\[
\frac{\partial b}{\partial e} (f(\theta;g), \theta; g)
\]

\[
= \lim_{\Delta e \to 0} \frac{1}{\Delta e} \left( f(\theta;g) - \sup \{d > 0 : \hat{v}(d;\theta, g(f(\theta;g)) - g'(f(\theta;g))\Delta e) = 0\} \right).
\]

\[
= h(f(\theta;g), g(f(\theta;g)), g'(f(\theta;g)), \theta),
\]

defining the function \( h \) and thus proving (41). \( \square \)

**Remark 1.** The function \( h(f(\theta), \hat{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta) = \frac{\partial b}{\partial e} (f(\theta), \theta; \hat{g} \circ f) \) needs to be computed to calculate the equilibrium \((f^*, g^*)\). The proof of Proposition 6 shows how to obtain this quantity numerically. Define the function \( \hat{v} : \mathbb{R}^+ \times \Theta \times \mathbb{R}^+ \to \mathbb{R}, \)

\[
(d, \theta, g) \mapsto \hat{v}(d; \theta, g)
\]

by

\[
\hat{v}(d; \theta, g) = \sup_{\tau \in \mathcal{T}_d} \mathbb{E}_d \left[ \int_0^\tau e^{-\lambda t} (d_t / \theta - C(d_t / g)) dt \right],
\]

where \( \mathcal{T}_d \) is the set of all stopping times with respect to the information generated by \( D \) with starting value \( d \) and \( \mathbb{E}_d \) is the corresponding expectation. In contrast to the function \( v \) defined in (14), the direct dependence on the minimum observed cash flow \( E \) is eliminated. Instead, a conventional optimal stopping problem in the observed cash flow \( D \) is given, parameterized by \( \theta \in \Theta \) and \( g \in \mathbb{R}^+ \). However, for \( e \leq f(\theta;g) \) we have \( v(d, e; \theta, g) = \hat{v}(d; \theta, g(e)), \) for \( (d, e) \in \mathcal{C} \) and \( \theta \in \Theta \). From the arguments in the proof of Proposition 6, \( g(f(\theta)) = \hat{g}(\theta) \) and \( (g^*)'(f(\theta)) = \hat{g}'(\theta) / f'(\theta) \) it
follows that
\[
\frac{\partial b}{\partial e}(f(\theta), \theta; g^*) = \lim_{\Delta e \to 0} \frac{1}{\Delta e} \left( f(\theta) - \sup\{d > 0 : \hat{\nu}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)} \Delta e) = 0 \} \right) \\
\approx \frac{1}{\Delta e} \left( f(\theta) - \sup\{d > 0 : \hat{\nu}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)} \Delta e) = 0 \} \right).
\]

for sufficiently small $\Delta e$, where the critical level $\sup\{d > 0 : \hat{\nu}(d; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)} \Delta e) = 0 \}$ is obtained by solving the free-boundary value problem associated optimal stopping problem with value function $\hat{\nu}(\cdot; \theta, \hat{g}(\theta) - \frac{\hat{g}'(\theta)}{f'(\theta)} \Delta e)$ numerically.

III.3 ODE Characterization of Equilibrium

**Proposition 7.** Given the setting of Proposition 3, denote by $(f^*, g^*)$ a fixed point of $T$. Suppose $f^*$, $g^*$, and $\phi$ are continuously differentiable, as well as the collection of solutions $(v(\cdot, \theta, g^*))_{\theta \in \Theta}$ of the boundary value problem (15-19) is sufficiently differentiable. Then $(f, \tilde{g}, \hat{g}) = (f^* \circ g^* \circ f^* \circ f^*)$ satisfies

\[
\begin{pmatrix}
  f'(\theta) \\
  \tilde{g}'(\theta) \\
  \hat{g}'(\theta)
\end{pmatrix}
= \begin{pmatrix}
  \frac{(1+\eta)^2 - 3(\eta-\mu)^2}{\sigma^2} \\
  \frac{f(\theta)^2/\theta}{C(f(\theta)/\tilde{g}(\theta)) - f(\theta)/\theta} \\
  \frac{1}{\sup_{\tilde{g}(\theta)=\hat{g}(\theta)}(f(\theta) - \hat{g}(\theta))}
\end{pmatrix}
\begin{pmatrix}
  f(\theta) \\
  \tilde{g}(\theta) \\
  \hat{g}(\theta)
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix},
\]

(47)
on $(\theta, \Theta)$ with initial condition

\[
\begin{pmatrix}
  f(\theta) \\
  \tilde{g}(\theta) \\
  \hat{g}(\theta)
\end{pmatrix}
= \Theta \begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3
\end{pmatrix},
\]

(48)
where the partial derivative of the boundary describing function $b(\cdot, \cdot; g^*)$ with respect to $e$ in $(f(\theta), \theta)$ is a function of $f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta)$ and $\theta$ i.e.

\[
\frac{\partial b}{\partial e}(f(\theta), \theta; g^*) = \tilde{h}(f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta), \theta),
\]

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for some function $\tilde{h}$, $(f_1^*, g_1^*) \in \mathbb{R}^2$ denotes the equilibrium of the perfect information case, i.e. $\Theta_1 = \{1\}$ and hence $D = X$, with $f_1^* = g_1^*$, which exists and is unique under the given assumptions, and $\Phi(\theta) = \int_0^\theta \phi(t) \, dt$, $\theta \in \Theta$.

**Proof of Proposition 7.** A fixed point $(f^*, g^*)$ of $T$, which exists by Proposition 3, satisfies by its very definition that $g^* = \mathcal{R} \cdot g(\cdot ; f^*)$ and $f^* = f(\cdot ; g^*)$, i.e. both strategies are their mutual (transformed) best responses. Corollary 1 of Proposition 5 yields the description of $\tilde{g}$ and $\hat{g}$, as well as Proposition 6 that of $f$, respectively. When looking at $\frac{\partial h}{\partial \epsilon}$ as given in Proposition 6, we see that the function $h$ describing $\frac{\partial h}{\partial \epsilon}$ in $(f(\theta), \theta)$ depends now on $\tilde{g}(\theta) = g^*(f(\theta))$ and $\tilde{g}'(\theta) = (g^*)'(f(\theta)) f'(\theta)$, where the latter is equivalent to $\frac{\tilde{g}(\theta)}{f'(\theta)} = (g^*)'(f(\theta))$, for $\theta \in \Theta$, and thus $\tilde{h}$ defined by

$$\tilde{h}(f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta), \theta) = h(f(\theta), \tilde{g}(\theta), \frac{\tilde{g}'(\theta)}{f'(\theta)}, \theta),$$

is a function of $f(\theta), \tilde{g}(\theta), f'(\theta), \tilde{g}'(\theta)$ and $\theta$ as claimed.

It remains to verify that the initial condition. Therefore, we focus on $\tilde{g}_f$ and $\hat{g}_f$ around the starting value $\theta$. By assumption, $f^*$ as well as $\phi$ are continuously differentiable, and hence by Corollary 1 of Proposition 5 we have $\tilde{g}_f'(\theta) = \tilde{g}_f'(\theta) = \frac{1}{f'}(\theta)$. Accordingly, $\tilde{g}_f'(\theta) < f'(\theta) \frac{\tilde{g}(\theta)}{f(\theta)} = f'(\theta)$, where the strict inequality follows from Lemma 2 implying $f' \geq l_f > 0$. By the assumed continuity, there exists an $\epsilon > 0$ such that $\tilde{g}' = \tilde{g}'$ on $[\theta, \theta + \epsilon]$, where $\epsilon \leq \bar{\theta} - \theta$. It follows that $\tilde{g}_f = \tilde{g}_f$ on $[\theta, \theta + \epsilon]$, since also $\tilde{g}(\theta) = \tilde{g}(\theta)$ by definition. This implies that $\tilde{g}_f \circ f^{-1} = \tilde{g}_f \circ f^{-1}$ on $f([\theta, \theta + \epsilon])$. Using Proposition 5, the best response $g(\cdot, f^*)$ and its transformation $\mathcal{R}(g(\cdot, f^*)) = g^*$ coincide on $f([\theta, \theta + \epsilon])$. Denote by $\mathbb{P}_\theta^{f^*}$ a modified prior, which is given by $\mathbb{P}_\theta^{f^*} = \mathbb{P}_\theta^f(\cdot | \bar{\theta} \leq \hat{\theta} + \epsilon)$, for $0 < \epsilon' \leq \epsilon$. Then $(f^*, g^*)$ restricted to $f^*([\theta, \theta + \epsilon])$ and $[\theta, \theta + \epsilon]$, respectively, is an equilibrium for the prior $\mathbb{P}_\theta^{f^*}$, for all $0 < \epsilon' \leq \epsilon$. Now, the best response operators are continuous in the sup-norm as shown in Proposition 3. As $\epsilon' \searrow 0$, we are tending to the perfect information case, here with known type $\bar{\theta}$, and hence the limit $(f^*(\theta), g^*(\theta))$ is an equilibrium of the perfect information case, here scaled by $\bar{\theta}$. The existence and uniqueness of the equilibrium in the perfect information case, i.e.: $\Theta_1 = \{1\}$ and hence $D = X$, follows from Theorem 1 and the working in Appendix C of Manso (2013). Note that Manso

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(2013) specifies the coupon function as step function, where our framework allows for a continuous coupon function $C$, which satisfies Assumption 2. However, using, for example, an approximating sequence of step functions $(C_n)_{n \geq 1}$ to our coupon function $C$ the results carry over. Further, denote by $f_1^*$ the firm’s default threshold in equilibrium. In order to transfer the result from the firm scale to the rating agency scale, we multiply the equilibrium $f_1^*$ by $\theta$ and the initial condition follows as claimed.

The differential equation satisfied by $(f, \tilde{g}, \hat{g}) = (f^*, g^* \circ f^*, g(\cdot; f^*) \circ f^*)$ in (47) and (48) of Proposition 7 can be used to obtain the fixed point $(f^*, g^*)$ constructively. The suggested ODE structure is somewhat more complicated, since characterization of the partial derivative $f'$ on the left hand side of (47) also involves the term $\tilde{h}(f(\theta), \tilde{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta)$ on the right hand side of (47), which depends on $f'$. Rewriting the first line of (47) as follows $H(f(\theta), \tilde{g}(\theta), f'(\theta), \hat{g}'(\theta), \theta) = 0$, is an implicit characterisation of $f'$. To ensure that this implicit characterization of $f'$ is well-defined, it is required that the partial derivative $H$ with respect to $f'$ is nonzero, such that we can invert this relation locally to back out $f'$.

**Proof of Proposition 4.** Denote $(f^*, g^*)$ a fixed point of $T$ given in Proposition 3. If $g(\cdot; f) = \mathcal{R} \circ g(\cdot; f^*)$ holds, then $(f^*, g^*)$ is an equilibrium. By Proposition 7 the latter is equivalent to $\tilde{g} = \hat{g}$ which is implied by $g' \leq \frac{f'}{\hat{g}}$, on $(\theta, \theta)$. □

**IV  Auxiliary Results**

From (12) we see that best response of the rating agency $g(\cdot; f)$ with respect to a firm strategy $f$ depends on the image of $f$, i.e. $f(\Theta) \subseteq \mathbb{R}_0^+$. In case $f(\Theta)$ is contained in a compact interval $[e, \bar{e}]$, which is bounded away from zero, i.e. $0 < e \leq \bar{e} < \infty$, then $g(\cdot; f) = Id$ on $[0, e]$ and $g(\cdot; f) = g(\bar{e})$ on $[\bar{e}, \infty)$. For this setting, the subsequent lemma shows that convergence in the sup norm $\|\cdot\|_\infty$ is preserved under the functional $\mathcal{R}$. Further, the function $\mathcal{R}(g; f)$ then acts on a compact interval $[\underline{e}_\mathcal{R}, \bar{e}_\mathcal{R}]$, with standard continuation outside, i.e. $\mathcal{R}(g) = Id$ on $[0, \underline{e}_\mathcal{R}]$ and $\mathcal{R}(g) = g(\bar{e}) = \text{constant on } [\bar{e}_\mathcal{R}, \infty)$.

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Lemma 4. Suppose that \( g, g' \in \mathcal{A}_g \) are continuous, non-decreasing with \( 0 \leq g, g' \leq \text{Id}, \) and there exists \( 0 < e \leq \bar{e} < \infty \) such that \( g = g' = \text{Id} \) on \( [0, e] \) as well as, \( g(e) = g(\bar{e}) = g'(e) = g'(\bar{e}), \) for \( e \in [\bar{e}, \infty) \). Then \( \mathcal{R}(g), \mathcal{R}(g') = \text{Id} \) on \( [0, e_{\mathcal{A}}] \) as well as, \( \mathcal{R}(g)(e) = g(\bar{e}) \) and \( \mathcal{R}(g')(e) = g'(\bar{e}), \) for \( e \in [\bar{e}, \infty) \), where \( e_{\mathcal{A}} = e \) and \( \bar{e}_{\mathcal{A}} = \frac{\bar{e}}{e} \), and

\[
\| \mathcal{R}(g) - \mathcal{R}(g') \|_\infty \leq 2 \frac{\bar{e}_{\mathcal{A}}}{e_{\mathcal{A}}} \| g - g' \|_\infty. \tag{49}
\]

Proof of Lemma 4. Take \( e \in [0, e_{\mathcal{A}}] \) with \( e_{\mathcal{A}} = e \), then \( \mathcal{R}(g)(e) = g(e) = e \) and \( \mathcal{R}(g')(e) = g'(e) = e \), hence \( |\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| = 0 \). Now, take \( e \in (e, \bar{e}) \). Noting that \( g(z)/z = g'(z)/z = 1 \) on \( (0, e] \) and in general \( g/\text{Id}, g'/\text{Id} \leq 1 \), we see that

\[
\mathcal{R}(g)(e) = e \inf\{g(z)/z : e \leq z \leq e\}, \quad \text{and} \quad \mathcal{R}(g')(e) = e \inf\{g'(z)/z : e \leq z \leq e\}.
\]

Without loss of generality assume \( \mathcal{R}(g')(e) \leq \mathcal{R}(g)(e) \) and write

\[
\mathcal{R}(g)(e) = e \frac{g(z_0)}{z_0}, \quad \text{and} \quad \mathcal{R}(g')(e) = e \frac{g'(z'_0)}{z'_0},
\]

where \( z_0, z'_0 \geq e \) are the respective minimizing arguments of the expressions above. These quantities exist due the continuity of \( g \) and \( g' \), but are perhaps not unique. Then by assumption and the optimality of \( z_0 \) we have

\[
e \frac{g'(z'_0)}{z'_0} = \mathcal{R}(g')(e) \leq \mathcal{R}(g)(e) = e \frac{g(z_0)}{z_0} \leq e \frac{g(z'_0)}{z'_0},
\]

and

\[
0 \leq \mathcal{R}(g)(e) - \mathcal{R}(g')(e) \leq e \frac{g(z'_0)}{z'_0} - e \frac{g'(z'_0)}{z'_0} \leq e \frac{e}{z_0} \| g - g' \|_\infty \leq \frac{\bar{e}}{e} \| g - g' \|_\infty.
\]
Finally, we have to check the case $e \in (\bar{e}, \infty)$. We see that
\[
\mathcal{R}(g)(e) = e \left( \inf_{e \leq z \leq \bar{e}} \{ g(z) / z \} \land \inf_{\bar{e} \leq z \leq e} \{ g(z) / z \} \right) \\
= \left( \frac{e}{\bar{e}} \mathcal{R}(g)(\bar{e}) \right) \land \left( e \inf \{ g(\bar{e}) / \bar{e} \land e \} \right) \\
= \left( \frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} \right) \land g(\bar{e}), \text{ for } e \geq \bar{e},
\]
and analogously
\[
\mathcal{R}(g')(e) = \left( \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} \right) \land g'(\bar{e}), \text{ for } e \geq \bar{e}.
\]
Without loss of generality assume that $\mathcal{R}(g')(\bar{e}) \leq \mathcal{R}(g)(\bar{e})$. Define
\[
e_0 = \inf \{ e \geq \bar{e} : \mathcal{R}(g)(e) = g(\bar{e}) \} = \frac{g(\bar{e})}{\mathcal{R}(g)(\bar{e})}, \text{ and}
\]
\[
e'_0 = \inf \{ e \geq \bar{e} : \mathcal{R}(g')(e) = g'(\bar{e}) \} = \frac{g'(\bar{e})}{\mathcal{R}(g')(\bar{e})}.
\]
Consider the case $e \geq e'_0$, then $\mathcal{R}(g')(e) = \mathcal{R}(g')(e'_0) = g'(\bar{e})$, and
\[
|\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| = \begin{cases} 
|g(\bar{e}) - g'(\bar{e})|, & \text{for } e \geq (e_0 \lor e'_0), \\
\frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} e - \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} e'_0 |, & \text{for } e'_0 \leq e < (e_0 \lor e'_0).
\end{cases}
\]
Focusing on $e'_0 \leq e < (e_0 \lor e'_0)$, recalling $\mathcal{R}(g')(\bar{e}) \leq \mathcal{R}(g)(\bar{e})$ and using that the assumption $g, g'$ are non-decreasing implies $\mathcal{R}(g)(\bar{e}) \geq e$ and $\mathcal{R}(g')(\bar{e}) \geq e$, respec-
tively, gives

\[ |\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| = \left| \frac{\mathcal{R}(g)(\bar{e})}{\bar{e}} e - \frac{\mathcal{R}(g')(\bar{e})}{\bar{e}} e' \right| \]

\[ = \frac{1}{\bar{e}} \left( (e - e_0) + (\mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e})) e' \right) \]

\[ \leq \frac{1}{\bar{e}} \left( \bar{e} (e_0 \lor e_0') - e_0' + (\mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e})) \frac{g'(\bar{e})}{\mathcal{R}(g')(\bar{e})} \bar{e} \right) \]

\[ \leq ((e_0 - e_0') \lor 0) + (\mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e})) \frac{\bar{e}}{\bar{e}} \]

\[ \leq \left( \frac{g'(\bar{e})}{\mathcal{R}(g')(\bar{e})} \left( \frac{\mathcal{R}(g')(\bar{e})}{\mathcal{R}(g)(\bar{e})} g(\bar{e}) - g'(\bar{e}) \lor 0 \right) \right) + \frac{\bar{e}^2}{\overline{e}^2} \| g - g' \|_\infty \]

\[ \leq \frac{\bar{e}}{\overline{e}} \left( (g(\bar{e}) - g'(\bar{e})) \lor 0 \right) + \frac{\bar{e}^2}{\overline{e}^2} \| g - g' \|_\infty \]

\[ \leq 2 \frac{\bar{e}^2}{\overline{e}^2} \| g - g' \|_\infty \]

And thus for all \( e \geq e_0' \) we have

\[ |\mathcal{R}(g)(e) - \mathcal{R}(g')(e)| \leq 2 \frac{\bar{e}^2}{\overline{e}^2} \| g - g' \|_\infty . \]

For \( \bar{e} \leq e \leq e_0' \), compute

\[ |\mathcal{R}(h)(y) - \mathcal{R}(h')(y)| \leq \left( \mathcal{R}(g)(\bar{e}) - \mathcal{R}(g')(\bar{e}) \right) \frac{e}{\bar{e}} \leq \frac{\bar{e}}{\overline{e}} \| g - g' \|_\infty \frac{e_0'}{\bar{e}} \]

\[ \leq \frac{\bar{e}}{\overline{e}} \| g - g' \|_\infty \frac{1}{\overline{g'(\bar{e})}} \frac{g'(\bar{e})}{\bar{e}} \leq \frac{\bar{e}}{\overline{e}} \| g - g' \|_\infty \frac{1}{\overline{g'(\bar{e})}} \frac{\bar{e} e \bar{e}}{\bar{e}} \]

\[ = \frac{\bar{e}^2}{\overline{e}^2} \| g - g' \|_\infty \].

Noting that \( e_0, e_0' \leq \bar{e} \mathcal{R} = \frac{\bar{e}^2}{\overline{e}} \) and verifying that \( \mathcal{R}(g) \) and \( \mathcal{R}(g') \) are constant and equal to \( g(\bar{e}) \) and \( g'(\bar{e}) \), respectively, finishes the proof. \( \square \)

In order to prove Proposition 2, the value function \( v(\cdot; \cdot; \theta, g) \) of the optimal
stopping problem of the firm in (6), with \( \theta \in \Theta \) and \( g \in A_C^g \) such that \( g \) is non-decreasing and bounded by \( Id \), has to be characterized. Note that (6) is on the rating agency-scale, using the imperfectly observed cash flow \( D \) and its running minimum \( E \). It is helpful to also consider the firm’s optimal stopping problem also on the firm-scale, i.e. \( X = E/\theta \) and \( Y = D/\theta \), for \( \theta > 0 \). For \((x,y) \in C\), define

\[
w(x,y; \theta, g) = \sup_{\tau \in \mathcal{F}_{(x,y)}} \mathbb{E}_{(x,y)} \left[ \int_0^\tau e^{-rt} (X_t - C(\theta X_t / g(\theta Y_t))) \, dt \right],
\]

where the firm cash flow \( X \) follows (1), its running minimum \( Y = (Y_t)_{t \geq 0} \) is given by \( Y_t = \min(Y_0, \inf_{0 \leq s \leq t} X_s) \), for \( t \geq 0 \), as well as \( g \in A_C^g \) is non-decreasing, bounded by \( Id \), and \( \theta \in \Theta \).

First properties of the value function \( w(\cdot, \cdot; \theta, g) \) defined in (50) are collected in the following Lemma. Therefore, it is useful to define the function \( g_\theta \) appearing inside the interest payment rate function \( C \) in (50) by \( g_\theta : \mathbb{R}^+_0 \to \mathbb{R}^+_0 \), \( y \mapsto g_\theta(y) = \frac{1}{\theta} g(\theta y) \).

**Lemma 5.** Denote \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \) the value functions specified in (14) and (50), respectively, for \( \theta > 0 \) and \( g \in A_C^g \), then \( g_\theta \in A_C^g \) and

\[
w(x,y; \theta, g) = v(\theta x, \theta y; \theta, g), \quad \text{for } (x,y) \in C,
\]

\[
w(x,y; \theta, g) = v(x,y; 1, g_\theta), \quad \text{for } (x,y) \in C.
\]

Moreover, for \( g \in A_C^g \) and \( 0 < \theta' \leq \theta \) it holds:

1. If \( g \) is non-decreasing, then \( g_\theta \) is non-decreasing, and \( \theta' g_\theta \leq g_\theta \).

2. If \( g/Id \) is non-increasing, then \( g_\theta/Id \) is non-increasing, and \( g_\theta \geq g_\theta \).

3. If \( g \leq Id \), then \( g_\theta \leq Id \).

**Proof of Lemma 5.** The equality (51) follows directly from the definition in (14), as \((D,E)\) is obtained from \((X,Y)\) by multiplying by \( \theta > 0 \). Equality (52) follows likewise, using the definition of \( g_\theta \) above and (50). To show part 1. observe that for
$0 < y' \leq y$, it follows $\theta y' \leq \theta y$, and by $g$ being non-decreasing, we have

$$g_\theta(y) = \frac{g(\theta y)}{\theta} \geq \frac{g(\theta y')}{\theta} = g_\theta(y'),$$

i.e. $g_\theta$ is non-decreasing. Now,

$$\theta g_\theta(y) = g(\theta y) \geq g(\theta' y) = \theta' g_\theta'(y'), \text{ for } y \geq 0,$$

since $g$ is non-decreasing and $\theta' y \leq \theta y$. For part 2., consider

$$\frac{g_\theta(y')}{y'} = \frac{g(\theta y')}{\theta y'} \geq \frac{g(\theta y)}{\theta y} = \frac{g_\theta(y)}{y}, \text{ for } y > 0,$$

since $g/Id$ is non-increasing and $\theta y \geq \theta y'$. Thus $g_\theta/Id$ is non-increasing. Further,

$$g_\theta'(y) = y \frac{g(\theta' y)}{\theta' y} \geq y \frac{g(\theta y)}{\theta y} = g_\theta(y), \text{ for } y > 0,$$

since $g/Id$ is non-increasing and $\theta' y \leq \theta y$. For part 3., write

$$g_\theta(y) = \frac{g(\theta y)}{\theta} \leq \frac{\theta y}{\theta} = y, \text{ for } y \geq 0,$$

since $g \leq Id$, hence $g_\theta \leq Id$.

The value functions $v$ and $w$ given in (14) and (50), respectively, are described by expectations conditioning on the starting values $(e, d)$ and $(x, y)$, respectively. For the subsequent analysis it is helpful to write the dependence on the starting value directly into the payoff function, which is possible since the driving processes $X$ and $E$, respectively, are geometric Brownian motions, see (1) and (2), respectively. Denote by $(\tilde{X}, \tilde{Y})$ the process $X$ defined in (14) with starting value 1 and $\bar{Y} = (\bar{Y}_t)_{t \geq 0}$.
its running minimum, i.e. \( \tilde{Y}_t = \inf_{0 \leq s \leq t} \tilde{X}_s \), for \( t \geq 0 \). Then

\[
v(d,e; \theta, g) = \sup_{\tau \in \mathcal{F}} \mathbb{E} \left[ \int_{0}^{\tau} e^{-rt} \left( \frac{d \tilde{X}_t}{\theta} - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) \right) dt \right], \tag{53}
\]

\[
w(x,y; \theta, g) = \sup_{\tau \in \mathcal{F}} \mathbb{E} \left[ \int_{0}^{\tau} e^{-rt} \left( x \tilde{X}_t - C(x \tilde{X}_t / g(\min(y, x \tilde{Y}_t))) \right) dt \right], \tag{54}
\]

where \( \mathcal{F} \) denotes the set of stopping times w.r.t. to the filtration generated by \((\tilde{X}, \tilde{Y})\).

To allow for straightforward calculations subsequently, we extend the interest payment rate function \( C \) from \([1, \infty]\) trivially to \([0, \infty]\) by setting \( C[0,1) = C(1) \). This representation allows to establish the following properties of \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \).

**Lemma 6.** Denote \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \) the value functions specified in (14) and (50), respectively, for \( \theta > 0 \) and \( g, g' \in \mathcal{A}^C \), then the following holds true:

1. \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \) are non-negative.
2. If \( 0 < \theta' \leq \theta \), then \( v(\cdot, \cdot; \theta', g) \geq v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta', g) \leq w(\cdot, \cdot; \theta, g) \).
3. If \( g' \leq g \), then \( v(\cdot, \cdot; \theta, g') \geq v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g') \geq w(\cdot, \cdot; \theta, g) \).
4. \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \) are non-increasing in \( e \) and \( y \), respectively, i.e.:

\[
v(d,e'; \theta, g) \geq v(d,e; \theta, g), \text{ for } 0 \leq e' \leq e < \infty,
\]

\[
w(x,y'; \theta, g) \geq w(x,y; \theta, g), \text{ for } 0 \leq y' \leq y < \infty.
\]

5. \( v(\cdot, \cdot; \theta, g) \) and \( w(\cdot, \cdot; \theta, g) \) are non-decreasing on rays starting in the origin, i.e.:

\[
v(d,e; \theta, g) \leq v(\lambda d, \lambda e; \theta, g), \text{ for } (d,e) \in \mathcal{C} \text{ and } \lambda \geq 1,
\]

\[
w(x,y; \theta, g) \leq w(\lambda x, \lambda y; \theta, g), \text{ for } (x,y) \in \mathcal{C} \text{ and } \lambda \geq 1.
\]

**Proof of Lemma 6.** Part 1. follows from \( \tau = 0 \). For the remainder of the proof recall the convention the interest payment rate function \( C \) is extended from \([1, \infty]\) trivially.

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to $[0, \infty]$ by setting $C|_{[0,1)} = C(1)$. For part 2., we focus on $v$ as given in (53) and compare the accumulated discounted net income stream until $\tau \in \tilde{T}$. For $0 < \theta' \leq \theta$, we have almost surely

$$d \tilde{X}_t / \theta' - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) \geq \tilde{X}_t / \theta - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t))) ,$$

and hence almost surely

$$\int_0^\tau e^{-rt} \left(d \tilde{X}_t / \theta' - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t)))\right) dt \geq \int_0^\tau e^{-rt} \left(\tilde{X}_t / \theta - C(d \tilde{X}_t / g(\min(e, d \tilde{Y}_t)))\right) dt .$$

The inequality is preserved by taking the expectation and the supremum over all $\tau \in \tilde{T}$, and the first assertion of part 2. follows. For $w$, we take (50). For $g/Id$ non-increasing, as assumed, part 2. of Lemma 5 gives that $0 < \theta' \leq \theta$ implies $g_{\theta'} \geq g_{\theta}$, and since $C$ is non-increasing we have

$$x \tilde{X}_t - C(x \tilde{X}_t / g_{\theta'}(\min(y, x \tilde{Y}_t))) \leq x \tilde{X}_t - C(x \tilde{X}_t / g_{\theta}(\min(y, x \tilde{Y}_t))) ,$$

and by similar arguments as before, i.e. integrating the discounted payoff stream over $[0, \tau]$ as well as noting the inequality is preserved by taking the expectation and the supremum over all $\tau \in \tilde{T}$, it follows that $w(x, y; \theta', g) \leq w(x, y; \theta, g)$, for all $(x, y) \in \mathcal{C}$, as claimed. To show part 3., observe that for $g' \leq g$ we have

$$C(a / g'(b)) \leq C(a / g(b)) , \text{ for } (a, b) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ ,$$

since $C$ is non-increasing, and the first assertion follows using similar arguments as for part 2. For $w$, we calculate

$$g_{\theta}'(y) = \frac{g_{\theta}'(\theta y)}{\theta} \leq \frac{g(\theta y)}{\theta} = g_{\theta}(y) , \text{ for } y > 0 .$$

Hence, $g'_{\theta} \leq g_{\theta}$ and the second assertion follows by identical arguments as the first assertion. To verify assertion 4, observe that $g$ is non-decreasing by assumption, and thus $g_{\theta}$ is also non-decreasing by part 1. of Lemma 5. Therefore, $g(b') \leq g(b)$ and

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\( g_\theta(b') \leq g_\theta(b) \), for \( 0 \leq b' \leq b \). Since \( C \) is non-increasing, it holds

\[
C(a/g(b')) \leq C(a/g(b)) \quad \text{and} \quad C(a/g_\theta(b')) \leq C(a/g_\theta(b)), \quad \text{for} \ a \geq 0 \quad \text{and} \quad 0 < b' \leq b.
\]

Applying the same arguments as in part 3. to the representations in (53) and (54) gives the claimed result. Now, part 5. is verified. Note that \( g/Id \) is non-increasing by assumption, which implies by part 2. of 5 that \( g \geq g_\lambda \), where we set \( 1 = \theta' \leq \theta = \lambda \), and

\[
\frac{a}{g(b)} \leq \frac{a}{g_\lambda(b)} = \frac{\lambda a}{g(\lambda b)}, \quad \text{for} \ (a, b) \in \mathbb{R}_0^+ \times \mathbb{R}^+.
\]  

(55)

Taking a look at the net income rate in (53) for \( v \) with starting value \( (\lambda d, \lambda e) \), where \((d, e) \in \mathcal{C}\) and \( \lambda \geq 1 \), we obtain

\[
\lambda d \bar{X}_t/\theta - C(\lambda e \bar{X}_t/g(\min(\lambda e, \lambda d \bar{Y}_t)))
\]

\[
= d \bar{X}_t/\theta - C(d \bar{X}_t/g(\min(e, d \bar{Y}_t))) + (\lambda - 1) d \bar{X}_t/\theta
\]

\[
+ C(d \bar{X}_t/g(\min(e, d \bar{Y}_t)) - C(\lambda d \bar{X}_t/g(\min(e, d \bar{Y}_t)))
\]

\[
\geq d \bar{X}_t/\theta - C(d \bar{X}_t/g(\min(e, d \bar{Y}_t))),
\]

since \((\lambda - 1) d \bar{X}_t/\theta\) is greater equal to zero due to \( \lambda \geq 1 \) \( C(d \bar{X}_t/g(\min(e, d \bar{Y}_t))) - C(\lambda d \bar{X}_t/g(\min(e, d \bar{Y}_t))) \geq 0 \) thanks to (55), set \( a = d \bar{X}_t \) and \( b = \min(e, d \bar{Y}_t) \).

And by similar arguments as before, i.e. integrating the discounted payoff stream over \([0, \tau]\) as well as noting the inequality is preserved by taking the expectation and the supremum over all \( \tau \in \mathcal{T} \), it follows that \( v(d, e; \theta, g) \leq v(\lambda d, \lambda e; \theta, g) \), for all \((d, e) \in \mathcal{C}\), as claimed. When considering \( w \), observe that from Lemma 5 part 2. it follows that \( g_\theta/Id \) is non-increasing. Now, the similar reasoning as for \( v \) applies and the proof is finished.

For a given rating agency strategy \( g \in \mathcal{G}_g^\mathcal{C} \) and \( \theta > 0 \), the early exercise region

\[
\mathcal{E}(\theta; g) = \{(d, e) \in \mathcal{C} : v(d, e; \theta, g) = 0\},
\]

(56)

allows us to characterize the best response of the firm \( \tau(\theta; g) \) as first hitting time. An
important subset of $\mathcal{E}(\theta; g)$ is that on the diagonal, which is identified with $\mathcal{D}(\theta; g)$ and the corresponding supremum $f(\theta; g)$, i.e.

$$\mathcal{D}(\theta; g) = \{d \in \mathbb{R}_0^+ : (d, d) \in \mathcal{E}(\theta; g)\}, \quad \text{and} \quad D(\theta; g) = \sup \mathcal{D}(\theta; g). \quad (57)$$

**Lemma 7.** Let $\mathcal{E}(\theta; g)$, $\mathcal{D}(\theta; g)$, and $D(\theta; g)$ be given by (56) and (57), respectively, for $\theta > 0$ and $g \in \mathcal{A}_g^C$. Then

$$\mathcal{D}(\theta; g) = [0, D(\theta; g)], \quad (58)$$

and for $(d, e) \in \mathcal{E}(\theta; g)$ it holds that

$$e \leq d \leq D(\theta; g). \quad (59)$$

**Proof of Lemma 7.** To see the first assertion, note that $v(0, 0; \theta, g) = 0$, and hence $\mathcal{D}(\theta; g)$ is non-empty. For $d \in \mathcal{D}(\theta; g)$, we have $d' \in \mathcal{D}(\theta; g)$, for $d' \in (0, d]$ by part 5. of Lemma 6 by setting $\lambda = d/d' \geq 1$. Further, we have for $d > \theta C$ that $v(d, d; \theta, g) > 0$, since then the income stream from not defaulting in $(d, d)$ is strictly positive. Accordingly, $\mathcal{D}(\theta; g)$ is a convex and bounded subset of $\mathbb{R}_0^+$. Since $v(\cdot, \cdot; \theta, g)$ is continuous, $\mathcal{D}(\theta; g)$ is also closed and $D(\theta; g) \in \mathcal{D}(\theta; g)$, and (58) follows. For the second assertion, take $(d, e) \in \mathcal{E}(\theta; g)$, then $(d, d) \in \mathcal{E}(\theta; g)$ by part 4. of Lemma 6, what is equivalent to $d \in \mathcal{D}$. Thus, $d \leq D(g; h)$. Since $(d, e) \in \mathcal{E}(\theta; g) \subseteq \mathcal{E}$ we have $e \leq d$ finishing the proof. \hfill \square

**Lemma 8.** The set $\mathcal{K}_f$ is convex and compact in $(C(\Theta, \mathbb{R}), \| \cdot \|_\infty)$. Moreover, $\mathcal{K}_f$ is uniformly bounded by $\overline{\theta} \overline{f}$ and uniformly Lipschitz continuous with Lipschitz $L_f = \overline{f}$.

**Proof of Lemma 8.** To see that $\mathcal{K}_f$ is convex, take $f, f' \in \mathcal{K}_f$, $\lambda \in [0, 1]$ and define $f^\lambda = \lambda f + (1 - \lambda) f'$. Now, $f^\lambda$ is continuous, since $f, f'$ are, hence $f^\lambda \in C(\Theta, \mathbb{R})$. For $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$ we have

$$f^\lambda(\theta) - f^\lambda(\theta') = \lambda f(\theta) + (1 - \lambda) f'(\theta) - \lambda f(\theta') - (1 - \lambda) f'(\theta')$$

$$= \lambda (f(\theta) - f(\theta')) + (1 - \lambda) (f'(\theta) - f'(\theta'))$$

$$\leq \lambda L_f (\theta - \theta') + \lambda L_f (\theta - \theta') = L_f (\theta - \theta').$$
Using similar reasoning, one verifies that all conditions of the definition of $\mathcal{K}_f$ in (31) hold for $f^\lambda$, and thus $f^\lambda \in \mathcal{K}_f$. Accordingly, $\mathcal{K}_f$ is convex. To see that $\mathcal{K}_f$ is compact it is by Arzela-Ascoli sufficient to show that $\mathcal{K}_f$ is closed, bounded and equicontinuous. To show that $\mathcal{K}_f$ is closed consider a sequence $(f_n)_{n \geq 1}$ in $\mathcal{K}_f$ that converges to some $f \in C(\Theta_\ast, \mathbb{R})$, i.e. $\lim_{n \to \infty} \| f_n - f \|_\infty = 0$. For $\theta, \theta' \in \Theta$ with $\theta' \leq \theta$ we have

$$
f(\theta) - f(\theta') \leq f_n(\theta) + \| f_n - f \|_\infty - f_n(\theta') + \| f_n - f \|_\infty \\
\leq L_f (\theta - \theta') + 2 \| f_n - f \|_\infty.
$$

This holds for all $n \geq 1$. As $n \to \infty$, we obtain $f(\theta) - f(\theta') \leq L_f (\theta - \theta')$. Using similar reasoning, one verifies that all conditions of the definition of $\mathcal{K}_f$ in (31) hold for $f$, and thus $f \in \mathcal{K}_f$. Accordingly, $\mathcal{K}_f$ is closed. That $\mathcal{K}_f$ is bounded follows immediately from the definition with uniform upper bound $\bar{\theta} f$. The equicontinuity of $\mathcal{K}_f$ is implied if all $f \in \mathcal{K}_f$ are Lipschitz continuous with a common Lipschitz constant $L_f$, which holds by the very definition of $\mathcal{K}_f$. Note that the common Lipschitz constant is given by $L_f = \bar{f}$.

**Lemma 9.** The set $\mathcal{K}_g$ is convex and compact in $(C(\Xi, \mathbb{R}_0^+), \| \cdot \|_\infty)$. Moreover, $\mathcal{K}_g$ is uniformly bounded by $\overline{\theta^2 f^2} / (\overline{\theta f})$ and uniformly Lipschitz continuous with Lipschitz constant $L_g = 1$.

**Proof of Lemma 9.** The proof follows along the same lines as that of Lemma 8 once it is shown that $\mathcal{K}_g$ is uniformly bounded and uniformly Lipschitz continuous. The uniform bound of $\overline{\theta^2 f^2} / (\overline{\theta f})$ follows from $g \leq Id$ for $g \in \mathcal{K}_h \subseteq (\Xi)$ and $\bar{\xi} = \theta^2 f^2 / (\theta f)$. For the uniform Lipschitz continuity, observe for $g \in \mathcal{K}_g$ and $e, e' \in \Xi$ with $e' \leq e$ that

$$0 \leq g(e) - g(e') = e g(e)/e - g(e') \\
\leq e g(e')/e' - g(e') = (e - e') g(e')/e' \leq (e' - e),$$

where the first step follows since $g \in \mathcal{K}_g$ is non-decreasing, and the second step from $g/Id$ is non-increasing, and the final step from $g \leq Id$. Accordingly, $g$ is Lipschitz continuous.
continuous with Lipschitz constant \( L_g = 1 \), which is common for all \( g \in \mathcal{K}_g \). Now, the remaining claims follow by the same arguments as in the proof of Lemma 8. \( \square \)