Life-Cycle Portfolio Choice with Imperfect Predictors

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Abstract

We study how imperfect predictors of stock returns affect life-cycle consumption and portfolio choice in the presence of undiversifiable labor income risk. Imperfect predictability forces investors to filter unobservable expected stock returns from realized predictive variables and stock returns and condition their decisions on the perceived conditional mean and variance of future stock returns. Recognizing the additional uncertainty from imperfect predictability, portfolios are more conservative than models with either perfect predictors or i.i.d. stock returns. The model therefore provides an explanation for why young stockholders might hold more conservative portfolios empirically, and quantifies the long run risks of investing in the stock market with imperfect predictability.

(JEL D14, G11, G17)

Key Words: Portfolio Choice over the Life Cycle, Stock Market Mean Reversion, Filtering, Stock Market Predictability, Imperfect Predictor.
1 Introduction

Optimal life-cycle portfolio choice is becoming increasingly important as households take control of their own financial decisions. A large recent literature shows the attractiveness of stocks for young investors when stock returns are i.i.d. because labor income can be thought of as an implicit riskless asset that the investor holds predominantly when young. Nevertheless, recent empirical studies provide some evidence supporting stock return predictability. In such predictive regression settings, the conditional expected stock return can be perfectly predicted. Popular predictors are the dividend/price ratio \( D/P \), the earnings per share \( EPS \) or the consumption-wealth ratio \( CAY \). Since these predictors themselves follow a persistent auto-regressive process (AR model), stock returns are mean reverting.

Given the evidence on stock market predictability, optimal portfolio choice in the presence of time varying stock returns is an active research area. In a life-cycle setting Michaelides and Zhang (2017) and Kraft et al. (2019) solve life-cycle models in which investors choose optimal consumption and asset allocation assuming variables exist that can perfectly predict expected stock returns. Perfect predictor models, however, seem restrictive because they assume that an observable predictor such as, for instance, the dividend/price ratio \( D/P \), the earnings per share \( EPS \) or the consumption-wealth ratio \( CAY \), follow a persistent auto-regressive process (AR model), stock returns are mean reverting.

\[
\begin{align*}
    r_{t+1} &= r_f + b\mu_t + z_{t+1} \\
    \mu_{t+1} &= a + \beta \mu_t + \varepsilon_{t+1}
\end{align*}
\]

\[\begin{bmatrix} z_{t+1} \\
\varepsilon_{t+1} \end{bmatrix} \sim \text{Normal}(0, \Omega)\]

where \( r_{t+1} \) denotes the real stock market return from time \( t \) to \( t+1 \), \( \mu_t \) is the predictor such as the dividend/price ratio at time \( t \), \( a \) and \( \beta \) are the regression’s intercept and slope coefficients of the predictor, \( r_f \) is the real risk free interest rate and \( z_{t+1} \) and \( \varepsilon_{t+1} \) are the white noises following a bi-variate normal distribution with mean of zero and covariance structure of \( \Omega \). When \( \beta = 0 \), this regression becomes the i.i.d. stock return model. Fama and French really focus on the importance of the D/P on long-time horizon. These observations show that the predictability of stock return is economically and statistically significant phenomenon that can not be dismissed. Fama and French (1989) is an excellent summary and example that documents and illustrates the time variation of expected stock returns.

\[\text{Kim and Omberg (1996), Breman et al. (1997), Brandt (1999), Campbell and Viceira (1999), Balduzzi and Lynch (1999), Campbell et al. (2001, 2003), and Wachter (2002) show that stock market risk premiums change materially with respect to the predictive factor(s) and analyze the implications for optimal portfolio choice.}\]
the dividend yield, can perfectly predict expected stock returns. This assumption can be criticized for data mining, non-robustness of test statistics and incorrect inference in small samples. Goyal and Welch (2008) re-examine the performance of different predictors and find that they offer weak predictive ability for expected stock returns, both in-sample and out-of-sample, consistent with Harvey (2017).\(^5\)

The relationship between the predictor and expected stock returns can be quite uncertain but can affect optimal portfolio choice in many different ways. For example, Xia (2001) shows how uncertainty about the predictability parameter in the predictive regression affects asset allocation, while Barberis (2000) shows that the predictability effects outweigh parameter uncertainty in long term strategic asset allocation. Pastor and Stambaugh (2009) more explicitly assume that the current expected stock return is unobservable and the predictor is imperfect and construct an imperfectly predictive system with noisy predictors to later argue (Pastor and Stambaugh (2012)) that the long run risks embedded in stock market investments are higher than previously thought.

How does the presence of such imperfect predictability affect optimal consumption and portfolio choice for a stockholder over the life cycle? Our contribution is to solve a life-cycle portfolio choice model with an imperfect predictor, jointly modeling an imperfect predictive system, liquidity constraints and non-diversifiable background labor income risk using Epstein-Zin (1989) preferences (hereafter, the imperfect predictor model). Our goal is to better understand how the imperfection in the predictive relation of stock returns affects saving and portfolio choice over the life cycle. In that sense the spirit of our contribution is probably closest to Pastor and Stambaugh (2012) who question the conventional wisdom that stocks are less risky in the long run due to the presence of various complications arising from imperfect predictability. Our contribution is to explicitly incorporate this logic in a quantitative life cycle portfolio choice setting.

Our key finding is that average portfolio allocations under the imperfect predictive

\(^5\)Ang and Bekaert (2007) also examine the predictive power of the dividend yield for forecasting the excess stock returns. They find that the univariate dividend yield regression provides a rather poor proxy to the true expected stock return.
system of stock returns are more conservative throughout the lifecycle than what either the perfect predictor or the i.i.d. stock returns model imply. This result substantially alters one of the main insights of models ignoring imperfect predictability. Specifically, such models predict that "stocks are for the young". With imperfect predictability, consistent with Pastor and Stambaugh (2012), stocks become more volatile in the long run, and therefore young households hold more conservative (balanced) portfolios.

This prediction of the imperfect predictor model is more consistent with empirical observation than either the i.i.d. stock returns or the perfect predictor models. When compared with the data from the U.S. Survey of Consumer Finances (SCF), the imperfect predictor model matches the data better than either the perfect predictor model or the i.i.d. stock returns model. Specifically, in the SCF data stockholder portfolios are balanced between bonds and stocks. Recently, Wachter and Yogo (2010) generate balanced portfolios through nonhomothetic utility over basic and luxury goods. In this paper, the balanced portfolio early in life arises due to the additional stock market uncertainty arising from imperfect predictability.

We limit our comparative statics across two extremes that capture the range of behavioral possibilities that exist. On one extreme, we compare the three models (i.i.d., perfect predictor and imperfect predictor) assuming the data generating process coincides with the data generating process for actual returns (that is, the i.i.d. investor makes decisions according to the i.i.d. model and the data generating process for stock returns follows the i.i.d. model). On the other extreme, we compare the three models assuming the data generating process for stock returns is from the imperfect predictive system but each investor makes decisions according to the three different models (for example, the i.i.d. investor makes decisions according to their expectation that returns are i.i.d.). We find that in either of these extreme assumptions the imperfect predictive system generates balanced portfolios throughout the life cycle.

Nevertheless, important differences arise across different comparisons. When the model coincides with the correct investor expectation about the stock return process,
the perfect predictor model generates the highest wealth accumulation among the three models. Stock market predictability generates aggressive portfolios that are rewarded with higher wealth accumulation over time. The imperfect predictability model generates less volatile portfolios that are not as accurate, and generates the second highest wealth accumulation while the i.i.d. model generates the lowest wealth accumulation among the three models.

The predictions change dramatically when the stock return process is generated by the imperfect predictive system but the investors believe they are facing either an i.i.d. stock returns or perfect predictability world. In that instance, the perfect predictability investors are penalized the most because they make aggressive asset allocation decisions that are proven wrong many times. They therefore accumulate the least amount of wealth across all models and wealth accumulation is substantially lower than the case when their perceptions and the correct model coincide. The i.i.d. model generates the second highest wealth accumulation in this instance, and the imperfect predictability model, as expected, delivers the highest wealth accumulation.

Our conclusions that portfolios are substantially more balanced over the life cycle with the imperfect predictor model relative to the i.i.d. stock returns model are further supported when investors are less risk averse or when investors are more impatient. In both cases the i.i.d. model predicts a complete portfolio specialization in stocks (Heaton and Lucas (2000) and Haliassos and Michaelides (2003)) almost throughout the working part of the life cycle, while this does not happen with the imperfect predictor model.

To better understand the predictions of the imperfect predictive model, we perform different comparative statics varying a number of different parameters. In understanding the results it is helpful to define a predictability and a mean reversion effect. The predictability effect controls how the conditional expected return is updated based on the realization of the noisy observed predictor and the mean reversion effect how the conditional expected return is updated based on the realization of observed stock returns.
The imperfect predictive system generates similar predictions to the perfect predictive system when the persistence of the unobserved factor is closer to the persistence of the observed factor. In those instances the parameters controlling the predictability effects in the filtering problem become more important, the unobserved factor becomes similar to the observed factor, and therefore the two models become more similar. Hence, a crucial determinant of how the imperfect predictability model behaves is how close the imperfect predictor is perceived to be relative to the observed predictor versus how close it is to the i.i.d. stock returns model. Those are the two extremes and the relative variances of the different variables determine the extent to which portfolios differ across models.

The paper is organized as follows. Section 2 explains the theoretical model and Section 3 discusses the calibration. Section 4 undertakes several comparative statics to understand the implications of the model and Section 5 evaluates hedging demands from the imperfect predictive system. Section 6 concludes.

2 The Model

2.1 Assumptions

2.1.1 Preferences

We denote adult age by \( t \) (\( t \in [20, 100] \)). The investor chooses the portfolio and consumption policies to maximize the following Epstein-Zin preferences:

\[
\begin{align*}
V_t &= \max_{(c_t, \alpha_t)} \left\{ (1 - \beta) C_t^{1-1/\psi} + \beta [R_t (V_{t+1})]^{1-1/\psi} \right\}^{1/(1-1/\psi)} \\
R_t (V_{t+1}) &= [E_t (p_{t+1} V_{t+1}^{1-\gamma} + b (1 - p_{t+1}) X_{t+1}^{1-\gamma})]^{1/(1-\gamma)}
\end{align*}
\]

where \( V_t \) is the continuation value at age \( t \), \( R_t \) is the uncertainty aggregator, \( X_{t+1} \) is the terminal wealth if the investor is dead at age \( t+1 \), \( \beta \) is the discount factor, \( \psi \) is the elasticity of inter-temporal substitution (hereafter, EIS), \( \gamma \) is the risk aversion
parameter, b is the strength of the bequest motive and $p_{t+1}$ is the conditional probability of surviving next period conditional on having survived until age t.

2.1.2 Labor Income Process

Following Cocco et al. (2005) and Carroll (1997), the labor income process before retirement is:

$$Y_{it} = Y^p_{it}U_{it}$$  \hspace{1cm} (2)

$$Y^p_{it} = \exp[g(t, Z_{it})]Y^p_{it-1}N_{it}$$  \hspace{1cm} (3)

where $g(t, Z_{it})$ is a deterministic function of age and household i's characteristics $Z_{it}$, $Y^p_{it}$ is a permanent component with innovation $N_{it}$ of household i's age t labor income, and $U_{it}$ is a transitory component of household i's age t labor income.

In equations (2) - (3), we assume that $ln(U_{it})$ and $ln(N_{it})$ are independent and identically distributed with mean $\{-0.5\sigma^2_u, -0.5\sigma^2_n\}$, and variances $\sigma^2_u$ and $\sigma^2_n$, respectively, with $ln(Y^p_{it})$ evolving as a random walk with a deterministic drift, $g(t, Z_{it})$. For simplicity, retirement is assumed to be exogenous and deterministic, with all households retiring in time period $K$, corresponding to age 65 ($K = 46$). Earnings in retirement ($t > K$) are given by $Y_{it} = \lambda Y^p_{iK}$, where $\lambda$ is the replacement ratio ($\lambda = 0.68$) of the last working period permanent component of labor income.

Durable goods, and in particular housing, can provide an incentive for higher spending early in life. We exogenously subtract a fraction of labor income every year allocated to durables (housing), and this fraction includes both rental and mortgage expenditures. This empirical process is taken from Gomes and Michaelides (2005) and is based on Panel Study Income Dynamics (PSID) data. We choose not to model explicitly the returns from housing following the empirical evidence (e.g., Cocco and Lopes (2015) and references therein) that households tend not to decumulate housing as fast as life-cycle models predict. A prominent explanation tends to be a psychological benefit from
continuing to own one’s house, an explanation that is consistent with the low observed demand for home equity conversion mortgages (Davidoff (2015)). For these reasons we do not explicitly model the potential effects of housing returns, and focus instead only on investments of liquid financial wealth for rich households (that empirically tend to be both stockholders and homeowners (Vestman (2018))).

2.1.3 Imperfect predictive system of stock returns

There are two assets in which the investor can invest, a risk-free asset and a risky diversified asset (stock). The risk free asset has a constant gross real return of $r_f$, and the risky asset has a gross real return $r_t$. Following Pastor and Stambaugh (2009), the expected stock returns are unobservable and therefore the investor must filter these expected stock returns from the other observable information. The observable information will be the realized stock return and the realization of a predictor variable such as the dividend yield. Denote by $(\mu_t, q_t, r_t)$ the unobservable expected stock return, the predictor and the stock return, respectively. Then, an imperfect predictive system can be defined as follows:

$$
\mu_{t+1} = \alpha_\mu + \phi_\mu \mu_t + \varepsilon_{t+1}
$$

$$
q_{t+1} = \alpha_q + \phi_q q_t + \upsilon_{t+1}
$$

$$
r_{t+1} = r_f + \mu_t + z_{t+1}
$$

where $\begin{bmatrix} \varepsilon_{t+1}, \upsilon_{t+1}, z_{t+1} \end{bmatrix} \sim Normal (0, \Omega)$ and $\Omega = \begin{bmatrix}
\sigma_\varepsilon^2 & \sigma_{\varepsilon \varepsilon} & \sigma_{\varepsilon z} \\
\sigma_{\varepsilon \varepsilon} & \sigma_{\upsilon \varepsilon} & \sigma_{\upsilon z} \\
\sigma_{\varepsilon z} & \sigma_{\upsilon z} & \sigma_z^2
\end{bmatrix}$.

The unobservable expected stock return ($\mu_t$) follows a simple AR(1) process described by equation (4). Equation (5) assumes that the predictor ($q_t$) follows an AR(1)
process, a typical assumption in the literature about stock return predictability. Equation (6) defines next period’s stock return \( (r_{t+1}) \) as a sum of the risk free rate \( (r_f) \), the unobservable expected stock return \( (\mu_t) \) and an innovation term (unexpected stock return, \( z_t) \). This model is consistent with a variety of economic models in which the expected return not only varies over time but also exhibits mean reversion.

Based on this imperfect predictive system, the investor must filter out \( \mu_t \) from the other observable variables \( (r_t, q_t) \). Applying a filtering algorithm (Tsay (2010), Chapter 11), the first two conditional expected moments of \( \mu_t \) can be rewritten as

\[
E(\mu_t|\{r_t, q_t\}) = E_r + \Sigma_{\mu[r,q]} \Sigma_{[r,q]}^{-1} \begin{bmatrix} r_t - r_f \\ q_t \end{bmatrix} - \begin{bmatrix} E_r \\ E_q \end{bmatrix}
\]

\[
Var(\mu_t|\{r_t, q_t\}) = \sigma_\mu^2 - \Sigma_{\mu[r,q]} \Sigma_{[r,q]}^{-1} \Sigma_{\mu[r,q]}'
\]

where \( \Sigma_{\mu[r,q]} = [\sigma_{\mu r}, \sigma_{\mu q}] \) and \( \Sigma_{[r,q]} = \begin{bmatrix} \sigma_r^2 & \sigma_r \sigma_q \\ \sigma_r \sigma_q & \sigma_q^2 \end{bmatrix} \).

Equations (7) and (8) can be further simplified as:

\[
E(\mu_t|\{r_t, q_t\}) = \hat{\mu}_t = E_r + \kappa_r [r_t - r_f - E_r] + \kappa_q [q_t - E_q]
\]

\[
Var(\mu_t|\{r_t, q_t\}) = Var(\hat{\mu}_t) = \sigma_\mu^2 - \kappa_r \sigma_{\mu r} - \kappa_q \sigma_{\mu q}
\]

where \( \kappa_r = \frac{\sigma_{\mu r}^2 - \sigma_{\mu q}^2 \sigma_{\mu q}}{\sigma_r^2 \sigma_q^2 - \sigma_{r q}^2}, \kappa_q = \frac{\sigma_{\mu q}^2 - \sigma_{r q}^2 \sigma_{\mu r}}{\sigma_r^2 \sigma_q^2 - \sigma_{r q}^2}, E_r = \frac{\alpha_r}{1-\phi_r}, E_q = \frac{\alpha_q}{1-\phi_q}, \sigma_r^2 = \sigma_\mu^2 + \sigma_\varepsilon^2, \sigma_q^2 = \frac{\sigma_\mu^2}{(1-\phi_r^2)}, \sigma_{\mu r} = \frac{\sigma_{\mu q}^2}{(1-\phi_q^2)}, \sigma_{\mu q} = \rho_{\mu \varepsilon} \sigma_\varepsilon \sigma_\mu + \frac{\phi_{\mu q} \sigma_{\mu q}^2}{(1-\phi_\mu \phi_q)}, \sigma_{r q} = \rho_{\mu \varepsilon} \sigma_\varepsilon \sigma_r + \frac{\phi_{r q} \sigma_{r q}^2}{(1-\phi_\mu \phi_q)} \).
expected stock return. Therefore, $\kappa_r$ measures the mean reversion effect. In contrast, a positive $\kappa_q$ measures the predictability effect because a positive shock to the current predictor value predicts an increase in next period’s expected stock return.

Similarly, the conditional variance of $\mu_t$ in equation (10) can be decomposed into three parts: the variance of unobservable expected stock returns ($\sigma_{\mu}^2$), the covariance between the unobservable expected stock returns and the realized stock returns ($\sigma_{\mu r}$), and the covariance between the unobservable expected stock returns and the predictor ($\sigma_{\mu q}$). If $\kappa_r$ is negative, and the covariance between unobservable expected stock returns and realized stock returns is negative, then the conditional variance of next period expected stock returns will be lower. Therefore, $\kappa_r$ again can be interpreted as measuring a mean reversion effect. In contrast, if $\kappa_q$ is positive, and the covariance between unobservable expected stock returns and the predictor ($\sigma_{\mu q}$) is positive, then the conditional variance of next period stock returns will again be lower. Therefore, we can interpret again $\kappa_q$ as measuring a predictability effect.

A few interesting and important special cases can arise from this imperfect predictive system under different parameter configurations. When $\rho_{\mu q} = 1$, $E_r = E_q$, $\sigma_{\mu} = \sigma_q$ and $\rho_{\mu r} = \rho_{rq}$, then the imperfect predictor is identical to the observed predictor and $\kappa_r = 0$ and $\kappa_q = 1$. Equations (9) and (10), therefore, become

$$\hat{\mu}_t = E(\mu_t | [r_t, q_t]) = q_t$$

(11)

$$\text{Var}(\mu_t | [r_t, q_t]) = 0$$

(12)

Equations (11) and (12) imply that $E_t(r_{t+1}) = r_f + q_t$, namely, the predictor perfectly predicts the expected stock return. Therefore, the imperfect predictive system ((4) - (6)) degenerates into the classical predictive regression used in Campbell and Shiller (1988a), Campbell and Viceira (1999), and Michaelides and Zhang (2017), among others. This is a special case of the model and we call this a perfect predictive (PP)
system. In contrast, if $|\rho_{vc}| < 1$ and $\rho_{vc} \neq 0$, the predictor ($q_t$), is not a perfect proxy for $\mu_t$, and the information from $r_t$ and $q_t$ enters the conditional expected stock return ($\mu_t$) according to (9) - (10) and the expected stock return is not completely determined by the observed predictor.

Similarly, the i.i.d. stock returns model is also a special case of this imperfect predictive system. When $E_r = E_q$, $\sigma_\mu = \sigma_q = 0$ and $\phi_\mu = \phi_q = 0.0$, then the imperfect predictor is identical to a model with i.i.d. stock returns with the same unconditional mean and variance in stock returns as in the imperfect predictive system. When we later compare the implications across different models we pay special attention to comparing these special cases with the baseline imperfect predictor model to improve our understanding of each model’s implications.

### 2.2 The Investor’s Optimization Problem

At the beginning of each year $t$, investor $i$ has a wealth $W_{i,t}$. Then, labor income $Y_{i,t}$ is realized. Following Deaton (1991), cash on hand $X_{i,t}$ can be defined as $X_{i,t} = W_{i,t} + Y_{i,t}$. Then, the investor must determine how much to consume, $C_{i,t}$ and how to invest the remaining savings in stocks $S_{i,t}$ and the risk free asset $B_{i,t}$. In the next period, before earning period $t+1$’s labor income, the wealth at $t+1$ is given by

$$W_{i,t+1} = S_{i,t} (1 + r_{t+1}) + B_{i,t} (1 + r_f) = (X_{i,t} - C_{i,t}) (\alpha_{i,t} (1 + r_{t+1}) + (1 - \alpha_{i,t}) (1 + r_f)),$$

where $S_{i,t}$ is the investment in the stock market in the previous period, $B_{i,t}$ is the investment in risk-free asset in the previous period and $\alpha_{i,t}$ is the share of wealth in stocks in the previous period and defined as $\alpha_{i,t} = \frac{S_{i,t}}{B_{i,t} + S_{i,t}}$.

The investor maximizes the household’s utility subject to the budget constraint and the constraints (2) through (5) with the non-negativity restrictions on $C_{i,t}$, $B_{i,t}$ and $S_{i,t}$. The non-negativity constraints on $B_{i,t}$ and $S_{i,t}$ guarantee the investors do not borrow against their future labor income or retirement wealth.

In this optimization problem, $\mu_t$ is unobservable and investors have to estimate it through (9) - (10) conditional on the observed information $(r_t, q_t)$ available at time $t$. 
The state variables of the investor’s problem are \( t, X_{i,t}, \hat{\mu}_t \) (given by equation (9)) and \( Y_{p,t} \), the control variables are \( C_{i,t} \) and \( \alpha_{i,t} \), and the policy functions are defined as \( C_{i,t} \left( X_{i,t}, Y_{p,t}, \hat{\mu}_t \right) \) and \( \alpha_{i,t} \left( X_{i,t}, Y_{p,t}, \hat{\mu}_t \right) \). Since the value function is homogeneous, we can normalize the investor’s cash on hand \( X_{i,t} \) by \( Y_{p,t} \), reducing the number of state variables by one. The policy functions, therefore, become \( c_{i,t} \left( x_{i,t}, \hat{\mu}_t \right) \) and \( \alpha_{i,t} \left( x_{i,t}, \hat{\mu}_t \right) \), where \( x_{i,t} = \frac{X_{i,t}}{Y_{p,t}} \).

### 2.3 Numerical Solution

It is useful to start out by writing first the optimization problem faced by the investor, if the true expected stock return \( (\mu_t) \) were observable. In that instance, the investor would be solving the following model:

\[
V_t (x_{i,t}, \mu_t) = \max_{(c_{i,t}, \alpha_{i,t})} \left\{ (1 - \beta)c_{i,t}^{1 - \frac{1}{\psi}} + \beta \left[ E_t \left( p_{t+1} \left( \frac{Y_{p,t}}{Y_{p,t+1}} V_{t+1} (x_{i,t+1}, \mu_{t+1}) \right)^{1 - \gamma} \right) \right]^{\frac{1}{\gamma}} + b (1 - p_{t+1}) \left( \frac{Y_{p,t}}{Y_{p,t+1}} x_{i,t+1} \right)^{1 - \gamma} \right\}^{\frac{1}{1 - \frac{1}{\psi}}}
\]

\[
\begin{align*}
\mu_{t+1} & = \alpha_\mu + \phi_\mu \mu_t + \varepsilon_{t+1} \\
r_{t+1} & = r_f + \mu_t + z_{t+1} \\
\ln (N_{t+1}) & = \mu_n + n_{t+1} \\
\ln (U_{t+1}) & = \mu_u + u_{t+1} \\
x_{i,t+1} & = \frac{Y_{p,t}}{Y_{p,t+1}} (x_{i,t} - c_{i,t}) (r_{t+1} \alpha_{i,t} + r_f [1 - \alpha_{i,t}]) + U_{i,t+1}
\end{align*}
\]

where \( c_{i,t} \) is the normalized consumption of household \( i \) at time \( t \), \( x_{i,t} \) is the normalized cash on hand of household \( i \) at time \( t \) and \( \alpha_{i,t} \) is the risky asset allocation of household \( i \) at time \( t \). The main exogenous state variable is the observed persistent expected stock return: this is a perfect predictive model.

We can write down the VAR model that generates stock returns in the perfect predictive model as follows:
Contrary to the perfect predictive model (and previous work in life-cycle portfolio choice models), the actual VAR generating stock returns is not fully observable, and, therefore, the VAR that is being approximated will be the one arising from the investor’s filtering problem. From the perspective of the investor, $\mu_t$ is unobservable, and the VAR needs to be transformed to the perceived VAR arising from the investors’ filtering problem. The filtering problem changes the optimization model as follows:

$$V_t(x_{i,t}, \hat{\mu}_t) = \max_{(c_{i,t}, \alpha_{i,t})} \left\{ \begin{array}{l}
(1 - \beta)c_{i,t}^{1-\frac{1}{\psi}} + \beta \left( E_t \left( p_{t+1} \left( \frac{Y_P}{Y_{P,t+1}} V_{t+1}(x_{i,t+1}, \hat{\mu}_{t+1}) \right)^{1-\gamma} \right) \right) \left( \frac{1}{\gamma} \right)^{1-\frac{1}{\psi}} \\
+ b(1 - p_{t+1}) \left( \frac{Y_P}{Y_{P,t+1}} x_{i,t+1} \right)^{1-\gamma} \right\}
\right. \left\{ \begin{array}{l}
q_{t+1} = \alpha_q + \phi_q q_t + v_{t+1} \\
r_{t+1} = r_f + \hat{\mu}_t + z_{t+1} \\
\hat{\mu}_{t+1} = \hat{\mu}_t + \kappa_r (r_{t+1} - r_t) + \kappa_q (q_{t+1} - q_t) \\
ln(N_{t+1}) = \mu_n + n_{t+1} \\
ln(U_{t+1}) = \mu_u + u_{t+1} \\
x_{i,t+1} = \frac{Y_P}{Y_{P,t+1}} (x_{i,t} - c_{i,t}) (r_{t+1} \alpha_{i,t} + r_f [1 - \alpha_{i,t}]) + U_{i,t+1}
\end{array} \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. \right. 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\footnote{When we later simulate the model, the actual VAR will be used to simulate returns, and the filtering analysis implemented on the true DGP to get back the correct policy function.}
among these grid points. Then, using these grid points from the discretization of (15),
we can construct next period’s return.

Finally, we iteratively apply a backward induction algorithm to solve the consumption
and investment policy functions of the optimization problem (15) from age T to
age 20.8

3 Calibration

3.1 Imperfect predictive system

To maintain comparability with the literature, we start using the parameters of the
imperfect predictive system estimated by Pastor and Stambaugh (2012) and provide
extensive comparative statics results later in the paper. We list these parameters in
Table 1.

7The temporary part of labor income \((\ln(U_t))\) is not correlated with the other variables. Its grid
points are, therefore, generated independently.

8We solve the model numerically using an Intel Xeon E5-2699 v3 2.3GHz RAM 256GB. To reduce
computational time, we parallelize this algorithm along the discrete state variables’ dimension using
OpenMP.
Table 1 shows the parameter choices for equations (4) - (6). \(E_r\) is the unconditional expectation of the risk premium, \(E_q\) is the unconditional expectation of the predictor, \(\phi_q\) is the persistence parameter of the predictor, \(\phi_\mu\) is the persistence parameter of the unobserved expected stock return process, \(\sigma_v\) is the standard deviation of the observable predictor’s innovations, \(\sigma_r\) is the standard deviation of stock returns and \(\sigma_\varepsilon\) is the standard deviation of the shock to unobservable factor. The correlation between the shock to the unobservable \(\mu_t\) and the stock return innovation is denoted by \(\rho_{z\varepsilon}\), the correlation between the stock return innovation and the predictor innovation is \(\rho_{vz}\), and the correlation between the shock to the unobservable \(\mu_t\) and the predictor innovation \(\rho_{v\varepsilon}\). \(R^2\) is the fraction of the variance in \(r_{t+1}\) explained by \(\mu_t \left( \frac{\text{Var}(\mu_t)}{\text{Var}(r_{t+1})} \right)\), implying that 
\[
\sigma_\varepsilon = \sqrt{R^2 \times \text{Var}(r_{t+1}) \times (1 - \phi_\mu^2)}.
\]

These estimated values have specific implications for the two key filtering parameters that affect the conditional mean and variance of stock returns given by the filtering equations (9) and (10). For these parameters \(\kappa_r\) is equal to -0.1, and therefore an unexpected increase in the stock return leads to a decrease in next period’s expected stock return, implying a mean reversion in stock returns. The correlation between the unobservable expected stock returns and actual stock returns is negative, and therefore this also means the conditional variance of next period expected returns will be lower. The other key parameter, \(\kappa_q\), is 2.95 for these choices and therefore captures a predictability effect with higher expected returns next period from (9).

### 3.2 Micro parameters

Even though empirical predictability studies are typically done on a monthly or quarterly frequency, we solve the model at an annual frequency to maintain comparability with the existing life-cycle portfolio literature. Carroll (1997) estimates the variances
of the idiosyncratic shocks using data from the PSID, and the baseline simulations use values close to those: 0.1 for $\sigma_u$ and 0.1 for $\sigma_n$. The deterministic component of labor income is identical to the values used by most life cycle papers, for example, Cocco et al. (2005), and this setting also facilitates comparisons between this model and its counterparts such as perfect predictor model and i.i.d. stock returns model. The relatively large estimate for the replacement ratio during retirement (68% of last working period labor income) arises from using both social security and private pension accounts to estimate the benefits in the PSID data and is consistent with not explicitly modeling tax-deferred saving through retirement accounts.

The baseline preference specification is taken to capture the observed behavior of stockholders. Gomes and Michaelides (2005) argue that this is well achieved, when using a coefficient of relative risk aversion ($\gamma$) equal to 5. The elasticity of inter-temporal substitution ($\psi$) is set to be 0.5. These choices are close to the empirical estimates for the EIS in Vissing-Jorgensen (2002) and the empirical preference parameter estimates in Gomes et al. (2009). The bequest parameter is set to 2.5 to capture the empirical observation that few rich stockholders die with zero financial assets. As to the discount rate, much macroeconomic research estimates this rate to be 1% per quarter or approximate 4% per year. In order to emphasize that the results in this paper does not stem from extreme assumptions about discount factor, $\beta$ in the baseline model is 0.96, which means the discount rate is assumed to be 4% per year. We do experiment with both risk aversion and discount factor parameters later on.

There is no estimate of the correlation between the innovations of the unobservable expected stock returns and the permanent, idiosyncratic earnings shocks to the labor income ($\rho_{nu}$) in the literature. We therefore set this correlation equal to zero. Angerer and Lam (2009) note that the correlation between the innovations of stock returns and transitory part of labor income ($\rho_{zu}$) does not empirically affect portfolios and this is consistent with the simulation results in life cycle models (Cocco et al. (2005)). We set this correlation at zero. Similarly, we also set $\rho_{nv}$ to zero. The correlation between the
permanent earning shocks to the labor income and the innovations of stock returns ($\rho_{zn}$) is set equal to 0.15 in the baseline model, which follows the same setting as Michaelides and Zhang (2017). Table 2 summarizes the parameter values used in the baseline model.

Table 2 presents the parameter choice used in the baseline model. The $\sigma_r$ is the standard error of the stock returns, $\sigma_n$ is the standard error of the permanent part of labor income, $\sigma_u$ is the standard deviation of the transitory component of labor income, $\gamma$ is the risk aversion, $r_f$ is the real risk free rate, $\psi$ is the elasticity of inter-temporal substitution, $b$ is the bequest motive, $\rho_{zu}$ is the correlation between the innovations of stock returns and the transitory component of labor income, $\rho_{en}$ is the correlation between the innovations of dividend yield process and the shocks to the permanent part of labor income, $\rho_{eu}$ is the correlation between the innovations of dividend yield process and the transitory component of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income. $\rho_{eu}$ is the correlation between the innovations of unobservable expected stock returns and the transitory component of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income, $\rho_{eu}$ is the correlation between the innovations of unobservable expected stock returns and the transitory component of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income.

$\rho_{zu}$ is the correlation between the innovations of dividend yield process and the shocks to the permanent part of labor income, $\rho_{en}$ is the correlation between the innovations of dividend yield process and the shocks to the permanent part of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income, $\rho_{en}$ is the correlation between the innovations of unobservable expected stock returns and the shocks to the permanent part of labor income.

$E[\ln (N_t)]$ is the expectation of logarithm of the permanent earning shocks to the labor income, $E[\ln (U_t)]$ is the expectation of logarithm of the transitory earning shocks to the labor income, and $\beta$ is the discount factor of the utility function.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_r$</td>
<td>0.18</td>
<td>$r_f$</td>
<td>0.1</td>
<td>$\rho_{zu}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\sigma_n$</td>
<td>0.1</td>
<td>$\sigma_u$</td>
<td>0.1</td>
<td>$\rho_{vu}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\rho_{zn}$</td>
<td>0.0</td>
<td>$\rho_{zn}$</td>
<td>0.15</td>
<td>$\rho_{en}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$b$</td>
<td>2.5</td>
<td>$\psi$</td>
<td>0.5</td>
<td>$\rho_{eu}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>5</td>
<td>$\beta$</td>
<td>0.96</td>
<td>$E[\ln (N_t)]$</td>
<td>-0.005</td>
</tr>
<tr>
<td>$E[\ln (U_t)]$</td>
<td>-0.005</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

3.3 SCF Data

We perform a indicative calibration to illustrate the empirical implications of the model. It is an indicative calibration because our main goal is to compare the model’s implications with two other key models: the i.i.d. stock returns and the perfect predictor model. We focus on two key moments: average financial wealth to labor income ratios and the average share of wealth in stocks.
The empirical portfolio holding data are based on the 2007 Survey of Consumer Finances (SCF). Financial wealth is the net worth of the household excluding housing (variable FIN in the public extract). We use wages and salaries, pension and unemployment insurance to construct labor income and compare the average wealth to labor income ratio to the model. For the share of wealth in stocks we focus on stockholders using the variable HEQUITY. Conditional on being a stockholder the average share of wealth in stocks is defined as $\alpha = \frac{equity}{(equity + bond)}$.

### 3.4 Model versus Data

We next compare the implications of this model relative to the empirically observed averages from the 2007 SCF. We emphasize again that this should be taken as an indicative calibration and should not be interpreted as our main contribution. The point of this calibration is to illustrate that the model generates reasonable mean wealth to income and shares of wealth allocated in the stock market over the life cycle using calibration choices that are reasonable.

The mean wealth to income ratio is higher in this model than in the data. This is to be expected given that the model is solved with parameter choices used mostly under the assumption that households use an i.i.d. stock returns model when making decisions. Our main point is that the wealth accumulation is not substantially higher than the empirical counterparts, especially during retirement. At the same time, investors hold balanced portfolios throughout the life cycle. One can argue that this is happening because we have higher wealth accumulation, which drives households towards riskless assets for diversification reasons. We will address this concern in what follows as we view the main contribution of our paper to better understand the perceived long run risk implications from stock market investing.
Table 3 compares the W/Y ratio and the mean share of wealth in stocks (α) between the 2007 U.S. Survey of Consumer Finances (SCF) and the baseline model.

<table>
<thead>
<tr>
<th>Age Group</th>
<th>W/Y (SCF)</th>
<th>W/Y (Model)</th>
<th>Empirical α (SCF)</th>
<th>Mean α (Model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>18-34</td>
<td>0.91</td>
<td>0.81</td>
<td>0.41</td>
<td>0.79</td>
</tr>
<tr>
<td>35-44</td>
<td>1.80</td>
<td>3.15</td>
<td>0.47</td>
<td>0.76</td>
</tr>
<tr>
<td>45-54</td>
<td>3.45</td>
<td>6.07</td>
<td>0.48</td>
<td>0.70</td>
</tr>
<tr>
<td>55-64</td>
<td>5.66</td>
<td>9.04</td>
<td>0.48</td>
<td>0.62</td>
</tr>
<tr>
<td>65-74</td>
<td>9.30</td>
<td>9.51</td>
<td>0.41</td>
<td>0.58</td>
</tr>
</tbody>
</table>

### 3.5 Conclusion

In a calibrated life cycle model where households are uncertain about the extent of stock market predictability, we show that portfolios are quite balanced throughout the life cycle despite a very high expected equity premium, and wealth accumulation is not unreasonably high by the end of the life cycle, even though we did not try to match these moments. These results should be interpreted as an indicative calibration as we view our primary contribution as understanding the long run stock market risk of imperfect predictive systems in a life cycle model that allows portfolio rebalancing. In the next sections we try to understand further the model predictions, and in particular how the predictions differ from models that are more widely used, namely an i.i.d. stock returns model and a perfect predictability model. Moreover, we perform different comparative statics experiments to build intuition and to further understand the importance of different model parameters that are objectively difficult to estimate or uniquely identify from the data, while their effects on saving and portfolio choice are not at all obvious without further analysis.
4 Understanding model predictions

4.1 Imperfect predictor versus other models

In comparing the implications of the imperfect predictor to other models (an i.i.d. or perfect predictor model for stock returns) we first need to take a stand on the data generating process for stock returns. The correct data generating process could be the one arising from the imperfect predictive system but the investor could, for example, be behaving according to the i.i.d. stock returns model because the investor might not be able to understand the subtleties associated with the imperfect system. We limit our comparative statics across two extremes that capture the range of behavioral possibilities that exist. On one extreme, we compare the three models (i.i.d., perfect predictor and imperfect predictor) assuming the data generating process coincides with the data generating process for actual returns (that is, the i.i.d. investor makes decisions according to the i.i.d. model and the data generating process for stock returns follows the i.i.d. model). On the other extreme, we compare the three models assuming the data generating process for stock returns is from the imperfect predictive system but each investor makes decisions according to the three different models.\footnote{We adjust returns slightly so that that unconditional moments for stock returns are identical across the three models for comparability.}

Figure 1 plots the average life-cycle profiles of wealth accumulation, consumption and share of wealth in stocks by simulating 10,000 individual life histories for the baseline preference parameters reported in Table 2 across the three different models and assuming that each investor in each model is making the correct assumption about the data generating process for stock returns. The i.i.d. stock returns model is the most well-known relative to the literature: stocks are for the young since labor income is viewed as an implicit riskless asset that is most plentiful early in life. The perfect predictor model means that the realization of the factor sends a clear signal to the asset allocation decision: higher (lower) factor realizations lead to higher (lower) expected returns and a higher (lower) share of wealth in stocks. As a result, the average share of
Wealth in stocks is below the i.i.d. model over most of the life cycle. Wealth accumulation on the other hand is higher for the perfect predictor model given the investor’s ability to time the market, something that shows up as the higher mean consumption across all models in Panel C.

The imperfect predictor model wealth accumulation (Panel A) is somewhere between the i.i.d. and perfect predictor models. The stock market is more uncertain in this world and therefore the investor does not take as wild positions in and out of the stock market as in the perfect predictor model. As a result, the average share of wealth in stocks (Panel B) is almost everywhere lower than the share of wealth in stocks implied by either the perfect predictor or the i.i.d. model. This is reflected in the wealth accumulation (Panel A) and consumption (Panel C) profiles with the i.i.d. model generating the lowest mean wealth (consumption), and the perfect predictor model the highest, with the imperfect predictor model in the middle. These results illustrate that even when using the correct model, investors recognizing imperfect predictability will hold more conservative portfolios over the life cycle than investors assuming stock returns are i.i.d. or investors assuming perfect predictability. In the case of perfect predictability this is even despite the fact that wealth accumulation is higher than that from the imperfect predictability model, which usually is associated with a lower share of wealth in stocks in these models due to the treatment of labor income as an implicit riskless asset. Therefore, this result is not driven by differences in wealth accumulation but rather by the fact that stock market risk is recognized to be higher under the imperfect (relative to the perfect) predictability model.

Figure 2 repeats the experiment but now assumes that the data generating process is the one associated with the imperfect predictive system, yet the three investors make decisions according to the three different models. As a result, only the imperfect predictor investor is using the correct model, the other two investors are using a misspecified model since they make decisions according to the i.i.d. and perfect predictor model, whereas the data are being generated according to the imperfect predictive model.
Figure 2, Panel A, shows that the imperfect predictive model generates the highest wealth accumulation, while the i.i.d. investor accumulates the second highest wealth and the perfect predictor investor the lowest wealth. Comparing the average shares of wealth in stocks in Panel B, we can see that the i.i.d. investor retains the highest exposure to the stock market. Nevertheless, the perfect predictor investor now has the lowest share of wealth in stocks. This is now partly associated with the fact that this investor is the most aggressive in their allocation but now this behavior is costly because it is wrong. As a result, compared to Figure 1, the wealth accumulation goes from the highest to the lowest across the models and falls by a large margin (peaking at around 12 in Figure 1 and now at around 7 in Figure 2).

4.2 Preference parameters

How robust are the conclusions that the perfect and imperfect predictor models generate more balanced portfolios over the lifecycle? In this subsection investigate the robustness of this conclusion when we change the discount factor and the relative risk aversion parameter across the three models (i.i.d., perfect and imperfect predictor) assuming the investor is using the correct data generating process for stock returns in each instance.

Figure 3 plots the mean life-cycle profiles when the discount factor is reduced from the baseline $\beta = 0.96$ to $\beta = 0.94$. Panel A shows that, as expected, the mean wealth accumulation is lower at every age relative to when the discount factor ($\beta$) is 0.96 (Figure 1). Nevertheless, for both the mean wealth level (Panel A) and the average share of wealth in stocks (Panel B) the conclusions are qualitatively identical to Figure 1. Moreover, the quantitative deviation from the i.i.d. model’s complete portfolio specialization of the share of wealth in stocks over large parts early in the life cycle is starker in Figure 3. That is, in this case, the share of wealth in stocks is at one over larger parts of the life cycle, whereas this does not happen for either the perfect or imperfect predictor cases. Once more, the imperfect predictor model generates the most balanced portfolios over the life cycle, illustrating the long run uncertainty associated
with the stock market in the presence of imperfect predictive systems.

We repeat the same experiment by reducing relative risk aversion from $\gamma = 5$ to $\gamma = 2$. Figure 4 compares the life-cycle profiles across the three models. Panel A shows that the mean wealth accumulation ordering stays the same over the life-cycle (perfect, imperfect and i.i.d. from highest to lowest). Interestingly, the mean wealth accumulation is not much reduced relative to Figure 1 despite the much weaker precautionary savings motive. This happens because, as shown in Panel B, the mean share of wealth in stocks is closer to one. In fact, for the i.i.d. model the mean share of wealth in stocks is closer to one over almost the whole of the working life cycle. Therefore, wealth accumulation can arise on average from the positive equity risk premium. It is interesting to note that for the other two models the average share of wealth is always balanced and away from one. The perfect predictor model is more balanced than the i.i.d. model, reflecting the portfolio re-allocation for high to low shares in stocks depending on the factor realization. The imperfect predictor is even more balanced and below the perfect predictor model reflecting the additional uncertainty that arises from the imperfect predictive model.

We conclude that the imperfect predictive model generates the most balanced portfolios across bonds and stocks throughout the life cycle is robust to changes in preference parameters. In fact, the divergence from the portfolio predictions of the i.i.d. stock returns model becomes even stronger.

4.3 Uncertainty about persistence of unobserved factor

The persistence of the unobservable expected stock returns ($\phi_\mu$) is still potentially difficult to accurately pin down. How do this parameter affect life-cycle saving and portfolio choice?

The parameter $\phi_\mu$ measures the persistence of the unobservable expected stock returns. This parameter is of interest because the predictor used in the predictive regression is often a highly persistent process in the classical literature such as Campbell
and Shiller (1988b), Fama and French (1988), Xia (2001) and Cochrane (2005). Figure 5 shows comparative statics effects when changing the persistence of the unobservable expected return ($\phi_\mu$) on mean wealth accumulation (Panel A) and the share of wealth in stocks (Panel B) fixing $\sigma_\varepsilon = 0.041$. The persistence varies from $\phi_\mu = 0.5$ (lower than the baseline model) to $\phi_\mu = 0.9$ (higher than the baseline model).

We find that mean wealth accumulation shifts down when the persistence of the unobservable expected return is lower, because the $R^2$ is so low that the investor is misled more frequently by the observable predictor and there is a higher probability to make wrong decisions. When $\phi_\mu = 0.9$, on the other hand, the persistence of the unobservable expected return is closer to that of the observable predictor and we therefore move closer to the perfect predictive model. In that instance, the wealth accumulation shifts towards the perfect predictor model (Panel A), and the same happens for the mean share of wealth in stocks (Panel B).

Figure 6 plots how the persistence of the unobserved expected stock return ($\phi_\mu$) affects the life-cycle profiles when instead we keep $R^2 = 0.1$. When $R^2$ is fixed, the standard error of the unobserved expected stock return innovation ($\sigma_\varepsilon$) decreases as the persistence of the unobserved expected stock return increases. The portfolio allocation shifts up during early life and mean wealth accumulation increases. This arises because the decrease of $\sigma_\varepsilon$ makes the predictor a more perfect proxy of the unobserved expected stock return. When $\phi_\mu$ goes up, the life-cycle profiles are close to the perfect predictor case.

The intuition behind why the share of wealth in stocks rises when the persistence of the unobserved factor rises can also be seen by comparing the conditional expected returns, standard deviation and Sharpe ratio as ($\phi_\mu$) rises. This is done in Table 4 that reports these moments conditional on a positive (negative) realized stock return and observed predictor in Panel A (Panel B). It can be seen that the increase in the conditional Sharpe ratio is rising for positive shocks and falling for negative shocks as persistence increases but overall, taking the averages across the two panels the Sharpe
ratio tends to be, on average, higher than for lower persistence parameters. An equivalent explanation involves comparing the effect of the predictor through $\kappa_q$ relative to $\kappa_r$ as $(\phi_\mu)$ rises in Table 4. We can see that the effect on asset allocation through $\kappa_q$ becomes more important than $\kappa_r$ as the persistence of the unobserved predictor rises because in those instances the unobserved predictor becomes more similar to the observed predictor and the model therefore becomes more similar to the perfect predictive model. This is illustrated by the absolute value of the ratio of $\kappa_q$ to $\kappa_r$ that gets maximized (in absolute value) when $(\phi_\mu)=0.9$ which coincides with the baseline value of the persistence $(\phi_q)$ of the observed predictor.

| $\phi_\mu$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{\tau_{t+1}|t}$ | Cond. Sharpe Ratio |
|-----------|-------------|-------------|----------------------|----------------|------------------------|-------------------|
| 0.1       | -0.1797     | 0.5812      | -3.23                | 0.044          | 0.1765                 | 0.2494            |
| 0.3       | -0.1594     | 1.0200      | -6.40                | 0.0486         | 0.1765                 | 0.2754            |
| 0.5       | -0.1318     | 1.7036      | -12.93               | 0.0557         | 0.1761                 | 0.3163            |
| 0.7       | -0.0875     | 2.9520      | -33.76               | 0.0686         | 0.1741                 | 0.3944            |
| 0.9       | 0.0287      | 6.0659      | -37.27               | 0.1009         | 0.1592                 | 0.6343            |

Table 4, Panel A, summarizes the effect of $\phi_\mu$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are 1%.

| $\phi_\mu$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{\tau_{t+1}|t}$ | Cond. Sharpe Ratio |
|-----------|-------------|-------------|----------------------|----------------|------------------------|-------------------|
| 0.1       | -0.1797     | 0.5812      | -3.23                | 0.036          | 0.1765                 | 0.2039            |
| 0.3       | -0.1594     | 1.0200      | -6.40                | 0.0314         | 0.1765                 | 0.1778            |
| 0.5       | -0.1318     | 1.7036      | -12.93               | 0.0243         | 0.1761                 | 0.1378            |
| 0.7       | -0.0875     | 2.9520      | -33.76               | 0.0114         | 0.1741                 | 0.0652            |
| 0.9       | 0.0287      | 6.0659      | -37.27               | -0.0209        | 0.1592                 | -0.1316           |

Table 4, Panel B, summarizes the effect of $\phi_\mu$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are -1%.
4.4 Uncertainty about persistence of predictor

Figure 7 depicts how the persistence of the observed predictor ($\phi_q$) affects the life-cycle profiles.\(^{10}\) Panel A shows that changing $\phi_q$ does not significantly change mean wealth accumulation. Panel B shows that the persistence of the predictor could have some effect on household financial decisions. When $\phi_q$ is lower than the persistence of the unobserved expected stock return ($\phi_\mu$), the mean share of wealth in stocks shifts down as $\phi_q$ is reduced. The effect of predictability is reduced with a lower persistence of the factor and the average share of wealth in stocks falls. Nevertheless, the average effects are quantitatively not very large, partly because in the imperfect predictive system the investor expects the unobserved expected return to be persistent.

Table 5 illustrates how $\kappa_r$ and $\kappa_q$, as well as the conditional stock return moments, vary with $\phi_q$. For positive realizations of the stock returns and the predictor, as $\phi_q$ falls both $\kappa_r$ and $\kappa_q$ rise and the effect on portfolio choice will depend on which rises faster.

To understand that we compute the ratio of the two and find that the ratio of $\kappa_q/\kappa_r$ is maximized when the persistence of the observed predictor ($\phi_q$) is the same as the persistence of the unobserved predictor ($\phi_\mu$). When the difference becomes larger, the effect of predictability gets reduced because there is higher uncertainty conditional on the realized stock and predictor values and the share of wealth in stocks is reduced.

\(^{10}\)Since the change of $\phi_q$ does not affect the standard deviation, we keep $R^2 = 0.1$. 
TABLE 5
Effect of persistence ($\phi_q$) of observed factor

Table 5 Panel A summarizes the effect of $\phi_q$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are 1%.

| $\phi_q$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{r_{t+1}t}$ | Cond. Sharpe Ratio |
|----------|-------------|-------------|----------------------|-----------------|-------------------|-------------------|
| 0.1      | 0.1817      | 10.980      | 60.44                | 0.1516          | 0.1747            | 0.8677            |
| 0.3      | 0.1308      | 9.7969      | 74.90                | 0.1393          | 0.1738            | 0.8015            |
| 0.5      | 0.0642      | 8.0424      | 125.35               | 0.1211          | 0.1730            | 0.6997            |
| 0.7      | -0.0072     | 6.0342      | -838.74              | 0.1003          | 0.1726            | 0.5811            |
| 0.9      | -0.0875     | 2.952       | -33.76               | 0.0686          | 0.1741            | 0.3944            |

Table 5 summarizes the effect of $\phi_q$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are -1%.

| $\phi_q$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{r_{t+1}t}$ | Cond. Sharpe Ratio |
|----------|-------------|-------------|----------------------|-----------------|-------------------|-------------------|
| 0.1      | 0.1817      | 10.980      | 60.44                | -0.0716         | 0.1747            | -0.41             |
| 0.3      | 0.1308      | 9.7969      | 74.90                | -0.0593         | 0.1738            | -0.341            |
| 0.5      | 0.0642      | 8.0424      | 125.35               | -0.0411         | 0.1730            | -0.237            |
| 0.7      | -0.0072     | 6.0342      | -838.74              | -0.0203         | 0.1726            | -0.117            |
| 0.9      | -0.0875     | 2.952       | -33.76               | 0.0114          | 0.1741            | 0.0652            |
5 Hedging demands from imperfect predictive system

The imperfect predictive system provides a rich set of correlations that can generate hedging demands whose quantitative effects have not been previously studied in the context of a quantitative life cycle model. We therefore now study the effect on asset allocation of the three main correlations in the imperfect system, namely the correlation in the innovations between the unobserved expected stock returns, the observed imperfect predictor and the stock return.

5.1 Correlation between unobservable expected stock return and stock return innovation

To investigate the importance of the correlation between the shock to unobservable expected stock returns and the innovation of stock returns ($\rho_{ze}$), we vary $\rho_{ze}$ from -0.9 to -0.65 and use the baseline model ($\rho_{ze} = -0.9$) as a comparison benchmark. Figure 8 plots the mean wealth accumulation (Panel A) and the mean share of wealth in stocks (Panel B) over the life cycle when $\rho_{ze}$ varies from the baseline to -0.7. We can see that there is a very small difference in the average asset allocation (Panel B), and a very small change in the total wealth accumulation (Panel A).

To further understand these results, we report in Table 6 the effect of this correlation on conditional moments. For a smaller $|\rho_{ze}|$, the mean reversion effect ($\kappa_r$) becomes weaker, while the predictability effect ($\kappa_q$) becomes stronger. This implies that these two effects go against each other, while the quantitative differences across the correlations are not very large. This small change in conditional moments provides an explanation why the results in Figure 8 are close to the baseline model.
TABLE 6

Effect of correlation between unobservable expected stock return and stock return innovation ($\rho_{ze}$)

Table 6, Panel A, summarizes the effect of $\rho_{ze}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are 1%.

| $\rho_{ze}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{r_{t+1}|t}$ | Cond. Sharpe Ratio |
|-------------|------------|------------|-----------------------|-----------------|----------------------|-------------------|
| -0.65       | -0.0325    | 3.0645     | -94.17                | 0.0703          | 0.1747               | 0.4026            |
| -0.8        | -0.0655    | 2.9970     | -45.76                | 0.0693          | 0.1744               | 0.3975            |
| -0.9        | -0.0875    | 2.9520     | -33.75                | 0.0686          | 0.1741               | 0.3944            |
| -0.95       | -0.0984    | 2.9295     | -29.76                | 0.0683          | 0.1739               | 0.3928            |

Panel B summarizes the effect of $\rho_{ze}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are -1%.

| $\rho_{ze}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[R_{t+1}|t]$ | $\sigma_{r_{t+1}|t}$ | Cond. Sharpe Ratio |
|-------------|------------|------------|-----------------------|-----------------|----------------------|-------------------|
| -0.65       | -0.0325    | 3.0645     | -94.17                | 0.0097          | 0.1747               | 0.0554            |
| -0.8        | -0.0655    | 2.9970     | -45.76                | 0.0107          | 0.1744               | 0.0613            |
| -0.9        | -0.0875    | 2.9520     | -33.75                | 0.0114          | 0.1741               | 0.0652            |
| -0.95       | -0.0984    | 2.9295     | -29.76                | 0.0117          | 0.1739               | 0.0672            |

5.2 Correlation between stock return and predictor innovation

Changing the correlation between the innovations of stock returns and the shocks to the predictor ($\rho_{zv}$) does not materially affect the mean wealth accumulation (Figure 9, Panel A), but does significantly change the portfolio allocation (Figure 9, Panel B). When $\rho_{zv}$ moves from the baseline value of -0.9 to -0.7, the predictability effect becomes stronger, while the mean reversion effect from observing the current return does not change quantitatively as much. This can be seen by comparing the ratio of $\kappa_q/\kappa_r$ in Table 7: as the correlation moves from -0.95 to -0.65, the absolute value of the ratio rises. This means that the mean reversion effect captured by $\kappa_r$ is changing faster and becoming stronger than the predictability effect captured by $\kappa_q$ and therefore the average share of wealth in stocks is reduced for a given age. Nevertheless, the quantitative magnitudes
on conditional expected returns and variances are not very strong, which explains the finding that average wealth accumulation does not change substantially.

### Table 7
Effect of correlation between stock returns and predictor innovation ($\rho_{zv}$)

Table 7, Panel A, summarizes the effect of $\rho_{zv}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are 1%.

| $\rho_{zv}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[\hat{R}_{t+1}|t]$ | $\sigma_{\hat{R}_{t+1}|t}$ | Cond. Sharpe Ratio |
|-----------|----------|-----------|------------------|----------------|----------------|-------------------|
| -0.65     | -0.1104  | 3.0542    | -27.67           | 0.0694         | 0.1736         | 0.4               |
| -0.8      | -0.0965  | 2.9856    | -30.94           | 0.0689         | 0.1739         | 0.3961            |
| -0.9      | -0.0875  | 2.9520    | -33.76           | 0.0686         | 0.1741         | 0.3944            |
| -0.95     | -0.0830  | 2.9387    | -35.42           | 0.0686         | 0.1741         | 0.3937            |

Panel B summarizes the effect of $\rho_{zv}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are -1%.

| $\rho_{zv}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[\hat{R}_{t+1}|t]$ | $\sigma_{\hat{R}_{t+1}|t}$ | Cond. Sharpe Ratio |
|-----------|----------|-----------|------------------|----------------|----------------|-------------------|
| -0.65     | -0.1104  | 3.0542    | -27.67           | 0.0106         | 0.1736         | 0.0608            |
| -0.8      | -0.0965  | 2.9856    | -30.94           | 0.0111         | 0.1739         | 0.0639            |
| -0.9      | -0.0875  | 2.9520    | -33.76           | 0.0114         | 0.1741         | 0.0652            |
| -0.95     | -0.0830  | 2.9387    | -35.42           | 0.0114         | 0.1741         | 0.0657            |

### 5.3 Correlation between unobservable expected stock return and predictor innovation

What happens when the correlation between the innovation of the predictor and the shock to the unobservable expected stock returns ($\rho_{v\varepsilon}$) varies? Figure 10, Panel A, plots the mean wealth accumulation over the life cycle, and Figure 10, Panel B, plots the mean share of wealth in stocks. When $\rho_{v\varepsilon}$ varies from 0.66 to 0.95 (baseline value is 0.9), the absolute value of $\kappa_q/\kappa_r$ rises, implying that the predictability effect becomes stronger because the correlation between the predictor and unobserved stock returns becomes stronger. Therefore, the predictability effect from the dividend yield becomes stronger, while the mean reversion effect only slightly decreases. Hence, as $\rho_{v\varepsilon}$ increases,
the portfolio choices from the imperfect predictor model tend toward the predictions from the perfect predictor model. In effect as this correlation increases, the imperfect predictor model comes closer to the perfect predictor model and the Figure 12 illustrates the speed with which this convergence takes place.

Table 8

Effect of correlation between unobservable expected stock return and predictor innovation ($\rho_{\varepsilon v}$)

Table 8, Panel A, summarizes the effect of $\rho_{\varepsilon v}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are 1%.

| $\rho_{\varepsilon v}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[\mathcal{R}_{t+1}|t]$ | $\sigma_{\mathcal{R}_{t+1}|t}$ | Cond. Sharpe Ratio |
|------------------------|------------|------------|---------------------|---------------------------|-----------------------------|-------------------|
| 0.66                   | -0.0893    | 2.0461     | -22.91              | 0.0506                    | 0.1765                      | 0.3375            |
| 0.8                    | -0.0884    | 2.5772     | -29.55              | 0.0649                    | 0.1752                      | 0.3704            |
| 0.9                    | -0.0875    | 2.9520     | -33.76              | 0.0686                    | 0.1741                      | 0.3944            |
| 0.95                   | -0.0881    | 3.1384     | -37.29              | 0.0705                    | 0.1734                      | 0.4065            |

Panel B summarizes the effect of $\rho_{\varepsilon v}$ on conditional expected returns, standard deviation and Sharpe ratio when the shocks to the observed predictor and stock return are -1%.

| $\rho_{\varepsilon v}$ | $\kappa_r$ | $\kappa_q$ | $\kappa_q/\kappa_r$ | $E[\mathcal{R}_{t+1}|t]$ | $\sigma_{\mathcal{R}_{t+1}|t}$ | Cond. Sharpe Ratio |
|------------------------|------------|------------|---------------------|---------------------------|-----------------------------|-------------------|
| 0.66                   | -0.0893    | 2.0461     | -22.91              | 0.0204                    | 0.1765                      | 0.1158            |
| 0.8                    | -0.0884    | 2.5772     | -29.55              | 0.0151                    | 0.1752                      | 0.0862            |
| 0.9                    | -0.0875    | 2.9520     | -33.76              | 0.0114                    | 0.1741                      | 0.0652            |
| 0.95                   | -0.0881    | 3.1384     | -37.29              | 0.0095                    | 0.1734                      | 0.0548            |

6 Hedging Demands from permanent income shocks

6.1 Correlation between predictor and permanent income innovation

When the correlation between the observed predictor and the permanent labor income innovation is positive (0.2) relative to the baseline of 0.0, Figure 11 shows that the wealth accumulation (Panel A) and the share of wealth in stocks (Panel B) rise. This
is because the predictor has a high negative correlation with the stock market return. Hence, when $\rho_{en}$ is positive, the labor income seems like a risk free asset squeezing the share of risk free assets in the portfolio. On the contrary, when this correlation is negative, the labor income acts as risky asset, leading to the reduction of the risky asset in the portfolio.

6.2 Correlation between unobservable expected stock return and permanent income innovation

When the correlation between the unobservable expected stock return and the permanent labor income innovation varies, the quantitative magnitudes on average portfolio choice do not vary substantially over the life cycle. The correlations vary from -0.2 to 0.2, consistent with the idea that this is a correlation between an idiosyncratic innovation and an aggregate one. It is perhaps intuitive that investors do not react substantially to a correlation between the unobserved expected stock return and the permanent income innovation in this formulation as this correlation varies.

7 Conclusions

In a quantitative life-cycle asset allocation model with undiversifiable labor income risk and liquidity constraints, we analyze the implications of an imperfect predictive system on optimal consumption and portfolio decisions. In the presence of an imperfect predictor of the unobservable expected stock returns, the optimal portfolio choice is shown to be more conservative than that predicted by either an i.i.d. stock returns or a perfect predictor model. Different from Wachter and Yogo (2010), who use the nonhomothetic utility over basic and luxury goods to generate balanced portfolios, this paper generates balanced portfolios by introducing imperfect predictability in stock returns. Imperfect predictability increases the risk from investing in the stock market, even in the presence of mean reversion and stock market predictability because of model
uncertainty. The investor’s filtering problem affects the expected conditional mean and variance of asset returns in ways that lead the household to hold safer portfolios on average. The results therefore corroborate the intuition in Pastor and Stambaugh (2012) that there might be more risk associated with long run investments in the stock market than an i.i.d. model or a perfect predictor model imply.
References


Figure 1, Panel A, presents the mean wealth over the life cycle for three different models. "Perfect" refers to the perfect predictor model, "i.i.d." to the i.i.d. stock returns model, and "imperfect" to the baseline imperfect predictor model. Panel B presents the mean share of wealth in stocks, and Panel C mean consumption. The baseline parameters are in Tables 1 and 2 in the main text. We use the correct data generating process for each model (that is, the i.i.d. investor faces an i.i.d. stock returns data generating process) and for comparability we use the same unconditional first and second moment of stock returns when comparing across models.
FIGURE 2

Comparison of life-cycle profiles across different models

Figure 2, Panel A, presents the mean wealth over the life cycle for three different models. "Perfect" refers to the perfect predictor model, "i.i.d." to the i.i.d. stock returns model, and "imperfect" to the baseline imperfect predictor model. Panel B presents the mean share of wealth in stocks, and Panel C mean consumption. The baseline parameters are in Tables 1 and 2 in the main text. We use the data generating process for the imperfect predictive model (that is, all investors face the data generating process associated with the imperfect predictive model).
FIGURE 3

Comparison of life-cycle profiles with more impatience

Figure 3, Panel A, presents the mean wealth over the life cycle for three different models when the
discount factor is reduced to $\beta = 0.94$ from the baseline $\beta = 0.96$. "Perfect" refers to the perfect
predictor model, "i.i.d." to the i.i.d. stock returns model, and "imperfect" to the baseline imperfect
predictor model. Panel B presents the mean share of wealth in stocks. The baseline parameters are in
Tables 1 and 2 in the main text. We use the correct data generating process for each model (that is,
the i.i.d. investor faces an i.i.d. stock returns data generating process) and for comparability we use
the same unconditional first and second moment of stock returns when comparing across models.
FIGURE 4

Comparison of life-cycle profiles with lower risk aversion $\gamma = 2$

Figure 4, Panel A, presents the mean wealth over the life cycle for three different models when the discount factor is reduced to $\gamma = 2$ from the baseline $\gamma = 5$. "Perfect" refers to the perfect predictor model, "i.i.d." to the i.i.d. stock returns model, and "imperfect" to the baseline imperfect predictor model. Panel B presents the mean share of wealth in stocks. The baseline parameters are in Tables 1 and 2 in the main text. We use the correct data generating process for each model (that is, the i.i.d. investor faces an i.i.d. stock returns data generating process) and for comparability we use the same unconditional first and second moment of stock returns when comparing across models.
FIGURE 5

Comparison of life-cycle profiles across different unobservable factor persistence parameters

Figure 5, Panel A, presents the mean wealth over the life cycle for three unobservable factor persistence parameters ($\phi_\mu = 0.5, \phi_\mu = 0.7$, and $\phi_\mu = 0.9$) with fixed $\sigma_e = 0.041$ for the baseline imperfect predictive system. Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 6

Comparison of life-cycle profiles across different unobservable factor persistence parameters with fixed $R^2 = 0.1$

Figure 6, Panel A, presents the mean wealth over the life cycle for three unobservable factor persistence parameters ($\phi_\mu = 0.5, \phi_\mu = 0.7,$ and $\phi_\mu = 0.9$) with fixed $R^2 = 0.1$. Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 7

Comparison of life-cycle profiles across different observable predictor persistence parameters with fixed $R^2 = 0.1$

Figure 7, Panel A presents the mean wealth over the life cycle for three observable predictor persistence parameters ($\phi_q = 0.3$, $\phi_q = 0.5$, and $\phi_q = 0.9$). Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 8

Life-cycle profiles across different correlations between unobservable factor and stock return innovation

Figure 8, Panel A presents the mean wealth over the life cycle for different correlations between the unobservable expected stock return and stock return innovation ($\rho_{ze} = -0.9$ or $-0.7$). Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 9

Life-cycle profiles across different correlations between stock return and predictor innovation

Figure 9, Panel A presents the mean wealth over the life cycle for different correlations between the stock return and the observable predictor innovation ($\rho_{zv} = -0.9$ or $-0.7$). Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 10
Life-cycle profiles across different correlations between unobservable factor and predictor innovation

Figure 10, Panel A presents the mean wealth over the life cycle for different correlations between the unobservable expected stock return and the predictor innovation ($\rho_{e\nu} = 0.9 \text{ or } 0.7$). Panel B presents the same comparative statics for the mean share of wealth in stocks.
FIGURE 11

Life-cycle profiles across different correlations between the observable predictor and the permanent income innovation

Figure 11, Panel A presents the mean wealth over the life cycle for different correlations between the observable predictor and the permanent income innovation ($\rho_{\text{pi}} = -0.2, 0.0 \text{ or } 0.2$). Panel B presents the same comparative statics for the mean share of wealth in stocks.

Panel A: Mean Normalized Wealth

Panel B: Mean Share of Wealth in Stocks