Inventory Risk and Dealer Competition in a Dealer Network*

Lixin Huang† Bin Wei‡

J. Mack Robinson College of Business Federal Reserve Bank of Atlanta

Georgia State University

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Abstract

We develop an inventory model of bidding and trading through a dealer network in over-the-counter markets. In the model upon arrival of a customer's order, dealers first bid against each other for the order, and then engage in bilateral trading to smooth inventory imbalances. We analyze and characterize the equilibrium for a general dealer network. We then focus on star networks to illustrate the competition between the peripheral dealer who has the best inventory position and the central dealer who has the best connectivity that allows him to better disperse the customer's order among other dealers. In sharp contrast with the existing inventory literature, we find that due to centrality the central dealer may still be in the best position to absorb the customer's sell (buy) order even though he has the largest (smallest) inventory across all dealers. (*JEL Classificaiton*: D81, D85, G11, G12)

Keywords: Dealer Network, Market Microstructure, Inventory, Liquidity

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†Finance Department, J. Mack Robinson College of Business, Georgia State University, 35 Broad Street, Suite 1235, Atlanta, GA 30303. E-mail: lxhuang@gsu.edu; Tel: (404) 413-7346.

‡Federal Reserve Bank of Atlanta, 1000 Peachtree Street N.E., Atlanta, Georgia 30309. E-mail: bin.we.@atl.frb.org; Tel: (404) 498–8913.

1 Introduction

The primary markets for bonds, derivatives, and other structured financial products are over-the-counter (OTC) markets, where participants trade through bilateral contracting. According to reports from the Securities Industry and Financial Markets Association (SIFMA), the average daily trading volume of bonds, including treasury, municipal, corporate, and agency bonds, exceeds $700 billion, while the average daily trading volume for stocks is about $200 billion. The OTC markets for bonds essentially consist of two parts: the inter-dealer market and the dealer-customer market. Further, not all dealers are the same; conversely, there is a small group of primary dealers who trade with a large number of smaller, non-primary dealers; that is, there is a significant asymmetry in dealers' connectivity. This connectivity asymmetry can have a great impact on inter-dealer and dealer-customer transactions. Dealers face a lot of risks, such as credit risk, liquidity risk, and counterparty risk. At the center of these various risks is a dealer's inventory exposure. Dealers make strategic decisions to optimize their inventory positions to reduce their risk exposure, and these strategic decisions determine transaction prices and volumes in OTC markets. As a matter of fact, dealers' inventory positions have been under spotlight following the dramatic drop in inventories during the 2008-2009 financial crisis.[1] In this paper, we study how the structure of the dealer network affects their ability to share the inventory risk, and consequently affects liquidity that dealers provide to customers.

Specifically, we extend a standard inventory model a la Stoll (1978) to the network setting. Dealers are risk averse and manage their inventory to achieve the optimal return-risk trade-off. The model consists of two periods. In the first period, dealers compete against each other for a customer's order, referred to as the "bidding game" in the paper. In the second period, dealers trade with each other through the network to smooth inventory, referred to as the "trading game" in the paper. We solve the model by backward induction. When

[1] See, for example, the artile "Beware of the Liquidity Traps in Bonds" in the Wall Street Journal on March 15, 2013.

dealers trade among themselves in the second period, their ability to smooth inventory is restricted by the structure of the dealer network, which only allows bilateral trades between directly connected dealers, as the OTC trades in the real world. The OTC market structure creates market power for dealers because how much they trade can influence the asset price. The market power leads to dealers' strategic trading decisions. We show that a dealer is less willing to absorb his trading partner's inventory when he is more risk-averse, the underlying asset is riskier, or there is less external liquidity supply. When dealers bid for the customer's order in the first period, the equilibrium is subgame perfect in the sense that dealers fully anticipate that the second-period trading game depends on the outcome of the first-period bidding contest. We show that the central dealer in a star-network has a bidding advantage and is more likely to win the customer's order. The central dealer's bidding advantage results from his better connectivity, which allows him to better disseminate the customer's order among other dealers. Nevertheless, as the central dealer's initial inventory position increases, the connectivity advantage can be offset by the inventory disadvantage, and the central dealer can lose the customer's order to a peripheral dealer. We illustrate the interplay of connectivity and inventory; in addition, we show that the customer gets better prices when there are more dealers participating in the dealer network.

There is a classic market microstructure literature on inventory risk that can justify the focus of our paper. Garman (1976) studies how a risk-neutral monopolistic dealer sets asset prices to maximize the expected profit under the constraint that his inventory does not explode. Amihud and Mendelson (1980) extend Garman (1976) to examine how the monopolistic dealer's inventory position determines the bid and ask prices under the assumption that inventory is exogenously bounded above and below the explosion level. Stoll (1978) explicitly models inventory risk by assuming a risk-averse dealer who requires a bid-ask spread in compensation for deviating from his optimal inventory position. Interestingly, Stoll (1978) finds that the dealer's inventory position only shifts the bid and ask prices around the true value of the asset, but does not change the width of the bid-ask spread. This result even holds in a multi-period model in Ho and Stoll (1981) and a multi-dealer model of Ho and Stoll (1983). Although our model is based on Stoll's (1978) risk-aversion model, the trading frictions caused by bilateral transactions in a network create market power for both sides involved in a transaction. As a result, the bid-ask spread is a combined outcome of market power and risk aversion. We believe this is a meaningful addition to the inventory literature.

In addition to the inventory literature, our paper fits into the growing literature on trading in the OTC market. Most of the papers are built on search models that assume random meeting and matching between customers and dealers. Since the seminal work by Duffie, Garleanu, and Pedersen (2005) that proposes a search framework to analyze OTC markets, the search-based model of OTC markets has been extended to include risk-averse agents (Duffie, Garleanu, and Pedersen (2007)), unrestricted asset holdings (Lagos and Rocheteau (2009)), and heterogeneous valuations (Hugonnier, Lester, and Weill (2014), Shen, Wei, and Yan (2015)). This strand of literature focus on search frictions in OTC markets, and typically does not explicitly model the dealer network.[2]

Our paper is also related to the literature that studies transactions in a fixed network (for example, see Kranton and Minehart (2001), Gale and Kariv (2007), Blume, Easley, Kleinberg and Tardos (2009), Manea (2011), Chang and Zhang (2015), Malamud and Rostek (2015), Neklyudov (2014), Neklyudov and Sambalaibat (2015), and Babus and Kondor (2016)). Our paper is most closely related to Babus and Kondor (2016) who study how OTC trades are affected by the dealer network structure. There are two key differences between our paper and Babus and Kondor (2016). First, while they focus on information diffusion under the assumption of risk-neutral dealers with private information, we focus on inventory risk with risk-averse dealers in the absence of information asymmetry. Second, they only consider trades among dealers; we take one step forward to consider how trades among dealers affect the trades between dealers and customers.

The real relevance of our paper lies in two aspects. First, inventory risk is a big concern

[2]One exception is Shen, Wei, and Yan (2015) that examines the endogenous creation of "intermediation chains", or a linear dealer network structure.

that has a critical impact on dealers' behavior in the real world; for example, see Lyons (1995) and Cao, Evans, and Lyons (2006) for the foreign exchange market, Manaster and Mann (1996) and Garleanu, Pedersen, and Poteshman (2008) for the futures market, Shachar(2014) for the credit default swap market, Dick-Nielsen (2013) and Randall (2015) for the corporate bond market. Second, the star-network we analyze in the paper is consistent with recent empirical studies that demonstrate the core-periphery network structure in the bond markets. It is well known that the primary dealers in the treasury bond market play a central role; Li and Schürhoff (2012) show that the municipal bond market has a persistent core-periphery structure. Di Maggio, Kermani, and Song (2015) show that the inter-dealer corporate bond market also has a similar core-periphery network structure. Our paper sheds light on how the core-periphery network structure affects inter-dealer inventory smoothing, which in turn impacts on the transactions between dealers and customers.

The rest of the paper proceeds as follows. In Section 2, we set up the general network model, define the equilibrium, and characterize the solutions. In Section 3, we analyze a general star network. We show that the asymmetry between the central dealer and peripheral dealers plays an important role in determining the equilibrium outcome. In Section 4, we illustrate the interplay between connectivity and inventory with a three-dealer example and then extend the analysis to a full complete network. The comparison between the star network and complete network highlights the role of connectivity in a dealer network. Section 5 offers conclusive remarks. The appendix includes the theoretical proofs.

2 Inventory Model in a General Network

We start with an inventory model that is embedded in a general network. There is one risk-free asset and one risky asset. The risk-free asset is cash, so the risk-free interest rate is assumed to be equal to zero. The payoff on the risky asset, $\widetilde{v}$, follows a normal distribution with mean equal to $\overline{v}$ and variance equal to σ^2; that is, $\widetilde{v} \sim N\left(\overline{v}, \sigma^2\right)$. There are $\mathcal{N} = N + 1 \geq 1$ dealers in the market, indexed by the set $\mathcal{I} = \{0, 1, \cdots, N\}$. Every pair of dealers can either be directly connected or indirectly connected through other dealers. Trades among dealers can only occur between directly connected dealers in a bilateral way. Specifically, the trading network is an undirected graph with $\mathcal{N}$ vertices (traders); we use g_i to denote the set of trading partners dealer i has and $m_i \equiv |g_i|$ to denote the number of dealer i's trading partners, respectively.

We assume that dealers are risk-averse with exponential utility. At the beginning, dealer $i \in \mathcal{I}$ is endowed with w_i units of the risk-free asset and x_i units of the risky asset; hence, x_i denotes dealer i's inventory position. The initial wealth distribution $\{w_i\}_{i=0}^N$ and the inventory distribution $\{x_i\}_{i=0}^N$ are publicly known.

Dealers first bid for the customer's order and then trade with directly connected partners to smooth their positions in the risky asset. There are two periods and three dates. The sequence of events evolves as follows:

- Date 0: A customer arrives and desires to trade z units of the risky asset. A positive z means a sell order from the customer, while a negative z means a buy order from the customer.[3] Dealers post bid or ask prices by taking into account the subsequent trades through the dealer network. The dealer who offers the best price—the lowest ask price or the highest bid price—wins the customer's order. When multiple dealers bid the best price, a random draw determines the winning dealer.

- Date 1: After the bidding game is over, dealers trade with directly connected partners in the dealer network to smooth inventory.

- Date 2: The payoff of the risky asset, $\widetilde{v}$, is realized and agents are paid off.

The equilibrium concept is the subgame perfect Nash equilibrium. We solve the model by backward induction: we first solve the equilibrium of the network trading game at date 1; afterwards, we go back to date 0 to solve the bidding game.

[3]Here the sign of z is considered from a dealer's perspective. When the customer sells (buys) the risky asset, the winning dealer's inventory increases (decreases) by z units.

2.1 Network trading game at date 1

Suppose that at date 1, before the network trading starts, dealer i owns w_i' units of the risk-free asset and x_i' units of the risky asset. If dealer i won the customer's order z at date 0, then his date-1 inventory and wealth becomes $x_i' = x_i + z$ and $w_i' = w_i - P^* z$, respectively, where P^* is the winning price paid in the transaction with the customer. Otherwise, without winning the customer's order, dealer i's inventory and wealth at date 1 remain unchanged; that is, $x_i' = x_i$ and $w_i' = w_i$.

For each trading partner of dealer i, denoted by dealer $j \in g_i$, dealer i buys q_{ij} units of the risky asset and pays a price p_{ij} per unit. After the transactions with all his connected partners, dealer i's inventory becomes $x_i' + \sum_{j \in g_j} q_{ij}$. At date 2, his terminal wealth is $w_i' + (x_i' + \sum_{j \in g_j} q_{ij})\widetilde{v} - \sum_{j \in g_i} p_{ij} q_{ij}$. The assumption of the exponential utility implies that dealer i's optimization problem is to maximize the following mean-variance function:

$$\max_{(q_{ij})_{j \in g_i}} w_i' + \left(x_i' + \sum_{j \in g_i} q_{ij}\right) \overline{v} - \sum_{j \in g_i} p_{ij} q_{ij} - \frac{1}{2}\gamma\sigma^2 \left(x_i' + \sum_{j \in g_i} q_{ij}\right)^2, \tag{1}$$

where γ is the risk-aversion coefficient. Trading with other partners leads to expected profits $\sum_{j \in g_i} (\overline{v} - p_{ij}) q_{ij}$, and in the meantime changes the dealer's inventory risk from $\frac{1}{2}\gamma\sigma^2 (x_i')^2$ to $\frac{1}{2}\gamma\sigma^2 \left(x_i' + \sum_{j \in g_i} q_{ij}\right)^2$. Each dealer trades optimally to maximize trading profits net of inventory costs.

We assume that all dealers trade simultaneously. Each dealer trades strategically with all his trading partners, taking as given all other dealers' trading strategies. Because trades can only occur bilaterally, a pair of directly connected dealers essentially form a local market, where each has the market power to affect the trading price. Following Babus and Kondor (2016), we focus on a linear trading equilibrium where each dealer's trading strategy is a linear function of his inventory position and the trading prices with all his trading partners. As shown by Vives (2011) and Babus and Kondor (2016), we need exogenous liquidity supply in each local market for the existence of a linear equilibrium.[4] For this purpose, we assume that there is a downward sloping liquidity supply $\beta_{ij}(p_{ij}-\overline{v})$, where β_{ij} is a negative constant. One way to interpret the exogenous liquidity is that there is limited arbitrage in the market: risk-neutral arbitrageurs buy the risky asset when $p_{ij}<\overline{v}$ and sell it when $p_{ij}>\overline{v}$; the maximum number of units they can buy or sell is proportional to the difference between the price and the expected value of the risky asset.

Let $\mathbf{p}_{g_i}=(p_{ij})_{j\in g_i}$ denote the vector of prices between dealer i and his trading partners. Dealer i's trading strategy in a linear trading equilibrium is a vector of functions of $q_{ij}(x_i';\mathbf{p}_{g_i})$ that maximizes his expected utility. We conjecture that dealer i's demand function in his transaction with dealer j is

$$q_{ij}(x_i';\mathbf{p}_{g_i})=b_{ij}x_i'+\textstyle\sum_{k\in g_i}c_{ij}^k(p_{ik}-\overline{v}). \tag{2}$$

Given the above conjecture and the market clearing condition

$$q_{ij}(x_i';\mathbf{p}_{g_i})+q_{ji}\left(x_j';\mathbf{p}_{g_j}\right)+\beta_{ij}(p_{ij}-\overline{v})=0, \tag{3}$$

we can solve dealer i's optimization problem and show that

$$\begin{aligned} q_{ij}(x_i';\mathbf{p}_{g_i}) \quad = \quad & \gamma\sigma^2\widetilde{c}_{ji}^i x_i'+\left(c_{ji}^i+\beta_{ij}\right)\left(1+\gamma\sigma^2\widetilde{c}_{ji}^i\right)(p_{ij}-\overline{v}) \\ & +\gamma\sigma^2\left(c_{ji}^i+\beta_{ij}\right)\sum_{k\in g_i,k\neq j}\widetilde{c}_{ki}^i(p_{ik}-\overline{v}), \end{aligned} \tag{4}$$

where the coefficients are derived by matching (4) with the conjectured form in (2), given by:

$$b_{ij} \quad = \quad \gamma\sigma^2\widetilde{c}_{ji}^i, \tag{5}$$

$$c_{ij}^j \quad = \quad \left(c_{ji}^i+\beta_{ij}\right)\left(1+\gamma\sigma^2\widetilde{c}_{ji}^i\right), \tag{6}$$

[4] Without the exogenous liquidity provision, there would be no linear equilibrium in the bilateral trading game where two traders submit demand schedules. See Kyle (1989) and Babus and Kondor (2016).

$$c_{ik}^{j} = \gamma\sigma^2 \left(c_{ji}^{i} + \beta_{ij}\right) \widetilde{c}_{ki}^{i} \text{ for } k \neq i, j, \tag{7}$$

$$\widetilde{c}_{ki}^{i} \equiv \frac{c_{ki}^{i} + \beta_{ik}}{1 - \gamma\sigma^2 \sum_{k \in g_i} \left(c_{ki}^{i} + \beta_{ik}\right)} \text{ for } k \neq i. \tag{8}$$

The equilibrium is determined by the solutions of c_{ik}^{j}, $i \in \mathcal{I}$, $j, k \in g_i$. For dealer i, there are m_i^2 such coefficients and the same number of equilibrium conditions given by equations (6) and (7). As a result, there are a total of $\sum_{i=0}^{N} m_i^2$ equations for the same number of coefficients c_{ik}^{j}.

Finally, by plugging the solution for q_{ij} into the market clearing condition (3), we have

$$\left(c_{ji}^{i} + c_{ij}^{j} + \beta_{ij}\right)\left(p_{ij} - \overline{v}\right) = -\begin{bmatrix} \gamma\sigma^2 \left(\widetilde{c}_{ji}^{i} x_i' + \widetilde{c}_{ij}^{j} x_j'\right) \\ +\gamma\sigma^2 \left(c_{ji}^{i} + \beta_{ij}\right) \sum\limits_{k \in g_i, k \neq j} \widetilde{c}_{ki}^{i} \left(p_{ik} - \overline{v}\right) \\ +\gamma\sigma^2 \left(c_{ij}^{j} + \beta_{ij}\right) \sum\limits_{k \in g_j, k \neq i} \widetilde{c}_{kj}^{j} \left(p_{jk} - \overline{v}\right) \end{bmatrix}. \tag{9}$$

For each paired traders, we have a linear price equation (9); the number of price equations is thus exactly equal to the number of equilibrium prices we want to solve for. From solving this system of linear equations, for each pair of connected dealers (i, j), we can express the price p_{ij} as a linear function of the date-1 inventories $\{x_k'\}_{k=0}^{N}$:

$$p_{ij} - \overline{v} = \sum_{k=0}^{N} d_{ij}^{k} x_k', \tag{10}$$

where the coefficients d_{ij}^{k} are endogenous functions of primitive model parameters.

Intuitively, if all dealers have zero inventory at date 1 (i.e., $x_i' = 0$ for any i), then all inter-dealer prices are equal to $\overline{v}$ and no trade occurs throughout the dealer network. Consequently, there must be a certain degree of order imbalance among dealers at date 1 in order for inter-dealer trade to take place. The order imbalance at date 1 can result from either the arrival of the customer order at date 0 or from initial inventory imbalance, or both.

We summarize the above results with the following proposition.

Proposition 1 *If a linear equilibrium exists in the OTC market, then every dealer's strategy is* $q_{ij}(x_i'; \mathbf{p}_{g_i}) = b_{ij}x_i' + \sum_{k\in g_i} c_{ij}^k (p_{ik} - \overline{v})$, $i \in \mathcal{I}$, $j, k \in g_i$. *The coefficients are determined in equations (5)-(8). The equilibrium prices* p_{ij} *are linear functions of the date-1 inventories as shown in (10).*

Proof. See Appendix. ■

Denote $\mathbf{x}' \equiv (x_0', \cdots, x_N')^T$ as the vector of date-1 inventories across all dealers. Hence, the equilibrium prices and trading quantities are linear functions of the date-1 inventories, which can be expressed as $p_{ij}^*(\mathbf{x}')$ and $(q_{ij}^*(\mathbf{x}'), q_{ji}^*(\mathbf{x}'))$ for any pair of connected dealers (i, j). Substituting the equilibrium prices and quantities into the objective function in (1), we obtain dealer i's value function at date 1, denoted by $u_i(w_i', \mathbf{x}')$ for $i \in \mathcal{I}$.

2.2 Bidding game at date 0

Now we go back to date 0 to characterize the equilibrium of the bidding game. Each dealer's optimized expected utility at date 1 depends on the exogenous coefficients—β_{ij}, γ, $\overline{v}$, σ^2—and the distribution of the date-1 inventory $\mathbf{x}'$. For simplicity, we omit the exogenous parameters and use $u_i(w_i', \mathbf{x}')$ to denote dealer i's optimized expected utility from the trading game, which is the value function of (1). For ease of exposition, we assume that the customer's order is a sell order (i.e., $z > 0$) hereafter; the case of a buy order can be similarly analyzed with the maximum bid price replaced by the minium ask price.

We can define the equilibrium of the date-0 bidding game for a sell order as follows.

Definition 1 *Upon the arrival of a sell order, an equilibrium of the bidding game at date 0 consists of bid prices* P_i, $i \in \mathcal{I}$, *such that:*

1. *The dealer who bids the highest price (i.e.,* $k^* \in \arg\max_{k\in\mathcal{I}} P_k$*) wins the sell order and buys the asset from the customer at the price* $P^* = P_{k^*}$*. In the case where multiple dealers bid the same highest price, we assume that the customer randomly selects and trades with one of these dealers.*

2. *The winning dealer k^* bids the best price to win the order; that is, there does not exist a price P'_{k^*} different from P_{k^*}, such that.*

$$P_{k^*} > P'_{k^*} \geq \max_{k \in \mathcal{I},\ k \neq k^*} P_k.$$

3. *Dealer k^* does not have the incentive to deviate to loose the customer's order to the dealer who bids the second best price:*

$$u_{k^*}(w_{k^*} - P_{k^*}z, \mathbf{x} + z\mathbf{e}_{k^*}) \geq u_{k^*}(w_{k^*}, \mathbf{x} + z\mathbf{e}_j),\ \ j \in \arg \max_{k \in \mathcal{I},\ k \neq k^*} P_k,$$

where $\mathbf{e}_k$ is defined as a vector of zeros except for the k_{th} element being one and $\mathbf{x} \equiv (x_0, \cdots, x_N)^T$ as the vector of initial inventories across all dealers at date 0.

4. *Other dealers do not have incentives to deviate to outbid the winning dealer to win the customer's order:*

$$u_i(w_i, \mathbf{x} + z\mathbf{e}_{k^*}) \geq u_i(w_i - P_{k^*}z, \mathbf{x} + z\mathbf{e}_i),\ \ \forall i \neq k^*.$$

The idea behind the bidding-game equilibrium is that each dealer posts a bid price, P_i, and the dealer who posts the highest bid price wins the customer's order (condition 1). The winning dealer only needs to quote a price that is slightly higher than the second-best price to win the order (condition 2). More importantly, because the winning dealer needs to share the customer's order with all the other dealers in the date-1 trading game, all the dealers will consider what happens in the trading game after the bidding game is over. Conditions 3 and 4 above capture the concept of subgame perfection. On one hand, the winning dealer prefers taking the customer's order directly at date 0 to receiving a part of it indirectly through other dealers at date 1 (condition 3); on the other hand, the non-winning dealers would rather get a part of the customer's order from the winning dealer at date 1 than outbid the winning dealer at date 0 (condition 4). Based on the conditions characterized above, the following bidding strategies constitute an bidding game equilibrium:

- Dealer k^* bids the highest price $P_{k^*} = \max_{i \in \mathcal{I}} P_i$.
- The second-best bid price is only one tick below the winning price; that is dealer k^{**} (the second-price bidder) bids the price $P_{k^{**}} = P_{k^*} - \epsilon$, where $k^{**} = \arg\max_{j \in \mathcal{I},\, j \neq k^*} P_j$ and $\epsilon > 0$ denotes the tick size and can be infinitesimally small.
- All the other dealers i ($i \neq k^*, k^{**}$) bid $P_i < P_{k^*} - \epsilon$.

Next, we use reservation prices to help characterize the bidding game equilibrium. For any dealer $i \in \mathcal{I}$, his reservation price Ψ_{ij} associated with dealer j ($j \neq i$) winning the bidding game is defined as

$$u_i(w_i - \Psi_{ij} z, \mathbf{x} + z\mathbf{e}_i) = u_i(w_i, \mathbf{x} + z\mathbf{e}_j).$$

That is, *conditional* on dealer j winning the order, the maximum price that dealer i is willing to pay in order to snatch the customer's order from dealer j is Ψ_{ij}. This implies that dealer j's winning price has to be greater than Ψ_{ij} for any $i \neq j$. As a result, for dealer j, bidding $P_j \geq \max_{i \neq j} \Psi_{ij}$ is sufficient for him to win the order because outbidding dealer j leads to a strictly lower utility for all the other dealers. Therefore, condition 4 in the equilibrium definition is equivalent to:

$$P_{k^*} \geq \max_{i \in \mathcal{I},\, i \neq k^*} \Psi_{ik^*}. \tag{11}$$

Condition 3 suggests that dealer k^*'s winning price P_{k^*} should be no greater than his reservation price $\Psi_{k^* k^{**}}$ associated with dealer k^{**} (i.e., the second-best bidder) winning the order; that is,

$$P_{k^*} \leq \Psi_{k^* k^{**}}. \tag{12}$$

Otherwise dealer k^* would rather lose the customer's order to dealer $P_{k^{**}}$.

The following lemma characterizes the sufficient and necessary condition for the existence of a bidding game equilibrium.

Lemma 1 *The sufficient and necessary condition for the existence of a bidding game equilibrium is that there exists at least one dealer, say dealer k^*, such that the following condition is satisfied:*

$$\max_{i \in \mathcal{I},\, i \neq k^*} \Psi_{ik^*} \leq \max_{j \in \mathcal{I},\, j \neq k^*} \Psi_{k^*j}. \tag{13}$$

The lemma can be proved as follows. First, the condition is a necessary condition, because otherwise Conditions (11) and (12) could not be satisfied simultaneously. Second, we can prove the sufficiency of the condition by constructing an equilibrium bidding strategy profile as follows. Dealer k^* bids the highest price $P_{k^*} = \max_{i \in \mathcal{I},\, i \neq k^*} \Psi_{ik^*}$. The second-price bidder is the dealer k^{**} such that dealer k^* has the highest reservation price conditional on dealer k^{**} winning the order; that is, $k^{**} = \arg\max_{j \in \mathcal{I},\, j \neq k^*} \Psi_{k^*j}$. The second-best bid price is one tick below P_{k^*} (i.e., $P_{k^{**}} = P_{k^*} - \epsilon$). All other dealers bid prices no greater than the second-best price; that is, $P_i < P_{k^*} - \epsilon$, for any $i \neq k^*, k^{**}$. It is trivial to check that no dealer has the incentive to deviate from the stipulated strategy.

Lemma 1 establishes the sufficient and necessary condition for the existence of the bidding game equilibrium. As long as Condition (13) is satisfied for some dealer, the equilibrium exists. However, it is not obvious that this condition always holds for at least one dealer. In the proposition below, we prove that this is indeed the case and hence there always exists a bidding game equilibrium.[5]

Proposition 2 *There always exists a bidding game equilibrium in which there exists $k^* \in \mathcal{I}$, such that Condition (13) is satisfied.*

[5] The bidding game is similar to but different from a first-price auction with perfect information. The key difference is that reservation prices are not exogenous; rather, they are endogenized depending on who win the customer's order. In general, there are multiple equilibria in a bidding game the second-best price bidder can raise the price above the highest reservation price associated with the winning dealer. We only consider the equilibrium where the winning bidder wins by bidding the highest reservation price associated with his winning the bidding game.

Proof. See Appendix. ■

As described in the introduction, the network trading game in Section 2.1 resembles Babus and Kondor (2016), but differs along an important dimension: while Babus and Kondor (2016) focus on information diffusion, we focus on risk aversion and inventory smoothing. The link between the dealer-customer trade and the inter-dealer trade is this paper's innovation. When dealers compete for customers' orders, they understand that ultimately the inventory is going to be shared by all the dealers in the network. In other words, the liquidity a dealer network provides to a customer can be divided into two components: the winning dealer of the bidding game provides the first-hand, direct liquidity to the customer; all other dealers provide the second-hand, indirect liquidity through the network trading game. As a result, the structure of the dealer network not only affects the equilibrium of the trading game at date 1, but has a critical impact on dealers' bidding strategies that determine the price the customer can receive at date 0.

Our model of bidding and trading in a dealer network sheds light on the interplay of connectivity and inventory in determining which dealer wins the customer's order, what price the customer receives, and how the customer's order is distributed within the dealer network. Absent a dealer network, the dealer with the best inventory position should win the customer's order. However, in the presence of a dealer network, it may no longer be true. Because the network allows for inventory sharing, dealers with better connectivity can better spread the customer's order among other dealers and hence may be better positioned to win the order. As a result, when the dealer with the best inventory position is not the one with the best connectivity, there is a competition between the dealer who has the best inventory position and the dealer who is best connected. To illustrate this competition, we need to be more specific about the network structure. For this purpose, we focus on star networks in the following section.

3 Star Network

We now focus on a specific type of network: a star network where a dealer is directly connected with a number of other dealers who are not mutually directly connected. Specifically, there are $N+1$ dealers where dealers $1, \cdots, N$ are only directly connected to dealer 0; we hence refer to dealer 0 as the "central dealer," and the other N dealers as "peripheral dealers." A star network is particularly interesting to study for the following reasons. First, it is very realistic as many over-the-counter markets have such a center-periphery network structure. Second, the model with a star network is very tractable to analyze. Third, there is a connectivity asymmetry: the central dealer is directly connected to all the N peripheral dealers, while peripheral dealers are only indirectly connected through the central dealer. It hence highlights the trade-off between connectivity and inventory: In case the central dealer doe not have the lowest inventory, who wins the customer's sell order, the peripheral dealer with the lowest inventory or the central dealer with the best connectivity?

3.1 Star network trading game

We first analyze the network trading equilibrium among all the dealers. We assume a symmetric structure such that the exogenous liquidity provision is the same between any central-peripheral pair; that is, the parameter β is a constant independent of the identity of a peripheral dealer. This assumption implies a symmetric equilibrium where all peripheral dealers adopt the same trading strategy against the central dealer.

For the ease of notation, we simply use p_i to denote the price in the transaction between the central dealer 0 and the peripheral dealer $i \in \{1, 2, \cdots, N\}$. In addition, we use lowercase letters for variables associated with peripheral dealers and uppercase letters for variables associated with the central dealer. For example, we use Q_i to denote the central dealer's demand in the transaction with the peripheral dealer i, and q_i the peripheral dealer's demand in the same transaction.

We conjecture that dealers adopt the following strategies in a linear equilibrium: the central dealer's demand is

$$Q_i = Bx_0' + C(p_i - \overline{v}) + D\sum_{k\neq i}(p_k - \overline{v}), \text{ for } i \in \{1, 2, \cdots, N\}, \tag{14}$$

and a peripheral dealer's demand is:

$$q_i = bx_i' + c(p_i - \overline{v}), \text{ for } i \in \{1, 2, \cdots, N\}. \tag{15}$$

First, consider peripheral dealer i's optimization problem:

$$\max_{q_i} w_i' + x_i'\overline{v} - (p_i - \overline{v}) q_i - \frac{1}{2}\gamma\sigma^2 (x_i' + q_i)^2.$$

From the market clearing condition $q_i + Q_i + \beta(p_i - \overline{v}) = 0$, we can derive the residual inverse demand function as follows:

$$p_i - \overline{v} = I_i - \frac{1}{C+\beta} q_i,$$

where $I_i \equiv -\frac{1}{C+\beta}\left[Bx_0' + D\sum_{k\neq i}(p_k - \overline{v})\right]$ denotes the intercept in the residual inverse demand function. When choosing the optimal trading quantity q_i, dealer i takes into account the price impact. That is, substituting the residual inverse demand function into his objective function yields the following first-order condition:

$$q_i = \frac{\gamma\sigma^2 x_i' + I_i}{\frac{2}{C+\beta} - \gamma\sigma^2} = \frac{\gamma\sigma^2 x_i' + (p_i - \overline{v})}{\frac{1}{C+\beta} - \gamma\sigma^2},$$

where in deriving the second equality we have used the fact that $I_i = (p_i - \overline{v}) + \frac{1}{C+\beta} q_i$. Comparing the above condition with the conjectured expression in (15), we obtain

$$b = \gamma\sigma^2 c \text{ and } c = \frac{C+\beta}{1 - \gamma\sigma^2 (C+\beta)} \tag{16}$$

Similarly, we can solve for the central dealer's optimal trading strategy $\{Q_i\}_{i=1}^N$ from the following optimization problem:

$$\max_{\{Q_i\}_{i=1}^N} w_0' + x_0'\overline{v} - \sum_{i=1}^N (p_i - \overline{v}) Q_i - \frac{1}{2}\gamma\sigma^2 \left(x_0' + \sum_{i=1}^N Q_i\right)^2,$$

subject to the residual inverse demand function: $(p_i - \overline{v}) = -\frac{1}{c+\beta}(bx_i' + Q_i)$. The first-order condition implies:

$$Q_i = (c+\beta)\left[(p_i - \overline{v}) + \gamma\sigma^2 \left(x_0' + \sum_{k=1}^N Q_k\right)\right].$$

Summing up both sides of the equation across i yields:

$$\sum_{i=1}^N Q_i = \widetilde{c}\left[\sum_{i=1}^N (p_i - \overline{v}) + N\gamma\sigma^2 x_0'\right],$$

where

$$\widetilde{c} \equiv \frac{c+\beta}{1 - N\gamma\sigma^2 (c+\beta)}. \tag{17}$$

Substituting the above expression into the first-order condition and comparing it with the conjectured expression in (14), we have

$$B = \gamma\sigma^2\widetilde{c},\ C = (c+\beta)(1+\gamma\sigma^2\widetilde{c}),\ \text{and } D = \gamma\sigma^2 (c+\beta)\widetilde{c}. \tag{18}$$

Combining the solution for c in (16) and the solution for C in (18), we have

$$\frac{c}{1+c\gamma\sigma^2} - \beta = (c+\beta)\left(1 + \frac{\gamma\sigma^2 (c+\beta)}{1 - N\gamma\sigma^2 (c+\beta)}\right). \tag{19}$$

The solution to (19) pins down the dealers' strategies in the equilibrium. As a matter of fact, the equilibrium exits and is unique.

Proposition 3 *There is a unique linear equilibrium where* $c \in \left(-\frac{1}{\gamma\sigma^2}, 0\right)$. *Moreover, the* *coefficient c is increasing in exogenous liquidity, β, the risk-aversion coefficient, γ, and the risk of the risky asset, σ^2.*

Proof. See Appendix. ■

We know that β is negative and measures the magnitude of outside liquidity. As β increases, it is closer to zero and exogenous liquidity provision reduces. The coefficient c can be viewed as the central dealer's willingness to share the other dealers' inventory. When the exogenous liquidity provision is low, there is less market competition, and hence the central dealer can ask for a higher price concession for agreeing to share other dealers' inventory. As a result, c is increasing in β. When the asset is more risky or the dealers are more risk averse, they are less willing to share the inventory; consequently c increases and is closer to zero.

As we mentioned above, a general star network does not allow peripheral dealers to share inventory directly—they can only do it indirectly through the central dealer. Consequently, the size of the network, N, has an effect on how dealers share inventory through the network.

Proposition 4 *As the number of peripheral dealers increases, each peripheral dealer is more willing to share other dealers' inventory, but the central dealer is less willing to share inventory; that is $\frac{\partial c}{\partial N} < 0$ but $\frac{\partial C}{\partial N} > 0$.*

Proof. See Appendix. ■

Now that we have solved the dealers' trading strategies, we proceed to solve the equilibrium prices and quantities by applying the market clearing conditions. Summing up the market clearing conditions $q_i + Q_i + \beta(p_i - \overline{v}) = 0$ across i, we have

$$\textstyle\sum_{i=1}^{N}(p_i - \overline{v}) = -\frac{N\gamma\sigma^2\widetilde{c}x_0' + b\sum_{i=1}^{N}x_i'}{c + \beta + \widetilde{c}} = -\frac{N\gamma\sigma^2\widetilde{c}x_0' + b\sum_{i=1}^{N}x_i'}{(c+\beta)(2 + N\gamma\sigma^2\widetilde{c})}.$$

Therefore, we can calculate the central dealer's total inventory following all inter-dealer transactions as follows:

$$\begin{aligned} & x_0' + \textstyle\sum_{i=1}^N Q_i \\ = & \left(1 + N\gamma\sigma^2\widetilde{c}\right) x_0' + \widetilde{c} \textstyle\sum_{i=1}^N (p_i - \overline{v}) \\ \equiv & \theta X', \end{aligned}$$

where

$$\begin{aligned} X' & \equiv x_0' - \frac{b}{2} \textstyle\sum_{i=1}^N x_i', \\ \theta & \equiv \frac{2\widetilde{c}}{(c+\beta)\left(2 + N\gamma\sigma^2\widetilde{c}\right)} = \frac{2}{2 - N\gamma\sigma^2 (c+\beta)}. \end{aligned}$$

The variable X' can be interpreted as a measure of *aggregate* inventory weighted by each dealer's connectivity. The variable θ is a constant that is between 0 and 1. Proposition 3 implies that θ is increasing in β; that is, the central dealer's final inventory exposure is greater when there is less external liquidity. Proposition 4 implies that θ is decreasing in N; meaning that the central dealer's final inventory exposure is smaller when there are more peripheral dealers.

The proposition below summarizes the equilibrium prices and quantities, which is a fully worked-out application of Proposition 1 in the star-network case.

Proposition 5 *In a star network, the equilibrium prices and quantities in the date-1 trading game are given by: for* $i \in \{1, 2, \cdots, N\}$,

$$p_i - \overline{v} = -\frac{\gamma\sigma^2}{2}\left(\frac{c}{c+\beta} x_i' + \theta X'\right), \tag{20}$$

and

$$Q_i = -\frac{\gamma\sigma^2 (c+\beta)}{2}\left(\frac{c}{c+\beta} x_i' - \theta X'\right), \tag{21}$$

$$q_i \quad = \quad \frac{\gamma\sigma^2 c}{2}\left(\frac{c+2\beta}{c+\beta}x_i' - \theta X'\right), \tag{22}$$

Proof. See Appendix. ■

As can been seen, the equilibrium prices and quantities are linear functions of the initial inventories of all the dealers. The price, p_i, as well as trading quantities, q_i and Q_i, all consist of two components, respectively associated with the *aggregate* inventory, X' and the peripheral dealer's inventory, x_i. If we have a closer look at the transaction between the central dealer and a peripheral dealer i, we make the following observations. First, the price solution shows that the coefficients on both components are negative, indicating that a greater inventory, no matter from whom, always leads to a smaller price. That said, the coefficients on the two components are different, capturing the overall supply-demand effect and the effect of asymmetric market power associated with the two directly connected dealers. Second, the central dealer's trading quantity, Q_i, is increasing in x_i but decreasing in X'. This means that the central dealer provides more liquidity to peripheral dealer i when the aggregate inventory smaller and dealer i is in greater demand for unloading inventory. Similarly, the peripheral dealer's trading quantity, q_i, is decreasing in x_i but increasing in X'.

We can now evaluate each dealer's maximal utility following inter-dealer transactions. First, for the central dealer, his maximal utility is given by:

$$\begin{aligned}
& w_0' + x_0'\overline{v} - \textstyle\sum_{i=1}^N (p_i - \overline{v})\, Q_i - \frac{1}{2}\gamma\sigma^2 \left(x_0' + \textstyle\sum_{i=1}^N Q_i\right)^2 \\
= \quad & w_0' + x_0'\overline{v} - \textstyle\sum_{i=1}^N \left[-\frac{\gamma\sigma^2}{2}\left(\frac{c}{c+\beta}x_i' + \theta X'\right)\right]\left[-\frac{\gamma\sigma^2(c+\beta)}{2}\left(\frac{c}{c+\beta}x_i' - \theta X'\right)\right] \\
& -\frac{1}{2}\gamma\sigma^2\theta^2 (X')^2 \\
= \quad & w_0' + x_0'\overline{v} - \left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta}\textstyle\sum_{i=1}^N (x_i')^2 - \frac{1}{2}\gamma\sigma^2\theta\,(X')^2 .
\end{aligned}$$

For peripheral dealer i, his maximal utility is given by:

$$\begin{aligned}
& w_i' + x_i'\overline{v} - (p_i - \overline{v})\, q_i - \frac{1}{2}\gamma\sigma^2 (x_i' + q_i)^2 \\
= \; & w_i' + x_i'\overline{v} - \left[-\frac{\gamma\sigma^2}{2}\left(\frac{c}{c+\beta}x_i' + \theta X'\right)\right]\left[bx_i' - \frac{\gamma\sigma^2 c}{2}\left(\frac{c}{c+\beta}x_i' + \theta X'\right)\right] \\
& -\frac{1}{2}\gamma\sigma^2\left((1+b)\, x_i' - \frac{\gamma\sigma^2 c}{2}\left(\frac{c}{c+\beta}x_i' + \theta X'\right)\right)^2 \\
= \; & w_i' + x_i'\overline{v} - \frac{1}{2}\gamma\sigma^2 (x_i')^2 - \frac{\gamma\sigma^2}{8} b\,(2+b)\left(\frac{c+2\beta}{c+\beta}x_i' - \theta X'\right)^2.
\end{aligned}$$

The last term in the peripheral dealer's maximal utility captures the gain from participating in the network trading game. Consider the case when X' is close to zero, then the peripheral dealer's gain is increasing in his own inventory exposure, x_i', meaning that the connection with the central dealer is an important channel for the peripheral dealer to disseminate the inventory risk. The central dealer's utility is increasing in the dispersion of peripheral dealers' inventories, measured by $\sum_{i=1}^{N}(x_i')^2$. The central dealer is the dealer of the dealers, who smooths inventory distribution through the whole network. When there is a greater dispersion in peripheral dealers' inventory exposure, the central dealer is more desirable to even out the dispersion; consequently, the central dealer can gain more from fulfilling this role.

So far we have derived the network trading equilibrium at date 1. We need to go back to date 0 to solve the bidding equilibrium.

3.2 Star network bidding game

Regarding the outcome of the bidding game at date 0, there are two scenarios: either the central dealer or one of the peripheral dealers wins the customer's order. By analyzing these two scenarios, we can derive the reservation price matrix $\Psi = (\Psi_{ij})_{i,j=0}^{N}$, which is essential for characterizing the bidding game equilibrium. The following proposition shows that the reservation prices are linear functions.

Proposition 6 *In a star network, the reservation prices are linear in z, x_i, and $X \equiv x_0 - \frac{\gamma\sigma^2 c}{2}\sum_{i=1}^{N} x_i$*

$$\begin{aligned}\Psi_{0i} &= \overline{v} + \psi_z z + \psi_x x_i + \psi_X X, \\ \Psi_{i0} &= \overline{v} + \psi'_z z + \psi'_x x_i + \psi'_X X, \\ \Psi_{ij} &= \overline{v} + \psi''_z z + \psi''_x x_i + \psi''_X X,\end{aligned}$$

where the expressions of the coefficients are given in the proof.

Proof. See Appendix. ■

The reservation price matrix characterized by Proposition 6 has several features. First, all the coefficients on the customer's order, z— ψ_z, ψ'_z, and ψ''_z—are all negative, implying that the transaction price is decreasing in the size of the customer's sell order. Intuitively, a larger sell order increases dealers' inventory exposure and dealers will ask for a higher premium for this exposure. Second, all the coefficients on the dealer i's initial inventory, x_i— ψ_x, ψ'_x, and ψ''_x—are all negative, implying that the price is decreasing in the dealer i's initial inventory. When a dealer's initial inventory is already very big, he is reluctant to accept a customer's order either directly or indirectly through order dealers, thus bid prices drop. Third, all the coefficients on X— ψ_X, ψ'_X, and ψ''_X—are all negative. The variable can be interpreted as a measure of aggregate inventory that accounts for the network asymmetry between the central dealer and all peripheral dealers. When the aggregate inventory is large, the dealer network as a whole is averse to an increase in inventory; consequently, the customer cannot get a good price for the sell order. These features suggest that the reservation prices can be ranked based on peripheral dealers' inventories, which in turn determines the bidding game equilibrium.

Proposition 7 *Without loss of generality, we sort all peripheral dealers increasingly by their* *initial inventories. The central dealer wins the customer's sell order if and only if*

$$x_0 \leq \frac{\psi_z - \psi_z'}{\psi_X' - \psi_X} z + \frac{\psi_x - \psi_x'}{\psi_X' - \psi_X} x_1 + \frac{\gamma\sigma^2 c}{2} \textstyle\sum_{i=1}^N x_i.$$

Proof. See Appendix. ■

The structure of the star network determines that there is an asymmetry between the central dealer and peripheral dealers: the central dealer is directly connected with all the peripheral dealers, while a peripheral dealer only directly connects with the central dealer. As a result, the central dealer has an advantage in spreading his inventory among all other dealers in the network. This connection advantage implies that the central dealer is in general more likely to win the customer's order. However, as the central dealer's inventory grows, his inventory exposure increases and he has an inventory disadvantage. The proposition shows that the central dealer wins the customer's order if the connection advantage overcomes the inventory disadvantage; otherwise, a peripheral dealer would win the customer's order. To illustrate the central dealer's advantage in connectivity, we can consider the case where all peripheral dealers have zero inventory.

Corollary 1 *When all the peripheral dealers have the same zero inventory, the central dealer wins the bidding game as long as his inventory is below a certain positive threshold.*

Proof. See Appendix. ■

One implication of Corollary 1 is that the central dealer always wins the customer's order when all dealers, central and peripheral, have zero inventory. After winning the customer's order, the central dealer shares it with all peripheral dealers. Intuitively, the more peripheral dealers there are, the better the central dealer can disseminate the customer's order. In turn, the central dealer is willing to bid a higher price for the customer's order. Not so obviously is the competition effect. When there are more dealers in a star network, the central dealer faces stronger competitions from peripheral dealers, which also raise the bid price received by the customer.

Proposition 8 *When all the dealers have the same zero inventory, as the number of peripheral dealers increases, the central dealer fulfills the customer's sell order at a higher bid price.*

Proof. See Appendix. ■

The above results illustrate that a star network's ability to absorb the customer's order is affected not only by the network structure, but also by the distribution of inventories. To illustrate the interplay between connectivity and inventory, we consider several special cases in the next section.

4 Connectivity vs. Inventory

The classic market microstructure literature on inventory risk (e.g., Stoll (1978)) generally argues that risk-averse agents require a bid-ask spread as compensation for deviating from their optimal inventory positions. This insight can be easily seen from the one-dealer special case without dealer network. Specifically, from the perspective of dealer 0 (the only dealer in this special case), trading with the customer leads to an expected profit of $z\left(\overline{v}-P\right)$, but also lower his utility by $\frac{1}{2}\gamma\sigma^2\left(x_0+z\right)^2$ due to the aversion to the inventory risk. Following Stoll (1978), the bid price P can be determined such that the dealer is indifferent between fulfilling the customer's order at price P and not trading with the customer:

$$u_0\left(w_0-Pz,x_0+z\right)=u_0\left(w_0,x_0\right),$$

implying:

$$P=\overline{v}-\gamma\sigma^2x_0-\frac{1}{2}\gamma\sigma^2z.$$

The price discount has two components. The first component is $-\gamma\sigma^2 x_0$, equal to the marginal cost due to inventory risk. An arbitrarily small order increases the dealer's inventory exposure and marginally reduces his utility by $\gamma\sigma^2 x_0$. The second component is $-\frac{1}{2}\gamma\sigma^2 z$, which captures the effect of the order size on the marginal cost associated with inventory risk. As z increases, the inventory exposure increases, and the marginal cost increases, and the dealer asks for a greater discount.

The existing inventory literature generally has not considered the effect of dealer network connectivity on inventory management. The key contribution of this paper is to explicitly model dealer network and examine its interplay with inventory risk in determining the equilibrium bid-ask spread. As we will show, connectivity in a dealer network has a huge impact on equilibrium outcome.

4.1 Role of Inventory

In the special case with two dealers (i.e., dealer 0 and dealer 1), we simply denote dealer i's demand as q_i and the inter-dealer price as p. Based on Proposition 5, we can show that: for $i = 0$ $(-i = 1)$ or 1 $(-i = 0)$

$$\begin{aligned} q_i &= bx_i' + c\left(p - \overline{v}\right) = \frac{b\beta}{2c+\beta}x_i' + \frac{bc}{2c+\beta}\left(x_i' - x_{-i}'\right), \\ p - \overline{v} &= -\frac{b}{2c+\beta}\left(x_0' + x_1'\right), \end{aligned}$$

where $b = \gamma\sigma^2 c$ and $c = \frac{-\beta - \sqrt{\beta^2 - 4\beta/(\gamma\sigma^2)}}{2} \in \left(-1/\left(\gamma\sigma^2\right), 0\right)$ increases with β.

Compared to the previous one-dealer case, the two dealers can share inventory risk through the inter-dealer transactions. In particular, the expression of q_i shows that each dealer's trade can be decomposed into two parts: the first part is $\frac{b\beta x_i'}{2c+\beta}$,which is a fraction of the inventory unloaded to the arbitrage trader; the second part is $\frac{bc(x_i'-x_{-i}')}{2c+\beta}$, which is the transaction with dealer 2 and depends on the inventory imbalance between the two dealers. When x_0' is equal to x_1', the second part disappears. Note that the inter-dealer price is equal to the fundamental value of the asset when $x'_0 + x'_1 = 0$. In this case, the external arbitrageur does not absorb any inventory from the two dealers. However, the market power prevents full inventory smoothing—both dealers keep a fraction of their initial inventory after trading.

The parameter β captures external liquidity supply. When β tends to zero (i.e., the arbitrage trader has a negligible presence), c tends to zero and the inter-dealer market essentially shuts down as $q_0 = q_1 = 0$. When β tends to $-\infty$ (i.e., the limit of arbitrage is removed), c tends to $-1/(\gamma\sigma^2)$ and b tends to -1; consequently, the second part essentially vanishes and the first part becomes $-x'_i$. If the arbitrageur has a very deep pocket, both dealers can unload almost all their inventories to the arbitrageur and barely rely on each other to share inventory. However, it is interesting to point out that the coefficient in the second part of q_i (i.e., $\frac{bc}{2c+\beta}$) is always negative, but has a U-shaped relationship with β: it tends to zero when β approaches $-\infty$ or 0, but achieves its minimum value when β takes an intermediate value. It is easy to understand what happens when β tends to $-\infty$: both dealers have more incentives to directly unload inventories to the arbitrageur and thus the second part in their demand is relatively less important. What happens when β tends to 0 is more subtle. When the arbitrageur has a very limited presence, the inter-dealer inventory sharing (i.e., the second part) would be more important; however, the two dealers are locked in their market power and do not trade substantially.

To facilitate the comparison with the three-dealer case studied later, we assume $x_0 > x_1$. Then based on Proposition 9 it is easy to show $\Psi_{01} < \Psi_{10}$, or dealer 1 wins the order. The optimal bid price is

$$P^* = \Psi_{01} = \left(\overline{v} - \gamma\sigma^2 (1+b) x_0\right) + \left(-\frac{1}{2}\gamma\sigma^2 \left(1 - b^2\right) z\right) + \left(\gamma\sigma^2 b (1+b) x_1\right).$$

The price discount increases in dealer 0's own inventory, x_0, his trading partner's inventory, x_1, and the order size, z. In one extreme case where β tends to zero, then b tends to zero and the price converges to the one in the one-dealer case as the inter-dealer trading through the network breaks down in the absence of the arbitrageur. In the other extreme case where β gradually decreases and tends toward $-\infty$, the external arbitrageur provides more and more liquidity and as a result complete inventory risk sharing is achieved as b converges to -1 and the price discount disappears because the winning dealer can pass the customer's order to the arbitrageur completely.

The two-dealer case is special because the two dealers are symmetric in connectivity. In addition, because there are no other dealers in the network, the connection between the two dealers is complete. As a matter of fact, we can generalize the main insight to a general complete network where all dealers are mutually directly connected. In this case where connections are symmetric and complete, we can conjecture that the dealer with the lowest inventory should be able to bid the best price and win the customer's order.[6] We rigorously prove that this conjecture is indeed true in the proposition below.

Proposition 9 *In a complete network the dealer with the lowest inventory wins the customer's sell order.*

Proof. See the Appendix. ■

The model with a complete network is a nice benchmark case to study. First, we are able to generalize the main insight from the classic inventory literature to an inventory model with a dealer network. It is worthwhile to point out that even in the benchmark case, the existence of a network structure still plays an important role in determining the equilibrium outcome. It is the completeness in combination with the symmetry in connectivity that makes it easy to generalize the main insight from the classic inventory literature. Second, the complete network can serve as a benchmark that allows us to frame the contrast with the case with a star network in the starkest terms to illustrate the effect of connectivity asymmetry. We illustrate the role of centrality in the next subsection.

[6] When the network is symmetric but incomplete, it is generally not the case that the dealer with the best inventory position wins the customer's order because the inventories of those connected dealers also have an effect on the bid prices.

4.2 Role of Centrality

The two-dealer case can is a (degenerate) special case of a star network where connectivity is symmetric. For any larger star network with at least three dealers, connectivity is no longer symmetric. To illustrate the role of centrality in a star network, we first extend the two-dealer case analyzed above to a three-dealer case. Specifically, we assume $x_0 \geq 0$, $x_1 = -x < 0$ (recall that we assumed $x_0 > x_1$ in the two dealer case), and $x_2 = x > 0$. In this three-dealer star network, both dealers 1 and 2 are connected to dealer 0 who is the central dealer, but, otherwise, are not directly connected.

Contrary to the two-dealer case, we can now show that despite the central dealer's greater inventory position ($x_0 > x_1$), he may still win the customer's order due to his centrality; that is, $\Psi_{10} < \Psi_{01}$. The key intuition is that even though dealer 1 has the lowest inventory, he still needs to trade through dealer 0 to share inventory and hence may not to be willing to take the customer's order.

Recall that the reservation prices are given by:

$$\begin{aligned}\Psi_{01} &= \overline{v} + \psi_z z + \psi_x x_1 + \psi_X x_0, \\ \Psi_{10} &= \overline{v} + \psi'_z z + \psi'_x x_1 + \psi'_X x_0.\end{aligned}$$

We proved that $\psi'_z - \psi_z \leq 0$ and $\psi'_X - \psi_X > 0$ in Corollary 1 and Proposition 8, respectively. Because $\psi'_z - \psi_z = \frac{1}{2}(\psi'_x - \psi_x) + \frac{1}{2}\left(1 - \frac{b}{2}\right)(\psi'_X - \psi_X)$, it must be true that $\psi'_x - \psi_x \leq 0$. Therefore, the condition $\Psi_{10} < \Psi_{01}$ holds as long as

$$x_0 + \frac{1}{2}\left(1 - \frac{b}{2}\right) z < \frac{\psi_x - \psi'_x}{\psi'_X - \psi_X}\left(-x + \frac{1}{2}z\right).$$

It is worthwhile to point out the following implications. First, when dealer 1's inventory is not too low (specifically, $x_1 = -x > -\frac{1}{2}z$), then the central dealer with a positive inventory can still win the order as long as his inventory is not too large and satisfies the above condition.

Second, when z is large enough, we can further show that even if the central dealer holds the largest inventory (i.e., $x_0 > x$), the above condition can still hold and hence the central dealer can still win the order. The reason is the following. Note that Corollary 1 implies that the condition should hold in the case where $x_0 = x = 0$; tha tis, $\frac{1}{2}\left(1 - \frac{b}{2}\right) < \frac{1}{2}\frac{\psi_x - \psi'_x}{\psi'_X - \psi_X}$. As a result, when z becomes sufficiently large such that $x_0 + \frac{\psi_x - \psi'_x}{\psi'_X - \psi_X} x < \frac{1}{2}\left(\frac{\psi_x - \psi'_x}{\psi'_X - \psi_X} - \left(1 - \frac{b}{2}\right)\right) z$, then the condition holds, implying the central dealer has the largest inventory but can still win the customer's order! This is a very striking result. The intuition is that when a very large order arrives, the central dealer becomes much more important because whoever receives the large order has a strong desire to share the inventory and if it is a peripheral dealer who receives the order, he has to rely on the central dealer to do the inventory risk sharing. As a result, a peripheral dealer is less willing to win the order, which is reflected in a low reservation price Ψ_{01}. On the other hand, as long as the central dealer's inventory is not too large, he is more willing to win the order compared to peripheral dealers. Therefore the condition may still hold even though the central dealer holds the largest inventory. This result suggests that the importance of centrality increases with the size of the order.

If the direct connection between the two peripheral dealers were established, then dealer 1 who has the lowest inventory would win the customer's order. This is in striking contrast with the previous case of a star network of three dealers. This comparison illustrates that centrality plays a very important role in determining the equilibrium outcome.

5 Conclusion

In this paper, we study how trades among dealers through a network affect their competition for the customer's order. We combine the classic literature on inventory with the burgeoning literature on network trading. The study has important relevance to the real world because OTC markets have a network structure and inventory is perhaps the most important concern for dealers who trade in OTC markets. We characterize the equilibrium of a general network and focus our analysis on star networks where a central dealer is connected with many peripheral dealers. We show that the central dealer has an advantage in connectivity that translates into his competitive strength in winning the customer's order. When the central dealer's inventory is low, he wins the customer's order and passes it to peripheral dealers. However, when the central dealer's inventory is high, a peripheral dealer wins the customer's order and indirectly passes it to other peripheral dealers through the central dealer. We show that the size of the network has a significant effect on how dealers share inventory. When there are more dealers connected to the network, the competition between dealers can better offset the frictions in the OTC market and facilitates inventory sharing, which leads to better prices offered to the customer.

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Appendix

Proof of Proposition 1. We only need to derive equations (5)-(7) as the rest of the proof is already stated in the text before the proposition. Given the conjectured trading strategies in equation (2) and the market clearing condition, the residual inverse demand function of dealer i in the transaction with dealer j is given by

$$\begin{aligned} p_{ij} - \overline{v} &= -\frac{b_{ji}x'_j + \sum\limits_{k \in g_j, k \neq i} c^k_{ji}\left(p_{jk} - \overline{v}\right) + q_{ij}}{c^i_{ji} + \beta_{ij}} \\ &\equiv I_{ij} - \frac{1}{c^i_{ji} + \beta_{ij}} q_{ij}, \end{aligned}$$

where

$$I_{ij} \equiv -\frac{1}{c^i_{ji} + \beta_{ij}} \left(b_{ji}x'_j + \sum_{k \in g_j, k \neq i} c^k_{ji}\left(p_{jk} - \overline{v}\right) \right).$$

Plugging the inverse demand function into dealer i's objective function, we can rewrite his optimization problem as:

$$\max_{(q_{ij})_{j \in g_i}} w'_i + \left(x'_i + \sum\nolimits_{j \in g_i} q_{ij}\right)\overline{v} - \sum\nolimits_{j \in g_i} \left(I_{ij} + \overline{v} - \frac{1}{c^{ji}_i + \beta_{ij}} q_{ij} \right) q_{ij} - \frac{1}{2}\gamma\sigma^2 \left(x'_i + \sum\nolimits_{j \in g_i} q_{ij}\right)^2.$$

The first order condition for q_{ij} is

$$0 = -I_{ij} + \frac{2q_{ij}}{c^{ji}_i + \beta_{ij}} - \gamma\sigma^2 \left(x'_i + \sum\nolimits_{k \in g_i} q_{ik}\right),$$

or

$$q_{ij} = \left(c^i_{ji} + \beta_{ij}\right) \left[p_{ij} - \overline{v} + \gamma\sigma^2 \left(x'_i + \sum\nolimits_{k \in g_i} q_{ik}\right) \right].$$

Summing up the above equation across $j \in g_i$ yields

$$\sum\nolimits_{j \in g_i} q_{ij} = \sum\nolimits_{j \in g_i} \widetilde{c}^i_{ji} \left[\left(p_{ij} - \overline{v}\right) + \gamma\sigma^2 x'_i \right],$$

By substituting the above expression back into the first order condition, we have

$$\begin{aligned}
q_{ij} &= \left(c_{ji}^{i}+\beta_{ij}\right)\left[p_{ij}-\overline{v}+\gamma\sigma^{2}\left(x_{i}'+\sum\nolimits_{k\in g_{i}}q_{ik}\right)\right] \\
&= \left(c_{ji}^{i}+\beta_{ij}\right)\left[p_{ij}-\overline{v}+\gamma\sigma^{2}\left(x_{i}'+\sum\nolimits_{k\in g_{i}}\widetilde{c}_{i}^{ki}\left[\left(p_{ik}-\overline{v}\right)+\gamma\sigma^{2}x_{i}'\right]\right)\right] \\
&= \left(c_{ji}^{i}+\beta_{ij}\right)\left[\left(1+\gamma\sigma^{2}\widetilde{c}_{i}^{ji}\right)\left(p_{ij}-\overline{v}\right)+\gamma\sigma^{2}\left(x_{i}'\left(1+\gamma\sigma^{2}\sum\nolimits_{k\in g_{i}}\widetilde{c}_{i}^{ki}\right)+\sum\nolimits_{k\neq j\in g_{i}}\widetilde{c}_{i}^{ki}\left(p_{ik}-\overline{v}\right)\right)\right] \\
&= \left(c_{ji}^{i}+\beta_{ij}\right)\left[\left(1+\gamma\sigma^{2}\widetilde{c}_{i}^{ji}\right)\left(p_{ij}-\overline{v}\right)+\frac{\gamma\sigma^{2}x_{i}'}{1-\gamma\sigma^{2}\sum\nolimits_{k\in g_{i}}\left(c_{i}^{ki}+\beta_{ik}\right)}+\gamma\sigma^{2}\sum\nolimits_{k\neq j\in g_{i}}\widetilde{c}_{i}^{ki}\left(p_{ik}-\overline{v}\right)\right] \\
&= c_{ij}^{j}\left(p_{ij}-\overline{v}\right)+\gamma\sigma^{2}\widetilde{c}_{ji}^{i}x_{i}'+\gamma\sigma^{2}\left(c_{ji}^{i}+\beta_{ij}\right)\sum\nolimits_{k\neq j\in g_{i}}\widetilde{c}_{ki}^{i}\left(p_{ik}-\overline{v}\right)
\end{aligned}$$

$$\begin{aligned}
0 &= q_{ij}\left(x_{i}';\mathbf{p}_{g_{i}}\right)+q_{ji}\left(x_{j}';\mathbf{p}_{g_{j}}\right)+\beta_{ij}\left(p_{ij}-\overline{v}\right) \\
&= c_{ji}^{i}\left(p_{ij}-\overline{v}\right)+\gamma\sigma^{2}\widetilde{c}_{ji}^{i}x_{i}'+\left(c_{ji}^{i}+\beta_{ij}\right)\gamma\sigma^{2}\sum_{k\in g_{i},k\neq j}\widetilde{c}_{ki}^{i}\left(p_{ik}-\overline{v}\right) \\
&\quad +c_{ij}^{j}\left(p_{ij}-\overline{v}\right)+\gamma\sigma^{2}\widetilde{c}_{ij}^{j}x_{j}'+\left(c_{ij}^{j}+\beta_{ij}\right)\gamma\sigma^{2}\sum_{k\in g_{j},k\neq i}\widetilde{c}_{kj}^{j}\left(p_{jk}-\overline{v}\right) \\
&\quad +\beta_{ij}\left(p_{ij}-\overline{v}\right),
\end{aligned}$$

which implies equations (5)-(7). ■

Proof of Proposition 2. We can prove this proposition by contradiction. Denote the column-wise maximum as follows:

$$\Psi_{k}\equiv\max_{i\neq k}\left\{\Psi_{ik}\right\}$$

Without loss of generality, dealers are ordered such that:

$$\Psi_{0}\geq\Psi_{1}\geq\cdots\geq\Psi_{N} \tag{23}$$

Suppose for any $k\in\{0,1,\cdots,N\}$,

$$\max_{i}\left\{\Psi_{i,k}\right\}>\max_{j}\left\{\Psi_{k,j}\right\}$$

First, it cannot be true that $\Psi_0 = \Psi_1 = \cdots = \Psi_N \equiv \Psi$. Otherwise, it implies:

$$\max_{i,j} \Psi_{i,j} = \max_k \Psi_k = \Psi$$

On the other hand, for any k, $\max_j \{\Psi_{k,j}\} < \Psi$, implying: $\max_k \max_j \{\Psi_{k,j}\} < \Psi$, a contradiction.

Second, therefore, there must exist k_0, such that:

$$\Psi_0 = \Psi_1 = ... = \Psi_{k_0} > \Psi_{k_0+1} \geq ... \geq \Psi_N$$

Denote:

$$i(k_0) = \arg\max_i \{\Psi_{i,k_0}\}$$

Then $\Psi_{i(k_0),k_0} = \Psi_{k_0}$. However, on the one hand, we have

$$\Psi_{k_0} = \Psi_{i(k_0),k_0} \leq \max_j \left\{\Psi_{i(k_0),j}\right\}$$

On the other hand, by assumption,

$$\max_j \left\{\Psi_{i(k_0),j}\right\} < \Psi_{i(k_0)} \leq \Psi_{k_0}$$

a contradiction. ■

Proof of Proposition 5. Given the result $\sum_{i=1}^N (p_i - \overline{v}) = -\frac{N\gamma\sigma^2\widetilde{c}x_0' + b\sum_{i=1}^N x_i'}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}$ and the market clearing condition $q_i + Q_i + \beta(p_i - \overline{v}) = 0$, we can derive the equilibrium price as

$$p_i - \overline{v} = -\frac{bx_i' + Bx_0' + D\sum_{k=1}^N (p_k - \overline{v})}{2(c+\beta)} = -\frac{1}{2(c+\beta)}\left[\frac{2B}{2+N\gamma\sigma^2\widetilde{c}}X' + bx_i'\right].$$

We can then substitute the expressions of $(p_i - \overline{v})$ and $\sum_{i=1}^N (p_i - \overline{v})$ into the conjectured expressions of Q_i and q_i in (14) and (15) to derive equations (21) and (22).

Next, we derive equation (19). Note that $c = \frac{1}{C+\beta} - \gamma\sigma^2$ implies $C = \frac{c}{1+\gamma\sigma^2 c} - \beta$, which together with the result $C = (c+\beta) + D = (c+\beta)\left(1 + \gamma\sigma^2 \frac{c+\beta}{1-\gamma\sigma^2 N(c+\beta)}\right)$ lead to equation (19).

We now prove that there exists a unique negative solution to equation (19) and the solution lies within the interval $(-1/(\gamma\sigma^2), 0)$. First, equation (19) can also be expressed as

$$\frac{c}{1+\gamma\sigma^2 c} - 2\beta - c = \gamma\sigma^2 \frac{(c+\beta)^2}{1-\gamma\sigma^2 N(c+\beta)}.$$

Let the left hand side of the equation be

$$LHS(c) \equiv \frac{c}{1+\gamma\sigma^2 c} - 2\beta - c,$$

and the right hand side of the equation be

$$RHS(c) \equiv \gamma\sigma^2 \frac{(c+\beta)^2}{1-\gamma\sigma^2 N(c+\beta)}.$$

Taking the first order derivative, we have

$$\begin{aligned}
\frac{\partial LHS(c)}{\partial c} &= \frac{1}{(1+\gamma\sigma^2 c)^2} - 1 \\
\frac{\partial^2 LHS(c)}{\partial c^2} &= \frac{-2\gamma\sigma^2}{(1+\gamma\sigma^2 c)^3} \\
\frac{\partial RHS(c)}{\partial c} &= \gamma\sigma^2 \frac{2(c+\beta) - \gamma\sigma^2 N (c+\beta)^2}{[1-\gamma\sigma^2 N(c+\beta)]^2} < 0.
\end{aligned}$$

When $c \in \left(-\frac{1}{\gamma\sigma^2}, 0\right)$, $LHS(c)$ is increasing and concave; $LHS(c)$ is decreasing. Because we have

$$LHS(0) = -2\beta > RHS(0) = \frac{\gamma\sigma^2\beta^2}{1-\gamma\sigma^2 N\beta}, \text{and}$$

$$LHS_+(-\frac{1}{\gamma\sigma^2}) = -\infty < 0 < RHS_+(-\frac{1}{\gamma\sigma^2}),$$

there is a unique intersection between $LHS(c)$ and $RHS(c)$ when $c \in \left(-\frac{1}{\gamma\sigma^2}, 0\right)$.

When $c \in \left(-\infty, -\frac{1}{\gamma\sigma^2}\right)$, we have

$$C = \frac{c}{1+\gamma\sigma^2 c} - \beta > 0.$$

So there cannot be solutions for $c > -\frac{1}{\gamma\sigma^2}$.

To prove the comparative static results, let

$$\frac{c}{1+\gamma\sigma^2 c} - 2\beta - c - \gamma\sigma^2 \frac{(c+\beta)^2}{1-\gamma\sigma^2 N (c+\beta)}.$$

We have proved $\frac{\partial F}{\partial c} = \frac{\partial LHS(c)}{\partial c} - \frac{\partial RHS(c)}{\partial c} > 0$. In addition, we have $\frac{\partial F}{\partial \beta} = -2 + \gamma\sigma^2 \frac{\gamma\sigma^2 N(c+\beta)^2 - 2(c+\beta)}{[1-\gamma\sigma^2 N(c+\beta)]^2} < 0$ because $0 < \gamma\sigma^2 \left[\gamma\sigma^2 N (c+\beta)^2 - 2(c+\beta)\right] < \left[1-\gamma\sigma^2 N (c+\beta)\right]^2$. And we also have $\frac{\partial F}{\partial \gamma\sigma^2} = \frac{-c^2}{[1+\gamma\sigma^2 c]^2} - \frac{(c+\beta)^2}{[1-\gamma\sigma^2 N(c+\beta)]^2} < 0$. ■

Proof of Proposition 4. Note that

$$\frac{\partial LHS}{\partial N} = \left(\frac{1}{(1+\gamma\sigma^2 c)^2} - 1\right)\frac{\partial c}{\partial N}$$

and

$$\begin{aligned} \frac{\partial RHS}{\partial N} &= \gamma\sigma^2 \left[\frac{2(c+\beta)}{1-\gamma\sigma^2 N(c+\beta)} + \frac{\gamma\sigma^2 N (c+\beta)^2}{(1-\gamma\sigma^2 N (c+\beta))^2}\right]\frac{\partial c}{\partial N} \\ &\quad + \left(\gamma\sigma^2\right)^2 \frac{(c+\beta)^3}{(1-\gamma\sigma^2 N(c+\beta))^2} \\ &= \left[2\gamma\sigma^2 \widetilde{c} + N\left(\gamma\sigma^2 \widetilde{c}\right)^2\right]\frac{\partial c}{\partial N} + \left(\gamma\sigma^2\right)\widetilde{c}^3 \end{aligned}$$

Therefore,

$$\frac{\partial c}{\partial N} = \frac{\left(\gamma\sigma^2\right)\widetilde{c}^3}{\frac{1}{(1+\gamma\sigma^2 c)^2} - 1 - 2\gamma\sigma^2\widetilde{c} - N\left(\gamma\sigma^2\widetilde{c}\right)^2} < 0$$

The inequality holds because the numerator is less than zero and the denominator is greater than zero ($\widetilde{c} < 0$ and $2 + N\gamma\sigma^2\widetilde{c} > 0$).Finally,

$$\frac{\partial C}{\partial N} = \frac{1}{(1+\gamma\sigma^2 c)^2}\frac{\partial c}{\partial N} < 0.$$

■

Proof of Proposition 6. First, we derive the reservation price Ψ_{0j} is defined by $u_0(w_0 - \Psi_{0j}z, \mathbf{x} + z\mathbf{e}_0) = u_0(w_0, \mathbf{x} + z\mathbf{e}_j)$, or

$$\begin{aligned} & w_0 - \Psi_{0j}z + (x_0 + z)\,\overline{v} - \left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta}\textstyle\sum_i x_i^2 - \frac{1}{2}\gamma\sigma^2\theta\,(X+z)^2 \\ = \; & w_0 + x_0\overline{v} - \left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta}\left(\textstyle\sum_i x_i^2 + (x_j+z)^2 - x_j^2\right) - \frac{1}{2}\gamma\sigma^2\theta\left(X - \frac{b}{2}z\right)^2, \end{aligned}$$

where we have used the facts that when the peripheral dealer $j \in \{1, \cdots, N\}$ wins the order, then $x_0' = x_0$, $x_j' = x_j + z$, and $x_i' = x_i$ for $i \in \{1, \cdots, N\}$ and $i \neq j$, implying $X' = x_0 - \frac{b}{2}\left(\sum_i x_i + z\right) = X - \frac{b}{2}z$ where $X \equiv x_0 - \frac{b}{2}\sum_i x_i$. As a result,

$$\begin{aligned} \Psi_{0j} &= \overline{v} + \frac{1}{z}\left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta}\left((x_j+z)^2 - x_j^2\right) + \frac{1}{2}\gamma\sigma^2\theta\frac{1}{z}\left[\left(X - \frac{b}{2}z\right)^2 - (X+z)^2\right] \\ &= \overline{v} + \left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta}(2x_j + z) - \frac{1}{2}\gamma\sigma^2\theta\left((2+b)\,X + \left(1 - \frac{b^2}{4}\right)z\right) \\ &= \overline{v} + \left[\left(\frac{\gamma\sigma^2}{2}\right)^2 \frac{c^2}{c+\beta} - \frac{1}{2}\gamma\sigma^2\theta\left(1 - \frac{b^2}{4}\right)\right]z + \frac{b^2}{2\,(c+\beta)}x_j - \frac{1}{2}\gamma\sigma^2\theta\,(2+b)\,X \\ &\equiv \overline{v} + \psi_z z + \psi_x x_j + \psi_X X. \end{aligned}$$

where

$$\begin{aligned} \psi_z &= \frac{1}{2}\psi_x + \frac{1}{2}\left(1 - \frac{b}{2}\right)\psi_X \\ \psi_x &= \frac{b^2}{2\,(c+\beta)} \\ \psi_X &= -\frac{1}{2}\gamma\sigma^2\theta\,(2+b) \end{aligned}$$

Next, we derive the reservation price Ψ_{i0} is defined by $u_i(w_i - \Psi_{i0}z, \mathbf{x} + z\mathbf{e}_i) = u_i(w_i, \mathbf{x} + z\mathbf{e}_0)$, or

$$\begin{aligned} & w_i - \Psi_{i0}z + (x_i + z)\,\overline{v} - \frac{1}{2}\gamma\sigma^2 (x_i + z)^2 - \frac{\gamma\sigma^2}{8} b\,(2+b) \left(\frac{c+2\beta}{c+\beta}(x_i+z) - \theta\left(X - \frac{b}{2}z\right)\right)^2 \\ = \; & w_i + x_i\overline{v} - \frac{1}{2}\gamma\sigma^2 x_i^2 - \frac{\gamma\sigma^2}{8} b\,(2+b)\left(\frac{c+2\beta}{c+\beta}x_i - \theta\,(X+z)\right)^2 \end{aligned}$$

where we have used the facts that when the central dealer 0 wins the order, then $x_0' = x_0 + z$, $x_i' = x_i$ for $i \in \{1, \cdots, N\}$, and $X' = x_0 + z - \frac{b}{2}\sum_i x_i = X + z$. As a result,

$$\begin{aligned} \Psi_{i0} = \; & \overline{v} + \frac{1}{z}\left(\frac{1}{2}\gamma\sigma^2 x_i^2 - \frac{1}{2}\gamma\sigma^2 (x_i+z)^2\right) \\ & + \frac{1}{z}\frac{\gamma\sigma^2}{8} b\,(2+b)\left[\left(\frac{c+2\beta}{c+\beta}x_i - \theta\,(X+z)\right)^2 - \left(\frac{c+2\beta}{c+\beta}(x_i+z) - \theta\left(X - \frac{b}{2}z\right)\right)^2\right] \\ = \; & \overline{v} - \frac{1}{2}\gamma\sigma^2 (2x_i + z) \\ & - \frac{\gamma\sigma^2}{8} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\left(\frac{c+2\beta}{c+\beta}(2x_i+z) - \theta\left(2X + \left(1 - \frac{b}{2}\right)z\right)\right) \\ = \; & \overline{v} + \left[-\frac{\gamma\sigma^2}{2} - \frac{\gamma\sigma^2}{8} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\left(\frac{c+2\beta}{c+\beta} - \theta\left(1 - \frac{b}{2}\right)\right)\right] z \\ & + \left[-\gamma\sigma^2 - \frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\frac{c+2\beta}{c+\beta}\right] x_i \\ & + \left[\frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\theta\right] X \\ \equiv \; & \overline{v} + \psi_z' z + \psi_x' x_i + \psi_X' X, \end{aligned}$$

where

$$\begin{aligned} \psi_z' &= \frac{1}{2}\psi_x' + \frac{1}{2}\left(1 - \frac{b}{2}\right)\psi_X' \\ \psi_x' &= \left[-\gamma\sigma^2 - \frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\frac{c+2\beta}{c+\beta}\right] \\ \psi_X' &= \left[\frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta} + \left(1+\frac{b}{2}\right)\theta\right)\theta\right] \end{aligned}$$

Lastly, we derive the reservation price Ψ_{ij} is defined by $u_i(w_i - \Psi_{ij} z, \mathbf{x} + z\mathbf{e}_i) = u_i(w_i, \mathbf{x} + z\mathbf{e}_j)$, or

$$\begin{aligned} & w_i - \Psi_{ij} z + (x_i + z)\,\overline{v} - \frac{1}{2}\gamma\sigma^2 (x_i+z)^2 - \frac{\gamma\sigma^2}{8} b\,(2+b) \left(\frac{c+2\beta}{c+\beta}(x_i+z) - \theta\left(X - \frac{b}{2}z\right)\right)^2 \\ = \; & w_i + x_i \overline{v} - \frac{1}{2}\gamma\sigma^2 x_i^2 - \frac{\gamma\sigma^2}{8} b\,(2+b)\left(\frac{c+2\beta}{c+\beta}x_i - \theta\left(X - \frac{b}{2}z\right)\right)^2 \end{aligned}$$

where we have used the facts that when the peripheral dealer $j \in \{1, \cdots, N\}$ wins the order, then $x_0' = x_0$, $x_j' = x_j + z$, and $x_i' = x_i$ for $i \in \{1, \cdots, N\}$ and $i \neq j$, implying $X' = x_0 - \frac{b}{2}\left(\sum_i x_i + z\right) = X - \frac{b}{2}z$ where $X \equiv x_0 - \frac{b}{2}\sum_i x_i$. As a result,

$$\begin{aligned} \Psi_{ij} &= \overline{v} + \frac{1}{z}\left(\frac{1}{2}\gamma\sigma^2 x_i^2 - \frac{1}{2}\gamma\sigma^2 (x_i+z)^2\right) \\ & \quad + \frac{1}{z}\frac{\gamma\sigma^2}{8} b\,(2+b)\left[\left(\frac{c+2\beta}{c+\beta}x_i - \theta\left(X - \frac{b}{2}z\right)\right)^2 - \left(\frac{c+2\beta}{c+\beta}(x_i+z) - \theta\left(X - \frac{b}{2}z\right)\right)^2\right] \\ &= \overline{v} - \frac{1}{2}\gamma\sigma^2 (2x_i + z) - \frac{\gamma\sigma^2}{8} b\,(2+b)\,\frac{c+2\beta}{c+\beta}\left(\frac{c+2\beta}{c+\beta}(2x_i+z) - 2\theta\left(X - \frac{b}{2}z\right)\right) \\ &= \overline{v} + \left[-\frac{\gamma\sigma^2}{2} - \frac{\gamma\sigma^2}{8} b\,(2+b)\,\frac{c+2\beta}{c+\beta}\left(\frac{c+2\beta}{c+\beta} + b\theta\right)\right] z \\ & \quad + \left[-\gamma\sigma^2 - \frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta}\right)^2\right] x_i + \left[\frac{\gamma\sigma^2}{4} b\,(2+b)\,\frac{c+2\beta}{c+\beta}\theta\right] X \\ &\equiv \overline{v} + \psi_z'' z + \psi_x'' x_i + \psi_X'' X, \end{aligned}$$

where

$$\begin{aligned} \psi_z'' &= \frac{1}{2}\psi_x'' - \frac{1}{2} b \psi_X'' \\ \psi_x'' &= -\gamma\sigma^2 - \frac{\gamma\sigma^2}{4} b\,(2+b)\left(\frac{c+2\beta}{c+\beta}\right)^2 \\ \psi_X'' &= \frac{\gamma\sigma^2}{4} b\,(2+b)\,\frac{c+2\beta}{c+\beta}\theta \end{aligned}$$

■

Proof of Proposition 7. Without loss of generality, assume that peripheral dealers' inventories are ordered as:

$$x_1 < x_2 \leq \cdots \leq x_N. \tag{24}$$

From the previous subsection, we have derived the reservation prices: for $i \in \{1, \cdots, N\}$,

$$\begin{aligned}
\Psi_{0i} &= \overline{v} + \psi_z z + \psi_x x_i + \psi_X X, \\
\Psi_{i0} &= \overline{v} + \psi'_z z + \psi'_x x_i + \psi'_X X, \\
\Psi_{ij} &= \overline{v} + \psi''_z z + \psi''_x x_i + \psi''_X X.
\end{aligned}$$

Because all the coefficients are negative, we have

$$\begin{aligned}
\Psi_{01} &> \Psi_{02} \geq \cdots \geq \Psi_{0N} \\
\Psi_{10} &> \Psi_{10} \geq \cdots \geq \Psi_{N0}
\end{aligned}$$

Note that: Ψ_{ij} $(j \neq 0)$, does not depend on j and decreases in i.

According to Proposition 2, if $\max_{i\in\{1,\cdots,N\}} \Psi_{i0} \leq \max_{i\in\{1,\cdots,N\}} \Psi_{0i}$, there exists an equilibrium in which the central dealer wins the order. Note that because of the above ordering for $\{\Psi_{0i}\}_{i=1}^N$ and $\{\Psi_{i0}\}_{i=1}^N$, the condition is equivalent to

$$\Psi_{01} \geq \Psi_{10} \tag{25}$$

or

$$\overline{v} + \psi_z z + \psi_x x_1 + \psi_X X \geq \overline{v} + \psi'_z z + \psi'_x x_i + \psi'_X X,$$

which can is equivalent to

$$x_0 \leq \frac{\psi_z - \psi'_z}{\psi'_X - \psi_X} z + \frac{\psi_x - \psi'_x}{\psi'_X - \psi_X} x_1 + \frac{\gamma\sigma^2 c}{2} \sum_i x_i.$$

■

Proof of Corollary 1. In a star network with all dealers having zero inventories, the reservation prices are given by $\Psi_{0i} = \overline{v} + \psi_z z$, $\Psi_{i0} = \overline{v} + \psi'_z z$, and $\Psi_{ij} = \overline{v} + \psi''_z$, for $i, j \in \{1, \cdots, N\}$.

To prove that the central dealer wins the order, we just need to prove $\psi'_z z \leq \psi_z z$ for $z > 0$, or $(\psi'_z - \psi_z) \leq 0$. After tedious algebra, we can show that

$$\psi'_z - \psi_z = L_1 + L_2 + L_3,$$

where

$$L_1 \equiv \beta \left[\left(\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2 + (N-1)\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} \right)^2 - \left(\frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right)^2 \right]$$
$$L_2 \equiv (N-1) \left[\frac{(\gamma\sigma^2 \widetilde{c})^2 (\gamma\sigma^2 c)^2}{4(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})^2} - \frac{(\gamma\sigma^2 \widetilde{c})^2}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})^2} \right]$$
$$L_3 \equiv \frac{1}{2}\gamma\sigma^2 \left[\left(1 + \frac{N\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}^2} \right)^2 - \left(1 + \gamma\sigma^2 c \left(1 + \frac{c}{2}\widetilde{\theta} \right) \right)^2 \right],$$

and

$$\widetilde{\theta} \equiv \gamma\sigma^2 \frac{1 + N\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} - \frac{1}{c+\beta} = \frac{\gamma\sigma^2}{2}\theta - \frac{1}{c+\beta}.$$

Because $\left(\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2+(N-1)\gamma\sigma^2 \widetilde{c}}{2+N\gamma\sigma^2 \widetilde{c}} \right) \geq \left(\frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2 \widetilde{c})} \right) > 0$, we have

$$\begin{aligned}
L_1 &\equiv \beta \left[\left(\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2 + (N-1)\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} \right)^2 - \left(\frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right)^2 \right] \\
&= \beta \left[\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2 + (N-1)\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} - \frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right] \times \\
&\quad \left[\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2 + (N-1)\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} + \frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right] \\
&\leq 2\beta \left[\frac{\gamma\sigma^2 c}{2(c+\beta)} \frac{2 + (N-1)\gamma\sigma^2 \widetilde{c}}{2 + N\gamma\sigma^2 \widetilde{c}} - \frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right] \frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \\
&= 2\beta \left[\frac{\gamma\sigma^2 \widetilde{c}}{(c+\beta)(2 + N\gamma\sigma^2 \widetilde{c})} \right]^2 \left[\frac{c\left[2 + (N-1)\gamma\sigma^2 \widetilde{c}\right]}{2\widetilde{c}} - 1 \right]
\end{aligned}$$

$$
\begin{aligned}
&= 2\beta \left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \frac{2c-2\widetilde{c}+(N-1)\gamma\sigma^2\widetilde{c}}{2\widetilde{c}} \\
&= \frac{\beta}{\widetilde{c}} \left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \frac{2c-2N\gamma\sigma^2 c(c+\beta)-2(c+\beta)+(N-1)\gamma\sigma^2 c(c+\beta)}{1-N\gamma\sigma^2(c+\beta)} \\
&= \frac{-\beta}{\widetilde{c}} \left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \frac{2\beta+(N+1)\,\gamma\sigma^2 c(c+\beta)}{1-N\gamma\sigma^2(c+\beta)} \\
&\leq \frac{-\beta}{\widetilde{c}} \left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \frac{2\beta-(N+1)\,\beta}{1-N\gamma\sigma^2(c+\beta)} \\
&= \frac{(N-1)\beta^2}{(c+\beta)} \left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2;
\end{aligned}
$$

$$
\begin{aligned}
L_2 &\equiv (N-1)\left[\frac{\left(\gamma\sigma^2\widetilde{c}\right)^2\left(\gamma\sigma^2 c\right)^2}{4(c+\beta)(2+N\gamma\sigma^2\widetilde{c})^2} - \frac{\left(\gamma\sigma^2\widetilde{c}\right)^2}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})^2}\right] \\
&= -(N-1)\left(1-\frac{\left(\gamma\sigma^2 c\right)^2}{4}\right)\frac{\left(\gamma\sigma^2\widetilde{c}\right)^2}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})^2};
\end{aligned}
$$

$$
\begin{aligned}
L_3 &\equiv \frac{1}{2}\gamma\sigma^2\left[\left(1+\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2}\right)^2 - \left(1+\gamma\sigma^2 c\left(1+\frac{c\widetilde{\theta}}{2}\right)\right)^2\right] \\
&= \frac{1}{2}\gamma\sigma^2\left[\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2} - \left(1+\gamma\sigma^2 c\left(1+\frac{c\widetilde{\theta}}{2}\right)\right)\right]\left[2+\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2} + \left(1+\gamma\sigma^2 c\left(1+\frac{c\widetilde{\theta}}{2}\right)\right)\right] \\
&\leq \gamma\sigma^2\left[\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2} - \left(1+\gamma\sigma^2 c\left(1+\frac{c\widetilde{\theta}}{2}\right)\right)\right]\left[1+\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2}\right] \\
&= \frac{2\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\left[\frac{N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}^2} - \left(1+\gamma\sigma^2 c\left(1+\frac{c\widetilde{\theta}}{2}\right)\right)\right] \\
&= \frac{2\left(\gamma\sigma^2\right)^2\widetilde{c}}{\left[(c+\beta)(2+N\gamma\sigma^2\widetilde{c})\right]^2}\left[(N-1)\,\widetilde{c}c\left(1+\frac{\gamma\sigma^2 c}{2}\right)+\beta\left(N\widetilde{c}-c\right)\right] \\
&\leq \frac{2\left(\gamma\sigma^2\right)^2\widetilde{c}}{\left[(c+\beta)(2+N\gamma\sigma^2\widetilde{c})\right]^2}\,(N-1)\,\widetilde{c}c\left(1+\frac{\gamma\sigma^2 c}{2}\right) \\
&= 2(N-1)\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 c\left(1+\frac{\gamma\sigma^2 c}{2}\right).
\end{aligned}
$$

So

$$
\begin{aligned}
L_1 + L_2 + L_3 \;\le\; & (N-1)\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \times \\
& \left[2c\left(1+\frac{\gamma\sigma^2 c}{2}\right) + \frac{\beta^2}{(c+\beta)} - (c+\beta)\left(1-\frac{(\gamma\sigma^2 c)^2}{4}\right)\right] \\
= \; & (N-1)\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \left[c-\beta+\gamma\sigma^2c^2+\frac{\beta^2}{(c+\beta)}+(c+\beta)\frac{(\gamma\sigma^2 c)^2}{4}\right] \\
= \; & (N-1)\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \left[\frac{c^2}{(c+\beta)}+\gamma\sigma^2c^2+(c+\beta)\frac{(\gamma\sigma^2 c)^2}{4}\right] \\
= \; & (N-1)\frac{c^2}{(c+\beta)}\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \left[1+\gamma\sigma^2(c+\beta)+(c+\beta)^2\frac{(\gamma\sigma^2)^2}{4}\right] \\
= \; & (N-1)\frac{c^2}{(c+\beta)}\left[\frac{\gamma\sigma^2\widetilde{c}}{(c+\beta)(2+N\gamma\sigma^2\widetilde{c})}\right]^2 \left[1+\frac{\gamma\sigma^2(c+\beta)}{2}\right]^2 \\
\le \; & 0
\end{aligned}
$$

Next, we show that the bid price $\psi_{i0} = \overline{v} + \psi_z' z$ increases in the number of peripheral dealers. We can show that:

$$\psi_z' = -\left(\frac{1}{2}\gamma\sigma^2 \cdot I + II\right)$$

where

$$
\begin{aligned}
I \;&\equiv\; \left(1+\gamma\sigma^2 c\right)^2 + \gamma\sigma^2 c^2\left(2+\gamma\sigma^2 c\right)\theta + \frac{1}{2}\gamma\sigma^2 c^3\left(1+\frac{\gamma\sigma^2 c}{2}\right)\theta^2 \\
II \;&\equiv\; -\left(c+\frac{1}{2}\gamma\sigma^2 c^2\right)\left(\gamma\sigma^2\frac{1+N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}}\right)^2
\end{aligned}
$$

To prove $\frac{\partial\psi_z'}{\partial N} > 0$, it is sufficient to prove $\frac{\partial I}{\partial N} < 0$ and $\frac{\partial II}{\partial N} < 0$, which are proved below.

$$
\begin{aligned}
\frac{\partial I}{\partial N} \;=\; & \left[2\left(1+\gamma\sigma^2 c\right)\gamma\sigma^2 + \gamma\sigma^2\left(4c+3\gamma\sigma^2c^2\right)\widetilde{\theta} + \frac{1}{2}\gamma\sigma^2\left(3c^2+2\gamma\sigma^2c^3\right)\widetilde{\theta}^2\right]\frac{\partial c}{\partial N} \\
& + \left[\gamma\sigma^2c^2\left(2+\gamma\sigma^2 c\right) + \gamma\sigma^2c^3\left(1+\frac{\gamma\sigma^2 c}{2}\right)\widetilde{\theta}\right]\frac{\partial\widetilde{\theta}}{\partial N}
\end{aligned}
$$

$$
\begin{aligned}
&= \gamma\sigma^2 \left\{ \left[\left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) \right] \left(\frac{1}{2}\widetilde{\theta}c+1\right)\frac{\partial c}{\partial N} + c^2\left(\widetilde{\theta}c+2\right)\left(1+\frac{\gamma\sigma^2}{2}c\right)\frac{\partial\widetilde{\theta}}{\partial N} \right\} \\
&= \gamma\sigma^2 \left(\frac{1}{2}\widetilde{\theta}c+1\right) \left\{ \left[\left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) \right] \frac{\partial c}{\partial N} + c^2\left(2+\gamma\sigma^2 c\right)\frac{\partial\widetilde{\theta}}{\partial N} \right\}.
\end{aligned}
$$

Recall $\theta \equiv \gamma\sigma^2 \frac{1+N\gamma\sigma^2\widetilde{c}}{2+N\gamma\sigma^2\widetilde{c}} - \frac{1}{c+\beta} = \frac{\gamma\sigma^2}{2}\theta - \frac{1}{c+\beta}$. We have

$$
\begin{aligned}
\frac{\partial\widetilde{\theta}}{\partial N} &= \frac{\gamma\sigma^2}{2}\frac{\partial\theta}{\partial N} + \frac{1}{(c+\beta)^2}\frac{\partial c}{\partial N} \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^2 (c+\beta) + \left(N\left(\frac{\gamma\sigma^2}{2}\theta\right)^2 + \frac{1}{(c+\beta)^2} \right)\frac{\partial c}{\partial N},
\end{aligned}
$$

and

$$
\frac{\partial I}{\partial N} = \gamma\sigma^2\left(\frac{1}{2}\widetilde{\theta}c+1\right) \left\{ \begin{array}{c} \left[\left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) \right]\frac{\partial c}{\partial N} \\ +c^2\left(2+\gamma\sigma^2 c\right)\left[\left(\frac{\gamma\sigma^2}{2}\theta\right)^2 (c+\beta) + N\left(\frac{\gamma\sigma^2}{2}\theta\right)^2 \frac{\partial c}{\partial N} + \frac{1}{(c+\beta)^2}\frac{\partial c}{\partial N} \right] \end{array} \right\}.
$$

Recall

$$
\frac{\partial c}{\partial N} = \frac{\left(\gamma\sigma^2\widetilde{c}\right)^2 (c+\beta)}{\frac{1}{\left(1+\gamma\sigma^2 c\right)^2} - 1 - 2\gamma\sigma^2\widetilde{c} - N\left(\gamma\sigma^2\widetilde{c}\right)^2}.
$$

We have

$$
\begin{aligned}
\left(\frac{\gamma\sigma^2}{2}\theta\right)^2 (c+\beta) &= \frac{\left(\gamma\sigma^2\widetilde{c}\right)^2 (c+\beta)}{\left(\widetilde{c}+c+\beta\right)^2} \\
&= \frac{\frac{1}{\left(1+\gamma\sigma^2 c\right)^2} - 1 - 2\gamma\sigma^2\widetilde{c} - N\left(\gamma\sigma^2\widetilde{c}\right)^2}{\left(\widetilde{c}+c+\beta\right)^2}\frac{\partial c}{\partial N} \\
&= \frac{\frac{1}{\left(1+\gamma\sigma^2 c\right)^2} - 1 - 2\gamma\sigma^2\widetilde{c}}{\left(\widetilde{c}+c+\beta\right)^2}\frac{\partial c}{\partial N} - NA^2\frac{\partial c}{\partial N},
\end{aligned}
$$

implying

$$
\frac{\partial I}{\partial N} = \gamma\sigma^2\left(\frac{1}{2}\widetilde{\theta}c+1\right) \left\{ \begin{array}{c} \left[\left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) \right] \\ +c^2\left(2+\gamma\sigma^2 c\right)\frac{\frac{1}{\left(1+\gamma\sigma^2 c\right)^2} - 1 - 2\gamma\sigma^2\widetilde{c}}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2\left(2+\gamma\sigma^2 c\right)}{(c+\beta)^2} \end{array} \right\} \frac{\partial c}{\partial N}.
$$

To prove $\frac{\partial I}{\partial N} < 0$, we just need to show that

$$\left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) + c^2\left(2+\gamma\sigma^2 c\right)\frac{\frac{1}{(1+\gamma\sigma^2 c)^2} - 1 - 2\gamma\sigma^2\widetilde{c}}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2\left(2+\gamma\sigma^2 c\right)}{\left(c+\beta\right)^2} \geq 0.$$

The expression on the left-hand side can be simplified as follows.

$$\begin{aligned}
& \left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) + c^2\left(2+\gamma\sigma^2 c\right)\frac{\frac{1}{(1+\gamma\sigma^2 c)^2} - 1 - 2\gamma\sigma^2\widetilde{c}}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2\left(2+\gamma\sigma^2 c\right)}{\left(c+\beta\right)^2} \\
= \; & \left(3+2\gamma\sigma^2 c\right)\widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right) + \left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C_i^i+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right] \\
= \; & \widetilde{\theta}c + 2\left(1+\gamma\sigma^2 c\right)\left(\widetilde{\theta}c+1\right) + \left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C_i^i+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right] \\
= \; & \frac{\gamma\sigma^2\widetilde{c}c}{\widetilde{c}+c+\beta} + 2\left(1+\gamma\sigma^2 c\right)\left[\frac{\beta}{\left(\widetilde{c}+c+\beta\right)} + \frac{\widetilde{c}\left[\gamma\sigma^2 c\left(c+\beta\right)+\beta\right]}{\left(\widetilde{c}+c+\beta\right)\left(c+\beta\right)}\right] \\
& + \left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right] \\
= \; & \frac{\gamma\sigma^2\widetilde{c}c + 2\left(1+\gamma\sigma^2 c\right)\beta}{\widetilde{c}+c+\beta} + 2\left(1+\gamma\sigma^2 c\right)\frac{\widetilde{c}\left[\gamma\sigma^2 c\left(c+\beta\right)+\beta\right]}{\left(\widetilde{c}+c+\beta\right)\left(c+\beta\right)} \\
& + \left(2+\gamma\sigma^2 c\right)\left[\frac{\left(c+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right].
\end{aligned}$$

Since $\gamma\sigma^2\widetilde{c}c + 2\left(1+\gamma\sigma^2 c\right)\beta \leq \gamma\sigma^2 c^2 + 2\left(1+\gamma\sigma^2 c\right)\beta < 0$, we have $\frac{\gamma\sigma^2\widetilde{c}c+2\left(1+\gamma\sigma^2 c\right)\beta}{\widetilde{c}+c+\beta} > 0$. It comes down to show that

$$\begin{gathered}
2\left(1+\gamma\sigma^2 c\right)\frac{\widetilde{c}\left[\gamma\sigma^2 c\left(c+\beta\right)+\beta\right]}{\left(\widetilde{c}+c+\beta\right)\left(c+\beta\right)} + \\
\left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right] \geq 0.
\end{gathered}$$

$C - c - \beta = \frac{c}{1+\gamma\sigma^2 c} - 2\beta - c > 0$ implies that $\gamma\sigma^2 c\left(c+\beta\right)+\beta < -\beta\left(1+\gamma\sigma^2 c\right)$, so we have

$$2\left(1+\gamma\sigma^2 c\right)\frac{\widetilde{c}\left[\gamma\sigma^2 c\left(c+\beta\right)+\beta\right]}{\left(\widetilde{c}+c+\beta\right)\left(c+\beta\right)} + \left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C+\beta\right)^2 - c^2 - 2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2} + \frac{c^2}{\left(c+\beta\right)^2}\right]$$

$$
\begin{aligned}
&> -2\left(1+\gamma\sigma^2 C\right)\frac{\widetilde{c}\beta\left(1+\gamma\sigma^2 c\right)}{\left(\widetilde{c}+c+\beta\right)\left(c+\beta\right)}+\left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}\right]\\
&= -2\left(1+\gamma\sigma^2 c\right)^2\frac{\left(2+N\gamma\sigma^2\widetilde{c}\right)\widetilde{c}\beta}{\left(\widetilde{c}+c+\beta\right)^2}+\left(2+\gamma\sigma^2 c\right)\left[\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}\right]\\
&\geq \frac{-4\widetilde{c}\beta}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}
\end{aligned}
$$

The last step follows from $\left(1+\gamma\sigma^2 c\right)^2\left(2+N\gamma\sigma^2\widetilde{c}\right)<2$, $\left(2+\gamma\sigma^2 c\right)>1$, and $\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}=\frac{\left(C+\beta\right)^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\left[\frac{c^2}{\left(c+\beta\right)^2}-\frac{c^2}{\left(\widetilde{c}+c+\beta\right)^2}\right]>0$.Since $C\leq c\leq\widetilde{c}$, we have

$$
\begin{aligned}
&\frac{-4\widetilde{c}\beta}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}\\
\geq\ &\frac{-4C\beta}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{\left(C+\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}\\
=\ &\frac{\left(C-\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\frac{c^2}{\left(c+\beta\right)^2}\\
=\ &\frac{\left(C-\beta\right)^2-c^2-2\gamma\sigma^2\widetilde{c}c^2}{\left(\widetilde{c}+c+\beta\right)^2}+\left[\frac{c^2}{\left(c+\beta\right)^2}-\frac{c^2}{\left(\widetilde{c}+c+\beta\right)^2}\right]\\
\geq\ &0\ .
\end{aligned}
$$

This concludes our proof of $\frac{\partial I}{\partial N}<0$. Next we prove $\frac{\partial II}{\partial N}<0$. Note that

$$
\begin{aligned}
\frac{\partial II}{\partial N} &= -\left(1+\gamma\sigma^2 c\right)\left(\frac{\gamma\sigma^2\theta}{2}\right)^2\frac{\partial c}{\partial N}-\left(c+\frac{1}{2}\gamma\sigma^2 c^2\right)2\left(\frac{\gamma\sigma^2\theta}{2}\right)\frac{\partial\left(\frac{\gamma\sigma^2\theta}{2}\right)}{\partial N}\\
&= -\left(1+\gamma\sigma^2 c\right)\left(\frac{\gamma\sigma^2\theta}{2}\right)^2\frac{\partial c}{\partial N}-\left(c+\frac{1}{2}\gamma\sigma^2 c^2\right)2\left(\frac{\gamma\sigma^2\theta}{2}\right)^3\left[\left(c+\beta\right)+N\frac{\partial c}{\partial N}\right]\\
&= -\left(\frac{\gamma\sigma^2\theta}{2}\right)^2\left\{\begin{array}{c}\left[\left(1+\gamma\sigma^2 c\right)+N\left(\frac{\gamma\sigma^2\theta}{2}\right)\left(2c+\gamma\sigma^2 c^2\right)\right]\frac{\partial c}{\partial N}\\ +\frac{\gamma\sigma^2\theta}{2}\left(c+\beta\right)\left(2c+\gamma\sigma^2 c^2\right)\end{array}\right\}.
\end{aligned}
$$

If $\left[\left(1+\gamma\sigma^2 c\right)+N\frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)\right]\leq 0$, then $\frac{\partial II}{\partial N}<0$ follows $\frac{\partial c}{\partial N}<0$ and $\left(c+\beta\right)\left(2c+\gamma\sigma^2 c^2\right)>0$. Otherwise, if $\left(1+\gamma\sigma^2 c\right)+N\frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)>0$, then because $\frac{\partial c}{\partial N}>-\gamma\sigma^2\left(c+\beta\right)^2$ (remember $\frac{\partial \widetilde{c}}{\partial N} = \frac{\frac{\partial c}{\partial N}+\gamma\sigma^2(c+\beta)^2}{[1-\gamma\sigma^2 N(c+\beta)]^2} > 0$), we have

$$\begin{aligned}
\frac{\partial II}{\partial N} &< \left(\frac{\gamma\sigma^2\theta}{2}\right)^2 \left\{ \begin{array}{c} \left[-\left(1+\gamma\sigma^2 c\right) - N\frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)\right]\left[-\gamma\sigma^2\left(c+\beta\right)^2\right] \\ -\frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)\left(c+\beta\right) \end{array} \right\} \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^2 (c+\beta) \left\{ \begin{array}{c} \left(1+\gamma\sigma^2 c\right)\gamma\sigma^2\left(c+\beta\right) \\ -\frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right] \end{array} \right\} \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^2 (c+\beta)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left[\left(1+\gamma\sigma^2 c\right)\gamma\sigma^2\widetilde{c} - \frac{\gamma\sigma^2\theta}{2}\left(2c+\gamma\sigma^2 c^2\right)\right] \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^3 (c+\beta)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left[\left(1+\gamma\sigma^2 c\right)\widetilde{c}\frac{2+N\gamma\sigma^2\widetilde{c}}{1+N\gamma\sigma^2\widetilde{c}} - \left(2c+\gamma\sigma^2 c^2\right)\right] \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^3 (c+\beta)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left[\left(1+\gamma\sigma^2 c\right)\left(\widetilde{c}+c+\beta\right) - \left(2c+\gamma\sigma^2 c^2\right)\right] \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^3 (c+\beta)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left[\left(1+\gamma\sigma^2 c\right)\left(\widetilde{c}+\beta\right) - c\right] \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^3 (c+\beta)\left(1+\gamma\sigma^2 c\right)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left(\widetilde{c}+\beta-\frac{c}{1+\gamma\sigma^2 c}\right) \\
&= \left(\frac{\gamma\sigma^2\theta}{2}\right)^3 (c+\beta)\left(1+\gamma\sigma^2 c\right)\left[1-N\gamma\sigma^2\left(c+\beta\right)\right]\left(\widetilde{c}-C_i^i\right) \\
&< 0.
\end{aligned}$$

■

Proof of Proposition 8. When all the peripheral dealers have zero inventory and the central dealer has inventory x_0, we have

$$\begin{aligned}
\Psi_{0i} &= \overline{v} + \psi_z z + \psi_X x_0, \\
\Psi_{i0} &= \overline{v} + \psi'_z z + \psi'_X x_0, \\
\Psi_{ij} &= \overline{v} + \psi''_z z + \psi''_X x_0.
\end{aligned}$$

The central dealer can win the order if and only if $\Psi_{i0} - \Psi_{0i} = \left(\psi'_z - \psi_z\right) z + \left(\psi'_X - \psi_X\right) x_0 \leq 0$. In Proposition 1, we have already proved $\left(\psi'_z - \psi_z\right) \leq 0$. If we can prove $\left(\psi'_X - \psi_X\right) > 0$, then $\Psi_{i0} - \Psi_{0i} \leq 0$ holds if and only if $x_0 \leq -\frac{(\psi'_z - \psi_z)}{(\psi'_X - \psi_X)} z$.

We now prove $(\psi_X - \psi'_X) < 0$. First, we can show that

$$\begin{aligned} \psi_X &= 2A_0 - A_1 \\ \psi'_X &= B_1 - 2B_0 \end{aligned}$$

where

$$\begin{aligned} A_0 &= -\frac{\gamma\sigma^2}{2}\theta \\ A_1 &= \frac{\gamma\sigma^2}{2} b\theta \\ B_0 &= -\frac{\gamma\sigma^2 b}{4}\left(1 + \frac{1}{2}b\right)\theta^2 \\ B_1 &= \frac{\gamma\sigma^2}{4} b^2 \left(1 + \frac{b}{2}\right)\theta^2 + \frac{\gamma\sigma^2}{2} b \left(1 + \frac{b}{2}\right) \frac{c + 2\beta}{c + \beta}\theta \end{aligned}$$

Note that

$$\begin{aligned} \psi_X - \psi'_X &= 2A_0 + 2B_0 - A_1 - B_1 \\ &= 2\left[-\frac{1}{2}\gamma\sigma^2\left(1 + \frac{N\gamma\sigma^2\widetilde{c}}{2 + N\gamma\sigma^2\widetilde{c}}\right)^2 + (N-1)\frac{\left(\gamma\sigma^2\right)^2 \widetilde{c}^2}{(c+\beta)\left(2 + N\gamma\sigma^2\widetilde{c}\right)^2}\right] \\ &\quad + \beta\frac{2\left(\gamma\sigma^2\right)^2}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} - \gamma\sigma^2\frac{\left(\gamma\sigma^2 c\right)^2}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} - \frac{\left(\gamma\sigma^2\right)^2 c}{2 - N\gamma\sigma^2(c+\beta)} \\ &\quad - \frac{\left(\gamma\sigma^2\right)^2 c}{2 - N\gamma\sigma^2(c+\beta)}\left(2 + \gamma\sigma^2 c\right)\left(1 + \frac{1}{2}c\widetilde{\theta}\right) \end{aligned}$$

Substituting $\widetilde{\theta} = \frac{\gamma\sigma^2}{2 - N\gamma\sigma^2(c+\beta)} - \frac{1}{c+\beta}$ into the above equation, we have

$$\begin{aligned} \psi_X - \psi'_X &= -\frac{4\gamma\sigma^2}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} + \frac{2(N-1)\left(\gamma\sigma^2\right)^2(c+\beta)}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} \\ &\quad + \beta\frac{2\left(\gamma\sigma^2\right)^2}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} - \frac{\left(\gamma\sigma^2\right)^3 c^2}{\left(2 - N\gamma\sigma^2(c+\beta)\right)^2} - \frac{\left(\gamma\sigma^2\right)^2 c}{2 - N\gamma\sigma^2(c+\beta)} \\ &\quad - \frac{\left(\gamma\sigma^2\right)^2\left(2c + \gamma\sigma^2 c^2\right)}{2 - N\gamma\sigma^2(c+\beta)} - \frac{1}{2}\frac{\left(\gamma\sigma^2\right)^2 c^2}{2 - N\gamma\sigma^2(c+\beta)}\left(\frac{\gamma\sigma^2}{2 - N\gamma\sigma^2(c+\beta)} - \frac{1}{c+\beta}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{4\gamma\sigma^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}+\frac{2\left(N-2\right)\left(\gamma\sigma^2\right)^2c}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}+\beta\frac{2N\left(\gamma\sigma^2\right)^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2} \\
&\quad -\frac{2\left(\gamma\sigma^2\right)^3c^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}-\frac{\left(\gamma\sigma^2\right)^2c}{2-N\gamma\sigma^2\left(c+\beta\right)} \\
&\quad -\frac{1}{2}\frac{\left(\gamma\sigma^2\right)^2c^2}{2-N\gamma\sigma^2\left(c+\beta\right)}\left(\frac{\gamma\sigma^2}{2-N\gamma\sigma^2\left(c+\beta\right)}-\frac{1}{c+\beta}\right) \\
&= -\frac{4\gamma\sigma^2\left(1+\gamma\sigma^2c\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}+\frac{2N\left(\gamma\sigma^2\right)^2\left(c+\beta\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}-\frac{2\left(\gamma\sigma^2\right)^3c^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2} \\
&\quad -\frac{\left(\gamma\sigma^2\right)^2c}{2-N\gamma\sigma^2\left(c+\beta\right)}-\frac{1}{2}\frac{\left(\gamma\sigma^2\right)^2c^2}{2-N\gamma\sigma^2\left(c+\beta\right)}\left(\frac{\gamma\sigma^2}{2-N\gamma\sigma^2\left(c+\beta\right)}-\frac{1}{c+\beta}\right)
\end{aligned}
$$

Because $c \geq N\widetilde{c}$, we have

$$
\begin{aligned}
&\frac{2N\left(\gamma\sigma^2\right)^2\left(c+\beta\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}-\frac{\left(\gamma\sigma^2\right)^2c}{2-N\gamma\sigma^2\left(c+\beta\right)} \\
\leq\ &\frac{2N\left(\gamma\sigma^2\right)^2\left(c+\beta\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}-\frac{N\left(\gamma\sigma^2\right)^2\left(c+\beta\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)\left(1-N\gamma\sigma^2\left(c+\beta\right)\right)} \\
=\ &N\left(\gamma\sigma^2\right)^2\left(c+\beta\right)\frac{2\left(1-N\gamma\sigma^2\left(c+\beta\right)\right)-\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2\left(1-N\gamma\sigma^2\left(c+\beta\right)\right)} \\
=\ &-\frac{N^2\left(\gamma\sigma^2\right)^3\left(c+\beta\right)^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2\left(1-N\gamma\sigma^2\left(c+\beta\right)\right)} \\
<\ &0
\end{aligned}
$$

It follows that:

$$
\begin{aligned}
\psi_X-\psi_X' &< -\frac{4\gamma\sigma^2\left(1+\gamma\sigma^2c\right)}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2}-\frac{2\left(\gamma\sigma^2\right)^3c^2}{\left(2-N\gamma\sigma^2\left(c+\beta\right)\right)^2} \\
&\quad -\frac{1}{2}\frac{\left(\gamma\sigma^2\right)^2c^2}{2-N\gamma\sigma^2\left(c+\beta\right)}\left(\frac{\gamma\sigma^2}{2-N\gamma\sigma^2\left(c+\beta\right)}-\frac{1}{c+\beta}\right) \\
&< 0
\end{aligned}
$$

because each term in the above inequality is negative. ■

Proof of Proposition 9. In the complete network, it is straightforward to show that $c_j^{ij}=c$ and $\widetilde{c}_j^{ij}=\widetilde{c}$ where $\widetilde{c}\equiv\frac{c+\beta}{1-N\gamma\sigma^2(c+\beta)}$ and c is the solution to $0=\beta+\gamma\sigma^2\widetilde{c}\left(c+\beta\right)$, or

$-\gamma\sigma^2 (c+\beta)(c-(N-1)\beta) = \beta.$

Next, from equation (9), we have

$$
\begin{aligned}
& -\gamma\sigma^2 \widetilde{c}\left(x_i' + x_j'\right) \\
= \ & (2c+\beta)(p_{ij}-\overline{v}) + \gamma\sigma^2 (c+\beta)\widetilde{c} \sum_{k\in g_i, k\neq j} (p_{ik}-\overline{v}) + \gamma\sigma^2 (c+\beta)\widetilde{c} \sum_{k\in g_j, k\neq i} (p_{jk}-\overline{v}) \\
= \ & \left(2c+\beta-2\gamma\sigma^2 (c+\beta)\widetilde{c}\right)(p_{ij}-\overline{v}) + \gamma\sigma^2 (c+\beta)\widetilde{c} \sum_{k\in g_i} (p_{ik}-\overline{v}) + \gamma\sigma^2 (c+\beta)\widetilde{c} \sum_{k\in g_j} (p_{jk}-\overline{v}) \\
= \ & (2c+3\beta)(p_{ij}-\overline{v}) - \beta \sum_{k\in g_i} (p_{ik}-\overline{v}) - \beta \sum_{k\in g_j} (p_{jk}-\overline{v}).
\end{aligned}
$$

Summing over $i\,(\neq j)$ yields:

$$
\begin{aligned}
& -\gamma\sigma^2 \widetilde{c}\left(\textstyle\sum_k x_k' + (N-1)\, x_j'\right) \\
= \ & (2c+3\beta) \sum_{k\in g_j} (p_{jk}-\overline{v}) - \beta \sum_{i\in g_j} \sum_{k\in g_i} (p_{ik}-\overline{v}) - N\beta \sum_{k\in g_j} (p_{jk}-\overline{v}) \\
= \ & (2c-(N-4)\beta) \sum_{k\in g_j} (p_{jk}-\overline{v}) - \beta \sum_{k\neq l} (p_{kl}-\overline{v})
\end{aligned}
$$

Further summing over j yields:

$$
-\gamma\sigma^2 \widetilde{c}\,(2N) \textstyle\sum_k x_k' = (2c-(2N-3)\beta) \displaystyle\sum_{k\neq l} (p_{kl}-\overline{v})
$$

Therefore,

$$
\sum_{k\neq l} (p_{kl}-\overline{v}) = -\frac{2N\gamma\sigma^2\widetilde{c}}{2c-(2N-3)\beta} \textstyle\sum_k x_k'
$$

and

$$
\begin{aligned}
& \sum_{k\in g_j} (p_{jk}-\overline{v}) \\
= \ & \frac{1}{(2c-(N-4)\beta)} \left[-\gamma\sigma^2\widetilde{c}\left(\textstyle\sum_k x_k' + (N-1)\, x_j'\right) + \beta \displaystyle\sum_{k\neq l} (p_{kl}-\overline{v})\right] \\
= \ & -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)} \left[(N-1)\, x_j' + \frac{(2c+3\beta)}{2c-(2N-3)\beta} \textstyle\sum_k x_k'\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&(p_{ij} - \overline{v}) \\
= \;& \frac{1}{2c+3\beta}\left[-\gamma\sigma^2\widetilde{c}\left(x_i' + x_j'\right) + \beta\sum_{k\in g_i}(p_{ik}-\overline{v}) + \beta\sum_{k\in g_j}(p_{jk}-\overline{v})\right] \\
= \;& -\frac{\gamma\sigma^2\widetilde{c}}{2c+3\beta}\left[\left(x_i'+x_j'\right) + \frac{\beta}{(2c-(N-4)\beta)}\left[(N-1)\left(x_i'+x_j'\right) + \frac{2(2c+3\beta)}{2c-(2N-3)\beta}\textstyle\sum_k x_k'\right]\right] \\
= \;& -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}\left[\left(x_i'+x_j'\right) + \frac{2\beta}{2c-(2N-3)\beta}\textstyle\sum_k x_k'\right]
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
q_{ij} &= \gamma\sigma^2\widetilde{c}x_i' + c(p_{ij}-\overline{v}) - \beta\sum_{k\in g_i, k\neq j}(p_{ik}-\overline{v}) \\
&= \gamma\sigma^2\widetilde{c}x_i' + (c+\beta)(p_{ij}-\overline{v}) - \beta\sum_{k\in g_i}(p_{ik}-\overline{v})
\end{aligned}
$$

Next, we determine the reservation price matrix $\Psi = (\Psi_{ij})_{i,j=0}^N$.

Recall that dealer i's date-1 utility is given by

$$
u_i(w_i', \mathbf{x}') = w_i' + x_i'\overline{v} - \textstyle\sum_{j\in g_i} q_{ij}(p_{ij}-\overline{v}) - \frac{1}{2}\gamma\sigma^2\left(x_i' + \textstyle\sum_{j\in g_i} q_{ij}\right)^2.
$$

where

$$
\begin{aligned}
q_{ij} &= \gamma\sigma^2\widetilde{c}x_i' + (c+\beta)(p_{ij}-\overline{v}) - \beta\sum_{k\in g_i}(p_{ik}-\overline{v}) \\
\sum_{k\in g_i}(p_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}\left[(N-1)x_i' + \frac{(2c+3\beta)}{2c-(2N-3)\beta}\textstyle\sum_k x_k'\right] \\
(p_{ij}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}\left[\left(x_i'+x_j'\right) + \frac{2\beta}{2c-(2N-3)\beta}\textstyle\sum_k x_k'\right]
\end{aligned}
$$

We determine the equilibrium prices and quantities in the following scenarios:

- if dealer i wins the order z, then $x_i' = x_i + z$, and

$$
\begin{aligned}
q_{ij} &= \gamma\sigma^2\widetilde{c}\,(x_i+z) + (c+\beta)\,(p_{ij}-\overline{v}) - \beta\sum_{k\in g_i}(p_{ik}-\overline{v}) \\
\sum_{k\in g_i} q_{ik} &= N\gamma\sigma^2\widetilde{c}\,(x_i+z) + (c-(N-1)\,\beta)\sum_{k\in g_i}(p_{ik}-\overline{v}) \\
\sum_{k\in g_i}(p_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(N-1)\,(x_i+z) + \frac{(2c+3\beta)}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k + z\right)\right] \\
(p_{ij}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(x_i+x_j) + z + \frac{2\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k + z\right)\right] \\
(p_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(x_i+x_k) + z + \frac{2\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k + z\right)\right] \\
\sum_{k\in g_i, k\neq j}(p_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(N-2)\,(x_i+z) - x_j + \frac{2c+\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k + z\right)\right]
\end{aligned}
$$

- if dealer $j\,(\neq i)$ wins the order z, then $x_i' = x_i$ and $x_j' = x_j + z$.

$$
\begin{aligned}
\widehat{q}_{ij} &= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)\,(\widehat{p}_{ij}-\overline{v}) - \beta\sum_{k\in g_i}(\widehat{p}_{ik}-\overline{v}) \\
&= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)\,(p_{ij}-\overline{v}) - \beta\left[\sum_{k\in g_i}(p_{ik}-\overline{v}) + \frac{(N-1)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z\right] \\
&= q_{ij} - \gamma\sigma^2\widetilde{c}z\left(1 + \frac{(N-1)\,\beta}{(2c-(N-4)\,\beta)}\right) \\
&= q_{ij} - \gamma\sigma^2\widetilde{c}z\frac{2c+3\beta}{2c-(N-4)\,\beta} \\
\widehat{q}_{ik} &= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)\,(\widehat{p}_{ik}-\overline{v}) - \beta\sum_{k\in g_i}(\widehat{p}_{ik}-\overline{v}) \\
&= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)\left[(p_{ik}-\overline{v}) + \frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z\right] \\
&\quad -\beta\left[\sum_{k\in g_i}(p_{ik}-\overline{v}) + \frac{(N-1)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z\right] \\
&= q_{ik} - \gamma\sigma^2\widetilde{c}z\left[1 - \frac{c+\beta}{2c-(N-4)\,\beta} + \frac{(N-1)\,\beta}{2c-(N-4)\,\beta}\right] \\
&= q_{ik} - \frac{(c+2\beta)\,\gamma\sigma^2\widetilde{c}}{2c-(N-4)\,\beta}z \\
\sum_{k\in g_i}(\widehat{p}_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(N-1)\,x_i + \frac{(2c+3\beta)}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k + z\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k\in g_i}(p_{ik}-\overline{v}) + \frac{(N-1)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z \\
(\widehat{p}_{ij}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(x_i+x_j)+z+\frac{2\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k+z\right)\right] \\
&= (p_{ij}-\overline{v}) \\
(\widehat{p}_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(x_i+x_k)+\frac{2\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k+z\right)\right] \\
&= (p_{ik}-\overline{v}) + \frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z
\end{aligned}
$$

We now turn to the determination of the reservation price Ψ_{ij}, which is defined by $u_i(w_i-\Psi_{ij}z,\mathbf{x}+z\mathbf{e}_i)=u_i(w_i,\mathbf{x}+z\mathbf{e}_j)$, or

$$
\begin{aligned}
& w_i+x_i\overline{v}-(\Psi_{ij}-\overline{v})\,z-\sum_{k\in g_i}q_{ik}\,(p_{ik}-\overline{v})-\frac{1}{2}\gamma\sigma^2\left(x_i+z+\sum_{k\in g_i}q_{ik}\right)^2 \\
= \; & w_i+x_i\overline{v}-\sum_{k\in g_i}\widehat{q}_{ik}\,(\widehat{p}_{ik}-\overline{v})-\frac{1}{2}\gamma\sigma^2\left(x_i+\sum_{k\in g_i}\widehat{q}_{ik}\right)^2
\end{aligned}
$$

or

$$
\begin{aligned}
(\Psi_{ij}-\overline{v})\,z &= \sum_{k\in g_i}\widehat{q}_{ik}\,(\widehat{p}_{ik}-\overline{v})+\frac{1}{2}\gamma\sigma^2\left(x_i+\sum_{k\in g_i}\widehat{q}_{ik}\right)^2 \\
&\quad -\sum_{k\in g_i}q_{ik}\,(p_{ik}-\overline{v})-\frac{1}{2}\gamma\sigma^2\left(x_i+z+\sum_{k\in g_i}q_{ik}\right)^2
\end{aligned}
$$

Because:

$$
\begin{aligned}
\widehat{p}_{ij} &= p_{ij} \\
\widehat{p}_{ik} &= p_{ik}+\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z, j\neq k\in g_i \\
\sum_{k\in g_i}(\widehat{p}_{ik}-\overline{v}) &= \sum_{k\in g_i}(p_{ik}-\overline{v})+\frac{(N-1)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z \\
q_{ij} &= \gamma\sigma^2\widetilde{c}\,(x_i+z)+(c+\beta)\,(p_{ij}-\overline{v})-\beta\sum_{k\in g_i}(p_{ik}-\overline{v}) \\
q_{ik} &= \gamma\sigma^2\widetilde{c}\,(x_i+z)+(c+\beta)\,(p_{ik}-\overline{v})-\beta\sum_{k\in g_i}(p_{ik}-\overline{v})\,, j\neq k\in g_i
\end{aligned}
$$

$$\begin{aligned}
\widehat{q}_{ij} &= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)(\widehat{p}_{ij} - \overline{v}) - \beta \sum_{k\in g_i} (\widehat{p}_{ik} - \overline{v}) \\
\widehat{q}_{ik} &= \gamma\sigma^2\widetilde{c}x_i + (c+\beta)(\widehat{p}_{ik} - \overline{v}) - \beta \sum_{k\in g_i} (\widehat{p}_{ik} - \overline{v}), j \neq k \in g_i
\end{aligned}$$

and

$$\begin{aligned}
\widehat{q}_{ij} &= q_{ij} - \gamma\sigma^2\widetilde{c}z - \beta\frac{(N-1)\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}z = q_{ij} - \frac{(2c+3\beta)\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}z \\
\widehat{q}_{ik} &= q_{ik} - \gamma\sigma^2\widetilde{c}z + (c+\beta)\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}z - \beta\frac{(N-1)\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}z, j \neq k \in g_i \\
&= q_{ik} - \frac{(c+2\beta)\gamma\sigma^2\widetilde{c}}{2c-(N-4)\beta}z \\
\sum_{k\in g_i} \widehat{q}_{ik} &= \sum_{k\in g_i} q_{ik} - \frac{(2c+3\beta)}{(2c-(N-4)\beta)}\gamma\sigma^2\widetilde{c}z - \frac{(N-1)(c+2\beta)}{2c-(N-4)\beta}\gamma\sigma^2\widetilde{c}z \\
&= \sum_{k\in g_i} q_{ik} - \frac{(N+1)c+(2N+1)\beta}{2c-(N-4)\beta}\gamma\sigma^2\widetilde{c}z
\end{aligned}$$

we have

$$\begin{aligned}
&(\Psi_{ij} - \overline{v})z \\
=\ & \sum_{k\in g_i} \widehat{q}_{ik}(\widehat{p}_{ik} - \overline{v}) - \sum_{k\in g_i} q_{ik}(p_{ik} - \overline{v}) \\
&+\frac{1}{2}\gamma\sigma^2\left(x_i + \sum_{k\in g_i}\widehat{q}_{ik}\right)^2 - \frac{1}{2}\gamma\sigma^2\left(x_i + z + \sum_{k\in g_i} q_{ik}\right)^2 \\
=\ & (\widehat{q}_{ij} - q_{ij})(p_{ij} - \overline{v}) + \sum_{k\in g_i, k\neq j}\widehat{q}_{ik}(\widehat{p}_{ik} - \overline{v}) - \sum_{k\in g_i, k\neq j} q_{ik}(p_{ik} - \overline{v}) \\
&+\frac{1}{2}\gamma\sigma^2\left(x_i + \sum_{k\in g_i} q_{ik} - \gamma\sigma^2\widetilde{c}z\frac{(N+1)c+(2N+1)\beta}{2c-(N-4)\beta}\right)^2 - \frac{1}{2}\gamma\sigma^2\left(x_i + \sum_{k\in g_i} q_{ik} + z\right)^2 \\
=\ & -\frac{(2c+3\beta)\gamma\sigma^2\widetilde{c}z}{(2c-(N-4)\beta)}(p_{ij} - \overline{v}) \\
&+\sum_{k\in g_i, k\neq j}\left(q_{ik} - \frac{\gamma\sigma^2\widetilde{c}(c+2\beta)}{2c-(N-4)\beta}z\right)\left(p_{ik} - \overline{v} + \frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\beta)}z\right) - \sum_{k\in g_i, k\neq j} q_{ik}(p_{ik} - \overline{v}) \\
&-\frac{1}{2}\gamma\sigma^2\left(2x_i + 2\sum_{k\in g_i} q_{ik} + z - \gamma\sigma^2\widetilde{c}z\frac{(N+1)c+(2N+1)\beta}{2c-(N-4)\beta}\right) \\
&\left(1 + \gamma\sigma^2\widetilde{c}\frac{(N+1)c+(2N+1)\beta}{2c-(N-4)\beta}\right)z
\end{aligned}$$

Note that:

$$
\begin{aligned}
& \sum_{k\in g_i,k\neq j}\left(q_{ik}-\frac{\gamma\sigma^2\widetilde{c}\,(c+2\beta)}{2c-(N-4)\,\beta}z\right)\left(p_{ik}-\overline{v}+\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z\right)-\sum_{k\in g_i,k\neq j}q_{ik}\,(p_{ik}-\overline{v}) \\
= \;& \frac{\gamma\sigma^2\widetilde{c}z}{(2c-(N-4)\,\beta)}\sum_{k\in g_i,k\neq j}q_{ik}-\frac{\gamma\sigma^2\widetilde{c}\,(c+2\beta)\,z}{2c-(N-4)\,\beta}\sum_{k\in g_i,k\neq j}(p_{ik}-\overline{v})-\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2\,z^2}{(2c-(N-4)\,\beta)^2} \\
= \;& \frac{\gamma\sigma^2\widetilde{c}z}{(2c-(N-4)\,\beta)}\left(\sum_{k\in g_i}q_{ik}-q_{ij}\right)-\frac{\gamma\sigma^2\widetilde{c}\,(c+2\beta)\,z}{2c-(N-4)\,\beta}\left(\sum_{k\in g_i}(p_{ik}-\overline{v})-(p_{ij}-\overline{v})\right) \\
& -\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2\,z^2}{(2c-(N-4)\,\beta)^2} \\
= \;& \frac{\gamma\sigma^2\widetilde{c}z}{(2c-(N-4)\,\beta)}\left[\sum_{k\in g_i}q_{ik}-(c+2\beta)\sum_{k\in g_i}(p_{ik}-\overline{v})-q_{ij}+(c+2\beta)\,(p_{ij}-\overline{v})\right] \\
& -\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2\,z^2}{(2c-(N-4)\,\beta)^2}
\end{aligned}
$$

Because

$$
\begin{aligned}
\sum_{k\in g_i}q_{ik} &= N\gamma\sigma^2\widetilde{c}\,(x_i+z)+(c-(N-1)\,\beta)\sum_{k\in g_i}(p_{ik}-\overline{v}) \\
q_{ij} &= \gamma\sigma^2\widetilde{c}\,(x_i+z)+(c+\beta)\,(p_{ij}-\overline{v})-\beta\sum_{k\in g_i}(p_{ik}-\overline{v})
\end{aligned}
$$

we have

$$
\begin{aligned}
& \sum_{k\in g_i,k\neq j}\left(q_{ik}-\frac{\gamma\sigma^2\widetilde{c}\,(c+2\beta)}{2c-(N-4)\,\beta}z\right)\left(p_{ik}-\overline{v}+\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}z\right)-\sum_{k\in g_i,k\neq j}q_{ik}\,(p_{ik}-\overline{v}) \\
= \;& \frac{\gamma\sigma^2\widetilde{c}z}{(2c-(N-4)\,\beta)}\left[\begin{array}{c} N\gamma\sigma^2\widetilde{c}\,(x_i+z)+(c-(N-1)\,\beta)\sum_{k\in g_i}(p_{ik}-\overline{v})-(c+2\beta)\sum_{k\in g_i}(p_{ik}-\overline{v}) \\ -\gamma\sigma^2\widetilde{c}\,(x_i+z)-(c+\beta)\,(p_{ij}-\overline{v})+\beta\sum_{k\in g_i}(p_{ik}-\overline{v})+(c+2\beta)\,(p_{ij}-\overline{v}) \end{array}\right] \\
& -\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2\,z^2}{(2c-(N-4)\,\beta)^2} \\
= \;& \frac{\gamma\sigma^2\widetilde{c}z\left[(N-1)\,\gamma\sigma^2\widetilde{c}\,(x_i+z)+\beta\,(p_{ij}-\overline{v})-N\beta\sum_{k\in g_i}(p_{ik}-\overline{v})\right]}{(2c-(N-4)\,\beta)}-\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2\,z^2}{(2c-(N-4)\,\beta)^2}
\end{aligned}
$$

Denote:

$$M=\gamma\sigma^2\widetilde{c}\frac{(N+1)\,c+(2N+1)\,\beta}{2c-(N-4)\,\beta}$$

Therefore,

$$\begin{aligned}
&(\Psi_{ij}-\overline{v})\\
=\;& -\frac{(2c+3\beta)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\,(p_{ij}-\overline{v})\\
&+\frac{\gamma\sigma^2\widetilde{c}\left[(N-1)\,\gamma\sigma^2\widetilde{c}\,(x_i+z)+\beta\,(p_{ij}-\overline{v})-N\beta\sum_{k\in g_i}(p_{ik}-\overline{v})\right]}{(2c-(N-4)\,\beta)}-\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2}{(2c-(N-4)\,\beta)^2}z\\
&-\gamma\sigma^2\,(1+M)\,x_i-\gamma\sigma^2\,(1+M)\sum_{k\in g_i}q_{ik}-\frac{1}{2}\gamma\sigma^2\left(1-M^2\right)z\\
=\;& -\frac{2\,(c+\beta)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\,(p_{ij}-\overline{v})+\frac{\gamma\sigma^2\widetilde{c}\left[(N-1)\,\gamma\sigma^2\widetilde{c}\,(x_i+z)-N\beta\sum_{k\in g_i}(p_{ik}-\overline{v})\right]}{(2c-(N-4)\,\beta)}\\
&-\gamma\sigma^2\,(1+M)\,(x_i+z)-\gamma\sigma^2\,(1+M)\left(N\gamma\sigma^2\widetilde{c}\,(x_i+z)+(c-(N-1)\,\beta)\sum_{k\in g_i}(p_{ik}-\overline{v})\right)\\
&-\left[\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2}{(2c-(N-4)\,\beta)^2}+\frac{1}{2}\gamma\sigma^2\left(1-M^2\right)-\gamma\sigma^2\,(1+M)\right]z\\
=\;& -\frac{2\,(c+\beta)\,\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\,(p_{ij}-\overline{v})\\
&-\left[\frac{N\beta\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}+\gamma\sigma^2\,(1+M)\,(c-(N-1)\,\beta)\right]\sum_{k\in g_i}(p_{ik}-\overline{v})\\
&+\left[\frac{(N-1)\,(\gamma\sigma^2\widetilde{c})^2}{(2c-(N-4)\,\beta)}-\gamma\sigma^2\,(1+M)\left(1+N\gamma\sigma^2\widetilde{c}\right)\right](x_i+z)\\
&-\left[\frac{(N-1)\,(c+2\beta)\,(\gamma\sigma^2\widetilde{c})^2}{(2c-(N-4)\,\beta)^2}+\frac{1}{2}\gamma\sigma^2\left(1-M^2\right)-\gamma\sigma^2\,(1+M)\right]z
\end{aligned}$$

Because

$$\begin{aligned}
\sum_{k\in g_i}(p_{ik}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(N-1)\,(x_i+z)+\frac{(2c+3\beta)}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k+z\right)\right]\\
(p_{ij}-\overline{v}) &= -\frac{\gamma\sigma^2\widetilde{c}}{(2c-(N-4)\,\beta)}\left[(x_i+x_j)+z+\frac{2\beta}{2c-(2N-3)\,\beta}\left(\textstyle\sum_k x_k+z\right)\right]
\end{aligned}$$

we have

$$\begin{aligned}
\left(\Psi_{ij}-\overline{v}\right) &= \frac{2\left(c+\beta\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}\left[\left(x_{i}+x_{j}\right)+z+\frac{2\beta}{2c-\left(2N-3\right)\beta}\left(\sum_{k}x_{k}+z\right)\right] \\
&\quad+\left[\frac{N\beta\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{\gamma\sigma^{2}\widetilde{c}\gamma\sigma^{2}\left(1+M\right)\left(c-\left(N-1\right)\beta\right)}{\left(2c-\left(N-4\right)\beta\right)}\right] \\
&\quad\times\left[\left(N-1\right)\left(x_{i}+z\right)+\frac{\left(2c+3\beta\right)}{2c-\left(2N-3\right)\beta}\left(\sum_{k}x_{k}+z\right)\right] \\
&\quad+\left[\frac{\left(N-1\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)}-\gamma\sigma^{2}\left(1+M\right)\left(1+N\gamma\sigma^{2}\widetilde{c}\right)\right]\left(x_{i}+z\right) \\
&\quad-\left[\frac{\left(N-1\right)\left(c+2\beta\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{1}{2}\gamma\sigma^{2}\left(1-M^{2}\right)-\gamma\sigma^{2}\left(1+M\right)\right]z \\
&\equiv \psi_{1}x_{i}+\psi_{2}x_{j}+\psi_{z}z+\psi_{X}\sum_{k}x_{k}
\end{aligned}$$

where

$$\begin{aligned}
\psi_{2} &= \frac{2\left(c+\beta\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}} \\
\psi_{1} &= \psi_{2}+\left(N-1\right)\left[\frac{N\beta\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{\gamma\sigma^{2}\widetilde{c}\gamma\sigma^{2}\left(1+M\right)\left(c-\left(N-1\right)\beta\right)}{\left(2c-\left(N-4\right)\beta\right)}\right] \\
&\quad+\left[\frac{\left(N-1\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)}-\gamma\sigma^{2}\left(1+M\right)\left(1+N\gamma\sigma^{2}\widetilde{c}\right)\right] \\
\psi_{X} &= \psi_{2}\frac{2\beta}{2c-\left(2N-3\right)\beta} \\
&\quad+\left[\frac{N\beta\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{\gamma\sigma^{2}\widetilde{c}\gamma\sigma^{2}\left(1+M\right)\left(c-\left(N-1\right)\beta\right)}{\left(2c-\left(N-4\right)\beta\right)}\right]\frac{\left(2c+3\beta\right)}{2c-\left(2N-3\right)\beta} \\
\psi_{z} &= \psi_{2}\left(1+\frac{2\beta}{2c-\left(2N-3\right)\beta}\right) \\
&\quad+\left[\frac{N\beta\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{\gamma\sigma^{2}\widetilde{c}\gamma\sigma^{2}\left(1+M\right)\left(c-\left(N-1\right)\beta\right)}{\left(2c-\left(N-4\right)\beta\right)}\right]\left[\left(N-1\right)+\frac{\left(2c+3\beta\right)}{2c-\left(2N-3\right)\beta}\right] \\
&\quad+\left[\frac{\left(N-1\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)}-\gamma\sigma^{2}\left(1+M\right)\left(1+N\gamma\sigma^{2}\widetilde{c}\right)\right] \\
&\quad-\left[\frac{\left(N-1\right)\left(c+2\beta\right)\left(\gamma\sigma^{2}\widetilde{c}\right)^{2}}{\left(2c-\left(N-4\right)\beta\right)^{2}}+\frac{1}{2}\gamma\sigma^{2}\left(1-M^{2}\right)-\gamma\sigma^{2}\left(1+M\right)\right]
\end{aligned}$$

Next, we prove $\psi_1 < \psi_2 < 0$. First, $\psi_2 = \frac{2(c+\beta)\left(\gamma\sigma^2\widetilde{c}\right)^2}{(2c-(N-4)\beta)^2} < 0$ is obvious.

Second, we just need to prove $\psi_1 - \psi_2 < 0$. Note that

$$\begin{aligned}
& \psi_1 - \psi_2 \\
= \; & (N-1)\left[\frac{N\beta\left(\gamma\sigma^2\widetilde{c}\right)^2}{(2c-(N-4)\beta)^2} + \frac{\gamma\sigma^2\widetilde{c}\gamma\sigma^2(1+M)(c-(N-1)\beta)}{(2c-(N-4)\beta)}\right] \\
& + \left[\frac{(N-1)\left(\gamma\sigma^2\widetilde{c}\right)^2}{(2c-(N-4)\beta)} - \gamma\sigma^2(1+M)\left(1+N\gamma\sigma^2\widetilde{c}\right)\right] \\
= \; & (N-1)\left[\frac{\left(\gamma\sigma^2\widetilde{c}\right)^2(2c+4\beta)}{(2c-(N-4)\beta)^2} + \frac{\gamma\sigma^2\widetilde{c}\gamma\sigma^2(1+M)(c-(N-1)\beta)}{(2c-(N-4)\beta)}\right] \\
& -\gamma\sigma^2(1+M)\left(1+N\gamma\sigma^2\widetilde{c}\right)
\end{aligned}$$

To prove $\psi_1 - \psi_2 < 0$, it is sufficient to prove the following results:

$$\begin{aligned}
1+M &> 0, \\
1+N\gamma\sigma^2\widetilde{c} &> 0, \\
(2c-(N-4)\beta) &< 0, \\
(c-(N-1)\beta) &< 0.
\end{aligned}$$

First, from $-\gamma\sigma^2(c+\beta)(c-(N-1)\beta) = \beta$, we have $(c-(N-1)\beta) < 0$.

Second, $(2c-(N-4)\beta) < 2(N-1)\beta - (N-4)\beta = (N+2)\beta < 0$.

Third, $1+N\gamma\sigma^2\widetilde{c} = \frac{1}{1-N\gamma\sigma^2(c+\beta)} > 0$.

Lastly,

$$\begin{aligned}
& 1+M \\
= \; & 1+\gamma\sigma^2\widetilde{c}\frac{(N+1)c+(2N+1)\beta}{2c-(N-4)\beta} \\
= \; & \frac{2c-(N-4)\beta+b((N+1)c+(2N+1)\beta)}{2c-(N-4)\beta} \\
= \; & \frac{2c-(N-4)\beta-(N+1)\beta+Nb\beta}{2c-(N-4)\beta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2c - (2N-4)\,\beta + (Nb+1)\,\beta}{2c - (N-4)\,\beta} \\
&> 0
\end{aligned}
$$

where the last inequality results from the fact that: $2c - (N-4)\,\beta < 0$, $(Nb+1)\,\beta < 0$, and

$$
\begin{aligned}
& 2c - (2N-4)\,\beta \\
=\ & (N-2)\,\beta - \sqrt{(N-2)^2\,\beta + 4\left((N-1)\,\beta^2 - \beta/\left(\gamma\sigma^2\right)\right)} - (2N-4)\,\beta \\
=\ & (2-N)\,\beta - \sqrt{(N-2)^2\,\beta + 4\left((N-1)\,\beta^2 - \beta/\left(\gamma\sigma^2\right)\right)} \\
<\ & 0
\end{aligned}
$$

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