Debt Overhang and Asymmetric Information in the Secondary Market *

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Abstract

We study an informational role of debt overhang when secondary asset markets suffer from information asymmetry. Because of debt overhang, the equityholders of firms with low asset qualities forgo profitable investment. This state-dependent investment decision provides partial information about the firms’ asset qualities to ill-informed asset buyers. Put differently, the agency friction in the primary debt market can alleviate the information friction in the secondary market, increasing the recovery value of assets in default. This informational role of debt overhang is large in bad times. Therefore, policies aimed at stimulating investment during recessions may actually lower the firm value. We provide empirical evidence consistent with our model predictions.

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1 Introduction

The government has used various policies to increase aggregate investment, especially during economic recessions. One of the main justifications for those policies is the presence of debt overhang. As first argued by Myers (1977), when a firm has an excessive amount of debt, the equityholders tend to forgo new investment projects even with positive net present value (NPV). This is because existing debtholders have priority over the payoffs from the new investment, while equityholders bear the investment cost. Accordingly, we may expect that the government can improve the firm value or even welfare, by reducing the investment cost and restoring firms’ incentive to undertake profitable investment.

In this paper, we challenge this common wisdom that government policies that seek to foster investment in positive-NPV projects necessarily increase the firm value. To this aim, we explore the debt overhang problem in the presence of asymmetric information in the secondary market for the assets of defaulted firms.

Our key observation is that a firm’s state-dependent investment decision partially informs about the firm’s asset quality to ill-informed asset buyers in the secondary market. This is because the equityholders of distressed firms forgo profitable investment due to debt overhang. Thus, if the government induces more firms to invest, the asset buyers have difficulty separating good firms from bad firms. That is, this policy may reduce the informational value contained in the firms’ investment decisions. As a result, the recovery value for defaulted firms might go down, which will lead to the lower firm value.

In general, secondary markets suffer information asymmetry between sellers and buyers.1 Specifically, in a credit market, once a firm defaults, its debtholders seize the firm’s assets and then try to liquidate the assets to potential buyers in the secondary market. But the debtholders are generally better informed about their assets than the potential buyers. This is because the debtholders monitor their firm’s performance while the firm is alive, whereas the potential buyers typically start collecting information about the firm after it defaults. Thus, especially when there is a time constraint on the auction process for asset sales, the information asymmetry will not be eliminated completely.

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Due to this asymmetric information, the secondary market often fails to facilitate efficient transfer of the assets of bankrupt firms. For instance, when Boston Generating, an energy company in the US, declared bankruptcy in 2010, the firm’s existing creditors rejected to sell the firm to the highest bidder in the first round of the auction.\(^2\) We may guess that the creditors did so because they believed even the highest bid was lower than the firm’s intrinsic value estimated by themselves. After that disagreement, the creditors went through a long legal procedure and were able to liquidate only some portion of the firm in the end.

Interestingly, the presence of debt overhang can mitigate this inefficiency. Put differently, the agency friction in the primary debt market can alleviate the information friction in the secondary market. To see why, imagine some indebted firm that has an asset or a plant, which will produce outputs at operating costs. Also, suppose that the quality of the asset, which determines the output size, fluctuates over time. Then, the firm’s equityholders will operate the asset only when the asset quality is high enough so that the outputs from the asset justify the debt payments as well as the operating costs.

As a result, potential asset buyers, who do not directly observe the firm’s asset quality, can partially infer whether the asset quality is high or low by observing the firm’s investment decision. More specifically, suppose that two firms with different asset qualities went bankrupt simultaneously for some reason. Then, if one firm had operated the asset prior to the default and the other firm had not, then the potential buyers can easily know that the former firm’s asset quality is higher. In this way, indebted firms’ state-dependent investment decisions play an informational role.

As such, some policymakers’ attempts to promote investment may make the individual firms’ investment decision less informative. That is, if the firms with even low asset qualities can easily invest, distinguish failed firms’ asset qualities based on their past investment decisions will be more difficult. Therefore, the recovery value of failed assets will be lower, which will cause negative feedback effects on the primary debt market.

Nonetheless, if the investment cost is so large that no firms choose to invest, the potential buyers cannot extract any information from the investment decisions, either. Hence, under this condition, the government can actually improve the underinvestment problem without

\(^2\) More information is available in the legal filings under Case Number 10-14419 (SCC).
worsening the informational value of investment.

To justify all the above arguments, we develop a credit-risk model with ex-post heterogeneous firms under the framework of Leland (1994). Each firm in the model makes an investment decision in the manner of Diamond and He (2014). We make the following two assumptions as the main departure from these papers. First, as expected, potential asset buyers in the secondary market cannot observe any individual firm’s asset quality, while equityholders and debtholders can do so. Second, every firm is exposed to a large profitability shock such as a big drop in the firm’s output price.

The second assumption is particularly important because, in the absence of the large shock, every firm defaults only when its asset quality hits a single endogenously determined threshold. Thus, the potential buyers can perfectly infer any defaulted firm’s asset quality. However, in our model, the potential buyers cannot do so because firms with different asset qualities may default simultaneously right after being hit by the large shock.

Given this information asymmetry, we model asset liquidation mechanism in the secondary market, using a signaling game. Specifically, we assume that debtholders are less skilled in managing an asset than potential buyers. This assumption makes sense because debtholders typically do not participate in making any crucial business decisions for their firms. However, the debtholders cannot fully utilize the gains from trade with the potential buyers because of the information asymmetry. To overcome this obstacle, the debtholders choose to retain some fraction of their asset to reveal the asset’s true quality as in Leland and Pyle (1977). This partial asset retention is certainly costly, but the debtholders are willing to do so to differentiate themselves from other debtholders.

Using this framework, we can rigorously study the above-mentioned informational role of debt overhang. That is, we can analyze how the firms’ investment decisions affect the recovery value of failed assets by examining the asset retention ratios chosen by the debtholders. Importantly, our model indeed shows that reducing the investment cost can hurt both debt and firm values by forcing some debtholders to retain a larger fraction of their assets in bankruptcy.

In fact, cutting the investment cost makes the credit market worse, especially when the large profitability shock is expected to arrive with a high probability. Put differently, the
adverse outcome of that policy is more likely to occur during recessions. Understanding this result is straightforward because if the profitability shock rarely occurs, the debtholders are less concerned about the information asymmetry in the secondary market. Similarly, the firms with low asset qualities are more sensitive to this informational channel than the firms with high asset qualities. This is because only the former firms will choose to default after being hit by the profitability shock.

We now provide two sets of reduced-form evidence suggesting the empirical relevance of our mechanism. First, we document that since 1990, the recovery rate in bankrupt firms in the economy is negatively correlated with the aggregate amount of investment in the prior year. This result is consistent with the prediction from our model where more firms investing deteriorates the informational value of investment on debt value. Second, we study the average net impact of the three different roles of investment on debt value by studying the decrease in investment costs due to investment tax credits in 2001 and 2008. We empirically document a slight positive impact on lower investment costs on debt value. The low statistical significance may be due to counteracting economic channels according to our model as well as low statistical power. However, a more interesting cross sectional impact of the decreases in investment cost is that firms with less tangible assets, interpreted as firms suffering more information asymmetry, saw decreases in the value of debt relative to those with more tangible assets.

Both of our theoretical and empirical results are directly related to policies that are designed to boost investment. In a recession, governments around the world seek to stimulate the economy. One such channel is to temporarily decrease investment costs, for example by increasing investment tax credits or allowing firms to immediately depreciate new tangible investment costs on their financial statements. Our results suggest that while a decrease in investment cost can be expected to increase the amount of investments, it may reduce firm value by reducing the information content of investments should a firm default. A further implication is that small firms with more information asymmetry may see a decrease in firm value, causing the fiscal policy of lower investment cost to exactly reduce the value of small

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3The recovery rate is calculated as the sales price of traded debt or a net present value calculation of assets that are redeployed to difference uses.
firms. The state of the economy, investment opportunities, and condition of the secondary markets should all be taken into consideration when deciding on such policies.

Our paper contributes to the literature as follows. After Myers (1977) addressed the debt overhang problem, many researchers have shown that debt overhang is a first-order friction that hinders both economic growth and economic recovery from recessions. For instance, Mello and Parsons (1992), Hennessy (2004), Hennessy et al. (2007), Moyen (2007), and Chen and Manso (2017) find that the agency cost of debt overhang is quantitatively significant. Reinhart et al. (2012) also considers the impact of debt overhang on the sovereign debt market.

Given the importance of this issue, researchers have studied how to resolve the debt overhang problem. Stulz and Johnson (1985) show that secured debt can alleviate the underinvestment problem. Hackbarth and Mauer (2012) find that jointly optimal capital and debt priority structure can mitigate the conflicts between equityholders and debtholders. Sundaresan et al. (2015) also analyze how debt priority structure affects the incentives for investment. Gertner and Scharfstein (1991), Titman and Tsyplakov (2007), and Diamond and He (2014) find some negative effects of short-term debt on investment. He (2011) analyzes the relationship between moral hazard and debt overhang. But none of these papers consider the interaction between debt overhang and asymmetric information in the secondary market as in our paper. Also, our paper is the first paper that argues that government policies aimed at increasing investment may adversely affect the debt market.

The paper is organized as follows. Section 2 develops the model. Section 3 solves the model. Section 4 discusses the model implications. Section 5 provides empirical evidence for our model predictions. Section 6 discusses further related issues. Section 7 concludes. Appendix includes technical arguments.

2 The Model

We build a model with ex-post heterogeneous firms by extending the framework of Leland (1994). Each firm in our model manages an asset-in-place, keeps servicing a fixed amount of debt, and has an additional investment project. The main departure from Leland (1994)
is that the secondary market for the assets of defaulted firms suffers from asymmetric information. Specifically, we make the following two assumptions. First, potential buyers in the secondary market do not observe the firm’s asset quality. Second, every firm is exposed to a large profitability shock such as a big drop in the firm’s output price.

Regarding the second assumption, in the absence of the large shock, even the uninformed potential buyers can rationally infer any failed firm’s asset quality, because every firm defaults only when its asset quality hits an endogenously determined threshold. However, in our model, the potential buyers cannot do so because firms with different asset qualities may default simultaneously after being by the large shock.

Let \( i \in [0, \infty) \) denote each individual firm in the economy.\(^4\) Time flows continuously. There are three types of agents: equityholders, debtholders, and potential asset buyers. All the agents are risk neutral and have a discount rate \( r \).

### 2.1 Firm Asset and Investment

Each firm \( i \) has an asset-in-place that produces after-tax cash flows \( A^i_t x^i_t dt \) over each time interval \([t, t + dt)\), where both \( A^i_t \) and \( x^i_t \) vary over time. Here, \( A^i_t \) denotes firm-specific profitability at time \( t \) such as the firm’s output price, while \( x^i_t \) denotes firm-specific asset quality at time \( t \).

The profitability \( A^i_t \) is publicly observable, but the asset quality \( x^i_t \) is privately observable only to the firm’s equityholders and debtholders. The first assumption makes sense because every market participant can easily observe the output price of any firm. Meanwhile, estimating the asset quality requires much more effort and time, especially for those potential buyers who do not closely monitor individual firms before they default. Thus, it is reasonable to assume that equityholders and debtholders have better information about their firms than potential asset buyers.

Of course, in practice, equityholders and debtholders may not have perfect knowledge of their firms. Moreover, there might be information asymmetry even between equityholders

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\(^4\)Rigorously, we can assume that there is only one firm in the economy, because we will focus on a separating equilibrium later. In this sense, any cross-sectional distribution of the firms does not matter in our model. By the same reason, we can interpret any idiosyncratic shocks as aggregate shocks in this model.
and debtholders. However, the main friction in our model is that potential buyers are less informed than equityholders and debtholders. To model this friction in a tractable way, we assume that both equityholders and debtholders know their firm’s asset quality perfectly.

Moreover, we implicitly assume that potential asset buyers cannot observe any firm’s equity value or debt value. We make this assumption to rule out a case in which potential buyers can back out the firm’s asset quality from its equity or debt prices. This assumption does not lose much generality because the equity and debt prices of even public firms tend to be noisy enough so that potential buyers still have difficulty extracting precise information about those firms, as in Grossman and Stiglitz (1980).

Let us now describe how $A_t^i$ and $x_t^i$ evolve over time. First, each firm $i$’s profitability switches between a good state and a bad state according to an idiosyncratic Poisson shock. Specifically, the good (resp. bad) state changes to the bad (resp. good) state with Poisson intensity $\phi_g$ (resp. $\phi_b$). In the good state, $A_t^i$ is normalized to 1, while in the bad state $A_t^i$ drops to $\gamma < 1$.

The firm’s asset quality evolves according to a geometric Brownian motion

$$\frac{dx_t^i}{x_t^i} = \mu_t^i dt + \sigma dZ_t^i,$$

where $\mu_t^i$ denotes the asset’s time-varying growth rate, $\sigma > 0$ denotes the asset volatility, and $Z_t^i$ is a standard idiosyncratic Brownian motion.

The growth rate $\mu_t^i$ is determined by the firm’s investment decision, which is controlled by the firm’s equityholders. We hereafter consider that each firm has a single representative equityholder. Specifically, as in Diamond and He (2014), the firm has an additional project such as a machine or plant. Also, at each point in time, the firm chooses whether to invest in this project. If the firm does not invest, the growth rate remains at $\mu_L$. But if the firm invests over a time interval $[t, t + dt)$, the growth rate increases to $\mu_H$ over that interval. Moreover, investing in this additional project incurs a flow cost $\kappa x_t^i dt$ to the firm’s equityholder. For simplicity, we further assume that this investment project is available to the firm only in the good state. From now, we will often omit the indexes $i$ and $t$ if there is no confusion.

In this circumstance, if the firm always invests in the additional project, the unlevered
firm value at time $t$ in the good state, $F^g(x_t)$, and that in the bad state, $F^b(x_t)$, satisfy

\[
\begin{align*}
F^g(x_t) &= E \left[ \int_t^\infty \phi_g e^{-(\phi_g(s-t))} \left( e^{-r(s-t)} F^b(x_s) + \int_t^s e^{-r(u-t)} \gamma x_u du \right) ds \right] \\
F^b(x_t) &= E \left[ \int_t^\infty \phi_b e^{-(\phi_b(s-t))} \left( e^{-r(s-t)} F^g(x_s) + \int_t^s e^{-r(u-t)} \gamma x_u du \right) ds \right],
\end{align*}
\]

which can be computed as

\[
\begin{align*}
F^g(x) &= \frac{(1-\kappa)(r+\phi_b-\mu_L)+\phi_g \gamma}{(r-\mu_H)(r+\phi_g+\phi_b-\mu_L)+\phi_g (\mu_H-\mu_L)} x \\
F^b(x) &= \frac{\gamma(r+\phi_g-\mu_H)+\phi_b(1-\kappa)}{(r-\mu_H)(r+\phi_g+\phi_b-\mu_L)+\phi_g (\mu_H-\mu_L)} x. 
\end{align*}
\]

(1)

Here, we assume that the coefficients for both $F^g(x)$ and $F^b(x)$ are positive and finite.

Moreover, to exclude uninteresting cases, we assume that the additional project always has positive net present value (NPV) by imposing the following parameter restrictions:

\[
\kappa < \frac{[(1-\kappa)(r+\phi_b-\mu_L)+\phi_g \gamma](\mu_H-\mu_L)}{(r-\mu_H)(r+\phi_g+\phi_b-\mu_L)+\phi_g (\mu_H-\mu_L)}. 
\]

(2)

That is, if the firm forgoes the investment once and only once over the time interval $[t, t+dt)$, then the firm can save the investment cost $\kappa x_t dt$, but the unlevered firm value in the good state is reduced by $F^g(x_t)(\mu_H-\mu_L)dt$. Thus, condition (2) ensures that every unlevered firm has an incentive to make the investment all the time.

However, because of the classical debt overhang problem, the firm may not always invest in the additional project, even though the project has positive NPV. This is because the equityholder pays the investment cost, but the debtholders have priority over the payoffs from the investment. In particular, when the asset quality is sufficiently low, the firm is expected to go bankrupt shortly and thus, the investment becomes less attractive to the equityholder. So, we can reasonably postulate that the equityholder uses a threshold-type investment strategy. That is, we aim to find some threshold $x_I$ such that the equityholder optimally chooses to invest at time $t$ if and only if the asset quality $x_t$ is larger than $x_I$. In this regard, we refer to the indicator function $1_{x_t \geq x_I}$ as the investment strategy of the firm.

Importantly, we assume that each firm’s investment decision $1_{x_t \geq x_I}$ is publicly observable, although the asset quality $x_t$ is not. In other words, even the potential buyers can see whether the firm operates the additional project at any given time $t$. As a result, the
potential buyers can rationally infer at least whether \( x_t \) is larger than \( x_I \). But we assume
that the potential buyers cannot observe the investment cost incurred to the firm, that is,
\( \kappa x_t 1_{x_t \geq x_I} \). Accordingly, the potential buyers still cannot back out the asset quality \( x_t \). The
above two assumptions are reasonable enough because most of investors can fairly easily
check whether a firm operates a new plant or not, but estimating the investment cost itself
is still challenging.

### 2.2 Firm Liability and Default

As in the standard Leland setup, the firm issues debt to earn a tax benefit. But, for simplicity,
we assume that the firm issues only perpetuity debt that pays a fixed amount of coupon \( c \).
Here, we again consider that the firm has a single representative debtholder. Also, the firm’s
equityholder has a deep pocket, but can default on the debt payment at any point in time.

Suppose \( \pi \in [0,1] \) is the corporate tax rate. Then, at each time \( t \), the firm with asset
quality \( x_t \) generates an after-tax net cash flow equal to

\[
\begin{align*}
&x_t - (1 - \pi)c - \kappa x_t 1_{x_t \geq x_I} & \text{in the good state} \\
&\gamma x_t - (1 - \pi)c & \text{in the bad state}.
\end{align*}
\]

The first term in each line denotes the after-tax cash flow produced from the asset, depending
on the profitability state. The second common term is the effective coupon payment to
the debtholder, where \( \pi c \) indicates the tax benefit. The third term in the first line is the
investment cost paid by the equityholder. Such a term does not appear in the second line
because the investment project is available to the firm only in the good state.

In case where the net cash flow is positive, the firm pays out all the positive profits as
dividends to the equityholder. Yet, when the net cash flow is negative, the equityholder
chooses whether to inject more money into the firm to absorb the operating losses or to
default on the debt payment.

In default, the entire control right over the firm asset is transferred to the debtholder.
As a result, the equity claim becomes worthless in bankruptcy. Reflecting this fact, the
equityholder optimally chooses to default when the equity value \( \text{or the continuation value} \)
of equity) hits zero.

In this circumstance, we can naturally postulate that the equityholder uses a threshold-type default strategy. That is, we will find two thresholds $x_D^g$ and $x_D^b$ such that the equityholder optimally defaults in the good (resp. bad) state whenever $x_t$ hits $x_D^g$ (resp. $x_D^b$). But there is another type of the default event: If the negative profitability shock hits the firm with asset quality $x \in [x_D^g, x_D^b]$, this firm immediately defaults because $x$ is less than $x_D^b$.

For clarification, in equilibrium, $x_I$ lies above $x_D^g$, meaning that the equityholder does not always invest in the additional project. But $x_I$ can be either larger or smaller than $x_D^b$. We determine all these equilibrium thresholds endogenously later.

### 2.3 Asset Liquidation

Once the firm defaults, the debtholder takes over the firm’s asset from the equityholder. Although the debtholder can manage the asset by herself afterwards, we assume that the debtholder is less skilled in operating the asset than the potential buyers in the secondary market. This assumption makes sense because debtholders generally do not have enough experience in making crucial business decisions for their firms. We also assume that the potential buyers have deep pockets as well. Therefore, the debtholder certainly wants to sell the entire asset immediately to utilize the gains from trade with potential buyers. However, what hinders the debtholder from doing so is the information asymmetry between the debtholder and potential buyers.

Specifically, we assume that every debtholder can extract only a fraction $\alpha$ of the cash flows from the asset, while every potential buyer can extract a fraction $\beta$ of the cash flows from the asset, where $0 \leq \alpha \leq \beta \leq 1$. Here, we have assumed $\beta \leq 1$ to obtain better quantitative results. What is crucial in our model is that $\alpha \leq \beta$. Moreover, simply following Diamond and He (2014), we assume that the additional project is no longer available after default.

Accordingly, the debtholder values the asset with quality $x$ as

$$\alpha S_{g,x} := \frac{\alpha(r + \gamma \phi_g + \phi_b - \mu_L)x}{r + \phi_g + \phi_b - \mu_L} \quad \text{and} \quad \alpha S_{b,x} := \frac{\alpha(\gamma(r + \phi_g - \mu_L) + \phi_b)x}{r + \phi_g + \phi_b - \mu_L}$$

(3)
in the good state and the bad state, respectively. These formulas can be similarly obtained as done in (1). Meanwhile, the potential buyers value the same asset as \( \beta S^g x \) and \( \beta S^b x \) in the good state and the bad state, respectively.

Now, let us consider three different default scenarios we have described above. First, each firm in the good state defaults if its asset quality hits \( x_D^g \). In this case, the potential buyers can rationally infer the firm’s true asset quality. Thus, the debtholder can sell the failed asset at the fair price \( \beta S^g x_D^g \). Second, each firm in the bad state defaults if its asset quality hits \( x_D^b \). The debtholder of this firm can also sell the failed asset at the fair price \( \beta S^b x_D^b \). But, in the third case in which a firm with asset quality \( x_I \in [x_D^g, x_D^b] \) defaults because of the profitability shock, the debtholder is not able to sell the failed asset at the fair price \( \beta S^b x_I \). This is because the potential buyers cannot precisely infer the firm’s asset quality.

To overcome this obstacle, the debtholder chooses to retain some fraction of her asset to reveal the asset’s true quality, as in Leland and Pyle (1977).\(^5\) This partial asset retention is costly to the debtholder because the potential buyers are better asset users. Nonetheless, the debtholder is willing to do so to distinguish herself from other debtholders. Throughout the paper, for tractability, we focus on a separating equilibrium in which each debtholder perfectly reveals her asset quality through partial asset retention.

Most importantly, the debtholder does not need to differentiate herself from the debtholders of all other firms hit by the negative profitability shock at the same time. The reason is that, as emphasized above, the potential buyers can infer at least whether the firm’s asset quality is larger than \( x_I \) from the firm’s investment decision made right before default.

As such, we first consider a case where \( x_D^g < x_I < x_D^b \). Also, from now, we call the debtholder, who holds an asset with quality \( x \), the debtholder of type \( x \). Then, the above argument says that, for all \( x \) and \( y \) such that \( x \in [x_D^g, x_I] \) and \( y \in [x_I, x_D^b] \), the debtholder of type \( x \) does not need to differentiate herself from the debtholder of type \( y \), and vice versa. Meanwhile, the debtholders, whose types belong to each of the subregions, \( [x_D^g, x_I] \) and \( [x_I, x_D^b] \), still attempt to separate themselves against each other.

In the other case where \( x_D^g < x_D^b < x_I \), the debtholders of the types belonging to \( [x_D^g, x_D^b] \)

\(^5\)In this paper, we do not assume that the debtholder can postpone liquidating her asset as in Daley and Green (2012). Embedding this dynamic signaling game into a credit-risk model seems to be an interesting topic for future research.
play a signaling game all together. This is because when $x^b_D < x_I$, the potential buyers cannot split defaulted firms into two groups based on their investment decisions made right before default.

3 Model Solutions

In this section, we first analyze the signaling game in the secondary market. We then pin down optimal default and investment strategies together with the equity and debt values.

3.1 Separating Equilibrium in the Secondary Market

This section characterizes a partial equilibrium in the secondary market for any given thresholds $\{x^g_D, x^b_D, x_I\}$. First, recall that each firm in the good (resp. bad) state defaults if its asset quality hits the default boundary $x^g_D$ (resp. $x^b_D$). In this case, the debtholder recovers $\beta S^g x^g_D$ (resp. $\beta S^b x^b_D$) by fully liquidating the failed asset at the fair price.

We thus now focus on the other default scenario caused by the negative profitability shock. To begin with, suppose that the debtholder of type $x \in [x^g_D, x^b_D]$ retains a fraction $f(x)$ of her asset to signal the asset quality. In equilibrium, the potential buyers correctly believe that the debtholder of type $x$ indeed retains the fraction $f(x)$ of her asset. We then consider the following two cases separately: (i) $x^g_D < x_I < x^b_D$ and (ii) $x^g_D < x^b_D < x_I$.

Case 1: In the first case, we need to split the debtholders into two subgroups as discussed above; that is, the debtholders, whose types belong to $[x^g_D, x_I)$, and the debtholders, whose types belong to $[x_I, x^b_D]$. We first consider the debtholders in the subgroup $[x^g_D, x_I)$.

Suppose the debtholder of type $x \in [x^g_D, x_I)$ mimics another debtholder of type $y \in [x^g_D, x_I)$. Then, the former debtholder is expected to earn $f(y)\alpha S^b x + (1 - f(y))\beta S^b y$ by managing the fraction $f(y)$ of her asset and liquidating the remaining fraction of the asset. Also, regarding off-equilibrium beliefs, we assume that the potential buyers assign the worst type $x^g_D$ to a debtholder who retains a fraction $\xi$ of her asset, where there is no $y \in [x^g_D, x_I)$ such that $\xi = f(y)$.
Thus, the debtholder of type \( x \in [x_D^g, x_I] \) solves the following maximization problem:

\[
\max_{y \in [x_D^g, x_I]} \alpha S^b x + (1 - f(y)) \beta S^b y.
\] (4)

Here, we can compute the optimal asset retention ratio \( f(x) \) explicitly. First, note that the debtholder of the worst type \( x_D^g \) can obviously sell the entire fraction of her asset due to the off-equilibrium beliefs. Thus, we have \( f(x_D^g) = 0 \). Second, in separating equilibrium, each debtholder maximizes her expected payoffs by truthfully revealing her asset type. Thus, \( f(x) \) must satisfy the following first-order condition (FOC) for the problem in (4):

\[
(\alpha - \beta) x f'(x) + \beta (1 - f(x)) = 0, \quad \forall x \in [x_D^g, x_I].
\] (5)

The solution to this equation subject to the boundary condition \( f(x_D^g) = 0 \) is given by

\[
f(x) = 1 - \left( \frac{x_D^g}{x} \right)^{\frac{\beta}{\beta - \alpha}}, \quad \forall x \in [x_D^g, x_I].
\] (6)

In Appendix 8.1, we show that this \( f(x) \), derived from the FOC, indeed solves the problem in (4). As expected, the formula implies that a debtholder with better quality asset retains more of her asset to differentiate herself from other debtholders with lower quality assets.

The debtholders in the other subgroup \([x_I, x_D^b]\) behave similarly. That is, by the same argument described above, the retention ratio \( f(x) \) for \( x \in [x_I, x_D^b] \) must be given by

\[
f(x) = 1 - \left( \frac{x_I}{x} \right)^{\frac{\beta}{\beta - \alpha}}.
\] (7)

That is, we have only replaced the term \( x_D^g \) in the denominator in (6) by \( x_I \).

The left panel in Figure 1 plots the retention ratio \( f(x) \) for all \( x \in [x_D^g, x_D^b] \). Note that \( f(x) \) jumps down at \( x_I \), although \( f(x) \) increases in \( x \) over each of the subregions, \([x_D^g, x_I]\) and \([x_I, x_D^b]\). This property will play a crucial role when we examine the informational role of investment.

**Case 2:** Now, consider the second case where \( x_D^g < x_D^b < x_I \). In this case, we need not separate the debtholders of defaulted firms into two groups. Thus, the same argument above
Figure 1: This figure plots the retention ratio $f(x)$. The left panel corresponds to the case where $x_D^g < x_I < x_D^b$. The right panel corresponds to the case where $x_D^g < x_D^b < x_I$.

simply implies that

$$f(x) = 1 - \left(\frac{x_D^g}{x}\right)^{\frac{\alpha}{\sigma}} , \quad \forall x \in [x_D^g, x_D^b].$$

(8)

The right panel in Figure 1 plots this retention ratio. In this case, $f(x)$ keeps increasing over the whole interval $[x_D^g, x_D^b]$.

To proceed further, let us define

$$R(x) = f(x)\alpha S^bx + (1 - f(x))\beta S^bx$$

(9)

as the recovery value for the type-$x$ debtholder’s asset. Obviously, the recovery value is higher when the debtholder retains less of her asset. Also, although the retention ratio $f(x)$ may not be monotone in $x$, depending on the position of $x_I$, the recovery value $R(x)$ always increases in $x$.

3.2 Equity Value

In this section, we compute the equity value by determining optimal default and investment strategies. Let $E^g(x)$ and $E^b(x)$ denote the arbitrage-free equity value in the good state and the bad state, respectively. Then, using a standard continuous-time technique, we derive that $E^g(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$rE^g(x) = x - (1 - \pi)c - \kappa 1_{x \geq x_I}x + \phi_g(E^b(x) - E^g(x)) + (\mu_L + \delta 1_{x \geq x_I})xE^g(x) + \frac{\sigma^2}{2}x^2E_{xx}^g(x),$$
subject to $E^g(x_D^g) = E^b_D(x_D^b) = 0$ and $E^g(x_I) = \frac{\sigma}{2}$, where $\delta = \mu_H - \mu_L$. The term in the left-hand side is the required return for the equityholder. In the right-hand side, the sum of the first three terms denotes the after-tax net cash flow. The fourth term is the expected net gain that is obtained when the negative profitability shock hits the firm. The last two terms explain how the fluctuating asset quality affects the equity value. In particular, the growth rate $\mu_L + \delta 1_{x \geq x_I}$ depends on the investment strategy.

Moreover, the first two boundary conditions, $E^g(x_D^g) = E^b_D(x_D^b) = 0$, correspond to the standard value-matching and smooth-pasting conditions, respectively, for an optimal default strategy. The other condition $E^g(x_I) = \frac{\sigma}{\delta}$ says that the equityholder is indifferent between investing and not investing when the firm’s asset quality equals $x_I$.

On the other hand, $E^b(x)$ satisfies that, for all $x \geq x_D^b$,

$$ rE^b(x) = \gamma x - (1 - \pi)c + \phi_b(E^g(x) - E^b(x)) + \mu_L x E^b(x) + \frac{\sigma^2}{2} x^2 E^b_{xx}(x), $$

subject to $E^b(x_D^b) = E^b_D(x_D^b) = 0$. We can understand this equation similarly as in the above. In addition, for each $x \in [x_D^g, x_D^b]$, we have $E^b(x) = 0$ because the firm with asset quality $x \in [x_D^g, x_D^b]$ immediately defaults after being hit by the negative profitability shock.

In Appendix 8.2, we solve for equilibrium thresholds $\{x_D^g, x_D^b, x_I\}$ in almost closed form. That is, for any conjectured thresholds $\{x_D^g, x_D^b, x_I\}$, we compute both $E^g(x)$ and $E^b(x)$ in closed form. We then calculate equilibrium thresholds $\{x_D^g, x_D^b, x_I\}$ numerically. For clarification, we can obtain equilibrium thresholds $\{x_D^g, x_D^b, x_I\}$ without solving for the debt value because every firm issues perpetuity debt only. The effect of maturity is not a primary interest of this paper.

### 3.3 Debt Value

Now, let $D^g(x)$ and $D^b(x)$ denote the arbitrage-free debt value in the good state and the bad state, respectively. Then, again, using a standard continuous-time technique, we derive that $D^g(x)$ satisfies

$$ rD^g(x) = c + \phi_g(D^b(x) - D^g(x)) + (\mu_L + \delta 1_{x \geq x_I}) x D^g(x) + \frac{\sigma^2}{2} x^2 D^g_{xx}(x), $$

(10)
subject to $D^g(x^g_D) = \beta S^g x^g_D$. The term in the left-hand side is the required return for the debtholder. In the right-hand side, the first term is the coupon payment. The second term is the expected net gain that is obtained when the negative profitability shock hits the firm. The other terms denote the change in the debt value due to the fluctuating asset quality. The boundary condition $D^g(x^g_D) = \beta S^g x^g_D$ corresponds to the recovery value for the firm whose asset quality hits $x^g_D$ in the good state.

Similarly, $D^b(x)$ satisfies that, for all $x \geq x^b_D$,

$$rD^b(x) = c + \phi_b(D^g(x) - D^b(x)) + \mu_L xD^b_x(x) + \frac{\sigma^2}{2}x^2D^b_{xx}(x),$$

subject to $D^b(x^b_D) = \beta S^b x^b_D$. Here, the boundary condition denotes the recovery value for the firm whose asset quality hits $x^b_D$ in the bad state. In addition, for each $x \in [x^g_D, x^b_D)$, the debt value $D^b(x)$ is equal to the recovery value $R(x)$ that is given by (9). Here, we do not set $D^b_D(x) = R(x)$ for $x = x^b_D$ because the point $x = x^b_D$ is negligible when calculating the HJB equation (10).

4 Model Implications

We now present our model implications. First, we analyze the informational role of debt overhang. We then study the policy implications of this informational channel. Specifically, we show that both debt and firm values may decrease when the investment cost is reduced. We also discuss other comparative statics results. From now, we will broadly interpret $1 - f(\cdot)$ and the recovery value $R(\cdot)$ as the secondary market liquidity.

4.1 Informational Role of Debt Overhang

In this section, we examine how the change in the investment threshold $x_I$ affects the asset retention ratio $f(x)$, by fixing the default thresholds $x^g_D$ and $x^b_D$. In fact, when equityholders have less incentive to invest, $x_I$ will become higher. In other words, a higher $x_I$ corresponds to a more severe debt overhang problem. In this sense, by fixing the default thresholds and changing $x_I$ only, we can isolate the informational effect of debt overhang on the secondary
market liquidity.

Let us first look at the left panel of Figure 2, which plots the asset retention ratio \( f(x; x_I) \) under three different thresholds \( x_I \in \{x_1^I, x_2^I, x_3^I\} \) such that \( x_1^I < x_2^I < x_3^I \). Specifically, the solid, dotted, and dashed lines plot \( f(x; x_I) \) for \( x_I \in \{x_1^I, x_2^I, x_3^I\} \), respectively. We also pick a particular point \( \bar{x} \) satisfying \( x_2^I < \bar{x} < x_3^I \). We then plot \( f(x; x_I) \) as a function of \( x_I \) in the right panel of Figure 2.

First, when the investment threshold increases from \( x_1^I \) to \( x_2^I \), the retention ratio \( f(x; x_2^I) \) becomes lower than \( f(x; x_1^I) \). This is because when \( x_I \) increases from \( x_1^I \) to \( x_2^I \), the debtholder of type \( \bar{x} \) needs to differentiate herself from the debtholders whose types are above \( x_2^I \) rather than above \( x_1^I \). As a result, the recovery value \( R(\bar{x}; x_I) \) increases as \( x_I \) increases from \( x_1^I \) to \( x_2^I \). This result is somewhat surprising because the increment in \( x_I \) corresponds to more severe debt overhang as mentioned above.

Applying the same argument, we can show that \( f(x; x_I) \) keeps decreasing in \( x_I \) until \( x_I \) reaches \( \bar{x} \). In particular, when \( x_I \) is at \( \bar{x} \), the retention ratio \( f(x; \bar{x}) \) simply becomes 0. The right panel of Figure 2 illustrates these results.

Second, if the investment threshold further increases to \( x_3^I \), which is larger than \( \bar{x} \), the retention ratio \( f(x; x_3^I) \) becomes higher than \( f(x; x_1^I) \). This is because when \( x_I \) is equal to \( x_3^I \), any individual firm with asset quality \( \bar{x} \) does not invest and thus, the debtholder of type \( \bar{x} \) needs to differentiate herself from the debtholders whose types are above \( x_3^I \) rather than above \( x_1^I \). As a result, the recovery value \( R(\bar{x}; x_I) \) decreases when \( x_I \) increases from \( x_1^I \) to \( x_3^I \).
The right panel of Figure 2 more sharply illustrates this result. That is, the retention ratio \( f(x; x_I) \) jumps up as soon as \( x_I \) exceeds \( \bar{x} \), but then remains constant even if \( x_I \) increases further. Here, \( f(x; x_I) \) stays constant for all \( x_I > \bar{x} \) because the default threshold \( x_D^b \) is fixed.

In sum, the retention ratio \( f(x; x_I) \) decreases as \( x_I \) increases from \( x_D^b \) to \( \bar{x} \), but then \( f(x; x_I) \) jumps up as soon as \( x_I \) exceeds \( \bar{x} \). This non-monotone response of \( f(x; x_I) \) has significant implications for government policies. That is, even if the government can effectively increase the equityholders’ incentive for investment, some debtholders will be worse off because if more firms invest, the potential buyers have more difficulty separating good firms from bad firms. Specifically, for instance, if \( x_I \) decreases from \( x_I^3 \) to \( x_I^1 \), the debtholders of type \( x \in [x_I^1, x_I^3] \) are better off, but the debtholders of type \( x \in [x_I^3, x_D^b] \) are worse off. This is because, as a response to the change in \( x_I \), the firms with asset quality \( x \in [x_I^1, x_I^3] \) now choose to invest.

Nonetheless, in case where \( x_I \) is even larger than \( x_D^b \), the government can benefit all debtholders by promoting more investment. The reason is that when \( x_I > x_D^b \), the potential buyers learn nothing useful from the firms’ investment decisions. Thus, as long as \( x_I \) falls below \( x_D^b \), all debtholders will be weakly better off.

In what follows, we will study the policy implications of this key mechanism by changing some of the model parameter values of our interest.

### 4.2 Investment Cost

We first examine how the investment cost affects the firms. At first glance, reducing the investment cost seems to obviously mitigate the debt overhang problem. But we will show that such a policy may benefit only equityholders, but not debtholders.

The change in the investment cost basically has three effects. First, the change in \( \kappa \) directly affects the net cash flows to the firms. Second, the change in \( \kappa \) alters the equityholders’ incentive to default. Third, the change in \( \kappa \) influences the equityholders’ incentive to invest. We call these three channels the NPV channel, default channel, and information channel, respectively.

Specifically, suppose \( \kappa \) is decreased. Then, first, the NPV-channel certainly increases the equity value because the equityholders are required to pay less for investment when \( \kappa \)
decreases. However, the debt value is not directly affected by this channel because the debtholders do not bear the investment cost.

Second, the default channel also pushes up the equity value because the equityholders choose to default later when the investment cost is lowered. But this channel can actually make the debt value either larger or smaller. On the one hand, the debt value can increase because the default risk is reduced when the default thresholds are lowered. On the other hand, when the default threshold in the good state, $x^g_D$, is lowered, the recovery value for the firms with asset quality $x \in [x^g_D, x_I)$ will decrease. This is because the asymmetric information problem for those firms deteriorates when $x^g_D$ decreases. Thus, the debt value might be pushed down. Nonetheless, under most of parameter values, this negative effect tends to be dominated by the above positive effect of the delayed default.

Third, the information channel also increases the equity value because the asset growth rate is pushed up earlier when the equityholders invest earlier. But this channel may increase or decrease the debt value as intensively discussed above.

We now ask when does the information channel matter? More specifically, when does the negative effect of the information channel become strong enough so that the debt and firm values can go down when $\kappa$ is lowered? To answer this question, first, note that the debtholders of the firms with lower asset qualities are more sensitive to this channel, because those firms are more exposed to the profitability-shock driven default risk. Second, the adverse effect of the information channel is more problematic when the arrival intensity of the negative profitability shock, $\phi_g$, is large. This is because when $\phi_g$ is small, the debtholders in the good state are not concerned about the asset recovery value that much.

In sum, while reducing the investment cost always increases the equity value, that policy may not raise the debt value mainly because of the information channel. Moreover, the magnitude of the informational effect will heavily depend on the arrival intensity of the negative profitability shock. In this regard, we will look at the effect of the change in $\kappa$ under two different sets of the parameter values. In the first set, we choose high $\phi_g$, which we interpret as the distressed times. In the second set, we choose low $\phi_g$, which we interpret as the normal times. We will show that the effect of reducing the investment cost will be quite different under these two conditions.
Table 1: Baseline Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Risk-free rate</td>
<td>$r = 7%$</td>
</tr>
<tr>
<td>Corporate tax rate</td>
<td>$\pi = 27%$</td>
</tr>
<tr>
<td>Productivity of the debtholders</td>
<td>$\alpha = 40%$</td>
</tr>
<tr>
<td>Productivity of the asset buyers</td>
<td>$\beta = 80%$</td>
</tr>
<tr>
<td>Growth rate without investment</td>
<td>$\mu_L = 0%$</td>
</tr>
<tr>
<td>Growth rate under investment</td>
<td>$\mu_H = 6%$</td>
</tr>
<tr>
<td>Asset volatility</td>
<td>$\sigma = 15%$</td>
</tr>
<tr>
<td>Arrival intensity for the positive profitability shock</td>
<td>$\phi_b = 0.4$</td>
</tr>
<tr>
<td>Investment cost</td>
<td>$\kappa = 6%$</td>
</tr>
<tr>
<td>Coupon payment</td>
<td>$c = 9$</td>
</tr>
</tbody>
</table>

4.2.1 Distressed Times

In this section, we assume that the arrival intensity for the negative profitability shock is high. Specifically, we choose $\phi_g = 2$, meaning that the shock is expected to arrive in 6 months on average. We choose the other parameter values as in Table 1. All these parameter values are consistent with those values chosen in the literature, including Miao (2005), Hackbarth et al. (2006), Chen (2010), Diamond and He (2014), and Chen and Manso (2017). In particular, we choose $\alpha = 40\%$ and $\beta = 80\%$ because the lowest and highest recovery rates are around 40\% and 80\%, respectively, according to Chen (2010).

Figure 3 shows the effect of a change in the investment cost on the equity value, debt value, and firm value. From the upper two panels, we confirm that the equity value decreases in the investment cost in both states. To examine the effect on the debt value, we consider the following two cases: (i) when $\kappa$ is reduced from 0.06 to 0.02 and (ii) when $\kappa$ is reduced from 0.15 to 0.06.

In the first case, let us first look at the effect on the debt value in the bad state, which is shown at the middle-right panel of Figure 3. In that panel, recall that the debt value $D^b(x)$ for $x \in [x_D^g, x_D^b)$ corresponds to the recovery value $R(x)$ itself. Note that when $\kappa = 0.06$, we have $x_D^g = 6.30$, $x_I = 7.65$, and $x_D^b = 8.36$. But when $\kappa = 0.02$, we have $x_D^g = 6.24$, $x_I = 6.39$, and $x_D^b = 8.28$.

The figure shows that $D^b(x)$ increases for every $x \in [7.65, 8.28)$ as $\kappa$ decreases from 0.06 to 0.02. This is because when $\kappa$ is reduced in that way, the debtholder of type $x \in [7.65, 8.28)$
Figure 3: This figure plots the effect of a change in the investment cost $\kappa$ under the choice of $\phi_g = 2.0$. The upper, middle, lower panels plot the equity, debt, and firm values, respectively. The left and right panels correspond to the good and bad states, respectively.

needs to differentiate herself from the debtholders of the type above 6.39 rather than above 7.65. Put differently, the retention ratio for the debtholders of the type above the old investment threshold, $x_I = 7.65$, increases, because more firms choose to invest after the investment cost is cut down. Basically, this scenario corresponds to the case in which the investment threshold $x^2_I$ is lowered to $x^1_I$ in Figure 2.

Meanwhile, $D^h(x)$ decreases for every $x \in [6.39, 7.65]$ as $\kappa$ decreases, although this pattern is not that visible in the figure. The reason is that when $\kappa$ decreases from 0.06 to 0.02, the debtholder of type $x \in [6.39, 7.65]$ only needs to differentiate herself from the debtholders of the types above 6.39 rather than above 6.30. In other words, the firms with asset qualities in [6.39, 7.65] now face less severe information asymmetry because the new investment threshold
$x_I = 6.39$ lies above the old default threshold $x_D^g = 6.30$. This scenario corresponds to the case where the investment threshold $x_I^3$ is lowered to $x_I^1$ in Figure 2.

Importantly, we can find this non-monotone effect on the debt value even in the good state as shown in the middle-left panel in Figure 3. This result says that when $\phi_g$ is high, the information channel can dominate the default channel. Specifically, recall that if the investment cost is reduced, the debt value can increase because the equityholders choose to default later. But the debtholders in the good state are also concerned about the future recovery value for failed assets. When $\phi_g$ is particularly large, those debtholders care about the recovery value more seriously. However, the recovery value may rather drop if the equityholders invest earlier. Thus, the debt value in the good state can drop as well, especially when $\phi_g$ is high.

We now consider the second case in which $\kappa$ is reduced from 0.15 to 0.06. In contrast to the previous case, the debt value in this case increases for all $x$ in both good and bad states, as in the two middle panels in Figure 3. To see why, note that when the investment cost is sufficiently large, the investment threshold lies above the default threshold in the bad state. Specifically, when $\kappa = 0.15$, we have $x_D^g = 6.30$, $x_D^b = 8.54$, and $x_I = 8.88$. As a result, the firms’ investment decisions do not carry any useful information and therefore, the recovery value for failed firms will be lowest possible. Thus, if $\kappa$ is reduced to 0.06 so that $x_I$ falls below $x_D^b$, the information channel increases the debt value unambiguously. Then, the firm value must increase as well.

Figure 4 illustrates the results in the above two cases in an alternative way. We first pick a particular point $x = 8$. We then plot both $D^g(x; \kappa)$ and $D^b(x; \kappa)$ for $\kappa \in [0.01, 0.15]$ in the top two panels. In the bottom panel, we also plot the optimal thresholds $x_D^g$, $x_D^b$, and $x_I$ for $\kappa \in [0.01, 0.15]$. Specifically, the bottom panel shows that the investment threshold $x_I$ coincides with $x = 8$ when $\kappa = 0.076$. Also, the point $x = 8$ always lies between $x_D^g$ and $x_D^b$ for any different $\kappa$. Thus, due to the information channel, the debt value $D^b(x; \kappa)$, which is the same as the recovery value, increases in $\kappa$ until $\kappa$ reaches 0.076. But then, $D^b(x; \kappa)$ jumps down as soon as $\kappa$ exceeds 0.076.

If $\kappa$ increases further beyond 0.076, $D^b(x; \kappa)$ is almost flat. Here, if the default threshold $x_D^g$ is hypothetically fixed, $D^b(x; \kappa)$ must be completely flat as discussed in the previous
Figure 4: The two panels on the top plot $D^g(x = 8; \kappa)$ and $D^b(x = 8; \kappa)$, respectively. The bottom panel plots the optimal thresholds for different values of $\kappa$.

section. But now, as $\kappa$ increases, $x^g_D$ increases as well. Thus, due to the minor informational role of the default channel, $D^b(x; \kappa)$ must increase to some extent. But the graph for $D^b(x; \kappa)$ suggests that such an effect is almost negligible.

The debt value in the good state $D^g(x; \kappa)$ exhibits a clear non-monotone pattern. That is, $D^g(x; \kappa)$ increases in $\kappa$ until $\kappa$ reaches 0.056 but then, decreases as $\kappa$ increases further. This result implies that when $\kappa$ is low so that the investment threshold lies below the point $x = 8$, the information channel tends to dominate the default channel. But when $\kappa$ is high so that the investment threshold lies above the point $x = 8$, the default channel tends to dominate the information channel. Of course, the assumption that $\phi_g$ is high also plays an important role here.
Figure 5: This figure plots the effect of a change in the investment cost $\kappa$ under the choice of $\phi_g = 0.3$. The upper, middle, lower panels plot the equity, debt, and firm values, respectively. The left and right panels correspond to the good and bad states, respectively.

4.2.2 Normal Times

We now consider an economy in the normal times. Specifically, we choose $\phi_g = 0.3$, meaning that the negative profitability shock is expected to arrive in 3.3 years on average. All other model parameters are taken from Table 1.

In this circumstance, the asymmetric information in the secondary market is not a big concern to the debtholders in the good state because the negative profitability shock is less likely to occur. As a result, when the investment cost is decreased, the default channel will dominate the information channel. Thus, the debt value in the good state will generally increase as the investment cost decreases.
Figure 6: This figure plots the comparative statics result for $\mu_H$ under the choice of $\phi_g = 2.0$.

Specifically, Figure 5 plots the effect of a change in $\kappa$ on the equity, debt, and firm values. The recovery value still exhibits the non-monotone response to some extent, as shown in the middle-right panel. Nonetheless, this negative effect of the information channel is not transmitted to the good state because $\phi_g$ is small. As a result, the debt value in the good state unambiguously increases as $\kappa$ decreases. In other words, the default channel dominates the information channel in this case.

### 4.3 Asset Growth Rate

In this section, we discuss the effect of a change in the asset growth rate. Here, we focus on the parameter $\mu_H$ because we can similarly analyze the effect of a change in $\mu_L$. Specifically, Figure 6 examines the effect of the change in the $\mu_H$ on the equity, debt, and firm values.
Again, we choose $\phi_g = 2.0$ and take all other parameter values from Table 1. Importantly, since higher $\mu_H$ fosters earlier investment, increasing $\mu_L$ can potentially decrease the debt value as in the comparative statics result regarding $\kappa$. Indeed, the recovery value $R(x)$ shows such a non-monotone pattern as in the middle-right panel in Figure 6. However, unlike in the case of $\kappa$, such a negative effect is not strong enough to be transmitted to the good state even under the choice of $\phi_g = 2.0$. See the middle-left panel of Figure 6.

We can understand this result as follows. Recall that when the investment cost is lowered, $D^g(x)$ may decrease only for those $x$ around the investment threshold. But when $\mu_H$ is increased, the firm’s asset quality $x$ itself is pushed up faster. As a result, the firm’s asset quality is less likely stay around the investment threshold for a long time. This positive effect offsets the above negative effect to some extent. Thus, the debt value in the good state tends to increase if $\mu_H$ increases. We find that this pattern holds for many other reasonable parameter values as well.

4.4 Asset Volatility

In this section, we present the effect of a change in asset volatility. We will basically show that the debt value may decrease in the asset volatility under some conditions.

Specifically, Figure 7 illustrates how the asset volatility $\sigma$ affects the equity, debt, and firm values. In this figure, we choose $\phi_g = 2.0$ and take all other parameter values from Table 1. We will briefly discuss later the case in which $\phi_g = 0.3$.

Note that the equity value increases in $\sigma$ in both good and bad states. This result is well known and comes from convexity of the equity value. A new result is that the debt value of a firm in the good state decreases in $\sigma$ if the firm’s asset quality is high, but increases in $\sigma$ if the firm’s asset quality is low. To understand this result, note that the debt value in the bad state, $D^b(x)$, jumps up at both $x_I$ and $x_D^b$ because the debtholder of type $x_I$ or $x_D^b$ can fully liquidate her asset. As a result, $D^b(x)$ becomes convex for those $x$ slightly below $x_I$ or $x_D^b$. Meanwhile, the debt value is concave when the firm’s asset quality is high enough as in the standard Leland setup.

But then, since $\phi_g$ is high, the debt value in the good state exhibits a similar pattern. As a result, $D^g(x)$ tends to increase in $\sigma$ when the firm’s asset quality is low, whereas $D^g(x)$
decreases in $\sigma$ when the firm’s asset quality is high. In other words, the debtholders of a firm, whose asset quality is slightly below $x_I$ or $x_D$, prefer a riskier project because a positive shock on that project can drastically change the information about the firm’s asset quality.

Meanwhile, when $\phi_g$ is low, the debt value in the good state is concave even for the firms with low asset qualities. Thus, $D^g(x)$ increases in $\sigma$ as in the standard Leland model. We omit to plot the graphs for this case.

5 Empirical Evidence

In this section, we provide two sets of suggestive evidence that the information content from investment decisions are empirically relevant. The first study presents aggregate reduced
form relations of aggregate investments and debt recovery rates the following year and the
second study uses a difference-in-differences approach to quantify the overall impact of lower
investment costs on debt value. Overall, we find that the empirical evidence is largely
consistent with predictions from the model, although more empirical research to disentangle
specific channels may be necessary.

First, using data from Moody’s Ultimate Recovery Rate database from 1990 to 2016 and
a corresponding aggregate measure of total investment relative to GDP for the United States,
we document that since 1990, the average recovery rate for a bankrupt firm is negatively
correlated with the aggregate amount of investment in the prior year. The relation is not
driven by particular outlier year such as the dot-com bubble episode or the financial crisis.
An increase in the aggregate investment to GDP of 1 percentage-point is associated with a 3.5
percentage-point decrease in the recovery rate next year; see Figure 8. Although this result
alone can be driven by other economic mechanisms, it is consistent with the prediction from
our model where more firms investing deteriorates the informational value of investment on
debt value, which manifest as a higher recovery rate based on observed debt traded prices.

Second, using a dataset on bond prices and firm fundamentals, we study the average
net impact of the NPV, default, and information channels of investment on debt value by
studying the decrease in investment costs due to investment tax credits in 2001 and 2008.
Unconditionally, we empirically document a slight positive impact on lower investment costs
on debt value. However, we find firms with less tangible assets saw decreases in the value of
debt relative to those with more tangible assets. Firms with more tangible assets likely have
less information asymmetry than those with less tangible assets because tangible capital
is likely easier to value. Our empirical findings are consistent with the model as when
investment costs decrease, more firms invest, reducing the informational value of investment.
Therefore, firms that suffer more information asymmetry see a smaller increase and are more
likely to see a decrease in their debt value.

To set up the differences-in-differences analysis, from 2000 through 2016, we use two

6The debt recovery rate of a bankrupt firm is defined as the recovered value of the credit instruments that
owners receive after the bankruptcy process. Recovery rates are estimated by both industry and academic
research papers, with the latter also sometimes using pre-default debt and credit default swaps to implicitly
estimate the implied recovery rate. See Eom et al. (2004), Bris et al. (2006), Guo et al. (2008), Davydenko
et al. (2012).
rounds of government interventions aimed at decreasing the investment cost. The first shock was in 2003 in response to the 2001 recession and the second was in response to the Financial Crisis. For both of these policies, the federal government allowed "bonus" depreciation on capital investments, which is effectively an investment tax credit which decreases investment costs. Our empirical analyses follows the set-up from Zwick and Mahon (2017), but we focus on the impact of the investment cost decrease on debt value, proxied by the value of corporate bonds\textsuperscript{7}.

To study the effect of reducing investment costs on corporate bond value, we use data on corporate bonds from TRACE and firm fundamentals from Compustat. As in Zwick and Mahon (2017), when studying the investment tax credits from 2003, we consider a window between 2001 to 2004. When studying investment tax credits in 2009, we use a window between 2008 and 2010. Our final sample comprises 190 unique firms with corporate bonds outstanding, with a total of 3,510 observations at the firm-month level. We define firms that are highly exposed as those in industries with high tangible capital investment, like in Zwick and Mahon (2017), using a similar threshold of 75% exposure for "highly exposed" firms.

\textsuperscript{7}We focus only on corporate bond value since we do not have data on a panel of bank loan market values. Also, as a verification exercise, we replicate the results on firm investments from investment tax credits in Zwick and Mahon (2017) to be 9.7 percent compared with their reported number of 10.4 percent.
For this exercise, to maximize data availability, we do not distinguish between actual bond traded price versus quotes from broker-dealers.

We find that on average, the bonus depreciation on capital investments led to an increase in debt value of about 90 basis points on its existing corporate bonds. In untabulated results, we also find that more profitable firms see a smaller increase in the debt value than less profitable firms. While the low statistical power and small sample size preclude additional robustness checks, these results are consistent with predictions of the model. In fact, the statistically insignificant results may also be due to counteracting effects of the cut on investment cost through the default boundary and information-liquidity mechanisms, discussed before.

A more interesting finding is that conditioning on a firm’s asset tangibility, we find that firms with less tangible assets actually saw a decrease in bond value relative to those with more tangible assets. This is also consistent with our model predictions, since firms suffering from more information asymmetry benefitted more from the information channel of investment. Once more firms invest in response to a decrease in investment costs, the information value of investment decreases.

Overall, the aggregate reduced form, overall firm-level empirical results, and cross-sectional firm-level results are consistent with predictions from the model.

Table 2: Investment Cost Decrease and Corporate Bond Values

<table>
<thead>
<tr>
<th></th>
<th>Full Sample (1)</th>
<th>2001 Shock (2)</th>
<th>2008 Shock (3)</th>
<th>Full Sample (1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Post × Exposed</td>
<td>0.009* (0.005)</td>
<td>0.005 (0.003)</td>
<td>0.010 (0.007)</td>
<td>-0.050** (0.021)</td>
</tr>
<tr>
<td>High Tangibility</td>
<td>-0.010 (0.016)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Post × Exposed × High Tangibility</td>
<td>0.058** (0.021)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Observations</td>
<td>3,510</td>
<td>1,502</td>
<td>2,008</td>
<td>3,510</td>
</tr>
<tr>
<td>Controls</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Fixed Effects</td>
<td>Firm + Year</td>
<td>Firm + Year</td>
<td>Firm + Year</td>
<td>Firm + Year</td>
</tr>
<tr>
<td>Cluster</td>
<td>Firm</td>
<td>Firm</td>
<td>Firm</td>
<td>Firm</td>
</tr>
</tbody>
</table>

*p < 0.1, ** p < 0.05, *** p < 0.01
6 Discussions

The key assumption that gives rise to the informational role of debt overhang is that outside investors are not directly informed of the firm’s asset quality, but can observe the firm’s investment decision. There are various reasons why investment decision is more transparent to outsiders than asset quality in reality. If the investment in the model is interpreted as an investment in tangible assets, whether the firm makes the investment is relatively easy for outsiders to verify through observing its physical presence. If the investment in the model is interpreted as operating costs to keep the firm to stay in a proper condition, whether the firm makes the investment can also be observed by comparing the operating practices of the firm with that of other firms in the same industry. On the contrary, a firm’s asset quality may be less observable to outsiders. First, it is difficult for outsiders to learn the asset quality of a private firm who does not publicize its financial statement. Second, even for a publicly listed firm whose financial statements are available to outsiders, its earnings would not perfectly reveal the underlying quality of its asset due to the existence of temporary factors that affect earnings as well as manager’s use of accounting manipulation.

In our model, a firm may default at a wide range of asset quality values after being hit by a negative profitability shock. The model’s main results will be robust when we instead have other types of shocks that drive a firm to instantaneous default. For example, consider an alternative setting where the firm is subject to a liquidity shock so that the firm may default with some probability whenever it needs to rely on drawing credit lines or issuing new equity to cover operating losses. In this setting, we still find that a firm’s debt value can be increasing in investment cost when the probability of the liquidity shock is large. Note that this setting has implications about the recent financial crisis. During the financial crisis, the simultaneous decrease in credit supply and stability of banks may lead more firms and investors to face funding problems. To the extent that unstable credit lines and frictional capital market may cause the firm to default, government policies that aim at boosting investment may decrease debt value and firm value through the information-liquidity channel, because firms’ higher incentive to invest exacerbates the information asymmetry in the secondary market.

In our model, debt overhang has an unintended positive effect as it improves liquidity in
the bankruptcy market through providing partial information about the firm’s asset quality to ill-informed outsiders. This result suggests that it is not always optimal for a firm to commit to always investing in positive NPV projects. From a survey of 392 CFOs, Graham and Harvey (2001) finds that most firms seem to commit to investing in profitable projects through the use of an ad hoc NPV rule. Our paper suggests that such commitment, if well expected by outside investors, would greatly reduce the informational content in a firm’s investment decision, and could reduce the value of the firm in recession times when the credit risk is high. A similar result holds when we consider an alternative setting where the investment becomes a negative NPV risk-shifting opportunity. In that setting, any restriction on the firm’s ability to invest, such as a debt covenant that restricts investment as a firm’s financial condition deteriorates, could suppress the information content in the firm’s risk-shifting behavior, which leads to a lower firm value under bad economic conditions.

Assuming asymmetric information in the bankruptcy market generates incentives for an information intermediary to enter and potentially increase social surplus. Credit rating agencies like Moody’s, Fitch, and S&P enter the market and provide evaluations of the quality of firms. Although credit ratings are useful for ranking firms ex ante, ex post they cannot be used to discern quality in bankruptcy. For one thing, the ratings cannot be used to perfectly distinguish exogenous bankruptcies from fundamental bankruptcies. For another, they are only updated periodically, and even within credit rating categories, there are differences in firm quality. Another type of institution that plays a role in reducing the asymmetric information in the bankruptcy market is the consulting company, who is employed by asset buyers to assess the value of a bankrupt or near-bankrupt firm. While it may be relatively easy to evaluate tangible assets with high deployability, it can be hard to provide an accuracy assessment on intangible assets that are less transparent and firm-specific. In that sense, our model predicts that the informational role of debt overhang tends to be larger for firms that have more hard-to-evaluate intangible assets.

In our model, we interpret a low parameter value of the probability of getting a negative profitability shock as the expansion phase, and a low parameter value of the probability as the recession phase in a business cycle. When the probability is high, every firm in the economy is more likely to default and enter into the illiquid bankruptcy market, therefore
the information contained in a firm’s investment decision becomes more valuable. Note that our interpretation regards the recession period as one where the negative profitability shock is very likely to happen for now and all future periods. Our main result will be robust, but quantitatively smaller, when the model is extended to allow for an endogenous process for the probability of getting a negative profitability shock to switch between a high and a low value over time.

7 Conclusion

We introduce a new information channel of debt overhang. When considering only debt overhang, policies that target an increase in investment is unambiguously beneficial to all agents and increase firm value. However, if the secondary asset market suffers from information asymmetry, increasing investment may decrease the informational value of debt overhang.

Because firms with higher asset quality in bankruptcy retain a larger fraction of assets to signal their quality to secondary asset buyers who are better at managing those firm assets, higher quality assets actually have less liquidity and do not realize the full gains from trade. However, prior to bankruptcy, equity decides whether a firm invests in a new project. Thus, for firms to invest in new projects, equity must expect a positive return. Therefore, in bankruptcy, secondary asset market buyers can indirectly infer a range of firm quality based on their investment decisions prior to default, giving rise to the information channel of debt overhang.

Empirically, the average recovery rate of bankrupt firms is lower if there was more investment in the year prior to bankruptcy. In a separate difference-in-differences methodology, although we show that two rounds of investment tax credits following recessions had slight positive impacts on debt value, our cross-sectional analyses suggest that those suffering more from information asymmetry, such as those with less tangible assets, may have actually saw a decrease in debt value. Altogether, both our theoretical and empirical results suggest that during times with poor economic conditions where information asymmetry is larger, policy-makers should first study the condition of the secondary market prior to introducing policy that aims to stimulate investment.
8 Appendix

8.1 Omitted Proof in Section 3.1

The retention strategy $f(x)$ given by (6) satisfies that, for any $y \in [x_D^g, x_I]$,

\[
 f(x)\alpha x + (1 - f(x))\beta x - [f(y)\alpha x + (1 - f(y))\beta y] \\
= \int_y^x \frac{\partial}{\partial z} [f(z)\alpha x + (1 - f(z))\beta z]dz \\
= \int_y^x [(\alpha - \beta)zf'(z) + \beta(1 - f(z)) + f'(z)\alpha(x - z)]dz \\
= \int_y^x f'(z)\alpha(x - z)dz, \text{ by the FOC (5),} \\
\geq 0, \text{ because } f'(z) \geq 0.
\]

Hence, the above retention strategy indeed solves the maximization problem in (4).

8.2 Almost Closed-Form Solutions

In this section, for any given \( \{x_D^g, x_I, x_D^b\} \), we first solve for the equity and debt values in closed form. We then compute equilibrium thresholds \( \{x_D^g, x_I, x_D^b\} \) numerically by solving the conditions satisfied by these thresholds in equilibrium.

First, consider the case of \( x_D^g < x_I < x_D^b \). In this case, we can solve for \( E^g(x) \) and \( E^b(x) \) using a standard technique for the system of ordinary differential equations. Specifically, consider the following matrix:

\[
 M = \begin{pmatrix}
 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1 \\
 2\sigma^{-2}(r + \phi_g) & -2\sigma^{-2}\phi_g & -2\sigma^{-2}\mu_H + 1 & 0 \\
 -2\sigma^{-2}\phi_b & 2\sigma^{-2}(r + \phi_b) & 0 & -2\sigma^{-2}\mu_L + 1
 \end{pmatrix}.
\]

Then, let \( \lambda_1 \) and \( \lambda_2 \) be the two negative eigenvalues for \( M \). We postpone to show that \( M \) has indeed two negative eigenvalues and two positive eigenvalues. Also, let \( v_{1g} \) and \( v_{1g} \) be the first two components of the eigenvector for \( \lambda_1 \). Similarly, let \( v_{2g} \) and \( v_{2g} \) be the first two components of the eigenvector for \( \lambda_2 \).
components of the eigenvector for $\lambda_2$. Then, the equity value is given by

$$E^g(x) = \begin{cases} 
E^{g,1}(x) = -\frac{(1-\pi)c}{r+\phi_g} + \frac{x}{r+\phi_g-\mu_L} + A_1x^m + A_2x^n, & \text{if } x_D^g \leq x \leq x_I \\
E^{g,2}(x) = -\frac{(1-\pi)c}{r+\phi_g} + \frac{(1-\pi)c}{r+\phi_g-\mu_H} + A_3x^m + A_4x^n, & \text{if } x_I \leq x \leq x_D^b \\
E^{g,3}(x) = -\frac{(1-\pi)c}{r+\phi_g} + F^g(x) + A_5v_1g x^{\lambda_1} + A_6v_2g x^{\lambda_2}, & \text{if } x_D^b \leq x 
\end{cases}$$

and

$$E^b(x) = \begin{cases} 
E^{b,1}(x) = 0, & \text{if } x \leq x_D^b \\
E^{b,2}(x) = -\frac{(1-\pi)c}{r} + F^b(x) + A_5v_1b x^{\lambda_1} + A_6v_2b x^{\lambda_2}, & \text{if } x_D^b \leq x,
\end{cases}$$

where

$$\begin{bmatrix} 
\eta_1 \\
\eta_2 \\
\eta_3 \\
\eta_4 
\end{bmatrix} = \sigma^{-2} \left( -\mu_L + \frac{\sigma^2}{2} \pm \sqrt{\left( \mu_L - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2(r + \phi_g)} \right),$$

$$\begin{bmatrix} 
\eta_3 \\
\eta_4 
\end{bmatrix} = \sigma^{-2} \left( -\mu_H + \frac{\sigma^2}{2} \pm \sqrt{\left( \mu_H - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2(r + \phi_g)} \right).$$

Now, note that $\{x_D^g, x_I, x_D^b, A_1, A_2, \cdots, A_6\}$ must satisfy the following nine conditions:

$$E^{g,1}(x_D^g) = 0, \quad E^{g,1}(x_D^b) = 0, \quad E^{g,1}(x_I) = E^{g,2}(x_I), \quad E^{g,1}(x_I) = E^{g,2}(x_I) = \frac{\kappa}{2},$$

$$E^{g,2}(x_D^b) = E^{g,3}(x_D^b), \quad E^{g,2}(x_D^b) = E^{g,3}(x_D^b), \quad E^{b,1}(x_D^b) = 0, \quad E^{b,2}(x_D^b) = 0.$$ 

We solve this system of equations numerically.

Let us now show that $M$ has two negative eigenvalues and two positive eigenvalues. Specifically, $f(\lambda) := \det(M - \lambda I)$ is given by

$$f(\lambda) = \lambda^2(-2\sigma^{-2}\mu_L + 1 - \lambda)(-2\sigma^{-2}\mu_L + 1 - \lambda) + \lambda 2\sigma^{-2}(r + \phi_b)(-2\sigma^2\mu_H + 1 - \lambda) + 
\lambda 2\sigma^{-2}(r + \phi_g)(-2\sigma^{-2}\mu_L + 1 - \lambda) + 4\sigma^{-4}(r + \phi_g)(r + \phi_b) - 4\sigma^{-4}\phi_g\phi_b.$$ 

Then, from some algebra, we have

$$f(\lambda) = [\lambda(-2\sigma^{-2}\mu_L + 1 - \lambda) + 2\sigma^{-2}r][\lambda(-2\sigma^{-2}\mu_H + 1 - \lambda) + 2\sigma^{-2}r] + 
2\sigma^{-2}\phi_b[\lambda(-2\sigma^{-2}\mu_L + 1 - \lambda) + 2\sigma^{-2}r] + 2\sigma^{-2}\phi_g[\lambda(-2\sigma^{-2}\mu_H + 1 - \lambda) + 2\sigma^{-2}r].$$

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Now, let $\lambda_H$ be the negative solution to

$$\lambda(-2\sigma^{-2}\mu_H + 1 - \lambda) + 2\sigma^{-2}r = 0.$$  

Also, let $\lambda_L$ be the positive solution to

$$\lambda(-2\sigma^{-2}\mu_L + 1 - \lambda) + 2\sigma^{-2}r = 0.$$  

Then, using the property that $\mu_L < \mu_H$, we know that

$$\lambda_H(-2\sigma^{-2}\mu_L + 1 - \lambda_H) + 2\sigma^{-2}r = 2\lambda_H\sigma^{-2}(\mu_H - \mu_L) < 0,$$

$$\lambda_L(-2\sigma^{-2}\mu_H + 1 - \lambda_L) + 2\sigma^{-2}r = -2\lambda_L\sigma^{-2}(\mu_H - \mu_L) < 0.$$  

This result implies

$$f(\lambda_H) < 0 \text{ and } f(\lambda_L) < 0.$$  

Also, note that $f(0) > 0$. Therefore, $f(\lambda)$ must have two negative roots and two positive roots.

On the other hand, the debt value is given by

$$D^g(x) = \begin{cases} 
D^{g,1}(x) = \frac{c}{r+\phi_g} + \frac{\phi_g\alpha S^g x}{r+\phi_g-\mu_L} + \frac{\phi_g(\beta-\alpha)S^b(x_D^g)^{1-\xi}}{r+\phi_g-\mu_L(1-\xi)+\frac{\phi_g(1-\xi)}{2}(1-\xi)} + B_1x^{\eta_1} + B_2x^{\eta_2}, & \text{if } x_D^g \leq x \leq x_I \\
D^{g,2}(x) = \frac{c}{r+\phi_g} + \frac{\phi_g\alpha S^g x}{r+\phi_g-\mu_H} + \frac{\phi_g(\beta-\alpha)S^b(x_I)^{1-\xi}}{r+\phi_g-\mu_H(1-\xi)+\frac{\phi_g(1-\xi)}{2}(1-\xi)} + B_3x^{\eta_1} + B_4x^{\eta_4}, & \text{if } x_I \leq x \leq x_D^b \\
D^{g,3}(x) = \frac{c}{r} + B_5v_{1g}x^{\lambda_1} + B_6v_{2g}x^{\lambda_2}, & \text{if } x_D^b \leq x,
\end{cases}$$

where $\xi = \frac{\beta}{\beta-\alpha}$ and

$$D^b(x) = \begin{cases} 
D^{b,1}(x) = R(x), & \text{if } x_D^g \leq x < x_D^b \\
D^{b,2}(x) = \frac{c}{r} + B_5v_{1b}x^{\lambda_1} + B_6v_{2b}x^{\lambda_2}, & \text{if } x_D^b \leq x.
\end{cases}$$
For clarification, the recovery value $R(x)$ is given by

$$R(x) = \begin{cases} 
1 - \left( \frac{x}{x}\right)^{\frac{\alpha}{\beta}} & \alpha S^b x + \left( \frac{x}{x}\right)^{\frac{\beta}{\beta}} \beta S^b x, \quad \text{if } x_D \leq x < x_I \\
1 - \left( \frac{x}{x}\right)^{\frac{\alpha}{\beta}} & \alpha S^b x + \left( \frac{x}{x}\right)^{\frac{\beta}{\beta}} \beta S^b x, \quad \text{if } x_I \leq x < x_D.
\end{cases}$$

Now, note that $\{B_1, B_2, \ldots, B_6\}$ must satisfy the following six conditions:

$$D^{g,1}(x_D^g) = \beta S^g x_D^g, \quad D^{g,1}(x_I) = D^{g,2}(x_I), \quad D^{g,2}(x_D) = D^{g,1}(x_D), \quad D^{g,2}(x_D^b) = D^{g,3}(x_D^b), \quad D^{b,2}(x_D^b) = \beta S^b x_D^b.$$

We can solve this system of linear equations in closed form.

Now, let us consider the case of $x_D^g < x_D^b < x_I$. In this case, we first consider the following matrix:

$$N = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
2\sigma^{-2}(r + \phi_g) & -2\sigma^{-2}\phi_g & -2\sigma^{-2}\mu_L + 1 & 0 \\
-2\sigma^{-2}\phi_b & 2\sigma^{-2}(r + \phi_b) & 0 & -2\sigma^{-2}\mu_L + 1
\end{pmatrix}.$$

We can show that $N$ has two negative real eigenvalues and two real positive eigenvalues similarly as above. Thus, let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be those four eigenvalues for $N$. Also, for each $i \in \{1, 2, 3, 4\}$, let $v_{ig}$ and $v_{ib}$ be the first two components of the eigenvector for $\lambda_i$.

Now, consider another matrix:

$$H = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
2\sigma^{-2}(r + \phi_g) & -2\sigma^{-2}\phi_g & -2\sigma^{-2}\mu_H + 1 & 0 \\
-2\sigma^{-2}\phi_b & 2\sigma^{-2}(r + \phi_b) & 0 & -2\sigma^{-2}\mu_L + 1
\end{pmatrix}.$$

Again, we can similarly show that $H$ has two negative eigenvalues and two positive eigenvalues. Thus, let $\{\lambda_5, \lambda_6\}$ be the two negative eigenvalues for $H$. Also, for each $i \in \{5, 6\}$, let $v_{ig}$ and $v_{ib}$ be the first two components of the eigenvector for $\lambda_i$. 38
Then, $E^g(x)$ is given by

\[
\begin{cases}
E^{g,1}(x) = -\frac{(1-\pi)c}{r+\phi g} + \frac{x}{r+\phi g - \mu L} + A_1 x^{\eta_1} + A_2 x^{\eta_2}, & \text{if } x^g_D \leq x \leq x^b_D \\
E^{g,2}(x) = -\frac{(1-\pi)c}{r} + C_1 x + A_3 v_{1g} x^{\lambda_1} + A_4 v_{2g} x^{\lambda_2} + A_5 v_{3g} x^{\lambda_3} + A_6 v_{4g} x^{\lambda_4}, & \text{if } x^b_D \leq x \leq x_I \\
E^{g,3}(x) = -\frac{(1-\pi)c}{r} + C_2 x + A_7 v_{5g} x^{\lambda_5} + A_8 v_{6g} x^{\lambda_6}, & \text{if } x_I \leq x
\end{cases}
\]

and $E^b(x)$ is given by

\[
\begin{cases}
E^{b,1}(x) = 0, & \text{if } x \leq x^b_D \\
E^{b,2}(x) = -\frac{(1-\pi)c}{r} + C_1 x + A_3 v_{1b} x^{\lambda_1} + A_4 v_{2b} x^{\lambda_2} + A_5 v_{3b} x^{\lambda_3} + A_6 v_{4b} x^{\lambda_4}, & \text{if } x^b_D \leq x \leq x_I, \\
E^{b,3}(x) = -\frac{(1-\pi)c}{r} + C_2 x + A_7 v_{5b} x^{\lambda_5} + A_8 v_{6b} x^{\lambda_6}, & \text{if } x_I \leq x
\end{cases}
\]

where

\[
C_{1g} = \frac{r + \phi b - \mu L + \phi g \gamma}{(r - \mu L)(r - \phi g + \phi g - \mu L)}, \quad C_{1b} = \frac{\gamma (r + \phi g - \mu L) + \phi b}{(r - \mu L)(r + \phi g + \phi b - \mu L)},
\]

\[
C_{2g} = \frac{(1 - \kappa)(r + \phi b - \mu L) + \phi g \gamma}{(r - \mu L)(r + \phi g - \mu L) + \phi g (r - \mu L)}, \quad C_{2b} = \frac{\gamma (r + \phi g - \mu H) + \phi b(1 - \kappa)}{(r - \mu L)(r + \phi g - \mu H) + \phi b(r + \phi g - \mu H)}.
\]

\[
\begin{bmatrix}
\eta_1 \\
\eta_2
\end{bmatrix} = \sigma^2 \left( -\mu L + \frac{\sigma^2}{2} \pm \sqrt{\left( \mu L - \frac{\sigma^2}{2} \right)^2 + 2\sigma^2 (r + \phi g)} \right).
\]

Again, $\{x^g_D, x_I, x^b_D, A_1, \ldots, A_8\}$ satisfy the following eleven conditions:

\[
\begin{align*}
E^{g,1}(x^g_D) &= 0, \quad E^{g,1}(x^g_D) = 0, \quad E^{g,1}(x^b_D) = E^{g,2}(x^b_D), \quad E^{g,1}(x^b_D) = E^{g,2}(x^b_D), \\
E^{g,2}(x_I) &= E^{g,3}(x_I), \quad E^{g,2}(x_I) = E^{g,3}(x_I) = \frac{\kappa}{\delta}, \quad E^{b,2}(x^b_D) = 0, \quad E^{b,2}(x^b_D) = 0, \\
E^{b,2}(x_I) &= E^{b,3}(x_I), \quad E^{b,2}(x_I) = E^{b,3}(x_I).
\end{align*}
\]

We solve this system of equations numerically.

Now, let us compute the debt value. Note that $D^g(x)$ is given by

\[
\begin{cases}
D^{g,1}(x) = \frac{c}{r+\phi g} + \frac{\phi g \alpha S^g x}{r+\phi g - \mu L} + \frac{\phi g (\beta - \alpha) S^g(x^g_D)^{\xi} x^{1-\xi}}{r+\phi g - \mu L (1-\xi) + \frac{c^2}{2} (1-\xi) \xi} + B_1 x^{\eta_1} + B_2 x^{\eta_2}, & \text{if } x^g_D \leq x \leq x^b_D \\
D^{g,2}(x) = \frac{c}{r} + B_3 v_{1g} x^{\lambda_1} + B_4 v_{2g} x^{\lambda_2} + B_5 v_{3g} x^{\lambda_3} + B_6 v_{4g} x^{\lambda_4}, & \text{if } x^b_D \leq x \leq x_I \\
D^{g,3}(x) = \frac{c}{r} + B_7 v_{5g} x^{\lambda_5} + B_8 v_{6g} x^{\lambda_6}, & \text{if } x_I \leq x,
\end{cases}
\]
where $\xi = \frac{\beta}{\beta - \alpha}$. Also, $D^b(x)$ is given by

$$
\begin{align*}
D^{b,1}(x) &= R(x), & \text{if } x_D^g \leq x < x_D^b \\
D^{b,2}(x) &= \frac{\xi}{\alpha} + B_3 v_{1b} x^{\lambda_1} + B_4 v_{2b} x^{\lambda_2} + B_5 v_{3b} x^{\lambda_3} + B_6 v_{4b} x^{\lambda_4}, & \text{if } x_D^b \leq x \leq x_I \\
D^{b,3}(x) &= \frac{\xi}{\alpha} + B_7 v_{5b} x^{\lambda_5} + B_8 v_{6b} x^{\lambda_6}, & \text{if } x_I \leq x,
\end{align*}
$$

Here, $R(x)$ is given by

$$
R(x) = \left[ 1 - \left( \frac{x_D^g}{x} \right)^{\frac{\beta}{\beta - \alpha}} \right] \alpha S^b x + \left( \frac{x_D^g}{x} \right)^{\frac{\beta}{\beta - \alpha}} \beta S^b x.
$$

Lastly, note that $\{B_1, \cdots, B_8\}$ satisfy the following eight conditions:

$$
\begin{align*}
D^{g,1}(x_D^g) &= \beta S^g x_D^g, & \quad D^{g,2}(x_D^g) &= \beta S^b x_D^b, & \quad D^{g,1}(x_D^b) &= D^{g,2}(x_D^b), & \quad D^{g,2}(x_I) &= D^{g,3}(x_I), \\
D^{g,2}(x_I) &= D^{g,3}(x_I), & \quad D^{b,2}(x_D^b) &= \beta S^b x_D^b, & \quad D^{b,2}(x_I) &= D^{b,3}(x_I), & \quad D^{b,2}(x_I) &= D^{b,3}(x_I).
\end{align*}
$$

We can solve this system of linear equations in closed form.
References


