Asset Pricing with Return Extrapolation*

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ABSTRACT

We present a new model of asset prices based on return extrapolation. The model is a Lucas-type general equilibrium framework, in which the agent has Epstein-Zin preferences and extrapolative beliefs. Unlike earlier return extrapolation models, our model can be brought to the data quantitatively. When the agent’s beliefs are calibrated to match survey expectations of investors, the model generates excess volatility and predictability of stock returns, a high equity premium, a low and stable risk-free rate, and a low correlation between stock returns and consumption growth. We compare our model directly to prominent rational models and document their different implications.

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In financial economics, there is growing interest in “return extrapolation,” the idea that investors’ beliefs about an asset’s future return are a positive function of the asset’s recent past returns. Models with return extrapolation have two appealing features. First, they are consistent with survey evidence on the beliefs of real-world investors. Second, they show promise in matching important asset pricing facts, such as volatility and predictability in the aggregate market, momentum and reversals in the cross-section, and bubbles (Barberis, Greenwood, Jin, and Shleifer (2015, 2017); Hong and Stein (1999)).

One limitation of existing models of return extrapolation, however, is that they can only be compared to the data in a *qualitative* way. Early models, such as Cutler, Poterba, and Summers (1990) and DeLong, Shleifer, Summers, and Waldmann (1990), highlight the conceptual importance of return extrapolation, but they are not dynamic consumption-based models and are not designed to match asset pricing facts quantitatively. Barberis et al. (2015) is a dynamic model that tries to make sense of both survey expectations and aggregate stock market prices. However, the simplifying assumptions in the model make it difficult to evaluate the model’s fit with the empirical facts. For instance, their model adopts a framework with constant absolute risk aversion (CARA) preferences and a constant interest rate. Under these assumptions, many ratio-based quantities that we study in asset pricing (e.g., the price-dividend ratio) do not have well-defined distributions in the model and therefore do not have properties that match what we observe in the data.

In this paper, we propose a new model of aggregate stock market prices based on return extrapolation that overcomes this limitation. The goal of the paper is to see if the model can match important facts about the aggregate stock market when the agent’s beliefs are calibrated to match survey expectations of investors, and to compare the model in a quantitative way to rational expectations models of the stock market.

We consider a Lucas economy in continuous time with a representative agent. The Lucas tree is a claim to an aggregate consumption process which follows a geometric Brownian motion. Besides the Lucas tree, there are two tradeable assets in the economy: the stock market and an instantaneous riskfree asset. The stock market is a claim to an aggregate dividend process whose growth rate

\footnote{Among others, Vissing-Jorgensen (2004), Bacchetta, Mertens, and van Wincoop (2009), Amromin and Sharpe (2013), Greenwood and Shleifer (2014), Kojien, Schmeling, and Vrugt (2015), and Kuchler and Zafar (2016) document that many individual and institutional investors have extrapolative expectations: they believe that the stock market will continue rising in value after a sequence of high past returns, and that it will continue falling in value after a sequence of low past returns.}
is positively correlated with consumption growth. The riskless asset is in zero net supply with its interest rate determined in equilibrium. The representative agent has Epstein-Zin preferences and extrapolative beliefs. She perceives that the expected growth rate of stock market prices is governed by a switching process between two regimes. If recent price growth of the stock market has been high, the agent thinks it is likely that the high-mean price growth regime is generating prices and therefore forecasts high price growth in the future. Conversely, if recent price growth has been low, the agent thinks that it is likely that the low mean-price growth regime is generating the prices and therefore forecasts low price growth in the future. The agent’s beliefs, together with her preferences, determine equilibrium asset prices.

We calibrate the agent’s beliefs to match the survey expectations of investors studied in Greenwood and Shleifer (2014). Specifically, we set the belief-based parameters such that for a regression of the agent’s expectations about future stock market returns on past twelve-month returns, the model produces a regression coefficient and a $t$-statistic that match the empirical estimates from surveys. Our parameter choice also allows the agent’s beliefs to match the survey evidence on the relative weight investors put on recent versus distant past returns when forming beliefs about future returns. Overall, the model generates a degree of extrapolative expectations for the agent that matches the empirical magnitude.

With the agent’s beliefs disciplined by survey data, the model quantitatively matches important facts about stock market prices and returns. The model generates significant excess volatility of stock market returns from the interaction between return extrapolation and Epstein-Zin preferences. Suppose the stock market has had high past returns. In such a case, return extrapolation leads the agent to forecast high future returns. Under Epstein-Zin preferences, the separation between the elasticity of intertemporal substitution and risk aversion gives rise to a strong intertemporal substitution effect. Therefore, the agent’s forecast of high future returns leads her to push up current prices significantly, generating excess volatility. A feedback loop emerges from this mechanism. If current returns are high, that makes the agent think that future returns will also be high, which leads her to push up prices, increasing current returns further, and so on. In general, there is a danger that this feedback loop could “explode.” Nonetheless, in the model, we assume the agent believes that the expected growth rate of stock market prices tends to switch over time from one regime to the other; she knows her optimism will decline in the future. As a result, the cumulative
impact of the feedback loop on investor expectations and asset prices is finite; the model remains stable.\(^2\)

The same mechanism that generates excess volatility, together with a strong degree of mean reversion in the agent’s expectations about stock returns, allows the model to produce the long-horizon predictability of stock market returns that we observe in the data. By assumption, the agent believes that her expectations about stock market returns will mean-revert. More important, the agent’s beliefs are incorrect: precisely when she thinks that the future price growth is high, future price growth tends to be low endogenously, causing the agent’s return expectations to mean-revert faster than what she perceives. As a result, when the price-dividend ratio is high—this is when the high past price growth of the stock market pushes up the agent’s expectation about future returns and hence her demand for the stock market—the agent’s return expectation tends to revert back to its mean, giving rise to low subsequent returns and hence the predictability of stock market returns using the price-dividend ratio.

The model also generates a high long-run equity premium. Three factors affect the long-run equity premium perceived by the agent. First, excess volatility interacts with the agent’s risk aversion and causes the perceived equity premium to increase. Second, return extrapolation gives rise to perceived persistence of the aggregate dividend process. Under Epstein-Zin preferences, this perceived persistence is significantly priced, pushing up the perceived equity premium. Finally, the separation between the elasticity of intertemporal substitution and the relative risk aversion helps to keep the equilibrium interest rate low and hence keep the equity premium high.

In addition, the true long-run equity premium is significantly higher than the perceived long-run equity premium. When the agent is optimistic about future price growth, her risk aversion limits the decline of the dividend yield component of the return expectation by counteracting the increase in the agent’s demand for the stock market; this makes it less likely for the true equity premium to be lower than the perceived equity premium. On the other hand, when the agent is pessimistic about future price growth, her risk aversion induces an increase of the dividend yield component of the return expectation—this is to offset the low perceived price growth component of the return expectation—by exacerbating the decrease in the agent’s demand for the stock market; this makes it

\(^2\)Models like Cutler et al. (1990) and Barberis et al. (2015) instead introduce fully rational investors in order to counteract the behavioral investors and preserve equilibrium.
more likely for the true equity premium to be higher than the perceived equity premium. Altogether, the asymmetry in the interaction between the agent’s beliefs and risk aversion implies that the true equity premium is more likely to be higher than the perceived equity premium. That is, the true long-run equity premium is above the perceived long-run equity premium.

In the model, the agent separately forms beliefs about dividend growth of the stock market and beliefs about aggregate consumption growth. Here, we assume that the bias in the agent’s beliefs about consumption growth only derives from the bias in her beliefs about dividend growth. Given the low correlation observed in the data between consumption growth and dividend growth, the bias in the agent’s beliefs about consumption growth is small, consistent with the lack of empirical evidence that investors have extrapolative beliefs about consumption growth. This assumption implies that the agent’s beliefs about stock market returns—they co-move strongly with the her beliefs about dividend growth—are not significantly affected by fluctuations in consumption growth, giving rise to the observed low correlation between stock market returns and consumption growth. Moreover, the agent’s approximately correct beliefs about consumption growth allow the model to generate low interest rate volatility.

Our model allows for different time-series properties of quantities as objectively measured by an outside econometrician and quantities as subjectively perceived by the agent. For instance, while the true equity premium and Sharpe ratio are countercyclical, the perceived equity premium and Sharpe ratio are both procyclical, consistent with the findings of Amromin and Sharpe (2013). The model also generates a negative equity premium when the agent expects high price growth of the stock market, a prediction that is consistent with the findings of Greenwood and Hanson (2013), Baron and Xiong (2015), Cassella and Gulen (2017), and Yang and Zhang (2016).

Recent empirical findings of de la O and Myers (2017) show that during periods when the price-dividend ratio of the U.S. stock market is high, expectations of aggregate dividend growth are much higher than the expectations of future stock market returns. As a result, changes in investors’ subjective expectations of future dividend growth explain the majority of stock market movements. Interestingly, even though our model is based on return extrapolation, it is actually consistent with these empirical results on cash flow expectations. When the past price growth of the stock market has been high, this has a positive effect not only on the agent’s beliefs about future price growth, but also on her beliefs about future dividend growth; indeed, her expectations
about dividend growth rise at a pace that exceeds her expectations about future dividend growth. By using the agent’s subjective expectations about stock market returns and dividend growth in a Campbell-Shiller decomposition, the model shows that changes in subjective expectations about future dividend growth explain most of the volatility of the price-dividend ratio. Importantly, the fact that prices in our model are mainly correlated with cash flow expectations is a consequence of the Campbell-Shiller accounting identity; return expectations, not cash flow expectations, are the cause of price movements. As a result, our model can make sense of both the empirical findings of de la O and Myers (2017) on cash flow expectations and the empirical findings of Greenwood and Shleifer (2014) on return expectations.

The model also has some limitations. Specifically, it generates an insufficiently persistent price-dividend ratio: when calibrated to the survey expectations data, the model predicts a persistence of the price-dividend ratio that is lower than its empirical value. In other words, to match the empirical persistence of the price-dividend ratio, investors need to form beliefs about future returns based on many years of past returns. However, the available survey evidence suggests that they focus on just the past year or two.

After presenting the model, we compare it to the standard rational expectations models of the aggregate stock market. As with the habit formation model of Campbell and Cochrane (1999), the long-run risks models of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2012), and the rare disasters models of Barro (2006), Gabaix (2012), and Wachter (2013), our model is developed in a Lucas economy with a representative agent. This model structure allows for a direct comparison between our model and models with rational expectations. Here, we focus on the long-run risks models on the grounds that these are the models most related to ours, and we document some different implications.

In Bansal and Yaron (2004), dividend growth and consumption growth share a stochastic yet persistent component. High past stock market returns are typically caused by positive shocks on this common component, which, given its persistence, implies high dividend growth moving forward, hence pushing up future raw returns. This mechanism gives rise to the agent’s extrapolative beliefs about future raw returns. However, precisely because dividend growth and consumption growth share a common persistent component, the comovement between the agent’s beliefs about stock

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3We provide a detailed explanation of this finding in Sections I and II.
market returns—they rationally drive returns—and her beliefs about consumption growth—they determine the interest rate in equilibrium—is high. In our model, however, the agent extrapolates past returns on the stock market, but extrapolates past consumption much less: the comovement between the agent’s beliefs about stock market returns and her beliefs about consumption growth is low. As a result, the agent in our model has extrapolative beliefs about both raw returns and excess returns, whereas the agent in Bansal and Yaron (2004) has extrapolative beliefs only about raw returns.

Furthermore, our model produces an equity premium and return volatility that do not vary significantly with respect to changes in the elasticity of intertemporal substitution. On the contrary, long-run risks models cannot generate a high equity premium with a low elasticity of intertemporal substitution. To see this model difference, we note that the agent’s beliefs in our model are much less persistent than the stochastic component of dividend and consumption growth in Bansal and Yaron (2004), allowing the equilibrium interest rate and hence the equity premium in our model to be less responsive to changes in the elasticity of intertemporal substitution. At the same time, the perceived dividend growth in our model depends more strongly on the agent’s beliefs about the price growth of the stock market, pushing up the perceived equity premium; as a comparison, dividend growth in Bansal and Yaron (2004) depends less strongly on the stochastic growth component. Finally, the true long-run equity premium in our model is above the perceived one, allowing the equity premium to be high even when the elasticity of intertemporal substitution is low.

We also compare our model to a model with fundamental extrapolation, the notion that some investors hold extrapolative expectations about the future dividend growth of the stock market. With fundamental extrapolation, high past dividend growth generates optimistic beliefs about future dividend growth and hence pushes up current prices, but higher prices do not further affect the agent’s beliefs about future dividends: the feedback loop described earlier is absent. As a result, for the same parameter values, the fundamental extrapolation model generates lower excess volatility and a lower average equity premium. This model comparison also points to a methodological contribution of the paper. While most belief-based models specify belief formation exogenously, the feedback loop in return extrapolation models involves endogenous belief formation. Our differential equations approach provides a solution to this greater modeling challenge.

Our paper adds to a new wave of theories that attempt to understand the role of belief formation
in driving the behavior of asset prices and the macroeconomy (Fuster, Hebert, and Laibson (2011); Choi and Mertens (2013); Alti and Tetlock (2014); Hirshleifer, Li, and Yu (2015); Barberis et al. (2015); Jin (2015); Ehling, Graniero, and Heyerdahl-Larsen (2015); Vanasco, Malmendier, and Pouzo (2015); Collin-Dufresne, Johannes, and Lochstoer (2016a,b); Greenwood, Hanson, and Jin (2016)). Our paper also adds to a growing literature on the source of stock price movements (Cochrane (2008); Cochrane (2011); Chen, Da, and Zhao (2013); de la O and Myers (2017)). Furthermore, it is related to theories of model uncertainty and ambiguity aversion such as Bidder and Dew-Becker (2016). These models typically assume that agents learn about the dynamic properties of the consumption process or the dividend process. As a result, they are closely linked to the fundamental extrapolation models in the behavioral finance literature, but do not match survey evidence on return expectations. Finally, our paper speaks to the debate between Bansal et al. (2012) and Beeler and Campbell (2012) which focuses on excess predictability: the notion that, in the long-run risks literature, future consumption and dividend growths are excessively predicted by current variables such as the price-dividend ratio (see also, Collin-Dufresne et al. (2016b)). Our model does not give rise to excess predictability: return extrapolation in the model only generates perceived but not true long-run risks, and therefore the true consumption and dividend growths remain unpredictable.

The paper proceeds as follows. In Section I, we lay out the basic elements of the model and characterize its solution. In Section II, we parameterize the model and examine its implications in detail. Section III provides a comparative statics analysis. Section IV discusses differences between our model and rational expectations models. Section V further compares the model to a model with fundamental extrapolation. Section VI concludes and suggests directions for future research. All technical details are in the Appendix.

I. The Model

In this section, we first describe the model setup and characterize its solution, and then derive equilibrium quantities that are important for understanding the implications of the model.
I.1. Model setup

We consider an infinite-horizon Lucas economy in continuous time with a representative agent. The Lucas tree is a claim to an aggregate consumption process. We assume it is a geometric Brownian motion

$$dC_t/C_t = g_C dt + \sigma_C d\omega^C_t,$$

and we denote the price of the Lucas tree at time \( t \) as \( P^C_t \).

Besides the Lucas tree, there are two other tradeable assets in the economy; they are the main focus of our analysis. The first asset is the stock market which is a claim to an aggregate dividend process given by

$$dD_t/D_t = g_D dt + \sigma_D d\omega^D_t;$$

we denote the price of the stock market at time \( t \) as \( P^D_t \). Both \( \omega^D_t \) and \( \omega^C_t \) are standard Brownian motions. We assume the instantaneous correlation between \( dD_t \) and \( dC_t \) is \( \rho \): \( \mathbb{E}_t[d\omega^D_t \cdot d\omega^C_t] = \rho dt \).

The second asset is an instantaneous riskless asset. This asset is in zero net supply, and its interest rate \( r_t \) is determined in equilibrium.

In terms of the agent’s preferences, we follow Epstein and Zin (1989, 1991) and assume that the agent has a recursive intertemporal utility

$$U_t = \left[ (1 - e^{-\delta dt})C_t^{1-\psi} dt + e^{-\delta dt} \left( \mathbb{E}_t[\tilde{U}_{t+dt}^{1-\gamma}] \right)^{(1-\psi)/(1-\gamma)} \right]^{1/(1-\psi)},$$

where \( \delta \) is the subjective discount rate, \( \gamma > 0 \) is the coefficient of relative risk aversion, and \( \psi > 0 \) is the reciprocal of the elasticity of intertemporal substitution. The superscript “\( \epsilon \)” is an abbreviation for “extrapolative” expectations: the certainty equivalence in (3) is computed under the representative agent’s subjective beliefs, which, as we specify later, incorporate the notion of return extrapolation. Note that when \( \psi \) equals \( \gamma \), (3) reduces to power utility.

\footnote{Since the aggregate consumption process in the model is exogenous, the dividend payment from the stock market does not further affect consumption. As a result, we can think of the stock market as an asset in zero net supply with a shadow price determined in equilibrium. This is a common assumption adopted by many other consumption-based models such as Campbell and Cochrane (1999) and Barberis, Huang, and Santos (2001).}
The subjective Euler equation, or the first-order condition, is

\[
E_t \left[ e^{-\delta(1-\gamma)dt/(1-\psi)} \left( \frac{\tilde{C}_{t+dt}}{C_t} \right)^{-\psi(1-\gamma)/(1-\psi)} \tilde{M}_{t+dt}^{(\psi-\gamma)/(1-\psi)} \tilde{R}_{j,t+dt} \right] = 1. \tag{4}
\]

Here \( \tilde{M}_{t+dt} \) is the gross return on the optimal portfolio held by the agent from time \( t \) to time \( t + dt \).

In a Lucas economy with a representative agent, the optimal portfolio in equilibrium is the Lucas tree itself, and therefore

\[
\tilde{M}_{t+dt} = \frac{\tilde{P}_{C,t+dt}^C + \tilde{C}_{t+dt}}{P_{C,t}^C} = \frac{\tilde{P}_{C,t+dt}^C + \tilde{C}_{t+dt}dt}{P_{C,t}^C} + o(dt). \tag{5}
\]

On the other hand, \( \tilde{R}_{j,t+dt} \) is the gross return on any tradeable asset \( j \) in the market from time \( t \) to time \( t + dt \); as mentioned above, the two tradeable assets we focus on are the stock market and the riskless asset.

We now turn to the key part of the model: the representative agent’s beliefs about stock market returns. According to surveys, real-world investors form beliefs about future stock market returns by extrapolating past returns (Vissing-Jorgensen (2004); Bacchetta et al. (2009); Amromin and Sharpe (2013); Greenwood and Shleifer (2014); Koijen et al. (2015); Kuchler and Zafar (2016)). One natural way to capture this notion of return extrapolation is through a regime-switching model. Specifically, we suppose that the agent believes that the expected growth rate of stock market prices is governed by

\[
(1 - \theta)g_D + \theta \tilde{\mu}_{S,t},
\]

where \( \tilde{\mu}_{S,t} \) is a latent variable which switches between a high value \( \mu_H \) in a high-mean price growth regime \( H \) and a low value \( \mu_L \) (\( \mu_L < \mu_H \)) in a low-mean price growth regime \( L \) with the following transition matrix\(^5\)

\[
\begin{pmatrix}
\tilde{\mu}_{S,t+dt} = \mu_H & \tilde{\mu}_{S,t+dt} = \mu_L \\
\tilde{\mu}_{S,t} = \mu_H \begin{pmatrix} 1 - \chi dt & \chi dt \\ \lambda dt & 1 - \lambda dt \end{pmatrix}
\end{pmatrix}.
\tag{6}
\]

Here \( \chi \) and \( \lambda \) are the intensities for the transitions of regime from \( H \) to \( L \) and from \( L \) to \( H \), respectively, and the parameter \( \theta \) (\( 0 \leq \theta \leq 1 \)) controls the extent to which the agent’s beliefs are

\(^5\)The models of Barberis, Shleifer, and Vishny (1998), Veronesi (1999), and Jin (2015) also adopt a regime-switching learning structure.
extrapolative: setting $\theta$ to zero gives the agent fully rational beliefs.

Given this perceived regime-switching model—this is not the true model—if recent stock market price growth has been high, the agent thinks it is likely that the high-mean price growth regime is generating prices and therefore forecasts high price growth in the future. Conversely, if recent price growth has been low, the agent thinks it is likely that the low-mean price growth regime is generating prices and therefore forecasts low price growth in the future. Formally, at each point in time, the agent computes the expected value of the latent variable $\tilde{\mu}_{S,t}$ given the history of past price growth: $S_t \equiv \mathbb{E}[\tilde{\mu}_{S,t}|\mathcal{F}_t^P]$. That is, she applies filtering theory (see, for instance, Lipster and Shiryaev (2001)) and obtains

$$dS_t = (\lambda \mu_H + \chi \mu_L - (\lambda + \chi)S_t)dt + (\sigma^D_{P,t})^{-1}\theta(\mu_H - S_t)(S_t - \mu_L)d\omega_t^e,$$

where $d\omega_t^e \equiv [dP^D_t/P^D_t - (1 - \theta)g_D dt - \theta S_t dt]/\sigma^D_{P,t}$ is a standard Brownian innovation term from the agent’s perspective. As a result, she perceives the evolution of the stock market price $P^D_t$ to be

$$dP^D_t/P^D_t = \mu^D,e_P(S_t)dt + \sigma^D_P(S_t)d\omega_t^e,$$

where

$$\mu^D,e_P(S_t) = (1 - \theta)g_D + \theta S_t.$$  

The agent’s expectation about price growth $\mu^D,e_P(S_t)$ is therefore a linear combination of a rational component $g_D$ and a behavioral component $S_t$; hereafter we call $S_t$ the sentiment variable.

In summary, the evolution of sentiment in (7) captures return extrapolation: high past price growth $dP^D_t/P^D_t$ pushes up the perceived shock $d\omega_t^e$, which leads the agent to raise her expectation of the sentiment variable $S_t$, causing her expectation about future price growth $\mu^D,e_P(S_t)$ to rise.$^6$

Although the subjective evolution of sentiment (7) is derived through optimal learning, the representative agent, it should be emphasized, does not hold rational expectations. With rational

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$^6$There are many ways to specify the evolution of $S_t$ in order to capture return extrapolation. We derive $S_t$ from a regime-switching model for two reasons. First, such a learning model captures base rate neglect, an important consequence of the representativeness heuristic (Tversky and Kahneman (1974)). To see this, note that the perceived regimes or states, $H$ and $L$, are not part of the true states of the economy. As a result, assigning positive probability weights to these regimes reflect the bias that the investor neglects the zero base rate associated with such regimes. Second, bounding $S_t$ by a finite range $(\mu_L, \mu_H)$ reduces the analytical difficulty of solving the model.
expectations, the agent will realize in the long run that the regime-switching model (6) is incorrect: she can look at the historical distribution of $d\omega^e_t$ and realize that it does not fit a normal distribution with a mean of 0 and a variance of $dt$. Instead, the agent in our behavioral model always believes that the regime-switching model is correct. In reality, it is possible that investors in the market learn over time that their mental model is incorrect. At the same time, new investors who hold extrapolative beliefs may continuously enter the market. The stable belief system in (6) is an analytically convenient way to capture these dynamics. Alternatively, if equations (6) and (7) represent the true data generating process, then the agent does hold rational expectations. In that case, the model becomes a fully rational model with incomplete information.\footnote{Information is incomplete in the sense that the agent does not directly observe the latent variable $\tilde{\mu}_{S,t}$.} We discuss the predictions of such a model in Section IV.

So far we have been focusing on the agent’s beliefs about stock market prices. These beliefs also have direct implications for the agent’s beliefs about dividend growth. If we write the perceived dividend process as

$$dD_t/D_t = g^e_D(S_t)dt + \sigma_D d\omega^e_t,$$  

we can connect the agent’s expectation about dividend growth $g^e_D(S_t)$ to her expectation about stock market price growth $\mu^{D,e}_P(S_t)$. To formally make this connection, we first observe that all the ratio-based quantities in our model (e.g., the price-dividend ratio of the stock market) are a function of the sentiment variable $S_t$; we can define $f(S_t) = P^D_t/D_t$. We then apply Ito’s lemma on both sides of the equation $f(S_t) = P^D_t/D_t$ and match terms to obtain

$$g^e_D(S_t) = \left(1 - \theta\right)g_D + \theta S_t - (f'/f)\mu^e_S(S_t) + \sigma^2_D - \sigma^2_P(S_t)\sigma_D - \frac{1}{2}(f''/f)(\sigma_S(S_t))^2,$$  

where

$$\sigma^2_P(S_t) = \frac{\sigma^2_D + 4\theta(\mu_H - S_t)(S_t - \mu_L)(f'/f)}{2} > \sigma_D.$$  

\footnote{Information is incomplete in the sense that the agent does not directly observe the latent variable $\tilde{\mu}_{S,t}$.}
Equation (11) highlights an “expectations transmission mechanism”: it says that the agent’s expectation about dividend growth equals the sum of her expectation about stock market price growth, her expectation about how the price-dividend ratio evolves with respect to changes in sentiment, and the Ito correction terms that are related to the agent’s risk aversion and the volatility of dividend growth, price growth, and changes in sentiment. In this way, the agent’s expectation about price growth affects her expectation about dividend growth.

It takes two steps to explain the economic intuition behind (11). First, iterating forward the Euler equation (4) with the stock market as the tradeable asset gives

$$\frac{P_t^D}{D_t} = \mathbb{E}_t^e \left[ \int_t^\infty e^{-\delta(1-\gamma)(s-t)/(1-\psi)} \left( \frac{\tilde{C}_s}{C_t} \right)^{-\psi(1-\gamma)/(1-\psi)} \tilde{M}_{t\to s}^{(\psi-\gamma)/(1-\psi)} \left( \frac{\tilde{D}_s}{D_t} \right) ds \right], \quad (13)$$

where $\tilde{M}_{t\to s}$ denotes the continuously compounded gross return for holding the Lucas tree from time $t$ to time $s$ ($s > t$).\(^8\) That is, the current price-dividend ratio of the stock market equals the agent’s subjective expectation of the sum of discounted future dividend growth. This present value formula (13) links the agent’s beliefs about future dividend growth to her beliefs about future price growth; the agent’s expectation about dividend growth—the left hand side of (11)—is tied to her expectation about price growth—the first term on the right hand side of (11).

We further explain why the agent’s expectation about dividend growth can be more responsive to changes in sentiment than her expectation about price growth. Under the regime-switching model, the agent perceives sentiment to be mean-reverting: $\mu_S(S_t)$ in (7) is a negative function of $S_t$. Given that sentiment drives the price-dividend ratio, the agent also perceives the price-dividend ratio to be mean-reverting. Furthermore, the price-dividend ratio is a positive function of sentiment, and therefore the price-dividend ratio is high when sentiment is high.\(^9\) Putting these together, the agent anticipates that the price-dividend ratio will decline from a high value when she expects high future price growth. That is, when the agent expects high future price growth, she also expects high dividend growth, at a pace that exceeds her expectations about future price growth.

To complete the description of the model, we need to further specify the agent’s beliefs about

\(^8\)The transversality condition holds in this economy as the stock market price is bounded by a finite range.

\(^9\)When sentiment is high, the agent’s expectation about future dividend growth is high. Equation (13) then implies that the current price-dividend ratio is high.
consumption growth. To do this, first note that with a local correlation of $\rho$ between consumption growth and dividend growth, we can rewrite the aggregate consumption process of (1) as

$$
dC_t/C_t = g_C dt + \sigma_C \left( \rho d\omega^D_t + \sqrt{1-\rho^2} d\omega^\perp_t \right),$$

(14)

where $\omega_t^\perp$ is a Brownian motion that is locally uncorrelated with $\omega^D_t$, the Brownian shock on dividends. We then assume that the agent perceives the consumption process as

$$
dC_t/C_t = g^{e}_C(S_t) dt + \sigma_C \left( \rho d\omega^{e}_t + \sqrt{1-\rho^2} d\omega^\perp_t \right).$$

(15)

That is, we replace the true Brownian shock on dividends $d\omega^D_t$ by the agent’s perceived Brownian shock $d\omega^{e}_t$ and factor the difference between these two Brownian shocks into $g^{e}_C(S_t)$, the agent’s subjective expectation about consumption growth. Conceptually, this amounts to assuming that the bias in the agent’s beliefs about consumption growth only comes from the bias in her beliefs about dividend growth.\(^\text{10}\) In doing so, we derive the agent’s expectation about dividend growth as

$$
g^{e}_C(S_t) - g_C = \rho \sigma_C \sigma^{-1}_D (g^e_D(S_t) - g_D).$$

(16)

Empirically, the correlation between consumption growth and dividend growth is low—$\rho$ is positive but low—and consumption growth is much less volatile than the dividend growth—$\sigma_C$ is much smaller than $\sigma_D$. As a result, (16) implies that the bias in the agent’s expectation about consumption growth—the difference between $g^{e}_C(S_t)$ and $g_C$—is small. This is in keeping with the lack of any evidence that investors have extrapolative beliefs about consumption growth.\(^\text{11}\) Moreover, the agent’s approximately correct beliefs about consumption growth allow the model to generate low interest rate volatility and a low correlation between consumption growth and stock market returns, both of which are consistent with the data (Campbell (2003); Hansen and Singleton (1982, 1983)).

\(^\text{10}\)For any alternative assumption, one needs to explain why the investor has incorrect beliefs about consumption above and beyond her incorrect beliefs about dividends.

\(^\text{11}\)Consistent with the way we model the agent’s expectations about consumption growth, Kuchler and Zafar (2016) find that survey expectations are “asset-specific”: respondents who become pessimistic about their employment situation after experiencing unemployment are not pessimistic about other economic outcomes, such as stock prices or interest rates. Similarly, Huang (2016) finds that investors who become optimistic about an industry’s future returns after having positive prior investment experience in the industry do not invest heavily in a dissimilar industry.
I.2. Model solution

The subjective Euler equation in (4) shows that, when pricing the stock market, the gross return from holding the Lucas tree is also part of the pricing kernel. This observation has two implications. First, both the price-dividend ratio $f(S_t) = P^D_t / D_t$ and the wealth-consumption ratio $P^C_t / C_t$ are functions of the sentiment variable $S_t$; we can define $l(S_t) \equiv P^C_t / C_t$. Second, the two functions $f$ and $l$ are interrelated through Euler equations, so they need to be solved simultaneously. Specifically, using the Euler equation to price the stock market—setting $\tilde{R}_{j,t+dt}$ in (4) to the gross return on the stock market—we obtain

$$0 = \left[ -\frac{(1-\gamma)}{1-\psi} \delta + \gamma g^C_C + g^D_D + [(f'/f) + \frac{\psi-\gamma}{1-\psi} (l'/l)]\mu^C_S + \frac{1}{2} [(f''/f) + \frac{\psi-\gamma}{1-\psi} (l''/l)]\sigma^2_S \right. \right.$$

$$\left. + \left( \gamma (\gamma+1) \frac{1}{2} \sigma^2_C + \frac{1}{2} \frac{\psi-\gamma}{1-\psi} 2 \psi^{-\gamma-1} (l'/l)^2 \sigma^2_S - \frac{(\psi-\gamma)}{1-\psi} \rho \sigma_C \sigma_S (l'/l) - \gamma \rho \sigma_C \sigma_D - \gamma \rho \sigma_C \sigma_S (f'/f) \right] \right.$$

$$\left. \left. + \left( \frac{\psi-\gamma}{1-\psi} \sigma_D \sigma_S (l'/l) + \frac{\psi-\gamma}{1-\psi} \sigma^2_S (l'/l) (f'/f) + \sigma_D \sigma_S (f'/f) + \frac{\psi-\gamma}{1-\psi} l^{-1} + f^{-1} \right) \right] \right. \right.$$

(17)

Similarly, using the Euler equation to price the Lucas tree—setting $\tilde{R}_{j,t+dt}$ in (4) to the gross return on the Lucas tree—we obtain

$$0 = \left[ -\frac{(1-\gamma)}{1-\psi} \delta - (\gamma - 1) g^D_D + \frac{\gamma(\gamma-1)}{2} \sigma^2_C + \frac{1-\gamma}{1-\psi} (l'/l) \mu^C_S + \frac{1-\gamma}{2(1-\psi)} (l''/l) \sigma^2_S \right. \right.$$

$$\left. + \frac{1}{2} \frac{1-\gamma}{1-\psi} \psi (l'/l)^2 \sigma^2_S + \frac{(1-\gamma)^2}{1-\psi} \rho \sigma_C \sigma_S (l'/l) + \frac{1-\gamma}{1-\psi} l^{-1} \right]. \right.$$

(18)

Substituting $\mu_S$ and $\sigma_S$ from (7), $g^D_D$ and $\sigma^D_D$ from (11) and (12), and $g^C_C$ from (16) into equations (17) and (18), we then obtain a system of two ordinary differential equations that jointly determines the evolutions of $f$ and $l$.\(^{12}\) The detailed derivation of (17) and (18) is in the Appendix.

Regarding the boundary conditions for solving the differential equations, note that, in (17) and (18), the second derivative terms are all multiplied by $\sigma_S$, and that $\sigma_S$ goes to zero as $S$ approaches either $\mu_H$ or $\mu_L$. As a result, $\mu_H$ and $\mu_L$ are both singular points, and therefore no boundary condition is required.

\(^{12}\)When $\theta = 0$, our model reduces to a fully rational benchmark. In this case, equations (17) and (18) lead to

$$f = \left[ \delta + \psi g_C - g_D - \frac{\gamma (\psi+1)}{2} \sigma^2_C + \gamma \rho \sigma_C \sigma_D \right]^{-1}, \quad l = \left[ \delta + (\psi - 1) g_C - \frac{\gamma (\psi-1)}{2} \sigma^2_C \right]^{-1}.$$
Equations (17) and (18) cannot be solved analytically. We apply a projection method with Chebyshev polynomials to solve them numerically. We leave the details of the numerical procedure to the Appendix.

I.3. Important equilibrium quantities

With the model solution at hand, we derive equilibrium quantities that are important for understanding the model implications. Specifically, we derive the dynamics of the interest rate, the objective and subjective expectations of stock market returns, and the steady-state distribution of the sentiment variable.

For the interest rate, we use the Euler equation in (4) to price the riskless asset—we set $\tilde{R}_{j,t+dt}$ to the gross return on the riskless asset $1 + r_t dt$—and obtain

$$r_t = \frac{1 - \gamma}{1 - \psi} \delta + \gamma g_C^e - \frac{\gamma(\gamma+1)}{2} \sigma_C^2 - \frac{\psi - \gamma}{1 - \psi} \times \left[ \frac{\mu_S^e - \gamma \rho \sigma_C \sigma_S}{l'} (l'/l) + \frac{1}{2} \sigma_S^2 (l''/l) \right].$$  \hspace{1cm} (19)

The interest rate is linked to the agent’s time preferences, her subjective expectation about consumption growth, precautionary saving, as well as how the wealth-consumption ratio $P_t^C/C_t$ responds to changes in sentiment.\(^\text{13}\)

To understand the risk-return tradeoff in the model, we compute, at each point in time, both the agent’s expectation about future stock market returns and the (objectively measured) rational expectation about future stock market returns. From equations (8) and (9), the log excess return on the stock market from time $t$ to time $t + dt$ is

$$r^D_{t+dt} = \ln(P_{t+dt}^D + D_{t+dt} dt) - \ln(P_t^D) - r_t dt$$

$$= [(1 - \theta)g_D + \theta S_t + f^{-1} - \frac{1}{2} (\sigma_D^p)^2 - r_t] dt + \sigma_D^p d\omega^e_t.$$  \hspace{1cm} (20)

Therefore, the agent’s expectation about the log excess return is

$$\mathbb{E}_t^e [r^D_{t+dt}] = (1 - \theta)g_D + \theta S_t + f^{-1} - \frac{1}{2} (\sigma_D^p)^2 - r_t,$$  \hspace{1cm} (21)

\(^{13}\)When $\theta = 0$, $r = \delta + \psi g_C - \frac{\gamma(\gamma+1)}{2} \sigma_C^2$. 

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and the subjective Sharpe ratio is \[\{(1 - \theta)g_D + \theta S_t + f^{-1} - \frac{1}{2}(\sigma_P^2)^2 - r_t\}/\sigma_D^2.\]

Next, to compute the rational expectation about the stock market return, we compare (2) with (10) and obtain a relationship between the true and perceived Brownian shocks

\[d\omega_t^e = d\omega_t^D - (g_D^e(S_t) - g_D)dt/\sigma_D.\] (22)

We substitute (22) into (20) and derive

\[r_{t+dt}^D = [(1 - \theta)g_D + \theta S_t + f^{-1} - \sigma_D^{-1}\sigma_P^2(g_D^e - g_D) - \frac{1}{2}(\sigma_P^2)^2 - r_t]dt + \sigma_P^2 d\omega_t^D.\] (23)

So the rational expectation about the log excess return on the stock market is

\[E_t[r_{t+dt}^D] = (1 - \theta)g_D + \theta S_t + f^{-1} - \sigma_D^{-1}\sigma_P^2(g_D^e - g_D) - \frac{1}{2}(\sigma_P^2)^2 - r_t.\] (24)

And the objectively measured Sharpe ratio of the stock market return is \[\{(1 - \theta)g_D + \theta S_t + f^{-1} - \sigma_D^{-1}\sigma_P^2(g_D^e - g_D) - \frac{1}{2}(\sigma_P^2)^2 - r_t\}/\sigma_D^2.\]

All the ratio-based quantities in this model such as the agent’s expectation about stock market returns and the interest rate are functions of the sentiment variable \(S_t\). Given this, to provide a statistical assessment of the model’s fit with the empirical facts, we also compute the steady-state distribution for the sentiment variable \(S_t\) as objectively measured by an outside econometrician. To that end, we first obtain the objective evolution of sentiment by substituting the change-of-measure equation (22) into the subjective evolution of sentiment in (7)

\[dS_t = [\mu_S(S_t) + \sigma_D^{-1}\sigma_S(S_t)(g_D - g_D^e(S_t))]dt + \sigma_S(S_t)d\omega_t^D.\] (25)

Compared to the subjective evolution of sentiment, the objective evolution exhibits a larger degree of mean reversion: the term \(\sigma_D^{-1}\sigma_S(S_t)(g_D - g_D^e(S_t))\) in (25) is a negative function of sentiment.

Denote the objective steady-state distribution for sentiment as \(\xi(S)\). Based on (25), we then
derive $\xi(S)$ as the solution to the Kolmogorov forward equation (the Fokker-Planck equation)

$$0 = \frac{1}{2} \frac{d^2}{dS^2} (\sigma_S^2(S)\xi(S)) - \frac{d}{dS} \left( (\mu_S(S) + \sigma_D^{-1} \sigma_S(S)(g_D - g_D'(S)))\xi(S) \right)$$

$$= (\sigma_S')^2 \xi + \sigma_S \sigma_S'' \xi + 2 \sigma_S \sigma_S' \xi' + \frac{1}{2} \sigma_S^2 \xi''$$

$$- \left[ (\mu_S') + \sigma_D^{-1} \sigma_S'(g_D - g_D') \right] \xi - \left[ \mu_S' + \sigma_D^{-1} \sigma_S(g_D - g_D') \right] \xi',$$

where $\sigma_S$ and $g_D'$ are from (7) and (11), respectively, and the expressions for $\sigma_S'$, $\sigma_S''$, $(\mu_S')'$ and $(g_D')'$ are provided in the Appendix. In addition, the steady-state distribution must integrate to one.

II. Model Implications

In this section, we examine the model implications. We begin by setting the benchmark values for the model parameters. We calibrate the agent’s beliefs to match the survey evidence documented in Greenwood and Shleifer (2014). We then look at two building blocks for the model implications: a set of important equilibrium quantities, each as a function of sentiment; and the steady-state distribution of sentiment. Finally, we discuss the model’s implications for asset prices.

II.1. Model parameterization

There are three types of parameters: asset parameters, utility parameters, and belief parameters. For the asset parameters, we set $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$. These values are consistent with those used in Campbell and Cochrane (1999), Barberis et al. (2001), Bansal and Yaron (2004), and Beeler and Campbell (2012).\(^{14}\) For the utility parameters, we set $\gamma$, the relative risk aversion parameter, to 10. As pointed out in Bansal et al. (2012) and Bansal and Yaron (2004), the long-run risks literature—a literature that depends significantly on the parameter values of Epstein-Zin preferences for its model implications—typically assigns a value of 10 or below for $\gamma$. Bansal and Yaron (2004), for instance, set $\gamma$ to either 10 or 7.5.\(^{15}\) For $\psi$, the reciprocal of the elasticity of intertemporal substitution, there exists a wide range of estimates

\(^{14}\)The parameter values for $g_C$ and $g_D$ are set such that both log consumption and log dividend grow at an annual rate of 1.84%; this rate is used in Barberis et al. (2001).

\(^{15}\)An estimate of 10 for $\gamma$ is also the maximum magnitude that Mehra and Prescott (1985) find reasonable.
in the asset pricing literature. The majority of previous papers suggests that \( \psi \) should be lower than one, but several other papers argue the opposite.\(^{16}\) Given this, we set \( \psi \) to 0.9, a value that implies an elasticity of intertemporal substitution slightly above one. We explain in Section IV that our model implications are quantitatively robust even with an elasticity of intertemporal substitution significantly lower than one. Finally, for \( \delta \), the subjective discount rate, we assign a value of 2%.

We now turn to the belief parameters. We set \( \mu_H \) and \( \mu_L \), the mean price growth in the high and low regimes, to 15% and \(-15\%\), respectively. As we will see later in this section, the probability of the agent’s return expectations approaching boundaries of \( \mu_H \) and \( \mu_L \) is approximately zero. As a result, our model implications are not very sensitive to the choice of \( \mu_H \) and \( \mu_L \). Next, we focus on \( \theta \), the parameter that controls the extent to which the representative agent is behavioral, and \( \chi \) and \( \lambda \), the perceived transition intensities between the high- and low-mean price growth regimes. We calibrate these parameters to match the survey expectations of investors studied in Greenwood and Shleifer (2014). Specifically, we set \( \theta = 0.5 \) and \( \chi = \lambda = 0.18 \) so that the agent’s beliefs match survey data along two dimensions.\(^{17}\) First, for a regression of the agent’s expectations about future stock market returns on past twelve-month returns, our parameter choice allows the model to produce a regression-coefficient and a \( t \)-statistic that match the empirical estimates from surveys. Second, our parameter choice also allows the agent’s beliefs to match the survey evidence on the relative weight investors put on recent versus distant past returns when forming beliefs about future returns. Below we examine these two dimensions in detail.

Empirically, Greenwood and Shleifer (2014) regress survey expectations about future stock market returns on past twelve-month cumulative raw returns across various survey expectations measures. They find that the regression coefficient is positive and statistically significant. To justify our parameter values for \( \theta \), \( \chi \), and \( \lambda \), we want to run the same regression in the context of the model. One caveat, however, is that we are uncertain about what survey respondents think the definition of return is. Does it include the dividend yield or not? And is it a raw return or an excess return? Given this caveat, we examine four measures of return expectations: \( \mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] \), the agent’s expectation about the percentage return on the stock market, \( \mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t \),

\(^{16}\)See Bansal et al. (2012) for a discussion of this point.

\(^{17}\)Recall that \( \theta = 0 \) means the agent is fully rational, whereas \( \theta = 1 \) means the agent is fully behavioral. 0.5 is a natural default value for \( \theta \): it implies that the representative agent is approximately an aggregation of rational and behavioral agents with equal weights.
the agent’s expectation about the percentage return in excess of the interest rate, \( E_t[dP_t^D/(P_t^D dt)] \),
the agent’s expectation about the price growth of the stock market, and \( E_t[dP_t^D/(P_t^D dt)] - r_t \), the
agent’s expectation about the price growth in excess of the interest rate. The latter two measures
are also plausible candidates because investors may not actively think about dividend yields when
answering survey questions.\(^{18}\)

[Place Table 1 about here]

Table 1 reports the regression coefficient, its \( t \)-statistic, the intercept, as well as the adjusted \( R^2 \)-
squared, when regressing each of the four measures of return expectations described above on either
the past twelve-month cumulative raw return or the current log price-dividend ratio, over a sample
of 15 years or 50 years. Each reported value—for instance, the regression coefficient—is averaged
over 100 trials, with each trial being a regression using monthly data simulated from the model.

Here we make two observations. First, the magnitude of the agent’s extrapolative beliefs about
future stock market returns matches the empirical values suggested by Greenwood and Shleifer
(2014). Regressing the agent’s expectation about future price growth (\( E_t[dP_t^D/(P_t^D dt)] \)) on past
twelve-month cumulative raw returns for a 15-year simulated sample, the regression coefficient is
4.0% with a Newey-West adjusted \( t \)-statistic of 8.4. Running the same regression for a 50-year
simulated sample, the regression coefficient is 4.0% with a \( t \)-statistic of 12.1. As a comparison, for
a 5-year sample of sparse data from the Michigan survey, the regression coefficient is 3.9% with a
\( t \)-statistic of 1.68; for a 15-year sample of data from the Gallup survey, the regression coefficient,
after some conversion, is 8% with a \( t \)-statistic of 8.81.

Second, by comparing the regression coefficients and the \( t \)-statistics across the four measures of
return expectations, we find that including the dividend yield in the calculation of return reduces
the regression coefficient by about a half, but does not significantly affect the \( t \)-statistic. Therefore,
even though we model return extrapolation as extrapolating past price growth, the agent also holds
extrapolative expectations about the total return. Furthermore, subtracting the interest rate from
the expectation of returns has little impact on the regression results because of low interest rate
volatility. In summary, across all four measures of return expectations, the agent extrapolates past

\(^{18}\) Hartzmark and Solomon (2017) provide empirical evidence for the idea that investors do not take dividend yield
into account when calculating returns. Barberis et al. (2015) also take this interpretation when calibrating their model
parameters to survey expectations.
stock market returns when forming expectations about future returns. In Section IV, we compare these regression results with those from the rational expectations models of Campbell and Cochrane (1999), Bansal and Yaron (2004), and Bansal et al. (2012).

Our parameter choice of $\theta$, $\chi$, and $\lambda$ is also disciplined by matching the agent’s beliefs with the survey evidence on the relative weight of recent versus distant past returns in determining investors’ return expectations. Specifically, we estimate the following non-linear least squares regression

$$
\text{Expectation}_t = a + b \sum_{j=1}^{n} w_j R^D_{(t-j\Delta t)\rightarrow(t-(j-1)\Delta t)} + \varepsilon_t
$$

(27)

using model simulations, where Expectation$_t$ is the agent’s time-$t$ expectation about stock market returns, $R^D_{(t-j\Delta t)\rightarrow(t-(j-1)\Delta t)}$ is the raw return from time $t - j\Delta t$ to $t - (j - 1)\Delta t$, and $w_j \equiv e^{-\phi(j-1)\Delta t}/\sum_{l=1}^{n} e^{-\phi(l-1)\Delta t}$. In Equation (27), each past realized return is assigned a weight. The weight decreases exponentially the further back we go into the past, and the coefficient $\phi$ measures the speed of this exponential decline. When $\phi$ is high, the agent’s expectation is determined primarily by recent past returns; when it is low, even distant past returns have a significant impact on the agent’s current expectation.

Table 2 reports the regression coefficient $a$, the intercept $b$, the adjusted $R$-squared, and most importantly, the parameter $\phi$. As before, we examine four measures of return expectations, $E_e[t(dP^D_t + D_t dt)/(P^D_t dt)]$, $E_e[t(dP^D_t)/(P^D_t dt)]$, $E_e[(dP^D_t + D_t dt)/(P^D_t dt)] - r_t$, and $E_e[t(dP^D_t)/(P^D_t dt)] - r_t$. Each reported value is averaged over 100 trials, with each trial being a regression using simulated data with a monthly frequency over either 15 years or 50 years. We set $\Delta t$, the time interval for each past return in (27), to 1/12 (one month), and we set $n$, the total number of past returns on the right hand side of (27), to 600.\textsuperscript{19}

Across the four measures of return expectations, the estimation of $\phi$ is stable: it is around 0.42. This value means that a monthly return three years ago is weighted about 25% as much as the most recent return; that is, the agent looks back a couple of years when forming beliefs about future returns. For comparison, Barberis et al. (2015) run the same regression (27) using survey

\textsuperscript{19}We choose $n = 600$ because further increasing $n$ has a minimal impact on the estimated values of the parameter $\phi$ and the adjusted $R$-squared.
data documented in Greenwood and Shleifer (2014); they estimate $\phi$ at a value of 0.44. We choose the values of $\theta$, $\chi$, and $\lambda$ such that the model generates about the same estimate of $\phi$ as surveys.

The literature has not reached consensus on the value of $\phi$. On the one hand, Greenwood and Shleifer (2014) and Kuchler and Zafar (2016) find that investor expectations depend only on recent returns. On the other hand, Malmendier and Nagel (2011, 2013) and Vanasco et al. (2015) suggest that distant past events may also play an important role when investors form beliefs. Reconciling this discrepancy is beyond the scope of the paper. Here, we provide two possible explanations. First, investors may simultaneously adopt two mechanisms when forming beliefs: one that focuses on recent past events such as daily stock market fluctuations, the other that focuses on infrequent but salient events such as a stock market crash. Second, the horizon over which investors form expectations may affect how far they look back into the past. Survey expectations are mainly about stock market returns over the next six to twelve months, and investors may therefore look back only a couple of years. Equity holdings, on the other hand, may span the next couple of decades, and investors may therefore look back a few decades when making decisions about equity holdings.

[Place Table 3 about here]

We summarize our default parameter values in Table 3. In Section III, we further provide a comparative statics analysis to examine the sensitivity of the model’s implications to changes in these parameter values.

II.2. Building blocks

We analyze the two building blocks for our model implications. First, we examine a set of important equilibrium quantities. We then look at the steady-state distribution of sentiment.

Figure 1 plots the price-dividend ratio of the stock market $f$, the volatility of stock market returns $\sigma^p_D$, the rational expectation about the log excess return $E[r^{D,e}]$ (the conditional equity premium), and the interest rate $r$, each as a function of the sentiment variable $S$.

[Place Figure 1 about here]

Figure 1 shows that the model generates substantial excess volatility: in the model, the volatility
of dividend growth is 11%, while the volatility of stock market returns is typically above 20%. This result highlights a strong interaction between return extrapolation and Epstein-Zin preferences. With return extrapolation, high past returns lead the agent to forecast high future returns. Under Epstein-Zin preferences, the separation between the elasticity of intertemporal substitution and risk aversion gives rise to a strong intertemporal substitution effect. As a result, the agent’s forecast of high future returns leads her to push up the current price significantly, generating excess volatility. The interaction between beliefs and preferences is quantitatively important. Without return extrapolation, Epstein-Zin preferences alone with i.i.d. dividend growth and consumption growth do not lead to any excess volatility. Without Epstein-Zin preferences—that is, setting both $\psi$ and $\gamma$ to 10 while keeping all the other parameter values unchanged—return extrapolation alone only generates 17% of the excess volatility.

Figure 1 also shows that the model generates a strong procyclical pattern for the price-dividend ratio of the stock market. As described in Section I, an expectations transmission mechanism leads the agent to forecast high future dividend growth after high past returns. Given that the stock market price is the sum of the agent’s expectations about discounted future dividends, high expectations about future dividend growth lead the agent to push up the current price, making the price-dividend ratio procyclical with respect to changes in sentiment.

In addition, the model generates a strong countercyclical pattern for the equity premium. Suppose the stock market has had high past returns. The agent’s expectation about future returns then increases, pushing up the stock market price relative to dividends. Given that the sentiment variable on average tends to revert back to its mean, future stock market prices tend to decrease, leading to low future returns. At the same time, the agent’s high expectation about future returns also makes her optimistic about future consumption growth, although to a much lesser extent. This in turn causes the agent to push up the equilibrium interest rate. Together, both a low (rational) expectation of stock market returns and a high interest rate contribute to a low equity premium during high-sentiment periods. Of these two forces, the first one dominates: moving $S_t$ from its mean to the top 25% percentile level leads to a total decrease of 9.7% for the equity premium, out of which 9.5% comes from the decrease in the expected log return of the stock market.

The second building block for the model implications is the objectively measured steady-state distribution of sentiment. Figure 2 plots this steady-state distribution. Under the true probability
measure, sentiment exhibits a strong degree of mean reversion for two reasons. First, the agent believes that sentiment will naturally revert: the agent thinks that the expected growth rate of stock market prices tends to switch over time from one regime to the other. Second, the agent’s expectations are incorrect: precisely when the agent thinks that the future price growth is high, future price growth tends to be low endogenously, causing sentiment to mean-revert faster than what the agent perceives. Altogether, the steady-state distribution has a mean of 2.0% and a standard deviation of 2.7%: the chance of sentiment approaching the extreme values of $\mu_H$ and $\mu_L$ is close to zero.

The mean reversion of sentiment helps to keep the model stable. Recall that the interaction between return extrapolation and Epstein-Zin preferences gives rise to a feedback loop: if the current price growth is high, the agent thinks that future price growth will also be high, which leads her to push up the current price, causing the current price growth to further rise, and so on. In general, there is a danger that this feedback loop could explode. However, in our model, we assume the agent believes that sentiment tends to revert back to its mean, so she will not become infinitely optimistic about future price growth of the stock market. Therefore, the cumulative impact of this feedback loop on investor expectations and asset prices remains finite.

II.3. Model implications on asset prices

We evaluate the model’s fit with empirical facts by analyzing its implications for asset prices. We begin with examining the model’s fit with the long-run properties of stock market prices and returns. Table 4 reports the model’s predictions for six important moments, and compares them side by side with the empirical values. In general, the model matches the facts: it generates a high equity premium, significant excess volatility, a Sharpe ratio similar to the empirical value, an interest rate that has a low level and low volatility, and a price-dividend ratio whose average level is close to the empirical one.

To understand these results, recall that the interaction between return extrapolation and
Epstein-Zin preferences give rise to a feedback loop, which in turn generates significant excess volatility. Three factors affect the perceived long-run equity premium. First and most intuitively, excess volatility interacts with the agent’s risk aversion and causes the perceived equity premium to increase. Second, return extrapolation gives rise to perceived persistence of both the aggregate dividend process and, to a lesser extent, the aggregate consumption process.\textsuperscript{20} Under Epstein-Zin preferences, this perceived persistence is significantly priced, pushing up the perceived equity premium. Finally, the separation between the elasticity of intertemporal substitution and the relative risk aversion keeps the equilibrium interest rate low and hence helps to keep the equity premium high.

The true long-run equity premium is higher than the perceived long-run equity premium. When the agent is optimistic about future price growth, her risk aversion limits the decline of the dividend yield component of the return expectation by counteracting the increase in the agent’s demand for the stock market; this makes it less likely for the true equity premium to be lower than the perceived equity premium. On the other hand, when the agent is pessimistic about future price growth, her risk aversion induces an increase of the dividend yield component of the return expectation by exacerbating the decrease in the agent’s demand for the stock market—this is to offset the low perceived price growth component of the return expectation, making it more likely for the true equity premium to be higher than the perceived equity premium. This asymmetry in the interaction between the agent’s beliefs and risk aversion implies that the true long-run equity premium is above the perceived long-run equity premium. Quantitatively, the true average equity premium is 4.9% when measured as the rational expectation of log excess returns; it is 8.6% when measured as the rational expectation of excess returns $E[(dP_t^D + D_t dt)/(P_t^D dt) - r_t]$. At the same time, the perceived long-run equity premium is 1.6% when measured as the subjective expectation of log excess returns $E^e[(dP_t^D + D_t dt)/(P_t^D dt) - r_t]$.

The model generates a negative equity premium during high-sentiment periods: the equity premium averaged over the top quartile of the sentiment distribution is $-13.05\%$. This implication is consistent with the recent empirical findings of Greenwood and Hanson (2013), Baron and Xiong\textsuperscript{20}

\textsuperscript{20}The agent is averse to persistent shocks when $\gamma > \psi$; with our choice of parameter values, this condition is satisfied.
(2015), Cassella and Gulen (2017), and Yang and Zhang (2016): these papers document that the expected excess return turns negative during high-sentiment periods. Note that in general, rational expectations models—for instance, long-run risks models and habit formation models—do not generate a negative equity premium at any time.\textsuperscript{21} In our model, however, subjective expectations and objective expectations of stock market returns differ significantly during high-sentiment or low-sentiment periods: when the sentiment level is high, the agent expects high stock market returns moving forward, but precisely because of her incorrect beliefs, future stock market returns are low on average, generating a negative equity premium.

We now further examine the model’s fit with the dynamic behavior of stock market prices. We start by investigating the source of stock market movements. Figure 3 plots both the agent’s expectation about stock market returns, $E^e[(dP^D_t + D_t dt)/(P^D_t dt)]$ and the agent’s expectation about dividend growth, $E^e[dD_t/(D_t dt)]$, each as a function of the sentiment variable $S$.

\textbf{[Place Figure 3 about here]}

The expectations transmission mechanism described before suggests an important pattern: the agent’s expectation about price growth is less responsive to changes in sentiment than her expectation about dividend growth. Note that the total return equals the sum of the price growth and the dividend yield, and that the dividend yield decreases as sentiment increases. Therefore, the agent’s expectation about stock market returns is even less responsive to changes in sentiment compared to her expectation about price growth. Quantitatively, Figure 3 suggests that a one-standard deviation (2.7\%) increase in sentiment from its mean (2.0\%) pushes up the agent’s expectation about stock market returns from 7.44\% to 8.22\%—a small increase of 0.78\%—while it pushes up the agent’s expectation about dividend growth from $-0.04\%$ to $6.48\%$—a much larger increase of $6.52\%$. Also, recall from Figure 1 that a one-standard deviation increase in sentiment from its mean pushes up the price-dividend ratio of the stock market from 19.16 to 21.75. Putting these results together, we observe that the movements in the price-dividend ratio primarily comes from changes in subjective expectations about future dividend growth.

To analyze the source of stock market movements statistically, we follow the procedure in\textsuperscript{21}Strictly speaking, a rational expectations model can still generate a negative equity premium if the stock market negatively correlates with some other risk factors in the agent’s portfolio and therefore serves as a diversification device.
Campbell and Shiller (1988) to decompose the log price-dividend ratio of the stock market:

\[ \ln(P_t^D / D_t) \approx \sum_{j=0}^{\infty} \xi^j \left( \Delta d_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)} - r^D_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)} \right) - (\ln(\xi) + (1 - \xi)\zeta)/(1 - \xi), \]

(28)

where \( \zeta \) is the in-sample average of the annual log dividend-price ratio, \( \xi = e^{-\zeta}/(\Delta t + e^{-\zeta}) \), \( \Delta d_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)} \) is the log dividend growth from time \( t+j\Delta t \) to \( t+(j+1)\Delta t \), and \( r^D_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)} \) is the log gross return over the same period.

Equation (28) says that the movement of price-dividend ratio comes from either the movement of future dividend growth or the movement of future returns. The first component is called “cash flow news” by empiricists, and the second component is called “discount rate news.” As pointed out by de la O and Myers (2017), there are two approaches to empirically address the relative importance of these two components. The first approach looks at future realized dividend growth and returns, and computes

\[ 1 \approx \frac{\text{Cov} \left( \sum_{j=0}^{\infty} \xi^j \Delta d_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)}, \ln(P_t^D / D_t) \right)}{\text{Var} \left( \ln(P_t^D / D_t) \right)} + \frac{\text{Cov} \left( -\sum_{j=0}^{\infty} \xi^j r^D_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)}, \ln(P_t^D / D_t) \right)}{\text{Var} \left( \ln(P_t^D / D_t) \right)}. \]

(29)

The first term on the right hand side of (29), \( \text{CF}_{\text{objective}} \), is the contribution of changes in cash flow news to stock market movements, and the second term, \( \text{DR}_{\text{objective}} \), is the contribution of changes in discount rate news to the stock market movements. These two terms approximately sum to one. By using future realized dividend growth and returns, this approach effectively imposes rational expectations. Most empirical studies that have conducted a Campbell-Shiller decomposition take this approach.

The second approach, on the other hand, takes subjective expectations on both sides of (28)
and computes

\[
1 \approx \frac{\text{Cov} \left( \sum_{j=0}^{\infty} \xi^j \Delta d_{t+(j+1)\Delta t}, \ell n \left( \frac{P_t^D}{D_t} \right) \right)}{\text{Var} \left( \ell n \left( \frac{P_t^D}{D_t} \right) \right)} + \frac{\text{Cov} \left( -\sum_{j=0}^{\infty} \xi^j r^D_{t+(j+1)\Delta t}, \ell n \left( \frac{P_t^D}{D_t} \right) \right)}{\text{Var} \left( \ell n \left( \frac{P_t^D}{D_t} \right) \right)}.
\]

This approach examines the relationship between stock market movements and subjective expectations about future stock market returns and future dividend growth. The first term on the right hand side of (30), \( CF_{\text{subjective}} \), is the contribution of changes in subjective expectations about cash flow news to stock market movements, and the second term, \( DR_{\text{subjective}} \), is the contribution of changes in subjective expectations about discount rate news to stock market movements.

[Place Table 5 about here]

Table 5 reports the four coefficients, \( CF_{\text{objective}} \), \( DR_{\text{objective}} \), \( CF_{\text{subjective}} \), \( DR_{\text{subjective}} \), as well as their corresponding adjusted \( R \)-squared. These coefficients and \( R \)-squared are calculated using 10,000 years of monthly data simulated from the model. By using future realized dividend growth and stock market returns and therefore imposing rational expectations, we obtain \( DR_{\text{objective}} = 0.98 \) with a \( R \)-squared of 0.209 and \( CF_{\text{objective}} = 0.02 \) with a \( R \)-squared of \( 1.2 \times 10^{-4} \). Therefore, the model replicates the empirical finding of the volatility test literature that the variation of the price-dividend ratio comes primarily from discount rate variation (see Cochrane (2008) and Cochrane (2011)).

On the other hand, by relaxing the rational expectations assumption and using the agent’s subjective expectations about dividend growth and returns, we obtain \( DR_{\text{subjective}} = -0.08 \) with a \( R \)-squared of 0.982 and \( CF_{\text{subjective}} = 1.08 \) with a \( R \)-squared of 0.984. This result draws a very different picture from earlier findings that use realized dividend growth and returns: changes in the agent’s subjective expectations about future cash flow news explain most of the variation of the price-dividend ratio, whereas changes in the rational expectations about future cash flow news do not. Empirically, de la O and Myers (2017) find that \( DR_{\text{subjective}} = -0.09 \) and \( CF_{\text{subjective}} = 1.09 \). These values match the theoretical values our model delivers.
Importantly, the fact that prices in our model are mainly correlated with cash flow expectations is a consequence of the Campbell-Shiller accounting identity; return expectations, not cash flow expectations, are the cause of price movements. As a result, our model can make sense of both the empirical findings of de la O and Myers (2017) on cash flow expectations and the empirical findings of Greenwood and Shleifer (2014) on return expectations.

Next, we examine the model’s implications for the predictability of stock market returns. Empirically, Campbell and Shiller (1988) and Fama and French (1988) document that a regression of future log excess returns on the current log price-dividend ratio gives a negative and significant regression coefficient. Moreover, the predictive power of the price-dividend ratio is greater when future returns are calculated over longer horizons.

[Place Table 6 about here]

Table 6 reports the regression coefficient $\beta_j$ and the adjusted $R^2$-squared for a regression of the log excess return of the stock market from time $t$ to time $t+j$ on the current log price-dividend ratio

$$r_{t\rightarrow t+j}^{D,e} = \alpha_j + \beta_j \ln(P_t^D / D_t) + \epsilon_{j,t}$$

over various time horizons $j$. We calculate the regression coefficients and the $R^2$-squared using 10,000 years of monthly data simulated from the model, and compare them side by side with the empirical values. Consistent with the data, $\beta_j$ is negative and its magnitude increases as the time horizon $j$ increases. The intuition for this model implication is as follows. A sequence of high past returns raises the agent’s expectation about future returns, which in turn pushes up the current price-dividend ratio. Since the agent’s expectation—the sentiment variable—tends to revert back to its mean, subsequent returns are low on average, giving rise to a negative regression coefficient in (31).

The magnitudes of the regression coefficient and the $R^2$-squared generated from the model are broadly consistent with the empirical values. One difference, however, is that in the model, the $R^2$-squared begins to decrease as the time horizon $j$ increases beyond three years, whereas in the data, the $R^2$-squared keeps rising over longer horizons. To understand this difference, recall that we calibrate the model to the survey expectations by setting $\theta$ to 0.5 and setting $\lambda$ and $\chi$ to 0.18: the
agent looks back a couple of years when forming beliefs about future returns. Given this parameter choice, the mean reversion of sentiment tends to occur over the first few years. Therefore, over longer horizons, no additional mean reversion in the agent’s beliefs contributes to the predictability of stock market returns.

We now study the autocorrelations of asset prices. Empirically, log excess returns exhibit negative autocorrelations at lags longer than one year (Poterba and Summers (1988)), and price-dividend ratios are highly persistent at short lags. Table 7 presents the theoretical values and empirical values for these autocorrelations. It shows that the model captures both facts, but only qualitatively: the negative autocorrelations of log excess returns are generated by the mean reversion of sentiment, whereas the persistence of the price-dividend ratio is driven by the persistence of sentiment. Quantitatively, however, the model generates a persistence for the price-dividend ratio that is much lower than the empirical value. The available survey evidence suggests that investors focus on just the past year or two when forming beliefs about future returns. However, to match the empirical persistence of the price-dividend ratio, the agent needs to form beliefs about future returns based on many years of past returns.

[Place Table 7 about here]

We leave a careful reconciliation of the survey expectations about stock market returns and the observed persistence of the price-dividend ratio for future research. One possibility, however, is to introduce an interaction between financial frictions and investor sentiment. As documented by Lopez-Salido, Stein, and Zakrajsek (2016), various types of financial frictions are empirically more persistent than investor sentiment. As a result, financial frictions can serve as a propagation mechanism to prolong the impact of survey expectations on asset prices.

Finally, we investigate the model’s implications for the correlation between stock market returns and consumption growth. Empirically, Hansen and Singleton (1982, 1983) document that this correlation is low. Nonetheless, most consumption-based asset pricing models generate a high, if not perfect, correlation between stock market returns and consumption growth. By imposing rational expectations, these models leave the consumption-based pricing kernels as the only source of stock market movements. In other words, stock market movements are driven by changes in consumption. As a result, the correlation between stock market returns and consumption growth
Table 8 reports both the model-implied values and the empirical values for the correlation between consumption growth and stock market returns. Consistent with the data, the model produces a low correlation: the correlation between annual log consumption growth and annual log excess returns is 0.19 in the model, and similarly it is 0.09 in the data. In comparison, the model of Campbell and Cochrane (1999) generates a correlation of 0.79, a much higher value. In our model, the agent extrapolates past price growth of the stock market, but extrapolates past consumption growth to a much lesser extent: the bias in the agent’s beliefs about consumption growth is small. In other words, the comovement between the agent’s beliefs about returns and her beliefs about consumption growth is low. As a consequence, the model produces a low correlation between consumption growth and stock market returns.

The model also generates a small but negative correlation between the current consumption growth and the stock market return in the subsequent period, an implication that is consistent with the data. Recall that consumption growth and dividend growth are positively correlated. If the current consumption growth is high, dividend growth is also high on average, which leads the agent to drive up the current price, increasing the current return and the current sentiment level. In the subsequent period, sentiment reverts towards its mean value, giving rise to a low stock market return.

We conclude this section by discussing the role of rational arbitrageurs. The model has a representative agent whose beliefs are biased. One natural question to ask is: if we introduce rational arbitrageurs, to what extent can they counteract the mispricing caused by the behavioral agent and therefore attenuate the significance of the model implications? Developing such a two-agent model is beyond the scope of the paper. However, three observations suggest that our model implications will remain intact after taking rational arbitrageurs into account.

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22 One exception is Barberis et al. (2001). They use “narrow framing”, the notion that investors may evaluate financial risks in isolation from consumption risks, to generate a low correlation between stock market returns and consumption growth. Specifically, they use power utility as the agent’s preferences over consumption, but use prospect theory developed by Kahneman and Tversky (1979) as the agent’s preferences over financial wealth.
First, in an economy with both rational and behavioral agents who have recursive preferences, the behavioral agents may eventually dominate the market: there is a positive probability that they take up most of the wealth in the economy in the long run. This is a key finding in Borovicka (2016). It suggests that our model’s implications can be the limiting implications of a model with both rational and behavioral agents in the initial period. Second, heterogeneity in investor beliefs is an additional source for excess volatility, so it can further amplify—rather than attenuate—our model implications.23 Lastly, as pointed out by Barberis et al. (2015), extrapolative expectations are persistent in a dynamic model, which means that the behavioral agents have persistently heightened demand for the stock market following high stock market returns. The persistence of this demand prevents near-future stock market returns from becoming too low, which reduces the incentive of the rational agents to counteract mispricing. In other words, the persistence of extrapolative expectations limits the overall impact of rational arbitrageurs on asset prices.

III. Comparative Statics

In this section, we examine the sensitivity of the model implications to changes in parameter values. We focus on parameters that either have dispersed estimates in the literature or cannot be directly observed from the data. Specifically, for the utility parameters, we study how changes in γ, the relative risk aversion parameter, and ψ, the reciprocal of the elasticity of intertemporal substitution, affect the equity premium and the volatility of stock market returns. For the belief parameters, we further look at how changes in θ affect the equity premium, the volatility of returns, the price-dividend ratio, and the average interest rate. We also examine how changes in θ, χ, and λ affect return predictability and the persistence of the price-dividend ratio.

III.1. Utility parameters

Figure 4 plots the long-run average of the equity premium and the volatility of stock market returns, each as a function of γ or ψ. The relative risk aversion is positively related to the equity premium but negatively related to the volatility of returns. Lower risk aversion naturally leads the agent to require a lower equity premium for risk compensation; reducing γ from 10 to 5, the

23See Xiong (2013) for more discussion about this amplification mechanism.
model still explains 75% of the observed equity premium. At the same time, lower risk aversion strengthens the feedback loop since it increases the agent’s demand for risky assets. Therefore, it leads to higher return volatility.

[Place Figure 4 about here]

Within the examined range, changes in the elasticity of intertemporal substitution does not significantly affect the average equity premium and the average volatility of returns. As a result, our model implications are quantitatively robust to changes in $\psi$. We provide a detailed explanation of this observation in the next section when we compare our model with the model of Bansal and Yaron (2004).

III.2. Belief parameters

Belief Parameters.—Figure 5a plots the long-run average of the equity premium, the volatility of stock market returns, the price-dividend ratio, and the interest rate, each as a function of $\theta$, the parameter that measures the extent to which the agent is behavioral. Setting $\theta$ to zero reduces the model to its rational benchmark. In this case, the equity premium is 0.23%; the return volatility equals the fundamental volatility of 11%; the price-dividend ratio stays constant at 135; and the interest rate stays at 2.35%. With $\theta = 0$, the model fails to match the long-run properties of the stock market. Increasing $\theta$ from zero allows the feedback loop described above to emerge, which generates excess volatility, pushes up the perceived equity premium, and significantly reduces the price-dividend ratio. At the same time, a higher $\theta$—and the agent’s more extrapolative beliefs about stock market returns—leads the agent to perceive a higher persistence in dividend growth, which, under Epstein-Zin preferences, is significantly priced, causing the perceived equity premium to further rise. Finally, a higher $\theta$ pushes the true equity premium further above the perceived one.

[Place Figure 5a about here]

Overall, an 1% increase in $\theta$ leads to a 0.09% increase in the equity premium and a 0.29% increase in the volatility of returns. On the other hand, the effect of $\theta$ on the interest rate is small. A higher $\theta$ increases the extrapolation bias in the agent’s beliefs about stock market returns, but
does not significantly affect her beliefs about consumption growth which determines the equilibrium interest rate.

Figure 5b examines how changes in $\theta$, $\chi$, and $\lambda$ affect the predictability of stock market returns and the persistence of the price-dividend ratio. Here the predictability of returns is measured by the slope coefficient for a regression of the next year’s log excess return on the current log price-dividend ratio, and the persistence of the price-dividend ratio is measured by the one-year autocorrelation of log price-dividend ratios.

Figure 5b shows that higher values of $\theta$, $\chi$, and $\lambda$ lead to stronger predictability of returns and a lower persistence of the price-dividend ratio. Higher values of $\chi$ and $\lambda$—that is, higher perceived transition intensities between the high- and low-mean price growth regimes—suggest that the agent focuses on a shorter history of past return realizations when forming beliefs about future returns. A higher value of $\theta$ has a similar implication: it suggests that the agent exhibits a stronger extrapolation bias, which means that the agent deviates more from a rational agent whose beliefs depend on both recent and distant past returns. In other words, with a higher $\theta$, the agent relies more heavily on recent past returns when forming beliefs about future returns. Therefore, higher values of $\theta$, $\chi$, and $\lambda$ all lead to a stronger degree of mean reversion in sentiment, which in turn gives rise to stronger predictability of returns and a lower persistence of the price-dividend ratio.

The comparative statics results in Figure 5b are consistent with recent empirical findings. At the aggregate level, Cassella and Gulen (2017) find that during periods when investors’ expectations about future returns depend on both recent and distant past returns, the price-dividend ratio does not strongly predict the next year’s return. Conversely, during periods when investors’ expectations depend primarily on recent past returns, the price-dividend ratio strongly predicts the next year’s return. In our model, higher values of $\theta$, $\chi$, and $\lambda$ lead to a higher $\phi$, and therefore the agent’s expectations about future returns depend more heavily on recent past returns. In the meantime, they also lead to stronger return predictability. Together, the model implies that when the agent forms beliefs based on a short history of past returns, the predictability of returns is strong. To give an example, increasing $\theta$ from 0.05 to 0.5 changes $\phi$ from 0.37 to 0.43. At the same time,
it changes $\beta_1$, the slope coefficient for a regression of the next year’s log excess return on the current log price-dividend ratio, from $-0.32$ to $-0.72$; the corresponding $R$-squared increases from 0.001 to 0.13. In the cross-section, Da, Huang, and Jin (2017) show that stocks associated with a larger extrapolation bias—beliefs of forecasters on these stocks depend more strongly on recent past returns—exhibit stronger return reversals. Applying our model to individual stocks gives the same prediction.

IV. Comparison with Rational Expectations Models

In this section, we provide a quantitative comparison between our model and several models with rational expectations. First, we look at a rational expectations model that is most analogous to our behavioral model, one in which the regime-switching process characterized in Section I represents the true data generating process. We then examine differences between our model and the rational expectations models of Bansal and Yaron (2004) and Bansal et al. (2012): we focus on the long-run risks models because these are the models most related to ours.

IV.1. The true regime-switching model

The rational expectations model most analogous to our model is the one that assumes the regime-switching process characterized in equations (6) and (7) is the true data generating process. In this case, the true evolution of the stock market price is

$$dP_t^D/P_t^D = [(1 - \theta)g_D + \theta \hat{\mu}_{S,t}]dt + \sigma_P^D(S_t)d\omega_t^P,$$

where $\omega_t^P$ is a standard Brownian motion. As with the behavioral model, the agent in this model does not directly observe the latent variable $\hat{\mu}_{S,t}$. Instead, she uses past stock market prices to form a Bayesian estimate of $\hat{\mu}_{S,t}$: $S_t = E[\hat{\mu}_{S,t}|F_t^P]$. That is, the perceived evolution of stock market price in (8) is fully rational. We further assume that the perceived dividend process (10) and the perceived consumption process (15) are also rational.

By construction, this rational expectations model shares the same equilibrium prices with our behavioral model: the solutions to the differential equations of (17) and (18) also apply to this model. Nonetheless, these two models have statistical properties that are significantly different.
One difference, for instance, lies in the models’ implications for the predictability of stock market returns.

[Place Table 9 about here]

Table 9 reports the regression coefficient $\beta_j$ and the adjusted $R$-squared for a regression of the log excess return of the stock market from time $t$ to time $t + j$ on the current log price-dividend ratio $\ln(P_t^D/D_t)$ over various time horizons $j$ (one to seven years), now using the true regime-switching model. Table 9 shows that the model fails to produce the predictability of stock market returns documented in Campbell and Shiller (1988) and Fama and French (1988): both the regression coefficients and the $R$-squared are close to zero. In contrast, Table 6 shows that the behavioral model produces the observed predictability of stock market returns.

[Place Figure 6 about here]

The lack of return predictability in the true regime-switching model is due to two reasons. First, with rational expectations, the agent’s beliefs about stock market returns are on average correct. Therefore, following high past price growth, the agent properly anticipates high future price growth, which pushes down the dividend yield in equilibrium, leading to flat returns in subsequent periods. Second, with rational expectations, the evolution of sentiment is more volatile and less mean-reverting compared to the one in the behavioral model. This further reduces the statistical significance of the return predictability regression. Figure 6 plots the evolution of sentiment in the true regime-switching model: the standard deviation of the sentiment distribution is 4.7%, almost twice as high as the standard deviation of sentiment in the behavioral model.

IV.2. The long-run risks models

Table 10 reports the regression coefficients and $t$-statistics when regressing four measures of rational expectations of return—raw return or excess return, with or without dividend yield—either on the past twelve-month return or on the current log price-dividend ratio. The regressions are based on simulated data from Bansal and Yaron (2004). Interestingly, their model generates, to some extent, extrapolative expectations about future raw returns: a regression of the agent’s expectations—this is also the rational expectations—about the next twelve-month total returns on
past twelve-month total returns yields a coefficient of 2.5% with a \( t \)-statistic of 2.4 for a 15-year simulated sample; the regression coefficient is 3.0% with a \( t \)-statistic of 3.8 for a 50-year simulated sample.

[Place Table 10 about here]

In Bansal and Yaron (2004), dividend growth and consumption growth share a stochastic yet persistent component. High past stock market returns are typically caused by positive shocks on this common component, which, given its persistence, implies high dividend growth moving forward, hence pushing up future raw returns. This mechanism gives rise to the agent’s extrapolative beliefs about future raw returns. Nevertheless, precisely because consumption growth and dividend growth share a common persistent component, high dividend growth tends to coincide with high consumption growth, which implies a high interest rate in equilibrium. That is, when raw returns are high, the interest rate is also high. As a result, the agent’s expectation about future excess returns is not extrapolative: regressing the agent’s expectation about future excess returns on past returns, the regression coefficient and the \( t \)-statistic are both close to zero.

These regression results highlight one fundamental difference between our model and the model of Bansal and Yaron (2004). In our model, the agent extrapolates past returns on the stock market, but extrapolates past consumption growth much less. The comovement between the agent’s beliefs about returns and her beliefs about consumption growth is low. On the contrary, in Bansal and Yaron (2004), the comovement between the agent’s beliefs about stock market returns—they rationally drive returns—and her beliefs about consumption growth—they determine the interest rate in equilibrium—is high. Therefore, the agent in our model has extrapolative beliefs about both raw returns and excess returns, whereas the agent in Bansal and Yaron (2004) has extrapolative beliefs only about raw returns.

Furthermore, as shown in Figure 4, our model produces an equity premium and return volatility that do not vary significantly with respect to changes in the elasticity of intertemporal substitution. On the contrary, the long-run risks models cannot generate a high equity premium with a low elasticity of intertemporal substitution. For instance, setting the elasticity of intertemporal substitution to 0.5, our model generates an equity premium of 7.1% (measured as the rational expectation of excess returns), while the model of Bansal and Yaron (2004) produces an equity premium between
1% and 2%. Given this contrast, our model does not face the challenge of defending a greater than one elasticity of intertemporal substitution.

To understand this model difference, first note that the sentiment variable, the state variable in our model, is much less persistent than the stochastic component of dividend and consumption growth in Bansal and Yaron (2004), allowing the equilibrium interest rate in our model to be less responsive to changes in the elasticity of intertemporal substitution. On the other hand, the perceived dividend growth in our model depends more strongly on the state variable of sentiment, leading to a higher perceived equity premium; in the model of Bansal and Yaron (2004), dividend growth depends less strongly on the stochastic growth component. In addition, the true average equity premium in our model is above the perceived one, allowing the equity premium to be high even when the elasticity of intertemporal substitution is low.

Table 11 repeats the regression analyses of Table 10 using the model of Bansal et al. (2012). Compared to the original model of Bansal and Yaron (2004), Bansal et al. (2012) introduces additional time variation in the long-run risks that further reduces the model’s ability to generate extrapolative expectations: when regressing the expectation of raw returns on past returns, the coefficient now becomes insignificant.

[Place Table 11 about here]

We complete our discussion in this section by making two remarks. First, the persistence introduced by the long-run risks models on consumption growth and dividend growth leads to excess predictability, the notion that future consumption and dividend growths are excessively predicted by current variables such as the price-dividend ratio (see Beeler and Campbell (2012) and Collin-Dufresne et al. (2016b) for a detailed discussion). Our model does not give rise to excess predictability: return extrapolation in the model only generates perceived but not true long-run risks, and therefore the true consumption and dividend growths remain unpredictable.

Second, the rational expectations models and our model generate different implications regarding the source of stock market movements. In models with rational expectations, the volatility of the price-dividend ratio comes from either the variation in discount rate news or the variation in

\footnote{Bansal and Yaron (2004) set their “leverage parameter” $\phi$ at 3.5. In comparison, with return extrapolation, our model effectively sets $\phi$ at 14.5, a much higher value.}
cash flow news: when using either the agent’s subjective expectations or the rational expectations, a Campbell-Shiller decomposition yields the same result. In our model, they yield very different results. When using rational expectations, the variation in discount rate news explains most of the volatility of the price-dividend ratio. However, when using the agent’s subjective expectations, changes in the expectations of cash flow news explain most of the volatility of the price-dividend ratio.

V. Fundamental Extrapolation

A literature in behavioral finance focuses on fundamental extrapolation, the notion that some investors hold extrapolative expectations about fundamentals such as dividend growth or GDP growth (Barberis et al. (1998); Fuster et al. (2011); Choi and Mertens (2013); Alti and Tetlock (2014); Hirshleifer et al. (2015)). In this section, we provide a quantitative comparison between our model and a model with fundamental extrapolation. To facilitate the comparison, we keep the two models almost identical. The only difference is that, in the model with fundamental extrapolation, sentiment is constructed from past dividend growth, whereas in the model with return extrapolation, sentiment is constructed from past price growth. Below we first briefly describe the key assumptions in this fundamental extrapolation model. We then discuss the model implications.

V.1. Model setup

With fundamental extrapolation, the agent believes that the expected growth rate of dividends—in contrast to the expected growth rate of stock market prices in the case of return extrapolation—is governed by $(1 - \theta)g_D + \theta \mu_{S,t}$, where $\mu_{S,t}$ is a latent variable that follows a regime-switching process described in Section I. The agent does not directly observe the latent variable $\mu_{S,t}$. Instead, she computes its expected value given the history of past dividend growth: $S_t \equiv \mathbb{E}[\mu_{S,t} | F^D_t]$. She then applies the optimal filtering theory and derives

$$dS_t = (\lambda \mu_H + \chi \mu_L - (\lambda + \chi)S_t)dt + \sigma_D^{-1} \theta (\mu_H - S_t)(S_t - \mu_L)d\omega^e_t$$

$$\equiv \mu_S^e(S_t)dt + \sigma_S(S_t)d\omega^e_t,$$

(33)
where $d\omega_t^e \equiv [dD_t/D_t - (1 - \theta)g_D dt - \theta S_t dt]/\sigma_D$ is a standard Brownian innovation term from the agent’s perspective. That is, she perceives the evolution of dividend as

$$dD_t/D_t = g_D^e(S_t)dt + \sigma_D d\omega_t^e,$$

where

$$g_D^e(S_t) = (1 - \theta)g_D + \theta S_t.$$

In other words, the agent’s expectation about dividend growth $g_D^e(S_t)$ is a linear combination of a rational component $g_D$ and a sentiment component $S_t$ constructed from past dividend growth. On the other hand, the perceived evolution of stock market price can be derived as

$$dP_t^D/P_t^D = \mu_{P^D, e}^D(S_t)dt + \sigma_P^D(S_t)d\omega_t^e,$$

where

$$\sigma_P^D(S) = \sigma_D + (f'/f)\sigma_D^{-1}\theta(\mu_H - S)(S - \mu_L),$$

$$\mu_{P^D, e}^D(S) = (f'/f)\mu_S^e + \frac{1}{2}(f''/f)\sigma_S^2 + (1 - \theta)g_D + \theta S - \sigma_P^2 + \sigma_D\sigma_P^D(S).$$

As before, $f$ is defined as the price-dividend ratio of the stock market.

As with the return extrapolation model, equations (17) and (18) determine $f$ and $l$, the price-dividend ratio and the wealth-consumption ratio, except that $\mu_S$, $\sigma_S$, $g_D^e$, $\mu_{P^D, e}^D$, and $\sigma_P^D$ are now from (33), (35), and (37).

**V.2. Model implications**

We first examine the model’s implications for investor expectations. Table 12 reports the regression coefficient, its $t$-statistic, the intercept, as well as the adjusted $R$-squared, when regressing the four measures of return expectations on either the past twelve-month cumulative raw returns or the current log price-dividend ratio. With fundamental extrapolation, the regression coefficient on past returns and the $t$-statistic are both close to zero.
Suppose past dividend growth has been high. On the one hand, it results in high past returns. On the other hand, fundamental extrapolation leads the agent to expect high dividend growth moving forward, but not high returns: the agent’s expectation of returns is determined by her perception of risk, which does not change significantly following high dividend growth. Therefore, a fundamental extrapolation model with a representative agent faces challenges in explaining survey expectations about returns. In contrast, our return extrapolation model is constructed to explain these survey expectations.25

Table 13 analyzes the model’s fit with the long-run properties of the stock market. Using the same parameters that allow the return extrapolation model to well explain the important moments of the stock market, the fundamental extrapolation model generates lower excess volatility and a much lower equity premium. Quantitatively, fundamental extrapolation produces 74.4% of excess volatility and 37.9% of the equity premium that return extrapolation produces.

This quantitative comparison highlights the importance of the feedback loop described above in matching asset pricing facts. With return extrapolation, the feedback loop emerges because extrapolative beliefs are applied to the stock market return, a variable that is endogenously determined in equilibrium. With fundamental extrapolation, however, the feedback loop is absent because extrapolative beliefs are applied to dividend growth, a variable that is exogenous in the model: high past dividend growth makes the agent optimistic about future dividend growth and therefore pushes up the current price, but the higher price does not further affect the agent’s beliefs about future dividend growth.

The feedback loop also points to a methodological contribution of the paper. Equation (33) shows that in a fundamental extrapolation model, sentiment $S$, the state variable that drives asset prices dynamics, can be exogenously specified without solving the equilibrium; this greatly simplifies the model. On the other hand, with return extrapolation, sentiment $S$ determines—and is endogenously determined by—asset prices. As a result, such a model requires solving beliefs and asset prices simultaneously, and therefore imposes a greater modelling challenge. Our numerical

---

25 A fundamental extrapolation model with heterogeneous agents—for instance, one with both an agent who extrapolates past dividend growth and an agent who is fully rational—can potentially generate extrapolative expectations of returns for the behavioral agent in the model. See the model of Ehling et al. (2015) as an example.
approach to solving a system of differential equations provides a solution to this challenge.

VI. Conclusion

We build a new return extrapolation model that can be brought to the data quantitatively. With the agent’s beliefs calibrated to fit the extrapolative expectations documented in surveys, the model matches the long-run properties of stock market prices: it generates a high average equity premium, significant excess volatility, a low average interest rate, low interest rate volatility, and a price-dividend ratio whose average level is similar to the empirical one. The model also matches the dynamic behavior of stock market prices: it produces the long-horizon predictability of stock market returns, and generates the observed low correlation between stock market returns and consumption growth. Overall, the model implications are quantitatively robust to changes in the elasticity of intertemporal substitution. In contrast, long-run risks models fail to generate a high equity premium with a low elasticity of intertemporal substitution.

Our analysis has left several important issues for future work. First, when calibrated with the survey expectations data, the model predicts a persistence of the price-dividend ratio that is much lower than its empirical value. To reconcile the survey expectations about stock market returns with the observed persistence of the price-dividend ratio, we need a deeper understanding about how investors form beliefs. Second, the extrapolation framework is closely linked to theories of model uncertainty. A careful investigation of this connection may produce useful insights to both literatures. Finally, our representative-agent model neglects an important channel that affects asset prices: the time-varying fraction of wealth held by behavioral agents. Explicitly incorporating rational agents into our framework can lead to additional implications.
Appendices

A. Derivation of the Differential Equations (17) and (18)

For the subjective Euler equation (4), setting $\tilde{R}_{j,t+dt}$, the return on the tradeable asset, to the gross return on the stock market, the equation becomes

\[\mathbb{E}_t e^{-\delta(1-\gamma)dt/(1-\psi)} \left( \frac{\tilde{C}_{t+dt}}{C_t} \right)^{-\psi(1-\gamma)/(1-\psi)} \left( \frac{\tilde{P}_{t+dt}^C + \tilde{C}_{t+dt}}{P_t^C} \right)^{(\psi-\gamma)/(1-\psi)} \left( \frac{\tilde{P}_{t+dt}^D + \tilde{D}_{t+dt}dt}{P_t^D} \right) = 1.\]  
(A.1)

Using Taylor expansion, (A.1) becomes

\[\mathbb{E}_t e^{-\delta(1-\gamma)dt/(1-\psi)} \left( \tilde{C}_{t+dt} \right)^{-\psi(1-\gamma)/(1-\psi)} \left( \tilde{P}_{t+dt}^C \right)^{(\psi-\gamma)/(1-\psi)} \tilde{P}_{t+dt}^D \left( 1 + \frac{\psi-\gamma}{1-\psi} \tilde{C}_{t+dt} dt + \frac{\tilde{D}_{t+dt}dt}{P_t^D} \right) = C_t^{-\psi(1-\gamma)/(1-\psi)} \left( P_t^C \right)^{(\psi-\gamma)/(1-\psi)} P_t^D.\]  
(A.2)

Rearranging terms gives

\[0 = \mathbb{E}_t \left[ d(\Theta C^{(\psi-\gamma)/(1-\psi)}l^{(\psi-\gamma)/(1-\psi)} Df) + \frac{\psi-\gamma}{1-\psi} \Theta C^{(\psi-\gamma)/(1-\psi)}l^{(2\psi-\gamma-1)/(1-\psi)} Df dt \right], \quad (A.3)\]

where $\Theta(C, t) \equiv e^{-\delta(1-\gamma)t/(1-\psi)}C^{-\psi(1-\gamma)/(1-\psi)}$. By Ito’s lemma, (A.3) leads to

\[0 = \mathbb{E}_t \left[ -\frac{\delta(1-\gamma)}{1-\psi} dt - \gamma(dC/C) + (dD/D) + (df/f) + \frac{\psi-\gamma}{1-\psi} \frac{\gamma(\gamma+1)}{2} (dC/C)^2 \right. \]

\[+ \frac{1}{2} \frac{\psi-\gamma}{1-\psi} \frac{2\psi-\gamma-1}{1-\psi} (dl/l)^2 - \frac{\gamma(\psi-\gamma)}{1-\psi} (dC/C)(dl/l) - \gamma(dC/C)(dD/D) - \gamma(dC/C)(df/f) \]

\[+ \frac{\psi-\gamma}{1-\psi} (dl/l)(dD/D) + \frac{\psi-\gamma}{1-\psi} (dl/l)(df/f) + (df/f)(dD/D) + \frac{\psi-\gamma}{1-\psi} l^{-1} dt + f^{-1} dt \quad \]

(A.4)

Using (7), (10), and (15) to further simplify (A.4) gives (17).

Setting $\tilde{R}_{j,t+dt}$ in (4) to the gross return on the Lucas tree, the subjective Euler equation (4)
becomes
\[
\mathbb{E}_t^e \left[ e^{-\delta(1-\gamma)dt/(1-\psi)} \left( \frac{C_{t+dt}}{C_t} \right)^{-\psi(1-\gamma)/(1-\psi)} \left( \frac{\tilde{P}^C_{t+dt} + \tilde{C}_{t+dt}}{\tilde{P}^C_t} \right)^{(1-\gamma)/(1-\psi)} \right] = 1. \tag{A.5}
\]

Rearranging terms yields
\[
0 = \mathbb{E}_t^e \left[ d(\Theta C^{(1-\gamma)/(1-\psi)}l^{(1-\gamma)/(1-\psi)}) + \frac{1-\gamma}{1-\psi} \Theta C^{(1-\gamma)/(1-\psi)}l^{(\psi-\gamma)/(1-\psi)} dt \right]. \tag{A.6}
\]

By Ito’s lemma, (A.6) leads to
\[
0 = \mathbb{E}_t^e \left[ -\frac{1-\gamma}{1-\psi} \delta dt - (\gamma - 1)(dC/C) + \frac{2(\gamma-1)}{2} (dC/C)^2 + \frac{1-\gamma}{1-\psi} (dl/l) \right. \\
\left. + \frac{1}{2} \frac{1-\gamma}{1-\psi} \psi^{-\gamma} (dl/l)^2 + \frac{(1-\gamma)^2}{1-\psi} (dC/C)(dl/l) + \frac{1-\gamma}{1-\psi} l^{-1} dt \right]. \tag{A.7}
\]

Using (7) and (15) to further simplify (A.7) gives (18).

\[\text{B. Steady-State Distribution for Sentiment}\]

Below we provide all the terms necessary for solving the Kolmogorov forward equation (26).

From the expression of \(\sigma_S\) in (7)
\[
\sigma_S' = \frac{\theta \sigma_P^D (\mu_H + \mu_L - 2S) - \theta (\mu_H - S)(S - \mu_L)(\sigma_P^D)'^2}{(\sigma_P^D)^2},
\]
\[
\sigma_S'' = \frac{\theta (\mu_H - S)(S - \mu_L)\{2[(\sigma_P^D)']^2 - \sigma_P^D(\sigma_P^D)''\}}{(\sigma_P^D)^3} - 2\theta \sigma_P^D(\sigma_P^D)'(\mu_H + \mu_L - 2S) + (\sigma_P^D)^2. \tag{B.1}
\]

For the expression of \(\mu_S^D\) in (7) and the expression of \(g_D^e\) in (11)
\[
(\mu_S^D)' = - (\lambda + \chi),
\]
\[
(g_D^e)' = \theta - \sigma_P^D (\sigma_P^D)' - \mu_S^D (f'/f) - \mu_S^D [f'^2/f - f'^2/f^2] \\
- \sigma_S^D (f''/f) - \frac{1}{2} \sigma_S^D [f''/f - f' f''/f^2]. \tag{B.2}
\]
where \( \sigma_P^D \) is from (11), and \((\sigma_P^D)'\) and \((\sigma_P^D)''\) are

\[
(\sigma_P^D)' = \frac{\theta(\mu_H + \mu_L - 2S)(f'/f) + \theta(\mu_H - S)(S - \mu_L)[f''/f - (f')^2/f^2]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}},
\]

\[
(\sigma_P^D)'' = \frac{-2\theta(\mu_H + \mu_L - 2S)(f'/f) + \theta(\mu_H - S)(S - \mu_L)[f''/f - (f')^2/f^2]}{[\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)]^{3/2}} + \frac{-2\theta f'/f + 2\theta(\mu_H - S)[f''/f - (f')^2/f^2]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}} + \frac{\theta(\mu_H - S)(S - \mu_L)[f''/f - 3(f'f'')/f^2 + 2(f')^3/f^3]}{\sqrt{\sigma_D^2 + 4\theta(\mu_H - S)(S - \mu_L)(f'/f)}}.
\]  

(C.3)

\[ \blacksquare \]

C. Numerical Procedure for Solving the Equilibrium

We use a projection method with Chebyshev polynomials to jointly solve the two differential equations (17) and (18). The value of the sentiment variable \( S \) ranges from \( \mu_L \) to \( \mu_H \), whereas the domain for Chebyshev polynomials is \([-1, 1]\). Therefore, we transform \( S \) to a new state variable \( z \)

\[
z \equiv aS + b, \quad \text{where} \quad a = \frac{2}{\mu_H - \mu_L}, \quad b = -\frac{\mu_H + \mu_L}{\mu_H - \mu_L},
\]  

(C.1)

and we define \( h(z) \equiv f(S(z)) \) and \( j(z) \equiv l(S(z)) \). Equations (17) and (18) can be rewritten as

\[
0 = \begin{bmatrix}
-(1-\gamma)\delta - \gamma g_C' + g_D' + [(h'/h) + \frac{\psi - \gamma}{1-\psi}(j'/j)]a\mu_S' + \frac{1}{2}[(h''/h) + \frac{\psi - \gamma}{1-\psi}(j''/j)]a^2\sigma_S^2 \\
+ \frac{\gamma+1}{2}\sigma_C^2 + \frac{1}{2}\frac{\psi - \gamma}{1-\psi}2\psi - \frac{\gamma - 1}{1-\psi}(a j'/j)^2\sigma_S^2 - \frac{\gamma(\psi - \gamma)}{1-\psi}\rho\sigma_C\sigma_S(a j'/j) - \gamma\rho\sigma_C\sigma_D - \gamma\rho\sigma_C\sigma_S(ah'/h) \\
+ \frac{\psi - \gamma}{1-\psi}\sigma_D\sigma_S(a j'/j) + \frac{\psi - \gamma}{1-\psi}\sigma_S^2 a^2(j'/j)(h'/h) + \sigma_D\sigma_S(ah'/h) + \frac{\psi - \gamma}{1-\psi}j^{-1} + h^{-1}
\end{bmatrix}
\]  

(C.2)

and

\[
0 = \begin{bmatrix}
-\frac{1-\gamma}{1-\psi}\delta - (\gamma - 1)g_C' + \frac{\gamma(\gamma - 1)}{2}\sigma_C^2 + \frac{1-\gamma}{1-\psi}(a j'/j)\mu_S' + \frac{1-\gamma}{2(1-\psi)}(a^2 j''/j)\sigma_S^2 \\
+ \frac{1}{2}\frac{1-\gamma}{1-\psi}(a j'/j)^2\sigma_S^2 + \frac{(1-\gamma)^2}{1-\psi}\rho\sigma_C\sigma_S(a j'/j) + \frac{1-\gamma}{1-\psi}j^{-1}
\end{bmatrix}.
\]  

(C.3)
We approximate $h$ and $j$ by

$$
\hat{h}(z) = \sum_{r=0}^{n} a_r T_r(z), \quad \hat{l}(z) = \sum_{r=0}^{m} b_r T_r(z),
$$

where $T_r(z)$ is the $r^{th}$ degree Chebyshev polynomial of the first kind.\textsuperscript{26} The projection method chooses the coefficients $\{a_r\}_{r=0}^{n}$ and $\{b_r\}_{r=0}^{m}$ so that the differential equations are approximately satisfied. One criterion for a sufficient approximation is to minimize the weighted sum of squared errors

$$
\sum_{i=1}^{N} \frac{1}{\sqrt{1-z_i^{2}}} \left[ \frac{1}{\nu} \left( \frac{1}{\nu} \right) \delta - \gamma g_C + g_D + [\hat{h}'/\hat{h}] + \psi - \frac{1}{1-\psi} (\hat{j}'/\hat{j}) a \mu_S + \frac{1}{2} [\hat{h}''/\hat{h}] + \psi - \frac{1}{1-\psi} (\hat{j}''/\hat{j}) a^2 \sigma^2 S \right] \right]^2_{z=z_i}
$$

$$
+ \sum_{i=1}^{N} \frac{1}{\sqrt{1-z_i^{2}}} \left[ \frac{1}{\nu} \left( \frac{1}{\nu} \right) \delta - (\gamma - 1) g_C + \frac{1}{2} (\nu - 1) \sigma^2 C + \frac{1}{2} (\nu - 1) \left( \frac{a \hat{j}'/\hat{j}) \mu_S + \frac{1}{2} (\nu - 1) a^2 \hat{j}''/\hat{j}) \sigma^2 S \right] \right]^2_{z=z_i}
$$

where $\{z_i\}_{i=0}^{N}$ are the $N$ zeros of $T_N(z)$. By the Chebyshev interpolation theorem, if $N$ is sufficiently larger than $n$ and $m$, and if the sum of weighted square in (C.5) is sufficiently small, the approximated functions $\hat{h}(z)$ and $\hat{l}(z)$ are sufficiently close to the true solutions.

For the numerical results in the main text, we set $m = 40$, $n = 40$, $N = 400$. We then apply the Levenberg-Marquardt algorithm and obtain a minimized sum of squared errors less than $10^{-11}$. The small size of the total error indicates convergence of the numerical solution. The solution is also insensitive to the choice of $n$, $m$, or $N$. Together, these findings suggest that the numerical solutions are a sufficient approximation for the true $h$ and $j$ functions.

The same numerical procedure is applied to solving the Kolmogorov forward equation (26). \textsuperscript{26}

\textsuperscript{26}See Mason and Handscomb (2003) for a detailed discussion of the properties of Chebyshev polynomials.
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Figure 1. Important Equilibrium Quantities Each as a Function of Sentiment. The figure plots the price-dividend ratio of the stock market $f$, the volatility of stock market returns $\sigma^D_P$, the rational expectation about the log excess return $\mathbb{E}[r^{D,e}]$ (the conditional equity premium), and the interest rate $r$, each as a function of the sentiment variable $S$. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Figure 2. Objectively Measured Steady-State Distribution of Sentiment. The figure plots the objective steady-state distribution of sentiment $\xi$ as a function of the sentiment variable $S$. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Figure 3. Agent’s Expectation about Stock Market Returns and about Dividend Growth. The dashed line plots the agent’s expectation about stock market returns, \( \mathbb{E}_t[(dP^D_t + D_t dt)/(P^D_t dt)] = (1 - \theta)g_D + \theta S_t + 1/f \), as a function of the sentiment variable \( S_t \). The solid line plots the agent’s expectation about dividend growth, \( \mathbb{E}_t[dD_t/(D_t dt)] \), as a function of the sentiment variable \( S_t \). The parameter values are: \( g_C = 1.91\% \), \( g_D = 2.45\% \), \( \sigma_C = 3.8\% \), \( \sigma_D = 11\% \), \( \rho = 0.2 \), \( \gamma = 10 \), \( \psi = 0.9 \), \( \delta = 2\% \), \( \theta = 0.5 \), \( \chi = 0.18 \), \( \lambda = 0.18 \), \( \mu_H = 15\% \), and \( \mu_L = -15\% \).
Figure 4. Comparative Statics: Utility Parameters. The upper panel plots the average equity premium $\mathbb{E}[r_{D,e}]$ and the average volatility of stock market returns $\sigma(r_{D,e})$, each as a function of $\gamma$, the relative risk aversion parameter. The lower panel plots the average equity premium $\mathbb{E}[r_{D,e}]$ and the average volatility of stock market returns $\sigma(r_{D,e})$, each as a function of $\psi$, the reciprocal of the elasticity of intertemporal substitution. The default values for $\gamma$ and $\psi$ are 10 and 0.9, respectively. The other parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Figure 5a. Comparative Statics: Belief Parameters (I). The figure plots the average equity premium $\mathbb{E}[r^{D,e}]$, the average volatility of stock market returns $\sigma(r^{D,e})$, the average price-dividend ratio $\exp(\mathbb{E}[\ln(P/D)])$, and the average interest rate $\mathbb{E}[r]$ (in percentage), each as a function of $\theta$, the parameter that controls the extent to which the agent is behavioral. The other parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Figure 5b. Comparative Statics: Belief Parameters (II). The figure plots the predictability of stock market returns and the persistence of the price-dividend ratio, each as a function of $\theta$, $\chi$, or $\lambda$. The predictability of returns is measured by the slope coefficient for a regression of the next year’s log excess return on the current log price-dividend ratio. The persistence of the price-dividend ratio is measured by the one-year autocorrelation of log price-dividend ratios. The default values for $\theta$, $\chi$, and $\lambda$ are 0.5, 0.18, and $-0.18$, respectively. The other parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Figure 6. Steady-State Distribution of Sentiment in the True Regime-Switching Model. The figure plots the steady-state distribution of sentiment $\xi$ as a function of the sentiment variable $S$ in the true regime-switching model. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Table 1. Investor Expectations.

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</tbody>
</table>

The table reports the regression coefficient and the t-statistic (in parenthesis), the intercept, as well as the adjusted $R$-squared, for regressing the agent’s expectation about future stock market returns either on the past twelve-month cumulative raw return $R_{t-12\rightarrow t}$ or on the current log price-dividend ratio $\ln(P_t/D_t)$, over a sample of 15 years or 50 years. In the top panel, the expectation measure for the first four columns is $\mathbb{E}_t[(dP_t^D + D_t dt)/(P_t^D dt)]$, and the expectation measure for the last four columns is $\mathbb{E}_t[dP_t^D/(P_t^D dt)]$. In the bottom panel, the expectation measure for the first four columns is $\mathbb{E}_t[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t$, and the expectation measure for the last four columns is $\mathbb{E}_t[dP_t^D/(P_t^D dt)] - r_t$. Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the model. The $t$-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 

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Table 2. Determinants of Investor Expectations.

<table>
<thead>
<tr>
<th></th>
<th>(\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)])</th>
<th>(\mathbb{E}_t^e[dP_t^D/(P_t^D dt)])</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi)</td>
<td>0.432</td>
<td>0.428</td>
</tr>
<tr>
<td>(a)</td>
<td>0.064</td>
<td>0.004</td>
</tr>
<tr>
<td>(b)</td>
<td>1.15</td>
<td>2.04</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>(R^2)</td>
<td>0.98</td>
<td>0.98</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>(\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t)</th>
<th>(\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\phi)</td>
<td>0.414</td>
<td>0.418</td>
</tr>
<tr>
<td>(a)</td>
<td>0.047</td>
<td>-0.013</td>
</tr>
<tr>
<td>(b)</td>
<td>0.70</td>
<td>1.58</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>(R^2)</td>
<td>0.92</td>
<td>0.97</td>
</tr>
</tbody>
</table>

The table reports the parameter \(\phi\), the intercept \(a\), the regression coefficient \(b\), and the adjusted \(R\)-squared, for running the non-linear least squares regression

\[
\text{Expectation}_t = a + b \sum_{j=1}^{n} w_j R_{(t-j\Delta t)\rightarrow(t-(j-1)\Delta t)} + \varepsilon_t,
\]

over a sample of 15 years or 50 years, where \(w_j = e^{-\phi(j-1)\Delta t}/\sum_{l=1}^{n} e^{-\phi(l-1)\Delta t}\). Here \(\Delta t = 1/12\) and \(n = 600\). In the top panel, the expectation measure for the first four columns is \(\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)]\), and the expectation measure for the last four columns is \(\mathbb{E}_t^e[dP_t^D/(P_t^D dt)]\). In the bottom panel, the expectation measure for the first four columns is \(\mathbb{E}_t^e[(dP_t^D + D_t dt)/(P_t^D dt)] - r_t\), and the expectation measure for the last four columns is \(\mathbb{E}_t^e[dP_t^D/(P_t^D dt)] - r_t\). Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the model. The \(t\)-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameter values are: \(g_C = 1.91\%\), \(g_D = 2.45\%\), \(\sigma_C = 3.8\%\), \(\sigma_D = 11\%\), \(\rho = 0.2\), \(\gamma = 10\), \(\psi = 0.9\), \(\delta = 2\%\), \(\theta = 0.5\), \(\chi = 0.18\), \(\lambda = 0.18\), \(\mu_H = 15\%\), and \(\mu_L = -15\%\).
Table 3. Parameter Values.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Variable</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Asset parameters:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Expected consumption growth</td>
<td>$g_C$</td>
<td>1.91%</td>
</tr>
<tr>
<td>Standard deviation of consumption growth</td>
<td>$\sigma_C$</td>
<td>3.80%</td>
</tr>
<tr>
<td>Expected dividend growth</td>
<td>$g_D$</td>
<td>2.45%</td>
</tr>
<tr>
<td>Standard deviation of dividend growth</td>
<td>$\sigma_D$</td>
<td>11%</td>
</tr>
<tr>
<td>Correlation between $dD$ and $dC$</td>
<td>$\rho$</td>
<td>0.2</td>
</tr>
<tr>
<td><strong>Utility parameters:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Relative risk aversion</td>
<td>$\gamma$</td>
<td>10</td>
</tr>
<tr>
<td>Reciprocal of $EIS$</td>
<td>$\psi$</td>
<td>0.9</td>
</tr>
<tr>
<td>Subjective discount rate</td>
<td>$\delta$</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Belief parameters:</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Degree of extrapolation</td>
<td>$\theta$</td>
<td>0.5</td>
</tr>
<tr>
<td>Perceived transition intensity from $H$ to $L$</td>
<td>$\chi$</td>
<td>0.18</td>
</tr>
<tr>
<td>Perceived transition intensity from $L$ to $H$</td>
<td>$\lambda$</td>
<td>0.18</td>
</tr>
<tr>
<td>Upper bound of sentiment</td>
<td>$\mu_H$</td>
<td>0.15</td>
</tr>
<tr>
<td>Lower bound of sentiment</td>
<td>$\mu_L$</td>
<td>–0.15</td>
</tr>
</tbody>
</table>
Table 4. Basic Moments.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Theoretical value</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity premium ($\mathbb{E}[r^{D,e}]$)</td>
<td>4.88%</td>
<td>3.90%</td>
</tr>
<tr>
<td>Return volatility ($\sigma(r^{D,e})$)</td>
<td>27.4%</td>
<td>18.0%</td>
</tr>
<tr>
<td>Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$)</td>
<td>0.20</td>
<td>0.22</td>
</tr>
<tr>
<td>Interest rate ($\mathbb{E}[r]$)</td>
<td>2.16%</td>
<td>2.92%</td>
</tr>
<tr>
<td>Interest rate volatility ($\sigma(r)$)</td>
<td>0.33%</td>
<td>2.89%</td>
</tr>
<tr>
<td>Price-dividend ratio ($\exp(\mathbb{E}[\ln(P/D)])$)</td>
<td>19.4</td>
<td>21.1</td>
</tr>
</tbody>
</table>

The table reports six important moments about stock market prices and returns: the long-run average of the equity premium (the rational expectation of log excess return, $\mathbb{E}[r^{D,e}]$), the average volatility of stock market returns (the volatility of log excess return, $\sigma(r^{D,e})$), the Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$), the average interest rate ($\mathbb{E}[r]$), interest rate volatility ($\sigma(r)$), and the average price-dividend ratio of the stock market ($\exp(\mathbb{E}[\ln(P/D)])$). The theoretical values for these moments are computed over the objectively measured steady-state distribution of sentiment $S$. The model parameters are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. For the empirical values, five out of six are from Campbell and Cochrane (1999); the empirical value for interest rate volatility is not reported in Campbell and Cochrane (1999), so we report the value from Beeler and Campbell (2012).
Table 5. Campbell-Shiller Decomposition.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>Realized dividend growth and returns</th>
<th>Subjective expectations about dividend growth and returns</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>Adjusted R-squared</td>
</tr>
<tr>
<td>$DR_{objective}$</td>
<td>0.98</td>
<td>0.209</td>
</tr>
<tr>
<td>$CF_{objective}$</td>
<td>0.02</td>
<td>$1.2 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

The table reports the four coefficients, $CF_{objective}$, $DR_{objective}$, $CF_{subjective}$, and $DR_{objective}$, defined in equations (29) and (30), as well as their corresponding adjusted R-squared. These coefficients and R-squared are calculated using 10,000 years of monthly data simulated from the model. At each point in time, for a given level of sentiment, subjective expectations about dividend growth and returns are calculated as the average values of 100 trials. Each trial is 50 years of monthly simulated data under the agent’s expectations with the given initial level of sentiment. For realized dividend growth and returns, both $\sum_{j=0}^{\infty} \xi^j \Delta d_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)}$ and $\sum_{j=0}^{\infty} \xi^j r^{D}_{(t+j\Delta t)\rightarrow(t+(j+1)\Delta t)}$ are approximated using 50 years of monthly simulated data. From the simulated data, $\xi = 0.9957$. The other parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 
Table 6. Return Predictability Regressions.

<table>
<thead>
<tr>
<th>Horizon (years)</th>
<th>Theoretical value</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>10× coefficient</td>
<td>Adjusted R-squared</td>
</tr>
<tr>
<td>1</td>
<td>-7.2</td>
<td>0.13</td>
</tr>
<tr>
<td>2</td>
<td>-9.5</td>
<td>0.16</td>
</tr>
<tr>
<td>3</td>
<td>-10.1</td>
<td>0.15</td>
</tr>
<tr>
<td>5</td>
<td>-10.6</td>
<td>0.13</td>
</tr>
<tr>
<td>7</td>
<td>-11.0</td>
<td>0.12</td>
</tr>
</tbody>
</table>

The table reports the regression coefficient $\beta_j$ and the adjusted $R$-squared for a regression of the log excess return of stock market from time $t$ to time $t+j$ on the current log price-dividend ratio $\ell \ln(P_t^D/D_t)$

$$r_{t\rightarrow t+j}^{D,e} = \alpha_j + \beta_j \ell \ln(P_t^D/D_t) + \varepsilon_{j,t},$$

where $j = 1, 2, 3, 5,$ and $7$ (years). The theoretical values are calculated using 10,000 years of monthly data simulated from the model. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).
Table 7. Autocorrelations of Log Excess Returns and Log Price-dividend Ratios.

<table>
<thead>
<tr>
<th>Lag (years)</th>
<th>Theoretical value</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\ell \ln(P^D/D) \cdot r_{D,e} \cdot \Sigma_{i=1}^{j} \rho(r_{t_{i+1}}^{D,e}, r_{t_{i-1}}^{D,e})$</td>
<td>$\ell \ln(P^D/D) \cdot r_{D,e} \cdot \Sigma_{i=1}^{j} \rho(r_{t_{i+1}}^{D,e}, r_{t_{i-1}}^{D,e})$</td>
</tr>
<tr>
<td>1</td>
<td>0.33</td>
<td>-0.28</td>
</tr>
<tr>
<td>2</td>
<td>0.11</td>
<td>-0.09</td>
</tr>
<tr>
<td>3</td>
<td>0.05</td>
<td>-0.02</td>
</tr>
<tr>
<td>5</td>
<td>0.00</td>
<td>-0.01</td>
</tr>
<tr>
<td>7</td>
<td>-0.02</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

The table reports, over various lags $j$, the autocorrelations of log price-dividend ratios and log excess returns, as well as the partial sum of the autocorrelations of log excess returns. The operator $\rho(x, y)$ computes the sample correlation between variable $x$ and variable $y$. The theoretical values are calculated using 10,000 years of monthly data simulated from the model; for each month, we compound the next 12 months of log excess returns to obtain an annual log excess return. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).
Table 8. Correlation between Consumption Growth and Stock Returns.

<table>
<thead>
<tr>
<th>Correlation</th>
<th>Theoretical value</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>way</td>
<td>monthly</td>
<td>quarterly</td>
</tr>
<tr>
<td>$\text{Corr}(r_{t \rightarrow t+1}, \ell n(C_{t-1}/C_{t-2}))$</td>
<td>-0.01</td>
<td>-0.02</td>
</tr>
<tr>
<td>$\text{Corr}(r_{t \rightarrow t+1}, \ell n(C_t/C_{t-1}))$</td>
<td>-0.01</td>
<td>-0.03</td>
</tr>
<tr>
<td>$\text{Corr}(r^{D,e}<em>{t \rightarrow t+1}, \ell n(C</em>{t+1}/C_t))$</td>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>$\text{Corr}(r^{D,e}<em>{t \rightarrow t+1}, \ell n(C</em>{t+2}/C_{t+1}))$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$\text{Corr}(r^{D,e}<em>{t \rightarrow t+1}, \ell n(C</em>{t+3}/C_{t+2}))$</td>
<td>-0.00</td>
<td>-0.00</td>
</tr>
</tbody>
</table>

The table reports correlations between log consumption growth and log excess returns of the stock market. The log consumption growth and log excess returns are computed at either a monthly, a quarterly, or an annual horizon. Correlations are either contemporaneous or with a lead-lag structure; for instance, at a monthly frequency, $\text{Corr}(r^{D,e}_{t \rightarrow t+1}, \ell n(C_{t+2}/C_{t+1}))$ is the correlation between the current monthly log excess return and the log consumption growth in the subsequent month. The theoretical values are calculated using 10,000 years of monthly data simulated from the model. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).
Table 9. Return Predictability Regressions in the True Regime-Switching Model.

| Horizon (years) | Theoretical value | | | | Empirical value | | |
|----------------|------------------|---|---|---|---|---|
|                | 10× coefficient  | 10³× Adjusted R-squared | 10× coefficient | Adjusted R-squared |
| 1              | 0.2              | 0.16                      | -1.3           | 0.04            |
| 2              | 0.4              | 0.31                      | -2.8           | 0.08            |
| 3              | 0.5              | 0.32                      | -3.5           | 0.09            |
| 5              | 0.8              | 0.54                      | -6.0           | 0.18            |
| 7              | 0.7              | 0.34                      | -7.5           | 0.23            |

The table reports the regression coefficient $\beta_j$ and the adjusted $R$-squared for a regression of the log excess return of stock market from time $t$ to time $t + j$ on the current log price-dividend ratio $\ln(P^D_t / D_t)$

$$r^D_{t\rightarrow t+j} = \alpha_j + \beta_j \ln(P^D_t / D_t) + \varepsilon_{j,t},$$

where $j = 1, 2, 3, 5, \text{ and } 7$ (years). The theoretical values are calculated using 10,000 years of monthly data simulated from the true regime-switching model described in Section IV. The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. The empirical values are from Campbell and Cochrane (1999).
Table 10. Investor Expectations in Bansal and Yaron (2004).

<table>
<thead>
<tr>
<th></th>
<th>Expectation of return</th>
<th>Expectation of return w/o divd.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{t-12 \rightarrow t}^D$</td>
<td>0.025 0.030</td>
<td>0.025 0.031</td>
</tr>
<tr>
<td></td>
<td>(2.4) (3.8)</td>
<td>(2.5) (4.0)</td>
</tr>
<tr>
<td>$\ln(P/D)$</td>
<td>0.068 0.067</td>
<td>0.069 0.067</td>
</tr>
<tr>
<td></td>
<td>(5.6) (7.6)</td>
<td>(5.8) (7.6)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.06 1.05</td>
<td>1.06 1.05</td>
</tr>
<tr>
<td></td>
<td>0.88 0.88</td>
<td>0.88 0.89</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td></td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.140 0.142</td>
<td>0.136 0.145</td>
</tr>
<tr>
<td></td>
<td>0.500 0.451</td>
<td>0.529 0.451</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Expectation of excess return</th>
<th>Expectation of excess return w/o divd.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$R_{t-12 \rightarrow t}^D$</td>
<td>-0.001 0.000</td>
<td>0.000 0.001</td>
</tr>
<tr>
<td></td>
<td>(-0.1) (0.1)</td>
<td>(-0.1) (0.2)</td>
</tr>
<tr>
<td>$\ln(P/D)$</td>
<td>-0.008 -0.012</td>
<td>-0.010 -0.011</td>
</tr>
<tr>
<td></td>
<td>(-0.8) (-1.3)</td>
<td>(-0.8) (-1.2)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.06 0.06</td>
<td>0.06 0.06</td>
</tr>
<tr>
<td></td>
<td>0.08 0.10</td>
<td>0.09 0.09</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td></td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.026 0.008</td>
<td>0.026 0.008</td>
</tr>
<tr>
<td></td>
<td>0.087 0.054</td>
<td>0.093 0.053</td>
</tr>
</tbody>
</table>

The table reports the regression coefficient and the $t$-statistic (in parenthesis), the intercept, as well as the adjusted $R$-squared, for regressing four measures of rational expectation of return—raw return or excess return, with or without dividend yield—either on the past twelve-month return or on the current log price-dividend ratio, over a sample of 15 years or 50 years. The conditional expectation of subsequent returns, the dependent variable in each regression, is computed by averaging realized returns across simulations over a twelve-month horizon for a given state of the economy. Each reported value is the estimator median over 1,000 trials, and each trial represents a regression using monthly data simulated from Bansal and Yaron (2004). The $t$-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameters take their default values from Table II and IV of Bansal and Yaron (2004).
### Table 11. Investor Expectations in Bansal et al. (2012).

<table>
<thead>
<tr>
<th></th>
<th>Expectation of return</th>
<th>Expectation of return w/o divd.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^D_{t-12\rightarrow t} )</td>
<td>0.006 (0.7)</td>
<td>0.006 (0.7)</td>
</tr>
<tr>
<td>( \ln(P/D) )</td>
<td>-0.006 (-0.3)</td>
<td>-0.006 (-0.2)</td>
</tr>
<tr>
<td>Constant</td>
<td>1.08</td>
<td>1.08</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.031 (0.021)</td>
<td>0.033 (0.020)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Expectation of excess return</th>
<th>Expectation of excess return w/o divd.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R^D_{t-12\rightarrow t} )</td>
<td>-0.003 (-0.3)</td>
<td>-0.004 (-0.5)</td>
</tr>
<tr>
<td>( \ln(P/D) )</td>
<td>-0.071 (-5.2)</td>
<td>-0.036 (-3.4)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.09</td>
<td>0.08</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.021 (0.010)</td>
<td>0.026 (0.008)</td>
</tr>
</tbody>
</table>

The table reports the regression coefficient and the \( t \)-statistic (in parenthesis), the intercept, as well as the adjusted \( R \)-squared, for regressing four measures of rational expectation of return—raw return or excess return, with or without dividend yield—either on the past twelve-month return or on the current log price-dividend ratio, over a sample of 15 years or 50 years. The conditional expectation of subsequent returns, the dependent variable in each regression, is computed by averaging realized returns across simulations over a twelve-month horizon for a given state of the economy. Each reported value is the estimator median over 1,000 trials, and each trial represents a regression using monthly data simulated from Bansal et al. (2012). The \( t \)-statistics are calculated using a Newey-West estimator with twelve-month lags. The parameters take their default values from Table 1 of Bansal et al. (2012).
Table 12. Investor Expectations in the Fundamental Extrapolation Model.

<table>
<thead>
<tr>
<th></th>
<th>$\mathbb{E}_t[(dP^D_t + D_t dt)/(P^D_t dt)]$</th>
<th>$\mathbb{E}_t[dP^D_t/(P^D_t dt)] - r_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{t-12 \rightarrow t}^D$</td>
<td>0.004 (0.21)</td>
<td>0.010 (1.15)</td>
</tr>
<tr>
<td></td>
<td>0.003 (0.63)</td>
<td>0.010 (2.04)</td>
</tr>
<tr>
<td>$\ell n(P/D)$</td>
<td>0.017 (0.88)</td>
<td>0.040 (2.08)</td>
</tr>
<tr>
<td></td>
<td>0.019 (1.62)</td>
<td>0.042 (3.52)</td>
</tr>
<tr>
<td>Constant</td>
<td>0.05 (0.49)</td>
<td>0.03 (0.71)</td>
</tr>
<tr>
<td>Sample size</td>
<td>15 yr. 50 yr.</td>
<td>15 yr. 50 yr.</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.14 (0.88)</td>
<td>0.17 (0.08)</td>
</tr>
<tr>
<td></td>
<td>0.04 (1.13)</td>
<td>0.08 (1.35)</td>
</tr>
<tr>
<td></td>
<td>0.24 (0.54)</td>
<td>0.31 (0.42)</td>
</tr>
<tr>
<td></td>
<td>0.10 (0.54)</td>
<td>0.22 (0.88)</td>
</tr>
</tbody>
</table>

The table reports the regression coefficient and the $t$-statistic (in parenthesis), the intercept, as well as the adjusted $R$-squared, for regressing the agent’s expectation about future stock market returns either on the past twelve-month cumulative raw return $R_{t-12 \rightarrow t}^D$ or on the current log price-dividend ratio $\ell n(P_t/D_t)$, over a sample of 15 years or 50 years. In the top panel, the expectation measure for the first four columns is $\mathbb{E}_t[(dP^D_t + D_t dt)/(P^D_t dt)]$, and the expectation measure for the last four columns is $\mathbb{E}_t[dP^D_t/(P^D_t dt)] - r_t$. In the bottom panel, the expectation measure for the first four columns is $\mathbb{E}_t[(dP^D_t + D_t dt)/(P^D_t dt)] - r_t$, and the expectation measure for the last four columns is $\mathbb{E}_t[dP^D_t/(P^D_t dt)] - r_t$. Each reported value is averaged over 100 trials, and each trial represents a regression using monthly data simulated from the fundamental extrapolation model described in Section V. The $t$-statistics are calculated using a Newey-West estimator with twelve-month lags.

The parameter values are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18$, $\mu_H = 15\%$, and $\mu_L = -15\%$. 

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### Table 13. Basic Moments in the Fundamental Extrapolation Model.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Theoretical value</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity premium ($\mathbb{E}[r^{D,e}]$)</td>
<td>1.85%</td>
<td>3.90%</td>
</tr>
<tr>
<td>Return volatility ($\sigma(r^{D,e})$)</td>
<td>20.4%</td>
<td>18.0%</td>
</tr>
<tr>
<td>Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$)</td>
<td>0.08</td>
<td>0.22</td>
</tr>
<tr>
<td>Interest rate ($\mathbb{E}[r]$)</td>
<td>2.27%</td>
<td>2.92%</td>
</tr>
<tr>
<td>Interest rate volatility ($\sigma(r)$)</td>
<td>0.23%</td>
<td>2.89%</td>
</tr>
<tr>
<td>Price-dividend ratio ($\exp(\mathbb{E}[\ell\ln(P/D)])$)</td>
<td>44.4</td>
<td>21.1</td>
</tr>
</tbody>
</table>

The table reports six important moments about stock market prices and returns: the long-run average of the equity premium (the rational expectation of log excess return, $\mathbb{E}[r^{D,e}]$), the average volatility of stock market returns (the volatility of log excess return, $\sigma(r^{D,e})$), the Sharpe ratio ($\mathbb{E}[r^{D,e}]/\sigma(r^{D,e})$), the average interest rate ($\mathbb{E}[r]$), interest rate volatility ($\sigma(r)$), and the average price-dividend ratio of the stock market ($\exp(\mathbb{E}[\ell\ln(P/D)])$). The theoretical values for these moments are computed over the objectively measured steady-state distribution of sentiment in the fundamental extrapolation model described in Section V. The model parameters are: $g_C = 1.91\%$, $g_D = 2.45\%$, $\sigma_C = 3.8\%$, $\sigma_D = 11\%$, $\rho = 0.2$, $\gamma = 10$, $\psi = 0.9$, $\delta = 2\%$, $\theta = 0.5$, $\chi = 0.18$, $\lambda = 0.18\%$, $\mu_H = 15\%$, and $\mu_L = -15\%$. For the empirical values, five out of six are from Campbell and Cochrane (1999); the empirical value for interest rate volatility is not reported in Campbell and Cochrane (1999), so we report the value from Beeler and Campbell (2012).