# PORTFOLIO SELECTION UNDER TIME DELAYS: A PIECEWISE DYNAMIC PROGRAMMING APPROACH

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Abstract. This paper solves the portfolio selection problems when the price process explicitly depends on historical information with time delays. New path-induced state variables characterizing the delayed information are constructed, together with the original state variables, to constitute a sufficient statistic for the dynamic portfolios. State variables are different for different investment horizons. As horizon increases, the number of state variables increases without bound. Thus, the optimal portfolio weight depends not only on the current values of the original state variables, but also on their historical paths. The method developed in the paper has many potential applications for important problems in economics and finance.

*Key words*: Asset allocation, time delays, sufficient statistic, stochastic delay differential equations, piecewise dynamic programming method.

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## 1. Introduction

Many long-standing and prominent phenomena in economics and finance, including time to produce (Ezekiel, 1938), time to build (Kydland and Prescott, 1982), short-run momentum (Jegadeesh and Titman, 1993), long-run reversal (De Bondt and Thaler, 1985 and Fama and French, 1988), post-earnings announcement drift (Bernard and Thomas, 1989, 1990) and long memory in volatility (Andersen, Bollerslev and Diebold, 2007), involve delayed information.[1] There are two key common features of these phenomena. First, current state variables depend on lagged state variables with certain time delays or moving averages of state variables with certain lengths of moving windows (path dependence). Second, one cannot transfer the systems to Markovian ones by introducing a finite number of state variables in a continuous-time setting.

These features, though intuitive, are mathematically complex. In fact, all the phenomena are naturally described by stochastic delay differential equations (SDDEs) which are only recently studied in mathematics literature. First, to define the processes, we need to specify a continuously infinite number of initial values, so SDDEs are truely non-Markovian with infinite dimensions in the sense that one cannot transfer SDDEs to Markovian systems by introducing a finite number of state variables. Second, there is even no Ito's formula for SDDEs.[2] SDDEs are more general than the standard Markov stochastic processes. They become Markovian when there is no time delay.

The dynamic portfolio selection literature in silent on the above phenomena with time delays because of the lack of tools and framework for analysis. Instead, many studies examine one-period (or static) portfolio problems, and the implications of delays on risk management for multiple periods are still unclear. In this paper, we solve the optimal portfolio choice problems under time delays.

The stochastic control problems with time delays involve infinite-dimensional non-Markovian systems and are conceptually much more difficult than the Markovian problems without delays. The standard dynamic programming method used by Merton (1971) does not apply because SDDEs are path-dependent and there is no Ito's

[1]In addition, some macroeconomic variables, such as inflation, are usually published with time delays, while their lagged values can still significantly forecast equity and bond returns (Fama, 1981, Campbell and Vuolteenaho, 2004 and Ludvigson and Ng, 2007, 2009). Furthermore, Hou and Moskowitz (2005) develop measures of firm price delay, which help explain many anomalies. Hong, Torous and Valkanov (2007) find evidence that stock markets react with a delay to information contained in industry returns about their fundamentals and that information diffuses only gradually across markets.

[2]Ito's formula is always applied to functions of the current values of some stochastic processes, as opposed to functions of the paths of the processes.

formula for them. Accordingly, three strands of literature have been developed. The first is to derive the reducible conditions under which the delay effect is effectively eliminated so that the dynamic programming method applies (Larssen and Risebro, 2003). However, the reducible conditions are extremely strict and hence the methods cannot be applied to portfolio selection problems with path-dependent stock price processes.[3] In addition, based on the duality relation between the anticipated backward stochastic differential equation (BSDE) and SDDE demonstrated in Peng and Yang (2009), Chen and Wu (2010) and Øksendal, Sulem and Zhang (2011) develop the maximum principle for the delayed control problems, which results in fully coupled forward-backward stochastic delay differential equations (FBSDDEs). However, currently no algorithm exists for solving FBSDDEs even numerically. Another strand of literature appeals to functional Ito calculus, while the big gap between the functional-form solutions and applications leads to unclear implications for practice (see, for example, Cont and Fournié, 2013 and references therein). To the best of our knowledge, there is no known example of the closed-form solutions of such systems in the extant literature.

To overcome the challenge, this paper develops a "piecewise dynamic programming method" by introducing new state variables and solves the control problem path by path. This method is applicable to general utilities. For CRRA utility, we solves the control problem up to the solutions to ordinary differential equations (ODEs). As far as we know, this is the first nontrivial closed-form solution for such systems.

More specifically, we show that the *original state variables* directly observed in the markets, are not a sufficient statistic for the dynamic portfolios and hence are unable to fully characterize the state of the economy.[4] To solve this problem, for a given investment horizon, we construct new *path-induced state variables* which, together with the original state variables, constitute a sufficient statistic for the dynamic portfolios. The path-induced state variables are weighted averages of historical state variables, and hence they do not introduce any new source of uncertainty. They are different for different investment horizons. The number of variables to constitute the sufficient statistic increases as the investment horizon increases. Effectively, we reduce the non-Markovian system to a *horizon-reducible Markovian system*. Because

[3] For example, Larssen (2002) and Larssen and Risebro (2003) show that the stochastic control problems for SDDEs can be reduced to a finite dimensional problems under the special conditions that time delays do not appear in both control variables and the value function, and parameters also have to satisfy certain equalities (Theorem 5.1 in Larssen and Risebro, 2003). Their methods cannot be applied to the portfolio selection problems with past-dependent underlying stock processes, because time delays affect the control variables in this case.

[4] Huang (1987) develops sufficient conditions under which the original state variables constitute a sufficient statistic for agents' dynamic portfolios.

horizon is unbounded, the number of state variables is also unbounded. In contrast, in the intertemporal portfolio theory, an investor's optimal portfolio is typically derived based on finite dimensional Markovian state variables, which constitute a sufficient statistic for the dynamic portfolio over time (Merton, 1971).

Although the system has infinite dimensions, for any finite investment horizon, the sufficient statistic only consists of a finite number of variables. Based on the new state variables, the dynamic programming method can apply. Effectively, we reduce infinite-dimensional partial differential equations (PDEs) to systems of ODEs.

The portfolio weights exhibit many new features which do not exist in Markovian framework. We show that investors base their decisions on historical paths over a delay-length look-back period, while they only need the current values of state variables when return is governed by Markovian systems. Because delays lead to path-dependent expected returns, the optimal portfolio can be very different for different paths.[5] The myopic demand only depends on the original state variables, while the intertemporal hedging demand also depends on the path-induced state variables. In contrast, in a standard Markovian setting, the myopic component and hedging component typically depend on the same and a finite number of state variables. The path dependence of the optimal portfolio further indicates that the portfolio weight may not be a monotonic function of horizon. In contrast, the dynamic strategies with Markov state variables typically have monotonically smooth horizon dependence.

An interesting question would be that when a state variable is only known with a delay (such as inflation), then if the associated risk is hedgeable? We find that delay gives rise to a new type of hedging demand (we will term it path hedging demand), which hedges the risk associated with delayed variables and significantly affects the portfolio weights. In contrast, the traditional portfolio theory with Markov processes suggests that the intertemporal hedging demand is completely caused by the state variables which are correlated with the wealth process. Due to the path hedging demand, different historical paths lead to different optimal portfolios.

The method developed in the paper has many potential applications for important problems in economics and finance. An direct application of our results is to study the portfolio choice problem when risk premium can be forecasted by delayed variables. These variables include some technical predictors involving moving averages of historical prices or trading volumes, some macroeconomic predictors which are published only with a time delay, short-run momentum and long-run reversal.

[5]Recent empirical studies start to exploit the path effect. For example, Da, Gurun and Warachka (2014) show that the profitability of momentum strategies depends on the shapes of historical price paths.

The paper also discusses a volatility timing problem where volatility exhibits long memory.

Many issues in macroeconomics typically involve time delays, such as time to produce (Ezekiel, 1938), time to build (Kydland and Prescott, 1982) and innovation implementation lags (Bambi, Gozzi and Licandro, 2014). In addition, many agents behavior are found to have effects on asset prices while are difficult to be modelled in the standard Markovian setup, such as the disposition effect, overextrapolation, prospect theory, mental accounting and overreaction/underreaction. They intrinsically involve time delays. To study the economic implication, the piecewise dynamic programming method developed in the paper may be needed.

The standard Markovian continuous-time equilibrium asset pricing models, pioneered by Merton (1973), are derived based on the premise that the state variable processes have finite dimensions so that agents' portfolios and equilibrium prices are spanned by finite dimensional state variables. Our method of constructing new state variables also sheds light on non-Markovian asset pricing theory with delayed information.

Bismut (1975) and Foldes (1978) are the first to study the portfolio selection problems under non-Markov processes. Under the assumption of complete market, the martingale approach is then developed and is able to solve the stochastic control problems under non-Markov processes (Harrison and Kreps, 1979, Pliska, 1986, Karatzas, Lehoczky and Shreve, 1987 and Cox and Huang, 1989). Li and Liu (2016) extend the martingale approach to study the optimal dynamic momentum strategy involving time delays and derive the closed-form solutions when the investment horizon is shorter than the time delay. The method developed in this paper also sheds light on the martingale approach in dealing with SDDEs for any investment horizons. In addition, our method does not rely on the premise of complete market,[6] a specific assumption to the area of asset pricing, and hence it applies to a wider range of problems outside financial markets.

The paper is organized as follows. Section 2 explores the modelling of time delays. Section 3 discusses the piecewise dynamic programming method, and Section 4 examines the CRRA utility. Section 5 presents some applications and Section 6 concludes. Calculation details are included in the appendices.

[6]The assumption of complete markets is strong. For example, markets are incomplete if stock returns exhibit stochastic volatility and there are no derivatives included in the set of securities, or if stock returns are predictable and the uncertainties associated with the predictors of stock returns cannot be replicated using existing securities.

## 2. The Model

The standard Markov processes which dominate economic models cannot capture time delays, an inherently non-Markovian feature. Instead, we model delays by using stochastic delay differential equations (SDDEs), which are just recently studied in mathematics literature and can exactly measure the delay effects.

Because SDDEs are relatively new to economics and finance literature, prior to developing price dynamics, we first discuss the modelling of time delays. In mathematics literature, there are two types of delays: discrete and distributed delays. For a process $X_t$, a discrete delay is characterized by a single historical value $X_{t-\tau}$, where $\tau$ is the length of time delay. Distributed delays refer to the time delays distributed over a set of time with a certain density: $\int_{t-\tau}^{t} g(s)X_s ds$, where $g(s)$ is the density. The most widely used distributed delays are the exponentially decayed weighted moving average (MA) over a time interval $[t-\tau, t]$

$$Y_t = \frac{k}{1-e^{-k\tau}} \int_{t-\tau}^{t} e^{-k(t-s)} X_s ds, \tag{2.1}$$

where $k$ measures the decaying rate of the weights on historical values of $X$. Especially, when $k = 0$, (2.1) becomes a standard MA

$$Y_t = \frac{1}{\tau} \int_{t-\tau}^{t} X_s ds. \tag{2.2}$$

When $k = \infty$, $Y_t$ reduces to $X_t$. In general, (2.1) can be expressed as a differential equation with time delay

$$dY_t = \frac{k}{1-e^{-k\tau}} \big[X_t - e^{-k\tau} X_{t-\tau} - (1-e^{-k\tau})Y_t\big] dt. \tag{2.3}$$

Especially, when $\tau = \infty$, the distributed delays become reducible and $Y_t$ is governed by an ODE, $dY_t = k(X_t - Y_t)dt$. When $\tau = 0$, $Y_t = X_t$. In economics and finance, the MA rules are important because many estimations and variables have the forms of (2.1). In this paper, we study this type of distributed delays, while our method can be applied to other types of distributed delays.

To capture different forms of the predictability of delayed information, we assume the price process of a risky asset satisfies

$$dP_t/P_t = (a + b_1 X_{t-\tau} + b_2 Y_t)dt + \sigma dZ_t. \tag{2.4}$$

The variable $X_t$ is observable at time $t$ while its lagged value $X_{t-\tau}$ or an MA of its lagged values $Y_t$ can forecast equity risk premium.

Model (2.4) captures many financial phenomena. Firstly, Hou and Moskowitz (2005) show that information is incorporated into prices with time delays and the length of delays is firm-specific. Secondly, many technical indicators extract unincorporated lagged information via MA of different variables, such as price and trading volume, and display statistically and economically significant predictive power to the equity risk premium (Brock, Lakonishok and LeBaron, 1992, Blume, Easley and O'Hara, 1994, Moskowitz, Ooi and Pedersen, 2012, and Neely, Rapach, Tu and Zhou, 2014). Thirdly, some macroeconomic variables are usually published with time delays, such as inflation, while their lagged values can still significantly forecast equity and bond returns (Ludvigson and Ng, 2007, 2009 and Rapach, Strauss and Zhou, 2010). In addition, some financial forecasting variables, such as, the dividend-price ratio and earnings-price ratio, are usually estimated based on MA rules over a historical time interval (Campbell and Shiller, 1988*a* and 1988*b*).

Before studying the optimal portfolio choice problem, we need to specify a process of $X_t$. Following the literature, we assume the predictor follows an Ornstein-Uhlenbeck process

$$dX_t = \kappa(\bar{X} - X_t)dt + \sigma_X dZ_t^X, \tag{2.5}$$

and $dZ_t dZ_t^X = \rho dt$. A more general case when $X_t$ follows SDDEs can be also studied using our method. The price model (2.3), (2.4) and (2.5) is characterized by a system of non-Markovian SDDEs. When $\tau = 0$, the price process (2.3)-(2.5) reduces to the Markovian system studied in Kim and Omberg (1996). In general, the non-Markovian delay feature characterized by (2.3)-(2.5) cannot be captured by the standard Markov processes.[7]

The following lemma states the existence and uniqueness of the solutions.

**Lemma 2.1.** *The system* (2.3), (2.4) *and* (2.5) *has a unique continuous adapted solution* $(P, X, Y)$ *for a given* $\mathcal{F}_0$*-measurable initial process* $X = \hat{X} : \Omega \to C([-\tau, 0], R)$ *and initial value* $P_0$*. Furthermore, if* $P_0 > 0$*, then* $P_t > 0$ *for* $t \geq 0$.[8]

Lemma 2.1 shows that, in order to define the process $(P, X, Y)$, we need an infinite-dimensional space of initial conditions $C([-\tau, 0], R)$, which is the set of continuous functions from $[-\tau, 0]$ to $R$. So the system (2.3)-(2.5) also has infinite dimensions. In contrast, in Merton (1971)'s Markovian setting, one needs to specify only the initial values of a finite number of state variables[9] to define the price process, and the model has finite dimensions.

The discretization of the model becomes the autoregressive moving average (ARMA) family of models. The Markov AR(1) process is corresponding to the case $\tau = 0$, while the non-Markovian AR(n), $n \geq 2$ and MA models are corresponding to $\tau > 0$.

[7]We study a single risky asset. The results of current paper can be easily extended to the case with multiple risky assets, multiple state variables or multiple time delays. In fact, our method is also applicable to prices with stochastic volatility as discussed in an application in Section 5.

[8]The proof follows from that of Theorem 2.2 in Chen and Wu (2010) and Theorem 1 in Arriojas, Hu, Mohammed and Pap (2007).

[9]Such variables could be price, predictors of returns, or volatility.

Different from the ARMA model which can be transferred to a Markovian system with higher but finite dimensions, SDDEs cannot be reduced to Markovian systems by introducing a finite number of state variables because they are truely non-Markovian with infinite dimensions.

## 3. The Optimal Control Problems and Solutions: A Piecewise Dynamic Programming Approach

The investor maximizes the following expected utility:

$$\max_{\{\phi_t\}_{t=0}^T,\{C_t\}_{t=0}^T} \mathbb{E}_0\left[\int_0^T \alpha e^{-\beta t}U(C_t) + (1-\alpha)e^{-\beta T}V(W_T)\right], \tag{3.1}$$

where $\phi_t$ is the portfolio weight of the risky asset, $C_t$ and $W_t$ are the consumption rate and wealth respectively financed by the trading strategy $\phi_t$, $U(\cdot)$ and $V(\cdot)$ are utility functions, $\beta$ is the subjective discount rate, and parameter $\alpha$ determines the relative importance of the intermediate consumption and the bequest. Assume that, apart from the stock (index), the investors asset menu also contains a riskless cash account with a constant riskless interest rate $r$. Then the wealth process satisfies

$$dW_t = \Big\{W_t\big[r + (a + b_1 X_{t-\tau} + b_2 Y_t - r)\phi_t\big] - C_t\Big\}dt + \sigma W_t \phi_t dZ_t.$$

Because SDDEs are path-dependent and there is no Ito's formula for them, we cannot use the standard dynamic programming method. To overcome this problem, we develop a *piecewise dynamic programming method* to solve the stochastic control problem path by path by introducing new state variables. The main idea is described as follows.

We first solve the problem for $0 \le T \le \tau$. In this case, the delayed variable $X_{u-\tau}$ for $u \le T$ is known at time 0 given the initial conditions. By constructing a new state variable $A_1^{(1)}$ summarizing the delayed information, $W, X, Y, A_1^{(1)}$ constitute a sufficient statistic for the indirect utility $J_t = J(t, W, X, Y, A_1^{(1)})$. Then we can apply the standard dynamic programming method.

Based on the solution for $0 \le T \le \tau$, we then solve the problem for $\tau \le T \le 2\tau$. Bellman's principle of optimality leads to $J_t = \max\big\{\mathbb{E}_t[J_{T-\tau}]\big\}$, where $J_{T-\tau}$ has been derived in the first step. So we transfer the problem into one with horizon shorter than $\tau$. In this case, $W, X, Y, A_1^{(1)}$ cannot constitute a sufficient statistic for the indirect utility because the delayed variables are not adapted to the filtration $\mathcal{F}_0$. To derive the dynamics of $J$, we construct new *path-induced state variables* $\tilde{B}^{(2)}$ and $A_1^{(2)}$ which, together with the *original state variables* $X, Y, W$, constitute a sufficient statistic for the optimal portfolios. Then we can apply the standard dynamic programming method.

Therefore, we overcome the problem that there is no Ito's formula with respect to the paths by introducing path-induced state variables, to which Ito's formula applies.

Similarly, we can solve the portfolios for $n\tau \le T \le (n+1)\tau$, $n = 2, 3, \cdots$ recursively by using forward induction steps of length $\tau$. Due to the path dependence, the solution has to be given piecewise.

This method is applicable to general utilities $U$ and $V$. For CRRA utility, the HJB equations can be solved up to the solutions to ODEs as shown in the next section.

## 4. CRRA Utility

Assume that the preferences of the investor can be represented by a CRRA utility index. The investor's optimization problem (3.1) becomes

$$\max_{\{\phi_t\}_{t=0}^T, \{C_t\}_{t=0}^T} \mathbb{E}_0\left[\int_0^T \alpha e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma} + (1-\alpha)e^{-\beta T}\frac{W_T^{1-\gamma}}{1-\gamma}\right],$$

where $\gamma > 0$ is the risk aversion coefficient (as well as being the inverse of the elasticity of intertemporal substitution).

4.1. **Investment Horizon Shorter Than $\tau$.** When $0 \le T \le \tau$, $X_{u-\tau}$ for all $u \le T$ is realized value of the predictors and is known at time 0. Let $J(t, W, X, Y)$ denote the indirect utility function. Bellman's principle of optimality leads to the following Hamilton-Jacobi-Bellman (HJB) equation (Merton, 1971) for $J$,

$$\begin{aligned}\max_{\phi, C}\Bigg\{&\frac{\partial J}{\partial t} + \frac{\partial J}{\partial W}W_t\big[r + (a + b_1X_{t-\tau} + b_2Y_t - r)\phi_t\big] - \frac{\partial J}{\partial W}C_t \\ &+ \frac{\partial J}{\partial X}\kappa(\bar{X} - X_t) + \frac{\partial J}{\partial Y}\Big(\frac{k}{1-e^{-k\tau}}X_t - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau} - kY_t\Big) \\ &+ \frac{\sigma^2}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\phi_t^2 + \frac{\partial^2 J}{\partial W \partial X}\sigma\sigma_X\rho W_t\phi_t + \frac{\sigma_X^2}{2}\frac{\partial^2 J}{\partial X^2} + \alpha e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}\Bigg\} = 0,\end{aligned} \tag{4.1}$$

with boundary condition

$$J(T, W, X, Y) = (1-\alpha)e^{-\beta T}\frac{W^{1-\gamma}}{1-\gamma}.$$

The optimal consumption, optimal portfolio weight and value function are summarized by the following proposition. We show the details in Appendix A.1.

**Proposition 4.1.** *When $0 \le T \le \tau$, the optimal consumption and optimal portfolio weight are given by*

$$\begin{aligned}C_t^* &= \alpha^{\frac{1}{\gamma}}W_t f^{-1}, \\ \phi_t^* &= \frac{1}{\gamma\sigma^2}\Big(a + b_1X_{t-\tau} + b_2Y_t - r + \gamma\sigma\sigma_X\rho\frac{\partial \ln f}{\partial X}\Big),\end{aligned} \tag{4.2}$$

*and the value function is given by*

$$J(t, W, X, Y) = e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\big[f(t, X, Y)\big]^{\gamma}, \tag{4.3}$$

*for* $0 \le t \le T$.

*The wealth-consumption ratio* $f$ *is given by*

$$f(t,X,Y) = \alpha^{\frac{1}{\gamma}} \int_t^T \hat{f}(u,X,Y)du + (1-\alpha)^{\frac{1}{\gamma}} \hat{f}(t,X,Y), \tag{4.4}$$

*where*

$$\hat{f}(t,X,Y) = \exp\left\{ \frac{A_{11,t}^{(1)}}{2} X_t^2 + A_{12,t}^{(1)} X_t Y_t + \frac{A_{22,t}^{(1)}}{2} Y_t^2 + A_{1,t}^{(1)} X_t + A_{2,t}^{(1)} Y_t + A_{3,t}^{(1)} \right\},$$

*and* $A_{ij,t}^{(1)}$ *and* $A_{i,t}^{(1)}$ *are governed by ODEs* (A.3) *in Appendix A.1.*

**Corollary 4.2.** *In particular, for an asset allocation problem (*$\alpha = 0$*),*

$$f = \exp\left\{ \frac{A_{11,t}^{(1)}}{2} X_t^2 + A_{12,t}^{(1)} X_t Y_t + \frac{A_{22,t}^{(1)}}{2} Y_t^2 + A_{1,t}^{(1)} X_t + A_{2,t}^{(1)} Y_t + A_{3,t}^{(1)} \right\},$$

*and the optimal portfolio weight is given by*

$$\phi_t^* = \frac{a + b_1 X_{t-\tau} + b_2 Y_t - r}{\gamma \sigma^2} + \frac{\sigma_X \rho}{\sigma} \left( A_{1i,t}^{(1)} X_t + A_{12,t}^{(1)} Y_t + A_{1,t}^{(1)} \right). \tag{4.5}$$

4.2. **Investment Horizon Longer Than** $\tau$**.** When $T \ge \tau$, different from the case when $T \le \tau$ in Subsection 4.1, the delayed variable $X_{u-\tau}$ becomesunknown at time 0 for $u \ge \tau$. In addition, $X_{u-\tau}$ cannot be spanned by $X_u$ and $Y_u$. So we need more state variables to constitute a sufficient statistic for the indirect utility function. In the following analysis, we will show that the state variables are different for different investment horizons. In fact, as horizon lengthens, the number of state variables increases without bound. Therefore, we have to solve the problem piecewise. For a given investment horizon, we will first introduce some path-induced state variables, which together with the original state variables, constitute a sufficient statistic. Then we can write down the HJB equation with respect to all the state variables.

For asset allocation problems without intermediate consumption ($\alpha = 0$), we can solve the optimal portfolio up to the solutions to ODEs for any investment horizons. The results are given piecewise and the proof follows mathematical induction. The results are summarized in the following proposition.

**Proposition 4.3.** *Assume* $\alpha = 0$ *and* $(n-1)\tau \le T \le n\tau$, $n = 1, 2, \cdots$. *For* $0 \le t \le T - (n-1)\tau$, *the optimal portfolio weight is given by*

$$\phi_t^* = \frac{a + b_1 X_{t-\tau} + b_2 Y_t - r}{\gamma \sigma^2} + \frac{\sigma_X \rho}{\sigma} \left( \sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)} \tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)} \right), \tag{4.6}$$

*and the value function is given by*

$$J_t = e^{-\beta t} \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left\{ \gamma \left( \frac{1}{2} \sum_{i=1}^{2^{n+1}-2} \sum_{j=1}^{2^{n+1}-2} A_{ij,t}^{(n)} \tilde{B}_{i,t}^{(n)} \tilde{B}_{j,t}^{(n)} + \sum_{i=1}^{2^{n+1}-2} A_{i,t}^{(n)} \tilde{B}_{i,t}^{(n)} + \tilde{B}_{2^{n+1}-1,t}^{(n)} + A_{2^{n+1}-1,t}^{(n)} \right) \right\}, \tag{4.7}$$

*where the coefficients $A^{(n)}$ are governed by ODEs*

$$\dot{A}^{(n)}_{ij,t} = \mathbb{F}^{(n)}_{ij}\Big(\big\{A^{(n)}_{kl,t}\big\}^{2^{n+1}-2}_{k,l=1}, t\Big),\ i,j = 1,\cdots,2^{n+1}-2,\ \textit{with } A^{(n)}_{kl,t} = A^{(n)}_{lk,t},$$
$$\dot{A}^{(n)}_{i,t} = \mathbb{F}^{(n)}_{i}\Big(X_{t-\tau}, \big\{A^{(n)}_{k,t}\big\}^{2^{n+1}-2}_{k=1}, t\Big),\ i = 1,\cdots,2^{n+1}-1,$$

*and the state variables $\tilde{B}^{(n)}_{1,t} = X_t$, $\tilde{B}^{(n)}_{2,t} = Y_t$, and $\tilde{B}^{(n)}_{i,t}$ for $i = 3,\cdots,2^{n+1}-1$ are governed by random ODEs*

$$\dot{\tilde{B}}^{(n)}_{i,t} = \mathbb{G}^{(n)}_{i}\Big(\big\{\tilde{B}^{(n)}_{j,t}\big\}^{2^n}_{j=1}, t\Big).$$

*Moreover,*

(i) *$\mathbb{F}$'s are governed by* (4.26) *with the following properties:*
  (iA) *$\mathbb{F}^{(n)}_{ij}$ is a quadratic function of $A^{(n)}_{kl,t}$, and terminal conditions are given by $A^{(n)}_{ij,T-(n-1)\tau} = constant$.*
  (iB) *$\mathbb{F}^{(n)}_{i}$ is a linear function of $X_{t-\tau}$ and $A^{(n)}_{k,t}$, for $i = 1,\cdots,2^{n+1}-2$, and a quadratic function of $X_{t-\tau}$ and $A^{(n)}_{k,t}$, for $i = 2^{n+1}-1$; and terminal conditions are given by $A^{(n)}_{i,T-(n-1)\tau} = 0$.*

(ii) *$\mathbb{G}$'s are governed by* (4.14) *and* (4.18) *with the following properties: $\mathbb{G}^{(n)}_i$ is a linear function for $i = 3,\cdots,2^{n+1}-2$; and a quadratic function for $i = 2^{n+1}-1$; and the initial conditions are given by $\tilde{B}^{(n)}_{i,T-n\tau} = 0$, for $i = 3,\cdots,2^{n+1}-1$.*

*Proof.* The proof follows from the mathematical induction.

When $n = 1$, Proposition 4.3 holds as shown in Corollary 4.2.[10]

Assume Proposition 4.3 holds for $(n-2)\tau \le T \le (n-1)\tau$, $n = 2,3,\cdots$. For $(n-1)\tau \le T \le n\tau$ and $0 \le t \le T-(n-1)\tau$,

$$\begin{aligned}
J_t &= \max_{\phi}\big\{\mathbb{E}_t[J_{T-(n-1)\tau}]\big\}\\
&= \max_{\phi}\Bigg\{e^{-\beta[T-(n-1)\tau]}\mathbb{E}_t\Bigg[\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Bigg\{\gamma\Bigg(\frac{1}{2}\sum_{i=1}^{2^n-2}\sum_{j=1}^{2^n-2}A^{(n-1)}_{ij,T-(n-1)\tau}\tilde{B}^{(n-1)}_{i,T-(n-1)\tau}\tilde{B}^{(n-1)}_{j,T-(n-1)\tau}\\
&\qquad+\sum_{i=1}^{2^n-2}A^{(n-1)}_{i,T-(n-1)\tau}\tilde{B}^{(n-1)}_{i,T-(n-1)\tau}+\tilde{B}^{(n-1)}_{2^n-1,T-(n-1)\tau}+A^{(n-1)}_{2^n-1,T-(n-1)\tau}\Bigg)\Bigg\}\Bigg]\Bigg\}\\
&= \max_{\phi}\Bigg\{e^{-\beta[T-(n-1)\tau]}\mathbb{E}_t\Bigg[\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Bigg\{\gamma\Bigg(\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}A^{(n-1)}_{ij,T-(n-1)\tau}\tilde{B}^{(n-1)}_{i,T-(n-1)\tau}\tilde{B}^{(n-1)}_{j,T-(n-1)\tau}\\
&\qquad+\sum_{i=1}^{2}A^{(n-1)}_{i,T-(n-1)\tau}\tilde{B}^{(n-1)}_{i,T-(n-1)\tau}+A^{(n-1)}_{2^n-1,T-(n-1)\tau}\Bigg)\Bigg\}\Bigg]\Bigg\}.
\end{aligned} \tag{4.8}$$

[10]Notice that $\tilde{B}^{(1)}_{3,t} = 0$ in this case.

The first equality follows from the Bellman's principle of optimality, the second equality follows from the assumption that Proposition 4.3 holds for $(n-2)\tau \le T \le (n-1)\tau$, and the last equality follows from the zero initial conditions $\tilde{B}^{(n-1)}_{i,T-(n-1)\tau} = 0$ for $i = 3, \cdots, 2^n - 1$.

In the last equation of (4.8), $A^{(n-1)}_{ij,T-(n-1)\tau}$ are deterministic (in fact, they are constant values) because $A^{(n-1)}_{ij,t}$ are governed by deterministic ODEs.

However, $A^{(n-1)}_{i,T-(n-1)\tau}$ are not $\mathcal{F}_t$-measurable. In fact, they are governed by backward random ODEs, so they depend on all $X_u$ for $u \in [T-n\tau, T-(n-1)\tau]$, which are not completely realized at time $t$. As a result, $A^{(n-1)}_{i,T-(n-1)\tau}$ lead to new state variables. In the following analysis, we first rewrite the last equation of (4.8) by introducing new state variables, then write down the HJB equation based on the new state variables together with the original state variables $X, Y$ and $W$, and finally derive the solutions of the HJB equation.

**Introducing New State Variables.** According to the assumption, $(\dot{A}^{(n-1)}_{1,t}, \cdots, \dot{A}^{(n-1)}_{2^n-2,t})^\top$ is a system of linear differential equations. We rewrite it as

$$\big(\dot{A}^{(n-1)}_{1,t}, \cdots, \dot{A}^{(n-1)}_{2^n-2,t}\big)' = \mathbf{G}^{(n)}\Big(\big\{A^{(n-1)}_{j,t}\big\}_{j=1}^{2^n-2}, t\Big) + \mathbf{H}^{(n)}(X_{t-\tau}, t), \tag{4.9}$$

where $\mathbf{G}^{(n)}$ is a homogeneous linear vector function of $A^{(n-1)}_{j,t}$ and $\mathbf{H}^{(n)}$ is a linear vector function of $X_{t-\tau}$. Denote by $\mathbf{\Phi}^{(n)}_t$ the basic solution matrix of the following homogeneous linear differential system,

$$\big(\dot{\hat{A}}^{(n-1)}_{1,t}, \cdots, \dot{\hat{A}}^{(n-1)}_{2^n-2,t}\big)' = \mathbf{G}^{(n)}\Big(\big\{\hat{A}^{(n-1)}_{j,t}\big\}_{j=1}^{2^n-2}, t\Big). \tag{4.10}$$

Standard ODE theory (Hale, 1969) implies that

$$\big(A^{(n-1)}_{1,t}, \cdots, A^{(n-1)}_{2^n-2,t}\big)' = -\int_t^{T-(n-2)\tau} \mathbf{\Phi}^{(n)}_t \Big(\mathbf{\Phi}^{(n)}_u\Big)^{-1} \mathbf{H}^{(n)}(X_{u-\tau}, u)du, \tag{4.11}$$

and hence

$$\begin{aligned}
&\big(A^{(n-1)}_{1,T-(n-1)\tau}, \cdots, A^{(n-1)}_{2^n-2,T-(n-1)\tau}\big)' \\
&= -\int_{T-(n-1)\tau}^{T-(n-2)\tau} \mathbf{\Phi}^{(n)}_{T-(n-1)\tau} \Big(\mathbf{\Phi}^{(n)}_u\Big)^{-1} \mathbf{H}^{(n)}(X_{u-\tau}, u)du \\
&= -\int_{T-n\tau}^{T-(n-1)\tau} \mathbf{\Phi}^{(n)}_{T-(n-1)\tau} \Big(\mathbf{\Phi}^{(n)}_{u+\tau}\Big)^{-1} \mathbf{H}^{(n)}(X_u, u+\tau)du.
\end{aligned} \tag{4.12}$$

Define

$$\big(\tilde{B}^{(n)}_{3,t}, \cdots, \tilde{B}^{(n)}_{2^n,t}\big)' = -\int_{T-n\tau}^{t} \mathbf{\Phi}^{(n)}_{T-(n-1)\tau} \Big(\mathbf{\Phi}^{(n)}_{u+\tau}\Big)^{-1} \mathbf{H}^{(n)}(X_u, u+\tau)du, \tag{4.13}$$

for $T-n\tau \le t \le T-(n-1)\tau$. Then $\tilde{B}_{i,t}^{(n)}$, $i=3,\cdots,2^n$ are forward and adapted to the filtration $\mathcal{F}_t$. They are governed by

$$\begin{aligned}&\big(\dot{\tilde{B}}_{3,t}^{(n)},\cdots,\dot{\tilde{B}}_{2^n,t}^{(n)}\big)' = -\mathbf{\Phi}_{T-(n-1)\tau}^{(n)}\Big(\mathbf{\Phi}_{t+\tau}^{(n)}\Big)^{-1}\mathbf{H}^{(n)}(X_t,t+\tau) \doteq \big(\mathbb{G}_3^{(n)}(X_t,t),\cdots,\mathbb{G}_{2^n}^{(n)}(X_t,t)\big)',\\ &\text{with}\quad \tilde{B}_{i,T-n\tau}^{(n)}=0, \qquad \tilde{B}_{i,T-(n-1)\tau}^{(n)} = A_{i-2,T-(n-1)\tau}^{(n-1)}.\end{aligned} \tag{4.14}$$

In addition, (4.11) and (4.13) imply the following relationship

$$\begin{aligned}\big(A_{1,t}^{(n-1)},\cdots,A_{2^n-2,t}^{(n-1)}\big)' &= -\int_{t-\tau}^{T-(n-1)\tau}\mathbf{\Phi}_t^{(n)}\Big(\mathbf{\Phi}_{u+\tau}^{(n)}\Big)^{-1}\mathbf{H}^{(n)}(X_u,u+\tau)du\\ =-\Bigg(\int_{T-n\tau}^{T-(n-1)\tau}-\int_{T-n\tau}^{t-\tau}\Bigg)&\mathbf{\Phi}_t^{(n)}\Big(\mathbf{\Phi}_{u+\tau}^{(n)}\Big)^{-1}\mathbf{H}^{(n)}(X_u,u+\tau)du\\ =\mathbf{\Phi}_t^{(n)}\Big(\mathbf{\Phi}_{T-(n-1)\tau}^{(n)}\Big)^{-1}&\big[\big(\tilde{B}_{3,T-(n-1)\tau}^{(n)},\cdots,\tilde{B}_{2^n,T-(n-1)\tau}^{(n)}\big)'-\big(\tilde{B}_{3,t-\tau}^{(n)},\cdots,\tilde{B}_{2^n,t-\tau}^{(n)}\big)'\big].\end{aligned} \tag{4.15}$$

Therefore, $\{A_{i,t}^{(n-1)}\}_{i=1}^{2^n-2}$ are transferred to $\{\tilde{B}_{j,t}^{(n)}\}_{j=3}^{2^n}$. Now we study $A_{2^n-1,T-(n-1)\tau}^{(n-1)}$ in (4.8), and rewrite it as

$$\begin{aligned}A_{2^n-1,T-(n-1)\tau}^{(n-1)} &= -\int_{T-(n-1)\tau}^{T-(n-2)\tau}\mathbb{F}_{2^n-1}^{(n-1)}(X_{u-\tau},A_{i,u}^{(n-1)}:i=1,\cdots,2^n-2,u)du\\ &= -\int_{T-n\tau}^{T-(n-1)\tau}\mathbb{F}_{2^n-1}^{(n-1)}(X_u,A_{i,u+\tau}^{(n-1)}:i=1,\cdots,2^n-2,u+\tau)du.\end{aligned} \tag{4.16}$$

Notice that $A_{i,u+\tau}^{(n-1)}$ is linear in $\{\tilde{B}_{j,T-(n-1)\tau}^{(n)}\}_{j=3}^{2^n}$ and $\{\tilde{B}_{j,u}^{(n)}\}_{j=3}^{2^n}$ implied by (4.15), and $\mathbb{F}_{2^n-1}^{(n-1)}$ is a quadratic function. So we can rewrite (4.16) as

$$\begin{aligned}A_{2^n-1,T-(n-1)\tau}^{(n-1)} =&\sum_{i=3}^{2^n}\sum_{j=3}^{2^n}C_{ij}^{(n)}\tilde{B}_{i,T-(n-1)\tau}^{(n)}\tilde{B}_{j,T-(n-1)\tau}^{(n)}\\ &+\sum_{i=3}^{2^n}\int_{T-(n)\tau}^{T-(n-1)\tau}\mathbb{G}_{i+2^n-2}^{(n)}\Big(\{\tilde{B}_{j,u}^{(n)}\}_{j=1}^{2^n},u\Big)du\tilde{B}_{i,T-(n-1)\tau}^{(n)}\\ &+\int_{T-(n)\tau}^{T-(n-1)\tau}\mathbb{G}_{2^{n+1}-1}^{(n)}\Big(\{\tilde{B}_{j,u}^{(n)}\}_{j=1}^{2^n},u\Big)du,\end{aligned} \tag{4.17}$$

where $\tilde{B}_{1,u}^{(n)}=X_u$, $\tilde{B}_{2,u}^{(n)}=Y_u$, for $T-n\tau\le u\le T-(n-1)\tau$; $C_{ij}^{(n)}$ are constants; $\mathbb{G}_{i+2^n-2}^{(n)}$, $i=3,\cdots,2^n$ is linear in $\{\tilde{B}_{j,u}^{(n)}\}_{j=1}^{2^n}$; and $\mathbb{G}_{2^{n+1}-1}^{(n)}$ is quadratic in $\{\tilde{B}_{j,u}^{(n)}\}_{j=1}^{2^n}$.

Define

$$\tilde{B}_{i+2^n-2,t}^{(n)} = \int_{T-n\tau}^{t}\mathbb{G}_{i+2^n-2}^{(n)}\Big(\{\tilde{B}_{j,u}^{(n)}\}_{j=1}^{2^n},u\Big)du, \qquad i=3,\cdots,2^n+1, \tag{4.18}$$

for $T-n\tau \le t \le T-(n-1)\tau$. Then $\tilde{B}^{(n)}_{i+2^n-2,t}$ is forward and adapted to the filtration $\mathcal{F}_t$, governed by

$$\begin{aligned}&\dot{\tilde{B}}^{(n)}_{i+2^n-2,t} = \mathbb{G}^{(n)}_{i+2^n-2}\Big(\big\{\tilde{B}^{(n)}_{j,t}\big\}_{j=1}^{2^n}, t\Big),\\ &\tilde{B}^{(n)}_{i+2^n-2,T-n\tau} = 0,\end{aligned} \tag{4.19}$$

and $\tilde{B}^{(n)}_{i+2^n-2,T-(n-1)\tau}$ is the coefficient of $\tilde{B}^{(n)}_{i,T-(n-1)\tau}$ in (4.17).

The state variables $\tilde{B}^{(n)}$ governed by random ODEs (4.14) and (4.18), together with the original state variables $X_t$, $Y_t$ and $W_t$, constitute a sufficient statistic for the indirect utility function. The last equation of (4.8) can be rewritten as

$$\begin{aligned}&J(t, W, \tilde{B}^{(n)}_i : i = 1, \cdots, 2^{n+1}-1)\\ =& \max_{\phi}\Big\{e^{-\beta[T-(n-1)\tau]}\mathbb{E}_t\Big[\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Big\{\gamma\Big(\frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2}A^{(n-1)}_{ij,T-(n-1)\tau}\tilde{B}^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{j,T-(n-1)\tau}\\ &+\sum_{i=1}^{2}\tilde{B}^{(n)}_{i+2,T-(n-1)\tau}\tilde{B}^{(n)}_{i,T-(n-1)\tau} + \sum_{i=3}^{2^n}\sum_{j=3}^{2^n}C^{(n)}_{ij}\tilde{B}^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{j,T-(n-1)\tau}\\ &+\sum_{i=3}^{2^n}\tilde{B}^{(n)}_{i+2^n-2,T-(n-1)\tau}\tilde{B}^{(n)}_{i,T-(n-1)\tau} + \tilde{B}^{(n)}_{2^{n+1}-1,T-(n-1)\tau}\Big)\Big\}\Big]\Big\}\\ \doteq& \max_{\phi}\Big\{e^{-\beta[T-(n-1)\tau]}\mathbb{E}_t\Big[\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Big\{\gamma\Big(\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2}C^{(n)}_{ij}\tilde{B}^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{j,T-(n-1)\tau}\\ &+\tilde{B}^{(n)}_{2^{n+1}-1,T-(n-1)\tau}\Big)\Big\}\Big]\Big\}.\end{aligned} \tag{4.20}$$

**HJB Equation and Its Solution.** The HJB equation for $(n-1)\tau \le T \le n\tau$ is given by

$$\begin{aligned}\max_{\phi}\Big\{&\frac{\partial J}{\partial t} + \frac{\partial J}{\partial W}W_t\big[r + (a + b_1X_{t-\tau} + b_2Y_t - r)\phi_t\big] + \frac{\partial J}{\partial X}\kappa(\bar{X} - X_t)\\ &+\frac{\partial J}{\partial Y}\Big(\frac{k}{1-e^{-k\tau}}X_t - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau} - kY_t\Big) + \frac{\sigma^2}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\phi_t^2\\ &+\frac{\partial^2 J}{\partial W\partial X}\sigma\sigma_X\rho W_t\phi_t + \frac{\sigma_X^2}{2}\frac{\partial^2 J}{\partial X^2} + \sum_{i=3}^{2^{n+1}-1}\frac{\partial J}{\partial\tilde{B}^{(n)}_i}\mathbb{G}^{(n)}_i\Big\} = 0,\end{aligned} \tag{4.21}$$

with boundary condition

$$\begin{aligned}&J_{T-(n-1)\tau}\\ =&e^{-\beta[T-(n-1)\tau]}\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Big\{\gamma\Big(\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2}C^{(n)}_{ij}\tilde{B}^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{j,T-(n-1)\tau} + \tilde{B}^{(n)}_{2^{n+1}-1,T-(n-1)\tau}\Big)\Big\}.\end{aligned} \tag{4.22}$$

$J$ is conjectured to have the form

$$
\begin{aligned}
&J(t, W, \tilde{B}_i^{(n)} : i = 1, \cdots, 2^{n+1} - 1) \\
=&e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\Big(\frac{1}{2}\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2} A_{ij,t}^{(n)}\tilde{B}_{i,t}^{(n)}\tilde{B}_{j,t}^{(n)} + \sum_{i=1}^{2^{n+1}-2} A_{i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + \tilde{B}_{2^{n+1}-1,t}^{(n)} + A_{2^{n+1}-1,t}^{(n)}\Big)\Big\},
\end{aligned}
\tag{4.23}
$$

where $A_{kl,t}^{(n)} = A_{lk,t}^{(n)}$. Using this conjecture, the optimal portfolio weight is given by

$$
\phi_t^* = \frac{a + b_1 X_{t-\tau} + b_2 Y_t - r}{\gamma\sigma^2} + \frac{\sigma_X\rho}{\sigma}\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big), \tag{4.24}
$$

and (4.21) becomes

$$
\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2} \dot{A}_{ij,t}^{(n)}\tilde{B}_{i,t}^{(n)}\tilde{B}_{j,t}^{(n)} + \sum_{i=1}^{2^{n+1}-2}\dot{A}_{i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + \dot{A}_{2^{n+1}-1,t}^{(n)} + \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma} \\
&+\frac{(1-\gamma)\sigma_X\rho}{\gamma\sigma}(a + b_1X_{t-\tau} + b_2\tilde{B}_{2,t}^{(n)} - r)\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big)\Big] \\
&+\frac{(1-\gamma)(a + b_1X_{t-\tau} + b_2\tilde{B}_{2,t}^{(n)} - r)^2}{\gamma^2\sigma^2} + \kappa(\bar{X} - \tilde{B}_{1,t}^{(n)})\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big) \\
&+\Big(\frac{k}{1-e^{-k\tau}}\tilde{B}_{1,t}^{(n)} - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau} - k\tilde{B}_{2,t}^{(n)}\Big)\Big(\sum_{i=1}^{2^{n+1}-2} A_{2i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{2,t}^{(n)}\Big) \\
&-\frac{(1-\gamma)\sigma^2}{2}\Big[\frac{a + b_1X_{t-\tau} + b_2\tilde{B}_{2,t}^{(n)} - r}{\gamma\sigma^2} + \frac{\sigma_X\rho}{\sigma}\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big)\Big]^2 \\
&+\frac{(1-\gamma)\sigma_X\rho}{\gamma\sigma}(a + b_1X_{t-\tau} + b_2\tilde{B}_{2,t}^{(n)} - r)\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big) \\
&+(1-\gamma)\sigma_X^2\rho^2\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big)^2 + \frac{\gamma\sigma_X^2}{2}\Big(\sum_{i=1}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\Big)^2 \\
&+\frac{\sigma_X^2}{2}A_{11,t}^{(n)} + \sum_{i=3}^{2^{n+1}-2}\mathbb{G}_i^{(n)}\Big(\big\{\tilde{B}_{j,t}^{(n)}\big\}_{j=1}^{2^n}, t\Big)\sum_{j=1}^{2^{n+1}-2}\Big(\sum_{j=1}^{2^{n+1}-2} A_{ij,t}^{(n)}\tilde{B}_{j,t}^{(n)} + A_{i,t}^{(n)}\Big) \\
&+\mathbb{G}_{2^{n+1}-1}^{(n)}\Big(\big\{\tilde{B}_{j,t}^{(n)}\big\}_{j=1}^{2^n}, t\Big) = 0,
\end{aligned}
\tag{4.25}
$$

implying that

$$
\begin{aligned}
\dot{A}_{ij,t}^{(n)} &= \mathbb{F}_{ij}^{(n)}\Big(\big\{A_{kl,t}^{(n)}\big\}_{k,l=1}^{2^{n+1}-2}, t\Big),\ i, j = 1, \cdots, 2^{n+1} - 2, \\
\dot{A}_{i,t}^{(n)} &= \mathbb{F}_i^{(n)}\Big(X_{t-\tau}, \big\{A_{k,t}^{(n)}\big\}_{k=1}^{2^{n+1}-2}, t\Big),\ i = 1, \cdots, 2^{n+1} - 1,
\end{aligned}
\tag{4.26}
$$

where $X_{t-\tau}$ is known at time $t$ and can be treated as a function of $t$. To save space, we do not specify the forms of $\mathbb{F}_{ij}^{(n)}$ which can be easily derived from (4.25) by matching the coefficients of $\tilde{B}_{i,t}^{(n)}\tilde{B}_{j,t}^{(n)}$ and $\tilde{B}_{i,t}^{(n)}$. It follows from (4.14) and (4.18) that $\mathbb{G}_i^{(n)}$ is a linear function for $i = 3, \cdots, 2^{n+1}-2$ and a quadratic function for $i = 2^{n+1}-1$. Therefore, $\mathbb{F}_{ij}^{(n)}$ is a quadratic function of $A_{kl,t}^{(n)}$, $\mathbb{F}_i^{(n)}$ is a linear function of $X_{t-\tau}$ and $A_{k,t}^{(n)}$, for $i = 1, \cdots, 2^{n+1}-2$, and a quadratic function of $X_{t-\tau}$ and $A_{k,t}^{(n)}$, for $i = 2^{n+1}-1$; and it follows from (4.22) and (4.23) that the terminal conditions are given by $A_{ij,T-(n-1)\tau}^{(n)} = 2C_{ij}^{(n)}$, and $A_{i,T-(n-1)\tau}^{(n)} = 0$.

Mathematical induction implies the completeness of the proof. □

From Proposition 4.3 we have several observations, which are novel in the stochastic control literature. First, we reduce the non-Markovian system to a *horizon-reducible Markovian system* and solve the infinite-dimensional stochastic control problems path by path. As far as we know, this is the first nontrivial closed-form solution for such systems.

Second, due to the path dependence, Proposition 4.3 shows that the optimal portfolio has different forms when the investment horizon $T$ locates in different time intervals with length of $\tau$.

The solutions can be solved iteratively. When $n = 1$, $\tilde{B}_{1,t}^{(1)} = X_t$ and $\tilde{B}_{2,t}^{(1)} = Y_t$, and $A^{(1)}$ is given by (A.3). Together with the relationships (4.14) and (4.19), $\tilde{B}^{(2)}$ can be determined. Then $A^{(2)}$ can be derived according to (4.26). Using this procedure, we can derive $A^{(n)}$ and $B^{(n)}$ for $n = 3, 4, \cdots$. It follows from (4.14) that $\tilde{B}^{(n)}$'s are determined by the basic solution matrix of a homogeneous linear ordinary differential system (4.10). Therefore, we solve the dynamic asset allocation problem up to the solutions to the systems of linear ordinary differential equations.

Third, the intertemporal hedging demand in Proposition 4.3 consists of two components. The first component $\sigma_X\rho/\sigma\big(A_{11,t}^{(n)}X_t + A_{12,t}^{(n)}Y_t\big)$ is caused by the original state variables. The second component $\sigma_X\rho/\sigma\big(\sum_{i=3}^{2^{n+1}-2} A_{1i,t}^{(n)}\tilde{B}_{i,t}^{(n)} + A_{1,t}^{(n)}\big)$ corresponding to path hedging is introduced by the path-induced state variables. In the optimal portfolio (4.6), $A_{1,t}^{(n)}$ utilizes the historical values of $X$ over the period $[-\tau, T-n\tau]$ as shown by (4.26), $\tilde{B}_{i,t}^{(n)}$, $i = 3, \cdots, 2^{n+1}-2$ utilizes the historical values of $X$ over the period $[T-n\tau, 0]$ as shown by (4.14) and (4.19), and $A_{ij}^{(n)}$ is governed by deterministic ODEs. Fig. 4.1 illustrates the time line for the formations of $A_1^{(n)}$ and $\tilde{B}_i^{(n)}$. The new path-induced state variables $A_1^{(n)}$ and $\tilde{B}_i^{(n)}$, $i = 3, \cdots, 2^{n+1}-2$ capture complementary historical information and, together with the original state variables, constitute a sufficient statistic for the optimal portfolio. **Do we need Fig. 4.1?** As a result, all the historical prices over $[-\tau, 0]$ are used by the optimal portfolio, and the infinite-dimensional problem is boiled down by introducing a finite number of new state variables.

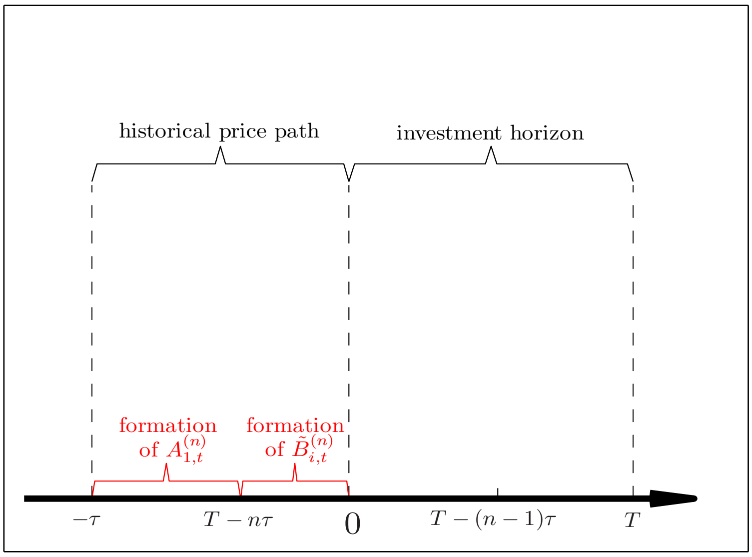


FIGURE 4.1. The time line for the information used to form $A^{(n)}_{1,t}$ and $\tilde{B}^{(n)}_{i,t}$, $i = 3, \cdots, 2^{n+1} - 2$, when $(n-1)\tau \le T \le n\tau$.

Fourth, the following corollary shows that the new path-induced state variables are weighted sums of historical state variables or squared historical state variables.

**Corollary 4.4.** *When $(n-1)\tau \le T \le n\tau$, the number of variable $\tilde{B}^{(n)}$'s is $2^n$, and $\tilde{B}^{(n)}$'s have the following forms,*

$$\tilde{B}^{(n)}_{1,t} = X_t, \qquad \tilde{B}^{(n)}_{2,t} = Y_t,$$

$$\tilde{B}^{(n)}_{i,t} = \int_{T-n\tau}^{t} \mathbb{H}^{(n)}_{i,1}(u)X_u + \mathbb{H}^{(n)}_{i,2}(u)Y_u + \mathbb{H}^{(n)}_{i,3}(u)du, \qquad i = 3, \cdots, 2^{n+1} - 2,$$

$$\tilde{B}^{(n)}_{2^{n+1}-1,t} = \int_{T-n\tau}^{t} \left[\mathbb{H}^{(n)}_{2^{n+1}-1,1}(u)X_u + \mathbb{H}^{(n)}_{2^{n+1}-1,2}(u)Y_u + \mathbb{H}^{(n)}_{2^{n+1}-1,3}(u)\right]^2 du,$$

*where $\mathbb{H}^{(n)}_{i,j}(u)$ are deterministic functions of time $u$.*

Corollary 4.4 shows that the number of variables $\tilde{B}^{(n)}$ increases as the investment horizon $T$ lengthens and falls into the next $\tau$-length time interval. There are $2^{n+1}-1$ $\tilde{B}^{(n)}$ when $(n-1)\tau \le T \le n\tau$.

The new state variables $\tilde{B}^{(n)}_i$, $i = 3, \cdots, 2^{n+1} - 2$ are weighted sums of historical state variables, while $\tilde{B}^{(n)}_{2^{n+1}-1}$ is weighted sums of the historical state variables and their squared values. Furthermore, the new path-induced state variables $\tilde{B}$'s are governed by random ODEs, not diffusion processes. Hence they do not introduce any new source of uncertainty,[11] and also do not affect the market completeness.

Fifth, although the portfolio weight and value function are given piecewise in (4.6)-(4.7), the following corollary shows that they are continuous in time.

**Corollary 4.5.** *The portfolio weight* (4.6) *and value function* (4.7) *are continuous functions of* $t$*.*

Finally, Proposition 4.3 shows that the optimal portfolio and the indirect utility depend on the path-dependent coefficients $A$'s and the path-induced state variables $\tilde{B}$, and hence they are also path dependent. Different historical paths lead to different optimal portfolios. The new variables $\tilde{B}_{i,t}^{(n)}$, $i = 3, \cdots, 2^{n+1} - 2$ and $A_{1,t}^{(n)}$ exploring the path effect have non-monotonic horizon dependence. This implies that the optimal portfolio weight should not be a monotonic function of investment horizon. In contrast, the dynamic strategies under Markov processes depend only on the current values of the state variables and typically have monotonically smooth horizon dependence.

## 5. Applications

### 5.1. Risk Premium Forecasting by Delayed Information.

A straightforward application of model (2.3)-(2.5) is to the case when risk premium can be forecasted by delayed predictors. When the predictor $X$ is price or trading volume, Brock, Lakonishok and LeBaron (1992), Blume, Easley and O'Hara (1994), and Neely, Rapach, Tu and Zhou (2014) show that these technical indicators based on lagged information display statistically and economically significant predictive power to the equity risk premium.

The predictor $X$ can also be some macroeconomic variables which can track changing macroeconomic conditions over business cycles, while are published only with a time delay. For example, Ludvigson and Ng (2007, 2009) and Rapach, Strauss and Zhou (2010), among others, find evidence that the lagged macroeconomic variables, such as inflation, can significantly forecast equity and bond returns.

In addition, some financial market variables, such as, the dividend-price ratio and earnings-price ratio, are usually estimated based on MA rules over a historical time interval (Campbell and Shiller, 1988*a*, 1988*b*).

[11]In other words, time delays increase the dimensionality of the system by introducing path dependence, while do not affect the martingale multiplicity, which is the continuous-time precise form of the discrete-time vague notion of "*the maximum number of 'dimensions of uncertainty' which could be resolved at any one time*" as detailed in Duffie and Huang (1985).

5.2. **Momentum.** When $X_t = \ln P_t$, $k = \infty$ and $b_1 = b_2$, model (2.4) reduces to a momentum model

$$dP_t/P_t = (a_1 + bm_t)dt + \sigma dZ_t, \qquad m_t = \frac{1}{\tau}\int_{t-\tau}^{t} \frac{dP_u}{P_u}, \tag{5.1}$$

where $a_1 = a - b_1\sigma^2\tau/2$, $b = b_1\tau$, and $m_t$ is the time series momentum variable documented in Moskowitz, Ooi and Pedersen (2012): "*the past 12-month excess return of each instrument is a positive predictor of its future return.*"

Li and Liu (2016) solve the asset allocation problem for $T - t \le \tau$ using the martingale approach. They show that the historical price paths play non-trivial roles and can qualitatively change the strategy. Our method developed in this paper provide a more general results for any investment horizons.

5.3. **Persistent Volatility.** The piecewise dynamic programming method can be also used to study delays in volatility.

To capture long memory in volatility, we rely on the parsimonious, yet empirically highly accurate, HAR-RV model (heterogenous AR model for the realized volatility) proposed by Corsi (2009), in which the forecast for the future volatility is a linear function of the current daily, weekly, and monthly realized volatilities. The HAR-RV model is capable of reproducing a remarkably slow volatility autocorrelation decay that is almost indistinguishable from that of a hyperbolic (long memory) pattern over most empirically relevant forecast horizons.

According to Andersen, Bollerslev and Diebold (2007) and Corsi (2009), the price $P_t$ at time $t$ of the risky asset with stochastic persistent volatility is given by

$$\begin{aligned} \frac{dP_t}{P_t} &= (r + \lambda V_t)dt + \sqrt{V_t}dB_t, \\ dV_t &= \Big(a + b_1V_t + b_2V_t^w + b_3V_t^m\Big)dt + \sigma_v\sqrt{V_t}dB_t^v, \end{aligned} \tag{5.2}$$

where

$$V_t^w = \int_{t-\tau}^{t} V_s ds, \quad V_t^m = \int_{t-\hat\tau}^{t} V_s ds, \quad 0 < \tau < \hat\tau, \tag{5.3}$$

$V_t$ is the observable variance measure following a square-root process with time delays,[12] and $B_t$ and $B_t^v$ are two Brownian motions with correlation of $\rho$. When $b_2 = b_3 = 0$, (5.2) reduces to the Heston (1993) model, and the corresponding asset allocation problem is examined by Liu (2007).[13] The discretization of the continuous-time model (5.2) at a daily frequency with $\tau$ =1 week and $\hat\tau$ = 1 month is consistent

[12]The term $\sqrt{V_t}$ in the volatility of $V_t$ is to guarantee the positiveness of $V_t$. In addition, this also leads to tractability of the optimal portfolio.

[13]The portfolio selection problems with stochastic volatility are also studied by Liu and Pan (2003) and Chacko and Viceira (2005), among others. However, the persistence, one of the most prominent feature of volatility, is not explicitly modelled in the asset allocation literature.

with the HAR-RV model of Eq. (10) in Andersen et al. (2007). Accordingly, we assume $\hat{\tau} = 4\tau$. The following proposition states the results for the asset allocation problem ($\alpha = 0$).

**Proposition 5.1.** *When $(n-1)\tau \leq T \leq n\tau$, the optimal portfolio weight is given by*

$$\phi_t^* = \frac{\lambda}{\gamma} + \rho\sigma_v A_{1,t}^{(n)}, \tag{5.4}$$

*where*

$$\begin{aligned} &\dot{A}_{1,t}^{(n)} + \frac{\sigma_v^2[\gamma + (1-\gamma)\rho^2]}{2}\Big(A_{1,t}^{(n)}\Big)^2 + \Big(b_1 + \frac{1-\gamma}{\gamma}\lambda\rho\sigma_v\Big)A_{1,t}^{(n)} \\ &\qquad + A_{2,t}^{(n)} + A_{3,t}^{(n)} - A_{2,t+\tau}^{(n-1)} - A_{3,t+4\tau}^{(n-4)} + \frac{\lambda^2(1-\gamma)}{2\gamma^2} = 0, \\ &\dot{A}_{2,t}^{(n)} + b_2 A_{1,t}^{(n)} = 0, \qquad \dot{A}_{3,t}^{(n)} + b_3 A_{1,t}^{(n)} = 0, \\ &\dot{A}_{4,t}^{(n)} + a\sum_{i=1}^{n} A_{1,t+(n-i)\tau}^{(i)} - A_{2,t}^{(n)}V_{t-\tau} - \sum_{i=1}^{4} A_{3,t+(i-1)\tau}^{(n+1-i)}V_{t-(5-i)\tau} + \frac{n}{\gamma}[(1-\gamma)r - \beta] = 0, \end{aligned} \tag{5.5}$$

*with terminal conditions $A_{i,T-(n-1)\tau}^{(n)} = A_{i,T-(n-1)\tau}^{(n-1)}$, $i = 1,2,3$, $A_{4,T-(n-1)\tau}^{(n)} = 0$.*

*The value function is given by*

$$J_t = \frac{W_t^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\big(A_{1,t}^{(n)}V_t + A_{2,t}^{(n)}V_t^w + A_{3,t}^{(n)}V_t^m + \tilde{B}_t^{(n)} + A_{4,t}^{(n)}\big)\Big\}, \tag{5.6}$$

*where*

$$\tilde{B}_t^{(n)} = \int_{T-n\tau}^{t}\Big\{a\sum_{i=1}^{n-1}A_{1,u+(n-i)\tau}^{(i)} - A_{2,u+\tau}^{(n-1)}V_u - \sum_{i=1}^{4}A_{3,u+i\tau}^{(n-i)}V_{u-(4-i)\tau} + \frac{n}{\gamma}[(1-\gamma)r-\beta]\Big\}du, \tag{5.7}$$

*and $A^{(m)} = \tilde{B}^{(m)} = 0$ for $m \leq 0$.*

Because of the special structure that the risk premium is proportional to the conditional variance, the portfolio weight is independent of the variance $V$, and hence is not path dependent. However the value function is path dependent. This special structure also leads to a finite number of state variables to constitute a sufficient statistic. But the state variables are different for different horizons.

## 6. Conclusion

This paper solves the optimal dynamic portfolios under time delays. The portfolio weights exhibit many new features which are not present in Markovian framework.

The methodology developed in this paper has many potential applications for important problems in economics and finance.

## Appendix A. Proofs

A.1. **Proof of Proposition 4.1.** We conjecture that the value function has the following form (Liu, 2007),

$$J(t,W,X,Y) = e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\big[f(t,X,Y)\big]^{\gamma}, \tag{A.1}$$

which leads to the following optimal consumption and optimal portfolio weight

$$\begin{aligned} C_t^* &= \alpha^{\frac{1}{\gamma}} W_t f^{-1}, \\ \phi_t^* &= \frac{1}{\gamma\sigma^2}\Big(a + b_1 X_{t-\tau} + b_2 Y_t - r + \gamma\sigma\sigma_X\rho\frac{\partial \ln f}{\partial X}\Big). \end{aligned} \tag{A.2}$$

Substituting (A.1)-(A.2) into (4.1), we obtain

$$\begin{aligned} &\frac{\partial f}{\partial t} + \Big[\frac{(1-\gamma)(a+b_1X_{t-\tau}+b_2Y_t-r)^2}{2\gamma^2\sigma^2} + \frac{1-\gamma}{\gamma}r - \frac{\beta}{\gamma}\Big] f \\ &+ \Big[\frac{1-\gamma}{\gamma}\frac{\sigma_X\rho}{\sigma}(a+b_1X_{t-\tau}+b_2Y_t-r) + \kappa(\bar{X}-X_t)\Big]\frac{\partial f}{\partial X} \\ &+ \Big(\frac{k}{1-e^{-k\tau}}X_t - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau} - kY_t\Big)\frac{\partial f}{\partial Y} + \frac{\sigma_X^2}{2}(\gamma-1)(1-\rho^2)f^{-1}\Big(\frac{\partial f}{\partial X}\Big)^2 + \frac{\sigma_X^2}{2}\frac{\partial^2 f}{\partial X^2} + \alpha^{\frac{1}{\gamma}} = 0, \end{aligned}$$

with terminal condition $f(T,X,Y) = (1-\alpha)^{\frac{1}{\gamma}}$.

Using Lemma 2 in Liu (2007), we obtain

$$f(t,X,Y) = \alpha^{\frac{1}{\gamma}}\int_t^T \hat{f}(u,X,Y)du + (1-\alpha)^{\frac{1}{\gamma}}\hat{f}(t,X,Y),$$

where $\hat{f}$ satisfies

$$\begin{aligned} &\frac{\partial \hat{f}}{\partial t} + \Big[\frac{(1-\gamma)(a+b_1X_{t-\tau}+b_2Y_t-r)^2}{2\gamma^2\sigma^2} + \frac{1-\gamma}{\gamma}r - \frac{\beta}{\gamma}\Big] \hat{f} \\ &+ \Big[\frac{1-\gamma}{\gamma}\frac{\sigma_X\rho}{\sigma}(a+b_1X_{t-\tau}+b_2Y_t-r) + \kappa(\bar{X}-X_t)\Big]\frac{\partial \hat{f}}{\partial X} \\ &+ \Big(\frac{k}{1-e^{-k\tau}}X_t - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau} - kY_t\Big)\frac{\partial \hat{f}}{\partial Y} + \frac{\sigma_X^2}{2}(\gamma-1)(1-\rho^2)\hat{f}^{-1}\Big(\frac{\partial \hat{f}}{\partial X}\Big)^2 + \frac{\sigma_X^2}{2}\frac{\partial^2 \hat{f}}{\partial X^2} = 0, \end{aligned}$$

with terminal condition $\hat{f}(T,X,Y) = 1$.

$\hat{f}$ is conjectured to have the form

$$\hat{f}(t,X,Y) = \exp\Big\{\frac{A_{11,t}^{(1)}}{2}X_t^2 + A_{12,t}^{(1)}X_tY_t + \frac{A_{22,t}^{(1)}}{2}Y_t^2 + A_{1,t}^{(1)}X_t + A_{2,t}^{(1)}Y_t + A_{3,t}^{(1)}\Big\}.$$

Using this conjecture, we obtain

$$
\begin{aligned}
&\dot{A}^{(1)}_{11,t} + \sigma_X^2\big[\gamma + (1-\gamma)\rho^2\big]\big(A^{(1)}_{11,t}\big)^2 - 2\kappa A^{(1)}_{11,t} + \frac{2k}{1-e^{-k\tau}}A^{(1)}_{12,t} = 0,\\
&\dot{A}^{(1)}_{12,t} + \sigma_X^2\big[\gamma + (1-\gamma)\rho^2\big]A^{(1)}_{11,t}A^{(1)}_{12,t} + \frac{(1-\gamma)\sigma_X\rho b_2}{\gamma\sigma}A^{(1)}_{11,t} - (k+\kappa)A^{(1)}_{12,t} + \frac{k}{1-e^{-k\tau}}A^{(1)}_{22,t} = 0,\\
&\dot{A}^{(1)}_{22,t} + \sigma_X^2\big[\gamma + (1-\gamma)\rho^2\big]\big(A^{(1)}_{12,t}\big)^2 + \frac{2(1-\gamma)\sigma_X\rho b_2}{\gamma\sigma}A^{(1)}_{12,t} - 2kA^{(1)}_{22,t} + \frac{(1-\gamma)b_2^2}{\gamma^2\sigma^2} = 0,\\
&\dot{A}^{(1)}_{1,t} + \sigma_X^2\big[\gamma + (1-\gamma)\rho^2\big]A^{(1)}_{11,t}A^{(1)}_{1,t} + \Big[\frac{(1-\gamma)\sigma_X\rho}{\gamma\sigma}(a + b_1X_{t-\tau} - r) + \kappa\bar{X}\Big]A^{(1)}_{11,t}\\
&\qquad - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau}A^{(1)}_{12,t} - \kappa A^{(1)}_{1,t} + \frac{k}{1-e^{-k\tau}}A^{(1)}_{2,t} = 0,\\
&\dot{A}^{(1)}_{2,t} + \sigma_X^2\big[\gamma + (1-\gamma)\rho^2\big]A^{(1)}_{12,t}A^{(1)}_{1,t} + \Big[\frac{(1-\gamma)\sigma_X\rho}{\gamma\sigma}(a + b_1X_{t-\tau} - r) + \kappa\bar{X}\Big]A^{(1)}_{12,t}\\
&\qquad - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau}A^{(1)}_{22,t} + \frac{(1-\gamma)\sigma_X\rho b_2}{\gamma\sigma}A^{(1)}_{1,t} - kA^{(1)}_{2,t} + \frac{(1-\gamma)b_2}{\gamma^2\sigma^2}(a + b_1X_{t-\tau} - r) = 0,\\
&\dot{A}^{(1)}_{3,t} + \frac{\sigma_X^2}{2}\big[\gamma + (1-\gamma)\rho^2\big]\big(A^{(1)}_{1,t}\big)^2 + \frac{\sigma_X^2}{2}A^{(1)}_{11,t} + \Big[\frac{(1-\gamma)\sigma_X\rho}{\gamma\sigma}(a + b_1X_{t-\tau} - r) + \kappa\bar{X}\Big]A^{(1)}_{1,t}\\
&\qquad - \frac{ke^{-k\tau}}{1-e^{-k\tau}}X_{t-\tau}A^{(1)}_{2,t} + \frac{1-\gamma}{2\gamma^2\sigma^2}(a + b_1X_{t-\tau} - r)^2 + \frac{(1-\gamma)r}{\gamma} - \frac{\beta}{\gamma} = 0,
\end{aligned}
\tag{A.3}
$$

with terminal conditions $A^{(1)}_{ij,T} = A^{(1)}_{i,T} = 0$.

A.2. **Proof of Corollary 4.4.** It follows from (4.13) and (4.18) that $\tilde{B}^{(n)}_{i,t}$ are weighted sums of historical state variables for $i = 3, \cdots, 2^{n+1}-2$, and a weighted sum of historical state variables and their squared values for $i = 2^{n+1}-1$. In addition, $\mathbb{H}^{(n)}_{i,j}(u)$, which are deterministic functions of time $u$, can be easily derived by matching the coefficients.

A.3. **Proof of Corollary 4.5.** When $0 \le t \le T-(n-1)\tau$, Proposition 4.3 shows that

$$
\begin{aligned}
J_{T-(n-1)\tau} =& e^{-\beta[T-(n-1)\tau]}\frac{W^{1-\gamma}_{T-(n-1)\tau}}{1-\gamma}\exp\Bigg\{\gamma\Bigg(\frac{1}{2}\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2}A^{(n)}_{ij,T-(n-1)\tau}\tilde{B}^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{j,T-(n-1)\tau}\\
&+\sum_{i=1}^{2^{n+1}-2}A^{(n)}_{i,T-(n-1)\tau}\tilde{B}^{(n)}_{i,T-(n-1)\tau} + \tilde{B}^{(n)}_{2^{n+1}-1,T-(n-1)\tau} + A^{(n)}_{2^{n+1}-1,T-(n-1)\tau}\Bigg)\Bigg\}.
\end{aligned}
\tag{A.4}
$$

When $T-(n-1)\tau \le t \le T-(n-2)\tau$, $J_t$ is given by (4.7) for $(n-2)\tau \le T \le (n-1)\tau$. In this case,

$$J_{T-(n-1)\tau} = e^{-\beta[T-(n-1)\tau]}\frac{W_{T-(n-1)\tau}^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\left(\frac{1}{2}\sum_{i=1}^{2^n-2}\sum_{j=1}^{2^n-2}A_{ij,T-(n-1)\tau}^{(n-1)}\tilde{B}_{i,T-(n-1)\tau}^{(n-1)}\tilde{B}_{j,T-(n-1)\tau}^{(n-1)}\right.\right.$$
$$\left.\left.+\sum_{i=1}^{2^n-2}A_{i,T-(n-1)\tau}^{(n-1)}\tilde{B}_{i,T-(n-1)\tau}^{(n-1)}+\tilde{B}_{2^{n+1}-1,T-(n-1)\tau}^{(n-1)}+A_{2^{n+1}-1,T-(n-1)\tau}^{(n-1)}\right)\right\}$$
$$=e^{-\beta[T-(n-1)\tau]}\frac{W_{T-(n-1)\tau}^{1-\gamma}}{1-\gamma}\exp\left\{\gamma\left(\sum_{i=1}^{2^{n+1}-2}\sum_{j=1}^{2^{n+1}-2}C_{ij}^{(n)}\tilde{B}_{i,T-(n-1)\tau}^{(n)}\tilde{B}_{j,T-(n-1)\tau}^{(n)}+\tilde{B}_{2^{n+1}-1,T-(n-1)\tau}^{(n)}\right)\right\},$$

where the first equality follows from Proposition 4.3 for $T-(n-1)\tau \le t \le T-(n-2)\tau$ and the second follows from (4.20). Using (4.20) and (4.23), the last equation equals the right hand side of (A.4). Therefore, $J_t$ is continuous at $t = T-(n-1)\tau$.

Similarly, $\phi_t^*$ is also continuous at $t = T-(n-1)\tau$.

A.4. **Proof of Proposition 5.1.** We rewrite (5.2) as

$$\begin{aligned}\frac{dP_t}{P_t} &= (r+\lambda V_t)dt + \sqrt{V_t}dB_t,\\ dV_t &= \left(a+b_1V_t+b_2V_t^w+b_3V_t^m\right)dt+\sigma_v\sqrt{V_t}dB_t^v,\\ dV_t^w &= (V_t-V_{t-\tau})dt,\\ dV_t^m &= (V_t-V_{t-4\tau})dt.\end{aligned} \tag{A.5}$$

The corresponding wealth process is given by

$$\frac{dW_t}{W_t} = (r+\lambda V_t\phi_t)dt + \phi_t\sqrt{V_t}dB_t. \tag{A.6}$$

The proof follows from the mathematical induction. When $n=1$, that is $T \le \tau$, $V_{u-\tau}$ and $V_{u-4\tau}$ are realized prices and are known at time 0. So the path is not a state variable. Let $J(t,W,V,V^w,V^m)$ denote the indirect utility function. The HJB equation is given by

$$\begin{aligned}\max_{\phi}\Bigg\{&\frac{\partial J}{\partial t}+\frac{\partial J}{\partial W}W_t(r+\lambda V_t\phi_t)+\frac{\partial J}{\partial V}(a+b_1V_t+b_2V_t^w+b_3V_t^m)\\ &+\frac{\partial J}{\partial V^w}(V_t-V_{t-\tau})+\frac{\partial J}{\partial V^m}(V_t-V_{t-4\tau})\\ &+\frac{1}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\phi_t^2V_t+\frac{\partial^2 J}{\partial W\partial V}\rho\sigma_vW_t\phi_tV_t+\frac{1}{2}\frac{\partial^2 J}{\partial V^2}\sigma_v^2V_t\Bigg\}=0,\end{aligned} \tag{A.7}$$

with boundary condition

$$J(T,W,V,V^w,V^m) = e^{-\beta T}\frac{W^{1-\gamma}}{1-\gamma}.$$

$J$ is conjectured to have the form

$$J(t,W,V,V^w,V^m)=e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\big(A^{(1)}_{1,t}V_t+A^{(1)}_{2,t}V^w_t+A^{(1)}_{3,t}V^m_t+A^{(1)}_{4,t}\big)\Big\}. \quad (A.8)$$

Using this conjecture, the optimal portfolio weight is given by

$$\phi^*_t=\frac{\lambda}{\gamma}+\rho\sigma_v A^{(1)}_{1,t}, \quad (A.9)$$

where

$$\dot A^{(1)}_{1,t}+\frac{\sigma_v^2[\gamma+(1-\gamma)\rho^2]}{2}\Big(A^{(1)}_{1,t}\Big)^2+\Big(b_1+\frac{1-\gamma}{\gamma}\lambda\rho\sigma_v\Big)A^{(1)}_{1,t}+A^{(1)}_{2,t}+A^{(1)}_{3,t}+\frac{\lambda^2(1-\gamma)}{2\gamma^2}=0,$$

$$\dot A^{(1)}_{2,t}+b_2A^{(1)}_{1,t}=0, \qquad \dot A^{(1)}_{3,t}+b_3A^{(1)}_{1,t}=0,$$

$$\dot A^{(1)}_{4,t}+aA^{(1)}_{1,t}-A^{(1)}_{2,t}V_{t-\tau}-A^{(1)}_{3,t}V_{t-4\tau}+\frac{1}{\gamma}[(1-\gamma)r-\beta]=0,$$

with terminal conditions $A^{(1)}_{i,T}=0$, $i=1,2,3,4$.

Assume Proposition 5.1 holds for $(n-1)\tau\le T\le n\tau$, $n=1,2,\cdots$. When $n\tau\le T\le(n+1)\tau$, Bellman's principle of optimality implies that

$$J_t=\max_{\phi}\big\{\mathbb{E}_t[J_{T-n\tau}]\big\}$$
$$=\max_{\phi}\bigg\{e^{-\beta(T-n\tau)}\mathbb{E}_t\bigg[\frac{W_{T-n\tau}^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\big(A^{(n)}_{1,T-n\tau}V_{T-n\tau}+A^{(n)}_{2,T-n\tau}V^w_{T-n\tau}+A^{(n)}_{3,T-n\tau}V^m_{T-n\tau}+A^{(n)}_{4,T-n\tau}\big)\Big\}\bigg]\bigg\}, \quad (A.10)$$

where $A^{(n)}_{i,T-n\tau}$, $i=1,2,3$ are constants governed by ODEs, and

$$A^{(n)}_{4,T-n\tau}=\int_{T-n\tau}^{T-(n-1)\tau}\bigg\{a\sum_{i=1}^{n}A^{(i)}_{1,u+(n-i)\tau}-A^{(n)}_{2,u}V_{u-\tau}-\sum_{i=1}^{4}A^{(n+1-i)}_{3,u+(i-1)\tau}V_{u-(5-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta]\bigg\}du$$
$$=\int_{T-(n+1)\tau}^{T-n\tau}\bigg\{a\sum_{i=1}^{n}A^{(i)}_{1,u+(n+1-i)\tau}-A^{(n)}_{2,u+\tau}V_u-\sum_{i=1}^{4}A^{(n+1-i)}_{3,u+i\tau}V_{u-(4-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta]\bigg\}du.$$

Define

$$\tilde B^{(n+1)}_t=\int_{T-(n+1)\tau}^{t}\bigg\{a\sum_{i=1}^{n}A^{(i)}_{1,u+(n+1-i)\tau}-A^{(n)}_{2,u+\tau}V_u-\sum_{i=1}^{4}A^{(n+1-i)}_{3,u+i\tau}V_{u-(4-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta]\bigg\}du.$$

Then

$$\dot{\tilde B}^{(n+1)}_t=a\sum_{i=1}^{n}A^{(i)}_{1,t+(n+1-i)\tau}-A^{(n)}_{2,t+\tau}V_t-\sum_{i=1}^{4}A^{(n+1-i)}_{3,t+i\tau}V_{t-(4-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta],$$

$\tilde B^{(n+1)}_{T-(n+1)\tau}=0$ and $\tilde B^{(n+1)}_{T-n\tau}=A^{(n)}_{4,T-n\tau}$. So (A.10) becomes

$$J_t=J\big(t,W,V,V^w,V^m,\tilde B^{(n+1)}\big)$$
$$=\max_{\phi}\bigg\{e^{-\beta(T-n\tau)}\mathbb{E}_t\bigg[\frac{W_{T-n\tau}^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\big(A^{(n)}_{1,T-n\tau}V_{T-n\tau}+A^{(n)}_{2,T-n\tau}V^w_{T-n\tau}+A^{(n)}_{3,T-n\tau}V^m_{T-n\tau}+\tilde B^{(n+1)}_{T-n\tau}\big)\Big\}\bigg]\bigg\},$$

The corresponding HJB equation is given by

$$\max_{\phi}\Bigg\{\frac{\partial J}{\partial t}+\frac{\partial J}{\partial W}W_t(r+\lambda V_t\phi_t)+\frac{\partial J}{\partial V}(a+b_1V_t+b_2V_t^w+b_3V_t^m)$$
$$+\frac{\partial J}{\partial V^w}(V_t-V_{t-\tau})+\frac{\partial J}{\partial V^m}(V_t-V_{t-4\tau})$$
$$+\frac{1}{2}\frac{\partial^2 J}{\partial W^2}W_t^2\phi_t^2V_t+\frac{\partial^2 J}{\partial W\partial V}\rho\sigma_vW_t\phi_tV_t+\frac{1}{2}\frac{\partial^2 J}{\partial V^2}\sigma_v^2V_t$$
$$\frac{\partial J}{\partial \tilde{B}^{(n+1)}}\Big(a\sum_{i=1}^{n}A^{(i)}_{1,t+(n+1-i)\tau}-A^{(n)}_{2,t+\tau}V_t-\sum_{i=1}^{4}A^{(n+1-i)}_{3,t+i\tau}V_{t-(4-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta]\Big)\Bigg\}=0,$$

with boundary condition

$$J(T,W,V,V^w,V^m)=e^{-\beta T}\frac{W^{1-\gamma}}{1-\gamma}.$$

By conjecturing

$$J_t=e^{-\beta t}\frac{W_t^{1-\gamma}}{1-\gamma}\exp\Big\{\gamma\big(A^{(n+1)}_{1,t}V_t+A^{(n+1)}_{2,t}V_t^w+A^{(n+1)}_{3,t}V_t^m+\tilde{B}^{(n+1)}_t+A^{(n+1)}_{4,t}\big)\Big\},$$

the optimal portfolio weight of the risky asset is given by

$$\phi_t^*=\frac{\lambda}{\gamma}+\rho\sigma_vA^{(n+1)}_{1,t},\tag{A.11}$$

where

$$\dot{A}^{(n+1)}_{1,t}+\frac{\sigma_v^2[\gamma+(1-\gamma)\rho^2]}{2}\Big(A^{(n+1)}_{1,t}\Big)^2+\Big(b_1+\frac{1-\gamma}{\gamma}\lambda\rho\sigma_v\Big)A^{(n+1)}_{1,t}$$
$$+A^{(n+1)}_{2,t}+A^{(n+1)}_{3,t}-A^{(n)}_{2,t+\tau}-A^{(n-3)}_{3,t+4\tau}+\frac{\lambda^2(1-\gamma)}{2\gamma^2}=0,$$
$$\dot{A}^{(n+1)}_{2,t}+b_2A^{(n+1)}_{1,t}=0,\qquad \dot{A}^{(n+1)}_{3,t}+b_3A^{(n+1)}_{1,t}=0,$$
$$\dot{A}^{(n+1)}_{4,t}+a\sum_{i=1}^{n+1}A^{(i)}_{1,t+(n+1-i)\tau}-A^{(n+1)}_{2,t}V_{t-\tau}-\sum_{i=1}^{4}A^{(n+2-i)}_{3,t+(i-1)\tau}V_{t-(5-i)\tau}+\frac{n}{\gamma}[(1-\gamma)r-\beta]=0,\tag{A.12}$$

with terminal conditions $A^{(n+1)}_{i,T-n\tau}=A^{(n)}_{i,T-n\tau}$ for $i=1,2,3$, and $A^{(n+1)}_{4,T-n\tau}=0$.

Mathematical induction implies the completeness of the proof.

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