EXTRAPOLATIVE ASSET PRICING

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Abstract. We study a Lucas economy where a representative agent extrapolates past returns. The setup allows extrapolation to be defined over returns, as in the survey data, instead of price changes as in most existing models. We solve for the equilibrium in closed form. Depending on the elasticity of intertemporal substitution (EIS), extrapolation can have negative impact on price and generate return momentum under the true measure, instead of reversal typically found in existing models. A negative bias correction for sentiment by the true measure leads to a correction premium whose level increases with EIS. Therefore, momentum is preserved when the correction is insufficient for small EIS, while reversal is revealed for large EIS. However, dividend-price ratio always positively predicts market return, independent of EIS. We further show that the financial market anomalies, including return predictability, generated by extrapolation may not be caused by overvaluation.

Key words: Extrapolation, momentum, price-dividend ratio, return predictability, elasticity of intertemporal substitution.

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1. Introduction

Return extrapolation, the expectation that market prices will continue rising (falling) after a sequence of high (low) past returns, plays an important role in investors' decision-making.[1] An increasing literature incorporates extrapolative expectation into asset pricing models and readily finds that extrapolation helps explain a large set of financial market anomalies.[2] However, it is puzzling that extrapolation always leads to return reversal in these models, even though the extrapolator believes past returns positively predict future returns. In this paper, we clarify the effect of extrapolation. We provide a mechanism whereby extrapolation has a negative impact on price and market returns exhibit momentum.

Most models of extrapolation use a setup of normal dividend and CARA preference, perhaps because of the mathematical tractability given other assumptions. This setup, in which percentage return and price-dividend ratio cannot be defined, does not have a direct correspondence with the empirical studies. In this paper, we study a Lucas (1978) economy where a representative extrapolator has CRRA preference and consumption follows a geometric Brownian motion. In this setup, prices are positive and percentage returns can be defined. This allows us to define extrapolation using returns; hence our model has a direct correspondence with the empirical studies. For example, Greenwood and Shleifer (2014) document that investors extrapolate market returns rather than price changes. The price-dividend ratio can be also defined in our setup, whose predictive ability for future returns is a major focus of the extrapolation literature.[3] The only departure from the standard asset pricing model is that the rational expectation is replaced by an extrapolative expectation in our model. We solve for the equilibrium in closed form that enables us to pinpoint the effects of extrapolation.

Extrapolators believe that market returns exhibit momentum in the sense that past returns can positively forecast future returns. That is, market returns have momentum under the extrapolative measure. It is natural to expect that market returns may also exhibit momentum under the true measure. However, it is puzzling that the existing models have shown that extrapolation always generates return reversal, e.g., De Long, Shleifer, Summers and Waldmann (1990), Hong and Stein (1999) and Barberis et al. (2015). Is it caused by either the counteraction of rational investors or overvaluation generated from extrapolation as claimed in the literature

[1]For example, Vissing-Jorgensen (2004), Bacchetta, Mertens and van Wincoop (2009), Barberis (2013), Amromin and Sharpe (2014), Greenwood and Shleifer (2014) and Kuchler and Zafar (2016) document that many individual and institutional investors have extrapolative expectations.

[2]For example, Fuster, Laibson and Mendel (2010), Fuster, Hebert and Laibson (2011), Choi and Mertens (2013) and Barberis, Greenwood, Jin and Shleifer (2015), Hirshleifer, Li and Yu (2015).

[3]For example, Barberis et al. (2015), Cassella and Gulen (2017) and Jin and Sui (2018).

(e.g., Barberis et al., 2015)? We resolve the conceptual issues and show that neither attributes to the return reversal. Both momentum and reversal can be generated by extrapolation depending on the extrapolator's EIS.

More specifically, extrapolation leads the extrapolator's expectation about dividend growth to positively depend on her expectation about future market returns, which is characterized by a weighted average of past returns and is termed sentiment. However, the dividend growth is independent of sentiment under the true measure. So there is a negative bias correction for sentiment by the true measure (used by an outside econometrician), leading to a correction premium that is negatively related to sentiment. The absolute value of the correction premium increases with EIS due to the substitution effect. Therefore, for a small EIS, the correction is not enough and momentum is preserved. For a large EIS, large correction leads to return reversal. This indicates that the return reversal is not caused by either rational investors or overvaluation as argued by Barberis et al. (2015). In fact, in our economy, returns exhibit reversal even though there is no rational investor who uses the true measure. On the other hand, there is no overvaluation in time series for the economies studied by both Barberis et al. (2015) and us as discussed later. Moreover, we show that momentum, which occurs for a small EIS, does not reverse in the long-run, confirming that the market is not overvalued.

Extrapolators have been found always generating a positive feedback (e.g., De Long et al., 1990; Cutler, Poterba and Summers, 1990; Hong and Stein, 1999; Barberis et al., 2015) due to the amplification mechanism in the sense that "*if good cash-flow news pushes the stock market up, this price increase feeds into extrapolators' expectations about future price changes, which then leads them to push the current price up even higher. However, this then further increases extrapolators' expectations about future price changes, leading them to push the current price still higher, and so on*" (Barberis et al., 2015, p. 7). Barberis et al. (2015) call this an *infinite feedback loop*. Although this sounds intuitive, we show that it is not always the case. For a small EIS, although prices always have a positive impact on her expectation about future market returns (by definition), expectation could in turn have a negative impact on prices, resulting in a *negative* feedback that however pulls down the price level relative to dividend.

The reason is as follows. The effects of extrapolation on price-dividend ratio depend on the agent's EIS, which determines the relative strength of income effect and intertemporal substitution effect. When the income effect dominates (a large EIS), the price-dividend ratio increases with sentiment, resulting in a positive feedback. However, when the substitution effect dominates (a small EIS), the price-dividend ratio decreases with sentiment. This differs from the popular argument in the literature that price is pulled back by the rational traders. In fact, a high extrapolator's expectation about future market returns leads her to expect both high discount rates and high dividend growth rates in the future. If the increase in discount rate dominates, then price-dividend ratio decreases, leading to a negative feedback. Put differently, an agent's optimal demand depends not only on the expected return, but also on intertemporal hedging, riskless rate and volatility.

Although whether returns exhibit momentum or reversal depends on EIS, we find that dividend-price ratio (D/P) always positively predicts future market return, independent of EIS. In fact, for a small EIS, D/P is positively related to sentiment, which is also positively related to expected return as discussed above. In this case, D/P positively predicts future return. For a large EIS, D/P still positively predicts future return because it is negatively related to sentiment, which is also negatively related to expected return. We also find that the predictive ability becomes stronger when the extrapolator relies more heavily on recent versus distant returns. This may be consistent with recent findings of Cassella and Gulen (2017).

Due to the infinite feedback loop, Barberis et al. (2015) (p. 18) argue that "*After a sequence of good cash-flow news, extrapolators cause the stock market to become overvalued*", and they use this "overvaluation" to explain many key results of their paper, such as excess volatility, negative autocorrelation of return and the return predictability by dividend-price ratio, consumption-wealth ratio and the change in aggregate consumption. Although the overvaluation-based explanation sounds intuitive, it may be misleading. On the one hand, there is no overvaluation in time series, given the fact that price is determined by a full equilibrium, which has taken into account all future effects of a current dividend change, and is correct by definition. On the other hand, we find that, when EIS is large, extrapolation leads to overvaluation relative to the rational benchmark in the sense that the price-dividend ratio is higher than that for the rational benchmark when sentiment is sufficiently high, consistent with Barberis et al. (2015). However, we also find that the price-dividend ratio under extrapolation does not necessarily converge to its rational-benchmark level in the long-run, implying that the extrapolation model and its rational benchmark may not be directly comparable with respect to price-dividend ratio. (We show that they can be comparable with respect to dividend growth.) Therefore, the stylized facts generated by their model may not be caused by overvaluation. In this paper, we provide understanding of the mechanisms in generating those stylized facts.[4]

The literature finds that the economy under extrapolation may not have an equilibrium because the "infinite feedback loop" may lead to price divergence (e.g.,

[4]In addition, one may be careful about using the term of "good cash-flow news", because good news (a positive shock) under the true measure could be bad news under the extrapolative measure, and vice versa.

De Long et al., 1990; Hong and Stein, 1999; Barberis et al., 2015). In fact, the riskless asset is assumed to be in perfectly elastic supply in their models for mathematical tractability. However, this masks the positive impact of sentiment on the riskless rate, enlarge extrapolators' demand for the risky asset when sentiment is high, and hence makes the equilibrium prone to explode. When the riskless rate is endogenized in our model, there always exists an equilibrium even in the presence of the "feedback loop" as long as the investment horizon is finite. With an infinite horizon, the equilibrium may not exist due to the transversality condition, which however takes effect even in the absence of extrapolators. In the presence of extrapolators, extrapolation affects the transversality condition, and stronger extrapolation leads to stricter condition. In this sense, extrapolation does not lead to the instability of equilibrium, rather, it does change the transversality condition.

Our model generates some other financial market anomalies. We find that sentiment is positively related to riskless rate but negatively related to the risk premium. More importantly, the risk premium becomes negative when sentiment is high regardless of EIS. This is consistent with the recent findings of Greenwood and Hanson (2013), Baron and Xiong (2017) and Cassella and Gulen (2017). The negative risk premium is caused by a negative bias correction by the rational expectation when sentiment is high.

In our model, volatility increases with sentiment regardless of EIS. Excess volatility occurs when the extrapolator's EIS is large, consistent with the existing literature showing that the positive feedback generates excess volatility (e.g., De Long et al., 1990; Barberis et al., 2015; and Jin and Sui, 2018). The negative feedback when the extrapolator's EIS is small in our model leads to lower return volatility than dividend volatility. We also find that extrapolation pulls down prices for a small EIS and pushes up prices for a large EIS.

The remainder of paper is organized as follows. Section 2 presents the economic setup and the rational benchmark. Under extrapolation, Section 3 solves for equilibrium and Section 4 discusses the properties of equilibrium. Section 5 concludes. Appendix provides the proofs.

2. Economic Setup and Rational Benchmark

2.1. **Economic Setup.** Consider a continuous-time Lucas (1978) economy with one consumption good and a representative agent with CRRA preference. There are two assets in the economy. One is a risky asset, the stock market, that is a claim to a continuous dividend stream. Assume that the dividend follows

$$dD_t = D_t(\mu_d dt + \sigma_d dZ_t), \tag{2.1}$$

where the dividend growth rate μ_d and volatility σ_d are constants, and Z_t is a standard Brownian motion under the objective measure. Under this setup, the price of the market is positive and its returns are well defined, allowing for a direct correspondence with the empirical studies. For example, our extrapolation will be defined over returns, as in the survey data, instead of price changes as in most existing models. The other asset is locally riskless with riskless rate r_{ft} that is determined in equilibrium. The riskless asset is in net zero supply.

The agent maximizes expected discounted utilities over consumption with subjective discount rate $\rho > 0$ by choosing consumption C_t and the fraction of wealth invested in the stock market ϕ_t:

$$\max_{\{C_t,\phi_t\}_{t=0}^T} \mathbb{E}^e\left[\int_0^T e^{-\rho t}\frac{C_t^{1-\gamma}}{1-\gamma}dt\right], \tag{2.2}$$

subject to the budget constraint

$$dW_t = \left[W_t\big(r_f + \phi_t(\mu_p^e - r_f)\big) - C_t\right]dt + W_t\phi_t\sigma_p dZ_t^e, \tag{2.3}$$

where $\mathbb{E}^e[\cdot]$ is the expectation under the agent's subjective probability measure $\mathbb{P}^e$, which is equivalent to the true measure $\mathbb{P}$. The superscript "e" is an abbreviation for "extrapolative expectation". T is horizon. Especially, the economy has an infinite horizon when $T \to \infty$. Our analytical results in this paper hold for both finite and infinite horizons. γ is the relative risk aversion coefficient, as well as the inverse of the agent's EIS, μ_p^e is the expected market return under $\mathbb{P}^e$ and σ_p is the volatility of market returns. When $\gamma = 1$, the utility in (2.2) is replaced by $\ln(C_t)$.

2.2. **Rational Benchmark.** Before studying extrapolative expectation, we first present the rational benchmark in which the agent is fully rational ($\mathbb{P}^e$=$\mathbb{P}$). We consider the limiting case $T \to \infty$, through which we can discuss the existence of equilibrium.[5] The following proposition characterizes the equilibrium, where the superscript "r" is an abbreviation for "rational benchmark".

Proposition 2.1. *(Rational benchmark.) Assume that the agent is fully rational and $T \to \infty$.*

(1) The equilibrium market price P_t satisfies

$$\frac{dP_t + D_t dt}{P_t} = \mu_p dt + \sigma_p dZ_t, \tag{2.4}$$

where

$$\mu_p = \rho + \gamma\left[\mu_d - \frac{(\gamma-1)\sigma_d^2}{2}\right] \equiv \mu^r, \qquad \sigma_p = \sigma_d, \tag{2.5}$$

[5] The equilibrium always exists for $T < \infty$.

the riskless rate is given by

$$r_f^r = \rho + \gamma \Big[\mu_d - \frac{(\gamma+1)\sigma_d^2}{2} \Big], \tag{2.6}$$

and the price-dividend ratio is given by

$$\Phi^r = \frac{1}{\rho + (\gamma - 1)(\mu_d - \gamma\sigma_d^2/2)}. \tag{2.7}$$

(2) The equilibrium exists if and only if

$$\rho + (\gamma - 1)\Big(\mu_d - \frac{\gamma\sigma_d^2}{2}\Big) > 0. \tag{2.8}$$

The relationship between price-dividend ratio Φ^r and dividend growth μ_d depends on the agent's EIS. Φ^r decreases with μ_d when EIS< 1 ($\gamma > 1$), and increases with μ_d when EIS> 1. We will show later that this EIS dependence determines the role played by extrapolation in the economy and also determines many key features of market returns, such as its series correlation and predictability by dividend-price ratio and sentiment.

The second part of Proposition 2.1 is due to transversality condition. The first γ in (2.8) is the inverse of the agent's EIS, and the term "$\gamma - 1$" reflects the relative strength of income and substitution effects. The second γ is a risk aversion coefficient, which disappears if there is no uncertainty ($\sigma_d = 0$). Equation (2.8) leads to two bounds of γ,

$$\begin{aligned} \overline{\gamma}^r &= \frac{\sigma_d^2/2 + \mu_d + \sqrt{(\sigma_d^2/2 - \mu_d)^2 + 2\rho\sigma_d^2}}{\sigma_d^2}, \\ \underline{\gamma}^r &= \frac{\sigma_d^2/2 + \mu_d - \sqrt{(\sigma_d^2/2 - \mu_d)^2 + 2\rho\sigma_d^2}}{\sigma_d^2}, \end{aligned} \tag{2.9}$$

between which the equilibrium exists. The superscript "r" in (2.9) is an abbreviation for "rational benchmark".

The upper bound $\overline{\gamma}^r$ arises as follow. The expected return has two components, the riskless rate and the risk premium. Proposition 2.1 shows that the risk premium is given by $\gamma\sigma_d^2$ while the riskless rate is given by $\rho + \gamma\mu_d - \gamma(\gamma+1)\sigma_d^2/2$. The term $\gamma\mu_d$ represents the intertemporal substitution and $-\gamma(\gamma+1)\sigma_d^2/2$ represents precautionary saving. While the risk premium increases linearly with γ, the precautionary saving component decreases quadratically with γ. When γ is large, the expected return decreases with γ. When γ is large enough, the precautionary saving can be very negative, thus the transversality condition is violated.

The lower bound $\underline{\gamma}^r$ depends on the relative level of dividend growth rate to subjective discount rate. To see this, we consider a limiting case $\gamma \to 0$, i.e., an infinite EIS. In this case, dividend growth rate have to be lower than discount rate; otherwise the perfect substitution leads to an infinite utility, and hence the equilibrium does not exist.

2.3. **CARA Utility.** Most papers on extrapolation use a setup with normally distributed dividend, an exogenous riskless rate and CARA utility, e.g., De Long et al. (1990), Hong and Stein (1999) and Barberis et al. (2015). We provide the rational benchmark for CARA preference in this subsection for comparison. Assume dividend growth is i.i.d. $dD_t = \mu_d dt + \sigma_d dZ_t$, the riskless rate r_f is exogenously given, and utility is given by

$$E_0\Big[-\int_0^\infty \frac{e^{-\rho t - AC_t}}{A} dt\Big],$$

where A is the absolute risk aversion coefficient. Barberis et al. (2015) show that the price of the risky asset is given by

$$P_t = \frac{\mu_d}{r_f^2} - \frac{A\sigma_d^2 Q}{r_f^2} + \frac{D_t}{r_f}, \tag{2.10}$$

where Q is a constant per-capita supply of the risky asset.

This setup cannot define percentage returns and price-dividend ratio, and hence does not have direct correspondence with the empirical studies, e.g., Greenwood and Shleifer (2014).[6]

The price-dividend ratio is measured by $P_t - D_t/r_f$ in Barberis et al. (2015), and (2.10) shows that it always increases with dividend growth rate μ_d. This is consistent with the case of large EIS ($\gamma < 1$) in our benchmark with CRRA preference. However, (2.7) also shows that price-dividend ratio decreases with μ_d in our benchmark for small EIS.[7] This difference between the two preferences will be preserved under extrapolation, and underlines the different pricing implications of extrapolation between Barberis et al.'s model and ours.

In fact, as shown in next section (Corollary 3.3), when the rational expectation in our rational benchmark is replaced by an extrapolative expectation, the relationships in Proposition 2.1 will still hold except that μ_d is replaced by the dividend growth rate under the extrapolative measure, which positively depends on sentiment. Therefore, how historical returns (sentiment) affect future return and price-dividend ratio through the expected dividend growth rate depends on EIS. In contrast, as in their rational benchmark, the price-dividend ratio in the extrapolation model of Barberis et al. (2015) depends also positively on the dividend growth rate under the extrapolative measure, by noting that both are positively related to sentiment as shown by Proposition 1 and (A16) in their paper respectively. As a result, they

[6] In addition, prices can be negative, the riskless asset is in perfectly elastic supply, and the risky asset has unlimited liability.

[7] The EIS for the CARA utility, given by $(AC_t)^{-1}$, is time-varying and depends on consumption.

find that overextrapolation always plays a destabilizing role and always generates positive feedback and return reversal.[8]

3. Asset Pricing under Extrapolation

In this section, we build a Lucas-type model with a representative agent who extrapolates past returns. The model is standard except that the rational expectation is replaced by an extrapolative expectation. This enables us to focus on and pin point the effects of extrapolation bias.

3.1. **Extrapolative Expectation.**

Motivated by the survey evidence analyzed by Vissing-Jorgensen (2004), Bacchetta et al. (2009), Barberis (2013), Greenwood and Shleifer (2014) and Cassella and Gulen (2017), we assume that the representative agent forms expectations about the future market return (scaled by volatility) by extrapolating past returns,

$$\mathbb{E}_t^e\left[\frac{dR_t}{\sigma_{pt}}\right] = (\alpha_0 + \alpha S_t)dt, \tag{3.1}$$

where $dR_t = (dP_t + D_t dt)/P_t$ is the instantaneous return of the market, σ_{pt} is return volatility, and S_t is defined as

$$S_t = \int_{-\infty}^{t} \kappa e^{-\kappa(t-u)} \frac{dR_u}{\sigma_{pu}}, \tag{3.2}$$

a weighted average of market returns (scaled by volatility) over all historical observations with exponentially decaying weights. We call S_t sentiment, which is endogenized. The extrapolator uses it as a predictor of future market return.

Equation (3.2) is equivalent to

$$dS_t = \kappa(\mu_p^e/\sigma_p - S_t)dt + \kappa dZ_t^e, \tag{3.3}$$

which also holds for the rational benchmark if the agent is fully rational ($\mathbb{P}^e$=$\mathbb{P}$). Using (3.1), (3.3) becomes

$$dS_t = \kappa\big[\alpha_0 - (1-\alpha)S_t\big]dt + \kappa dZ_t^e. \tag{3.4}$$

So S_t mean-reverts to $\alpha_0/(1-\alpha)$ under the extrapolative measure.

In (3.2), $\kappa > 0$ measures the decaying rate of the weights on historical returns. An agent with a higher decaying rate relies more heavily on recent versus distant returns when predicting future market returns. In particular, when $\kappa \to \infty$, all the

[8] In fact, the extant models of extrapoalation either using a CARA utility or directly motivating the myopic extrapolator's demand show that overextrapolation always plays a destabilizing role and generates return reversal, e.g., De Long et al. (1990), Cutler, Poterba and Summers (1990), Hong and Stein (1999) and Barberis et al. (2015). As a result, these models of extrapolation "$\cdots$ *instead introduce fully rational investors in order to counteract the behavioral investors and preserve equilibrium*" (Jin and Sui, 2018).

weights go to the current return. When $\kappa \to 0$, the agent puts equal weights on all historical returns.

Coefficient α measures the level of extrapolation. For $\alpha > 0$, the agent believes that returns have momentum and we call her extrapolator in this paper by following the literature (e.g., Barberis et al., 2015). For $\alpha < 0$, the agent has a contrarian expectation. When $\alpha = 0$, the agent does not extrapolate historical returns. In this case, the agent is optimistic for $\alpha_0 > \mu^r/\sigma_d$ in the sense that her unconditional expectation on return-volatility ratio is greater than that in the rational benchmark; she is pessimistic for $\alpha_0 < \mu^r/\sigma_d$, and she behaves fully rationally for $\alpha_0 = \mu^r/\sigma_d$.

Coefficient α_0 measures the optimism/pessimism of the agent. In this paper, we choose

$$\alpha_0 = (1-\alpha)\mu^r/\sigma_d. \tag{3.5}$$

That is, the agent has a correct unconditional expectation about return-volatility ratio on average. Under this specification, the degree of rationality is completely determined by α. When $\alpha = 0$, the agent is fully rational; and when $\alpha = 1$, sentiment follows a random walk under the extrapolative measure (e.g., Barberis et al., 2015). Unless specified otherwise, in this paper, we choose $0 < \alpha < 1$ to study extrapolation.[9] However, our general analytical results also hold when the agent is a contrarian ($\alpha < 0$) or when the agent's irrationality only lies in beliefs ($\alpha = 0$ and $\alpha_0 \neq (1-\alpha)\mu^r/\sigma_d$).

The model of extrapolation (3.1)-(3.2) follows that in Barberis et al. (2015) who consider that the agents extrapolate price changes. Instead, we model extrapolation in terms of percentage return. This is in line with the evidence on how investors form expectations about future market returns documented in Greenwood and Shleifer (2014). Model (3.1)-(3.2) is consistent with the "belief in the law of small numbers" of Tversky and Kahneman (1971) and with the representativeness of Tversky and Kahneman (1974). People who believe in the law of small numbers draw general conclusions about the underlying data generating process by relying too heavily on relatively small sequences of data. As a consequence, extrapolators believe that recent high (low) returns are more likely to be followed by high (low) returns.

We assume that the agent extrapolates the return scaled by volatility in (3.1)-(3.2) is for tractability.[10] Our main results on momentum, reversal, return predictability

[9]Li and Liu (2018) show that, in a special case of our setting when volatility is a constant and S_t is an equally weighted average of historical returns (scaled by volatility), the stationary condition for the market return process is $|\alpha| < 1$.

[10]This assumption leads to a homogeneous partial differential equation (PDE) of price-dividend ratio, which has a closed-form solution. Alternatively, if we consider that the agent extrapolates return, then the price-dividend ratio will be governed by a nonlinear PDE, which unlikely has a closed-form solution. However, numerical simulations show that our main results of momentum, reversal, return predictability by dividend-price ratio and excess volatility still hold in this case.

by dividend-price ratio and excess volatility do not alter if we alternatively assume that the agent extrapolates return. In fact, the variation in volatility in our model is small for typical parameters, and hence plays a marginal role as to be shown later.

3.2. **Equilibrium.** This section derives the equilibrium.

Definition 3.1. An equilibrium is a set of processes $\{P_t, S_t, r_{ft}\}_{t=0}^{T}$ and of consumption and investment policies $\{C_t^*, \phi_t^*\}_{t=0}^{T}$ such that consumption and investment policies solve dynamic optimization problem (2.2) for the agent, given processes $\{P_t, S_t, r_{ft}\}_{t=0}^{T}$, and the markets for consumption and for both securities clear, that is, $C_t^* = D_t$ and $\phi_t^* = 1$ for $t \in [0, T]$.

The agent's value function solves the following Hamilton-Jacobi-Bellman (HJB) equation (Merton, 1971):

$$
\begin{aligned}
0 = \max_{C,\phi} \Big\{ & e^{-\rho t} \frac{C^{1-\gamma}}{1-\gamma} + \frac{\partial J}{\partial t} + \big[W\big(r_f + \phi(\sigma_p(\alpha_0 + \alpha S) - r_f)\big) - C\big] \frac{\partial J}{\partial W} \\
& + \kappa\big[\alpha_0 - (1-\alpha)S\big] \frac{\partial J}{\partial S} + \frac{1}{2} W^2 \phi^2 \sigma_p^2 \frac{\partial^2 J}{\partial W^2} + \kappa W \phi \sigma_p \frac{\partial^2 J}{\partial W \partial S} + \frac{\kappa^2}{2} \frac{\partial^2 J}{\partial S^2} \Big\},
\end{aligned} \tag{3.6}
$$

with boundary condition $J_T = 0$. The value function J has the form (Liu, 2007):

$$
J(S, W, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1-\gamma} \big[\Phi(S,t)\big]^{\gamma}. \tag{3.7}
$$

Substituting it into (3.6) and using FOC, we obtain

$$
C^* = W \Phi^{-1}, \qquad \phi^* = \frac{\sigma_p(\alpha_0 + \alpha S) - r_f}{\gamma \sigma_p^2} + \frac{\kappa \Phi^{-1}}{\sigma_p} \frac{\partial \Phi}{\partial S}. \tag{3.8}
$$

Equation (3.8) shows that $\Phi = W/C (= P/D)$ is the wealth-consumption ratio, or price-dividend ratio. Substituting (3.8) into (3.6), we obtain a PDE of Φ:

$$
\begin{aligned}
& \frac{\partial \Phi}{\partial t} + \frac{\kappa^2}{2} \frac{\partial^2 \Phi}{\partial S^2} + \frac{\gamma - 1}{2} \kappa^2 \Phi^{-1} \Big(\frac{\partial \Phi}{\partial S}\Big)^2 + \kappa\big[\alpha_0 - (1-\alpha)S - (\gamma - 1)\sigma_p\big] \frac{\partial \Phi}{\partial S} \\
& - \Big[\Big(1 - \frac{1}{\gamma}\Big)\sigma_p(\alpha_0 + \alpha S) - \frac{(\gamma-1)\sigma_p^2}{2} + \frac{\rho}{\gamma}\Big] \Phi + 1 = 0,
\end{aligned} \tag{3.9}
$$

with boundary condition $\Phi(S, T) = 0$. Applying Ito's lemma to $\Phi = P/D$, we obtain

$$
dD_t = D_t \Big[\mu_d^e dt + \Big(\sigma_p - \kappa \frac{\partial \ln \Phi}{\partial S}\Big) dZ_t^e\Big], \tag{3.10}
$$

where

$$
\mu_d^e(S,t) = \frac{\sigma_d}{\gamma}(\alpha_0 + \alpha S) - \frac{\rho}{\gamma} + \kappa \frac{\partial \ln \Phi}{\partial S} \Big(\frac{\alpha_0 + \alpha S}{\gamma} - \sigma_d\Big) + \frac{(\gamma - 1)\sigma_d^2}{2}. \tag{3.11}
$$

By matching dividend volatility in (2.1) and (3.10), we have

$$
\sigma_p(S,t) = \sigma_d + \kappa \frac{\partial \ln \Phi}{\partial S}. \tag{3.12}
$$

Using market clearing condition, $\phi^* = 1$, we have

$$r_f(S,t) = \sigma_p(\alpha_0 + \alpha S - \gamma\sigma_d). \tag{3.13}$$

It implies that the market price of risk under $\mathbb{P}^e$ is always a constant: $\theta = \gamma\sigma_d$. The expected dividend growth rate under the subjective measure (3.11) reduces to

$$\mu_d^e = \sigma_p\frac{\alpha_0 + \alpha S}{\gamma} - \frac{\rho}{\gamma} - \sigma_d\sigma_p + \frac{(\gamma+1)\sigma_d^2}{2}. \tag{3.14}$$

It shows that the dividend growth is sentiment-dependent under the extrapolative measure, rather than i.i.d. as under the true measure. The extrapolator's bias expectation about dividend growth is due to the fact that the expectations about dividend growth and return determine each other.

By substituting (3.12) into (3.9), we obtain

$$\frac{\partial\Phi}{\partial t} + \frac{\kappa^2}{2}\frac{\partial^2\Phi}{\partial S^2} + \frac{\kappa}{\gamma}\big[(\alpha-\gamma)S + \alpha_0\big]\frac{\partial\Phi}{\partial S} - \Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_d(\alpha_0+\alpha S) - \frac{(\gamma-1)\sigma_d^2}{2} + \frac{\rho}{\gamma}\Big]\Phi + 1 = 0, \tag{3.15}$$

with boundary condition $\Phi(S,T) = 0$. Following Liu (2007), we define $\widehat{\Phi}(S,t)$ such that

$$\Phi = \int_t^T \widehat{\Phi}(S,u)du.$$

Then $\widehat{\Phi}$ satisfies

$$\frac{\partial\widehat{\Phi}}{\partial t} + \frac{\kappa^2}{2}\frac{\partial^2\widehat{\Phi}}{\partial S^2} + \frac{\kappa}{\gamma}\big[(\alpha-\gamma)S + \alpha_0\big]\frac{\partial\widehat{\Phi}}{\partial S} - \Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_d(\alpha_0+\alpha S) - \frac{(\gamma-1)\sigma_d^2}{2} + \frac{\rho}{\gamma}\Big]\widehat{\Phi} = 0, \tag{3.16}$$

with boundary condition $\widehat{\Phi}(S,T) = 1$. By conjecturing that

$$\widehat{\Phi}(S,t) = e^{A_t S + B_t},$$

and substituting it into (3.16), we obtain

$$\begin{aligned} &A' + \kappa\Big(\frac{\alpha}{\gamma} - 1\Big)A - \alpha\sigma_d\Big(1 - \frac{1}{\gamma}\Big) = 0, \\ &B' + \frac{\kappa^2}{2}A^2 + \frac{\kappa\alpha_0}{\gamma}A - \Big(1 - \frac{1}{\gamma}\Big)\sigma_d\alpha_0 - \frac{\rho}{\gamma} + \frac{(\gamma-1)\sigma_d^2}{2} = 0, \end{aligned} \tag{3.17}$$

with terminal conditions $A_T = B_T = 0$, where the operator " $'$ " denotes the derivative with respect to t. The solution of (3.17) is given by (3.19).

Proposition 3.2. *The equilibrium market price satisfies*

$$\frac{dP_t + D_t dt}{P_t} = \mu_p^e dt + \sigma_p dZ_t^e,$$

where

$$\mu_p^e = \sigma_p(\alpha_0 + \alpha S_t), \qquad \sigma_p = \sigma_d + \kappa \frac{\int_t^T A_u e^{A_u S_t + B_u} du}{\int_t^T e^{A_u S_t + B_u} du}, \tag{3.18}$$

and

$$\begin{aligned} A_t =& -(\gamma-1)\alpha\sigma_d \frac{1-e^{-\kappa(\gamma-\alpha)(T-t)/\gamma}}{\kappa(\gamma-\alpha)}, \\ B_t =& -\frac{\gamma(\gamma-1)\alpha\sigma_d}{\kappa(\gamma-\alpha)^2}\Big[\frac{(\gamma-1)\alpha\sigma_d}{\gamma-\alpha} - \frac{\alpha_0}{\gamma}\Big]\Big[1-e^{-\kappa(\gamma-\alpha)(T-t)/\gamma}\Big] \\ &+ \frac{\gamma(\gamma-1)^2\alpha^2\sigma_d^2}{4\kappa(\gamma-\alpha)^3}\Big[1-e^{-2\kappa(\gamma-\alpha)(T-t)/\gamma}\Big] - B^*(\alpha)(T-t), \\ B^*(\alpha) =& \frac{\rho}{\gamma} + \frac{(\gamma-1)\alpha_0\sigma_d}{\gamma-\alpha} - \frac{\gamma(\gamma-1)(\alpha^2-2\alpha+\gamma)\sigma_d^2}{2(\gamma-\alpha)^2}. \end{aligned} \tag{3.19}$$

The riskless rate is given by

$$r_f = \sigma_p(\alpha_0 + \alpha S_t - \gamma\sigma_d), \tag{3.20}$$

and the price-dividend ratio is given by

$$\Phi = \int_t^T e^{A_u S_t + B_u} du. \tag{3.21}$$

Proposition 3.2 summarizes the equilibrium under the extrapolative measure, which has closed-form solutions. Because the expectations about return and about dividend growth determine each other, the equilibrium described in Proposition 3.2 could be a rational one if (3.10) is the true data generating process for dividend under the true measure. Proposition 3.2 provides a general result that also applies to the rational benchmark. In fact, Proposition 3.2 reduces to Proposition 2.1 when $\alpha = 0$. In addition, the riskless rate in (3.20) is positively related to sentiment.

The extrapolator thinks that dividend growth depends on sentiment and that sentiment predicts future returns. This gives rise to the dependence of price-dividend ratio on sentiment in (3.21), and further relates price-dividend ratio to future returns. This generalizes the case with the i.i.d. expected return summarized in Proposition 2.1.

If EIS< 1 ($\gamma > 1$), then A_t in (3.19) is an increasing function of t, and A_t is bounded: $-\frac{(\gamma-1)\alpha\sigma_d}{\kappa(\gamma-\alpha)} < A_t \le 0$. If EIS$> 1$, then A_t is a decreasing function of t and $0 \le A_t \le A_0$, where $A_0 = (1-\gamma)\alpha\sigma_d \frac{1-e^{-\kappa(\gamma-\alpha)/\gamma T}}{\kappa(\gamma-\alpha)}$. Notice that A_0 is bounded for $\gamma \in (\alpha, 1)$; while A_0 can be unbounded for $\gamma \le \alpha$ given that T can be unbounded in general. This will affect the existence of equilibrium, as to be detailed in Section 4.

Equation (3.21) further shows that

$$\frac{\partial \Phi}{\partial S} = \int_t^T A_u e^{A_u S_t + B_u} du \begin{cases} < 0, & \text{if} \quad \text{EIS} < 1, \\ > 0, & \text{if} \quad \text{EIS} > 1. \end{cases} \tag{3.22}$$

As in the rational benchmark (2.7), the price-dividend ratio (3.21) reflects the relative strength between income and substitution effects. After high realized returns (or high sentiment), price-dividend ratio decreases (increase) if EIS<1 (EIS>1). This underlies the different implications between Barberis et al.'s model and ours.

Despite the presence of extrapolation, Proposition 3.2 shares the same message to Proposition 2.1 for the rational benchmark with respect to the impact of the expected dividend growth rate on equilibrium. In fact, it follows from (3.13) and (3.14) that

$$r_f = \rho + \gamma \Big[\mu_d^e - \frac{(\gamma+1)\sigma_d^2}{2} \Big], \tag{3.23}$$

which is consistent with the rational benchmark relationship (2.6). The extrapolator's expectation about the future return implies her expectation about dividend growth; hence the expected dividend growth in (2.6) is replaced by μ_d^e.

Corollary 3.3. *To the leading order of κ, the expected return and volatility satisfy*

$$\mu_p^e = \rho + \gamma \Big[\mu_d^e - \frac{(\gamma-1)\sigma_d^2}{2} \Big], \qquad \sigma_p = \sigma_d, \tag{3.24}$$

the riskless rate is given by (3.23)*, and the price-dividend ratio is given by*

$$\Phi = \frac{1 - e^{-[\rho + (\gamma-1)(\mu_d^e - \gamma\sigma_d^2/2)](T-t)}}{\rho + (\gamma-1)(\mu_d^e - \gamma\sigma_d^2/2)}. \tag{3.25}$$

Corollary 3.3 further shows that, when κ is small, the equilibrium price and riskless rate under extrapolation share the same forms as those in the rational benchmark if we replace the expected dividend growth rate by μ_d^e. When $T \to \infty$, the price-dividend ratio (3.25) is also the same as (2.7) except that μ_d is replaced by μ_d^e. This underlines the comparison with the model of Barberis et al. (2015) with respect to the impact of expected dividend growth in the rational benchmarks in Section 2.3.

3.3. **Expectation.** Past returns affect extrapolator's expectations about future market returns—price impact on expectation (characterized by sentiment S_t). Price impact is always positive in the sense that an extrapolator becomes more optimistic after a sequence of high past returns. On the other hand, investor's expectations in turn feed into asset prices—expectation impact on price (measured by α). If the expectation impact is also positive, together with the positive price impact, there is a positive feedback, which is called "infinite feedback loop" in Barberis et al. (2015). However, Proposition 3.2 and Corollary 3.4 imply that expectation impact may not be always positive, depending on the agent's EIS.

More specifically, the state price density is given by $\xi_t = e^{-\rho t} D_t^{-\gamma}$. So the price-dividend ratio satisfies

$$\Phi_t = \mathbb{E}_t^e\Big[\int_t^T \frac{\xi_v}{\xi_t}\frac{D_v}{D_t}dv\Big] = \mathbb{E}_t^e\Big[\int_t^T e^{-\rho(v-t)}\Big(\frac{D_v}{D_t}\Big)^{1-\gamma}dv\Big]. \tag{3.26}$$

Equation (3.26) shows that it is the difference between γ and 1 that determines the dominance of the substitution effect and the income effect. This is caused by the trade-off between the conditional state price density ξ_v/ξ_t and the dividend growth.

High past returns (high sentiment) lead the extrapolator to expect high dividend growth, or high values of D_v/D_t in (3.26), reflecting the positive price impact. In this case, a small EIS ($\gamma > 1$) decreases the price-dividend ratio, leading to a negative expectation impact, while a large EIS increases the price-dividend ratio, leading to a positive expectation impact.

3.4. **Bias Correction under the True Measure.** It follows from (2.1) and (3.10) that Z_t^e and Z_t satisfy the relationship

$$dZ_t^e = (\mu_d - \mu_d^e)/\sigma_d dt + dZ_t. \tag{3.27}$$

Using it, the market return under the true measure (used by an outside econometrician) is characterized by the following proposition.

Proposition 3.4. *Under the true measure $\mathbb{P}$, the market price satisfies*

$$\frac{dP_t + D_t dt}{P_t} = \mu_p dt + \sigma_p dZ_t, \tag{3.28}$$

where

$$\begin{aligned} \mu_p(S,t) &= \sigma_p(\psi_0 + \psi S_t), \\ \psi(\sigma_p) &= \alpha\Big(1 - \frac{\sigma_p}{\gamma\sigma_d}\Big), \\ \psi_0(\sigma_p) &= \alpha_0\Big(1 - \frac{\sigma_p}{\gamma\sigma_d}\Big) + \frac{\mu^r}{\gamma\sigma_d} + \sigma_p - \sigma_d, \end{aligned} \tag{3.29}$$

and σ_p is given by (3.18)*, and sentiment follows*

$$dS_t = \kappa\big[\psi_0 - (1-\psi)S_t\big]dt + \kappa dZ_t. \tag{3.30}$$

If EIS$<$ 1 (EIS$>$ 1), then $\psi > 0$ ($\psi < 0$) and hence expected return under the true measure depends positively (negatively) on sentiment. By comparing (3.4) and (3.30), the extrapolator thinks that S_t is more persistent than the actual process. Therefore, there is a negative correction in the expected return by the true measure. Because the expectations on dividend growth and on return determine each other, an extrapolator thinks that the dividend process, or the aggregate consumption process, is more persistent than the actual process as shown in (3.11).

Especially, when EIS$= 1$, the agent is myopic and the following corollary characterizes the equilibrium. Denote by $r_{pt} = \int_{t-\Delta}^{t} \frac{dP_u + D_u du}{P_u}$ the market return during $[t - \Delta, t]$. The serial correlation of market returns at lag h is defined as $ACF(h) = cov(r_{p,t+h\Delta}, r_{pt})/var(r_{pt})$.

Corollary 3.5. *If EIS$= 1$, then*

$$\mu_p^e = \sigma_d(\alpha_0 + \alpha S_t), \quad \sigma_p = \sigma_d, \quad r_f = \sigma_d(\alpha_0 + \alpha S_t - \sigma_d), \quad \Phi = \rho^{-1}\big[1 - e^{-\rho(T-t)}\big]. \tag{3.31}$$

In this case, market return under the true measure is i.i.d.:

$$\frac{dP_t + D_t dt}{P_t} = (\mu_d + \rho)dt + \sigma_d dZ_t,$$

and the series correlation of returns at any lag h is zero:

$$ACF(h) = 0.$$

Interestingly, Corollary 3.5 shows that, although the extrapolator believes sentiment is able to predict market returns, the equilibrium turns out to be independent of sentiment under the true measure. So market returns can behave very differently under two equivalent measures: they have trend under the extrapolative measure, while become i.i.d. under the true measure. In fact, when the representative agent has a log preference, the price-dividend ratio and volatility become independent of sentiment because the effect of sentiment happens to be completely offset by bias correction, resulting in an i.i.d. return process under the true measure.

When EIS$\neq 1$, price-dividend ratio depends on sentiment. Propositions 3.2 and 3.4 show that the relationships of sentiment S with price-dividend ratio Φ and expected return μ_p depend on the agent's EIS, which determines the relative strength of income and substitution effects. Although market returns always exhibit momentum under the extrapolative measure, they may end up with either momentum or reversal under the true measure, depending on the dominance between the two effects.

Proposition 3.6. *An outside econometrician who uses the true measure always negatively corrects the extrapolator's bias expectation about the future return:*

$$\mu_p = \mu_p^e + \eta(S_t, t), \tag{3.32}$$

where

$$\eta = \frac{\sigma_p}{\gamma\sigma_d}\big[\rho - \sigma_p(\alpha_0 + \alpha S_t)\big] + \frac{\sigma_p}{\sigma_d}\Big(\mu_d - \frac{\gamma+1}{2}\sigma_d^2\Big) + \sigma_p^2 \tag{3.33}$$

has a negative relationship with S_t.

The deviation η characterizes a correction premium, reflecting the bias correction under the true measure. An extrapolator expects a high future return and a high dividend growth rate when sentiment is high. However, under the true measure (used by an outside econometrician), the expected return and dividend growth should not be as high as expected by the extrapolator, leading to a negative correction premium. The absolute value of the correction premium increases with EIS due to the substitution effect, and also increases with the level of sentiment. Therefore, a small EIS leads to a small correction and hence, under the true measure, high sentiment predicts high future returns. However, a large EIS leads sentiment to negatively predict future returns.

In all, the extrapolator reacts to past returns and then affects prices in two dimensions: *extrapolation* that forms her expectation about future returns and *EIS* that determines the degree of response. As to be shown in the next section, a large EIS leads to a large response that produces overreaction to past returns (or to sentiment) and subsequent return reversal. We also find it is possible to have underreaction to past returns and to produce momentum if her EIS is small.

4. Properties of Equilibrium

In this section, we study the properties of equilibrium. We find that the effects of extrapolation on the expectation impact, existence of equilibrium, momentum, and return predictability by dividend-price ratio and sentiment depend crucially on the agent's EIS. We also show that the model generates a large set of empirical regularities observed in financial markets.

4.1. **Momentum and Reversal.** Short-run momentum and long-run reversal in returns are two of most prominent financial market anomalies, and have been extensively documented in the literature, e.g., Jegadeesh and Titman (1993) and Moskowitz, Ooi and Pedersen (2012) for momentum; Fama and French (1988*b*) and Poterba and Summers (1988) for reversal; and Cutler, Poterba and Summers (1991) for both.

Corollary 4.1. *To the order of α, return volatility and the expected market returns under $\mathbb{P}^e$ and $\mathbb{P}$ are given, respectively, by*

$$\begin{aligned} \sigma_p &= \sigma_d\Big[1-\Big(1-\frac{1}{\gamma}\Big)\alpha a_{1t}\Big], \\ \mu_p^e &= \alpha\sigma_d S_t + \alpha_0\sigma_d\Big[1-\Big(1-\frac{1}{\gamma}\Big)\alpha a_{1t}\Big], \\ \mu_p &= \Big(1-\frac{1}{\gamma}\Big)\alpha\sigma_d S_t + a_{2t}, \end{aligned} \tag{4.1}$$

where $a_{1t} \in [0, 1)$ *and* a_{2t} *are deterministic functions of* t *given by* (A.2) *in Appendix A.2. In this case, the serial correlation of market returns at lag* h *satisfies*

$$ACF(h) \begin{cases} > 0 & \text{if} \quad EIS < 1; \\ = 0 & \text{if} \quad EIS = 1; \\ < 0 & \text{if} \quad EIS > 1, \end{cases} \tag{4.2}$$

for $\alpha > 0$*, and sign of the serial correlation has the opposite pattern for* $\alpha < 0$*. The absolute value of series correlation decreases with* h*.*

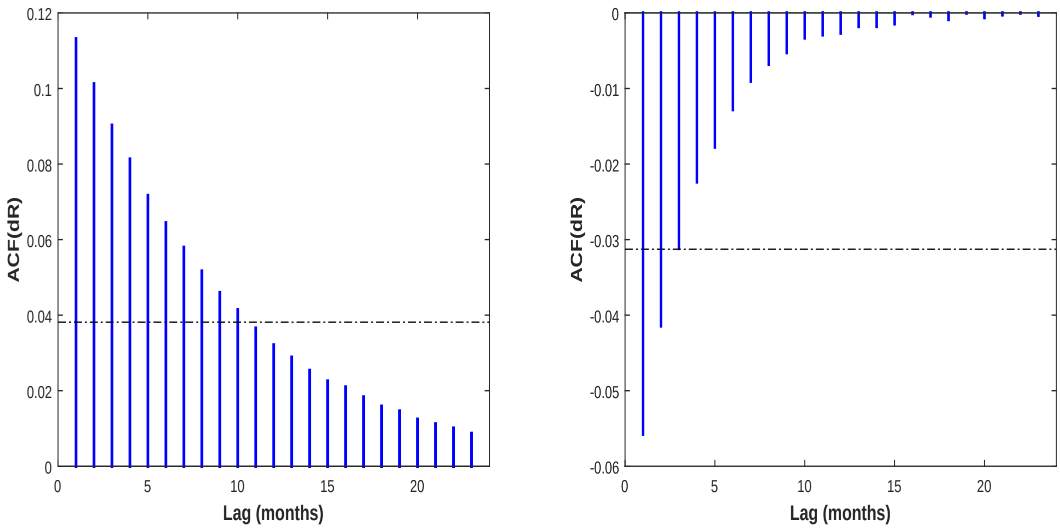
Corollary 4.1 highlights the impact of EIS on momentum and reversal. Under the extrapolative measure $\mathbb{P}^e$, high realized returns, or high sentiment, always lead to high expected return (by definition). Under the true measure $\mathbb{P}$, Corollary 4.1 shows that the relationship between expected return and sentiment, however, depends on EIS. The expected return is positively related to sentiment for a small EIS ($\gamma > 1$) and negatively related to sentiment for a large EIS ($\gamma < 1$). As a result, although the extrapolator always believes that market returns exhibit momentum, the market returns may exhibit reversal under the true measure if the extrapolator has a large EIS. In addition, one can verify that the returns in another limiting case of Corollary 3.3 have the same serial correlation (4.2) as in Corollary 4.1.

The reason is as follow. The extrapolator's expectation about the future return implies that her expectation about dividend growth also positively depends on sentiment. However, the dividend growth is independent of sentiment under the true measure. So there is a negative bias correction for sentiment by the true measure, leading to a correction premium that is negatively related to sentiment as shown in Proposition 3.6. The level of the correction premium increases with EIS. Therefore, for a small EIS, the correction is not enough and momentum is preserved. For a large EIS, large correction leads to return reversal, which is found in most models of extrapolation.

Fig. 4.1 numerically verifies the momentum and reversal. It shows that the series correlations of returns for lag ranging from 1 month to 2 years are positive for $\gamma > 1$ and negative for $\gamma < 1$.[11] In addition, the left panel shows that returns are positively

[11]Unless specified otherwise, in this paper, we set $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$ (e.g., Basak and Cuoco, 1998; Chabakauri, 2015), and $T = 100$ in annual terms. According to equation (3) in Greenwood and Shleifer (2014), the ratio of the weights on returns in quarter t and $t-1$ equals λ, so κ in our paper and λ satisfy $e^{\kappa/4} = 1/\lambda$. We set $\kappa = 2.3$, corresponding to $\lambda = 0.56$ which is the average of the estimates for λ reported in their Table 4. We also choose $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$. That is, the agent has a correct unconditional expectation about return-volatility ratio on average. Under this specification, the degree of rationality is completely determined by α. When $\alpha = 0$, the agent is fully rational; and when $\alpha = 1$, sentiment follows a random walk under the extrapolative measure (e.g., Barberis et al., 2015). In most simulations, we set $\alpha = 0.5$.

correlated for large lags and hence the momentum does not reverse in the long-run. This implies that the market is not overvalued but correctly priced in our economy.



(a) $\gamma = 5$ (b) $\gamma = 0.8$

FIGURE 4.1. Series correlations of returns for lag ranging from 1 month to 2 years, together with the 95% confidence bounds. The results are based on Monte Carlo simulations. Here $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $\kappa = 2.3$, $T = 100$, $\alpha = 0.5$ and $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$.

Barberis et al. (2015) find that their model can also generate momentum by putting some delay in the reaction to past prices when extrapolators forming expectations. We show that our model can generate momentum for small EIS even without the assumption of delayed reaction. Our mechanism in generating momentum is also different from that of underreaction proposed in Hong and Stein (1999). In fact, the underreaction is caused by the gradual diffusion of information. However, there is no underreaction to information in our model because price is determined by a full equilibrium where a dividend shock (under the true measure) is immediately and correctly priced in our economy without any requirements for future corrections.

4.2. **Does Extrapolation Always Lead to A Positive Feedback?** The extant literature shows that an extrapolator always leads to a positive feedback (e.g., De Long et al., 1990; Cutler et al., 1990; Hong and Stein, 1999; Barberis et al., 2015) in the sense that "*if good cash-flow news pushes the stock market up, this price increase feeds into extrapolators' expectations about future price changes, which then* *leads them to push the current price up even higher. However, this then further increases extrapolators' expectations about future price changes, leading them to push the current price still higher, and so on*" (Barberis et al., 2015, p. 7).

Proposition 3.2 implies that this "infinite feedback loop" occurs only for EIS>1 when the income effect dominates. In this case, after a sequence of high past returns, the extrapolator expects a high future return. The income effect increases price-dividend ratio, resulting in a positive feedback.

As shown in Section 3.3, EIS determines the dominance of the substitution effect and the income effect. For a small EIS, although prices always have a positive impact on her expectation about future market returns (by definition), the impact of expectation on prices could be negative due to the dominance of substitution effect, leading to a *negative* feedback. As a result, extrapolation tends to pull down, rather than push up, the price level relative to dividend.

In other words, a high extrapolator's expectation about future market returns leads her to expect both high discount rates and high dividend growth rates in the future. If the increase in discount rate dominates (substitution effect dominates income effect), then price-dividend ratio decreases, leading to a negative feedback as indicated by (3.26). It is easier to understand this under the true measure, where dividend growth is constant and expected return decreases when the substitution effect dominates. Put differently, an agent's optimal demand depends not only on the expected return, but also on intertemporal hedging, riskless rate and volatility. Therefore, the feedback, as well as return auto-correlations, can have the opposite patterns under the true measure and under the extrapolative measure.

The mechanism in generating positive (negative) feedback has been reflected by the rational benchmark in Section 2.2. When the agent is fully rational, Proposition 2.1 shows that high expected dividend growth decreases price-dividend ratio when $\gamma > 1$. When the agent is an extrapolator, after high realized returns (high S_t), she expects high dividend growth (see, (3.14)) and hence pulls down current price.

4.3. **Existence of Equilibrium.** The literature further argues that the "infinite feedback loop" may lead to price divergence and hence extrapolators tend to destabilize the equilibrium. For example, Jin and Sui (2018) claim that "*In general, there is a danger that this feedback loop could "explode". $\cdots$ Models like Cutler et al. (1990) and Barberis et al. (2015) instead introduce fully rational investors in order to counteract the behavioral investors and preserve equilibrium.*" In this section, we study whether extrapolation leads to the nonexistence of equilibrium.

Corollary 4.2. *(Existence of equilibrium.) (1) When $T < \infty$, the equilibrium characterized in Proposition 3.2 exists for $\gamma > 0$.*[12]

(2) When $T = \infty$, the equilibrium exists if and only if

$$\gamma > \alpha \quad \textit{and} \quad B^*(\alpha) > 0, \tag{4.4}$$

where $B^*(\alpha)$ is given by (3.19). *If the agent is fully rational ($\alpha = 0$ and $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$), then condition* (4.4) *reduces to* (2.8) *in Proposition 2.1.*

In the CARA setup used in the literature, the riskless asset is usually assumed to be in perfectly elastic supply for mathematical tractability. However, this makes the equilibrium prone to explode. If we instead consider that the riskless rate is endogenously determined in the equilibrium, then the riskless rate increases with sentiment. This decreases extrapolators' demand for the risky asset when sentiment is high, and hence prevents their expected utility from approaching infinity. In this case, the equilibrium tends to be preserved.[13] In fact, the first part of Corollary 4.2 shows that, when the riskless rate is endogenously determined and percentage return is well defined, the equilibrium always exists even with a positive feedback as long as the investment horizon is finite ($T < \infty$). Therefore, the intuition that, due to the "infinite feedback loop", sufficiently strong extrapolation should lead to price divergence may not hold.

For an infinite-horizon economy ($T = \infty$), the equilibrium exists under condition (4.4). Otherwise, because of the transversality condition, which is affected by extrapolation (α), the economy loses equilibrium when coefficients A and/or B governed by (3.17) approach infinity.

[12]When $T < \infty$, the equilibrium does not exist for an infinite EIS ($\gamma = 0$). In this case, state price density become $e^{-\rho t}$, and hence both risk-free rate and expected market return are constant:

$$\mu_p = r_f = \rho, \tag{4.3}$$

implying that there is no equilibrium if the agent believes a time-varying expected return, such as the extrapolative expectation (3.1). In fact, S_t is a weighted average of historical returns and cannot be a constant, so (4.3) cannot hold.

[13]There are three markets in the economy studied by Barberis et al. (2015), namely, the markets for the risky asset, for the riskless asset and for the consumption good. The risky asset market is assumed to clear, while the remaining two markets fail to clear. Especially, if we set the rational-benchmark riskless rate in (2.10) as $r_f = \rho + A\mu_d - \frac{A^2\sigma_d^2}{2}$, which is determined by the state price density $e^{-\rho t - AD_t}$, one can verify that the riskless asset market also clears. In this special case with rational agents only, the riskless rate happens to be a constant as assumed in Barberis et al. (2015). But their results for the general case in the presence of extrapolators cannot be consistent with the equilibrium determined by the state price density, which implies that the riskless rate depends on sentiment and cannot be a constant any more. More issues caused by the exogenous riskless rate are discussed in Loewenstein and Willard (2006).

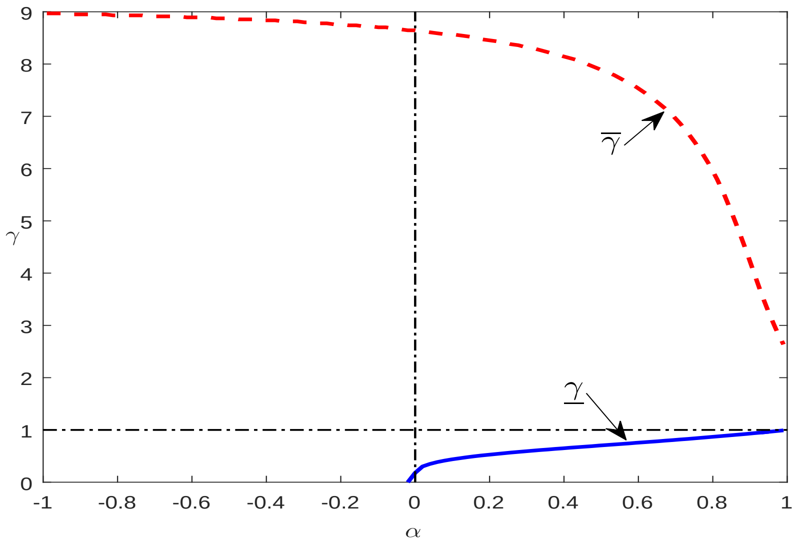
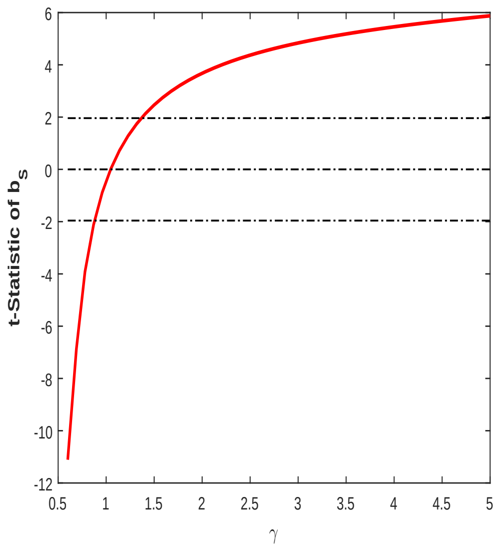


FIGURE 4.2. This figure illustrates the upper and lower boundaries for γ, between which equilibrium exists. Here $\rho = 0.02$, $\mu_d = 0.025$, $\sigma_d = 0.08$, $\kappa = 2.3$, $T = 100$, $\alpha = 0.5$ and $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$.

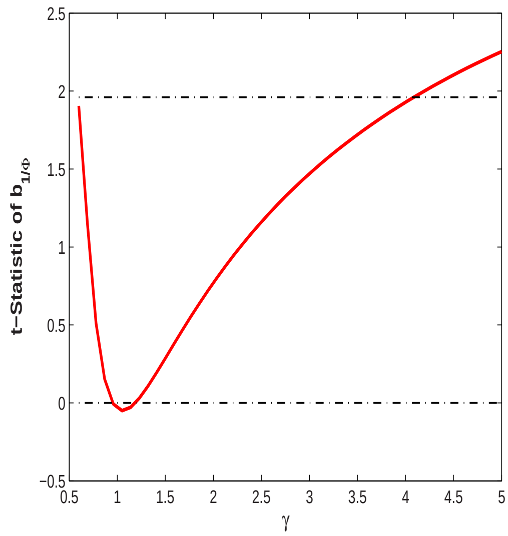
We numerically study the condition (4.4). Fig. 4.2 illustrates that, for an infinite-horizon economy, there are two bounds of γ, between which equilibrium exists. Due to the transversality condition, the economy may not have equilibrium even if there is no extrapolation ($\alpha = 0$), consistent with the rational benchmark in Section 2.2. The upper bound $\overline{\gamma}$ decreases with α while the lower bound $\underline{\gamma}$ increases with α. Therefore, the economy tends to lose equilibrium when extrapolation is strong (large α), consistent with Barberis et al. (2015). In this case, the infinite investment horizon allows the extrapolator to have infinitely many opportunities to explore feedback and hence may lead to nonexistence of equilibrium.

In contrast, a contrarian ($\alpha < 0$) tends to preserve equilibrium. There can be even no lower bound for γ when $\alpha < 0$.

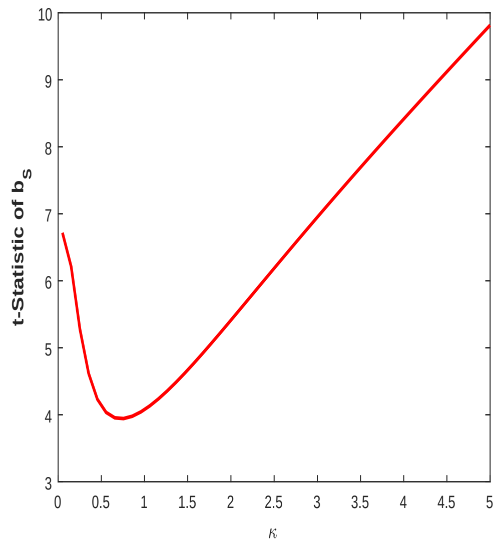
4.4. **Return Predictability by Dividend-Price Ratio and Sentiment.** Proposition 3.2 shows that the effects of extrapolation on dividend-price ratio depend on the investor's EIS. When EIS< 1, (3.21) shows that dividend-price ratio ($1/\Phi$) is positively related to sentiment, which is also positively related to expected return. In this case, dividend-price ratio positively predicts future return. When EIS> 1, dividend-price ratio still positively predicts future return because dividend-price ratio is negatively related to sentiment, which is also negatively related to future return. This is consistent with the findings of Campbell and Shiller (1988) and Fama and French (1988a).



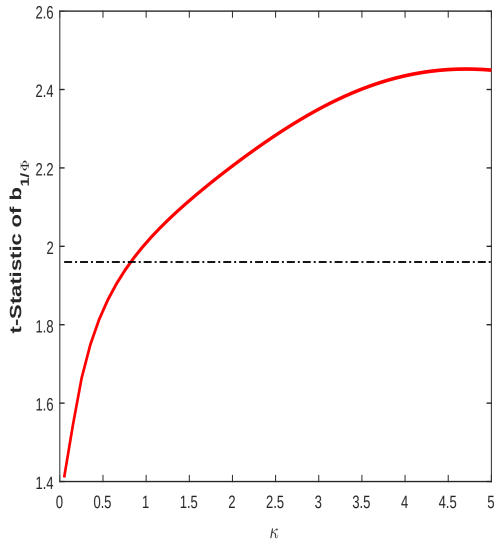
(a) $X = S$



(b) $X = 1/\Phi$



(c) $X = S$



(d) $X = 1/\Phi$

FIGURE 4.3. We regress model generated monthly market returns on different predictors: $dR_t = a + b_X X_t + \varepsilon_{t+dt}$ for $X = S$ and $X = 1/\Phi$. Panels (a) and (b) plot the t-statistics of the coefficients b_S and $b_{1\Phi}$ respectively as functions of γ. Standard errors are adjusted for heteroskedasticity and serial correlation according to Newey and West (1987). Here $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $T = 100$, $\alpha = 0.5$, $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$, $\kappa = 2.3$ for Panels (a) and (b), and $\gamma = 5$ for (c) and (d).

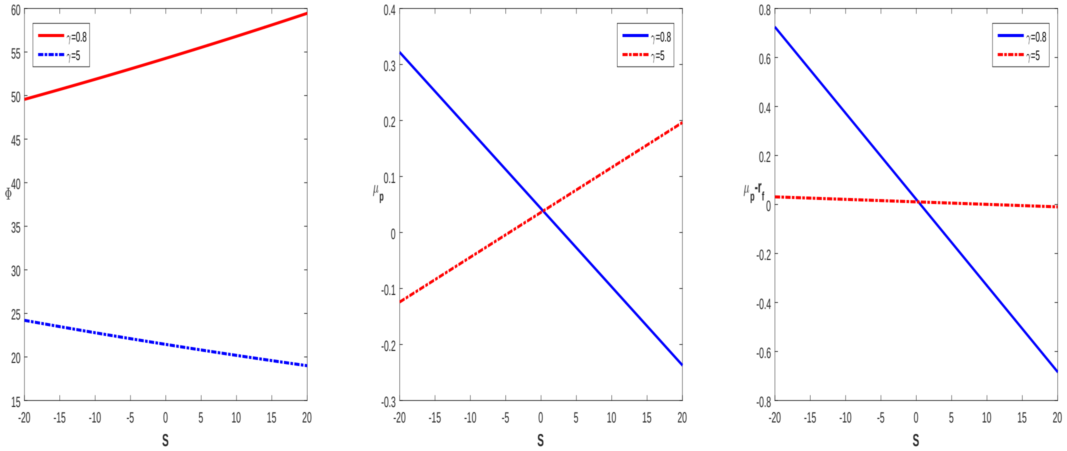
Fig. 4.3 verifies the predictive ability of sentiment and dividend-price ratio using model-generated data. We regress the market return on different predictors (with monthly frequency): $dR_t = a + b_X X_t + \varepsilon_{t+dt}$ for $X = S$ and $X = 1/\Phi$. Fig. 4.3 reports the t-statistics of the coefficients b_S and $b_{1/\Phi}$ respectively as functions of γ. Panel (a) shows that sentiment S positively (negatively) predicts future return for $\gamma > 1$ ($\gamma < 1$). In Panel (b), dividend-price ratio $1/\Phi$ positively predicts future return no matter if $\gamma > 1$ or $\gamma < 1$. The predictive ability is more statistically significant for large γ. Panel (a) also verifies momentum for large γ and reversal for small γ discussed in Section 4.1.

By choosing $\gamma = 5$, Panels (c) and (d) illustrates the impact of the decaying rate κ on return predictability. Panel (c) shows that sentiment has strong predictive ability for all $\kappa \in [0, 5]$. More important, Panel (d) suggests that dividend-price ratio can significantly predict future market returns only when κ is large, with which the extrapolator relies heavily on recent versus distant returns when predicting future market returns. This may be consistent with the recent findings of Cassella and Gulen (2017).[14]

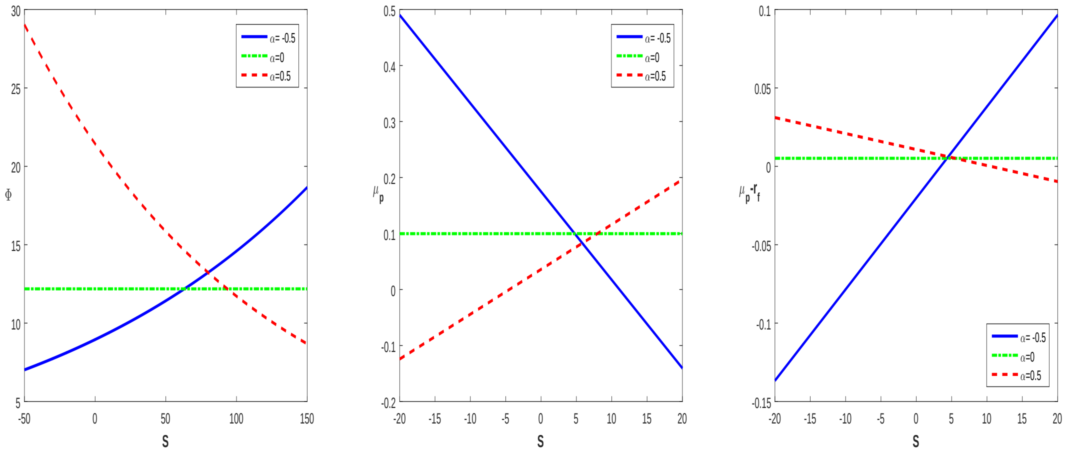
4.5. **Negative Risk Premium for High Sentiment.** Proposition 3.2 shows that, under the extrapolative measure $\mathbb{P}^e$, risk premium is always positive ($\mu_p^e - r_f = \gamma\sigma_d\sigma_p > 0$). However, risk premium under the true measure $\mathbb{P}$ could be negative. Fig. 4.4 illustrates the impact of sentiment on the equilibrium for different EIS and different α. The right panels show that, under extrapolation ($\alpha > 0$), for both $\gamma > 1$ and $\gamma < 1$, risk premium can be negative when sentiment is high. This is consistent with the recent findings of Greenwood and Hanson (2013), Baron and Xiong (2017) and Cassella and Gulen (2017).

This can be understood from the bias correction between the true measure and the extrapolative measure. As shown in Proposition 3.6, when sentiment is high, the extrapolator with a biased expectation expects a higher risk premium than that under the true measure, leading to the negative dependence on sentiment of the bias correction of risk premium under the true measure. As a result, sufficiently high sentiment can generate a negative risk premium as illustrated in Fig. 4.4. In this case, her myopic demand is negative; while her positive total demand ($\phi^* = 1$) is

[14] By developing a measure of the degree of extrapolative weighting (DOX), which captures the tendency of investors to overweight recent returns, Cassella and Gulen (2017) find that price-scaled variables predict future market returns only when DOX is high, while these variables hold no predictive ability at short horizons when DOX is low.



(a) Price-dividend ratio (b) Expected return Under $\mathbb{P}$ (c) Risk premium Under $\mathbb{P}$



(d) Price-dividend ratio (e) Expected return Under $\mathbb{P}$ (f) Risk premium Under $\mathbb{P}$

FIGURE 4.4. The impact of EIS and α on price-dividend ratio, expected return and risk premium. Here $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $\kappa = 2.3$, $T = 100$, $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$, $\alpha = 0.5$ (for the upper panels) and $\gamma = 5$ (for the lower panels).

due to positive intertemporal hedging.[15] Fig. 4.4 also verifies that risk premium is negatively related to sentiment regardless of EIS.

If $\gamma > 1$, Fig. 4.4 (d) shows that the price-dividend ratio for $\alpha > 0$ is higher (lower) than that for the rational benchmark ($\alpha = 0$) when S is sufficiently small (large). In this sense, extrapolation leads to overvaluation relative to the rational benchmark when S is sufficiently small. However, as discussed above, there is no

[15] In a dynamic asset allocation problem, Li and Liu (2018) show that an investor with CRRA preference and $\gamma > 1$ may long a risky asset with negative momentum to effectively home-make an asset with return reversal.

overvaluation in time series for our economy, therefore the relative overvaluation to the rational benchmark cannot be corrected in the long-run. The price-dividend ratios follow the opposite pattern if $\gamma < 1$.

In addition, the left and middle panels verify the relationships of sentiment with dividend-price ratio and expected return under the true measure respectively as discussed in Sections 4.2-4.4. They also verify that the positive feedback of Barberis et al. (2015) occurs when $\gamma < 1$, but a negative feedback occurs when $\gamma > 1$.

4.6. **Volatility.** According to the accounting identity of $P = D\Phi$, the return volatility equals the sum of dividend volatility and the volatility of price-dividend ratio: $\sigma_p = \sigma_d + \kappa \frac{\partial \ln \Phi}{\partial S}$. The sign of the last term, reflecting positive/negative expectation impact, is determined by EIS as shown in Proposition 3.2. Then we have the following corollary on volatility under extrapolation.

Corollary 4.3. *Return volatility is always positive: $\sigma_p > 0$, and satisfies*

$$\sigma_p \begin{cases} > \sigma_d, & \text{if} \quad EIS > 1, \\ = \sigma_d, & \text{if} \quad EIS = 1, \\ < \sigma_d, & \text{if} \quad EIS < 1. \end{cases}$$

In addition, σ_p increases with S_t.

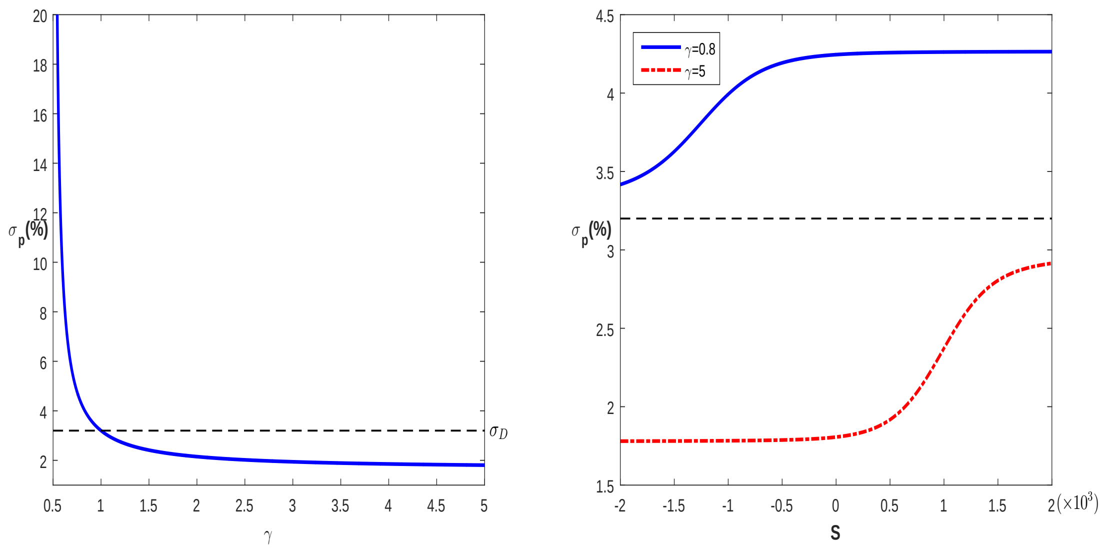
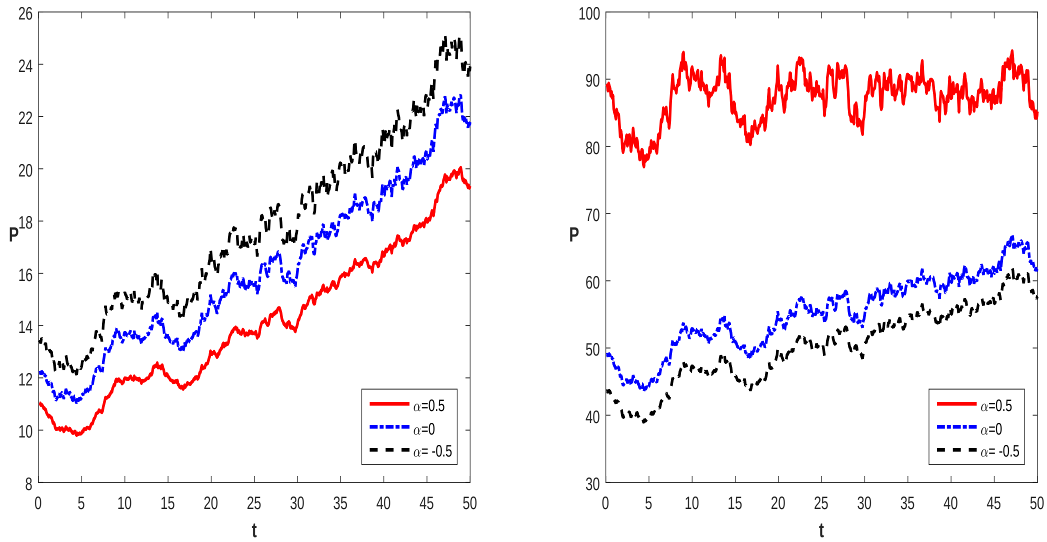


FIGURE 4.5. The impact of EIS and sentiment on volatility. Here $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $T = 100$, $\alpha = 0.5$, $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$, $\kappa = 2.3$ and $S_t = 0.1$ (for the left panel).

Corollary 4.3 shows that excess volatility (LeRoy and Porter, 1981; Shiller, 1981) occurs for EIS> 1 due to the positive feedback, consistent with Barberis et al. (2015)

and Jin and Sui (2018). The negative feedback for EIS<1 leads to lower market volatility relative to that of dividend. This also illustrated in the left panel of Fig. 4.5. In addition, Corollary 4.3 also shows that volatility increases with sentiment regardless of EIS and the right panel of Fig. 4.5 verifies this.



(a) $\gamma = 5$ (b) $\gamma = 0.8$

FIGURE 4.6. With the same typical dividend process, the price processes under the rational expectation ($\alpha = 0$), extrapolative expectation ($\alpha > 0$) and contrarian expectation ($\alpha < 0$) respectively. Here $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $\kappa = 2.3$, $T = 100$ and $\alpha_0 = \mu^r/\sigma_d$.

4.7. **Prices.** In this subsection, we study the impact of the level of extrapolation α on price level. We choose a fixed α_0, which is independent of α, so that a change in α does not affect optimism/pessimism (α_0). With the same typical dividend process, Fig. 4.6 illustrates the impact of α on prices (or price-dividend ratios Φ). Prices under rational expectation is higher (lower) than those under extrapolative expectation (contrarian expectation) for a small EIS ($\gamma > 1$). In fact, it follows from (A.1) that $A_tS + B_t$ decreases with α, and hence the price-dividend ratio also decreases with α according to (3.21). Price levels follow the opposite pattern for a large EIS ($\gamma < 1$).

This is consistent with the negative (positive) feedback for $\gamma > 1$ ($\gamma < 1$) discussed in Section 4.2. In fact, when $\gamma > 1$, Section 4.2 shows that after high sentiment, the extrapolator has a high expected return, leading to low price-dividend ratio. Notice that sentiment is in general positive because it is a weighted average of historical returns. An increase in α increases the expected return under the extrapolative expectation, resulting in low price-dividend ratio, see the left panel of Fig. 4.6.

4.8. **The Long-Run Properties.** In this section, we study the steady-state distributions of sentiment S and price-dividend ratio Φ for the limiting case with an infinite horizon $T = \infty$.[16]

Note that (3.3) also holds for the case when the agent is fully rational ($\mathbb{P}^e$=$\mathbb{P}$), that is,

$$dS_t = \kappa(\mu^r/\sigma_d - S_t)dt + \kappa dZ_t. \tag{4.5}$$

Therefore, in the rational benchmark, the mean value of the steady-state distribution of S under the true measure equals μ^r/σ_d. When the agent is an extrapolator, it follows from (3.4) and (3.5) that

$$dS_t = (1-\alpha)\kappa(\mu^r/\sigma_d - S_t)dt + \kappa dZ^e_t, \tag{4.6}$$

and hence the mean value of the steady-state distribution of S under the extrapolative measure also equals μ^r/σ_d. That is, the extrapolator has a correct expectation about the long-run mean of S.

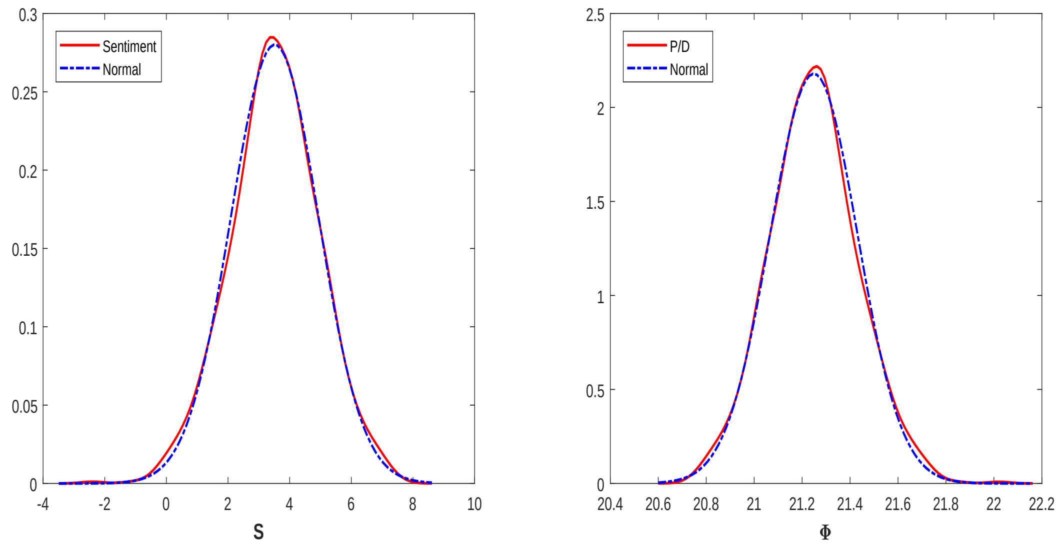


FIGURE 4.7. The left panel plots the steady-state distribution of S under the true measure (the red solid line), comparing with the normal distribution with the same mean and variance (the blue dash-dotted line). The right panel plots the distribution of Φ corresponding to the steady-state distribution of S. Here $\gamma = 5$, $\rho = 0.02$, $\mu_d = 0.018$, $\sigma_d = 0.032$, $\kappa = 2.3$, $\alpha = 0.5$ and $\alpha_0 = (1-\alpha)\mu^r/\sigma_d$.

[16]This limiting case is studied in Appendix B. In this paper, we focus on the case with a finite horizon $T < \infty$, in which the discussions also apply to the limiting case.

Fig. 4.7 illustrates the steady-state distributions of S and Φ under the true measure, which are derived using Fokker-Planck equation. Because S has marginal effect on volatility σ_p, both distributions in Fig. 4.7 are close to normal distributions. As a result, the mean value $\bar{\Phi}$ of the steady-state distribution of Φ is very close to $\Phi(\bar{S})$, the price-dividend ratio when S equals its long-run mean $\bar{S}$.

Interestingly, the long-run mean of Φ (= 21.25) is much higher than Φ^r (= 12.23). Numerical simulations (not reported here) show that this is also true when $\gamma < 1$. To understand this, we study the long-run mean of S. For simplicity, we assume σ_p is a constant given that it has little variation as S varies. Then (3.30) shows that the long-run mean of S under the true measure equals

$$\frac{\psi_0}{1-\psi} = \frac{\mu^r}{\sigma_d} + \frac{\sigma_d - \sigma_p}{1-\alpha(1-\frac{\sigma_p}{\gamma\sigma_d})}\frac{r_f^r}{\gamma\sigma_d^2}. \tag{4.7}$$

If $\gamma < 1$, (4.7) shows that $\psi_0/(1-\psi)$ is greater than μ^r/σ_d, the long-run mean of S (under the true measure) in the rational benchmark. According to (B.4), Φ increases with S when $\gamma < 1$, so $\bar{\Phi} > \Phi^r$. Similarly, if $\gamma > 1$, $\bar{\Phi}$ is also higher than Φ^r due to the facts that the long-run mean of S is less than μ^r/σ_d and that Φ decreases with S in this case. Therefore, the price-dividend ratio under extrapolation does not converge to its rational-benchmark level in the long-run, even though the extrapolator is assumed to have a correct expectation about return in the long-run (3.5). Therefore, the extrapolation model and its rational benchmark may not be directly comparable with respect to price-dividend ratio. In other words, the overvaluation relative to the rational benchmark may not necessarily be corrected in the long-run. However, Corollary 3.3 shows that they can be, to some extent, comparable with respect to dividend growth, through which we discuss model implications in our paper.

It follows from (3.21) that, for EIS> 1, $\Phi \to \infty$ ($\Phi \to 0$) when $S \to \infty$ ($S \to -\infty$). This implies that Φ is greater than its long-run mean under the true measure when $S > \bar{S}$. In this sense, the extrapolator with EIS> 1 overreacts to S. Similarly, the extrapolator with EIS< 1 underreacts to S.

4.9. **Contrarian Expectation.** When $\alpha < 0$, the agent has a contrarian expectation. The above analytical results show that the contrarian expectation tends to have the oppositive effect to the extrapolative expectation. For example, the lower panels of Fig. 4.4 show that a contrarian ($\alpha < 0$) has exactly the opposite effects on equilibrium relative to an extrapolator ($\alpha > 0$). In addition, a contrarian tends to preserve the equilibrium (Fig. 4.2). She generates excess volatility if $\gamma > 1$ as demonstrated in Appendix A.4.

5. Conclusion

By introducing an extrapolative expectation into an otherwise standard asset pricing model, we find that the effects of extrapolation depend on the agent's EIS. Different from the literature, we show that, for a small EIS, extrapolation leads to a negative feedback and generates momentum. We also find that extrapolation does not lead to the nonexistence of equilibrium, rather, it does change the transversality condition. We resolve some conceptual issues by showing that the above phenomena are not due to either the counteraction of rational investors or overvaluation generated from extrapolation, but caused by the interaction between extrapolation and EIS. The model is able to generate many other features of market returns.

Our model with a representative agent can be easily extended to one with both rational investors and extrapolators in the market. In this case, heterogeneity may lead to one more state variable—consumption share—in addition to sentiment. We leave this for future research.

Appendix A. Proofs

A.1. **Proof of Corollary 3.3.** To the leading order of κ, A and B in (3.19) become

$$\begin{aligned} A_t =& -\frac{\gamma-1}{\gamma}\alpha\sigma_d(T-t), \\ B_t =& \Big(-\frac{\rho}{\gamma}-\frac{\gamma-1}{\gamma}\alpha_0\sigma_d+\frac{\gamma-1}{2}\sigma_d^2\Big)(T-t), \end{aligned}$$

and (3.18) shows that $\sigma_p=\sigma_d$. Accordingly, Φ reduces to (3.25).

It follows from (3.13) and (3.23) that

$$\mu_p^e=\rho+\gamma\Big[\mu_d^e-\frac{\gamma\sigma_d^2-\sigma_d(2\sigma_p-\sigma_d)}{2}\Big],$$

which, to the leading order of κ, reduces to (3.24).

A.2. **Proof of Corollary 4.1.** To the order of α, A_t and B_t become

$$\begin{aligned} A_t =& -\frac{(\gamma-1)\alpha\sigma_d}{\gamma\kappa}\big[1-e^{-\kappa(T-t)}\big], \\ B_t =& -\frac{(\gamma-1)\alpha\alpha_0\sigma_d}{\gamma^2\kappa}\big[1-e^{-\kappa(T-t)}\big]+\Big[\frac{(\gamma-1)\sigma_d^2}{2}-\frac{(\gamma-1)\alpha_0\sigma_d}{\gamma}\Big(1+\frac{\alpha}{\gamma}\Big)-\frac{\rho}{\gamma}\Big](T-t). \end{aligned} \tag{A.1}$$

Substituting (A.1) into (3.18) and (3.28), σ_p, μ_p^e and μ_p are, to the order of α, given by (4.1), where

$$\begin{aligned} a_{1t} =& 1-\frac{a_0}{a_0+\kappa}\frac{1-e^{-(a_0+\kappa)(T-t)}}{1-e^{-a_0(T-t)}}, \\ a_{2t} =& -\frac{\gamma-1}{\gamma}\alpha\Big[\alpha_0\sigma_d+\mu_d+\frac{\rho}{\gamma}-\frac{(\gamma+1)\sigma_d^2}{2}\Big]a_{1t}+\mu_d+a_0, \end{aligned} \tag{A.2}$$

and

$$a_0 = (\gamma - 1)\alpha_0\sigma_d/\gamma - (\gamma - 1)\sigma_d^2/2 + \rho/\gamma.$$

Because $a'_{1t} < 0$, we have $0 = a_{1T} \leq a_{1t} \leq a_{10} = 1 - \frac{a_0}{a_0+\kappa}\frac{1-e^{-(a_0+\kappa)T}}{1-e^{-a_0 T}} < 1$.

To determine the sign of return series correlation, we only need to study $cov(r_{p,t+h\Delta}, r_{pt})$. Because S_t is, to the order of α, given by

$$S_t \approx \int_0^t \kappa e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(t-u)}\Big[\alpha_0\Big(1 - \frac{\sigma_{pu}}{\gamma\sigma_d}\Big) + \sigma_{pu} + \frac{\mu_d}{\sigma_d} + \frac{\rho}{\gamma\sigma_d} - \frac{(\gamma+1)\sigma_d}{2}\Big]du$$
$$+ S_0 e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)t} + \kappa e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(t-u)} dZ_u,$$

together with (3.28), we have

$$\begin{aligned}
&cov(r_{p,t+h\Delta}, r_{pt})\\
=&cov\Big(\int_{t+(h-1)\Delta}^{t+h\Delta}\int_0^u \kappa\sigma_{pu}\psi_u e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u-v)}dZ_v du, \int_{t-\Delta}^{t}\int_0^u \kappa\sigma_{pu}\psi_u e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u-v)}dZ_v du + \sigma_{pu}dZ_u\Big)\\
=&\kappa f(h,t) f_0(t),
\end{aligned}$$

where

$$f(h,t) = \int_{t-\Delta}^{t} \sigma_{p,u+h\Delta}\psi_{u+h\Delta} e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)(u+h\Delta)}du,$$
$$f_0(t) = \int_{t-\Delta}^{t} \sigma_{pu}\Big[\kappa\alpha\Big(1 - \frac{\sigma_{pu}}{\gamma\sigma_d}\Big)e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\frac{e^{2\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u} - 1}{2\kappa(1+\frac{1-\gamma}{\gamma}\alpha)} + e^{\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\Big]du.$$

Because $a_{1t} > 0$, we have $1 - \frac{\sigma_{pu}}{\gamma\sigma_d}$ when $\gamma > 1$. In addition, it is easy to verify that $\sigma_{pu} > 0$. So $f_0(t) > 0$ when $\gamma > 1$. We rewrite $f_0(t)$ as

$$f_0(t) = \int_{t-\Delta}^{t} \sigma_{pu}\Big\{\Big[1 - \frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_d} - 1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)}\Big]e^{\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u} + \frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_d} - 1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)}e^{-\kappa(1+\frac{1-\gamma}{\gamma}\alpha)u}\Big\}du.$$

If $\gamma < 1$, then

$$0 < \frac{\alpha(\frac{\sigma_{pu}}{\gamma\sigma_d} - 1)}{2(1+\frac{1-\gamma}{\gamma}\alpha)} = \frac{(1-\gamma)\alpha(1+\frac{\alpha}{\gamma}a_{1u})}{2\gamma(1+\frac{1-\gamma}{\gamma}\alpha)} < \frac{(1-\gamma)\alpha}{\gamma + (1-\gamma)\alpha} < 1,$$

implying that $f_0(t) > 0$. In all, we have $f_0(t) > 0$, and hence the sign of $cov(r_{p,t+h\Delta}, r_{pt})$ is completely determined by $f(h,t)$, whose sign is determined by ψ_u. Notice that $\psi_u = \alpha\frac{\gamma-1}{\gamma}\big(1 + \frac{\alpha a_{1u}}{\gamma}\big)$, which is positive if $\gamma > 1$ and negative if $\gamma < 1$. So the series correlations of return are positive for all lag h if $\gamma > 1$ and negative for all h if $\gamma < 1$. If $\gamma = 0$, then both expected return and volatility in (4.1) are deterministic, and hence the series correlations are 0 for all lags.

In addition, $\partial f(h,t)/\partial h < 0$, implying that the series correlation decreases as lag increases.

A.3. **Proof of Corollary 4.2.** If $T < \infty$, then $\gamma = 0$ violates the transversality condition according to the first equation of (3.19). If $T = \infty$, then coefficients A and/or B increase to infinity when $\gamma \leq \alpha$ or $-B^*(\alpha) > 0$.

Corollary 4.2 can also be proved by directly examining the infinite-horizon economy in Appendix B.

A.4. **Proof of Corollary 4.3.** Equation (3.19) implies that if $\gamma < 1$, then $A_t > 0$ and $\sigma_p > \sigma_d > 0$; if $\gamma = 1$, then $A_t = 0$ and $\sigma_p = \sigma_d > 0$; and if $\gamma > 1$, then $A_t < 0$ and $\sigma_p < \sigma_d$.

If $\gamma > 1$, to prove $\sigma_p > 0$, we need to show that

$$\int_t^T e^{A_u S + B_u} du > \alpha(\gamma - 1) \int_t^T \frac{1 - e^{-\kappa(\gamma-\alpha)/\gamma(T-u)}}{\gamma - \alpha} e^{A_u S + B_u} du.$$

It suffice to show that

$$1 > \alpha(\gamma - 1) \frac{1 - e^{-\kappa(\gamma-\alpha)/\gamma(T-u)}}{\gamma - \alpha}. \tag{A.3}$$

In fact, $\alpha \leq 1$ leads to $\frac{\alpha(\gamma-1)}{\gamma-\alpha} \leq 1$, which further leads to (A.3).

In addition, it follows from (3.12) that

$$\begin{aligned}
\frac{\partial \sigma_p}{\partial S} &= \kappa \Phi^{-2} \Big[\frac{\partial^2 \Phi}{\partial S^2} \Phi - \Big(\frac{\partial \Phi}{\partial S} \Big)^2 \Big] \\
&= \kappa \Phi^{-2} \Big[\int_t^T A_u^2 e^{A_u S + B_u} du \int_t^T e^{A_u S + B_u} du - \Big(\int_t^T A_u e^{A_u S + B_u} du \Big)^2 \Big] > 0.
\end{aligned}$$

Appendix B. A Limiting Case with Infinite Horizon

For an infinite-horizon economy ($T \to \infty$), the HJB equation (3.6) still holds except that the boundary condition becomes $\mathbb{E}_t^e[J_T] \to 0$ as $T \to \infty$. The value function J has the form (Liu, 2007):

$$J(S, W, t) = e^{-\rho t} \frac{W^{1-\gamma}}{1-\gamma} \big[\Phi(S) \big]^\gamma,$$

where Φ is not an explicit function of time t anymore. Therefore, equations (3.8)-(3.23) still hold, but PDE (3.15) becomes an ODE

$$\frac{\kappa^2}{2} \frac{\partial^2 \Phi}{\partial S^2} + \frac{\kappa}{\gamma} \big[(\alpha - \gamma) S + \alpha_0 \big] \frac{\partial \Phi}{\partial S} - \Big[\Big(1 - \frac{1}{\gamma} \Big) \sigma_d (\alpha_0 + \alpha S) - \frac{(\gamma - 1)\sigma_d^2}{2} + \frac{\rho}{\gamma} \Big] \Phi + 1 = 0.$$

We define $\hat{\Phi}(S, t)$ such that

$$\Phi = \int_0^\infty \hat{\Phi}(S, u) du. \tag{B.1}$$

Then $\hat{\Phi}$ satisfies

$$-\frac{\partial\widehat{\Phi}}{\partial t}+\frac{\kappa^2}{2}\frac{\partial^2\widehat{\Phi}}{\partial S^2}+\frac{\kappa}{\gamma}\big[(\alpha-\gamma)S+\alpha_0\big]\frac{\partial\widehat{\Phi}}{\partial S}-\Big[\Big(1-\frac{1}{\gamma}\Big)\sigma_d(\alpha_0+\alpha S)-\frac{(\gamma-1)\sigma_d^2}{2}+\frac{\rho}{\gamma}\Big]\widehat{\Phi}=0, \tag{B.2}$$

with $\hat{\Phi}(S,0)=1$ and $\hat{\Phi}(S,\infty)=0$.

It can be verified that (4.4) is the condition for existence of equilibrium. Under this condition, the solution to (B.2) is given by

$$\widehat{\Phi}(S,t)=e^{a_tS+b_t},$$

where

$$\begin{aligned}
a_t=&-(\gamma-1)\alpha\sigma_d\frac{1-e^{-\kappa(\gamma-\alpha)t/\gamma}}{\kappa(\gamma-\alpha)},\\
b_t=&-\frac{\gamma(\gamma-1)\alpha\sigma_d}{\kappa(\gamma-\alpha)^2}\Big[\frac{(\gamma-1)\alpha\sigma_d}{\gamma-\alpha}-\frac{\alpha_0}{\gamma}\Big]\Big[1-e^{-\kappa(\gamma-\alpha)t/\gamma}\Big]\\
&+\frac{\gamma(\gamma-1)^2\alpha^2\sigma_d^2}{4\kappa(\gamma-\alpha)^3}\Big[1-e^{-2\kappa(\gamma-\alpha)t/\gamma}\Big]-B^*t,
\end{aligned} \tag{B.3}$$

and B^* is given by (3.19). Equations (B.1) and (B.3) show that

$$\frac{\partial\Phi}{\partial S}=\int_0^\infty a_ue^{a_uS_t+b_u}du\begin{cases}<0, & \text{if}\quad \gamma>1,\\ >0, & \text{if}\quad \gamma<1.\end{cases} \tag{B.4}$$

Proposition B.1 summarizes the equilibrium.

Proposition B.1. *The equilibrium market price satisfies* $(dP_t+D_tdt)/P_t=\sigma_p(\alpha_0+\alpha S_t)dt+\sigma_pdZ_t^e$, *where*

$$\sigma_p=\sigma_d+\kappa\frac{\int_0^\infty a_ue^{a_uS_t+b_u}du}{\int_0^\infty e^{a_uS_t+b_u}du},$$

and a_t, b_t *are given by* (B.3), *the riskless rate is given by* $r_f=\sigma_p(\alpha_0+\alpha S_t-\gamma\sigma_d)$, *and the price-dividend ratio is given by* $\Phi=\int_0^\infty e^{a_uS_t+b_u}du$.

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