Fundamental Disagreement about Monetary Policy and the Term Structure of Interest Rates*

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Abstract

Forecasters disagree about the future path of monetary policy, particularly in the long-run. We propose an affine term structure model in which investors hold heterogeneous beliefs about the long-run level of rates. As they trade government bonds at equilibrium prices, they implicitly disagree about their risk-return tradeoff and engage in speculative trading. Our model fits U.S. Treasury yields and the short rate paths predicted by different groups of professional forecasters very well. We show that 1) a perceived slow-moving drift in the long-run level of the short rate is important in generating long-run disagreement about the policy rate; 2) almost half of the variation in term premiums is driven by disagreement about the policy rate; 3) disagreement affects term premiums through investors' heterogeneous responses to asymmetric signals as well as through endogenous wealth fluctuations.

Keywords: Disagreement, Heterogeneous Beliefs, Noisy Information, Speculation, Survey Forecasts, Yield Curve, Term Premium.
JEL Classification Codes: D83, D84, E43, G10, G12.

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1 Introduction

Bond yields reflect investors' expectations about the future path of short rates as well as their attitudes towards risk. Most term structure models specify these two components of interest rates for a representative investor. While this perspective provides a reasonable starting point for many analyses, it may mask important dynamics among investors and thus fail to provide a complete account of the driving forces behind bond yields. In this paper, we propose and estimate a term structure model which explicitly incorporates differences in beliefs about future short rates.

It has been widely documented that economic agents hold heterogeneous beliefs about future macroeconomic outcomes. This is not only true for households and firms, but also for professional forecasters who arguably are among the best informed economic agents. In fact, as their reputation and business models depend on it, professional forecasters have a strong incentive to provide accurate predictions. This notwithstanding, a prior literature has documented that forecasters disagree considerably about the near-term outlook for key macroeconomic indicators such as real growth, inflation, and short term interest rates.[1] While the literature has proposed different rationales for why forecasters disagree, informational frictions appear to provide one of the most promising explanations (Coibion and Gorodnichenko (2012)).

To determine the fair value of longer-term bonds, investors need to make forecasts of future short rates far into the future. Interestingly, as shown by Andrade et al. (2016), professional forecasters disagree much more about the medium and longer-term outlook of short rates than about the near term. In other words, the term structure of disagreement about future short rates is upward sloping. This is in contrast to real output growth and inflation for which the term structures of disagreement are downward sloping or flat, respectively. Andrade et al. (2016) show that the different shapes of the term structure of disagreement can be rationalized in a model with informational frictions where the state variables follow a vector autoregression with slow-moving long-run means which agents filter from the imperfectly observed data. As their information sets differ, agents in their model infer different levels of the time-varying long-run means and thus disagree about the fundamentals to which the economy will eventually converge.

In this paper, we incorporate this idea into a term structure model. Our model features two types of agents. They both perfectly observe the level, slope, and curvature factors of the yield curve which capture all of the comovement among bond yields. While the slope

[1] See, e.g., Mankiw, Reis and Wolfers (2003), Lahiri and Sheng (2008), Patton and Timmermann (2010), Dovern, Fritsche and Slacalek (2012), and Andrade and Le Bihan (2013).

and curvature factors are stationary, the level factor has a time-varying long-run mean which itself follows a random walk. The two agents receive private noisy signals which, along with the perfectly observed yield curve factors, they use to filter the drift. While investors in our model have different beliefs about the long-run level of rates, we assume that they have identical preferences and perceive the same volatility of shocks. Still, as they trade bonds at equilibrium prices, their pricing kernels and hence their perceived risk-return trade-off of bonds differ.

Our approach closely follows Xiong and Yan (2010). However, we adapt their framework in several important ways. First, while in our model investors disagree about the expected path of nominal short rates, Xiong and Yan (2010) consider heterogeneous beliefs about the inflation target. Yet, as shown in Andrade et al. (2016), disagreement about the inflation target is not sufficient to explain the long-run disagreement about the short rate. We therefore explicitly model disagreement about future nominal short rates. Second, we depart from Xiong and Yan (2010) by assuming that the two agents in our model observe different signals rather than having different priors about the informativeness of the flow of public signals. This makes our model more in line with the above mentioned literature on informational frictions. Third, and most importantly, we embed our model with information-driven heterogeneous beliefs about the long-run level of rates into the affine term structure framework of Joslin, Singleton and Zhu (2011). This allows us to asses its empirical validity and study the implications of heterogeneous policy expectations for term premiums. As in Joslin, Singleton and Zhu (2011), in our model the dynamics of the pricing factors under the risk-neutral measure follow a stationary vector autoregression. However, under the physical measure the pricing factors follow a non-stationary vector autoregression. Specifically, the level factor has a long-run mean that evolves according to a random walk. Moreover, as they observe private signals both investors in our model have individual beliefs about that long-run mean. Accordingly, their forecasts of the policy rate differ at all horizons.

We fit our model using zero coupon Treasury yields as well as the term structure of survey forecasts of the federal funds rate for three different hypothetical investors: the consensus forecaster from the Blue Chip Financial Forecasts (BCFF) survey, as well as the forecasts given by the top and bottom ten average responses of the Blue Chip survey. The difference between the latter is a measure of the disagreement among forecasters about future short rates (see Andrade et al. (2016)). Our model fits yields and the three survey forecast paths of the short rate very well. The model implies that investors expecting higher future short rates perceive term premiums to be negative on average for most maturities. This is in contrast to investors predicting short rates to be low whose implied average term premium ranges from about 50 basis points at the one year maturity to about two percent for the ten year Treasury.

The representative investor's term premium broadly moves in line with that implied by the Adrian, Crump and Moench (2013) model which does not involve survey forecasts. Our model also produces reasonable maximal Sharpe ratios.

Similar to Xiong and Yan (2010), we can show that the law of motion of the pricing kernel of the representative agent is well approximated by a weighted average of the pricing kernel dynamics of the two investors, with the weights given by their shares of total wealth. In addition, the expected short rate of the representative investor is the wealth-weighted average of the short rate expectations of the two investors. The same holds for the perceived long-run mean of the level factor. As the relative wealth ratio in our model directly depends on the differences in perceived prices of risk which themselves are driven by differences in beliefs about the long-run level of rates, we can characterize the evolution of the relative wealth ratio as a function of estimated model quantities. This allows us to decompose changes in the term premium of the representative agent into three different sources of variation. The first is a common response of investors to changes in yields that would also be present in a representative agent economy. The second reflects investors' heterogeneous repricing of risk in response to their private signals. The third arises because of endogenous wealth fluctuations. In our estimated model, the latter two disagreement-driven sources of variation account for about 40 % of the variation in term premiums on average.

Our paper is related to the small but growing literature on bond pricing with heterogeneous beliefs. Ehling et al. (2016) consider a model in which investors with habit formation utility disagree about the distribution of inflation, not just expected inflation. This disagreement induces heterogeneity in investors' consumption and investment decisions and on average raises real and nominal bond yields. They further document empirically that inflation disagreement has a strong effect on real and nominal bond yields over and above the impact of expected inflation, consistent with their theoretical model. Buraschi and Whelan (2016) study the interactions between risk aversion and disagreement. In their model heterogeneous beliefs do not arise because of different signals, but because agents have different views about the (constant) long-run growth rate of consumption and because their perceptions of the correlation of shocks differs. They find that disagreement has larger effects on equilibrium bond prices when risk aversion is low. Barillas and Nimark (2016) build a model of the term structure in which investors with heterogeneous information sets form higher-order expectations about the beliefs of all other investors. Equilibrium bond prices then reflect a speculative component which depends on investors' beliefs about the error that the average investor makes when predicting future short rates. Their model suggests that the speculative component explains a sizable fraction of the variation in U.S. Treasury yields. Barillas and Nimark (2015) generalize this model to allow for richer price of risk specifications as used in the empirical term structure literature. In their model, investors observe heterogeneous signals of the state variables driving bond yields. They forecast the forecasts of other investors and engage in speculative trading. In equilibrium, individual investors' prices of risk then reflect idiosyncratic signals, higher order expectations of the true state variables, as well as investor-specific expectations of maturity-specific shocks. Importantly, in their model the pricing factors follow stationary vector autoregressions under both the risk-neutral and the physical measure implying that investors do not disagree about short rates in the long-run. This is in stark contrast to the evidence provided in Andrade et al. (2016) and the findings documented in this paper. Our paper is also related to the term structure literature using survey information in the model estimation. For example, Kim and Wright (2005) and Piazzesi, Salomao and Schneider (2015) use consensus survey forecasts to discipline the time-series dynamics under the physical measure. Giacoletti, Laursen and Singleton (2016) build a dynamic term structure model in which a representative investors updates her beliefs about future bond yields. They find that when this updating is conditioned on the dispersion in bond yield forecasts, the model produces substantially smaller forecast errors. We provide a structural interpretation to their findings by explicitly modeling investors' reactions to noisy signals and studying the term premium dynamics induced by relative wealth fluctuations.

Our paper is structured as follows. Section 2 provides some facts about short rate disagreement and term premiums. In Section 3, we describe our model with heterogeneous beliefs which gives rise to long-run or fundamental disagreement about short rates. In Section 4, we introduce this model into a standard affine term structure framework. Section 5 presents the estimation results and Section 6 provides some robustness checks. Section 7 concludes.

2 A First Look at the Data

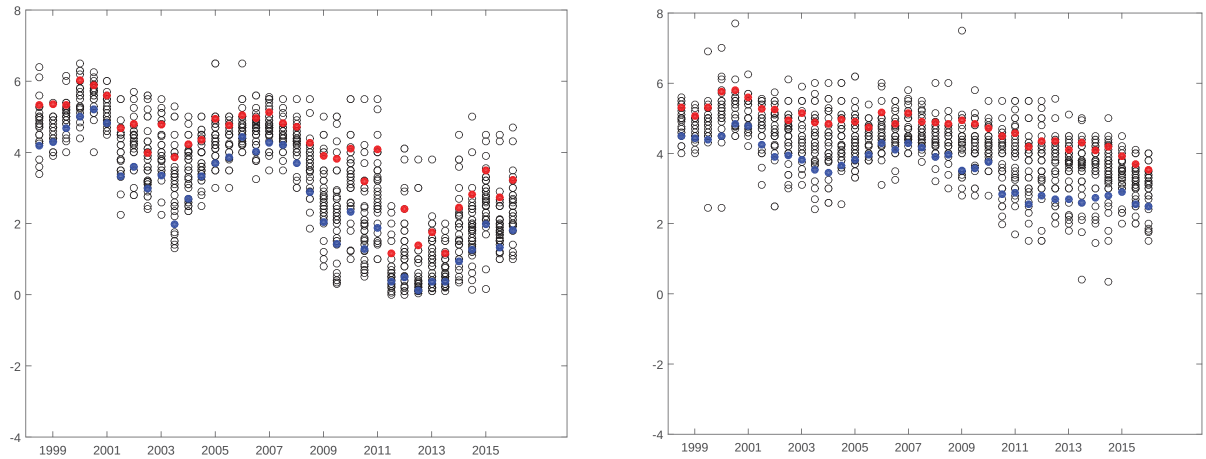
In this section, we motivate our subsequent analysis by providing some stylized facts on disagreement about future policy rates and term premiums. Our results are based on the Blue Chip Economic Indicators (BCEI) and the Blue Chip Financial Forecasts (BCFF) surveys. The BC surveys have been conducted monthly since the early 1980s. They ask two partly overlapping panels about 40 professional forecasters from a wide range of institutions including broker-dealers, banks, and economic consulting firms to provide forecasts of the quarterly average of a variety of economic and financial variables for specific calendar quarters in the future. Since the mid 1980s, the surveys have also biannually been collecting forecasts from 2 years as far as 7-to-11 years ahead. While the surveys publish the individual forecasts for horizons up to eight quarters into the future at a monthly frequency, they only report three quantities for the biannual forecasts of horizons of two years and above. These are the average across all forecasters, which we label the "consensus forecast", as well as the average of the top ten and bottom ten responses for a given forecasted variable at a given horizon. While the estimated term structure model with disagreement in Sections 4 and 5 relies on these three forecast series for the federal funds rate from the BCFF survey, we set the stage in this section by providing information also on individual longer term three-month Treasury bill forecasts from the BCEI survey. To the best of our knowledge, no such individual longer-term forecast data has previously been studied in the literature.

Our analysis is motivated by Andrade et al. (2016) who show that the term structure of disagreement about future short rates is upward sloping. This implies that while forecasters agree to a large extent about monetary policy in the near-term, they have strongly opposing views about the medium and longer-term policy outlook. For example, Andrade et al. (2016) document that the top ten and bottom ten average forecasts of the federal funds rate over horizons from two years up to 7-to-11 years ahead differ by around two percentage points over the 1986-2013 sample. This implies a substantial amount of disagreement about monetary policy at medium and longer-term horizons and is in sharp contrast to disagreement about real GDP growth or CPI inflation for which the term structures of disagreement are downward sloping or flat, respectively. Moreover, the level of long-run disagreement about real growth and inflation, while non-negligible, is quite a bit lower than that for the policy rate, especially at very long horizons. Andrade et al. (2016) interpret the dispersion of beliefs about the long-run as disagreement about the fundamentals of the economy. They show that fundamental disagreement about the short term policy rate is largely, but not entirely explained by disagreement about the perceived inflation target and the long-run real rate of growth.

In the remainder of this section, we expand on the results in Andrade et al. (2016) by showing that fundamental disagreement about short term interest rates i) is not driven by outlier predictions; ii) is a persistent phenomenon in the sense that individual forecasters tend to see high or low future short rates across all forecast horizons; iii) implies sizable fundamental disagreement about term premiums; and iv) is strongly correlated with the term premium perceived by the consensus forecaster and that implied by a standard term structure model.

Figure 1 shows the time series and cross-section of individual forecasters' predictions for the three-month Treasury bill at horizons of two and 7-11 years into the future. The figures show that while individual longer-term forecasts broadly move together there is a considerable degree of disagreement among forecasters. Specifically, they disagree by as much as six percentage points about the level of the three-month TBill two years out (left-hand chart). The strong disagreement is particularly pronounced just after the start of the large-

Figure 1: Disagreement about short rates at medium and long horizons



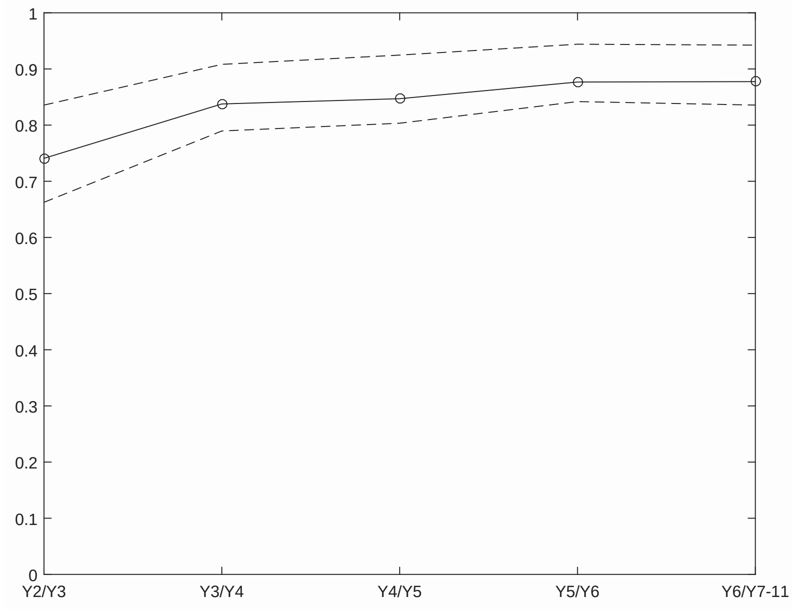
Notes:
This figure plots individual forecasts from the Blue Chip Economic Indicators survey for the three-month Treasury Bill at forecast horizons two and 7-11 years into the future. The red and blue dots represent the top and bottom ten average responses, respectively. The sample is from 1999-2016.

scale asset purchase programs by the Federal Reserve in 2009, but drops considerable when calendar-based forward guidance was introduced in the summer of 2011. The figure also shows that the width of the forecast distribution as measured by the difference between the top ten and bottom ten average responses is wide and varies considerably over time.

The right-hand chart of Figure 1 shows the predictions of the three-month TBill of the same individuals in the long-run. As expected, there is less of a cyclical element in these forecasts. That said, the chart also shows that the entire distribution of long-run forecasts of the short rate has trended down over the sample. This strongly suggests that the long-run level of the short rate is perceived to vary over time, consistent with Andrade et al. (2016). This feature of forecasters' beliefs will be a central element of our modeling strategy. Interestingly, while there clearly is a strong common element in the individual forecasts, the distribution at this very long horizon is also quite wide. This indicates that forecasters disagree to a considerable degree about the long-run (fundamental) value of the short term interest rate. Quite strikingly, at the end of our sample some forecasters believe the long-run value of the TBill will remain below two percent while others see it go back to a level of around four percent. These heterogeneous assessments likely reveal sharply different views of the equilibrium state of the economy.

As it is inherently difficult to predict far into the future, one might worry that individual forecasters' responses are to some extent arbitrary and do not necessarily reflect their views of the world. While we do not observe the names of individual forecasters in our sample of long-term predictions, we are able to trace their forecast paths at any given point in time.

Figure 2: Consistency in individual beliefs across horizons



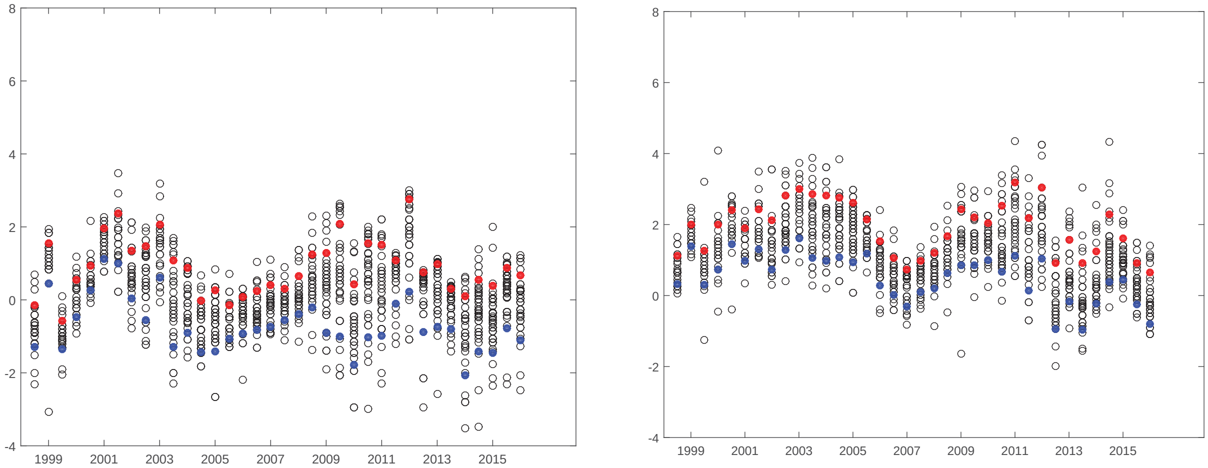
Notes:
This figure plots the rank correlation among individual forecasts from the Blue Chip Economic Indicators (BCEI) survey for the three-month Treasury Bill at adjacent forecast horizons between two and 7-11 years into the future. The dashed lines provide the 5th and 95th probability bands. The sample is from 1999-2016.

We can thus check whether the individual medium to long-run predictions are consistent in the sense that they reveal a particular forecaster believing in higher or lower future rates. To this end, we rank the individual forecasts at all medium to long-run horizons and then compute the rank correlation between two adjacent horizons. This gives us a sense of the probability that a forecaster who believes in low rates (compared to other forecasters) at, say, the four-year ahead horizon also believes in low rates at the five-year horizon.

Figure 2 shows the rank correlations across forecasters and their 90 percent confidence interval for adjacent medium to long-term forecast horizons. At all horizons, these correlations are large and precisely estimated. Maybe not surprisingly, the rank correlations are somewhat lower around 70 percent at medium-term horizons suggesting that individual forecasts are to some degree driven by different views about the state of the business cycle and the corresponding monetary stance at these horizons. That said, for longer forecast horizons the rank correlations increase further and reach almost 90 percent at the six year and 7-11 year ahead horizon. This implies that individual forecasts are highly consistent across horizons and likely reflect different fundamental views about the economy.

The term premium is defined as the difference between the yield on a government bond and the average short rate expected to prevail over the life of the bond. Since we observe survey

Figure 3: Disagreement about term premiums at medium and long horizons



Notes:
This figure plots forward term premiums implied by individual forecasts of the three-month Treasury Bill from the Blue Chip Economic Indicators (BCEI) survey at forecast horizons 1-2 and 7-11 years into the future. The red and blue dots represent the top and bottom ten average responses, respectively. The sample is from 1999-2016.

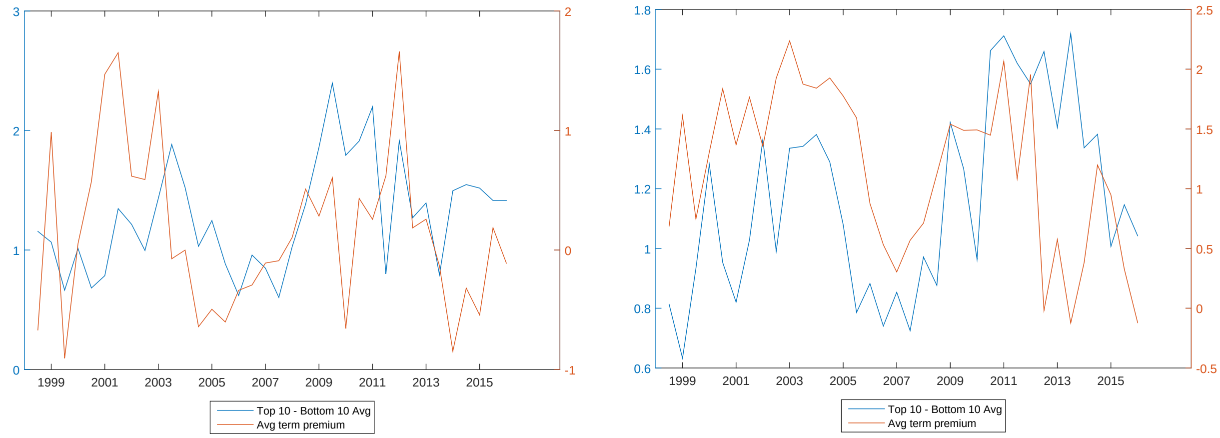
participants' individual forecast paths for the short rate, we can compute their perceived individual term premiums for various forward horizons. Figure 3 displays the evolution of forward term premiums implied by the individual TBill forecasts for the one-to-two year and 7-11 year forward horizons. Not surprisingly, as yields are perfectly observed these term premiums are simply mirror images of the short rate forecasts shown in Figure 1 above.

The figure clearly shows that the assessment of the compensation that long-term bond investors command differ widely across forecasters. Moreover, especially in the latter part of the sample quite a few survey participants see term premiums in negative territory, possibly suggesting that they view longer term Treasuries as hedges against adverse states of the economy. As before, while individuals' views about term premiums are quite heterogeneous, the top and bottom ten average predictions appear to represent well the dispersion of beliefs across the forecaster distribution.

In the remaining sections of the paper we will develop and estimate a term structure model with heterogeneous beliefs about the fundamental or long-run level of the short rate. The model predicts that the term premium as implied by the average or consensus belief is correlated with the disagreement among individual forecasters about the average future short rate. The reason is that in equilibrium different beliefs will induce speculative trading and relative wealth fluctuations which in turn affect the marginal pricing of risk in the economy.

Figure 4 displays the one-two year and the 7-11 year forward term premium implied by the consensus belief as well as the difference between the top and bottom ten average

Figure 4: Consensus term premium and disagreement about short rates



Notes:
This figure plots forward term premiums implied by the consensus forecast along with the difference between the top and bottom ten average forecasts for the three-month Treasury Bill from the Blue Chip Economic Indicators (BCEI) survey at forecast horizons 1-2 and 7-11 years into the future. The sample is from 1999-2016.

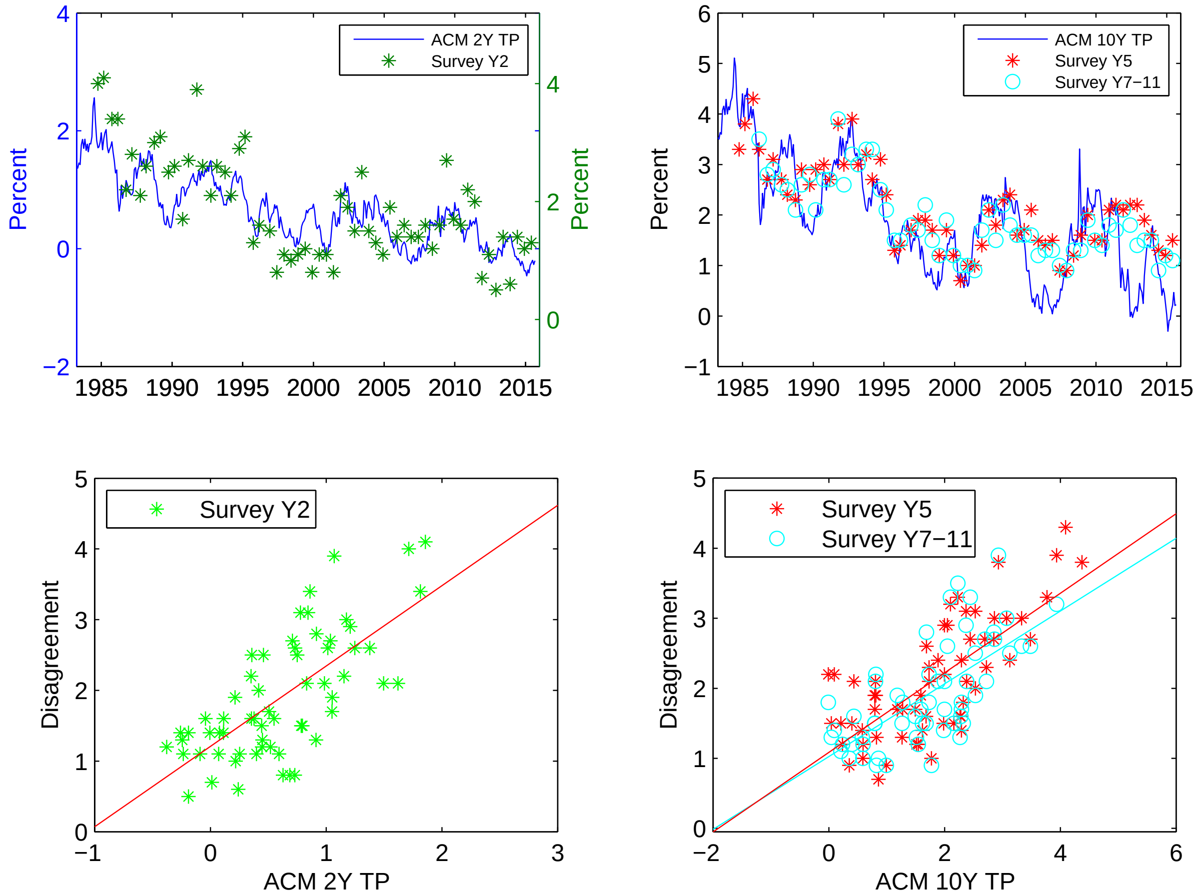
forecasts of the short rate for the corresponding horizon. The charts clearly show that the two measures comove at both horizons, providing some suggestive evidence that the mechanism highlighted in the introduction and further detailed in the remainder of the paper is at work.

The positive correlation between a measure of the disagreement about future short rates and the term premium shown above is not restricted to the term premium implied by the consensus forecast. The upper panel of Figure 5 shows the time series of the two and ten-year Treasury term premium obtained from the Adrian, Crump and Moench (2013) (ACM) model.[2] This no-arbitrage term structure model uses the first five principal components of Treasury yields as pricing factors and does not include survey forecasts in the estimation. We superimpose the difference between the top and bottom ten average responses at the two-year horizon and the five and 7-11 year ahead horizons, respectively. The charts show a strong comovement of statistical term premiums and short rate disagreement at both horizons. This becomes even more apparent when considering scatter plots of the same series in the bottom panel of Figure 5. Both survey-based and statistical term premiums are strongly correlated with measures of longer-term disagreement about the short rate.

In sum, the results provided in this section show that individual forecasters' views about future short rates differ quite substantially at all forecast horizons including the very long-run. Moreover, the long-run level of short rates as perceived by individual forecasters drifts slowly over time. We have further seen that the top and bottom ten average forecasts represent

[2]See https://www.newyorkfed.org/research/data_indicators/term_premia.html.

Figure 5: Disagreement about short rates and term premiums



Notes:
This figure plots disagreement measures calculated using survey forecasts and ACM term premiums obtained from the model described in Adrian, Crump and Moench (2013). The upper two charts display the distance between top-10 and bottom-10 average forecasts of the federal funds rate obtained from the Blue Chip Financial Forecasts (BCFF). One- to four-quarter ahead, five-year ahead and long-range (7-11 years) survey forecasts are used. The lower two charts compare two-year and ten-year ACM term premiums with different disagreement measures. Asterisks and circles in the charts are long-horizon forecasts from surveys conducted biannually.

well the differences in beliefs across individuals. Individuals' forecasts across horizons are strongly correlated suggesting that these forecasts reflect different fundamental views about the economy. Finally, forecast disagreement comoves strongly with different measures of term premiums.

In what follows, we build a term structure model which features two agents who hold different long-run views of the level of interest rates based on their private signals and the observed dynamics of yield curve factors. We bring the model to the data using the term structure of short rate forecasts for the top and bottom ten average as well as the consensus forecaster as inputs. We then use the model to illustrate how disagreement about future short rates can induce term premium dynamics.

3 A Simple Model With Fundamental Disagreement

In a complete market with heterogeneous beliefs about payoffs, agents take speculative positions and trade assets at equilibrium prices. Hence, they feature different stochastic discount factors and disagree about assets' risk-return tradeoff. In this section, we develop a model that has these features. Our modeling approach is inspired by Xiong and Yan (2010) who show that in an economy with two groups of investors with different beliefs about the inflation target, the equilibrium price of a nominal bond is the wealth-weighted average of each group's valuation of the bond in *hypothetical homogeneous economies* in which only one type of investor is present. We adapt the framework of Xiong and Yan (2010) in several important ways. First, instead of letting investors disagree about the inflation target, we assume that investors directly disagree about the expected path of nominal short rates. More specifically, we assume that the nominal short rate is driven by a small number of factors which agents observe. One of the factors, call it the level of interest rates, features a time-varying long-run mean which evolves according to a random walk. The agents do not observe the long-run mean, but need to filter it from the observed factors as well as their own private signals. As shown in Andrade et al. (2016), differences in beliefs about the long-run mean resulting from informational frictions such as noisy or sticky information are vital to explaining the term structure of disagreement about the federal funds rate which is upward-sloping.

We also depart from Xiong and Yan (2010) by assuming that the two groups of agents in our model observe different signals rather than having different priors about the informativeness of the flow of public signals. Finally, we embed our model with information-driven heterogeneous beliefs about the long-run level of rates into the affine term structure framework of Joslin, Singleton and Zhu (2011) in order to be able to asses its empirical validity and study the implications of heterogeneous policy expectations for term premiums.

In the remainder of this section, we first introduce a simple two-agent endowment economy in which speculative trading will lead to endogenous relative wealth fluctuations. We then discuss a model with information-driven heterogeneous beliefs about the long-run mean of a state variable and discuss its properties relative to a hypothetical economy with homogenous beliefs.

3.1 Model

Following Xiong and Yan (2010), we study a standard endowment economy of Lucas (1978) with two types of agents. We show that in this model the relative wealth ratio between agents depends on their beliefs about the stochastic discount factor.

Both agents have the same utility function which is denoted by $u(\cdot)$ and only depends on consumption c^i for $i = A, B$. We denote ρ as the time preference parameter and $D(t)$ as the endowment at time t. Applying the martingale technique of Pliska (1997), each group's dynamic optimization problem can be written as a static one at time zero:

$$\max_{c_i} E_0^i\Big[\sum_{t=0}^{\infty}(1+\rho)^{-t}u(c_t^i)\Big], \tag{3.1}$$

subject to

$$E_0^i\Big[\sum_{t=0}^{\infty} M_t^i c_t^i\Big] \leq \alpha_i E_0^i\Big[\sum_{t=0}^{\infty} M_t^i D(t)\Big], \tag{3.2}$$

where M_t^i is the nominal pricing kernel and the budget constraint is determined by the initial fraction of group-i's endowment α_i. With power utility preference $u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$, the first order conditions for the optimal consumption stream are

$$(1+\rho)^{-t}\frac{1}{\Pi_t c_t^{i\gamma}} = \varsigma_i M_t^i, \tag{3.3}$$

where γ is the risk aversion parameter, Π_t is the price index at time t and $\varsigma_i > 0$ is the Lagrange multiplier associated with the initial budget constraint. The investor follows the optimal nominal consumption policy

$$\Pi_t c_t^i = \kappa_t^i \Pi_t W_t^i, \tag{3.4}$$

where W_t^i is group-i's real wealth at time t, and κ_t^i is the share of wealth that group-i consumes at time t, which itself is closely related to the risk aversion parameter.[3]

3.2 Relative Wealth Dynamics

We are interested in measuring fluctuations in relative wealth which are caused by speculative trading. We define $\eta_A^B(t)$ as the ratio of pricing kernels between group-A and group-B

[3]The standard solution for κ_t^i in the case power utility is given in Samuelson (1969) and Merton (1971), which needs not be constant over time. However, when $\gamma = 1$, we have log utility and κ is a constant for both groups.

investors. Hence, $\eta_A^B(t)$ can be thought of as a change of measure from the beliefs of agent B to the beliefs of agent A. The nominal consumption policy (3.4) and Equation (3.3) imply that

$$\eta_A^B(t) = \frac{\varsigma_A}{\varsigma_B}\frac{M_t^A}{M_t^B} = (\frac{c_t^B}{c_t^A})^\gamma = \mathcal{H}_t(\frac{W_t^B}{W_t^A})^\gamma, \tag{3.5}$$

where $\mathcal{H}_t = (\frac{\kappa_t^B}{\kappa_t^A})^\gamma$. To ensure the absence of arbitrage opportunities, the relative wealth ratio needs to be positive at all times, as this quantity acts as the Radon-Nikodym derivative between the two groups' probability measures, see Appendix A. We will show later that with this condition the representative agent's short rate expectations are always bounded by two groups' beliefs.

Equation (3.5) shows that changes in the relative wealth ratio between groups are induced by changes in the group-specific pricing kernels. Therefore, disagreement about the pricing kernel causes relative wealth fluctuations and affects the equilibrium price that both groups trade at.

3.3 Beliefs in Two Economies

Having established that variations in the relative wealth ratio arise due to variations in the two groups' discount factors, we aim to discuss the difference between a homogeneous economy and an economy with heterogeneous beliefs. Sections 3.3.1 and 3.3.2 show how agents update their beliefs about the long-run mean in a homogenous and heterogenous economy and discuss the source of disagreement in the latter. In Section 3.3.3, we then introduce three benchmark beliefs – the econometrician's, the representative agent's and the average belief, in order to highlight the distinctions between the two economies. These beliefs are helpful in understanding the empirical evidence and the underlying mechanism associated with disagreement, which pave a way for the estimation of our term structure model with disagreement.

3.3.1 Homogeneous Economy

In a homogeneous economy, we assume that the pricing kernel of the representative agent depends on a number of state variables or risk factors $\mathbf{X}_t$. We further assume that one state variable $X_t \in \mathbf{X}_t$ follows an autoregressive (AR) process with a slow-moving drift

$$X_t = (1 - \beta_X)\mu_t + \beta_X X_{t-1} + \sigma_X \varepsilon_t^X, \tag{3.6}$$

where β_X is the autoregressive coefficient, μ_t is the long-run mean of the factor, σ_X is its volatility, and ε_t^X is standard Gaussian white noise. The long-run mean μ_t follows a random walk, but is not observable to the agent:

$$\mu_t = \mu_{t-1} + \sigma_\mu \varepsilon_t^\mu, \tag{3.7}$$

where σ_μ is the volatility of the drift and ε_t^μ is standard Gaussian white noise independent of ε_t^X.

The agent aims to make forecasts about the future evolution of the state variables. As she doesn't observe μ_t, she acts as an econometrician and filters it from the observed state variables. We assume that her information set at time t is $\{X_u\}_{u=0}^t$, and that her prior belief about μ_0 has a Gaussian distribution. Accordingly, her posterior beliefs about μ_t are also Gaussian. We denote the posterior distribution about μ_t by

$$\mu_t | \{X_u\}_{u=0}^t \sim N(\hat{\mu}_t^H, \hat{P}_t),$$

where $\hat{\mu}_t^H$ and $\hat{P}_t$ are the mean and variance of the posterior distribution, respectively. Applying standard filtering techniques as in Harvey (1990), we have the following steady-state solution:

$$\hat{\mu}_{t+1|t}^H = \hat{\mu}_{t|t}^H = \hat{\mu}_{t|t-1}^H + \frac{(1-\beta_X)\bar{P}}{Q} \hat{\varepsilon}_X^H(t), \tag{3.8}$$

where Q is a function of β_X, σ_X and σ_μ.[4] The posterior variance $\bar{P}$ is the positive root of the *algebraic Riccati equation*:

$$\frac{(1-\beta_X)^2 \bar{P}^2}{Q^2} - \sigma_\mu^2 = 0,$$

and $\hat{\varepsilon}_X^H(t)$, the scaled forecast errors of X_t, are given by

$$\hat{\varepsilon}_X^H(t) = \frac{1}{Q}[X_t - (1-\beta_X)\hat{\mu}_{t|t-1}^H - \beta_X X_{t-1}],$$

which is also Gaussian. Equation (3.8) can be rewritten as

$$\hat{\mu}_{t+1|t}^H = \hat{\mu}_{t|t}^H = \hat{\mu}_{t|t-1}^H + \frac{\sigma_\mu}{Q}[X_t - (1-\beta_X)\hat{\mu}_{t-1}^m - \beta_X X_{t-1}], \tag{3.9}$$

where $\frac{\sigma_\mu}{Q}$ is the learning gain. We will use this updating rule of the econometrician in a

[4]Note that in steady state, we have $Q^2 = (1-\beta_X)^2 \bar{P} + \sigma_X^2$.

homogenous economy as a point of reference for the updating rule of a representative investor as well as an econometrician in the heterogeneous economy. We derive those in the following section.

3.3.2 Economy with Heterogeneous Beliefs

In an economy with heterogeneous beliefs, we assume there are two groups of investors, group A and group B, who may hold different beliefs about the informativeness of a flow of signals about the pricing factor X_t. We write the group-i's belief in the following state-space representation:

$$X_t = (1-\beta_X)\mu_t^i + \beta_X X_{t-1} + \sigma_X \varepsilon_t^{X,i}, \tag{3.10}$$

$$\mu_t^i = \mu_{t-1}^i + \sigma_\mu \varepsilon_t^{\mu,i}, \tag{3.11}$$

The main difference between Equations (3.7) and (3.11) is that the innovations in μ_t^i can be correlated with private signals independent of the forecasts errors of X_t. Specifically, following Andrade et al. (2016), we assume that the two groups of investors observe noisy signal flows which they believe to be informative. More formally, we assume there are private signals S_t^i, which investors in this economy believe are partially correlated with the fundamental shock to μ_t and hence are informative for their updating of μ_t^i. We write down the belief of group-i ($i \in \{A, B\}$) about the data generating process (DGP) of the private signal S_t^i:

$$S_t^i = \phi \varepsilon_t^{\mu,i} + \sqrt{1-\phi^2}\varepsilon_t^{S,i}, \tag{3.12}$$

where the parameter $\phi \in [0,1]$ measures the perceived correlation between the private signal S_t^i and the shock $\varepsilon_t^{\mu,i}$ to the long-run mean. Note that $\varepsilon_t^{S,i} \sim N(0,1), i \in \{A, B\}$, is a standard Gaussian process independent of all the Gaussian processes introduced earlier. Unlike S_t^i, which is private information, ϕ is public knowledge and hence common to both groups.

Again, using the filtering method in Harvey (1990), we obtain

$$\begin{aligned} \hat{\mu}_{t+1|t}^i = \hat{\mu}_{t|t}^i &= \hat{\mu}_{t|t-1}^i + \hat{u}_t^{\mu,i} \\ &= \hat{\mu}_{t|t-1}^i + \sqrt{1-\phi^2}\sigma_\mu \hat{\varepsilon}_X^i(t) + \phi\sigma_\mu S_t^i, \end{aligned} \tag{3.13}$$

and the forecast errors of X_t perceived by group-i investors are given by

$$\hat{\varepsilon}_X^i(t) = \frac{1}{Q}[X_t - (1-\beta_X)\hat{\mu}_{t|t-1}^i - \beta_X X_{t-1}]. \tag{3.14}$$

Note that our *a priori* specification of perceived correlation ϕ plays a pivotal role here, as it controls to what extent noisy information affects investors' beliefs about the slow-moving drift. At its core, the reduced form representation (3.12) nests the standard rational expectations and heterogeneous beliefs approaches. In the standard full information rational expectations case described by Aumann (1976), i.e. $\phi = 0$, there will be no disagreement as investors know all relevant information is incorporated in observed prices. Therefore, they disregard private signals and use observed X_t to infer the law of motion of μ_t^i. If $\phi = 1$, this specification becomes Andrade et al. (2016)'s "noisy information" model, where S_t^i is perfectly correlated with the innovations in μ_t^i. In this case, investors agree to disagree and rely only on their own private information to update their beliefs. Following Banerjee (2011), the case $\phi \in (0, 1)$ can be interpreted as one of "relative over-confidence", since each group of investors believe their private signal, though noisy, is more informative than the signals of others. In other words, the common knowledge assumption about the signal informativeness is relaxed: investors are uncertain about the informativeness of other investors' signals and also use prices to update their beliefs. Note that in a "noisy rational expectations" model if prices are exogenously determined and investors are price takers, the signal extraction problem is also in line with our specification.[5]

To summarize, in our heterogeneous economy, noisy signals that are perceived to be correlated with the fundamental shock represent the key source of disagreement. Agents use their heterogeneous signals to learn about the long-run mean of the state variable and make different forecasts of the short rate.

3.3.3 Benchmark Beliefs

We next introduce the econometrician's belief to understand the difference between the above two economies. An econometrician is an outside observer who observes the state variables and has full knowledge of the structure of the economy, but does not know the beliefs of the different groups of investors. In the homogeneous economy, as the pricing factors are observable to the econometrician, her belief always tracks the true DGP described by Equation (3.7).

While the representative agent has the same belief as the econometrician in the homogeneous economy, this is not necessarily true for an economy with heterogeneous beliefs. Hence, while one can always construct a representative investor who will replicate the equilibrium

[5] In this case, investors do not have incentives to expect others' behavior. If prices are endogenously determined and higher-order expectations play a role, we need to solve a fixed point problem to obtain the equilibrium pricing functions, where prices are embodied to infer partial information of other investors, see for example, He and Wang (1995), Allen, Morris and Shin (2006) and Barillas and Nimark (2015). The signal extraction problem becomes more complicated but a linear representation is still possible, see Kasa, Walker and Whiteman (2014).

in a heterogeneous beliefs economy, the belief process of this representative agent is not a sufficient statistic for the risk-return tradeoffs in the economy. The reason is that the law of motion of the pricing kernel of the representative agent is a weighted average of the corresponding pricing kernel dynamics of the two agents, where the weights fluctuate with their relative wealth, as shown in the proposition below.

Proposition 1. *A representative agent, who has the same power utility preference as those investors in the heterogeneous economy described in Section 3.3.2, can be constructed. The representative agent's belief about the nominal pricing kernel is denoted by M_t^R. Applying first order approximations, the law of motion of M_t^R is approximately the adjusted-wealth-weighted average belief of group-A and group-B investors:*

$$\ln(\frac{M_{t+1}^R}{M_t^R}) \approx w_t^A \ln(\frac{M_{t+1}^A}{M_t^A}) + w_t^B \ln(\frac{M_{t+1}^B}{M_t^B}), \tag{3.15}$$

where the risk-aversion-adjusted wealth weights are $w_t^A = \frac{1}{\eta_A^B(t)^{\frac{1}{\gamma}}+1}$ and $w_t^B = 1 - w_t^A$. The approximation is exact in the continuous time limit.

As we have mentioned before, $\eta_A^B(t)$, which can also be considered as a change of measure, is positively bounded. Therefore, the representative agent's pricing kernel is always bounded by the two groups' pricing kernels. Proposition 1 tells us that we can use the weighted average of investors' belief about the pricing kernel in order to approximately characterize the representative agent's pricing kernel.[6] The above proposition gives us highly tractable solutions for the prices of different kinds of assets. By iterating forward, it is easy to see that $\ln(\frac{M_{t+h}^R}{M_t^R}) \approx w_t^A \ln(\frac{M_{t+h}^A}{M_t^A}) + w_t^B \ln(\frac{M_{t+h}^B}{M_t^B})$.

Note that this relation holds for any state and thus also when both agents are risk-neutral, i.e. when market prices of risk for both investors are zero. Hence, given Proposition 1, we can show that the representative agent's belief about future short rates is also a wealth-weighted average of the two groups' beliefs. The following proposition formalizes this point.

Proposition 2. *In an economy with heterogeneous beliefs described in Section 3.3.2, Proposition 1 implies that the expected short rate at $t+1$ is*

$$E_t^R[r_{t+1}] \approx w_t^A E_t^A[r_{t+1}] + w_t^B E_t^B[r_{t+1}]. \tag{3.16}$$

We next show that under mild assumptions the average or "consensus" belief of the two investors closely tracks the econometrician's belief in the heterogeneous economy. This results

[6] In an extreme case when the risk aversion level goes to infinity, there will be no relative wealth fluctuation and the representative agent's belief is simply the average of two groups' beliefs.

is important as we use consensus forecasts of future short rates to inform estimation of the econometrician's belief in our empirical implementation.

Proposition 3. *In an economy with heterogeneous beliefs described in Section 3.3.2 based on the specification in Equations (3.12), (3.11) and (3.13), the average belief of group-A and group-B investors follows the law of motion:*

$$\hat{\mu}_t^m = \frac{1}{2}(\hat{\mu}_t^A + \hat{\mu}_t^B) = \hat{\mu}_{t-1}^m + \sqrt{(1-\phi^2)}\frac{\sigma_\mu}{Q}[X_t - (1-\beta_X)\hat{\mu}_{t-1}^m - \beta_X X_{t-1}] \tag{3.17}$$

if the following conditions are satisfied:

$$S_t^A = -S_t^B, \tag{3.18}$$

$$\frac{1}{2}(\hat{\mu}_0^A + \hat{\mu}_0^B) = \hat{\mu}_0^m. \tag{3.19}$$

This agent thus follows the same Bayesian learning process as in a homogeneous economy described in Section 3.3.1 but with scaled learning gains.

The conditions in the above proposition imply the signal flows S_t^i perceived by two groups are in opposite directions but of the same magnitude. Hence, the signals observed by the two investors are not used for updating the drift perceived by the consensus agent. With Proposition 3 it is easy to see this representation nests the econometrician's belief. Comparing Equations (3.17) and (3.9), we see that the learning gain of the consensus agent in the economy with heterogeneous beliefs is equal to that of the econometrician in either economy scaled by $\sqrt{1-\phi^2}$. Hence, when the perceived correlation ϕ is zero, the consensus agent boils down to the econometrician as the noisy information is believed to be truly uninformative.

For small but positive values of ϕ, the consensus and the econometrician's learning gains will be very similar. While we do not observe ϕ directly, empirical evidence on the strong predictive power of (long-run) consensus forecasts of the short rate for future bond yields (see, e.g., Kim and Wright (2005) and Van Dijk et al. (2014)) suggest small values of ϕ. In our empirical implementation, we set $\phi = 0.1$, which implies a scale parameter $\sqrt{1-\phi^2} = 0.995$). This value of ϕ generates a level of disagreement very similar to that observed in the data. We assess the robustness of the above conditions in Section 6. In the following sections, we use the term *econometrician's belief* interchangeably with the belief of the consensus forecaster (i.e., the average belief).[7]

[7]Note that proposition 3 is similar but not identical to Proposition 1 in Xiong and Yan (2010) in which they show that when both agents react in opposite ways to a common signal, the average belief always exactly tracks the econometrician's. This allows them to isolate disagreement-driven effects from other effects such as erroneous beliefs or underestimation of risk. Here, we use similar arguments to justify our use of consensus forecast data to inform estimates of the econometrician's belief.

4 A GATSM with Fundamental Disagreement

In this section, we show how to incorporate the model with heterogeneous beliefs about the future short rate presented above into a Gaussian affine term structure model (GATSM). Specifically, we employ the framework of Joslin, Singleton and Zhu (2011) and adapt it in the following way. As in Joslin, Singleton and Zhu (2011), we assume that yields are affine functions of three pricing factors which are portfolios of yields that have an interpretation as level, slope, and curvature of bond yields. We also specify the dynamics of the pricing factors under the risk-neutral measure as a stationary vector autoregression. We depart from Joslin, Singleton and Zhu (2011) by assuming different dynamics of the pricing factors under the physical measure. Specifically, we assume that the three pricing factors follow a vector autoregression with stationary roots. While the slope and curvature factors are assumed to have a constant long-run mean, we assume that the level has a time-varying mean which itself follows a random walk. We further assume that agents have different beliefs about that long-run level of rates. In the following, we introduce the individual pieces of this specification which will form the basis for our empirical analysis in the next section.

We first specify the dynamics of the pricing kernel. In our model, markets are complete. Market completeness implies the existence of a unique equivalent martingale measure (or risk-neutral measure) to each group. We denote ξ_t^i as the Radon-Nikodym derivative, which converts the risk-neutral measure to the physical measure. Assume ξ_t^i follows the log-normal process

$$\xi_{t+1}^i = \xi_t^i \exp(-\frac{1}{2}\lambda_t^{i\prime}\lambda_t^i - \lambda_t^{i\prime}\varepsilon_{t+1}^i), \tag{4.1}$$

where λ_t^i is a vector of market prices and ε_{t+1}^i are the sources of risk as perceived by group-i investors, for $i \in \{A, B\}$. The nominal pricing kernel of investor i then takes the form

$$\frac{M_{t+1}^i}{M_t^i} = \exp(-r_t)\frac{\xi_{t+1}^i}{\xi_t^i} = \exp(-r_t - \frac{1}{2}\lambda_t^{i\prime}\lambda_t^i - \lambda_t^{i\prime}\varepsilon_{t+1}^i), \tag{4.2}$$

where r_t is the nominal short rate which is observed by both groups of investors. Note that here we follow the vast literature on affine term structure models by specifying the pricing kernel exogenously, rather than deriving it from the utility function in Section 3.1. As shown in Hördahl, Tristani and Vestin (2006) and Piazzesi (2010), this pricing kernel specification is fully consistent with standard time-separable utility when prices of risk are constant. Time-varying prices of risk could be obtained by applying higher-order approximations to the steady state or assuming exogenously time-varying second moments.

From Equation (4.2), the law of motion of the relative pricing kernel ratio is given by

$$\ln(\frac{\eta_A^B(t+1)}{\eta_A^B(t)}) = -\ln(\frac{M_{t+1}^B}{M_t^B}) + \ln(\frac{M_{t+1}^A}{M_t^A}) = \frac{1}{2}\lambda_t^{B\prime}\lambda_t^B + \lambda_t^{B\prime}\varepsilon_{t+1}^B - \frac{1}{2}\lambda_t^{A\prime}\lambda_t^A - \lambda_t^{A\prime}\varepsilon_{t+1}^A, \quad (4.3)$$

with the initial condition $\eta_A^B(0) = \frac{\varsigma_A}{\varsigma_B}$.

4.1 Pricing Restrictions

Our specification implies that log bond prices are affine in the vector of risk factors $\mathbf{X}_t$:

$$\ln p_t^{(n)} = \mathcal{A}_n + \mathcal{B}_n{}'\mathbf{X}_t + \epsilon_t^{(n)}, \quad (4.4)$$

where $\epsilon_t^{(n)}$ are the log yield pricing errors. Employing a normalization scheme proposed by Joslin, Singleton and Zhu (2011), the discrete-time evolution of the risk factors $\mathbf{X}_t$ under $\mathbb{Q}$ (risk-neutral measure) is governed by the following equation

$$X_{t+1} = C(k_\infty^{\mathbb{Q}}) + J(\lambda^{\mathbb{Q}})\mathbf{X}_t + \Sigma_{XX}^{1/2} e_{X,t+1}^{\mathbb{Q}}, \quad (4.5)$$

where $C(k_\infty^{\mathbb{Q}})$ and $J(\lambda^{\mathbb{Q}})$ are risk-neutral parameters, and Σ_{XX} is the variance-covariance matrix of the pricing factors. Note that $k_\infty^{\mathbb{Q}}$ determines the unconditional mean of the short rate under $\mathbb{Q}$, $C(k_\infty^{\mathbb{Q}})$ is a vector of the same length as $\mathbf{X}_t$ with the first entry being $k_\infty^{\mathbb{Q}}$ (and other entries being zero), and $J(\lambda^{\mathbb{Q}})$ is a diagonal coefficient matrix. See Appendix B for details.

We assume that each agent i beliefs the pricing factors to evolve according to

$$\mathbf{X}_{t+1} = \boldsymbol{\alpha}_t^{X,i} + \boldsymbol{\beta}^X \mathbf{X}_t + \Sigma_{XX}^{1/2}\varepsilon_{t+1}^i, \quad i = A, B, M, \quad (4.6)$$

under the physical measure $\mathbb{P}$, where $\boldsymbol{\alpha}_t^{X,i}$ and $\boldsymbol{\beta}^X$ are coefficients, see Appendix C for details. The first entry of vector $\boldsymbol{\alpha}_t^{X,i}$ collects the long-run mean $(1-\beta_X)\mu_t^i$ of the level factor as perceived by agent i, see also Equation (3.6). The above equation is thus a generalization of the setup in Sections 3.3.1 and 3.3.2 to multiple factors where only one is assumed to have a shifting endpoint. Note that since the long-run mean as perceived by investor i is evolving according to a random walk, the level factor is non-stationary (with a drifting component) under $\mathbb{P}$ while all the risk factors are stationary under $\mathbb{Q}$. As discussed in Joslin, Singleton and Zhu (2011), this framework explicitly allows for different degrees of stationarity under the two measures.

Given these ingredients, we have the following recursive linear restrictions for the bond pricing parameters:

$$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{B}'_{n-1} C(k_\infty^{\mathbb{Q}}) + \frac{1}{2}\mathcal{B}'_{n-1}\Sigma_{XX}\mathcal{B}_{n-1}, \tag{4.7}$$

$$\mathcal{B}'_n = \mathcal{B}'_{n-1} J(\lambda^{\mathbb{Q}}) - \mathbf{1}', \tag{4.8}$$

$$\mathcal{A}_0 = 0, \quad \mathcal{B}'_0 = \mathbf{0}. \tag{4.9}$$

We assume the market prices of risk are essentially affine as suggested in Duffee (2002), which means $\lambda_t^i = \Sigma_{XX}{}^{-1/2}(\lambda_{0,t}^{X,i} + \lambda_1^X \mathbf{X}_t), i = A, B$. With this assumption, we have $C(k_\infty^{\mathbb{Q}}) = \boldsymbol{\alpha}_t^{X,i} - \lambda_{0,t}^{X,i}$ and $J(\lambda^{\mathbb{Q}}) = \boldsymbol{\beta}^X - \lambda_1^X$, so

$$\begin{aligned} E_t^{\mathbb{Q}}[\mathbf{X}_{t+1}] &= C(k_\infty^{\mathbb{Q}}) + J(\lambda^{\mathbb{Q}})\mathbf{X}_t \\ &= (\boldsymbol{\alpha}_t^{X,A} - \lambda_{0,t}^{X,A}) + (\boldsymbol{\beta}^X - \lambda_1^X)\mathbf{X}_t \\ &= (\boldsymbol{\alpha}_t^{X,B} - \lambda_{0,t}^{X,B}) + (\boldsymbol{\beta}^X - \lambda_1^X)\mathbf{X}_t, \end{aligned} \tag{4.10}$$

By rearranging the terms in the last equality, Equation (4.10) suggests

$$(1 - \beta^L)(\mu_{t-1}^A - \mu_{t-1}^B) = \sigma_L^X(\lambda_t^{X_L,A} - \lambda_t^{X_L,B}). \tag{4.11}$$

From Equation (4.11), it is easy to see that as disagreement about the slow-moving drift $(\mu_{t-1}^A - \mu_{t-1}^B)$ increases, so does the difference in market prices of risk as perceived by the two investors, while the quantities of risk perceived by the two investors are the same. Hence, when investors strongly disagree about the future path of short rates their assessment of market prices of risk also differs widely. In other words, in our model disperse beliefs about future short rates and about the level of risk premiums are two sides of the same coin.

The above restrictions are silent about the belief of the representative agent (RA). Looking beyond the pricing restrictions, we find two channels through which disagreement can cause changes in market prices of risk of the RA. First assume that the two groups' beliefs about market prices of risk are unchanged. Then, the realization of returns would favor investors whose belief was more aligned with the true evolution of short rates and thus cause wealth to flow to this group of investors. At the same time, however, shifts in the RA's belief can also be caused by investors' heterogeneous reactions to exogenous shocks driving their pricing kernels and thus their risk-return assessment. In Section 5.5, we will discuss both channels in detail.

4.2 Estimation with the Kalman Filter

We can cast the above set of equations in simple state space form which will path the way for our estimation of the model using the Kalman filter. The measurement equation is

$$\begin{bmatrix} \mathbf{y}_t^o \\ \mathbf{y}_t^{E,A} \\ \mathbf{y}_t^{E,B} \\ \mathbf{y}_t^{E,M} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^X \\ \mathbf{A}_X^{E,A} \\ \mathbf{A}_X^{E,B} \\ \mathbf{A}_X^{E,M} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^X & 0 & 0 & 0 \\ \mathbf{B}_X^{E,A} & B_\mu^{E,A} & 0 & 0 \\ \mathbf{B}_X^{E,B} & 0 & B_\mu^{E,B} & 0 \\ \mathbf{B}_X^{E,M} & 0 & 0 & B_\mu^{E,M} \end{bmatrix} \times \begin{bmatrix} \mathbf{X}_t \\ \mu_t^A \\ \mu_t^B \\ \mu_t^M \end{bmatrix} + e_t, \tag{4.12}$$

where the first few entries of e_t are yield pricing errors $\epsilon_t^{(n)}$. The transition equation is

$$\begin{bmatrix} \mathbf{X}_t \\ \mu_t^A \\ \mu_t^B \\ \mu_t^M \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}^X \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\beta}^X & 0 & 0 & l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{X}_{t-1} \\ \mu_{t-1}^A \\ \mu_{t-1}^B \\ \mu_{t-1}^M \end{bmatrix} + u_t. \tag{4.13}$$

In the above measurement equation, the longer-term average short rate forecasts at time t for horizons from $h1$ months to $h2$ months ahead can be expressed as

$$\begin{aligned} E_t^i[r^{h1,h2}] &= E_t^i[\frac{r_{t+h1} + r_{t+h1+1} + ... + r_{t+h2-1}}{h2-h1}] \\ &= [1\ 1\ 1] \times E_t^i[\frac{\sum_{i=h1}^{h2-1} \mathbf{X}_{t+i}}{h2-h1}] \\ &= [1\ 1\ 1] \times \left(\boldsymbol{\alpha}_{h1,h2}^X + [\boldsymbol{\beta}_{h1,h2}^X\ l_{h1,h2}] \times [\mathbf{X}_t; \mu_t^i]\right), \quad i = A, B, M, \end{aligned}$$

where the slow-moving drifts come from the physical dynamics in different groups' beliefs.[8] In short, some of the coefficients in the measurement equation are nonlinear functions of the coefficients in the transition equation as they link observable survey expectations to model-implied forecasts of the short rate for different agents. Details are provided in Appendix C.

Assuming Gaussian measurement errors and innovations to the transition equation, our model can be estimated using the Kalman filter. In addition to the measurement and

[8] By iterating the transition equation forward we can obtain $\boldsymbol{\alpha}_{h1,h2}^X$ and $[\boldsymbol{\beta}_{h1,h2}^X\ l_{h1,h2}]$, which are essentially functions of $\boldsymbol{\alpha}^X$, $\boldsymbol{\beta}^X$ and forecast horizons $h1$ and $h2$.

transition error variances, the unknown parameters are $(\Sigma_{XX}, \lambda^{\mathbb{Q}}, k_{\infty}^{\mathbb{Q}})$ controlling the risk-neutral dynamics of the pricing factors, and $(\boldsymbol{\alpha}^X, \boldsymbol{\beta}^X)$ controlling the dynamics of the pricing factors under the physical measure. μ_t^M captures the econometrician's belief and - following our discussion in the previous section - can be considered as a good proxy for the objective DGP. Note that the model-implied estimates of the econometrician's belief will not affect the model-implied estimates of the beliefs of the two groups of investors A and B, as their perceived drifts μ_t^A and μ_t^B are filtered using observations on the full path of their respective short rate forecasts and as the innovations to the perceived drifts are assumed independent.

Note further that while the level factor is fully spanned by the cross-section of bond yields, its time-varying long-run mean μ_t as perceived by different investors is not. At the same time, the different perceived long-run means bear information about future expected short rates and expected bond returns. Hence, we can interpret the long-run means as perceived by different investors as "hidden" or "unspanned" factors in the spirit of Duffee (2011*b*) and Joslin, Priebsch and Singleton (2014). In contrast to these papers which use observable macroeconomic variables, here we use observable information on survey forecasts to identify the unspanned factors.[9]

Given the parameter estimates, we can directly solve for the representative agent's belief at any point in time. By applying Proposition 2, the drift in the representative agent's belief is given by the adjusted-wealth-weighted average of the two investors, i.e. $\mu_t^R = w_t^A \mu_t^A + w_t^B \mu_t^B$, where

$$\begin{aligned} w_t^A &= \frac{1}{\eta_A^B(t)^{\frac{1}{\gamma}} + 1}, \; w_t^B = 1 - w_t^A, \\ \text{and } \ln\left(\frac{\eta_A^B(t+1)}{\eta_A^B(t)}\right) &= \frac{1}{2}\lambda_t^{B\prime}\lambda_t^B + \lambda_t^{B\prime}\varepsilon_{t+1}^B - \frac{1}{2}\lambda_t^{A\prime}\lambda_t^A - \lambda_t^{A\prime}\varepsilon_{t+1}^A. \end{aligned} \tag{4.14}$$

We use $\gamma = 6$ in our baseline analysis, but note here that our results are robust to a range of risk aversion values $\gamma \in [2, 10]$, that are deemed reasonable in the literature, see Bliss and Panigirtzoglou (2004).

5 Empirical Results

In this section, we present the empirical results of our analysis. We start by describing the data used. We then document that the model fits Treasury yields and survey forecasts of future short rates very well. Finally, we analyze the relative importance of the different

[9] This is similar in spirit to Chernov and Mueller (2012) who also filter an unspanned factor from survey forecasts of inflation.

driving forces of term premiums in our model with heterogenous beliefs about the policy rate.

5.1 Data

We jointly estimate our model using zero-coupon Treasury yields as well as survey expectations of short rates for different groups of investors. We obtain the latter from the *Blue Chip Financial Forecasts* (BCFF) survey. Specifically, we use two- and four-quarter-ahead, one-to-two, four-to-five year-ahead, and long-term forecasts which cover horizons between six and ten or seven and eleven years into the future, depending on when the survey was taken. The short-term forecasts are observed monthly and this is our frequency of observation also for Treasury yields. The medium-term and long-term forecasts are observed biannually. The missing monthly observations in between biannual survey observations are easily dealt with in our Kalman filter estimation. The BCFF provides medium and long-term forecasts for three different cross-sectional averages of the forecaster distribution: the average across all responses (the "consensus" forecast), the average of the top-10 responses and the average of the bottom-10 responses. Following the result in Section 1, we use the consensus forecast as a measurement to pin down the econometrician's belief. We further employ the top-10 and bottom-10 average responses as proxies representing two investors at the opposite spectrum of the belief distribution about future short rates. As discussed in Andrade et al. (2016), the difference between the top-10 and bottom-10 average responses is closely correlated with common measures of forecaster disagreement, such as the cross-sectional standard deviation or the interquartile range.

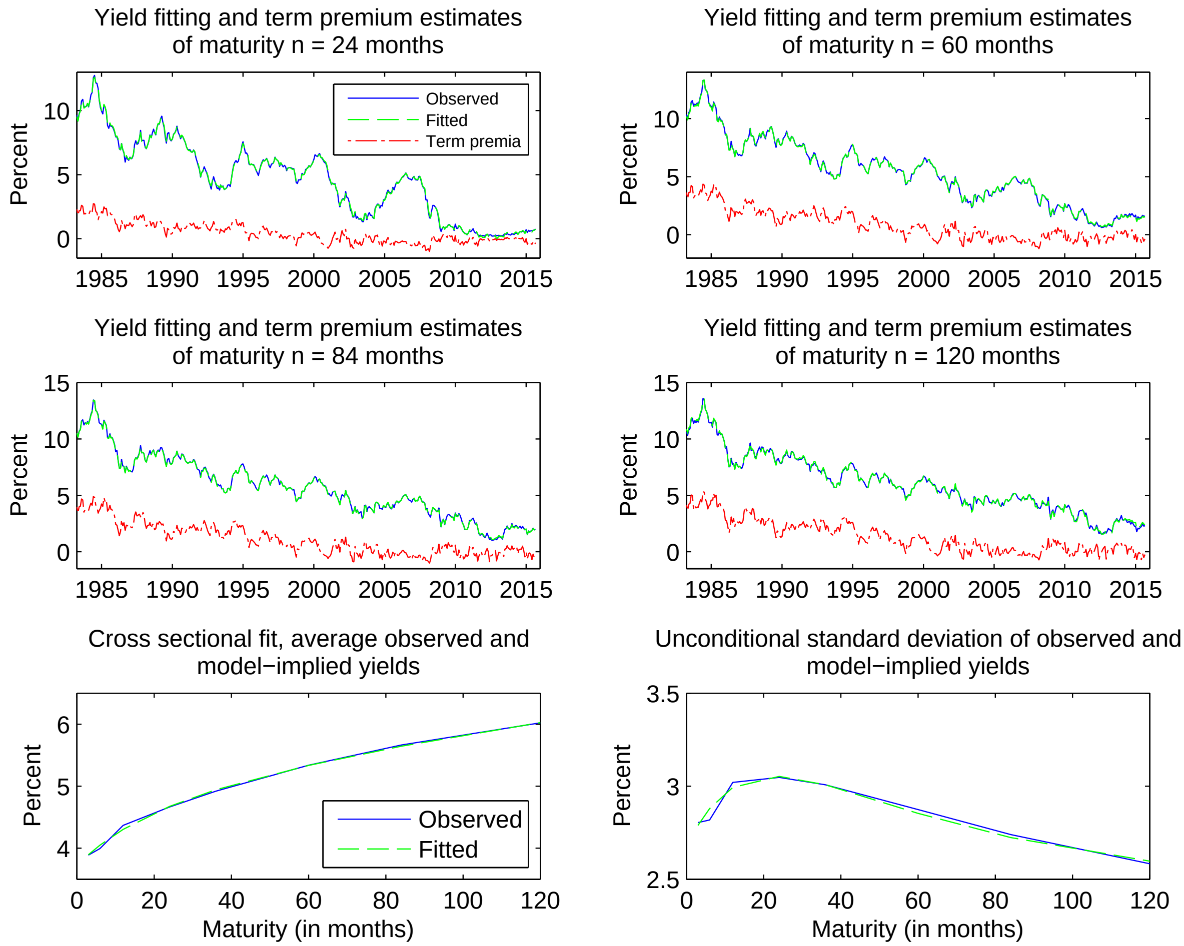
We obtain zero coupon Treasury yields from Gürkaynak, Sack and Wright (2007) (GSW henceforth).[10] The GSW zero coupon yields are based on fitted Nelson-Siegel-Svensson curves, the parameters of which are published along with the estimated zero coupon curve. We use these parameters to back out the cross-section of zero-coupon yields for maturities up to ten years, using end-of-month values. In our estimation, we use $N = 8$ Treasuries with maturities $n = 3, 6, 12, 24, 36, 60, 84, 120$ months. Our sample period is $1983:03 - 2015:08$ for a total of 400 monthly observations.

5.2 Model Fit

Our model fits both yields and survey forecasts of the short rate very precisely, as displayed in Figure 6 which shows the time series and cross section of observed and model-implied yields. The average of yield pricing errors is no more than 5 basis points in absolute value and is thus well in line with previous studies. The bottom two panels of Figure 6 provide a

[10]See http://www.federalreserve.gov/econresdata/researchdata.htm

Figure 6: Time-series and cross-sectional fit of yields

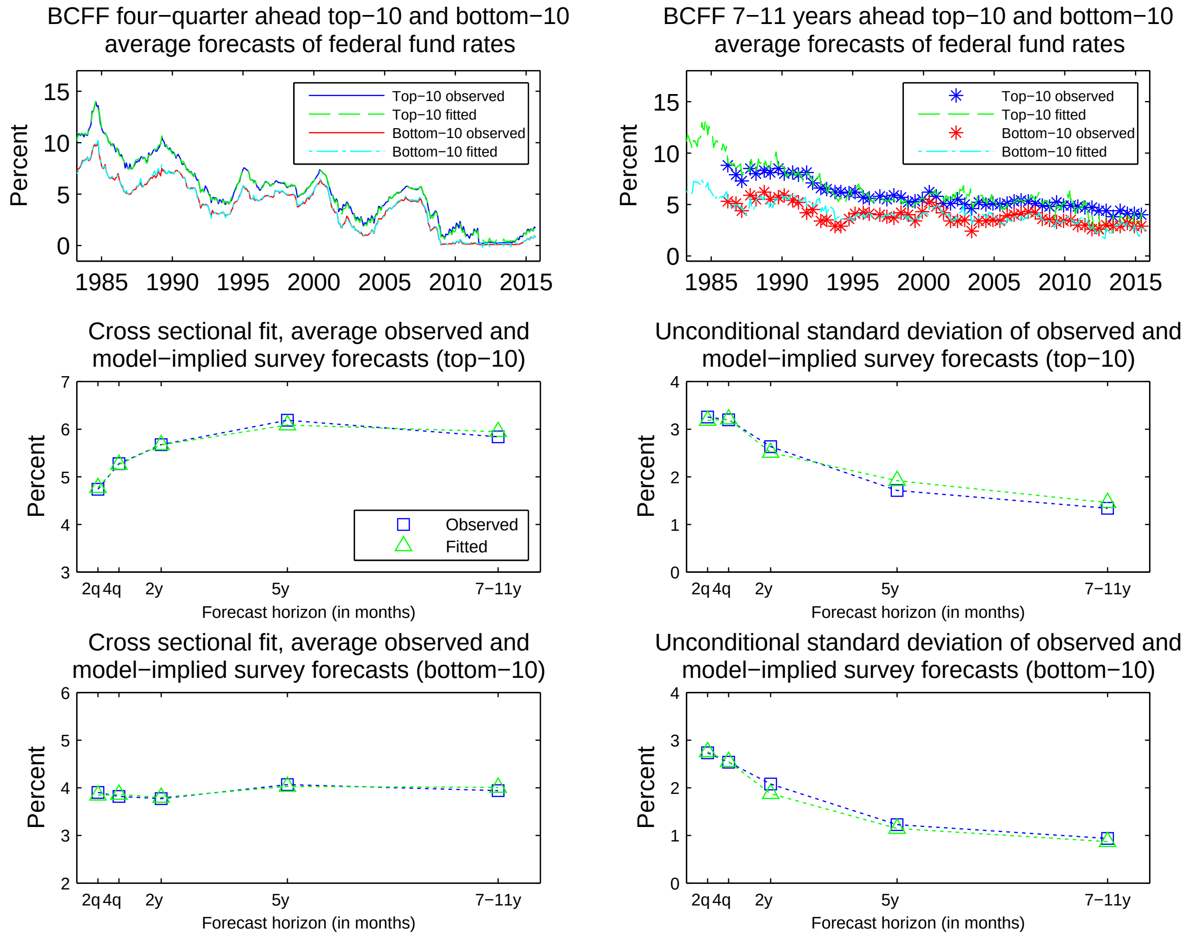


Notes:
This figure provides plots of observed and model-implied yields. Observed yields are displayed by solid lines, dashed lines correspond to model-implied yields. Dash-dotted lines show model-implied term premiums for two-, five-, seven- and ten-year maturities in the upper four charts. The bottom two panels plot unconditional averages and standard deviations of observed yields against those implied by the model.

plot of the unconditional mean and standard deviation of yields across maturities as observed and fitted by the model. The charts show that the model fits both moments well.

We next turn to the model fit of survey forecasts of the policy rate. The top two charts in Figure 7 show the observed and fitted top-10 and bottom-10 average survey forecasts of the federal funds rate, where actual values are plotted by solid lines. These two charts document that with only the perceived long-run mean of the level factor being different across investors, our model is able to capture the substantial time variation in two groups' disagreement about future short rates. The bottom four panels of Figure 7 provide a plot of unconditional first and second moments of two groups' survey forecasts as observed and fitted by the model, again documenting that the model fits survey forecasts at all horizons quite precisely.

Figure 7: Time-series and cross-sectional fit of survey forecasts

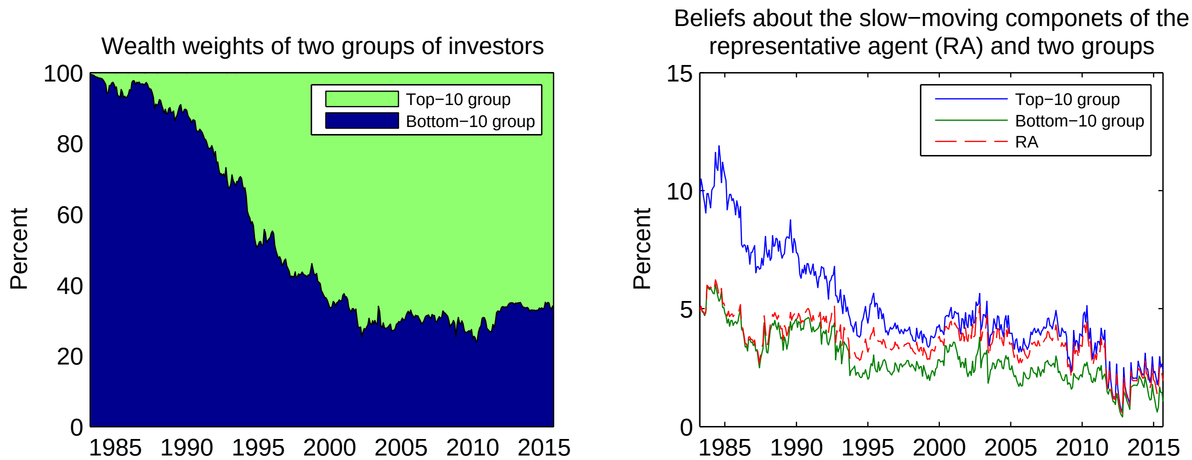


Notes:
This figure provides plots of observed and model-implied survey forecasts of the fed funds rate. Observed survey forecasts are displayed as solid lines, dashed lines correspond to model-implied survey forecasts. The top two charts show the Blue Chip Financial Forecasts (BCFF) four-quarter ahead and long-range (7-11 years) top-10 and bottom-10 average forecasts of the federal funds rate. Asterisks in the top right chart are long-term forecasts which are observed biannually. The bottom four panels plot unconditional means and standard deviations of survey forecasts of the top-10 and bottom-10 average responses against those implied by the model.

5.3 Disagreement and Relative Wealth Dynamics

Having shown that our model fits both yields and survey data on future short rates precisely, we now study how disagreement about monetary policy affects term premiums and the pricing of risk. Figure 8 visualizes the degree of belief heterogeneity about the long-run mean of the short rate as well as the evolution of the relative wealth ratio among the two groups of investors. The left-hand chart displays the wealth weights of two groups of investors as

Figure 8: Heterogeneous beliefs



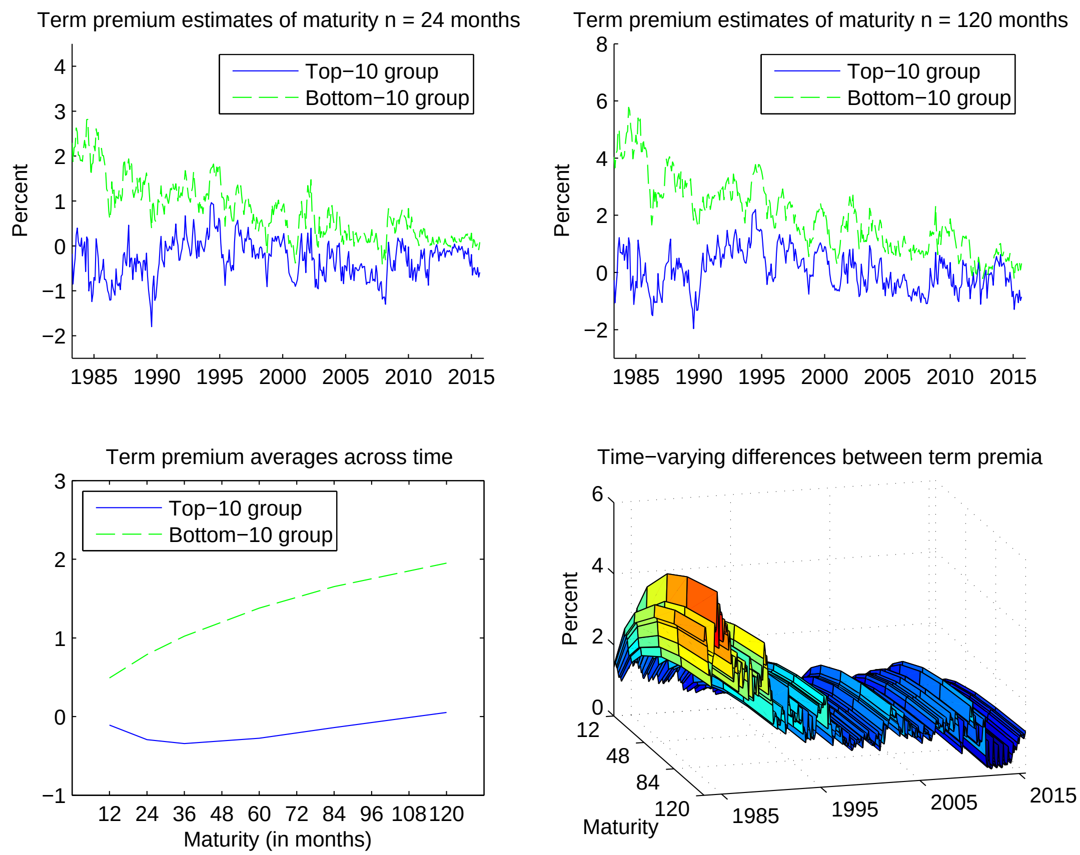
Notes:
This figure provides graphs exhibiting the slow-moving components in different beliefs and wealth weights. The left chart sets out the wealth weights of two groups of investors $\frac{w_A}{w_A+w_B}$ and $\frac{w_B}{w_A+w_B}$, i.e. top-10 (A) and bottom-10 (B) average groups. The right-hand chart plots the estimates of representative agent's and the two investors' beliefs about the long-run mean, i.e., $\hat{\mu}_t^R, \hat{\mu}_t^A, \hat{\mu}_t^B$, where $\hat{\mu}_t^R = w_t^A \hat{\mu}_t^A + w_t^B \hat{\mu}_t^B$.

provided in Equation (3.5).[11] The chart shows that the group expecting higher future short rates and lower term premiums is gradually gaining market power, as the wealth distribution shifts towards the top-10 investor over time. This is surprising, as short rates have seen a secular decline over the sample period considered and thus on average forecasts of investors that were more pessimistic about (in the sense that they expected lower) future short rates on average were closer to the actual outcome. On possible reason for this counterintuitive finding is the following. Investors betting lower future short rates and thus expecting higher returns on long-term bonds tend to invest a larger fraction of their wealth in long-term bonds. While long-term bonds have larger average returns than short-term bonds, they are also much more risky. In fact, as shown e.g. in Van Binsbergen and Koijen (2017), short-term Treasuries have considerably higher Sharpe ratios than long-term Treasuries. As a consequence, investors that tend to roll over short-term bonds rather than holding long-term bonds gradually accumulate more wealth than the investors putting a larger fraction of their wealth in long-term bonds.

The right-hand chart shows the evolution of the long-run means as perceived by the top-10 and the bottom-10 investors, in addition to the model-implied long-run mean of the representative investor. Recall that the latter is itself a wealth-weighted average of the two

[11]Note that the initial relative weight ratio $\eta_A^B(0)$ is undefined. We calibrate this value by minimizing the average of the squared differences between μ_t^R and μ_t^M, as μ_t^M is considered a noisy measure of the true DGP. The difference is given by $\frac{1}{\mu_t^B - \mu_t^A}(\mu_t^R - \mu_t^M)$, which is weighted by the level of disagreement $\frac{1}{\mu_t^B - \mu_t^A}$. When disagreement is low, the difference between μ_t^R and μ_t^M is considered more informative and will be given a larger weight.

Figure 9: Term premiums perceived by two groups of investors



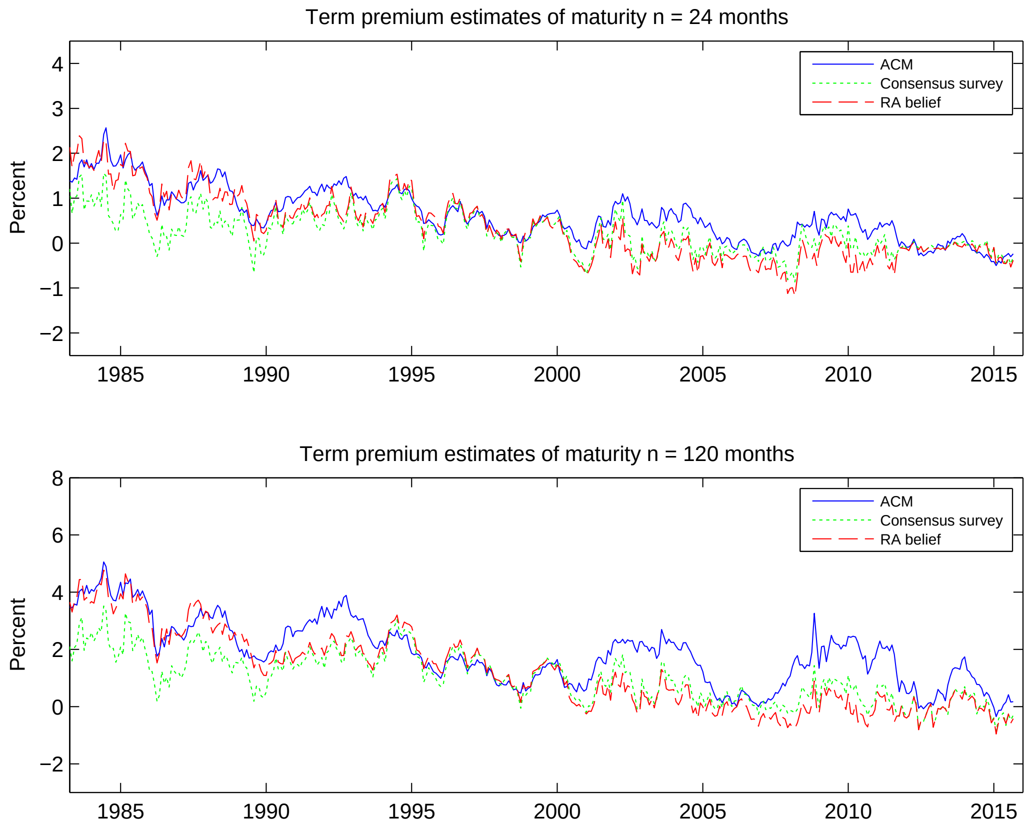
Notes:
This figure plots the term premiums implied by the top-10 and bottom-10 groups' beliefs about future short rates. The upper panels plot the the term premium estimates for two- and ten-year treasury notes. The lower left panel plots the sample averages of term premium estimates in two groups' beliefs for different maturities. The lower right panel displays time-varying differences across maturities between two groups' beliefs.

investors' beliefs. As the relative wealth ratio is tilted towards the bottom-10 investor at the beginning of the sample, the representative agent's belief about the long-run mean of the policy rate is initially very similar to that investor. However, as the top-10 investors gradually increases their wealth over time, the representative investor behaves more and more similar to an investor expecting higher future short rates and thus lower term premiums on long-term bonds.

5.4 Disagreement and Term Premiums

Having established that the expected long-run mean of rates of the two groups of investors has converged somewhat over time, it is instructive to also compare the term premiums as implied by the different beliefs. The upper panel of Figure 9 shows these term premiums

Figure 10: Term Premium Estimates



Notes:
This figure provides plots of the term premium estimates for two- and ten-year treasury notes. The term premium estimates of the representative agent (RA) are plotted as dashed lines. The dotted lines correspond to the term premium of the consensus survey forecaster, and solid lines to ACM term premiums obtained from the model described in Adrian, Crump and Moench (2013).

for the two and ten-year maturity, respectively. As one can see, the top-10 average investor expecting high future short rates implicitly perceived term premiums on two-year notes to hover between minus one and one percent over the last 30-years. Similarly, this investor perceived ten-year Treasury term premiums to fluctuate around zero in a somewhat wider range. In stark contrast, the bottom-10 average investor expecting low future policy rates, has term premiums consistently positive across time and maturity, declining from about three (six) percent at the two-year (ten-year) maturity in the early 1980s to just below one percent for both maturities at the end of the sample.

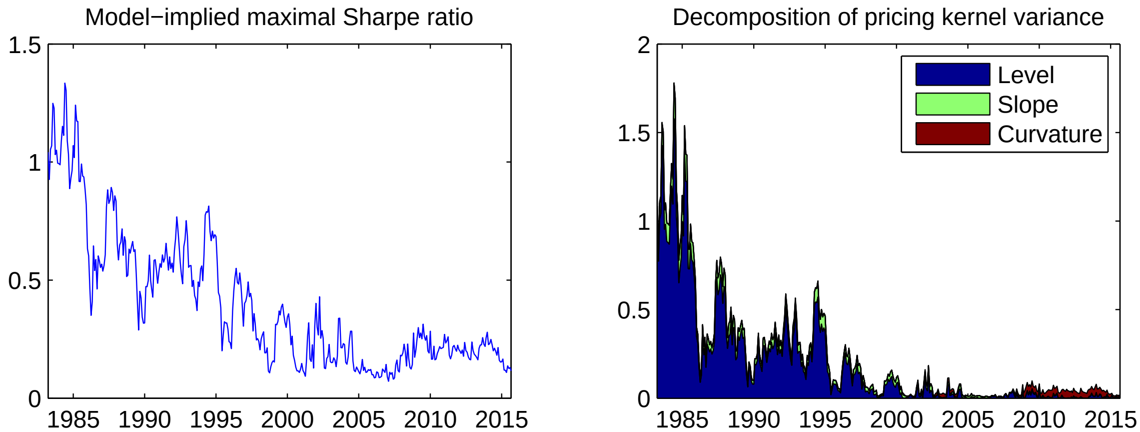
The bottom-left panel of Figure 9 provides the time series average of the implied term premiums across maturities for the two agents. While the bottom-10 average investor has an upward sloping term structure of term premiums ranging between 50 basis points at the one-year maturity and 200 basis points at the ten-year maturity, the top-10 investor essentially sees term premiums on average slightly negative across all except the very long maturities. Hence, to this investor Treasuries provide insurance for which she is willing to pay a premium. The bottom-right chart shows the differences between the two term premium estimates across time and maturities, reinforcing that these differences have been much less pronounced since around the year 2000, in line with the above finding that the relative wealth ratio has been much more stable in the second half of our sample.

Given these differences in term premiums across investors and fluctuations in the relative wealth ratio, what is the term premium of the representative investor in this heterogeneous beliefs economy? Figure 10 provides time series of the two-year (top panel) and ten-year (bottom panel) term premium estimates, as perceived by the representative investor as dashed lines. We compare these to two other measures of the term premium. The first is the term premium from the consensus forecaster (dotted lines), the second is based on the term structure model by Adrian, Crump and Moench (2013) which does not use any survey information (solid lines). The three estimates broadly move together but behave differently in some episodes. The term premium estimates of the consensus forecaster are quite low in the 1980s, falling into negative territory for the two-year maturity. This is due to the fact that short rate expectations of the consensus forecaster remained high for some time after the Volcker disinflation period. In contrast, the term premium perceived by the representative investor was substantially higher as the investors expecting lower future short rates held relatively more wealth during this period. The representative agent's estimates are very similar to the ACM term premium before 2000. However, in the years after 2000 both the consensus forecaster and the representative agent perceive the term premium to be lower than that implied by the ACM model. This is because survey-based forecasts of future short rates remained fairly high during this period while the short rate path implied by the statistical ACM model was substantially lower. Interestingly, at the two-year maturity all three term premium estimates behave quite similarly after mid 2011 which is when the FOMC announced that it would keep the fed funds rate exceptionally low "at least through mid-2013", marking the Committee's first use of date-based forward guidance.[12] This date-based forward guidance seems to have played an important role in stabilizing short-term yields also through term premiums alike, as shown in the upper panel of Figure 10.

While the above charts show that the term premium implied by the representative investor in the heterogeneous belief economy has properties similar to other common measures of term premiums, we further assess the plausibility of our estimates by computing the conditional maximal Sharpe ratio implied by the model at any given point in time. This analysis follows

[12]In January 2012, the FOMC replaced "mid-2013" with "late-2014" and in September 2012, replaced "late-2014" with "mid-2015".

Figure 11: Maximal Sharpe Ratio and Pricing Kernel Decomposition



Notes:
This figure shows the maximal attainable Sharpe ratio of the representative investor and a variance decomposition of her pricing kernel. The left-hand chart plots the time series of the maximal Sharpe ratio $\sqrt{\lambda_t{}'\lambda_t}$. The right-hand chart decomposes $\lambda_t{}'\lambda_t$ into its components associated with the three pricing factors.

Duffee (2010) who suggested that implausibly large Sharpe ratios can indicate misspecification of models with weakly spanned factors. Following Duffee (2010), we obtain the model-implied maximal attainable Sharpe ratio as $\sqrt{\text{Var}_t(\ln M_{t+1})} = \sqrt{\lambda_t'\lambda_t}$ where λ_t denotes the vector of prices of risk perceived by the representative investor. The maximal Sharpe ratio implied by our model is at a reasonable level, as shown in the left panel of Figure 11. The peak in the maximal Sharpe ratio is below 1.5, the sample average is around 0.5, and it has been hovering around 0.25 during the last ten years of the sample.

In our model, as in the vast majority of term structure models, exposures of individual bond maturities on the risk factors are constant over time. Accordingly, all variations in term premiums are due to variations in the market prices associated with the models' risk factors. We can assess the economic significance of the three pricing factors in our model by calculating their contribution to the time variation of the representative agent's pricing kernel. Following Adrian, Crump and Moench (2013), we decompose the conditional variance of the pricing kernel into the contributions due to each price of risk according to

$$\text{Var}_t(\ln M_{t+1}) = \lambda_t'\lambda_t = \sum_j \lambda_{j,t}^2, \quad j = L, S, C. \tag{5.1}$$

As the right-hand chart in Figure 11 shows, it is primarily time variation in the price of level risk that contributes to the variability of the pricing kernel. This is consistent with the evidence in Cochrane and Piazzesi (2008) and in Adrian, Crump and Moench (2013) who also find that level risk is the predominant source of variation in expected excess bond returns.

5.5 Parsing the Channels

We now turn to the main question of our analysis: what is the impact of disagreement about future monetary policy on the term structure of interest rates and on term premiums in particular? Specifically, we assess to what extent term premiums are driven by differences in beliefs that trigger speculative trading, or through movements in the wealth distribution as a consequence of past disagreement.

In Section 4.1, we have shown that the market prices of risk of the representative investor are affected by disagreement through two channels. We can label these channels as *exogenous* and *endogenous* where the exogenous impact derives from the heterogeneous responses of investors to their private (noisy) signals, and the endogenous channel corresponds to disagreement-induced changes in the relative wealth distribution as a consequence of speculative trading.

In our model, as in Joslin, Priebsch and Singleton (2014), the (scaled) market prices of risk are linear in the risk factors:

$$\Sigma^{1/2}\lambda_t = \lambda_{0,t} + \lambda_1 \mathbf{X}_t.$$

Since $\lambda_{0,t} = \boldsymbol{\alpha}_t^X - C(k_\infty^{\mathbb{Q}}) = (w_t^A \boldsymbol{\alpha}_t^{X,B} + w_t^B \boldsymbol{\alpha}_t^{X,B}) - C(k_\infty^{\mathbb{Q}}) = w_t^A \lambda_{0,t}^A + w_t^B \lambda_{0,t}^B$, the jth element in vector $\Sigma^{1/2}\lambda_t$ is given by

$$\lambda_t^j = e_j \Sigma^{1/2}\lambda_t = e_j(\lambda_1 \mathbf{X}_t + w_t^A \lambda_{0,t}^A + w_t^B \lambda_{0,t}^B), \qquad j = L, S, C, \tag{5.2}$$

where e_j is selection vector. We thus see that the market prices of risk in the representative agent's belief are weighted averages of two groups' beliefs. Therefore, changes in λ_t^j are driven by three sources, which can be formally written in the following first-order approximation:

$$\Delta\lambda_t^j \approx \underbrace{e_j \lambda_1 \Delta \mathbf{X}_t}_{\text{Common response}} + \underbrace{\underbrace{e_j(w_t^A \Delta\lambda_{0,t}^A + w_t^B \Delta\lambda_{0,t}^B)}_{\text{Heterogeneous response}} + \underbrace{e_j(\lambda_{0,t}^A \Delta w_t^A + \lambda_{0,t}^B \Delta w_t^B)}_{\text{Wealth effect}}}_{\text{Disagreement-driven}}. \tag{5.3}$$

The first source is the one that would also be present in a homogenous beliefs economy. It is simply the common response of investors to changes in yields as captured by changes in the model's pricing factors. The other two sources are disagreement-driven. The first reflects the heterogeneous responses of investors' risk attitudes to their private signals about future short rates. The second arises because of endogenous wealth fluctuations. Fixing investors' risk attitudes, any change in the relative wealth ratio will induce changes in the representative agent's belief. It is worth noting that as these two disagreement-driven effects interact with each other, the relation between term premiums and measures of disagreement about future short rates is not constant, in line with the empirical evidence in Giacoletti, Laursen and Singleton (2016) who detect a time-varying impact of disagreement on expected excess returns.

Based on the above decomposition for market prices of risk, we can perform a similar decomposition for changes in expected one-month excess holding period returns of a Treasury with maturity n:

$$\Delta E_t[rx_{t+1}^{(n-1)}] \approx \mathcal{B}_{n-1}\lambda_1\Delta\mathbf{X}_t + \mathcal{B}_{n-1}(w_t^A\Delta\lambda_{0,t}^A + w_t^B\Delta\lambda_{0,t}^B) + \mathcal{B}_{n-1}(\lambda_{0,t}^A\Delta w_t^A + \lambda_{0,t}^B\Delta w_t^B),$$

where $\mathcal{B}_{n-1}$ is the vector of loadings of the log price of a bond with maturity n on the pricing factors X derived above. By the same token, the decomposition for changes of the term premium of a Treasury with remaining maturity of n months is given by:

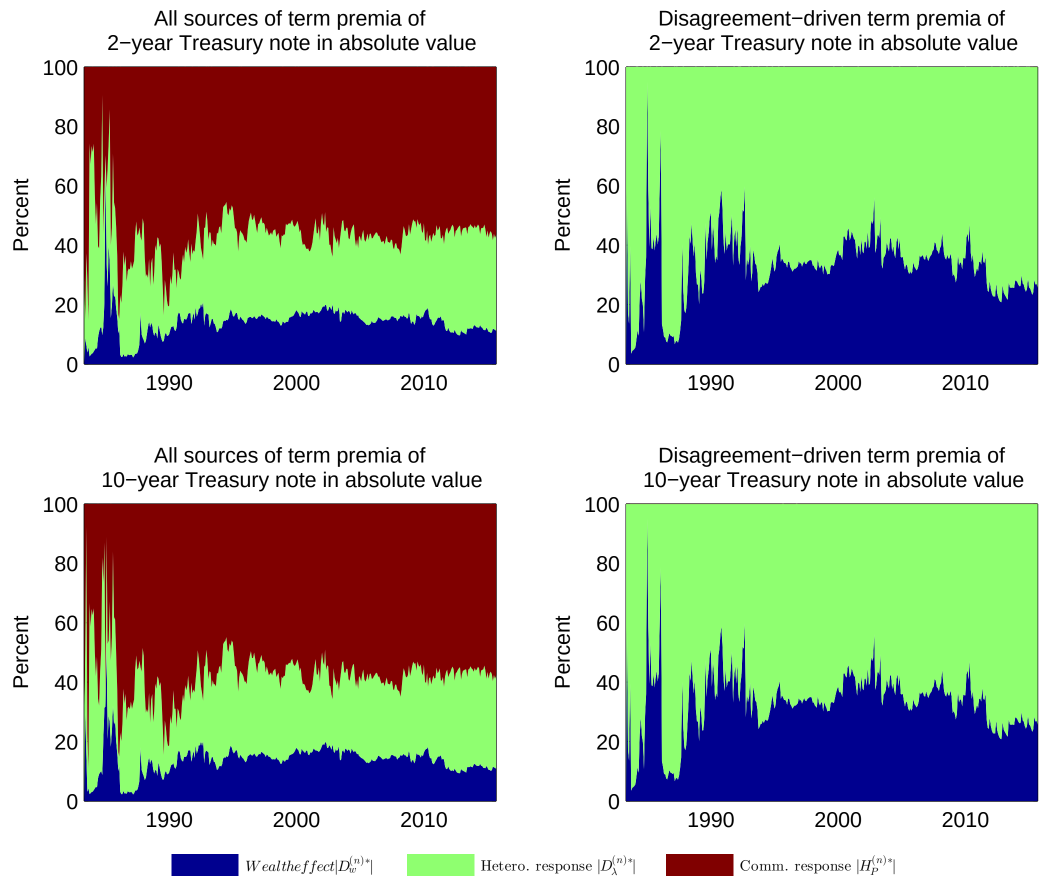
$$\begin{aligned} \Delta TP_t^{(n)} &= \Delta\Big(\frac{E_t[rx_{t+1}^{(n-1)} + rx_{t+2}^{(n-2)} + ... + rx_{t+n-2}^{(2)}]}{n-1}\Big) \\ &\approx \underbrace{B_n^{H*}\lambda_1\Delta\mathbf{X}_t}_{\text{Common response}} + \underbrace{\underbrace{B_n^{D*}(w_t^A\Delta\lambda_{0,t}^A + w_t^B\Delta\lambda_{0,t}^B)}_{\text{Heterogeneous response}} + \underbrace{B_n^{D*}(\lambda_{0,t}^A\Delta w_t^A + \lambda_{0,t}^B\Delta w_t^B)}_{\text{Wealth effect}}}_{\text{Disagreement-driven}} \\ &= H_{X,t}^{(n)} + D_{\lambda,t}^{(n)} + D_{w,t}^{(n)} \end{aligned} \tag{5.4}$$

where $\Delta TP_t^{(n)}$ can again be decomposed into the three different sources of variation.

Intuitively, the wealth effect arises because investors disagree about expected excess returns and therefore choose different portfolio allocations. Given the previous period's portfolio, the realization of returns then changes the relative market power of the two groups of investors, which in turn affects the term premium of the representative investor. The heterogeneous response to private signals does not affect the wealth distribution contemporaneously. However, as investors receive private signals about future short rates and thus term premiums, these heterogenous signals will have an impact on the representative investor's term premium as long as as the relative wealth ratio is different from one.

Figure 12 displays the relative importance of the three sources of variation of term premiums in our estimated model. Specifically, we compute the absolute values of the three terms in Equation (5.4) and report their relative shares. The left panels of Figure 12 provide this decomposition for the two and ten-year maturities, respectively. They show that a sizable share of the variation of term premiums is disagreement-driven, accounting for on average more than 40% of their monthly changes. Not surprisingly, the term premium variation arising from disagreement is particularly pronounced in the early 1980s when beliefs about

Figure 12: Decomposition of variation in term premiums



Notes:
This figure provides decompositions of the changes in term premiums of two and ten-year Treasury notes. The left-hand side panels decompose the cumulative variation $\sum_{k=1}^{t} \Delta TP_k^{(n)}$ into three components in absolute value $|\sum_{k=1}^{t} H_{\mathcal{P},k}^{(n)}|$, $|\sum_{k=1}^{t} D_{w,k}^{(n)}|$ and $|\sum_{k=1}^{t} D_{\lambda,k}^{(n)}|$. The right-hand side panels compare only the magnitudes of $|\sum_{k=1}^{t} D_{w,k}^{(n)}|$ and $|\sum_{k=1}^{t} D_{\lambda,k}^{(n)}|$.

future short rates were the most dispersed. In the right-hand panels of Figure 12 we further decompose the variation of term premiums driven by disagreement into the heterogeneous belief and the relative wealth effects for the same two maturities. The results show that the relative importance of the relative wealth channel varies substantially over time, but is on average dominated by the heterogeneous risk attitudes channel over much of the sample.

6 Robustness

In this section, we document the robustness of our results with respect to a few modeling choices. We first show that the parameter ϕ, which determines the relative importance of private signals in the formation of the two investors' beliefs, generates levels of disagreement consistent with the data. We then compare the serial correlation of forecast errors about future short rates implied by our model with those implied by a model without shifting endpoints.

Recall from the discussion in Section 3.3.2 that an econometrician in the heterogeneous beliefs economy would not observe the two agents' signals S_t^i. Hence, the parameter ϕ is not identified. We choose to set $\phi = 0.1$ which, according to Proposition 3, implies that the consensus investor's belief is a close approximation of the econometrician's belief. We confirm that this value generates a degree of disagreement consistent with the data in the following way. We generate $10,000$ simulations of a hypothesized economy using the parameter estimates of our model, the calibrated value of ϕ, and the sample variance of the three model factors. In each simulation, the two investors receive their private noisy signal and observe new realizations of the yield curve factors. They use both to filter the long-run mean of the level factor, and employ the latter along with the observed yield curve factors to forecast future short rates. We show this relatively low degree of informational frictions can generate substantial disagreement as observed in the data.[13]

We then test whether the conditions of Proposition 3 are realistic. First, we find that Condition (3.18) is not strictly binding. Even when the noisy signals observed by the two groups of investors cannot be perfectly averaged out, our results still hold. As long as the noisy signals have opposite signs, as they should have in order to generate beliefs that diverge relative to the common information embedded in yields themselves, the average belief will be very close to the econometrician's belief. We validate this point by constructing a simulation exercise similar to the above. The two groups now observe, rather than mirrored signal flows (with different signs) generated from the same Gaussian distribution, two separated signal flows generated respectively from two independent truncated Gaussian distributions. We find that in this simulation exercise, the difference with respect to the case where the two signals are exactly offsetting is just a few basis points. Second, our results are not sensitive to the initial value given in Condition (3.19). This is because investors learn quickly from new information. To check this, we conduct simulations with a range of initial values, and find that even after a training sample of only five years, the dynamics of the average investor's belief are hardly changed by altering the initial values of the long-run mean as perceived by the two investors.

[13]The disagreement about the drift in our simulations is around 2% on average, which matches the average difference in long-term short rate forecasts in the BCFF survey. Moreover, in the simulations the econometrician's belief is almost indistinguishable from the consensus belief. The sample length is 400 observations for each simulation, which is equal to the sample size of our data ($1983:03 - 2015:08$), and the initial value is the same as in our model estimation.

Last, we assess whether agents make serially correlated forecast errors of short rates in our simulations. This is not the case. In fact, both the econometrician and the consensus forecaster produce unbiased forecasts, as they both consider the persistence of the slow-moving component. However, there are persistent differences between the prediction of the econometrician (or the consensus forecaster) in our model and her counterpart from a model with a constant mean. The sample autocorrelations of the differences are around 0.9 for one lag and 0.4 for twelve lags. This highlights the importance of specifying a shifting endpoint in the VAR model that is used to forecast the short rate.

7 Conclusion

Bond investors disagree about the future path of policy rates, and particularly about their long-run level. Accordingly, they disagree about the risk-return tradeoff of longer-term bonds and engage in speculative trading. This induces shifts in their relative wealth which, in turn, affects the marginal pricing of risk in the economy. Hence, term premiums as perceived by an econometrician observing only yields partly reflect disagreement-driven changes in the marginal pricing of risk.

In this paper, we have formalized this intuition in an affine term structure model with heterogeneous beliefs. In our model investors perfectly observe the level, slope, and curvature of the yield curve but receive different private signals about the long-run level of rates. Our model fits yields and survey forecasts of future short rates very well. It generates sizable movements in the relative wealth ratio and implies subjective and objective term premiums which are in line with other estimates. The model further implies reasonable Sharpe ratios. We use the model to show that a sizable fraction of the variation of term premiums is disagreement-driven.

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Appendix A Proofs of Propositions

A.1 Proposition 1

To replicate the price dynamics in the heterogeneous-investor economy, the representative investor's stochastic discount factor should be the same as that of investors in each group (say, group-A) after adjusting for the difference in the probability measures. Let η_t^R denote the change of measure from the representative agent's measure to group-A investors' measure, then the representative investor's real pricing kernel m_t^R should have the following property in any future state:

$$m_t^A = \eta_t^R m_t^R, \tag{A.1}$$

where $m_t^A = (1+\rho)^{-t}\frac{1}{D_t^{A\gamma}}$ and $m_t^R = (1+\rho)^{-t}\frac{1}{D_t^{R\gamma}}$ with the market clearing condition $c_t^A + c_t^B = D_t^A + D_t^B = D_t^R = D_t$. Note that similar to Equation (A.1), we have the following equation for the nominal pricing kernel:

$$M_t^A = \eta_t^R M_t^R, \tag{A.2}$$

where $M_t^A = m_t^A/\Pi_t$, $M_t^R = m_t^R/\Pi_t$.

Equation (A.2) implies $u'(c_t^A) = \eta_t^R u'(c_t^R)$. With power utility, we can show that $\eta_t^R = \frac{\varsigma_A}{\varsigma_R}\frac{M_t^A}{M_t^R} = \frac{c_t^{R\gamma}}{c_t^{A\gamma}}$, and similarly, $\eta_A^B(t) = \frac{\varsigma_A}{\varsigma_B}\frac{M_t^A}{M_t^B} = \frac{c_t^{B\gamma}}{c_t^{A\gamma}}$, where $\varsigma_i, i = R, A, B$ are initial conditions. By substituting these into $c_t^R = c_t^A + c_t^B$ and after some algebra, we obtain

$$\eta^R(t) = (\eta_A^B(t)^{\frac{1}{\gamma}} + 1)^{\gamma}. \tag{A.3}$$

With first order approximations, it further implies

$$
\begin{aligned}
\ln(\frac{\eta^R(t+1)}{\eta^R(t)}) &= \gamma \ln(\frac{\eta_A^B(t+1)^{\frac{1}{\gamma}}+1}{\eta_A^B(t)^{\frac{1}{\gamma}}+1}) \\
&\approx \gamma \frac{(\eta_A^B(t+1)^{\frac{1}{\gamma}}+1)-(\eta_A^B(t)^{\frac{1}{\gamma}}+1)}{\eta_A^B(t)^{\frac{1}{\gamma}}+1} \\
&= \gamma \frac{\eta_A^B(t+1)^{\frac{1}{\gamma}}-\eta_A^B(t)^{\frac{1}{\gamma}}}{\eta_A^B(t)^{\frac{1}{\gamma}}} \frac{\eta_A^B(t)^{\frac{1}{\gamma}}}{\eta_A^B(t)^{\frac{1}{\gamma}}+1} \\
&\approx \gamma \ln(\frac{\eta_A^B(t+1)^{\frac{1}{\gamma}}}{\eta_A^B(t)^{\frac{1}{\gamma}}}) \frac{\eta_A^B(t)^{\frac{1}{\gamma}}}{\eta_A^B(t)^{\frac{1}{\gamma}}+1} \\
&= \ln(\frac{\eta_A^B(t+1)}{\eta_A^B(t)}) \frac{\eta_A^B(t)^{\frac{1}{\gamma}}}{\eta_A^B(t)^{\frac{1}{\gamma}}+1},
\end{aligned} \tag{A.4}
$$

From Equation (4.3) we have:

$$
\ln(\frac{\eta_A^B(t+1)}{\eta_A^B(t)}) = \ln(\frac{M_{t+1}^A}{M_t^A}) - \ln(\frac{M_{t+1}^B}{M_t^B}). \tag{A.5}
$$

Similarly, we can obtain

$$
\ln(\frac{\eta^R(t+1)}{\eta^R(t)}) = \ln(\frac{M_{t+1}^A}{M_t^A}) - \ln(\frac{M_{t+1}^R}{M_t^R}). \tag{A.6}
$$

Using Equations (A.4), (A.5) and (A.6), we easily get Approximation (3.15) in Proposition 1, where $w_t^A = \frac{1}{\eta_A^B(t)^{\frac{1}{\gamma}}+1}$ and $w_t^B = 1 - w_t^A$. It is trivial to prove the approximation is exact in continuous time limit.

Lemma 1. *If X_T is a random variable to be realized at time $T > t$ and $E_t^B[X_T] < \infty$, then $E_t^B[X_T] = E_t^A[\frac{\eta_A^B(T)}{\eta_A^B(t)} X_T]$.*

For any random variable X_T with $E_t^B[X_T] < \infty$, we can define $Y_T = \frac{u'(c_t^B)}{u'(c_T^B)} X_T$. Suppose there is a financial security that is a claim to the cash flow Y_T. Then group-B investors' valuation of this security is

$$
E_t^B[e^{-\rho(T-t)} \frac{u'(c_T^B)}{u'(c_t^B)} Y_T] = e^{-\rho(T-t)} E_t^B[\frac{u'(c_T^B)}{u'(c_t^B)} Y_T] = e^{-\rho(T-t)} E_t^B[X_T].
$$

Similarly, group-A investors' valuation of this security is

$$E_t^A[e^{-\rho(T-t)}\frac{u'(c_T^A)}{u'(c_t^A)}Y_T] = e^{-\rho(T-t)}E_t^B[\frac{u'(c_T^A)}{u'(c_t^A)}Y_T]$$
$$= e^{-\rho(T-t)}E_t^B[\frac{u'(c_T^A)/u'(c_T^B)}{u'(c_t^A)/u'(c_t^B)}X_T]$$
$$= e^{-\rho(T-t)}E_t^B[\frac{\eta_A^B(T)}{\eta_A^B(t)}X_T].$$

where the last equality follows from Equation (3.5). Since group-A and group-B investors should have the same security valuation in equilibrium, we must have

$$E_t^B[X_T] = E_t^A[\frac{\eta_A^B(T)}{\eta_A^B(t)}X_T].$$

This lemma shows that the wealth ratio between the two groups can also act as the Radon-Nikodym derivative between the two groups' probability measures. Extending this lemma, we can now prove that *with power utility, the equilibrium bond price is a weighted average of artificial bond prices.* When $\gamma = 1$, the equilibrium price is a weighted average of two groups' beliefs in hypothetical economies, which is shown below. when $\gamma > 1$ (but being an integer), the equilibrium price is still a weighted average of artificial bond prices after a binomial expansion, but the expression is more complicated; interested readers can refer to Ehling et al. (2016) for details.

An example with log utility The time-t nominal price of an asset, which provides a single nominal payoff X_T at time T, is given by

$$P_X(t) = p_t E_t^A[e^{-\rho(T-t)}\frac{u'(c_T^A)}{u'(c_t^A)}\frac{X_T}{P_T}] = p_t E_t^A[e^{-\rho(T-t)}\frac{c_t^A}{c_T^A}\frac{X_T}{P_T}].$$

With the market clearing condition $c_t^A + c_t^B = D_t$ and after some algebra, we have

$$c_t^A = \frac{1}{1+\eta_A^B(t)}D_t, \quad c_t^B = \frac{\eta_A^B(t)}{1+\eta_A^B(t)}D_t,$$

which leads to

$$\begin{aligned}
P_X(t) &= p_t E_t^A[e^{-\rho(T-t)}\frac{c_t^A}{c_T^A}\frac{X_T}{P_T}] \\
&= p_t E_t^A[e^{-\rho(T-t)}\frac{1+\eta_A^B(T)}{1+\eta_A^B(t)}\frac{D_t}{D_T}\frac{X_T}{P_T}] \\
&= \frac{1}{1+\eta_A^B(t)} p_t E_t^A[e^{-\rho(T-t)}\frac{D_t}{D_T}\frac{X_T}{P_T}] + \frac{\eta_A^B(t)}{1+\eta_A^B(t)} p_t E_t^A[e^{-\rho(T-t)}\frac{\eta_A^B(T)}{\eta_A^B(t)}\frac{D_t}{D_T}\frac{X_T}{P_T}] \\
&= \frac{1}{1+\eta_A^B(t)} p_t E_t^A[e^{-\rho(T-t)}\frac{D_t}{D_T}\frac{X_T}{P_T}] + \frac{\eta_A^B(t)}{1+\eta_A^B(t)} p_t E_t^B[e^{-\rho(T-t)}\frac{D_t}{D_T}\frac{X_T}{P_T}].
\end{aligned}$$

Therefore, with log utility, the equilibrium bond price is a weighted average of two groups' beliefs in hypothetical economies, i.e., $P_X(t) = w_t^A P_X^A(t) + w_t^B P_X^B(t)$, where w_t^i is the time-t wealth share of group-i investors, and $P_X^i(t)$ is the nominal price of the asset in a hypothetical economy, in which only group-i investors are present.

A.2 Proposition 2

Proposition 2 is a direct application of Proposition 1. The proof of Proposition 1 implies

$$\ln(\frac{M_{t+h}^R}{M_t^R}) \approx w_t^A \ln(\frac{M_{t+h}^A}{M_t^A}) + w_t^B \ln(\frac{M_{t+h}^B}{M_t^B}), \tag{A.7}$$

where $w_t^A = \frac{1}{\eta_A^B(t)^{\frac{1}{\gamma}}+1}$ and $w_t^B = 1 - w_t^A$. This holds for any states of market prices of risk. By setting market prices of risk of two groups' investors to zero, the future payoff is discounted by the expectation of future short-term interest rates. In this case, the pricing kernels become corresponding risk-neutral rates, i.e., forecasts of (average) future short-term interest rates. The representative agent's risk-neutral rates are weighted averages of two group's beliefs, as we have the following relationship for pricing kernels:

$$\ln(e^{r_t+E_t^R[r_{t+1}]}) \approx w_t^A \ln(e^{r_t+E_t^A[r_{t+1}]}) + w_t^B \ln(e^{r_t+E_t^B[r_{t+1}]}), \tag{A.8}$$

which directly leads to Equation (3.16).

A.3 Proposition 3

The heuristics are straightforward. With the initial condition Equation (3.19), we need to prove that the law of motion of $\frac{1}{2}(\hat{\mu}_t^A + \hat{\mu}_t^B)$ tracks the law of motion of $\hat{\mu}_t^M$. Basd on the specification in Equations (3.12), (3.11) and (3.13) and condition (3.18), we can show that $\frac{1}{2}(\Delta\hat{\mu}_t^A + \Delta\hat{\mu}_t^B) = \Delta\hat{\mu}_t^M$, where Δ is the forward difference operator and $\Delta\hat{\mu}_t^M$ is following the Bayesian updating process specified in Equation (3.8). In words, in an economy with heterogeneous beliefs, the agent is following the same Bayesian updating process as the representative agent in a homogeneous economy, with learning gains scaled by $\sqrt{1-\phi^2}$. Therefore, we have Proposition 3.

Appendix B Normalization Scheme

To describe the disagreement between two groups of investors, we firstly write down their physical dynamics respectively:

$$\mathcal{P}_t^A = K_{0\mathcal{P},t}^{\mathbb{P},A} + K_{1\mathcal{P}}^{\mathbb{P}}\mathcal{P}_{t-1} + \sqrt{\Sigma_{\mathcal{P}\mathcal{P}}}\varepsilon_t^{\mathbb{P},A}, \tag{B.1}$$

$$\mathcal{P}_t^B = K_{0\mathcal{P},t}^{\mathbb{P},B} + K_{1\mathcal{P}}^{\mathbb{P}}\mathcal{P}_{t-1} + \sqrt{\Sigma_{\mathcal{P}\mathcal{P}}}\varepsilon_t^{\mathbb{P},B}, \tag{B.2}$$

where $\mathcal{P}_t$ are portfolios of yields as pricing factors, $K_{1\mathcal{P}}^{\mathbb{P}}$ and $\Sigma_{\mathcal{P}\mathcal{P}}$ are the coefficient matrix and covariance matrix common to two groups, $K_{0\mathcal{P},t}^{\mathbb{P},i}$ is a vector of time-varying drifts perceived by group-i, and $\varepsilon_t^{\mathbb{P},A}, \varepsilon_t^{\mathbb{P},B}$ are standard Gaussian innovations.

We can see that investors disagree about the slow-moving drift, which is the only source of disagreement that is related to conditional forecasts. Specifically, the disagreement about future short rates and term premium dynamics are both captured by the belief dispersion about drifts. It is worth emphasizing that the drift vector $K_{0\mathcal{P},t}^{\mathbb{P},i}$ in group-i's belief is an explicit linear function of a slow-moving component μ_t^i so that $K_{0\mathcal{P},t}^{\mathbb{P},i} = K_{0\mathcal{P}}^{\mathbb{P}}(\mu_t^i)$. This specification is motivated by Andrade et al. (2016) and Dovern (2015), where they suggest disagreement about an underlying trend is an important ingredient to explaining disagreement about the long-term outlook of the policy rate. We will show later a slow-moving component is enough to characterize the disagreement measures at differ short rate forecast horizons.

Our term structure model in the representative agent's belief can be written as:

$$\mathcal{P}_t = K_{0\mathcal{P}}^{\mathbb{Q}} + K_{1\mathcal{P}}^{\mathbb{Q}}\mathcal{P}_{t-1} + \sqrt{\Sigma_{\mathcal{P}\mathcal{P}}}\varepsilon_t^{\mathbb{Q}}, \tag{B.3}$$

$$\mathcal{P}_t = K_{0\mathcal{P},t}^{\mathbb{P}} + K_{1\mathcal{P}}^{\mathbb{P}}\mathcal{P}_{t-1} + \sqrt{\Sigma_{\mathcal{P}\mathcal{P}}}\varepsilon_t^{\mathbb{P}}, \tag{B.4}$$

$$r_t = \delta_0 + \delta_1\mathcal{P}_t, \tag{B.5}$$

where risk-neutral parameters $K_{0\mathcal{P}}^{\mathbb{Q}}, K_{1\mathcal{P}}^{\mathbb{Q}}$ and short rate parameters δ_0, δ_1 are determined by a parameter set $\Theta^{\mathbb{Q}} \equiv (\Sigma_{XX}, \lambda^{\mathbb{Q}}, k_{\infty}^{\mathbb{Q}})$, and the drift vector $K_{0\mathcal{P},t}^{\mathbb{P}} = K_{0\mathcal{P}}^{\mathbb{P}}(\mu_t)$ of the objective DGP is controlled by a drifting term μ_t. We use the consensus survey forecasts to estimate a drift μ_t^M as a noisy measure of μ_t, such that $\mu_t = \mu_t^M + \varepsilon_t^M$. For the sake of parsimony, here we employ a normalization scheme proposed by Joslin, Singleton and Zhu (2011) to estimate our model.

Under the normalization scheme of Joslin, Singleton and Zhu (2011), we have a risk-neutral parameter set $\Theta^{\mathbb{Q}} \equiv (\Sigma_{XX}, \lambda^{\mathbb{Q}}, k_{\infty}^{\mathbb{Q}})$. Let $\mathbf{X}_t$ denote a set of risk factors with

$$r_t = 1'\mathbf{X}_t, \tag{B.6}$$

$$\mathbf{X}_{t+1} = C(k_{\infty}^{\mathbb{Q}}) + J(\lambda^{\mathbb{Q}})\mathbf{X}_t + \Sigma_{XX}^{1/2} e_{X,t+1}^{\mathbb{Q}}. \tag{B.7}$$

Joslin, Singleton and Zhu (2011) show that there exists a unique rotation of $\mathbf{X}_t$ so that the factors are portfolios of bond yields:

$$\mathcal{P}_t = v(\Theta^{\mathbb{Q}}, W) + L(\lambda^{\mathbb{Q}}, W)\mathbf{X}_t, \tag{B.8}$$

where W denote weights used to construct factor-mimicking portfolios such that the latent states are portfolios of yields.[14] That is, $\mathcal{P}_t = y_t \cdot W'$. It can be shown that the parameters controlling the risk neutral dynamics ($K_{0\mathcal{P}}^{\mathbb{Q}}, K_{1\mathcal{P}}^{\mathbb{Q}}, \delta_0, \delta_1$ and $\Sigma_{\mathcal{P}\mathcal{P}}$) are all functions of the elements in $\Theta^{\mathbb{Q}}$, with transformations given by

$$K_{0\mathcal{P}}^{\mathbb{Q}} = LC - LJL^{-1}v + v \tag{B.9}$$

$$K_{1\mathcal{P}}^{\mathbb{Q}} = LJL^{-1}, \tag{B.10}$$

$$\delta_0 = -1'L^{-1}v, \tag{B.11}$$

$$\delta_1 = (L^{-1})'1, \tag{B.12}$$

$$\Sigma_{\mathcal{P}\mathcal{P}} = L\Sigma_{XX}L'. \tag{B.13}$$

The physical dynamics can be written in a similar normalized form

$$\mathbf{X}_{t+1} = C(\mu_t) + J(\lambda^{\mathbb{P}})\mathbf{X}_t + \Sigma_{XX}^{1/2}\varepsilon_{t+1}, \tag{B.14}$$

where we impose the economic restrictions in Section 3.3.1 such that the coefficient matrix is diagonal. Applying the same transformations above, it is easy to get $(K_{0\mathcal{P},t}^{\mathbb{P}}, K_{1\mathcal{P}}^{\mathbb{P}})$.

[14] We choose the portfolio weights similar to Duffee (2011*a*) to reduce fitting errors. The portfolios can be interpreted as empirical Level, Slope and Curvature.

Appendix C State-Space Representation

Our model can be written in a state-space representation in terms of principal components $\mathcal{P}_t$

$$\begin{bmatrix} \mathbf{y}_t^o \\ \mathbf{y}_t^{E,A} \\ \mathbf{y}_t^{E,B} \\ \mathbf{y}_t^{E,M} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{\mathcal{P}} \\ \mathbf{A}_t^{E,A} \\ \mathbf{A}_t^{E,B} \\ \mathbf{A}_t^{E,M} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^{\mathcal{P}} \\ \mathbf{B}^{E,A} \\ \mathbf{B}^{E,B} \\ \mathbf{B}^{E,M} \end{bmatrix} \times \mathcal{P}_t + e_t, \tag{C.1}$$

$$\mathcal{P}_{t+1} = K_{0\mathcal{P},t}^{\mathbb{P}} + K_{1\mathcal{P}}^{\mathbb{P}} \mathcal{P}_t + \sqrt{\Sigma_t^{\mathcal{P}\mathcal{P}}} \varepsilon_t^{\mathbb{P}}, \tag{C.2}$$

where $\mathbf{y}_t^o$ is a vector of zero coupon yields, $\mathbf{y}_t^{E,i}, i = A, B, M$ is a vector of group-i's survey forecasts of future short rates,[15] $\mathbf{A}^{\mathcal{P}}$ and $\mathbf{B}^{\mathcal{P}}$ are explicit functions of $\Theta^{\mathbb{Q}} \equiv (\Sigma_{XX}, \lambda^{\mathbb{Q}}, k_{\infty}^{\mathbb{Q}})$, and $(\mathbf{A}_t^{E,i}, \mathbf{B}_t^{E,i})$ can be mapped from short rate parameters and group-i's physical dynamics. Note that $[\mathbf{B}^{\mathcal{P}}, \mathbf{B}^{E,A}, \mathbf{B}^{E,B}, \mathbf{B}^{E,A}]'$ is the coefficient matrix that is time-homogeneous, but $\mathbf{A}_t^{E,A} = \mathbf{A}^E(\mu_t^A)$, $\mathbf{A}_t^{E,A} = \mathbf{A}^E(\mu_t^B)$ and $\mathbf{A}_t^{E,M} = \mathbf{A}^E(\mu_t^M)$, as linear functions of slow-moving drifts, are time-varying. We assume the slow-moving component μ_t^i follows a random walk

$$\mu_t^i = \mu_{t-1}^i + u_t^i, \ i = A, B, M. \tag{C.3}$$

Applying the invariant transformations in Joslin, Singleton and Zhu (2011), we rewrite the model in terms of normalized factors $\mathbf{X}_t$. The transition equation is

$$\begin{bmatrix} \mathbf{X}_t \\ \mu_t^A \\ \mu_t^B \\ \mu_t^M \end{bmatrix} = \begin{bmatrix} \boldsymbol{\alpha}^X \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\beta}^X & 0 & 0 & l \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \mathbf{X}_{t-1} \\ \mu_{t-1}^A \\ \mu_{t-1}^B \\ \mu_{t-1}^M \end{bmatrix} + u_t. \tag{C.4}$$

For description we assume there are three pricing factors and the slow-moving components are one-dimensional, and then the dynamics about $\mathbf{X}_t$ in the representative agent's belief are

[15] Specifically, $\mathbf{y}_t^{E,M}$ is a vector of consensus survey forecasts of future short rates, while $\mathbf{y}_t^{E,A}$ and $\mathbf{y}_t^{E,B}$ are top- and bottom-10 average survey forecasts.

given by

$$\begin{bmatrix} X_t^L \\ X_t^S \\ X_t^C \end{bmatrix} = \begin{bmatrix} 0 \\ \alpha^S \\ \alpha^C \end{bmatrix} + \begin{bmatrix} \beta^L & 0 & 0 & (1-\beta^L) \\ 0 & \beta^S & 0 & 0 \\ 0 & 0 & \beta^C & 0 \end{bmatrix} \times \begin{bmatrix} X_{t-1}^L \\ X_{t-1}^S \\ X_{t-1}^C \\ \mu_{t-1}^M \end{bmatrix} + u_t^X. \tag{C.5}$$

In this case, $C(\mu_t) = [(1-\beta^L)\mu_t, \alpha^S, \alpha^C]'$, $\boldsymbol{\beta}^X = J(\lambda^{\mathbb{P}}) = \text{diag}(\beta^L, \beta^S, \beta^C)$, and $\boldsymbol{\alpha}_t^{X,i} = C(\mu_t^i) = [(1-\beta^L)\mu_t^i, \alpha^S, \alpha^C]'$, $i = A, B, M$. With Equation (B.8) we can have the physical dynamics of the unnormalized factors $\mathcal{P}_t$. Note that with this parameterization the slow-moving component μ_t enters the system as a drift of the factor X_t^L.

The measurement equation is

$$\begin{bmatrix} \mathbf{y}_t^o \\ \mathbf{y}_t^{E,A} \\ \mathbf{y}_t^{E,B} \\ \mathbf{y}_t^{E,M} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^X \\ \mathbf{A}_X^{E,A} \\ \mathbf{A}_X^{E,B} \\ \mathbf{A}_X^{E,M} \end{bmatrix} + \begin{bmatrix} \mathbf{B}^X & 0 & 0 & 0 \\ \mathbf{B}_X^{E,A} & B_\mu^{E,A} & 0 & 0 \\ \mathbf{B}_X^{E,B} & 0 & B_\mu^{E,B} & 0 \\ \mathbf{B}_X^{E,M} & 0 & 0 & B_\mu^{E,M} \end{bmatrix} \times \begin{bmatrix} \mathbf{X}_t \\ \mu_t^A \\ \mu_t^B \\ \mu_t^M \end{bmatrix} + e_t. \tag{C.6}$$

Firstly, we have $A_n^X = -\frac{1}{\mathcal{A}_n}, B_n^X = -\frac{1}{\mathcal{B}_n}$, where $A_n^X \in \mathbf{A}^X, B_n^X \in \mathbf{B}^X$. $\mathcal{A}_n$ and $\mathcal{B}_n$ can be obtained from the recursions:

$$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{B}_{n-1}' C(k_\infty^{\mathbb{Q}}) + \frac{1}{2}\mathcal{B}_{n-1}' \Sigma_{XX} \mathcal{B}_{n-1}, \tag{C.7}$$

$$\mathcal{B}_n' = \mathcal{B}_{n-1}' J(\lambda^{\mathbb{Q}}) - \mathbf{1}', \tag{C.8}$$

$$\mathcal{A}_0 = 0, \quad \mathcal{B}_0' = \mathbf{0}. \tag{C.9}$$

Note that with the above recursions and the equation $\mathcal{P}_t = y_t \cdot W'$ we are able to obtain the parameters in the mapping $\mathcal{P}_t = v(\Theta^{\mathbb{Q}}, W) + L(\lambda^{\mathbb{Q}}, W)\mathbf{X}_t$.

We can rewrite the physical dynamics of group-i investors' belief as $\mathbf{X}_t^i = \boldsymbol{\beta}_i \mathbf{X}_{t-1}^i + u_t^i$, where $\mathbf{X}_t^i$ is a vector of states, and the constant term is omitted for ease of presentation. Then the h-period-ahead factor forecasts are given by $E_t^i[\mathbf{X}_{t+h}^i] = \boldsymbol{\beta}_i^h \mathbf{X}_t^i$. We thus have $E_t^i[\mathbf{y}_{t+h}^{(1)}] = \text{e}_i E_t^i[\mathbf{X}_{t+h}^i] = \text{e}_i \boldsymbol{\beta}_i^h \mathbf{X}_t^i$, where e_i denotes a selection vector with ones in the entries corresponding to pricing factors and zeros elsewhere.

Appendix D Additional Results

Table 1: Model Parameter Estimates

$k_\infty^{\mathbb{Q}}$	2.59E-04	(7.16E-5)				
	X^L		X^S		X^C	
$\lambda^{\mathbb{Q}}$	0.000	(1.34E-6)	-0.033	(1.46E-2)	-0.087	(2.22E-2)
diag(Σ_{XX})	1.28E-05	(6.63E-6)	2.88E-05	(1.40E-5)	3.52E-05	(1.87E-5)
$\boldsymbol{\alpha}^X$			4.34E-04	(2.30E-4)	2.00E-04	(7.98E-5)
$\boldsymbol{\beta}^X$	0.922	(2.80E-2)	0.962	(1.38E-2)	0.891	(2.70E-2)
	A		B		M	
σ_{short}	1.39E-03	(6.59E-4)	1.08E-03	(5.02E-4)	9.58E-04	(3.45E-4)
σ_{long}	2.09E-03	(8.25E-4)	1.62E-03	(9.81E-4)	1.44E-03	(9.75E-4)
σ_y	9.96E-04	(3.11E-4)				
chol(Σ_μ)	1.62E-08	(4.45E-7)	-1.78E-03	(6.39E-4)	3.53E-03	(3.56E-4)
			2.85E-06	(1.53E-6)	2.02E-03	(8.69E-4)
					8.04E-06	(6.82E-6)

Notes: This table reports parameter estimates for our affine term structure model. The sample period is 1983:03-2015:08, and standard errors are reported in parentheses. σ_y is the standard deviation of bond yield observational errors. σ_{short} and σ_{long} denote observational error standard deviations of short-horizon forecasts (less than one year) and long-horizon forecasts, respectively. M, A and B denote respective beliefs of the medium forecaster, top-10 group and bottom-10 group. Other parameters are defined in Appendix B.