The Value of Scattered Information

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We analyze a model in which the value of a security is comprised of multiple distinct parts and private information about these pieces is scattered among investors. We show that as information is scattered into smaller, distinctively informative pieces, endogenous information acquisition activity can increase, even if the acquisition cost does not decrease. Our paper generalizes Grossman-Stiglitz (1980) for an arbitrary number of distinct pieces of information and demonstrates that, when information is scattered among investors, information free-riding can be alleviated. Our model also provides new insights on information markets with an information monopolist.

KEYWORDS: asymmetric information; multiple dimensions of uncertainty; information acquisition; information monopolist.

JEL CLASSIFICATIONS: D82, G14.

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I. Introduction

A fundamental principle in finance is that the price of a security is driven by the value of underlying assets. However, there are often many distinct pieces that comprise the value of a financial security, such as the value associated with outcomes in different locations (e.g., for a multinational), or different industries (e.g., for a conglomerate), or the supply and demand of different products and technologies, and so on. Thus investors—in different locations, with industry-specific knowledge, or specific product familiarity—will learn about distinct pieces of information about a security's value and, in the process, information becomes *scattered*. It's not simply that an investor acquires imprecise information or that information about one underlying asset is dispersed; rather, the piece of information acquired is wholly distinct from what others learn about, whatever the precision of information may be. As a growing literature demonstrates, this scattering of information is natural and pervasive in modern financial markets[1], especially in an era of increasing corporate diversification and globalization, but our understanding of the foundations of its impact on asset prices has been very limited. What is the value of learning about something distinctly informative when information is scattered among the trading population but traders can infer some of the collective wisdom of other traders from the price of the security? How does this value change as information is more widely scattered into smaller and smaller pieces?

To answer these questions, we generalize the seminal model of Grossman and Stiglitz (1980), in a novel way, from having one piece of uncertainty in the asset value that investors can learn about to having an arbitrary number of n distinct pieces. Our main finding is the surprising discovery that as information is scattered into smaller and smaller distinctly informative pieces, even if the acquisition cost per piece does not decrease, the overall endogenous information acquisition activity in the economy can actually *increase*. This re-

[1]Value-relevant firm-specific information is fragmented both geographically [García and Norli (2012), Bernile, Kumar, and Sulaeman (2015)] and across industries [Menzly and Ozbas (2010)]. A large empirical and theoretical literature demonstrates that information is slowly reflected in prices and segmented information is implicit in many of these studies. So, evidence of slow aggregation tends to be evidence of fragmented information.

lationship holds despite the fact that the informativeness of each piece and of the market price continually *decreases*. These findings have new implications for the study of markets for information. We show that a monopolistic information seller may prefer to scatter information among the investor population and, in doing so, indirectly alleviate the information free-riding problem in financial markets.

To better understand the type of information structures and trading environment captured by our model and the novelty of our results, consider the following example. Suppose that there is a farm for sale with multiple growing locations. Trading opens today, but there is an opportunity for investors to learn more about the farm. Suppose that strategic investors for effort cost c can learn about the value of *one-half* of the farm's units. For example, if the farm has four distinct growing locations (say A, B, C, and D), investors could fully inspect either the first two (A and B) or the second two (C and D) of those locations, for cost c. About half of the investors who choose to become privately-informed will learn fully about the first two locations while the other half will learn fully about the other two locations. Hence, information about the total asset value will be scattered into $n = 2$ distinctly informative pieces ($\{A, B\}$ or $\{C, D\}$) among those investors who choose to become informed. Naturally, these privately-informed investors will have an informational advantage in trading tomorrow relative to investors who did not exert the effort to acquire private information.[2]

Now, hold everything so far fixed except suppose that information is *more scattered* than before, say, into $n = 4$ distinct pieces. That is, investors can only fully inspect one of the four growing locations (only one of A, B, C, or D), *but for the same cost c as before*. In this case, an investor can only learn privately about the value of *one-fourth* of the units. Would an investor who would have marginally chosen to become informed about one-half of the units in the previous case still choose to become informed about only one-fourth of the units in the present case, all else held equal? Surprisingly the answer can be yes. In fact, under mild

[2] In line with Grossman and Stiglitz (1980) assume prices are not fully revealing of all private information. This assumption holds in our formal model.

conditions, such investors will even be willing to pay *more* to observe a smaller piece when information is more finely scattered (i.e., larger n).

What is driving the main result? One would expect that one piece of information, in an economy with a higher degree of scattered information, but obtained at the same cost, would be less useful as it contains less information. However, traders' decisions about becoming informed are not solely based on the informativeness of their signals, but more importantly depend on how much of their information "leaks" to the uninformed via market prices and how much additional profit traders can generate from the signals relative to remaining uninformed.

In the baseline case of Grossman and Stiglitz (1980) (i.e., $n = 1$ in our model), the trade-off in becoming informed is between (1) costly reduction of uncertainty about this one piece and (2) costless partial inference from a noisy price, made possible by information leakage. However, if information is scattered into several pieces—holding the acquisition cost per piece fixed—this baseline trade-off takes on a competing dynamic as n increases: both the *informativeness of each piece* and the *information leakage* to uninformed investors declines in n, at *varying rates*. Leakage of information declines more rapidly for low n than for high n. For example, going from one to two pieces doubles the inference problem for the uninformed while going from 100 to 101 pieces only worsens the inference problem by 1%. The benefit of less information leakage outweighs the reduction in informativeness, resulting in a larger number of informed traders. Eventually, for large enough n, the trade-off reverses until informativeness is so low it is no longer feasibly attractive and nobody acquires information.

Much of economists' understanding of the value of asymmetric information derives from Grossman and Stiglitz (1980), who address the fundamental issue of how costly information acquisition can be supported in financial markets in which the price can reveal some or all of that information. However, this foundational work, and the preponderance of the vast literature that builds upon it, restricts attention to only *one* piece of uncertainty in the asset value that market participants can learn about. A small subset of these studies employ a model with multiple pieces of uncertainty that investors can learn about, including Paul (1993), Subrahmanyam and Titman (1999), Goldman (2005), Yuan (2005), Kondor (2012), and Goldstein and Yang (2015).

Goldstein and Yang (2015) is perhaps the closest related study.[3] However, there are important differences that distinguish our paper. Goldstein and Yang (2015) hold the number of pieces of information n fixed at $n = 2$ and do comparative statics on the information production along those two dimensions. In our paper, investors produce information on one of an arbitrary number of n pieces of information and we do comparative statics with respect to n. Outside of our paper, none have analyzed more than $n = 2$ pieces of uncertainty except for Kondor (2012), who considers $n = 3$ pieces of uncertainty but restricts attention to the case of *two informationally distinct* groups of investors (since one of the three pieces of uncertainty is observed by *both* groups) and does not consider the information acquisition problem. Interestingly, the model of Grossman and Stiglitz (1980) is nested as a special case of our model when $n = 1$. Furthermore, we provide closed-form expressions for the informativeness of the price, the informativeness of each piece of information, the fraction of investors who choose to become informed, and the threshold n^* for the degree of scattered information at which the fraction of investors attains a maximum over all $n \geq 1$.

Our analysis is relevant to the study of information markets in which a financial intermediary sells information to risk-averse investors. In the framework of Grossman and Stiglitz (1980), informed investors learn perfectly about the potentially observable part of the overall asset value. However, researchers have demonstrated that a monopolistic information seller would not necessarily distribute information in this way. Rather, the seller has financial incentive to intentionally alter information about an asset by injecting artificial noise, if such actions are feasible.[4] We provide a novel alternative strategy for the seller that also improves the seller's revenue relative to the mechanism implied by Grossman and

[3]We discuss how some of our modeling choices are informed by their results in Section II.B.
[4]Admati and Pfleiderer (1986); García and Sangiorgi (2011).

Stiglitz (1980). The strategy is to split the information about the value of the asset into precise and distinctly informative pieces, and scatter them among the trading population.

This strategy highlights how our main result—that scattered information may increase overall information acquisition—can alleviate the problem of "information free-riding" in financial markets. This information leakage problem, which is at the heart of the Grossman and Stiglitz (1980) paradox, is that price conveys the information of privately informed investors, which depresses the willingness of investors to produce costly information since investors could instead "free ride" on the information in the market price. Suppose an information monopolist who charges price c for information were to split the available information signal into the sum of two independent pieces and offer to each investor for the same price c only one of these pieces selected at random. Then, under mild conditions, more investors would choose to become informed than under the original case in which the signal is not split. This incentive to split the information indirectly reduces the number of information free-riders. This effect continues with further splitting until some degree beyond which it would reverse.

In Section II, we present the model and our main result. In Section III, we consider the case in which an information monopolist sells scattered information among the investor population. We conclude the text in Section IV. Proofs are deferred to the Appendix A. The online appendix Appendix B contains additional supporting material.

II. Model and Analysis

Our model begins with the simplest case in which investors are endowed with their information, which is naturally scattered among the investor population. Next, we consider the case in which every agent has access to one piece of private information that can be acquired at some cost. In this case, the allocation of information is an equilibrium result of each investor choosing whether or not to exert the effort cost to produce the information. Last, we analyze the value of scattered information and discuss our main result.

II.A. The environment

Assets: There are two assets in the financial market: one risk-free asset (a bond) and one risky asset (a stock). The bond's price is normalized to 1 and its payoff is 1. The stock's price is endogenously determined by market clearing and its payoff is

$$\widetilde{u} = \mu + \widetilde{v} + \widetilde{\varepsilon}, \tag{1}$$

where μ is a constant; $\widetilde{v}$ is the sum of $n \in \{1, 2, \ldots\}$ potentially observable pieces of uncertainty,

$$\widetilde{v} = \widetilde{v}_1 + \widetilde{v}_2 + \cdots + \widetilde{v}_n, \tag{2}$$

which are independent and identically distributed (iid) mean-zero normal random variables each with precision τn, $\tau > 0$, (equivalently, variance $\frac{1}{\tau n}$); and $\widetilde{\varepsilon}$ is unobservable uncertainty, which is a mean-zero normal random variable with precision $\tau_\varepsilon > 0$ that is independent of the n pieces:

$$\widetilde{v}_i \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \frac{1}{\tau n}\right), \quad \widetilde{\varepsilon} \sim \mathcal{N}\left(0, \frac{1}{\tau_\varepsilon}\right), \quad \mathbf{Cov}(\widetilde{v}_i, \widetilde{\varepsilon}) = 0. \tag{3}$$

Equations (1) and (2) show that for larger n, the stock's payoff depends on more pieces of uncertainty. However, (3) ensures that overall uncertainty in the risky asset remains fixed such that $\widetilde{u} \sim \mathcal{N}\left(\mu, \frac{1}{\tau} + \frac{1}{\tau_\varepsilon}\right)$, which does not depend on n. This construction allows us to analyze the implications of multiple pieces of uncertainty without other factors contaminating our results. The supply of the risky asset is $\overline{x} > 0$, which we normalize to $\overline{x} \equiv 1$.

Traders: In this economy, there is a unit continuum of traders, each with constant absolute risk aversion (CARA) utility with risk-aversion parameter $\gamma > 0$ and initial wealth W_0.[5] To prevent fully revealing prices, there are also some liquidity traders (noise traders), whose demand is a random variable $\widetilde{x} \sim \mathcal{N}\left(0, \frac{1}{\tau_x}\right)$, which is independent of $\{\widetilde{v}_i\}_{i=1}^n$ and $\widetilde{\varepsilon}$.

Scattered Information: Fraction $\lambda \in [0, 1]$ of strategic traders are (privately) informed

[5]Our results are independent of W_0, a consequence of CARA utility.

and fraction $1-\lambda$ are (privately) uninformed. We first assume λ is exogenous and later we endogenize it. Each of the λ informed traders are equally likely to perfectly observe precisely one of the n potentially observable pieces of uncertainty. We discuss this model of scattered information in greater detail in Section II.B, including how it can be endogenized naturally. Put differently, for any i, $\frac{\lambda}{n}$ of the continuum are informed about $\widetilde{v}_i$ and uninformed about any other piece. In this sense, information about the stock's payoff is scattered among the informed trading population. For larger n, information is more widely scattered into smaller pieces. Accordingly, we call n the *degree of scattered information*.

Economy: The structure of the economy and all constants are common knowledge.

II.B. Modeling scattered information

We model scattered information by focusing on the case in which each investor observes at most one of the n distinct pieces that contribute to the payoff. A similar restriction is employed in the $n = 2$ settings of Paul (1993) and Goldstein and Yang (2015), who both appeal to "the spirit of Hayek's view that one of the most important functions of the price system is the decentralized aggregation of information and that no one person or institution can process all information relevant to pricing" (Paul, 1993, p. 1477). This restriction abstracts from certain complexities that could characterize access to information, but it retains the core ingredient of limited access that underlies the notion of scattered information.

In the analysis that follows, we focus on the case in which the information is equally split among the n groups of informed investors (i.e., for any i, $\frac{\lambda}{n}$ are informed about $\widetilde{v}_i$ and uninformed about any other piece). We do this for several reasons. First, it can naturally be endogenized. Suppose investors can acquire exactly one of the n pieces of information, each at a common cost. Any investor who chooses to become informed would prefer the piece that the fewest other investors have acquired since it would be the most difficult to learn about via the market price. As each informed investor acquires the smallest piece in circulation, the proportion informed about each piece naturally equalizes across all n pieces. We prove this result in the Online Appendix.

Second, this case provides a *consistent* allocation scheme for comparing aggregate results *across different values of* n versus analyzing varying allocation schemes among pieces for a fixed n. As the number of pieces n changes, the relative allocation of each piece remains uniform across pieces while the overall total volatility of all pieces remains fixed, permitting comparable results across dimension n.

Third, a key component of our analysis is the reduction in price informativeness as n increases. Price informativeness, as well as other quantities of interest, depend on how information is allocated among investors. Goldstein and Yang (2015) show that price informativeness is highest when information is split equally between two groups of informed investors. This case presents the highest incentive for investors to remain uninformed and learn from an informative price—a force operating against greater private information acquisition. By analogy, our focus is on the case in which information is equally split among n groups of informed investors so that price informativeness is high and any implied increase in information acquisition is conservative among the set of possible, more general allocations of each piece of information among investors.

Fourth, an equal split is also the natural result of randomizing which piece of information an investor receives upon choosing to become informed. This randomization is consistent with the reasoning of Subrahmanyam and Titman (1999, p. 1047), that "...when an investor pays to receive information, there is some uncertainty about what he will receive. Two investors expending the same resources on information collection are likely to receive...different signals." One consistent interpretation of our setup is that traders are randomly endowed with information production technology regarding one of the n pieces and can choose to exert effort cost c to employ that technology. Another interpretation is that for cost c, an investor can learn about one piece of information, randomly selected from among the n. For example, if information is scattered geographically, then an investor who chooses to become informed can learn about the most local piece but other pieces are too remote or too costly to learn about. In Section III, we also consider the interpretation that investors acquire information in an information market from an information monopolist who sets the cost c and chooses whether to scatter the information equally among the investor population.

II.C. Equilibrium concept and characterization

As in Grossman and Stiglitz (1980) and related literature, we consider the rational expectations equilibrium (REE). In equilibrium, each strategic trader maximizes expected utility given an information set and the market price is determined by the market-clearing condition. Because of the symmetric nature of each $\widetilde{v}_i$, we can look for the following linear equilibrium that weights the contribution of each piece symmetrically:

$$\widetilde{p} = \alpha\widetilde{v} + \alpha_x\widetilde{x} + \alpha_0,$$

where $\alpha, \alpha_x, \alpha_0$ are left to determine.[6] Let $X_{\text{inf}}(\widetilde{v}_i, \widetilde{p})$ denote the demand of traders informed about $\widetilde{v}_i$ and $X_{\text{uninf}}(\widetilde{p})$ denote the demand of uninformed traders who only condition on price. In a CARA-Normal environment under a linear pricing rule, strategic traders' demands are also linear in their information sets. So the market-clearing condition,

$$\sum_{i=1}^{n} \frac{\lambda}{n} X_{\text{inf}}(\widetilde{v}_i, \widetilde{p}) + (1-\lambda)X_{\text{uninf}}(\widetilde{p}) + \widetilde{x} = 1, \tag{4}$$

which equates aggregate traders' net demand plus liquidity trading with exogenous supply ($\overline{x} \equiv 1$) for every realization of the economy, determines a linear price. Moreover, Proposition 1 shows that this linear equilibrium is unique.

Proposition 1 *There exists a unique symmetric linear REE, in which $\widetilde{p} = \alpha\widetilde{v} + \alpha_x\widetilde{x} + \alpha_0$. The coefficients are given as a function of $(\mu, \lambda, n, \gamma, \tau, \tau_\varepsilon, \tau_x)$ in the appendix.*

[6] The weights on each $\widetilde{v}_i$ are equal by symmetry, so the linear equilibrium form reduces as follows: $\widetilde{p} = \alpha_1\widetilde{v}_1 + \ldots + \alpha_n\widetilde{v}_n + \alpha_x\widetilde{x} + \alpha_0 = \alpha(\widetilde{v}_1 + \ldots + \widetilde{v}_n) + \alpha_x\widetilde{x} + \alpha_0 = \alpha\widetilde{v} + \alpha_x\widetilde{x} + \alpha_0$.

II.D. Trading intensity

Let $I = \frac{\alpha}{\alpha_x}$ denote the *trading intensity* on any piece of information.[7] Proposition 2 characterizes the trading intensity in terms of the parameters of the economy.

Proposition 2 *The trading intensity I varies with the fraction of informed traders $\lambda \in [0,1]$ uniquely in equilibrium according to the following polynomial:*

$$\lambda = n\gamma I\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon \tau} I^2 \left(1 - \frac{1}{n}\right) + \frac{1}{\tau}\left(1 - \frac{1}{n}\right)\right]. \tag{5}$$

Since the right hand side of (5) is an increasing function of I, for an economy with a higher fraction of information traders, trading intensity would be higher, holding all else equal.

II.E. Price informativeness

Price informativeness is defined as the reciprocal of the variance of $\widetilde{u}$ conditional on the price: $\frac{1}{\mathbf{Var}[\tilde{u}|\tilde{p}]}$. This variance captures the residual uncertainty uninformed traders face after learning from the price. Proposition 2 indicates that the trading intensity of privately informed investors impacts the equilibrium price function since the equilibrium price coefficients depend on I: $\alpha = I\alpha_x$. Lemma 3 characterizes the dependence of price informativeness on trading intensity I.

Lemma 3 *Price informativeness is an increasing function of trading intensity I:*

$$\frac{1}{\mathbf{Var}[\tilde{u}|\tilde{p}]} = \frac{1}{\frac{1}{\tau_\varepsilon} + \frac{1}{\tau + \tau_x I^2}}. \tag{6}$$

Thus, an environment with a higher I also has a higher price informativeness.

[7] We call I the *trading intensity* because it can be shown that $I = \frac{\lambda}{n}\frac{\partial}{\partial \widetilde{v}_i} X_{\text{inf}}(\widetilde{v}_i, p)$. Because there are $\frac{\lambda}{n}$ of strategic traders informed about $\widetilde{v}_i$ and a unit increase in $\widetilde{v}_i$ causes a $\widetilde{v}_i$-informed investor to trade $\frac{\partial}{\partial \widetilde{v}_i} X_{\text{inf}}(\widetilde{v}_i, p)$ more stock, the impact for the group would be $\frac{\lambda}{n}\frac{\partial}{\partial \widetilde{v}_i} X_{\text{inf}}(\widetilde{v}_i, p)$.

II.F. Endogenous information acquisition equilibrium

So far, we have assumed that informed agents were exogenously determined. For the remainder of the paper, we consider the situation in which traders must choose between becoming informed or not before trading takes place. Specifically, we consider the case in which traders learn about exactly one piece $\widetilde{v}_i$, chosen at random among the n, for an effort cost c. Interpretation and discussion of this set-up is in Section II.B.

Our first result, Lemma 4, concerns the willingness to pay to become informed. By Proposition 1, the trading game has a unique linear REE, so we can calculate the ex-ante utilities that an uninformed and informed trader expect to obtain. The willingness to pay is the cost that makes a trader indifferent between becoming informed or remaining uninformed.

Lemma 4 *Given $(\lambda, n, \gamma, \tau, \tau_\varepsilon, \tau_x)$, the willingness to pay to become informed is given by*

$$\frac{1}{2\gamma}\log\frac{\mathbf{Var}[\widetilde{u}|\widetilde{p}]}{\mathbf{Var}[\widetilde{u}|\widetilde{v}_i,\widetilde{p}]} = \frac{1}{2\gamma}\log\frac{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}}{\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}}. \tag{7}$$

Lemma 4 shows that the willingness to pay is proportional to, in log scale, the relative improvement in the precision of prediction of the asset value from obtaining a piece of private information. Recall that an uninformed trader's uncertainty about the risky asset is given by the variance of $\widetilde{v}$ conditional on the price. An informed trader who knows about $\widetilde{v}_i$ can improve this prediction, in which case, the uncertainty about the risky asset is given by the variance of $\widetilde{v}$ conditional on the trader's information and price. By becoming informed, a trader benefits from a more precise prediction.

We now turn around the situation in Lemma 4 and consider the cost of acquiring information as exogenous. Given a cost of acquiring information, some traders choose to become informed and some do not. After each trader's information acquisition decision, the exogenous trading game considered preivously takes place. Proposition 5 shows that, for any parameters, we can identify the unique fraction of traders who become informed.

Proposition 5 *For any parameters $(\mu, c, n, \gamma, \tau, \tau_\varepsilon, \tau_x)$, there exists a unique information market equilibrium in which there are $\lambda \in [0,1]$ fraction of traders who choose to become informed, where*

$$\lambda = \begin{cases} 1, & 0 < c < \underline{c}, \\ \widehat{\lambda} := n\gamma \widehat{I}\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon \tau}\widehat{I}^2\left(1-\frac{1}{n}\right) + \frac{1}{\tau}\left(1-\frac{1}{n}\right)\right], & \underline{c} \le c \le \overline{c}, \\ 0, & c > \overline{c}, \end{cases} \tag{8}$$

and trading intensity

$$I = \begin{cases} \overline{I}, & 0 < c < \underline{c}, \\ \widehat{I} := \sqrt{\frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c} - [\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}} - \frac{n\tau}{2\tau_x(n-1)} - \frac{\tau+\tau_\varepsilon}{2\tau_x}}, & \underline{c} \le c \le \overline{c}, \\ 0, & c > \overline{c}, \end{cases} \tag{9}$$

where $\overline{I} > 0$ and $0 < \underline{c} < \overline{c}$ are positive constants with closed-form expressions in terms of the parameters $(n, \gamma, \tau, \tau_\varepsilon, \tau_x)$ given in the appendix.[8]

We provide a sketch of the proof and defer the details to the appendix. First, in equilibrium, the cost c must be equal to the willingness to pay to be informed as in (7) of Lemma 4. So, I must satisfy

$$c = \frac{1}{2\gamma}\log\frac{\frac{1}{\tau_\varepsilon} + \frac{1}{\tau+\tau_x I^2}}{\frac{1}{\tau_\varepsilon} + \frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}} \tag{10}$$

for $c > 0$.[9] Isolating the non-negative root of I^2 in (10) leads to the closed-form expression

$$\widehat{I} := \sqrt{\frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c} - [\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}} - \frac{n\tau}{2\tau_x(n-1)} - \frac{\tau+\tau_\varepsilon}{2\tau_x}}, \tag{11}$$

which is a strictly decreasing function of c (with $\lim_{c\downarrow 0}\widehat{I} = +\infty$) that decreases in c until it reaches $\widehat{I} = 0$ at positive constant $\overline{c}$.[10] By (5) of Proposition 2, trading intensity of $\widehat{I}$ implies

[8] If $n = 1$, the expressions for $\widehat{\lambda}$ in (8) and $\widehat{I}$ in (9) are not well defined. So when $n = 1$, we use the continuous extension by taking the limit as $n \to 1$, in which case $\widehat{I} = \sqrt{\frac{\tau_\varepsilon}{\tau_x(e^{2\gamma c}-1)} - \frac{\tau}{\tau_x}}$ and $\widehat{\lambda} = \frac{\gamma}{\tau_\varepsilon}\sqrt{\frac{\tau_\varepsilon}{\tau_x(e^{2\gamma c}-1)} - \frac{\tau}{\tau_x}}$.

[9] All traders would choose to become informed in the trival case in which $c = 0$ so we restrict attention to positive acquisition cost $c > 0$.

[10] Technically, beyond cost $\overline{c}$, the root is imaginary. However, if c is too large, then no trader chooses to become

an informed trading population fraction of $\widehat{\lambda} := n\gamma\widehat{I}\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon\tau}\widehat{I}^2\left(1-\frac{1}{n}\right) + \frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]$. But (5) implies λ is strictly increasing in I and imposes an upper bound on I such that λ does not exceed 1. So, for sufficiently small c ($0 < c \leq \underline{c}$), I is a capped version of $\widehat{I}$ with upper bound $\overline{I}$ such that $\lambda = \widehat{\lambda} = 1$ at $I = \widehat{I} = \overline{I}$.

II.G. The value of scattered information

Having established the unique information acquisition equilibrium, we can now study the effect of scattered information. As we have shown, given any parameters $(\mu, c, n, \gamma, \tau, \tau_\varepsilon, \tau_x)$, we can find the fraction of informed traders λ. We also found the trading intensity I (and hence price informativeness by Lemma 3), for given cost $c > 0$ of acquiring a piece of information. Now, we analyze how information acquisition and trading intensity vary with n, the degree of scattered information.

Our first results, Proposition 6 and Corollary 7, concern the effect of scattered information on trading intensity and price informativeness, respectively.

Proposition 6 *For given $(\mu, c, \gamma, \tau, \tau_\varepsilon, \tau_x)$, trading intensity I is strictly decreasing in the degree of scattered information n for $c < \overline{c}$, and otherwise flat at level $I = 0$.*

Proposition 6 shows that the higher the degree of scattered information, the lower the intensity of trading on each piece of private information. This result is intuitive because each piece of information is less informative about the stock's value, so less uncertainty is resolved by becoming informed and hence informed traders respond less intensely to their private signals.

Corollary 7 *For given $(\mu, c, \gamma, \tau, \tau_\varepsilon, \tau_x)$, price informativeness is decreasing in the degree of scattered information n for $c < \overline{c}$, and otherwise flat at level $\frac{1}{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau}}$.*

Corollary 7 shows that the informativeness of the price also decreases as the degree of scattered information increases. This is intuitive because the decreased information content

informed and so no information gets embedded into the price. Thus, $I = 0$ for $c \geq \overline{c}$ and hence imaginary roots are not applicable.

of price comes from the decrease in trading intensity of informed traders, which impounds less information into the market price. So in parallel to Proposition 6, the trading response of uninformed traders also decreases as the degree of scattered information increases because uninformed traders face greater uncertainty in the stock's value.

As indicated in Proposition 6, trading intensity decreases as information is split into smaller pieces scattered among the investor population. This result alone might seem to indicate that information acquisition would also decrease. However, Corollary 7 indicates that the uninformed trader also obtains less information and therefore, might benefit from now acquiring private information. Our next results show the net effect of these two considerations on information acquisition.

Proposition 8 *For given $(\mu, c, \gamma, \tau, \tau_\varepsilon, \tau_x)$, let*

$$\widehat{c} := \frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{2\tau}\right), \tag{12}$$

$$n^* := \frac{\tau_\varepsilon + 2(e^{2\gamma c}-1)(\tau_\varepsilon+\tau)}{2(e^{2\gamma c}-1)(\tau_\varepsilon+2\tau)}, \tag{13}$$

$$\overline{n} := \frac{\tau_\varepsilon e^{2\gamma c}}{(\tau_\varepsilon+\tau)(e^{2\gamma c}-1)}. \tag{14}$$

If $c < \widehat{c}$, then

1. $1 < n^* < \overline{n}$;
2. $\widehat{\lambda}$ *is strictly increasing on* $n \in [1, n^*)$;
3. $\widehat{\lambda}$ *is strictly decreasing on* $n \in (n^*, \overline{n})$.

If $c \geq \widehat{c}$, then $\widehat{\lambda}$ is non-increasing on $n \in [1, \overline{n}]$.

Except for the extreme cases of too low or too high information acquisition costs, $\lambda = \widehat{\lambda}$. Accordingly, Proposition 8 focuses on how $\widehat{\lambda}$ varies with n. Our next results will explicitly handle the trivial cases of extreme costs and carry the results of Proposition 8 over to λ itself.

The first thing to notice in Proposition 8 is that if the cost c is not too large ($c < \widehat{c}$)—so that, for example, investors are potentially interested in acquiring information—then for low to moderate values of n, overall information acquisition in the economy actually *increases* as information becomes more scattered. Then beyond some degree of scattering n^* the effect reverses and information acquisition decreases. Subsequent results will clarify behavior beyond $\overline{n}$ and for cases of extreme costs.

Lemma 9 *For given* $(\mu, c, \gamma, \tau, \tau_\varepsilon, \tau_x)$*, for all* $c > 0$ *there exists a* $\widehat{\tau_x} > 0$ *such that* $c > \underline{c}$ *for all* $n \geq 1$ *if* $\tau_x > \widehat{\tau_x}$*. If* $\tau_x > \tau \frac{\gamma^2}{{\tau_\varepsilon}^2}$*, then* $\underline{c} < \widehat{c}$ *for all* $n \geq 1$*.*

Lemma 9 addresses the situation of acquisition costs that are too low. The problem is that if the cost is too low, then we have the trivial situation in which all investors choose to become informed ($\lambda = 1$), so that there is no sensitivity to n. Lemma 9 shows that there are many ways to avoid the trivial condition of costs being too low. The cost per piece is not too small if any of the following are true: (1) liquidity trading is not too volatile (τ_x sufficiently large), or (2) unobservable uncertainty in the stock's payoff is not too large (τ_ε sufficiently large), or (3) potentially observable uncertainty in the stock's payoff is sufficiently large (τ sufficiently small), or (4) the appetite for risk is sufficiently high (γ sufficiently small). Lemma 9 also shows that largest value at which the cost reaches this trivial state can be made arbitrarily small by applying any combination of these four conditions.

With Lemma 9, we can extend Proposition 8 to a obtain our main result, Theorem 10.

Theorem 10 *For given* $(\mu, c, \gamma, \tau, \tau_\varepsilon, \tau_x)$*, if* $\tau_x > \tau \frac{\gamma^2}{{\tau_\varepsilon}^2}$*, then for any* $c \in (\underline{c}, \widehat{c})$*,*

1. $1 < n^* < \overline{n}$*;*
2. λ *is strictly increasing on* $n \in [1, n^*)$*;*
3. λ *is strictly decreasing on* $n \in (n^*, \overline{n})$*; and*
4. $\lambda = 0$ *for* $n \geq \overline{n}$*.*

If $c \geq \widehat{c}$, then λ is non-increasing on all $n \geq 1$.

Holding the cost c of acquiring one piece of information fixed, the condition $c < \widehat{c}$ in Theorem 10 specifies that the cost is not too large, and $c > \underline{c}$ specifies that the cost is not too small to avoid the trivial case of full participation. The condition $\tau_x > \tau \frac{\gamma^2}{\tau_\varepsilon{}^2}$ guarantees that this lower bound on cost is sufficiently low. Under these mild conditions, the degree of scattered information at which information participation is largest is $n^* > 1$ [Theorem 10.1]. Hence, the fully concentrated case $n = 1$—the classic Grossman and Stiglitz (1980) setting—does not entail the largest participation.

Theorem 10.2 shows surprisingly that for an increase in the degree of scattered information $n < n^*$, more traders want to become informed! This is surprising because one would expect that one piece of information, in an economy with a higher degree of scattered information, but obtained at the same cost, would be less useful as it contains less information. However, traders' decisions about becoming informed are not solely based the informativeness of their signals, but more importantly depend on how much of their information leaks to the uninformed via market prices and how much additional profit traders can generate from the signals relative to remaining uninformed.

In the baseline case of Grossman and Stiglitz (1980) (i.e., $n = 1$ in our model), all privately-informed investors learn about a single common piece of uncertainty in the payoff. The trade-off in becoming informed is between (1) costly reduction of uncertainty about this one piece and (2) costless partial inference from a noisy price, made possible by "information leakage" from trading activity of those who choose to become informed. However, if information is scattered into several pieces—holding the acquisition cost per piece fixed—this basic trade-off takes on a competing dynamic as n increases: both the *informativeness of each piece* and the *information leakage* to uninformed investors declines in n, at varying rates.

It is straightforward that the informativeness of each piece declines simply because each piece, being smaller, resolves less uncertainty in the overall value. To see why information leakage declines, consider the following. For $n = 1$, there is a single signal embedded in the market price as a result of informed trading. The uninformed must disentangle that signal from the noise in the price. For $n = 2$, suppose that the mass of informed traders is unchanged from the $n = 1$ case. Then there are two distinct signals embedded into the price, each of which is of lower strength than the single signal in the previous case because, (1) the mass of informed traders is split into two separate populations of half the size and (2) each signal is less informative. The uninformed must now disentangle two weaker signals from the noise, reducing information leakage. So relative to the previous $n = 1$ case, a higher mass of informed traders *can potentially* be supported because their incentive to acquire information is less hampered by information leakage. For general n, the uninformed must disentangle n progressively weaker signals and hence information leakage progressively declines, again potentially supporting more informed traders.

Leakage of information declines more rapidly for low n than for high n. For example, going from one to two pieces doubles the inference problem for the uninformed while going from 100 to 101 pieces only worsens the inference problem by 1%. If the acquisition cost per piece is not too high so that reduced informativeness can be feasibly attractive, then for low values of n, the benefit of less information leakage outweighs the reduction in informativeness, resulting in a larger number of informed traders.

Theorem 10.3 shows that eventually, for larger n the mass of informed investors reaches a maximum (near n^*) and then the reverse effect begins to occur as fewer investors elect to spend c on private information. Ultimately, for large enough n, participation completely ceases [Theorem 10.4].

Note that we have assumed throughout our analysis, that the cost of acquiring information c is fixed. It may be more natural to think that the cost per piece would be smaller when there are a larger number of pieces since each piece is smaller and, hence, plausibly easier to acquire. Nevertheless, if the cost per piece goes down as n increases, then our main result is only strengthened because, in equilibrium, even more traders would participate at a lower cost. Therefore our results represent a lower bound on these surprising implications of the value of scattered information.

Theorem 10.2 also shows that scattered information can be a natural economic force that reduces information "free riding". Recall that an uninformed traders can utilize the information in the price function to make trading decisions and, in some sense, free ride on informed traders' information, which is impounded into the price when informed traders trade. It is the free riding problem that leads to the impossibility of an efficient market in Grossman and Stiglitz (1980). So, in an economy with a higher degree of scattered information, free riding is naturally alleviated relative to the baseline case of Grossman and Stiglitz (1980)—more traders have incentive to acquire information themselves.

III. Information markets

Here, we consider an application of the main result in an environment in which private information about the payoff is sold at some price to investors by a monopolistic seller who determines the degree of information scattering. That is, instead of being endowed with information or with costly information production technology, investors could obtain information from an information intermediary such as an information monopolist.

In a standard environment that does not consider the role of scattered information, Admati and Pfleiderer (1986) show that an information monopolist would prefer to sell to investors noisy, conditionally independent signals [e.g., signals such as those in Hellwig (1980), Verrecchia (1982)] rather than precise signals [e.g., signals such as those in Grossman and Stiglitz (1980)]. In these studies, there is only $n = 1$ piece of uncertainty in the asset value that investors can learn about. Sales of information about this one piece take the form of individualized signals that are rendered conditionally independent by means of manufactured noise. Signals are different, but only because of the noise, not because of learning about different pieces of the fundamental. In the context of our model, they consider signals of the form $\widetilde{v}+\widetilde{\eta}_j$ (rather than just $\widetilde{v}$) for each investor $j \in [0,1]$, where the independent noise terms $\{\widetilde{\eta}_j\}_{j\in[0,1]}$ are added by the seller. [11]

Since the signal structure of Admati and Pfleiderer (1986) is assumed common knowledge, investors are aware that the information provider adulterates their signal. In contrast, we consider the case in which the information monopolist can split up information into n pieces and sell separate signals of each piece that are *noiseless* and *un*conditionally independent to n groups of investors.[12] In this case, different investors obtain uncontaminated information about different pieces of the value of the asset, but no artificial noise is added so no investor is intentionally misled about the value of a piece. Our results show that the seller may prefer to divide up information into small *valid* signals and scatter them among the investor population.

Proposition 11 *An information monopolist may prefer to sell scattered information ($n \geq 2$). In addition, the optimal degree of scattered information is finite.*

To see why the first part of Proposition 11 is true, consider first the case in which the information monopolist cannot select n but can only choose the cost $c \equiv c_n$. By Proposition 8, participation level λ is uniquely determined for any cost c that the seller chooses, and so the seller selects c to maximize total revenue, $c\lambda$. For $n = 1$, let c_1 be the revenue maximizing price of information. Such a price induces λ_1 informed traders for a total information acquisition revenue of $c_1\lambda_1$. Now consider the $n = 2$ case. Theorem 10 shows that at the same price $c_2 = c_1$, but for access to just one of $n = 2$ split pieces of the original information, there can be more informed traders: $\lambda_2 > \lambda_1$. Thus, the total revenue can go up: $c_2\lambda_2 > c_1\lambda_1$. Hence, $n = 2$ may dominate $n = 1$ in which $n = 1$ cannot be an equilibrium if the information monopolist is able to select n. This argument continues to apply for larger

[11] García and Sangiorgi (2011) study a general information structure, in which η_j can be correlated. Fixing the number of informed traders, the optimal correlation they find coincides with our scattered information structure. They have a corner solution for the optimal degree of scattered information. As we will show in Proposition 11, the optimal degree of scattered information is interior in our setting.

[12] We maintain the assumptions of Admati and Pfleiderer (1986) that all investors are identical, investors cannot resell their information, the information structure is common knowledge, and the seller either does not participate in trading or is an infinitesimally small trader.

n, until n reaches n^*. In sum, the information seller may have incentive to scatter noiseless information among the investor population.

Our result brings new implications for information sales. Admati and Pfleiderer (1986) have focused on one piece of common information that is possibly dispersed with noise among the investor population. We have shown that scattered information provides an alternative strategy for the seller to increase revenues. In practice, information is distributed to traders in a variety of ways. We rely on the observation that newpapers tend to cover local or domestic news as our empirical evidence.

IV. Conclusion

In this paper, we have analyzed the value of scattered information by generalizing the seminal financial market model of Grossman and Stiglitz (1980) to an arbitrary n number of dimensions of uncertainty in the fundamental value of the risky asset. Our analysis centers around the information acquisition decisions that investors face of whether or not to costly acquire one of several distinct pieces of information about the asset. Our approach is tractable and permits closed-form expressions for the informativeness of the price, the informativeness of each piece of information, the fraction of investors who choose to become informed in equilibrium, and the threshold n^* for the degree of scattered information at which the fraction of investors attains a maximum over all $n \geq 1$.

We find the surprising result that as information is more widely scattered into smaller distinctly informative pieces, overall information production can increase even when the information production cost per piece is fixed. Moreover, this phenomenon is even stronger if information production costs per piece decrease in n, a natural scenario. This phenomenon identifies a natural economic force that attenuates the information free-riding problem. We show that an information monopolist may prefer to sell scattered information to the investor population instead of selling concentrated, noisy information.

APPENDIX A. OMITTED PROOFS FROM THE TEXT

Proof of Proposition 1. The payoff $\widetilde{u}$, piece $\widetilde{v}_i$, and the price $\widetilde{p}$ are jointly normally distributed:

$$\begin{pmatrix} \widetilde{u} \\ \widetilde{v}_i \\ \widetilde{p} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ 0 \\ \alpha_0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\tau}+\frac{1}{\tau_\varepsilon} & \frac{1}{\tau n} & \frac{\alpha}{\tau} \\ \frac{1}{\tau n} & \frac{1}{\tau n} & \frac{\alpha}{\tau n} \\ \frac{\alpha}{\tau} & \frac{\alpha}{\tau n} & \frac{\alpha^2}{\tau}+\frac{\alpha_x^2}{\tau_x} \end{pmatrix}\right).$$

Normal-normal conditioning gives $\mathbf{E}[\widetilde{u}|\widetilde{v}_i,\widetilde{p}] = \mu + \frac{\frac{\alpha_x^2}{\tau_x}\widetilde{v}_i + \frac{\alpha}{\tau}(1-\frac{1}{n})(\widetilde{p}-\alpha_0)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}(1-\frac{1}{n})}$; $\mathbf{Var}[\widetilde{u}\,|\widetilde{v}_i,\widetilde{p}] = \frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}$; $\mathbf{E}[\widetilde{u}|\widetilde{p}] = \mu + \frac{\frac{\alpha}{\tau}(\widetilde{p}-\alpha_0)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}}$; and $\mathbf{Var}[\widetilde{u}|\widetilde{p}] = \frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}$. A trader informed about v_i solves the program $X_{\text{inf}}(\widetilde{v}_i,\widetilde{p}) := \arg\max_x \mathbf{E}[-\exp(-\gamma[W_0+x(\widetilde{u}-p)])\,|\,\widetilde{v}_i,\widetilde{p}]$ and therefore submits the following demand:

$$X_{\text{inf}}(\widetilde{v}_i,\widetilde{p}) = \frac{\mathbf{E}[\widetilde{u}|\widetilde{v}_i,\widetilde{p}]-\widetilde{p}}{\gamma\mathbf{Var}[\widetilde{u}|\widetilde{v}_i,\widetilde{p}]} = \frac{\mu + \frac{\frac{\alpha_x^2}{\tau_x}\widetilde{v}_i + \frac{\alpha}{\tau}(1-\frac{1}{n})(\widetilde{p}-\alpha_0)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}(1-\frac{1}{n})} - \widetilde{p}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)}.$$

An uninformed trader observes only the price, solves the program $X_{\text{uninf}}(\widetilde{p}) := \arg\max_x \mathbf{E}[-\exp(-\gamma[W_0+x(\widetilde{u}-p)])\,|\,\widetilde{p}]$, and therefore submits the following demand:

$$X_{\text{uninf}}(\widetilde{p}) = \frac{\mathbf{E}[\widetilde{u}|\widetilde{p}]-\widetilde{p}}{\mathbf{Var}[\widetilde{u}|\widetilde{p}]} = \frac{\mu + \frac{\frac{\alpha}{\tau}(\widetilde{p}-\alpha_0)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}} - \widetilde{p}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)}.$$

The market clearing condition [equation (4)] states that $\sum_{i=1}^n \frac{\lambda}{n}X_{\text{inf}}(\widetilde{v}_i,\widetilde{p}) + (1-\lambda)X_{\text{uninf}}(\widetilde{p}) + \widetilde{x} = \overline{x}$, for all realizations of $\{\widetilde{v}_i\}_{i=1}^n$ and $\widetilde{x}$. Applying $\widetilde{p} = \alpha\widetilde{v}+\alpha_x\widetilde{x}+\alpha_0$ yields

$$\sum_{i=1}^n X_{\text{inf}}(\widetilde{v}_i,\widetilde{p}) = \frac{n\mu + \frac{\frac{\alpha_x^2}{\tau_x}\sum_{i=1}^n\widetilde{v}_i + n\frac{\alpha}{\tau}(1-\frac{1}{n})(\alpha\widetilde{v}+\alpha_x\widetilde{x})}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}(1-\frac{1}{n})} - n(\alpha\widetilde{v}+\alpha_x\widetilde{x}+\alpha_0)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)}$$

and

$$X_{\text{uninf}}(\widetilde{p}) = \frac{\mu + \frac{\frac{\alpha}{\tau}(\alpha\widetilde{v}+\alpha_x\widetilde{x})}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}} - (\alpha\widetilde{v}+\alpha_x\widetilde{x}+\alpha_0)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)}.$$

Collecting terms on the market clearing equation gives:

$$\begin{aligned}
\widetilde{v}: \quad & \frac{\lambda}{n}\frac{\frac{\frac{\alpha_x^2}{\tau_x}+n\frac{\alpha}{\tau}\left(1-\frac{1}{n}\right)(\alpha)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}\left(1-\frac{1}{n}\right)} - n(\alpha)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} + (1-\lambda)\frac{\frac{\frac{\alpha}{\tau}(\alpha)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}} - (\alpha)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} = 0,\\
\widetilde{x}: \quad & \frac{\lambda}{n}\frac{\frac{n\frac{\alpha}{\tau}\left(1-\frac{1}{n}\right)(\alpha_x)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}\left(1-\frac{1}{n}\right)} - n(\alpha_x)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} + (1-\lambda)\frac{\frac{\frac{\alpha}{\tau}(\alpha_x)}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}} - (\alpha_x)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} + 1 = 0,\\
\text{Constant}: \quad & \frac{\lambda(\mu-\alpha_0)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} + \frac{(1-\lambda)(\mu-\alpha_0)}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} = \overline{x}.
\end{aligned}$$

Comparing the equations of collected terms corresponding to $\widetilde{v}$ and $\widetilde{x}$ gives:

$$\frac{\lambda}{n}\frac{\frac{\frac{\alpha_x^2}{\tau_x}}{\frac{\alpha_x^2}{\tau_x}+\frac{\alpha^2}{\tau}\left(1-\frac{1}{n}\right)}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}\right)} = \frac{\alpha}{\alpha_x} =: I. \tag{A1}$$

Denote I for $\frac{\alpha}{\alpha_x}$, then the above equation simplifies to $\frac{\lambda}{n}\frac{\frac{1}{1+\frac{\tau_x}{\tau}I^2\left(1-\frac{1}{n}\right)}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}\right)} = I$, and thus,

$$\lambda = n\gamma I\left[\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}I^2\left(1-\frac{1}{n}\right)+\frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]. \tag{A2}$$

Note the right-hand-side of (A2) is an increasing function of I. So we know there exists a unique $I \in [0,+\infty)$ that satisfies (A2). Once we find the value of I, we can solve for α from the collected terms associated with $\widetilde{v}$:

$$\alpha = \frac{\frac{\lambda}{n}\frac{\frac{\frac{1}{\tau_x}+\frac{I^2}{\tau}n\left(1-\frac{1}{n}\right)}{\frac{1}{\tau_x}+\frac{I^2}{\tau}\left(1-\frac{1}{n}\right)}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}\right)}+(1-\lambda)\frac{\frac{\frac{I^2}{\tau}}{\frac{1}{\tau_x}+\frac{I^2}{\tau}}}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}\right)}}{\frac{\lambda}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}\right)}+\frac{1-\lambda}{\gamma\left(\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}\right)}}.$$

Then, $\alpha_x = \frac{\alpha}{I}$, and the collected constant terms imply

$$\alpha_0 = \mu - \frac{\gamma\overline{x}}{\frac{\lambda}{\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}}+\frac{1-\lambda}{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}}}. \tag{A3}$$

Since I is uniquely determined, the linear equilibrium is unique. ∎

Proof of Proposition 2. The result follows immediately from (A1) and (A2) in the Proof of Proposition 1. ∎

Proof of Lemma 3. Price informativeness can be written as $\frac{1}{\mathbf{Var}[\widetilde{u}|\widetilde{p}]} = \frac{1}{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}} = \frac{1}{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}}$, where the first equality follows from the reciprocal of $\mathbf{Var}[\widetilde{u}|\widetilde{p}]$ in the proof of Proposition 1, and the second equality follows from $I = \frac{\alpha}{\alpha_x}$. It is straightforward to verify that the final expression is an increasing function of I. ∎

Proof of Lemma 4. Consider a trader who is deciding whether to become informed. We apply Lemma 12 of the online appendix, which shows for any $\widetilde{X} \sim \mathcal{N}(a, b^2)$ that $\mathbf{E}\left[e^{-\widetilde{X}^2}\right] = \frac{1}{\sqrt{1+2b^2}}e^{-\frac{a^2}{1+2b^2}}$, to compute the following ex-ante utilities. The ex-ante expected utility of remaining uninformed is given by

$$\begin{aligned}
&\mathbf{E}\{\max_x \mathbf{E}[-\exp(-\gamma(W_0 + x(\tilde{u}-\tilde{p}))|\tilde{p}]\} \\
&= \mathbf{E}\left\{-\exp\left(-\gamma W_0 - \frac{(\mathbf{E}[\tilde{u}|\tilde{p}]-\tilde{p})^2}{2\mathbf{Var}[\tilde{u}|\tilde{p}]}\right)\right\} \\
&= -\exp(-\gamma W_0)\frac{1}{\sqrt{1+\frac{\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}{\mathbf{Var}[\tilde{u}|\tilde{p}]}}}\exp\left(-\frac{\left(\mathbf{E}\frac{\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]}{\sqrt{2\mathbf{Var}[\tilde{u}|\tilde{p}]}}\right)^2}{1+\frac{\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}{\mathbf{Var}[\tilde{u}|\tilde{p}]}}\right) \\
&= -\exp(-\gamma W_0)\frac{1}{\sqrt{1+\frac{\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}{\mathbf{Var}[\tilde{u}|\tilde{p}]}}}\exp\left(-\frac{\frac{(\mu-\alpha_0)^2}{2\mathbf{Var}[\tilde{u}|\tilde{p}]}}{1+\frac{\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}{\mathbf{Var}[\tilde{u}|\tilde{p}]}}\right) \\
&= -\exp(-\gamma W_0)\frac{\sqrt{\mathbf{Var}[\tilde{u}|\tilde{p}]}}{\sqrt{\mathbf{Var}[\tilde{u}|\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}}\exp\left(-\frac{\frac{(\mu-\alpha_0)^2}{2}}{\mathbf{Var}[\tilde{u}|\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]]}\right).
\end{aligned}$$

Being informed at cost c has ex-ante expected utility given by

$$\begin{aligned}
&\mathbf{E}\{\max_x \mathbf{E}[-\exp(-\gamma(W_0 - c + x(\tilde{u}-\tilde{p}))|\tilde{v}_i,\tilde{p}]\} \\
&= \mathbf{E}\left\{-\exp\left(-\gamma(W_0-c) - \frac{(\mathbf{E}[\tilde{u}|\tilde{v}_i,\tilde{p}]-\tilde{p})^2}{2\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]}\right)\right\} \\
&= -\exp(-\gamma(W_0-c))\frac{\sqrt{\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]}}{\sqrt{\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{v}_i,\tilde{p}]]}}\exp\left(-\frac{\frac{(\mu-\alpha_0)^2}{2}}{\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{v}_i,\tilde{p}]]}\right).
\end{aligned}$$

The informed ex-ante utility must equal the uninformed ex-ante utility in order for the trader to be willing to pay c to become informed. Hence, to show the first equation of the lemma, we need to prove that $\mathbf{Var}[\tilde{u}|\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]] = \mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{v}_i,\tilde{p}]]$. The left-hand side is $\mathbf{Var}[\tilde{u}|\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]] = \mathbf{E}[\mathbf{Var}[\tilde{u}|\tilde{p}]]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{p}]] = \mathbf{Var}[\tilde{u}-\tilde{p}]$, where the first equality follows since $\mathbf{Var}[\tilde{u}|\tilde{p}]$ is a constant, and the second equality follows by the law of total variance. By the same reasoning, the right-hand side also equals $\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{v}_i,\tilde{p}]] = \mathbf{E}[\mathbf{Var}[\tilde{u}|\tilde{v}_i,\tilde{p}]]+\mathbf{Var}[\mathbf{E}[\tilde{u}-\tilde{p}|\tilde{v}_i,\tilde{p}]] = \mathbf{Var}[\tilde{u}-\tilde{p}]$. Thus, the willingness to pay for becoming informed is given by $c = \frac{1}{2\gamma}\log\frac{\mathbf{Var}[\tilde{u}|\tilde{p}]}{\mathbf{Var}[\tilde{u}|v_i,\tilde{p}]} = \frac{1}{2\gamma}\log\frac{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}}{\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x\left(\frac{\alpha}{\alpha_x}\right)^2}} = \frac{1}{2\gamma}\log\frac{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}}{\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}}$, where the second inequality follows from the expressions for $\mathbf{Var}[\tilde{u}|\tilde{p}]$ and $\mathbf{Var}[\tilde{u}|v_i,\tilde{p}]$ in the proof of Proposition 1, and the third inequality follows from $I=\frac{\alpha}{\alpha_x}$. ∎

Proof of Proposition 5. In equilibrium, c must equal to the willingness to pay for becoming informed, so by Lemma 4, I must satisfy $c=\frac{1}{2\gamma}\log\frac{\frac{1}{\tau_\varepsilon}+\frac{1}{\tau+\tau_x I^2}}{\frac{1}{\tau_\varepsilon}+\frac{1}{\frac{n}{n-1}\tau+\tau_x I^2}}$. Rearranging this equation, I must satisfy

$$\left(\frac{I^4}{\tau^2}\left(1-\frac{1}{n}\right)+\frac{1}{\tau_x}\frac{I^2}{\tau}\left(2-\frac{1}{n}\right)+\frac{1}{\tau_x{}^2}\right)\left(e^{2\gamma c}-1\right)\left(\frac{1}{\tau}+\frac{1}{\tau_\varepsilon}\right)$$
$$=e^{2\gamma c}\left(\frac{I^4}{\tau^3}\left(1-\frac{1}{n}\right)+\frac{I^2}{\tau^2\tau_x}+\frac{1}{\tau\tau_x{}^2 n}\right)-\frac{I^2}{\tau^2\tau_x}-\frac{I^4}{\tau^3}\left(1-\frac{1}{n}\right),$$

which is a quadratic equation of I^2. If $c\le\overline{c}$, where

$$\overline{c}:=\frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n}\right), \tag{A4}$$

then this equation has a non-negative root

$$\widehat{I}^2:=\frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c}-[\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}}-\frac{n\tau}{2\tau_x(n-1)}-\frac{\tau+\tau_\varepsilon}{2\tau_x}, \tag{A5}$$

whose derivation is detailed in Lemma 13 in the online appendix. Its square root $\widehat{I}$ is given by the closed-form expression

$$\widehat{I}:=\sqrt{\frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c}-[\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}}-\frac{n\tau}{2\tau_x(n-1)}-\frac{\tau+\tau_\varepsilon}{2\tau_x}}. \tag{11}$$

If $n=1$, the above expression is not well defined. So when $n=1$, we use the continuous extension by taking the limit as $n\downarrow 1$, in which case $\widehat{I}=\sqrt{\frac{\tau_\varepsilon}{\tau_x(e^{2\gamma c}-1)}-\frac{\tau}{\tau_x}}$. By (5) of Proposition 2, $\widehat{I}$ implies an informed trading population fraction of $\widehat{\lambda}:=n\gamma\widehat{I}\left[\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\widehat{I}^2\left(1-\frac{1}{n}\right)+\frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]$. But (5) implies λ is strictly increasing in I and imposes an upper bound on I such that λ does not exceed 1. In Lemma 14 in the online appendix, we show that $\widehat{I}$ is a strictly decreasing function of c until it reaches $\widehat{I}=0$ at positive constant $\overline{c}$ and $\lim_{c\downarrow 0}\widehat{I}=+\infty$. So, by

Lemma 14 for sufficiently small c ($0 < c \le \underline{c}$), I is a capped version of $\widehat{I}$ with unique upper bound $\overline{I}$ such that $\lambda = \widehat{\lambda} = 1$ at $I = \widehat{I} = \overline{I}$:

$$I = \begin{cases} \overline{I}, & 0 < c < \underline{c}, \\ \widehat{I} := \sqrt{\frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c} - [\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}} - \frac{n\tau}{2\tau_x(n-1)} - \frac{\tau+\tau_\varepsilon}{2\tau_x}}, & \underline{c} \le c \le \overline{c}, \\ 0, & c > \overline{c}. \end{cases}$$

where $\overline{I}$ is the unique positive constant satisfying

$$n\gamma\overline{I}\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2\left(1-\frac{1}{n}\right) + \frac{1}{\tau}\left(1-\frac{1}{n}\right)\right] = 1, \tag{A6}$$

$\underline{c}$ is the corresponding unique value of c such that $\widehat{I} = \overline{I}$, and $\overline{c}$ is given by (A4) above. The defining relation of $\overline{I}$ in (A6) is a cubic equation in $\overline{I}$ with closed-form real root for $n > 1$ given by the cubic-root formula:

$$\overline{I} = \frac{2^{1/3}\left(9\tau_\varepsilon\gamma^2(n-1)^2\tau\tau_x{}^2 + \zeta\right)^{2/3} - 2\cdot 3^{1/3}\gamma^2(n-1)(\tau_\varepsilon(n-1)+n\tau)\tau_x}{6^{2/3}\gamma(n-1)\tau_x(9\tau_\varepsilon\gamma^2(n-1)^2\tau\tau_x{}^2+\zeta)^{1/3}}, \tag{A7}$$

where $\zeta := \sqrt{3}\gamma^2\sqrt{(n-1)^3\tau_x{}^3(4\gamma^2(\tau_\varepsilon(n-1)+n\tau)^3 + 27\tau_\varepsilon{}^2(n-1)\tau^2\tau_x)}$. Setting $\widehat{I} = \overline{I}$ yields: $\underline{c} = \frac{1}{2\gamma}\log\left(1 + \frac{\tau}{\tau+\overline{I}^2\tau_x}\cdot\frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n+\overline{I}^2(n-1)\tau_x}\right) > 0$. Note that for $n \ge 1$ that $\frac{\tau}{\tau+\overline{I}^2\tau_x}\cdot\frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n+\overline{I}^2(n-1)\tau_x} < \frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n}$ since $\overline{I}^2\tau_x > 0$ and $\overline{I}^2(n-1)\tau_x \ge 0$. Therefore, $\underline{c} < \overline{c}$. Moreover, the defining relation of $\overline{I}$ in (A6) implies that $\gamma\overline{I} = \frac{\tau\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n+\overline{I}^2(n-1)\tau_x}$, which leads to the following expression for $\underline{c}$:

$$\underline{c} = \frac{1}{2\gamma}\log\left(1 + \frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x}\right). \tag{A8}$$

Note that if $n = 1$, then $\overline{c} = \frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{\tau}\right)$, $\overline{I} = \frac{\tau_\varepsilon}{\gamma}$, and $\underline{c} = \frac{1}{2\gamma}\log\left(1 + \frac{\tau_\varepsilon}{\tau+\frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x}\right)$.

Finally, from (5), the fraction λ of investors who choose to become informed is: $\lambda =$

$\min\left\{1, n\gamma I\left[\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}I^2\left(1-\frac{1}{n}\right)+\frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]\right\}$, which is equivalent to $\lambda = \begin{cases} 1, & 0<c<\underline{c}, \\ \widehat{\lambda}, & \underline{c}\le c\le \overline{c}, \\ 0, & c>\overline{c}, \end{cases}$ where $\widehat{\lambda} := n\gamma\widehat{I}\left[\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\widehat{I}^2\left(1-\frac{1}{n}\right)+\frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]$, since $\widehat{I}=I$ unless $\widehat{I}$ exceeds $\overline{I}$ in which case $\lambda=1$ or unless $c>\overline{c}$ in which $\lambda=I=0$. By Proposition 1, the trading game has a unique linear REE. So there exists a unique information market equilibrium in which there are $\lambda\in[0,1]$ fraction of traders who choose to be informed. ∎

Proof of Proposition 6 Let

$$\widehat{I}(n) := \sqrt{\frac{\sqrt{\frac{h(n)}{A}}-k(n)}{2\tau_x(n-1)}}, \tag{A9}$$

where

$$h(n) := [\tau+(n-1)\tau_\varepsilon]^2(A+1)-[\tau-(n-1)\tau_\varepsilon]^2, \tag{A10}$$

$$k(n) := n\tau+(n-1)(\tau+\tau_\varepsilon), \tag{A11}$$

and $A := e^{2\gamma c}-1$. Note that $\widehat{I}(n)=\widehat{I}$ of (9) and (11). In Lemma 15 of the online appendix, we show that the slope of $\widehat{I}(n)$ is given by: $\widehat{I}'(n) = -\tau\frac{(2+A)\tau_\varepsilon(n-1)+A\left(\tau-\sqrt{\frac{h(n)}{A}}\right)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$. Now we show that $\widehat{I}(n)$ is strictly decreasing in n on $n\ge 1$. Define $g(n) := (2+A)\tau_\varepsilon(n-1)+A\left(\tau-\sqrt{\frac{h(n)}{A}}\right)$ such that $\widehat{I}'(n) = -\tau\frac{g(n)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$. Since $c>0$, then $A>0$ and the following inequality holds for all $n>1$: $4(1+A)(n-1)^2{\tau_\varepsilon}^2>0$. Add $2\tau\tau_\varepsilon A(2+A)(n-1)+\tau^2A^2+(n-1)^2{\tau_\varepsilon}^2A^2$ to both sides of the above inequality and group terms on each side to obtain: $[(2+A)(n-1)\tau_\varepsilon+\tau A]^2 > A\,h(n)$. Since both sides of the above inequality are positive for $n>1$, taking the positive square root of both sides retains the inequality: $(2+A)(n-1)\tau_\varepsilon+\tau A > \sqrt{A}\sqrt{h(n)}$. Organizing terms to the left-hand side yields $g(n)\equiv(2+A)\tau_\varepsilon(n-1)+A\left(\tau-\sqrt{\frac{h(n)}{A}}\right)>0$, and so $g(n)>0$ on $n>1$. Since the denominator of $\widehat{I}(n)$ is positive on $n>1$, we have $\widehat{I}'(n)<0$ on $n>1$. For $n=1$ we have zero-by-zero division in $\widehat{I}'(n)$ so we apply L'Hospital's rule to obtain: $\widehat{I}'(1) = -\frac{{\tau_\varepsilon}^2(1+A)}{2\tau\tau_x A^2\sqrt{\frac{\tau_\varepsilon-\tau A}{\tau_x A}}} < 0$.

The defining condition for $\overline{I}$ in (A6) can be written as: $\gamma\overline{I}\left[\frac{n}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2(n-1)+\frac{1}{\tau}(n-1)\right]=1$.

Implicit partial differentiation of $\overline{I}$ with respect to n in this condition reveals that $\overline{I}$ is also strictly decreasing in n. Hence, I is strictly decreasing in n for $c < \overline{c}$, and otherwise flat at $I = 0$. ∎

Proof of Corollary 7. Follows immediately from Lemma 3 and Proposition 6. ∎

Proof of Proposition 8. Let

$$\widehat{\lambda}(n) := n\gamma \widehat{I}(n)\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon \tau}\widehat{I}^2(n)\left(1-\frac{1}{n}\right) + \frac{1}{\tau}\left(1-\frac{1}{n}\right)\right], \tag{A12}$$

where $\widehat{I}(n)$ is as in (A9) of Proposition 6. Note that $\widehat{\lambda}(n) = \widehat{\lambda}$ in (8) of Proposition 5 for $0 < c \leq \overline{c}$, where $\overline{c}$ is given by (A4). For $c > \overline{c}$, $\widehat{I}(n)$ and $\widehat{\lambda}(n)$ become imaginary and $I = \lambda = 0$, so we restrict attention to $c \leq \overline{c}$. Equivalently, given c, let $\overline{n}$ be defined as the largest n such that $c = \overline{c}$. Since $\overline{c}$ is strictly decreasing in n, $\overline{n}$ is well-defined and unique. Moreover, $n \leq \overline{n}$ if and only if $c \leq \overline{c}$. Solving for $\overline{n}$ from (A4) gives: $\overline{n} = \frac{\tau_\varepsilon}{\tau_\varepsilon+\tau}\frac{e^{2\gamma c}}{e^{2\gamma c}-1}$. So, we restrict attention to $n \leq \overline{n}$ for which $\widehat{\lambda}(n)$ is real. In Lemma 16 and Lemma 17 in the online appendix we show that $\widehat{\lambda}(n) = \frac{\gamma \widehat{I}(n)}{2\tau\tau_\varepsilon}\left[\tau + (n-1)\tau_\varepsilon + \sqrt{[\tau+(n-1)\tau_\varepsilon]^2 + \frac{4\tau}{A}(n-1)\tau_\varepsilon}\right]$ and $\widehat{\lambda}'(n) = \gamma\frac{[3-4n-2A(n-1)]\tau + [1-2A(n-1)]\left[(n-1)\tau_\varepsilon + \sqrt{\frac{h(n)}{A}}\right]}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$, respectively, where $h(n) := [\tau+(n-1)\tau_\varepsilon]^2(A+1) - [\tau-(n-1)\tau_\varepsilon]^2$ and $A := e^{2\gamma c}-1$. Now, define the numerator of the fraction in the above expression for $\widehat{\lambda}'(n)$ to be $r(n) := [3-4n-2A(n-1)]\tau + [1-2A(n-1)]\left[(n-1)\tau_\varepsilon + \sqrt{\frac{h(n)}{A}}\right]$. Note that $\widehat{\lambda}'(n) = 0$ only if $r(n) = 0$. The numerator $r(n)$ has exactly two roots: $n^* = 1$ and $n^* = \frac{\tau_\varepsilon + 2A\tau_\varepsilon + 2A\tau}{2A\tau_\varepsilon + 4A\tau}$. For $n = 1$ we have zero-by-zero division in $\widehat{\lambda}'(n)$ so we apply L'Hospital's rule to obtain: $\widehat{\lambda}'(1) = \gamma\frac{(1+A)(\tau_\varepsilon - 2\tau A)}{2\tau\tau_x A^2\sqrt{\frac{\tau_\varepsilon - \tau A}{\tau_x A}}} > 0$ for $\tau_\varepsilon > 2\tau A$. Thus, $n^* = \frac{\tau_\varepsilon + 2A\tau_\varepsilon + 2A\tau}{2A\tau_\varepsilon + 4A\tau}$ is the unique root of $\widehat{\lambda}'(n)$ and $n^* > 1$ for $\tau_\varepsilon > 2\tau A$. Since $\widehat{\lambda}'(n)$ is strictly positive at $n = 1$ and has a unique root at $n^* > 1$, then $\widehat{\lambda}'(n) > 0$ for all $n \in [1, n^*)$. Now, consider $n = 1 + \frac{1}{2A}$. Note that $1 + \frac{1}{2A} - n^* = \frac{\tau(1+A)}{A(2\tau+\tau_\varepsilon)} > 0$ so $n^* < 1 + \frac{1}{2A}$. Now, $r\left(1+\frac{1}{2A}\right) = -2\left(1+\frac{1}{A}\right) < 0$, and since $r(n)$ has no roots beyond n^*, then $r(n) < 0$ for $n > n^*$. Therefore, $\widehat{\lambda}'(n) < 0$ for $n > n^*$.

Conversely, if $\tau_\varepsilon \leq 2\tau A$, then $n^* \leq 1$ and so any root of $\widehat{\lambda}'(n)$ is at or below 1 because $\widehat{\lambda}'(1) \leq 0$. Hence, $\widehat{\lambda}$ is non-increasing for $n \in [1, \overline{n}]$.

Finally, we show that $n^* < \overline{n}$ for $\tau_\varepsilon > 2\tau A$. First, $A+1 < 1+\frac{\tau_\varepsilon}{2\tau} < \left(1+\frac{\tau_\varepsilon}{2\tau}\right)\left(1+\frac{\tau_\varepsilon}{\tau}\right)$, where the first inequality follows from $\tau_\varepsilon > 2\tau A$. Next, expanding the product after the last inequality and rearranging terms implies $(A+1)\frac{\tau^2}{(\tau_\varepsilon+2\tau)(\tau_\varepsilon+\tau)} < \frac{1}{2}$. Applying the identity $\frac{\tau^2}{(\tau_\varepsilon+2\tau)(\tau_\varepsilon+\tau)} = \frac{\tau_\varepsilon+\tau}{\tau_\varepsilon+2\tau} - \frac{\tau_\varepsilon}{\tau_\varepsilon+\tau}$ and rearranging terms gives $(A+1)\frac{\tau_\varepsilon+\tau}{\tau_\varepsilon+2\tau} - \frac{1}{2} < (A+1)\frac{\tau_\varepsilon}{\tau_\varepsilon+\tau}$. Dividing both sides by A and simplifying yields $n^* \equiv \frac{\tau_\varepsilon+2A(\tau_\varepsilon+\tau)}{2A(\tau_\varepsilon+2\tau)} < \frac{(A+1)\tau_\varepsilon}{A(\tau_\varepsilon+\tau)} \equiv \overline{n}$. ∎

Proof of Lemma 9. Note that $\underline{c} \equiv \frac{1}{2\gamma}\log\left(1+\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x}\right) < \frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{2\tau}\right) \equiv \widehat{c}$ if and only if $\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x} < \frac{\tau_\varepsilon}{2\tau}$. We first show that this latter inequality is satisfied weakly for $\tau = \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$ for all $n \geq 1$. Then, we show that it is satisfied strictly for $\tau < \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$. The left-hand side of the condition is non-decreasing in n for $\tau \geq \overline{I}^2\tau_x$:

$$\frac{\partial}{\partial\overline{I}}\left(\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x}\right) = \frac{\gamma\left(\tau-\overline{I}^2\tau_x\right)}{\left(\tau+\overline{I}^2\tau_x\right)^2} \geq 0 \quad \text{for} \quad \tau \geq \overline{I}^2\tau_x. \tag{A13}$$

The defining condition of $\overline{I}$ in (A6) can be written as: $\gamma\overline{I}\left[\frac{n}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2(n-1)+\frac{1}{\tau}(n-1)\right] = 1$. Implicit partial differentiation of $\overline{I}$ with respect to n in this condition reveals that $\overline{I}$ is strictly decreasing in n:

$$\frac{\partial\overline{I}}{\partial n} = -\overline{I}\frac{\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2+\frac{1}{\tau}}{\frac{n}{\tau_\varepsilon}+3\frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2(n-1)+\frac{1}{\tau}(n-1)} < 0, \quad \text{for} \quad n \geq 1. \tag{A14}$$

Equations (A13) and (A14) imply $\frac{\partial}{\partial n}\left(\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x}\right) \leq 0$ for $n \geq 1$ if $\tau \geq \overline{I}^2\tau_x$. Moreover, (A14) implies $\overline{I} \leq \frac{\tau_\varepsilon}{\gamma}$ for $n \geq 1$ because $\overline{I} = \frac{\tau_\varepsilon}{\gamma}$ at $n = 1$, and hence $\tau \geq \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$ implies $\tau \geq \overline{I}^2\tau_x$. Therefore, if $\tau \geq \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$, then the condition $\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x} \leq \frac{\tau_\varepsilon}{2\tau}$ is satisfied for all $n \geq 1$ if it is satisfied for $n = 1$. At $n = 1$ the left-hand side of this condition becomes $\frac{\gamma\frac{\tau_\varepsilon}{\gamma}}{\tau+\frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x} = \frac{\tau_\varepsilon}{\tau+\frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x}$, which is less than or equal to $\frac{\tau_\varepsilon}{2\tau}$ for $\tau \leq \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$. Therefore, if $\tau = \frac{\tau_\varepsilon{}^2}{\gamma^2}\tau_x$, then the condition $\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x} \leq \frac{\tau_\varepsilon}{2\tau}$ is satisfied for all $n \geq 1$.

The defining relation of $\overline{I}$ in (A6) can be rearranged to show for $n > 1$ that $\tau + \overline{I}^2\tau_x =$

$\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon$, which plugged into (A8) reveals an equivalent expression for $\underline{c}$:

$$\underline{c} = \frac{1}{2\gamma}\log\left(1+\frac{\gamma\overline{I}}{\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon}\right). \tag{A15}$$

So, for $n>1$, the following two inequalities are equivalent: $\frac{\gamma\overline{I}}{\tau+\overline{I}^2\tau_x} \leq \frac{\tau_\varepsilon}{2\tau}$ and $\frac{\gamma\overline{I}}{\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon} \leq \frac{\tau_\varepsilon}{2\tau}$. The left-hand side of the latter inequality is strictly increasing in $\overline{I}$ for $n>1$ because its numerator is positive and strictly increasing in $\overline{I}$ and its denominator is positive and strictly decreasing in $\overline{I}$: $\frac{\partial}{\partial\overline{I}}\left(\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon\right) = -\frac{\tau_\varepsilon\tau}{\gamma\overline{I}^2(n-1)} < 0$, for $n>1$. Note that the denominator is positive because $\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon = \tau+\overline{I}^2\tau_x > 0$ and $\frac{\tau_\varepsilon}{\gamma\overline{I}}-1$ is well-defined for $n>1$ since $\overline{I}<\frac{\tau_\varepsilon}{\gamma}$ for $n>1$. Implicit partial differentiation of $\overline{I}$ with respect to τ_x in the defining condition (A6) reveals that $\overline{I}$ is strictly decreasing in τ_x:

$$\frac{\partial\overline{I}}{\tau_x} = -\frac{\frac{1}{\tau_\varepsilon\tau}\overline{I}^3(n-1)}{\frac{n}{\tau_\varepsilon}+3\frac{\tau_x}{\tau_\varepsilon\tau}\overline{I}^2(n-1)+\frac{1}{\tau}(n-1)} < 0, \quad \text{for} \quad n>1. \tag{A16}$$

Hence, the left-hand side of the condition $\frac{\gamma\overline{I}}{\frac{\tau}{n-1}\left(\frac{\tau_\varepsilon}{\gamma\overline{I}}-1\right)-\tau_\varepsilon} \leq \frac{\tau_\varepsilon}{2\tau}$ is strictly decreasing in τ_x for $n>1$. At $n=1$, the equivalent condition is $\frac{\tau_\varepsilon}{\tau+\frac{\tau_\varepsilon^2}{\gamma^2}\tau_x} \leq \frac{\tau_\varepsilon}{2\tau}$, whose left-hand side is also strictly decreasing in τ_x. Moreover, the left-hand sides of both equivalent conditions can be made arbitrarily close to zero for large enough τ_x since $\lim_{\tau_x\to\infty}\overline{I}=0$ and $\lim_{\tau_x\to\infty}\frac{\tau_\varepsilon}{\tau+\frac{\tau_\varepsilon^2}{\gamma^2}\tau_x}=0$. Therefore, if the condition is satisfied for some τ_x, then it is satisfied for larger τ_x. The above argument showed it is satisfied for all n at $\tau = \frac{\tau_\varepsilon^2}{\gamma^2}\tau_x$. Hence, the condition $\underline{c}<\widehat{c}$ is satisfied strictly for all n if $\tau<\frac{\tau_\varepsilon^2}{\gamma^2}\tau_x$, and the lower bound $\underline{c}$ is arbitrarily close to zero for large enough τ_x. ∎

Proof of Theorem 10. This theorem follows immediately from Proposition 8 and Lemma 9 since $\lambda=\widehat{\lambda}$ for $c\in(\underline{c},\overline{c})$ and $\lambda=0$ for $n\geq\overline{n}$, which occurs for $c\geq\overline{c}$, as shown in the proof of Proposition 8. ∎

Proof of Proposition 11. We show that optimal degree of scattered information is finite. Recall from Proposition 5 that if $c = \underline{c}$, all traders would acquire information. To maximize revenues, the information seller would select $c \geq \underline{c}$. Then the optimal degree of scattered information must be bounded by:

$$n^* = \frac{\tau_\varepsilon + 2(e^{2\gamma c} - 1)(\tau_\varepsilon + \tau)}{2(e^{2\gamma c} - 1)(\tau_\varepsilon + 2\tau)} \leq \frac{\tau_\varepsilon + 2(e^{2\gamma \underline{c}} - 1)(\tau_\varepsilon + \tau)}{2(e^{2\gamma \underline{c}} - 1)(\tau_\varepsilon + 2\tau)}. \tag{A17}$$

From equation (A7) and (A8), we know that both $\overline{I}$ and $\underline{c}$ are of the same order of $\frac{1}{n^{1/3}}$. Using Taylor expansion on the right hand side of equation (A17) implies that

$$n^* \leq \frac{\tau_\varepsilon + 2(e^{2\gamma \underline{c}} - 1)(\tau_\varepsilon + \tau)}{2(e^{2\gamma \underline{c}} - 1)(\tau_\varepsilon + 2\tau)} \leq K{n^*}^{1/3},$$

where K is some coefficient that is independent of n^*. Then, n^* must be bounded above, i.e. the optimal degree of scattered information must be finite. ∎

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B.A. *Endogenizing Information Acquisition*

In this subsection, we address why we model scattered information as in Section II.B. By endogenizing information acquisition, we show that information is equally split among the n groups of informed investors (i.e., for any i, $\frac{\lambda}{n}$ are informed about $\widetilde{v}_i$ and uninformed about any other piece).

Suppose investors can acquire exactly one of the n pieces of information, each at a common cost, c. Let λ_i denote the equilibrium fraction of traders who acquire $\widetilde{v}_i$. Let $\widetilde{p} = \sum_i \alpha_i v_i + \alpha_x \widetilde{x} + \alpha_0$ be the equilibrium price function, where $\alpha_i > 0$ are constants. Let $X_i(\widetilde{v}_i, \widetilde{p})$ denote the $\widetilde{v}_i$ informed traders' demand function. We need to show that $\lambda_1 = \cdots = \lambda_n$.

First, notice that Lemma 4 still holds. [13] So we know that the variance reductions by acquiring any one piece $\widetilde{v}_i$ are same:

$$\frac{\mathbf{Var}[\widetilde{u}|\widetilde{p}]}{\mathbf{Var}[\widetilde{u}|\widetilde{v}_1, \widetilde{p}]} = \cdots = \frac{\mathbf{Var}[\widetilde{u}|\widetilde{p}]}{\mathbf{Var}[\widetilde{u}|\widetilde{v}_n, \widetilde{p}]},$$

or equivalently,

$$\mathbf{Var}[\widetilde{u}|\widetilde{v}_1, \widetilde{p}] = \cdots = \mathbf{Var}[\widetilde{u}|\widetilde{v}_n, \widetilde{p}].$$

Normal-normal conditioning gives

$$\begin{aligned}
\mathbf{Var}[\widetilde{u}|\widetilde{v}_1, \widetilde{p}] &= \mathbf{Var}[\widetilde{u} - \widetilde{v}_1|\widetilde{v}_1, \widetilde{p}] \\
&= \mathbf{Var}(\widetilde{u} - \widetilde{v}_1) - \frac{\mathbf{Var}(\widetilde{v}_1)\mathbf{Cov}(\widetilde{u} - \widetilde{v}_1, \widetilde{p})^2}{\mathbf{Var}(\widetilde{v}_1)\mathbf{Var}(\widetilde{p}) - \mathbf{Cov}(\widetilde{v}_1, \widetilde{p})^2} \\
&= \frac{n-1}{n\tau} - \frac{\left(\frac{\alpha_2 + \cdots + \alpha_n}{n\tau}\right)^2}{\frac{\alpha_2^2 + \cdots + \alpha_n^2}{n\tau} + \frac{\alpha_x^2}{\tau_x}}.
\end{aligned}$$

[13] It is sufficient to notice that $\mathbf{Cov}(\mathbf{E}[\widetilde{u}|\widetilde{p}], \widetilde{p}) = \mathbf{Cov}(\widetilde{u}, \widetilde{p}) = \mathbf{Cov}(\mathbf{E}[\widetilde{u}|\widetilde{v}_i, \widetilde{p}], \widetilde{p})$.

Thus,

$$\frac{\left(\frac{\alpha_2+\cdots+\alpha_n}{n\tau}\right)^2}{\frac{\alpha_2^2+\cdots+\alpha_n^2}{n\tau}+\frac{\alpha_x^2}{\tau_x}}=\cdots=\frac{\left(\frac{\alpha_1+\cdots+\alpha_{n-1}}{n\tau}\right)^2}{\frac{\alpha_1^2+\cdots+\alpha_{n-1}^2}{n\tau}+\frac{\alpha_x^2}{\tau_x}}\equiv K$$

If $n=2$, it is clear that $\alpha_1=\alpha_2$. From now on, suppose $n>2$. Recall that if $\frac{A}{B}=\frac{C}{D}$, then $\frac{A}{B}=\frac{A-C}{B-D}$. Above equations imply that

$$K=\frac{2(\alpha_1+\cdots+\alpha_n)-\alpha_i-\alpha_j}{(\alpha_i+\alpha_j)n\tau},\forall i\neq j$$

Thus, $\alpha_i+\alpha_j$ must be independent of i and j. It must be that $\alpha_1=\cdots=\alpha_n$.

Since the equilibrium price function is symmetric (i.e., $\alpha_1=\cdots=\alpha_n$), we have

$$\frac{\partial X_1(\widetilde{v}_1,\widetilde{p})}{\partial\widetilde{v}_1}=\cdots=\frac{\partial X_n(\widetilde{v}_n,\widetilde{p})}{\partial\widetilde{v}_1}.$$

From market's clearing condition, we know that

$$\lambda_i\frac{\partial X_i(\widetilde{v}_i,\widetilde{p})}{\partial\widetilde{v}_i}=\frac{\alpha_i}{\alpha_x}.$$ [14]

Thus, we obtain that $\lambda_1=\cdots=\lambda_n$. So information is equally split among the n groups of informed investors.

B.B. *Omitted Proofs and Calculations From Appendix A*

Lemma 12 *Let $X\sim\mathcal{N}(a,b^2)$. Then,* $\mathbf{E}\left[e^{-X^2}\right]=\frac{1}{\sqrt{1+2b^2}}e^{-\frac{a^2}{1+2b^2}}$.

[14]Interested readers are referred to Goldstein and Yang (2015)'s equation (9) for an intuitive explanation.

Proof of Lemma 12.

$$\begin{aligned}\mathbf{E}\left[e^{-X^2}\right] &= \int_{-\infty}^{\infty} e^{-x^2}\cdot\frac{1}{\sqrt{2\pi}b}e^{-\frac{(x-a)^2}{2b^2}}dx\\ &= \frac{e^{a^2}}{\sqrt{1+2b^2}}\int_{-\infty}^{\infty} e^{-2ax}\cdot\frac{1}{\sqrt{2\pi}b'}e^{-\frac{(x-a)^2}{2b'^2}}dx\\ &= \frac{e^{a^2}}{\sqrt{1+2b^2}}\mathbf{E}[e^{-2aX'}],\end{aligned}$$

where

$$b' \equiv \frac{b}{\sqrt{1+2b^2}} \quad \text{and} \quad X' \sim \mathcal{N}\left(a,(b')^2\right).$$

Now,

$$e^{a^2}\mathbf{E}\left[e^{-2aX'}\right] = e^{a^2-2a(a)+\frac{1}{2}(-2a)^2(b')^2} = e^{a^2(2(b')^2-1)} = e^{-\frac{a^2}{1+2b^2}}. \quad \blacksquare$$

Lemma 13 *If $c \le \bar{c} \equiv \frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n}\right)$, then the following quadratic equation of I^2,*

$$\begin{aligned}&\left(\frac{I^4}{\tau^2}\left(1-\frac{1}{n}\right)+\frac{1}{\tau_x}\frac{I^2}{\tau}\left(2-\frac{1}{n}\right)+\frac{1}{{\tau_x}^2}\right)\left(e^{2\gamma c}-1\right)\left(\frac{1}{\tau}+\frac{1}{\tau_\varepsilon}\right)\\ &\qquad = e^{2\gamma c}\left(\frac{I^4}{\tau^3}\left(1-\frac{1}{n}\right)+\frac{I^2}{\tau^2\tau_x}+\frac{1}{\tau{\tau_x}^2 n}\right)-\frac{I^2}{\tau^2\tau_x}-\frac{I^4}{\tau^3}\left(1-\frac{1}{n}\right),\end{aligned}$$

has non-negative root

$$\hat{I}^2 := \frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c}-[\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}}-\frac{n\tau}{2\tau_x(n-1)}-\frac{\tau+\tau_\varepsilon}{2\tau_x}.$$

Proof of Lemma 13. Collecting terms to the left-hand side and multiplying through by ${\tau_x}^2$ gives:

$$\begin{aligned}&\left(\frac{I^2\tau_x}{\tau}\right)^2\left(1-\frac{1}{n}\right)\left(e^{2\gamma c}-1\right)\frac{1}{\tau_\varepsilon}\\ &\qquad +\frac{I^2\tau_x}{\tau}\left(e^{2\gamma c}-1\right)\left[\left(1-\frac{1}{n}\right)\frac{1}{\tau}+\left(2-\frac{1}{n}\right)\frac{1}{\tau_\varepsilon}\right]+\left(e^{2\gamma c}-1\right)\left(\frac{1}{\tau}+\frac{1}{\tau_\varepsilon}\right)-\frac{e^{2\gamma c}}{\tau n}=0.\end{aligned}$$

From the quadratic formula, we obtain:

$$\frac{I^2\tau_x}{\tau} = \frac{-\left(e^{2\gamma c}-1\right)\left[\left(1-\frac{1}{n}\right)\frac{1}{\tau}+\left(2-\frac{1}{n}\right)\frac{1}{\tau_\varepsilon}\right] \pm \sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)^2\left[\left(1-\frac{1}{n}\right)\frac{1}{\tau}+\left(2-\frac{1}{n}\right)\frac{1}{\tau_\varepsilon}\right]^2 \\ -4\left(1-\frac{1}{n}\right)\left(e^{2\gamma c}-1\right)\frac{1}{\tau_\varepsilon}\left[\left(e^{2\gamma c}-1\right)\left(\frac{1}{\tau}+\frac{1}{\tau_\varepsilon}\right)-\frac{e^{2\gamma c}}{\tau n}\right]\end{array}}}{2\left(1-\frac{1}{n}\right)\left(e^{2\gamma c}-1\right)\frac{1}{\tau_\varepsilon}}.$$

Simplifying:

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)\tau_\varepsilon+(2n-1)\tau] \pm \sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)^2[(n-1)\tau_\varepsilon+(2n-1)\tau]^2 \\ -4(n-1)\left(e^{2\gamma c}-1\right)\tau\left[\left(e^{2\gamma c}-1\right)(\tau+\tau_\varepsilon)n-e^{2\gamma c}\tau_\varepsilon\right]\end{array}}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau] \pm \sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)^2[(n-1)(\tau+\tau_\varepsilon)+n\tau]^2 \\ -4(n-1)\left(e^{2\gamma c}-1\right)\tau\left[\left(e^{2\gamma c}-1\right)(\tau+\tau_\varepsilon)n-e^{2\gamma c}\tau_\varepsilon\right]\end{array}}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau] \pm \sqrt{e^{2\gamma c}-1}\sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau]^2 \\ -4(n-1)\tau\left[\left(e^{2\gamma c}-1\right)(\tau+\tau_\varepsilon)n-e^{2\gamma c}\tau_\varepsilon\right]\end{array}}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau] \pm \sqrt{e^{2\gamma c}-1}\sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)-n\tau]^2 \\ -4(n-1)\tau\left[-e^{2\gamma c}\tau_\varepsilon\right]\end{array}}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau] \pm \sqrt{e^{2\gamma c}-1}\sqrt{\begin{array}{c}\left(e^{2\gamma c}-1\right)[(n-1)\tau_\varepsilon-\tau]^2 \\ +4(n-1)e^{2\gamma c}\tau_\varepsilon\tau\end{array}}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau] \pm \sqrt{e^{2\gamma c}-1}\sqrt{e^{2\gamma c}[(n-1)\tau_\varepsilon+\tau]^2-[(n-1)\tau_\varepsilon-\tau]^2}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}$$

The numerator is real for all $n \geq 1$ since the terms inside the square roots are positive for all $n \geq 1$. Of the two roots given by the above expression, only the (+) root is possibly non-negative. Selecting this root gives:

$$I^2 = \frac{-\left(e^{2\gamma c}-1\right)[(n-1)(\tau+\tau_\varepsilon)+n\tau]+\sqrt{e^{2\gamma c}-1}\sqrt{e^{2\gamma c}[(n-1)\tau_\varepsilon+\tau]^2-[(n-1)\tau_\varepsilon-\tau]^2}}{2(n-1)\left(e^{2\gamma c}-1\right)\tau_x}.$$

If $c \leq \overline{c} \equiv \frac{1}{2\gamma}\log\left(1+\frac{\tau_\varepsilon}{\tau_\varepsilon(n-1)+\tau n}\right)$, then this root is non-negative (so its square root will be real) and simplifies to:

$$\widehat{I}^2 := \frac{\sqrt{[\tau+(n-1)\tau_\varepsilon]^2 e^{2\gamma c}-[\tau-(n-1)\tau_\varepsilon]^2}}{2\tau_x(n-1)\sqrt{e^{2\gamma c}-1}} - \frac{n\tau}{2\tau_x(n-1)} - \frac{\tau+\tau_\varepsilon}{2\tau_x}.$$

$\blacksquare$

Lemma 14 *$\widehat{I}$ is a strictly decreasing function of c until it reaches $\widehat{I}=0$ at positive constant $\overline{c}$ and* $\lim_{c\downarrow 0}\widehat{I}=+\infty$.

Proof of Lemma 14. Let

$$\widehat{I}(A) := \sqrt{\frac{\sqrt{\frac{h(A)}{A}}-k}{2\tau_x(n-1)}},$$

where

$$h(A) := [\tau+(n-1)\tau_\varepsilon]^2(A+1)-[\tau-(n-1)\tau_\varepsilon]^2,$$

$$k := n\tau+(n-1)(\tau+\tau_\varepsilon),$$

and

$$A := e^{2\gamma c}-1.$$

The slope of $\widehat{I}(A)$ is given by:

$$\widehat{I}'(A) = \frac{1}{2\widehat{I}}\frac{\frac{1}{2}\frac{1}{\sqrt{\frac{h(A)}{A}}}\frac{Ah'(A)-h(A)}{A^2}}{2\tau_x(n-1)} = -\frac{1}{2\widehat{I}}\frac{\tau_\varepsilon\tau}{A^2\tau_x\sqrt{\frac{h(A)}{A}}} < 0.$$

Since A is strictly increasing in c, $A'(c) = 2\gamma e^{2\gamma c} > 0$, and $\widehat{I}$ is strictly decreasing in A, then $\widehat{I}$ is strictly decreasing in c. Note that $\widehat{I} = 0$ at $\overline{c}$ and above cost $\overline{c}$ the square root is imaginary. However, if c is too large, then no trader chooses to become informed and so no information gets embedded into the price. Thus, $\widehat{I} = 0$ for $c \geq \overline{c}$ and hence imaginary roots are not applicable.

For $n = 1$, $\widehat{I} = \sqrt{\frac{\tau_\varepsilon}{\tau_x(e^{2\gamma c}-1)} - \frac{\tau}{\tau_x}}$, so $\lim_{c\downarrow 0}\widehat{I} = +\infty$. For $n > 1$, since $\lim_{c\downarrow 0} A(c) = 0$ and $h(0) = 4(n-1)\tau_\varepsilon\tau > 0$, then $\lim_{c\downarrow 0}\widehat{I} = +\infty$. ∎

Lemma 15 *As in Proposition 6, let* $\widehat{I}(n) := \sqrt{\frac{\sqrt{\frac{h(n)}{A}} - k(n)}{2\tau_x(n-1)}}$, *where* $h(n) := [\tau + (n-1)\tau_\varepsilon]^2(A+1) - [\tau - (n-1)\tau_\varepsilon]^2$; $k(n) := n\tau + (n-1)(\tau + \tau_\varepsilon)$; *and* $A := e^{2\gamma c} - 1$. *The slope of* $\widehat{I}(n)$ *is given by* $-\tau\frac{(2+A)\tau_\varepsilon(n-1)+A\left(\tau - \sqrt{\frac{h(n)}{A}}\right)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$.

Proof of Lemma 15.

$$
\begin{aligned}
\widehat{I}'(n) &= \frac{1}{2\widehat{I}(n)} \frac{2\tau_x(n-1)\left(\frac{1}{2}\frac{\frac{h'(n)}{A}}{\sqrt{\frac{h(n)}{A}}} - k'(n)\right) - 2\tau_x\left(\sqrt{\frac{h(n)}{A}} - k(n)\right)}{4\tau_x{}^2(n-1)^2} \\
&= \frac{(n-1)\left(\frac{1}{2}h'(n) - Ak'(n)\sqrt{\frac{h(n)}{A}}\right) - \left(h(n) - Ak(n)\sqrt{\frac{h(n)}{A}}\right)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} \\
&= \frac{\frac{1}{2}h'(n)(n-1) - h(n) + A[k(n) - k'(n)(n-1)]\sqrt{\frac{h(n)}{A}}}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} \\
&= \frac{-\tau^2 A - \tau(A+2)(n-1)\tau_\varepsilon + A\tau\sqrt{\frac{h(n)}{A}}}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} = -\tau\frac{(2+A)\tau_\varepsilon(n-1) + A\left(\tau - \sqrt{\frac{h(n)}{A}}\right)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}.
\end{aligned}
$$

∎

Lemma 16 $\widehat{\lambda}(n) := n\gamma\widehat{I}(n)\left[\frac{1}{\tau_\varepsilon} + \frac{\tau_x}{\tau_\varepsilon\tau}\widehat{I}^2(n)\left(1-\frac{1}{n}\right) + \frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]$ *of Proposition 8, where* $\widehat{I}(n)$ *is as in (A9), is equivalently given by*

$$
\widehat{\lambda}(n) = \frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[\tau + (n-1)\tau_\varepsilon + \sqrt{[\tau + (n-1)\tau_\varepsilon]^2 + \frac{4\tau}{A}(n-1)\tau_\varepsilon}\right].
$$

Proof of Lemma 16. By substitution for $\widehat{I}^2(n)$ from (A9), $h(n)$ from (A10), and $k(n)$ from (A11) from the proof of Proposition 6 into the definition of $\widehat{\lambda}(n)$ in (A12) of Proposition 8, and simplification:

$$\begin{aligned}
\widehat{\lambda}(n) &:= n\gamma\widehat{I}(n)\left[\frac{1}{\tau_\varepsilon}+\frac{\tau_x}{\tau_\varepsilon\tau}\widehat{I}^2(n)\left(1-\frac{1}{n}\right)+\frac{1}{\tau}\left(1-\frac{1}{n}\right)\right]\\
&= \frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[2\tau n+2\tau_x\frac{\sqrt{\frac{h(n)}{A}}-k(n)}{2\tau_x(n-1)}(n-1)+2\tau_\varepsilon(n-1)\right]\\
&= \frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[2\tau n+\sqrt{\frac{h(n)}{A}}-k(n)+2\tau_\varepsilon(n-1)\right]\\
&= \frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]\\
&= \frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[\tau+(n-1)\tau_\varepsilon+\sqrt{[\tau+(n-1)\tau_\varepsilon]^2+\frac{4\tau}{A}(n-1)\tau_\varepsilon}\right].
\end{aligned}$$

∎

Lemma 17 *The slope of $\widehat{\lambda}(n)$ of Proposition 8 (and Lemma 16) is given by*

$$\widehat{\lambda}'(n)=\gamma\frac{[3-4n-2A(n-1)]\tau+[1-2A(n-1)]\left[(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}},$$

where $\widehat{I}(n)$ is from (A9) and $h(n)$ is from (A10).

Proof of Lemma 17

$$\lambda'(n)=\frac{\gamma\widehat{I}(n)}{2\tau\tau_\varepsilon}\left[\tau_\varepsilon+\frac{\frac{1}{2}\frac{h'(n)}{A}}{\sqrt{\frac{h(n)}{A}}}\right]+\frac{\gamma\widehat{I}'(n)}{2\tau\tau_\varepsilon}\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]$$

$$\widehat{\lambda}'(n) = \frac{\gamma}{2\tau\tau_\varepsilon}\frac{\tau_\varepsilon\sqrt{\frac{h(n)}{A}}+\frac{1}{2}\frac{h'(n)}{A}}{\sqrt{\frac{h(n)}{A}}}\frac{4\tau_x A(n-1)\widehat{I}^2(n)}{4\tau_x A(n-1)\widehat{I}(n)} - \frac{\gamma}{2\tau_\varepsilon}\left(\frac{(2+A)\tau_\varepsilon(n-1)+A\left(\tau-\sqrt{\frac{h(n)}{A}}\right)}{4\tau_x A(n-1)^2\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}\right)\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]$$

$$\widehat{\lambda}'(n) = \frac{\gamma}{2\tau\tau_\varepsilon}\frac{4\tau_x(n-1)\frac{\sqrt{\frac{h(n)}{A}}-k(n)}{2\tau_x(n-1)}\left(\tau_\varepsilon A\sqrt{\frac{h(n)}{A}}+\frac{1}{2}h'(n)\right)}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} - \frac{\gamma}{2\tau_\varepsilon(n-1)}\frac{\left[(2+A)\tau_\varepsilon(n-1)+A\tau-A\sqrt{\frac{h(n)}{A}}\right]\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \frac{\gamma}{\tau\tau_\varepsilon}\frac{\left(\sqrt{\frac{h(n)}{A}}-k(n)\right)\left(\tau_\varepsilon A\sqrt{\frac{h(n)}{A}}+\frac{1}{2}h'(n)\right)}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} - \frac{\gamma}{2\tau_\varepsilon(n-1)}\frac{\left[2(n-1)\tau_\varepsilon+A\left[\tau+(n-1)\tau_\varepsilon-\sqrt{\frac{h(n)}{A}}\right]\right]\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \frac{\gamma}{\tau\tau_\varepsilon}\frac{\left(\sqrt{\frac{h(n)}{A}}-k(n)\right)\left(\tau_\varepsilon A\sqrt{\frac{h(n)}{A}}+\frac{1}{2}h'(n)\right)}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} - \frac{\gamma}{2\tau_\varepsilon(n-1)}\frac{2(n-1)\tau_\varepsilon\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]+A\left[\tau+(n-1)\tau_\varepsilon\right]^2-h(n)}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \frac{\gamma}{\tau\tau_\varepsilon}\frac{\left(\sqrt{\frac{h(n)}{A}}-k(n)\right)\left(\tau_\varepsilon A\sqrt{\frac{h(n)}{A}}+\frac{1}{2}h'(n)\right)}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}} - \frac{\gamma}{2\tau_\varepsilon(n-1)}\frac{2(n-1)\tau_\varepsilon\left[\tau+(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]-4(n-1)\tau\tau_\varepsilon}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \gamma\frac{\frac{1}{\tau}\left(\sqrt{\frac{h(n)}{A}}-k(n)\right)\left(A\sqrt{\frac{h(n)}{A}}+\frac{1}{2}\frac{h'(n)}{\tau_\varepsilon}\right)-\left[(n-1)\tau_\varepsilon+\sqrt{\frac{h(n)}{A}}\right]+\tau}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \gamma\frac{\frac{1}{\tau}\left(h(n)-\frac{h'(n)}{2\tau_\varepsilon}k(n)\right)+\tau-(n-1)\tau_\varepsilon+\frac{1}{\tau}\left(\frac{h'(n)}{2\tau_\varepsilon}-Ak(n)-\tau\right)\sqrt{\frac{h(n)}{A}}}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \gamma\frac{\frac{1}{\tau}\left(h(n)-\frac{h'(n)}{2\tau_\varepsilon}k(n)\right)+\tau-(n-1)\tau_\varepsilon+[1-2A(n-1)]\sqrt{\frac{h(n)}{A}}}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

$$\widehat{\lambda}'(n) = \gamma\frac{\tau[3-4n+2A(1-n)]+[1-2A(n-1)](n-1)\tau_\varepsilon+[1-2A(n-1)]\sqrt{\frac{h(n)}{A}}}{4\tau_x A(n-1)\widehat{I}(n)\sqrt{\frac{h(n)}{A}}}$$

∎