Intertemporal Substitution, Precautionary Saving, and The Currency Premium

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Abstract

Engel (2016, AER) points out that existing exchange rate models cannot explain two empirical regularities simultaneously. The first is the short premium puzzle, which states that the currency premium (short premium) is positively correlated with the interest rate differential of foreign over home countries. The second, seemingly paradoxically, states that the expected currency premium over a long horizon (long premium) is negatively correlated with the interest rate differential.

We build a model with two features that are key to resolving Engel's paradox. First, the consumption mean linearly depends on consumption volatility with a positive coefficient λ. This introduces an intertemporal substitution component into the interest rate that positively depends on consumption volatility. Second, the expected consumption variance depends on consumption volatility in addition to consumption variance itself. With this setup, our model can solve the two puzzles simultaneously.

Economically speaking, for the short premium, the precautionary saving effect (higher consumption variance causes investors to save more) dominates on average, which places an upper bound on λ. For the long premium, the intertemporal substitution effect (higher consumption volatility implies higher consumption growth) dominates on average, which provides a lower bound on λ. We show that there is a range for λ such that both bounds are satisfied, thus resolving Engel's paradox.

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1 Introduction

Currency markets are extremely important in economics and finance. First, currency markets are some of the most liquid financial markets. Second, exchange rates are one of the most important financial prices because of their key roles in international trade. The study of currency markets has advanced our understanding of both economic theory and asset pricing models. Our paper contributes to this literature by reconciling two seemingly paradoxical features of currency markets.

There are two empirical regularities in currency markets. First, the currency premium over short horizons ("short currency premium" or "currency premium") is positively correlated with the interest rate differential between foreign and home countries. This is called the "forward premium puzzle" or "interest parity puzzle" (see Fama (1984)). Second, the expected future currency premium over long horizon (long currency premium) and the sum of expected future currency premiums over all horizons (cumulative currency premium) are negatively correlated with the interest rate differential. This is called the excess co-movement puzzle (see e.g. Evans (2012)). Engel (2016) points out that these two empirical regularities constitute a paradox because existing models cannot account for them simultaneously.

There is no economic principle that dictates the signs of the above two correlations; rather, they depend on the detailed properties of asset pricing models. We build a model with two features that are both absent from the literature. First, we assume that the consumption mean linearly depends on the consumption volatility (consumption volatility in the consumption mean) with a positive coefficient λ. This implies that the interest rate has a positive dependence on the consumption volatility through its intertemporal substitution component, in addition to negative dependence on the consumption variance through its precautionary saving component (standard in the literature). Second, expected consumption variance depends on consumption volatility in addition to consumption variance itself. This allows the consumption volatility to dominate the expected consumption variance in the long horizon.

In our model, when the home consumption variance is high, the home interest rate can be high or low depending on the strength of the intertemporal substitution effect, which is characterized by λ. For the short premium, if λ is not too large, the precautionary saving effect dominates on average, and the home country interest rate and the home country risk premium are negatively correlated. This leads to, *ceteris paribus*, a positive correlation between the interest rate differential (which depends negatively on the home interest rate) and the currency premium. Thus resolving short premium puzzle places an upper bound on λ. For the long premium, if λ is not too small, the intertemporal substitution effect dominates on average, and the home country interest rate and the home country long premium are positively correlated. This leads to, *ceteris paribus*, a negative correlation between the interest rate differential and the long currency premium. Thus resolving long premium puzzle places a lower bound on λ. We show that there is a range for λ such that both bounds are satisfied and thus resolving Engel's paradox.

The two features of our model have been explored in the literature. For example, consumption volatility in the consumption mean, is documented empirically in Bekaert and Liu (2004). The other feature, which centers on the consumption volatility process, holds for many processes except affine processes. As an alternative, Engel (2016) assumes consumption variance has two independent components. The stochastic volatility model in our paper is first proposed by Stein and Stein (1991). Our model leads to a tractable term structure of interest rates as in Constantinides (1992).

Prior studies have relied on richer features of preference, such as habit formation, recursive utility and long-run risks (see, for example, Verdelhan (2010), Colacito and Croce (2011), Bansal and Shaliastovich (2013), Colacito and Croce (2013), and Lustig et al. (2015)). Engel (2016) shows that none of these models can account simultaneously for the two puzzles described above. In our paper, the preference is the standard Constant Relative Risk Aversion (CRRA) utility but the consumption process is assumed to have the above two additional features absent from standard models.

The forward premium puzzle implies that high-interest currencies tend to appreciate, as documented in Hansen and Hodrick (1980), Bilson (1980), and Fama (1984), among others. The literature attempting to solve this anomaly is vast: see, e.g., Frankel and Engel (1984), Backus et al. (1993), Froot and Thaler (1990), Backus et al. (2001), and Brennan and Xia (2006).

The excess co-movement puzzle implies that the covariance of the level of real exchange rate and the real interest rate differential is higher than would be implied by uncovered interest rate parity (Engel (2016), Evans (2012)). Studies addressing the excess co-movement in exchange rate levels are well known in the international economics literature (see, e.g. Frankel and Meese (1987), Rogoff (1996), Engel and West (2004), Engel and West (2006), Alquist and Chinn (2008), Mark (2009), Faust et al. (2007), Andersen et al. (2007), Clarida and Waldman (2008), Bacchetta and van Wincoop (2010), and Evans (2012)). This puzzle leads to more excessive overshooting than the classical Dornbusch model and Mundell-Fleming model suggest (see also Dornbusch (1976) and Frankel (1979)).

2 Engel's Paradox

In this section, we introduce the puzzles studied in Engel (2016) and Engel's paradox.

We denote the home country by the superscript h and the foreign country by the superscript f. We express the real exchange rate S_t in home currency per unit of foreign currency and use r_t^f and r_t^h to denote foreign and home real interest rates respectively. In this paper, we only study the real exchange rate and the real interest rate. We refer to the real exchange rate and the real interest rate as "exchange rate" and "interest rate", respectively.

We follow the definition in Engel (2016), where currency excess return is defined in log form:

$$\rho_{t+1} = s_{t+1} - s_t + r_t^f - r_t^h \ , \tag{2.1}$$

where $s_t = \ln S_t$ is the log of the real exchange rate. We define the currency premium ("short premium") as the conditional expected currency excess return:

$$E_t[\rho_{t+1}] = E_t[s_{t+1}] - s_t + r_t^f - r_t^h \ .$$

The expected currency premium over horizon j, ($j \in \{0, 1, 2, \cdots\}$) is given by $E_t[\rho_{t+j+1}]$. For large j, we refer to it as the "long premium". Finally, we use the term "cumulative premium" to refer to the cumulative expected currency premium

$$\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}] \ .$$

Uncovered interest parity (UIP) implies that the short premium is zero. Thus, if UIP holds, both the long premium and the cumulative premium should also be zero:

$$\begin{aligned} E_t[\rho_{t+j+1}] &= 0, \qquad \textit{for all } j, \\ \sum_{j=0}^{\infty} E_t[\rho_{t+j+1}] &= 0 \ . \end{aligned}$$

Empirical tests find that the short, long, and cumulative premium are all nonzero. However, the covariance of the short premium with the interest rate differential has the opposite sign from the covariance of the long premium (as well as the cumulative premium) with the interest rate differential, as pointed out by Engel (2016).

On the one hand, the covariance between the short premium and the interest rate differential is positive,

$$cov\left[E_t[\rho_{t+1}], r_t^f - r_t^h\right] > 0 \ . \qquad (\textit{short premium puzzle}) \tag{2.2}$$

We term this the short premium puzzle for the real exchange rate. A stronger version of the short premium puzzle is the forward premium puzzle

$$cov\left[E_t[s_{t+1} - s_t], r_t^f - r_t^h\right] > 0 \ . \qquad (\textit{forward premium puzzle}) \tag{2.3}$$

Both puzzles are well documented in empirical studies.

On the other hand, the covariance between the long premium and the interest rate differential is negative in the data, that is, for large j,

$$cov\left[E_t[\rho_{t+j+1}], r_t^f - r_t^h\right] \ < \ 0. \qquad (\textit{long premium puzzle}) \tag{2.4}$$

We term this the long premium puzzle. Finally, we consider the cumulative premium puzzle, which is even stronger than the long premium puzzle. Empirical studies find that the covariance of the cumulative premium and the interest rate differential is negative,

$$cov\Big[\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}], r_t^f - r_t^h\Big] \ < \ 0 \ . \qquad (\textit{excess comovement puzzle}) \tag{2.5}$$

This puzzle is closely related to the excess co-movement puzzle described in the literature.

The short premium puzzle implies that a high-interest-rate currency tend to have a higher premium. This is a violation of uncovered interest rate parity (UIP). The risk-based explanation for the short premium puzzle explains the short premium as compensation for risk, see e.g. Backus et al. (2001) and Brennan and Xia (2006). The forward premium puzzle provides an explanation for the empirical regularity that the high-interest-rate currencies tend to appreciate.

The excess co-movement puzzle has important implication for the level of exchange rate. As shown in Engel (2016), by telescoping (2.1) forward, the level of exchange rate can be expressed as the difference between the cumulative expected interest rate differential and the cumulative currency premium, if we assume that the exchange rate, interest rate differential and short currency excess return are all stationary,

$$s_t - \lim_{j\to\infty} E_t[s_{t+j}] = E_t \sum_{j=0}^{\infty} \left[r^f_{t+j} - r^h_{t+j} - \overline{r^f - r^h} \right] - E_t \sum_{j=0}^{\infty} [\rho_{t+j+1} - \overline{\rho}] \ , \tag{2.6}$$

where $\overline{r^f - r^h}$ is the unconditional mean of the interest rate differential and $\overline{\rho}$ is the unconditional mean of the excess return.

Taken together with equation (2.6), the inequality (2.5) states that the covariance between the exchange rate level and the interest rate differential is higher than the covariance between the cumulative expected currency premium and the interest rate differential, which is the covariance under UIP. This higher covariance implies that the real exchange rate level is excessively volatile - in other words, the exchange rate levels exhibit excess co-movement, implying higher over-shooting than the classical Dornbusch and Mundell-Fleming models propose (see e.g. Engel (2016)).

The inequalities (2.4) and (2.5), seem to contradict inequalities (2.2) and (2.3). Engel (2016) shows that a variety of models - including recursive utility, habit formation, long-run risk, and others - cannot simultaneously accommodate, the forward premium puzzle and the excess co-movement puzzle. He refers to this fact as a paradox. In this paper, we refer to the pair of inequalities (2.2) and (2.4) as Engel's paradox in long premium formulation, and term the inequalities (2.2) and (2.5) Engel's paradox in cumulative premium formulation.

3 A Two-Country Exchange Rate Model

3.1 Utility Functions

We consider a two-country economy and assume that a representative agent exists in each country. The aggregate consumption of each agent can be viewed as the consumption bundles of goods allocated as in Colacito and Croce (2013).[2]

[2] In general, representative households from different countries have common factors in their consumption processes. We can model the common factor as C^c_t; we then use C^h_t and C^f_t to represent

We assume that the aggregate consumption processes for each country, $i \in \{h, f\}$, are independent with identical parameters. The independence of C_t^h and C_t^f can be viewed as the allocation result in which agents have complete home bias in consumption. The exchange rate exists in this scenario as the limiting case - see Colacito (2006).

Similar to Engel (2016), we assume that all parameters for home and foreign countries are identical. This assumption allows the model to be symmetric between home and foreign countries so that the same pattern can be observed on both sides of the currency investments. This assumption is frequently used in the prior literature.

We assume that the representative agent in each country $i \in \{h, f\}$ has a constant relative risk aversion utility

$$\sum_{t=0}^{\infty} \mathrm{E}_0 \left[e^{-\beta t} \frac{C_t^{i\,1-\gamma}}{1-\gamma} \right] ,$$

where C_t^i is the aggregate consumption for agent i, $\beta > 0$ is the subjective discount coefficient, and $\gamma > 0$ is the risk-aversion coefficient. We assume β and γ are identical across both countries. Engel (2016) studies the currency premium using the Epstein and Zin (1989) recursive utility. In this paper, a time-additive preference is sufficient to jointly solve the two puzzles of interest, so the recursive utility is not needed.

3.2 Aggregate Consumption Process

As is standard in the literature, we assume that the aggregate consumption process for each country is exogenously given. We denote, for country $i \in \{h, f\}$, $c_t^i = \ln C_t^i$ to be the log-aggregate consumption for the corresponding representative household. We assume that the difference in log-aggregate consumption $c_{t+1}^i - c_t^i$ (which we will term consumption growth in this paper) satisfies the following process:

$$c_{t+1}^i - c_t^i = \mu_{ct}^i + \sigma_{ct}^i \varepsilon_{ct+1}^i , \tag{3.1}$$

where the conditional mean μ_{ct}^i of the consumption growth is given by

$$\mu_{ct}^i = \lambda(x_t^i + \theta) + (h - 1/2)(x_t^i + \theta)^2 , \tag{3.2}$$

and the conditional "volatility" σ_{ct}^i of the consumption growth is given by

$$\sigma_{ct}^i = (x_t^i + \theta) \ ,$$

the idiosyncratic portion of consumption in each country, so that the aggregate consumption for each agent $i \in \{h, f\}$, is $C_t^i C_t^c$. Since the change in the exchange rate is the ratio of the foreign pricing kernel over the home pricing kernel, the common factor will be canceled out when studying the exchange rate. If we also assume that C_t^c is independent of C_t^i, C_t^c will be canceled out when we study the interest rate differential, making it irrelevant to the study of the puzzles in this paper. Thus, in this paper, we ignore the common factor C_t^c.

where λ, h, and θ are constants. We assume that x_t^i follows an AR(1) process

$$x_{t+1}^i = \varphi x_t^i + \sigma \varepsilon_{xt+1}^i \ . \tag{3.3}$$

We also assume that the innovations ε_{ct+1}^i and ε_{xt+1}^i are i.i.d, following a $N(0,1)$ distribution across country and time. The constants φ and σ satisfy $0 < \varphi < 1$ and $\sigma > 0$.

Note that because x_t^i is normal, $(x^i + \theta)$ can be negative; thus, consumption volatility equals $|x_t^i + \theta|$. Without loss of any generality, we assume $\theta > 0$. When $\theta > 0$ is large enough, $x_t^i + \theta$ is positive with probability close to 1, and we will refer to $x_t^i + \theta$ as consumption volatility. This model of stochastic volatility is first used in Stein and Stein (1991).

Note that the consumption mean μ_{ct}^i depends on both conditional volatility $x_t^i + \theta$ and conditional variance $(x_t^i + \theta)^2$. As will be shown in this paper, the conditional volatility term in μ_{ct}^i, $\lambda(x_t^i + \theta)$, leads to a positive correlation between r_t and the risk premium through the intertemporal substitution component of the interest rate and is key to resolving Engel's paradox. This type of consumption volatility in consumption mean model is documented empirically in the literature - see, e.g., Bekaert and Liu (2004).

The parameter h is not crucial for our results, but provides some flexibility without sacrificing tractability. The consumption mean in Engel (2016) does not depend on consumption volatility, which is its key difference from our formulation.

Engel (2016) uses two variance processes with different AR(1) coefficients to generate two decay modes. Our setup generates two decay modes as long as θ is positive.

We could also extend our model by assuming x_t^i to be the square of the OU process, so that $x_t^i + \theta$ is guaranteed to be positive. In this case, the economic intuition remains unchanged. In this paper, we assume x_t^i follows the Ornstein-Uhlenbeck (OU) process (3.3) for simplicity and tractability.

3.3 Pricing Kernels

For each country $i \in \{h, f\}$, the pricing kernel with the consumption bundles of country i as the numeraire can be written as

$$\pi_{t+1}^i = e^{-\beta} e^{-\gamma(c_{t+1}^i - c_t^i)} = e^{-\beta} e^{-\gamma(\mu_t^{ic} + \sigma_{ct}^i \varepsilon_{ct+1}^i)} \ . \tag{3.4}$$

We assume the market to be complete so that the pricing kernel is unique. The pricing kernel of home country π_{t+1}^h is different from the pricing kernel of foreign country π_{t+1}^f because the numeraire is different (see Brennan and Xia (2006)).

Note that the pricing kernel of the home country,

$$\pi_{t+1}^h = \exp\{-\beta - \gamma(\mu + \lambda(x_t^h + \theta) + (h - 1/2)(x_t^h + \theta)^2 + (x_t^h + \theta)\varepsilon_{ct+1}^h)\} \ ,$$

depends only on the home country risk ε_{ct+1}^h; thus only the home country risk is priced by the home country pricing kernel, with a market price of risk $x_t^h + \theta$. The foreign country risk, in contrast, is not priced by the home country pricing kernel: ε_{ct+1}^f does not appear in the pricing kernel π_{t+1}^h.

4 Interest Rates and Interest Rate Differential

From the pricing kernel, we can determine the one-period real interest rate for country $i \in \{h, f\}$ as follows:

$$\begin{aligned} r_t^i &= -\ln E_t[\pi_{t+1}^i] \\ &= \beta + \gamma\mu_{ct}^i - \frac{\gamma(1+\gamma)}{2}\sigma_{ct}^{i2} \\ &= \beta + \gamma\mu + \gamma\lambda(x_t^i + \theta) - \gamma\left(\frac{1+\gamma}{2} - h\right)(x_t^i + \theta)^2 \quad . \end{aligned} \tag{4.1}$$

The interest rate r_t^i can be decomposed into two components: the precautionary saving component, which is represented[3] by $-\frac{\gamma(\gamma+1)}{2}\sigma_{ct}^{i2}$, and the intertemporal substitution component, $\gamma\mu_{ct}^i$.

The precautionary saving component of the interest rate in our model is

$$-\frac{\gamma(\gamma+1)}{2}\sigma_{ct}^{i2} \;=\; -\frac{\gamma(\gamma+1)}{2}(x_t^i + \theta)^2 \ . \tag{4.2}$$

which negatively depends on the conditional variance $(x_t^i + \theta)^2$. The intertemporal substitution component of the interest rate

$$\gamma\mu_{ct}^i = \gamma\mu + \gamma\lambda(x_t^i + \theta) + \gamma h(x_t^i + \theta)^2 \ , \tag{4.3}$$

where γ is the inverse of the elasticity of intertemporal substitution (EIS), depends on conditional volatility $x_t^i + \theta$. When $\lambda > 0$, this component implies that the interest rate positively depends on conditional volatility. Similarly, when $h > 0$, this term of the interest rate positively depends on conditional variance.

The interest rate r_t^i can also be written as

$$r_t^i = \left(\gamma\lambda - 2\theta\gamma\left(\frac{1+\gamma}{2} - h\right)\right) x_t^i - \gamma\left(\frac{1+\gamma}{2} - h\right) x_t^{i2} + \Theta \ , \tag{4.4}$$

where Θ represents the remaining constants. This equation shows that r_t^i is a non-monotonic function of x_t^i.

As a side note, the above interest rate leads to a tractable term structure of interest rates, which was first studied by Constantinides (1992).

5 Currency Premium and Country Premium

Since we assumed the market to be complete, the exchange rate is proportional to the ratio of pricing kernels of the two countries. The log growth rate of the real exchange rate is

$$s_{t+1} - s_t = \ln \pi_{t+1}^f - \ln \pi_{t+1}^h \ ,$$

[3] Note that the precautionary saving component depends on $(\gamma + 1)$ which is proportional to the prudence (the third derivative of the utility function, see Leland (1968) and Kimball (1990)).

where for $i \in \{h, f\}$, π^i_{t+1} is the pricing kernel with the consumption bundles of country i as numeraire. This relationship has been studied in Backus et al. (2001) and Brennan and Xia (2006). As mentioned earlier, if the pricing kernels have a common component, it will be canceled in the difference.

5.1 Currency Premium

An investment in foreign currency can be compared to an investment in a stock that pays dividends continuously. In this analogy, the exchange rate S_t corresponds to the stock price, and the foreign exchange rate r^f_t corresponds to the continuous compound dividend yield of the stock.

The one-period excess return on the investment in foreign currency in our model is

$$\begin{aligned} R_{t+1} &= \frac{S_{t+1} e^{r^f_t - r^h_t}}{S_t} = \frac{\pi^f_{t+1}}{\pi^h_{t+1}} e^{r^f_t - r^h_t} \\ &= \exp\left\{ \frac{1}{2}\gamma^2 \sigma^{h2}_{ct} + \gamma\sigma^h_{ct}\varepsilon^h_{ct+1} - \frac{1}{2}\gamma^2\sigma^{f2}_{ct} - \gamma\sigma^f_{ct}\varepsilon^f_{ct+1} \right\} \quad . \end{aligned}$$

From the above equation, R_{t+1} has home country risk ε^h_{ct+1} with exposure (beta coefficient) $\gamma\sigma^h_{ct}$, and foreign country risk ε^f_{ct+1} with exposure (beta coefficient) $-\gamma\sigma^f_{ct}$. However, only home country risk is priced, because the home pricing kernel has a market price of risk $\gamma\sigma^h_{ct}$ for home country risk ε^h_{ct+1} and a market price of risk 0 for the foreign country risk ε^f_{ct+1} Thus, the risk premium is

$$(\gamma\sigma^h_{ct}) \cdot (\gamma\sigma^h_{ct}) + 0 \cdot (-\gamma\sigma^f_{ct}) = (\gamma\sigma^h_{ct})^2$$

and

$$E_t[R_{t+1}] = e^{\gamma^2\sigma^{h2}_{ct}} \quad .$$

We define the country premium of country $i \in \{h, f\}$ as

$$\nu^i_t = \gamma^2\sigma^{i2}_{ct} \quad . \tag{5.1}$$

The log currency premium can be then expressed in terms of the country premium:

$$\begin{aligned} E_t[\rho_{t+1}] &= E_t[\ln R_{t+1}] \\ &= \nu^h_t - \frac{\nu^h_t + \nu^f_t}{2} = \frac{1}{2}(\nu^h_t - \nu^f_t) \quad , \end{aligned} \tag{5.2}$$

where $-\frac{\nu^h_t + \nu^f_t}{2}$ is due to Jensen's effect. The logarithmic form makes the risk premium symmetric.

In our model, assuming the distribution of the consumption process (3.1), the country premium (5.1) for country $i \in \{h, f\}$ is

$$\nu_t^i = \gamma^2 \ (x_t^i + \theta)^2 \quad . \tag{5.3}$$

The country premium is proportional to the conditional variance of consumption growth. Thus, it is negatively proportional to the precautionary saving component of the interest rate (equation (4.2)), $-\frac{\gamma(\gamma+1)}{2}(x_t^i + \theta)^2$.

5.2 Term Structure of the Country Premium

We now study the term structure of the expected country premium, $E_t[\nu_{t+j}^i]$, of j-periods in the future.

Lemma 5.1. *For country $i \in \{h, f\}$, the term structure of the expected country premium $j \in \{0, 1, 2, \cdots\}$ periods in the future is*

$$E_t[\nu_{t+j}^i] = \gamma^2 \ \varphi^{2j} x_t^{i2} + 2 \ \gamma^2 \theta \ \varphi^j x_t^i + \Theta_1 \quad , \tag{5.4}$$

where $\Theta_1 = \frac{\gamma^2}{2} \left[\frac{1-\varphi^{2j}}{1-\varphi^2} + \theta^2 \right]$ is a constant.

We term $E_t[\nu_{t+j}^i]$ the long country premium for large j, and $\sum_{j=0}^{\infty} E_t[\nu_{t+j}^i - \Theta_1]$ the cumulative country premium.

Note that when $\theta \neq 0$, ignoring the constant term Θ_1, which is irrelevant to the correlation, $E_t[\nu_{t+j}]$ has two terms x_t^i and x_t^{i2} that decay as functions of j, φ^j and φ^{2j}, respectively.

The term x_t^i in the expected future variance is one of the key features of our model. In the literature, consumption variance is modeled as a CIR process or a square root process, which implies that expected future variance depends only on itself. This is one reason previous models fail to jointly solve the forward premium puzzle and the excess co-movement puzzle in Engel (2016).

6 Resolution of Engel's Paradox

In this section, we study two formulations of Engel's paradox.

Engel's paradox states that the sign of the covariance of the expected currency premium with the interest rate differential changes between the short horizon and the long horizon. In our one-state-variable setup, either the interest rate, the expected currency premium, or both needs to be non-monotonic functions of the state variable in order to resolve Engel's paradox, as shown in the following proposition.

Proposition 6.1. *If the interest rate r and the expected country premium are both monotonic functions of x, then the covariance between the expected currency premium and the interest rate differential has the same sign for all horizons j.*

In our paper, x_t has a mean of 0, thus most of its probability mass is concentrated near 0. When $\theta > 0$ is large, the probability mass for $x_t + \theta < 0$ is small, so both $x_t + \theta$ and $(x_t + \theta)^2$ are effectively increasing in x_t. The expected country premium, given by equation (5.4), is a linear function of $x_t + \theta$ and $(x_t + \theta)^2$ with positive coefficients, and thus is increasing in x_t. However, the interest rate $r(x_t)$ given by (4.4) depends both on $x_t + \theta$ and $-(x_t + \theta)^2$, which are generated by the intertemporal substitution effect and the precautionary saving effect respectively, and is thus a non-monotonic function of x_t. This non-monotonic dependence of the interest rate on x_t is key to resolving Engel's paradox.

Note that the covariance between the expected currency premium and the interest rate differential is related to the covariance between the country premium and the interest rate,

$$\begin{aligned} \operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \frac{1}{2}\operatorname{cov}\left(E_t[\nu_{t+j}^h] - E_t[\nu_{t+j}^f],\ r_t^f - r_t^h\right) \\ &= -\operatorname{cov}\left(E_t[\nu_{t+j}^h],\ r_t^h\right) . \end{aligned}$$

The first equality follows from the independent consumption assumption between countries, while the second equality follows from the independence and the identical distribution assumptions.

From the equations for the interest rate (4.4) and the country premium (5.4), we can prove the following result.

Lemma 6.2. *The term structure of the covariance between the expected currency premium and the interest rate differential over horizon $j \in \{0, 1, 2, \cdots\}$ is*

$$\begin{aligned} &\operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) \\ &= \gamma^3 \left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j}\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\ \gamma^3 \left[2\theta^2 \left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\ \varphi^j\ \frac{\sigma^2}{1-\varphi^2} \quad . \end{aligned} \tag{6.1}$$

The proof of this lemma is given in Appendix A. This lemma will be used repeatedly below.

For a better understanding of equation (6.1), we first consider some special cases.

Corollary 6.3. *If $\theta = 0$ and $\lambda \neq 0$, the covariance between the expected future currency premium and the interest rate differential is positively proportional to the currency premium:*

$$\begin{aligned} \operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \gamma^3 \left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j} \frac{2\sigma^4}{(1-\varphi^2)^2} \\ &= \varphi^{2j}\operatorname{cov}\left(\rho_{t+1},\ r_t^f - r_t^h\right) \end{aligned}$$

for all $j \in \{0, 1, 2, \cdots\}$.

When $\theta = 0$, the expected country premium, $E_t[\nu^i_{t+j}] = x^{i2}_t \varphi^{2j}$, is proportional to the country premium itself. This is true in widely used affine models of stochastic variance. For example, affine models of stochastic variance are used in Engel (2016). In this case, the above covariance has the same sign for all j and thus Engel paradox cannot be resolved.

Corollary 6.4. *If $\lambda = 0$ and $\theta \neq 0$, the covariance of the expected future currency premium with the interest rate differential is positively proportional to $\left(\frac{1+\gamma}{2} - h\right)$:*

$$\begin{aligned}
&\operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r^f_t - r^h_t\right) \\
&= \gamma^3 \left(\frac{1+\gamma}{2} - h\right)\ \varphi^{2j} \frac{2\sigma^4}{(1-\varphi^2)^2} + 4\gamma^3\theta^2 \left(\frac{1+\gamma}{2} - h\right)\ \varphi^j \frac{\sigma^2}{1-\varphi^2} \\
&= \gamma^3 \left[\varphi^{2j}\ \frac{2\sigma^4}{(1-\varphi^2)^2} + 4\theta^2\ \varphi^j \frac{\sigma^2}{1-\varphi^2}\right] \left(\frac{1+\gamma}{2} - h\right),
\end{aligned}$$

for all $j \in \{0, 1, 2, \cdots\}$.

In this case, even though the expected currency premium has two decay modes, the covariance still has the same sign for all j. This is because the interest rate negatively depends on consumption variance and does not depend on consumption volatility.

Engel (2016) considers an extensive list of models, such as recursive utility, habit formation, delayed overshooting, and long-run risks. In these models, the consumption growth processes are affine. Two decay modes for the expected currency premium can be achieved by assuming multiple factors of volatility processes. However, in these models, the consumption mean does not depend on consumption volatility; thus, there is no intertemporal substitution effect. Due to Corollary 6.4, the covariance of the expected currency premium with the interest rate differential remains positive or negative for all horizons, depending on the sign of $\left(\frac{1+\gamma}{2} - h\right)$.

6.1 Short Premium Puzzle

The short premium puzzle (2.2) states that the short premium is positively correlated with the interest rate differential. Noting that the covariance between the short premium $E_t[\rho_{t+1}]$ and the interest rate differential is given by (6.1) with $j = 0$, we have the following property.

Proposition 6.5. *The short premium puzzle is solved if*

$$\lambda\theta < \left(2\theta^2 + \frac{\sigma^2}{1-\varphi^2}\right)\left(\frac{1+\gamma}{2} - h\right) \quad . \tag{6.2}$$

To understand this condition intuitively, note that the interest rate r_t can be written as a non-monotone function of x_t,

$$r_t = -\gamma\left(\frac{\gamma+1}{2} - h\right)(x_t - x_r)^2,$$

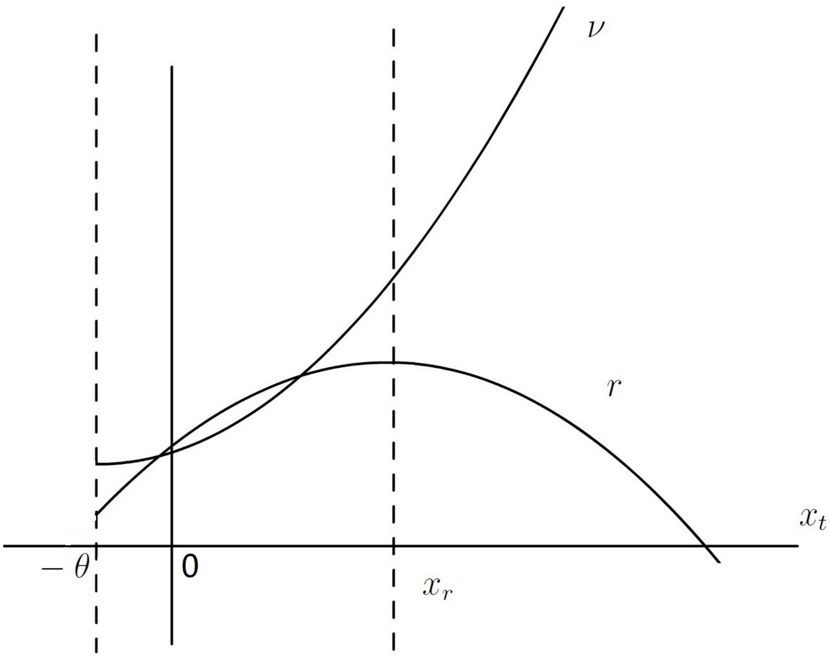

Figure 1: $r(x_t)$ and $\nu(x_t)$ are both non-monotone in x_t. For $x_t \in (-\theta, x_r)$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$, both $r(x_t)$ and $\nu(x_t)$ increase with x_t and thus increase with each other. For $x_t > x_r$ or $x_t < -\theta$, $r(x_t)$ decrease with $\nu(x_t)$. Thus the unconditional covariance between the two is negative if λ is small enough. Note that the region $x_t < -\theta$ has negligible probability mass if $\theta \gg 0$.

where $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. The country premium ν_t with $\nu_t = \gamma^2(x_t + \theta)^2$ is also a non-monotone function of x_t. These two functions of x_t are plotted in Figure 1.

Given our assumptions that $\theta > 0$ and $\lambda > 0$, and if we further assume $h < \frac{\gamma+1}{2}$, then we have $x_r > -\theta$. Note that for $x_t \in (-\theta, x_r)$, r_t and ν_t are positively correlated and the intertemporal substitution effect dominates. In contrast, outside this interval, r_t and ν_t are negatively correlated and the precautionary saving effect dominates. The unconditional covariance between r_t and ν_t is the average of the covariances for all x_t. When λ approaches 0, the interval $(-\theta, x_r)$ becomes the zero set, and the covariance between r_t and ν_t is negative. In contrast, when λ increases, the interval also increases and the covariance becomes positive. Thus, for the unconditional covariance to be negative, λ cannot be too large, as specified by equation (6.2).

6.2 Long Premium Paradox

We now study the long premium puzzle and provide conditions for resolving the long premium formulation of Engel's paradox. The long premium puzzle states that for large j, the expected future premium must be negatively correlated with the interest rate differential.

Proposition 6.6. *The condition for solving the long premium puzzle is*

$$\lambda\theta \; > \; 2\theta^2 \left(\frac{1+\gamma}{2} - h \right). \tag{6.3}$$

Putting conditions (6.2) and (6.3) together, we have the following proposition which specifies the condition for the resolution of the weak form of Engel's paradox.

Proposition 6.7. *The simultaneous resolution of the short premium puzzle and the long premium puzzle requires the following conditions:*

$$2\theta^2 \left(\frac{1+\gamma}{2} - h \right) < \lambda\theta < \left(2\theta^2 + \frac{\sigma^2}{1-\varphi^2} \right) \left(\frac{1+\gamma}{2} - h \right) . \tag{6.4}$$

The first and second inequalities are the conditions for solving the long and short premium puzzles, respectively. When both inequalities are satisfied, the long formulation of Engel's paradox is resolved.

Taken together, conditions (6.2) and (6.3) imply the following corollary.

Corollary 6.8. *To simultaneously resolve the short premium puzzle and the long premium puzzle, for terms in the interest rate that depends on $(x_t + \theta)^2$, the precautionary saving component must dominates the intertemporal substitution component*

$$h < \frac{1+\gamma}{2} . \tag{6.5}$$

In many existing models, consumption growth follows affine processes and the consumption mean depends on consumption variance only. Thus, both the intertemporal substitution component and the precautionary saving component depend on consumption variance only. In this setup, the condition for the precautionary saving effect to dominate the intertemporal substitution effect is equation (6.5). This condition is assumed in Engel (2016). In our model, the consumption mean depends on both the consumption volatility and the consumption variance, so the condition in (6.5) is insufficient. In stead, condition (6.4) is needed.

Note that the consumption variance $(x_t + \theta)^2$ appears in r_t as

$$-\gamma\left(\frac{1+\gamma}{2} - h\right)(x_t+\theta)^2 \quad .$$

Here, $\gamma h(x_t + \theta)^2$ is from the intertemporal substitution component while $-\frac{\gamma(1+\gamma)}{2}$ is from the precautionary saving component. Equation (6.5) states that, for terms that depends on $(x_t+\theta)^2$, the precautionary saving component dominates the intertemporal substitution component.

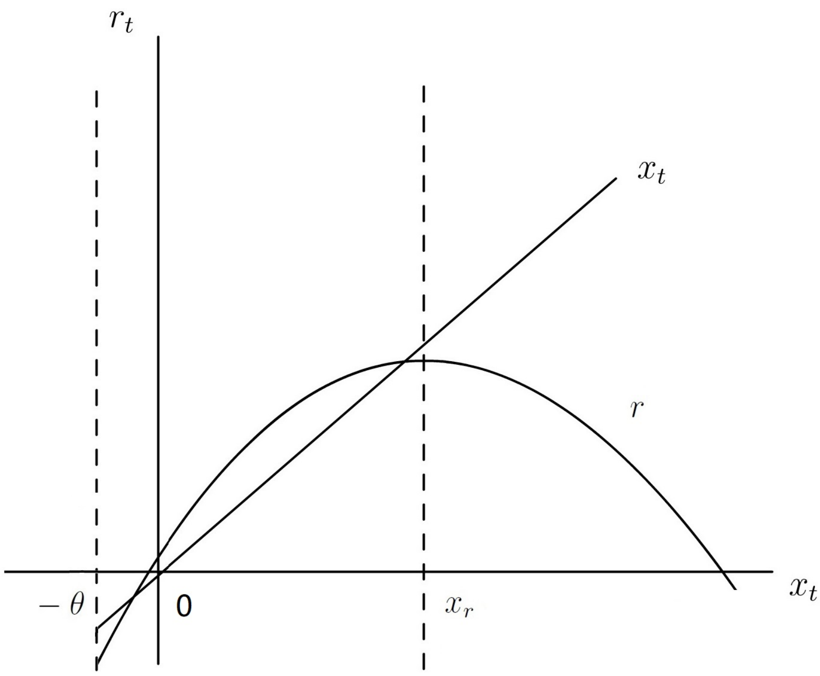

Figure 2: The long country premium is proportional to x_t. $r(x_t)$ is non-monotone in x_t. It increases with x_t for $x_t \leq x_r$ and decrease for $x_t > x_r$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. The over all correlation between $r(x_t)$ and x_t is positive if $x_r > 0$.

The expected country premium $\mathrm{E}_t[\nu_{t+j}]$ for large j is dominated by x_t (see equation (5.4),which is plotted in Figure 2 together with the interest rate r_t). Note that r_t

and $\mathrm{E}_t[\nu_{t+j}]$ are positively correlated for $x_t < x_r$ and the intertemporal substitution effect dominates in this interval; in contrast, r_t and ν_t are negatively correlated and the precautionary saving effect dominates if $x_t > x_r$. The unconditional covariance between r_t and ν_t is the average of the covariances for all $x_t \in (-\infty, \infty)$. Thus, the unconditional covariance is positive if and only if $x_r > 0$, which is equivalent to equation (6.3).

6.3 Cumulative Premium Paradox

In this subsection, we study the cumulative premium puzzle and provide conditions for resolving the cumulative premium formulation of Engel's paradox. Using equation (5.4), one can readily show that the cumulative country premium is

$$\sum_{j=0}^{\infty} E_t[\nu^i_{t+j} - \Theta_1] = \gamma^2 \, \frac{1}{1-\varphi^2} x_t^{i2} + 2\,\gamma^2\theta \, \frac{1}{1-\varphi} x_t^i \quad . \tag{6.6}$$

Using the above equation, we can prove the following property.

Lemma 6.9. *The covariance between the cumulative expected future currency premium and the interest rate differential is*

$$\mathrm{cov}\left(\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}], r_t^f - r_t^h\right) = \mathrm{cov}\left(\sum_{j=0}^{\infty}(E_t[\nu^h_{t+j}] - E_t[\nu^f_{t+j}]), r_t^f - r_t^h\right)$$
$$= \gamma^3 \left(\frac{1+\gamma}{2} - h\right) \, \frac{1}{1-\varphi^2} \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\gamma^3 \left[2\theta^2 \left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right] \, \frac{1}{1-\varphi} \frac{\sigma^2}{1-\varphi^2} \ .$$

The proof of this lemma is given in Appendix A. The cumulative premium puzzle given in equation (2.5) can be solved accordingly.

Proposition 6.10. *The condition for solving the cumulative premium puzzle is*

$$\frac{1}{2}\left(\frac{1+\gamma}{2} - h\right) \, \frac{1}{1-\varphi^2} \frac{2\sigma^2}{1-\varphi^2} \ < \ \left[\lambda\theta - 2\theta^2\left(\frac{1+\gamma}{2} - h\right)\right] \, \frac{1}{1-\varphi} \quad . \tag{6.7}$$

Together with condition (6.2), we have the following proposition which specifies the condition for the resolution of the strong form of Engel's paradox.

Proposition 6.11. *The simultaneous resolution of the short premium puzzle and the cumulative premium puzzle requires the following conditions:*

$$\left[2\theta^2 + \frac{\sigma^2}{(1+\varphi)(1-\varphi^2)}\right]\left[\frac{1+\gamma}{2} - h\right] < \lambda\theta < \left[2\theta^2 + \frac{\sigma^2}{1-\varphi^2}\right]\left[\frac{1+\gamma}{2} - h\right]. \tag{6.8}$$

The first and second inequalities are the conditions for solving the cumulative puzzle and the short premium puzzle respectively. When both inequalities are satisfied, the cumulative formulation of Engel's paradox is resolved.

Condition (6.8) implies the following corollary.

Corollary 6.12. *To simultaneously resolve the short premium puzzle and the cumulative premium puzzle, for terms that depends on $(x_t + \theta)^2$, the precautionary saving component must dominates the intertemporal substitution component:*

$$h < \frac{1+\gamma}{2} \ . \tag{6.9}$$

This condition states that the precautionary effect should dominates the intertemporal substitution effect in terms of consumption variance. This is required to solve the short premium puzzle, Engel (2016).

Note that (6.8) is stronger than (6.4), which is expected. By using a telescoping sum, it is possible to show that the cumulative premium is related to the level of exchange rate S_t. The negative correlation between the cumulative expected currency premium and the interest rate differential leads to a level of volatility exceeding that predicted by the UIP or Dornbusch models.

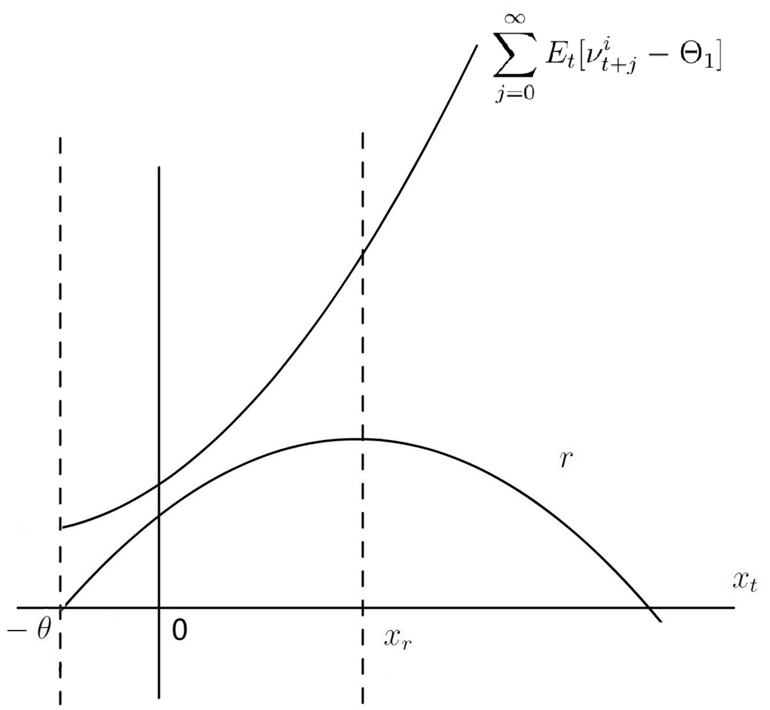

Figure 3: $r(x_t)$ and $\sum_{j=0}^{\infty} E_t[\nu_{t+j}](x_t)$ are both non-monotone in x_t. Both $r(x_t)$ and $\sum_{j=0}^{\infty} E_t[\nu_{t+j}](x_t)$ increase with x_t, and thus, increase with each other if $-\theta(1+\varphi) < x_t < x_r$, $x_r = -\theta + \frac{\lambda}{\gamma+1-2h}$. For $x_t > x_r$ or $x_t < -\theta(1+\varphi)$, $r(x_t)$ decreases with $\sum_{j=0}^{\infty} E_t[\nu_{t+j}](x_t)$. Thus, the unconditional covariance between the two is positive if λ is large enough.

The interest rate $r_t = -\gamma(\frac{\gamma+1}{2} - h)(x_t - x_r)^2$, as above. The cumulative country premium $\sum_{j=0}^{\infty} \mathrm{E}_t[\nu_{t+j}]$ is given by equation (6.6) as a function of x_t. These two functions of x_t are plotted in Figure 3. Note that r_t and $\sum_{j=0}^{\infty} \mathrm{E}_t[\nu_{t+j}]$ have a positive covariance for $x_t \in (-(1+\varphi)\theta, x_r)$ and the intertemporal substitution effect dominates[4]. In contrast, r_t and ν_t have a negative covariance outside this interval and the precautionary saving effect dominates. The unconditional covariance between r_t and ν_t is the average of the covariances for all $x_t \in (-\infty, \infty)$.

7 Economic Intuition

It is worthwhile to discuss the economic intuition. Let us begin with two economic mechanisms behind our solution to the currency puzzles.

First, note that the interest rate is given by

$$r_t = \beta + \gamma\mu + \gamma\lambda(x_t+\theta) - \gamma\left(\frac{1+\gamma}{2} - h\right)(x_t+\theta)^2 \ ,$$

which positively depends on the consumption volatility $x_t+\theta$ through the intertemporal substitution effect and negatively depends on the consumption variance $(x_t+\theta)^2$ through the precautionary saving effect. When $x_t+\theta$ is large, $(x_t+\theta)^2$ dominates $x_t+\theta$, so the precautionary effect dominates and the interest rate decreases with consumption variance. When $x_t+\theta$ is small, $x_t+\theta$ dominates $(x_t+\theta)^2$, so the intertemporal substitution effect dominates and the interest rate increases with consumption volatility. The competition mechanism of these two effects makes the interest rate a nonmonotonic function of $x_t+\theta$ and is the key to resolving Engel's paradox.

Second, the currency premium of the simple return, $\frac{S_{t+1}}{S_t}e^{r_t^f}$, is

$$E_t\left(\frac{S_{t+1}}{S_t}e^{r_t^f - r_t^h}\right) = e^{\gamma^2\sigma_{ct}^{h2}} \ ,$$

which depends on home consumption variance $\gamma^2\sigma_{ct}^{h2}$ but not on foreign consumption variance. Thus, only home risk is priced. The currency premium for the log return, which is the focus of this paper, is

$$E_t[\rho_{t+1}] = E_t\left(\ln\left[\frac{S_{t+1}}{S_t} - r_t^h + r_t^f\right]\right) \ .$$

It has two components,

$$E_t[\rho_{t+1}] = \gamma^2\sigma_{ct}^{h2} - \frac{\gamma^2}{2}(\sigma_{ct}^{h2} + \sigma_{ct}^{f2}) \ ,$$

where the first term is compensation for risk, and the second term, which is due to Jensen's effect, is not. The currency premium can be written as the differential of home and foreign country premiums:

$$E_t[\rho_{t+1}] = \frac{1}{2}(\nu_t^h - \nu_t^f) \ .$$

[4]Given our assumption that $\theta > 0$ and $\lambda > 0$, and further assuming $h < \frac{\gamma+1}{2}$, one can show that $x_r > -\theta$.

To study the short premium puzzle, note that the covariance of the currency premium and the interest rate differential can be written as

$$\begin{aligned}\operatorname{cov}\left(E_t[\rho_{t+1}],\ r_t^f - r_t^h\right) &= \frac{1}{2}\operatorname{cov}\left(\nu_t^h - \nu_t^f,\ r_t^f - r_t^h\right) \\ &= -\frac{1}{2}\left(\operatorname{cov}\left(\nu_t^h,\ r_t^h\right) + \operatorname{cov}\left(\nu_t^f,\ r_t^f\right)\right).\end{aligned}$$

The log return makes the above covariance symmetric between the home and foreign countries. We remark that the covariance $\operatorname{cov}\left(E_t[\rho_{t+1}],\ r_t^f - r_t^h\right)$ is not due to the relative sizes of the home and foreign interest rates (i.e., the sign of their differential). Rather, it depends on $\operatorname{cov}(\nu_t^h, r_t^h)$ and $\operatorname{cov}(\nu_t^f, r_t^f)$. The sign of these two covariances depend on whether the intertemporal substitution effect or the precautionary saving effect dominates. This is determined by λ. When λ is small, the precautionary saving effect dominates on average. In this case, higher consumption variance leads to higher precautionary saving and thus lower interest. This mechanism creates the positive correlation between the short premium and the interest rate differential and is the risk-based explanation of the short premium puzzle in the literature.

On the other hand, for the long (cumulative) premium puzzle, the covariance of the long (cumulative) premium and the interest rate differential can be written as

$$\begin{aligned}\operatorname{cov}\left(E_t[\rho_{t+j+1}],\ r_t^f - r_t^h\right) &= \frac{1}{2}\operatorname{cov}\left(E_t[\nu_{t+j}^h] - E_t[\nu_{t+j}^f],\ r_t^f - r_t^h\right) \\ &= -\frac{1}{2}\left(\operatorname{cov}\left(E_t[\nu_{t+j}^h],\ r_t^h\right) + \operatorname{cov}\left(E_t[\nu_{t+j}^f],\ r_t^f\right)\right).\end{aligned}$$

When λ is large, the intertemporal substitution effect dominates on average. Higher consumption volatility implies higher consumption growth and thus higher interest. This mechanism creates the negative correlation between the long premium and the interest rate differential.

In this paper, we show that there is a range for λ such that the above two mechanisms hold simultaneously, thus resolving Engel's paradox.

One major advantage of using the currency premium for the log return is greater ease in studying the level of currency. If s_t is stationary (we extend our model to a stationary version below), $E_t[s_{t+\infty}]$ is a constant. By using a telescoping sum (Engel (2016)), we have

$$E_t[s_{t+\infty}] - s_t = \sum_{j=1}^{\infty} E_t[r_{t+j}^h - r_{t+j}^f] + \sum_{j=1}^{\infty} E_t[\rho_{t+j}]\ .$$

When uncovered interest parity holds, $E_t[\rho_{t+j}] = 0$,

$$E_t[s_{t+\infty}] - s_t = \sum_j E_t[r_{t+j}^h - r_{t+j}^f]\ .$$

This implies that a higher home interest rate r_t^h has lower s_t and thus a higher value of home currency. When $E_t[\rho_{t+j}] \neq 0$ and furthermore, $\operatorname{cov}(E_t[\rho_{t+j}], r_t^f - r_t^h) < 0$, a high

r^h implies an even higher level of home currency than implied by uncovered interest parity. This is the excess co-movement puzzle.

7.1 Stationary Exchange Rate

Engel shows that if the exchange rate is stationary, the cumulative expected currency premium is linked to the excess co-movement of the level of exchange rate. Furthermore, Engel (2016) empirically documents that the exchange rate is stationary. However, in our model (as well as in the main model in Engel), the exchange rate is not stationary. Below, we extend our model so that the exchange rate is stationary and the main economic intuition remains unchanged.

We assume that the consumption growth for each country $i \in \{h, f\}$ is similar to that in the benchmark model (3.1) with an additional term $-kc_t^i$,

$$c_{t+1}^i - c_t^i = -kc_t^i + \mu_t^i + \lambda(x_t^i + \theta) + (h - 1/2)(x_t^i + \theta)^2 + (x_t^i + \theta)\varepsilon_{ct+1}^i \ , \tag{7.1}$$

where $0 < k < 1$ is a constant. The term $-kc_t^i$ makes the process c_t^i stationary with mean reversion coefficient $\phi = 1 - k$.

In this case, the exchange rate satisfies

$$s_{t+1} - s_t = \ln \pi_{t+1}^f - \ln \pi_{t+1}^h = \gamma[(c_{t+1}^h - c_t^h) - (c_{t+1}^f - c_t^f)] \ \ ,$$

which implies that, up to an additive constant,

$$s_t = \gamma(c_t^h - c_t^f) \ .$$

Thus, s_t is mean reverting with mean reversion coefficient ϕ because c_t^h and c_t^f are stationary with mean reversion coefficient ϕ.

Note that both the country premium and the precautionary saving component of the interest rate are the same as those in our benchmark model (3.1). The intertemporal substitution component only differs from our benchmark model by the additional term $-kc_t^i$. If k is small[5], its impact on the covariance term structure and the conditions in Proposition 6.11 is negligible.

The cumulative premium is linked to the level of exchange rate by a telescoping sum, as shown in Engel (2016). The cumulative premium puzzle implies that the level of exchange rate is more volatile than is implied by UIP. As a result, on average, currency overshooting is higher than predicted by the Dornbusch model, which assumes a zero risk premium.

7.2 Forward Premium Puzzle

The forward premium puzzle states that the slope coefficient of the Fama regression is greater than zero, and implies that a high interest currency tends to appreciate. Because inequality (2.3) implies inequality (2.2), the forward premium puzzle is stronger than the short premium puzzle.

[5]Engel (2016) documents that the real exchange rate is mean reverting with slow speed (e.g., $k = -0.02$ for Canada and $k = -0.04$ for the UK).

Proposition 7.1. *The condition for solving the forward premium puzzle is*

$$0 < \left[h - \frac{1}{2}\right]\left[\frac{1+\gamma}{2} - h\right]\frac{2\sigma^2}{1-\varphi^2} - \left[2\theta\left(\frac{1}{2} - h\right) - \lambda\right]\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]. \tag{7.2}$$

Furthermore, if $h < \frac{1+\gamma}{2}$ *and* $2\theta\left(\frac{1+\gamma}{2} - h\right) < \lambda$*, the inequality (7.2) becomes*

$$h \ > \ \frac{1}{2} + \frac{\left[2\theta\left(\frac{1}{2} - h\right) - \lambda\right]\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]}{\left(\frac{1+\gamma}{2} - h\right)\frac{2\sigma^2}{1-\varphi^2}} \ . \tag{7.3}$$

As shown in Proposition 6.7 and Corollary 6.8, the assumption $h < \frac{1+\gamma}{2}$ and $2\theta\left(\frac{1+\gamma}{2} - h\right) < \lambda$ are needed to resolve Engel's paradox in long premium formulation.

8 Conclusion

While prior studies have emphasized the importance of the precautionary saving effect in understanding the currency premium over short horizons (e.g., the forward premium puzzle), we find that the intertemporal substitution effect plays a key role in understanding the currency premium over long horizons (e.g., the excess co-movement puzzle).

We build a model in which the interest rate positively depends on the consumption volatility and negatively depends on the consumption variance. The positive dependence on the consumption volatility is generated through intertemporal substitution, since we assume that the consumption mean positively depends on the consumption volatility. The negative dependence on the consumption variance is generated by precautionary saving, as proposed in the literature. Furthermore, in our model expected consumption variance depends on the consumption volatility as well as the consumption variance. Under this setup, the interest rate differential is positively correlated with the short currency premium but negatively correlated with the long currency premium (and cumulative currency premium). Thus, our paper resolves the paradox raised by Engel (2016).

Our results suggest that the consumption mean should depend positively on consumption volatility, which has not been widely used in the literature. Furthermore, expected consumption variance should depend on current consumption volatility in addition to current consumption variance. One alternative is a multiple factors model of consumption variance processes with different mean reversion coefficients.

Appendix A: Proof of Proposition and Lemmas

Proof of Lemma 6.2

From equations (5.2), (5.3), and (5.4), the expected log currency premium over horizon j is

$$\begin{aligned} E_t[\rho_{t+j+1}] &= \frac{1}{2} E_t\left[\nu^h_{t+j} - \nu^f_{t+j}\right] \\ &= \frac{1}{2}\gamma^2\varphi^{2j}(x^{h2}_t - x^{f2}_t) + \gamma^2\theta\varphi^j(x^h_t - x^f_t) \; . \end{aligned}$$

The interest rate is given in equation (4.4). The interest rate differential between the foreign and home countries is

$$r^f_t - r^h_t = \gamma\left(\frac{1+\gamma}{2} - h\right)\; \left[x^{h2}_t - x^{f2}_t\right] + \gamma\left[2\theta\left(\frac{1+\gamma}{2} - h\right) - \lambda\right]\; \left[x^h_t - x^f_t\right] \quad . (8.1)$$

Because x^h_t and x^f_t are independent, the covariance between the expected future currency premium and the interest rate differential with different time horizons $j \in \{0, 1, 2, \cdots\}$ is

$$\begin{aligned} &\text{cov}\left(E_t[\rho_{t+j+1}],\; r^f_t - r^h_t\right) \\ &= \frac{1}{2}\gamma^3\left(\frac{1+\gamma}{2} - h\right)\; \varphi^{2j} Var\left[x^{h2}_t - x^{f2}_t\right] + \gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\varphi^j\; Var\left[x^h_t - x^f_t\right] \\ &= \gamma^3\left(\frac{1+\gamma}{2} - h\right)\; \varphi^{2j}\; \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\;\gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\; \varphi^j\; \frac{\sigma^2}{1-\varphi^2} \quad . \end{aligned}$$

For the last equality, we use the moments of x_t as shown in Appendix B.

Proof of Proposition 6.5

The covariance (6.1) when j = 0 is

$$\begin{aligned} &\text{cov}\left(E_t[\rho_{t+j+1}],\; r^f_t - r^h_t\right) \\ &= \gamma^3\left(\frac{1+\gamma}{2} - h\right)\; \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\;\gamma^3\left[2\theta^2\left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right]\; \frac{\sigma^2}{1-\varphi^2} \quad . \end{aligned}$$

Because $\gamma > 0$ and $0 < \varphi < 1$, the fact that the above covariance is positive yields condition (6.2).

Proof of Lemma 6.9

From equations (5.2), (5.3) and (6.6), the cumulative expected future currency premium is

$$E_t\left[\sum_{j=0}^{\infty}\rho_{t+j+1}\right] = \frac{1}{2}\gamma^2\frac{1}{1-\varphi^2}(x^{h2}_t - x^{f2}_t) + \gamma^2\theta\frac{1}{1-\varphi}(x^h_t - x^f_t) \; .$$

The interest rate differential between the foreign and home countries is given in (8.1).

Because x_t^h and x_t^f are independent, from the moments in Appendix B, the covariance between the cumulative expected future currency premium and the interest rate differential is

$$
\begin{aligned}
&\operatorname{cov}\left(\sum_{j=0}^{\infty} E_t[\rho_{t+j+1}], r_t^f - r_t^h\right) \\
&= \frac{1}{2}\gamma^3 \left(\frac{1+\gamma}{2} - h\right) \frac{1}{1-\varphi^2} Var\left[x_t^{h2} - x_t^{f2}\right] + \gamma^3 \left[2\theta^2 \left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right] \frac{1}{1-\varphi} Var\left[x_t^h - x_t^f\right] \\
&= \gamma^3 \left(\frac{1+\gamma}{2} - h\right) \frac{1}{1-\varphi^2} \frac{2\sigma^4}{(1-\varphi^2)^2} + 2\,\gamma^3 \left[2\theta^2 \left(\frac{1+\gamma}{2} - h\right) - \lambda\theta\right] \frac{1}{1-\varphi} \frac{\sigma^2}{1-\varphi^2} \quad .
\end{aligned}
$$

For the last equality, we use the moments of x_t as shown in Appendix B.

Appendix B: The Moments of x_t

Theorem 8.1. *Assuming that x_t follows the OU process (3.3) and that μ_t follows the mean reverting process*

$$
\begin{aligned}
x_{t+1} &= \varphi x_t + \sigma e_{t+1}^x \\
\mu_{t+1} &= -a x_t - b x_t^2 + \phi \mu_t + (x_t + v)\varepsilon_{t+1}^c \quad ,
\end{aligned}
$$

we have the following results:

$$
\begin{aligned}
E_t[x_{t+j}] &= \varphi^j x_t \\
E_t[x_{t+j}^2] &= \varphi^{2j} x_t^2 + \sigma^2 \frac{1-\varphi^{2j}}{1-\varphi^2} \\
E[x_t] &= 0 \\
E[x_t^2] &= \frac{\sigma^2}{1-\varphi^2} \\
E[x_t^3] &= 0 \\
E[x_t^4] &= \frac{3\sigma^4}{(1-\varphi^2)^2} \\
var[x_t] = E[x_t^2] &= \frac{\sigma^2}{1-\varphi^2} \\
var[x_t^2] &= \frac{2\sigma^4}{(1-\varphi^2)^2} \\
cov[x_t, \mu_t] &= -\frac{a\sigma^2}{1-\varphi^2} \frac{\varphi}{1-\phi\varphi} \\
cov[x_t^2, \mu_t] &= -\frac{b\sigma^4}{(1-\varphi^2)^2} \frac{\varphi^2}{1-\phi\varphi^2}
\end{aligned}
$$

for $j \in \{0, 1, \cdots\}$.

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