A Theory of Intermediated Investment with Hyperbolic Discounting Investors

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Abstract

We study the role of financial intermediaries in providing liquidity for economic agents with hyperbolic discounting. We show that in a competitive market with financial intermediaries making zero profits, sophisticated agents are offered with a contract of perfect commitment, while naïve ones will be attracted by a contract that offers seemingly attractive return in the long run, but introduces a discontinuous penalty for early withdrawal. In competitive equilibrium with partially naïve types, Pareto Optimality is not achieved. When types are private information, naïve depositors early withdraw and cross-subsidize sophisticated ones. If contracts are linear or a secondary market for long-term deposit contract opens for trading, welfare of partially naïve agents will be improved. We show that arbitrage-free contracts offered by financial intermediaries allow for a unique term structure of interest rates, which contains a premium for naïveté.
1. Introduction

Investment is a complex intertemporal decision involving tradeoffs among costs and benefits occurring at different times, which not only affect one's health, wealth, and happiness, but may also determine the economic prosperity of nations. Due to their complexity and market frictions for investment, financial intermediaries are involved in most of the investment activities.

One important role of financial intermediaries, like banks, is to provide depositors insurance against preference shocks, which traditional insurance market cannot provide because the shocks are private information (Diamond and Dybvig, 1983). A growing literature on savings suggests that agents have time-inconsistent taste for immediate gratification, and are often naïve about this taste (Frederick, Loewenstein and O’Donoghue, 2002). This taste for immediate gratification also generates "preference shocks" that create liquidity needs. This paper intends to study the role of financial intermediaries when preference shocks are generated by time-inconsistent preferences.

Our model builds on the three-date model in Diamond and Dybvig (1983) and assumes an illiquid long-term asset which yields low return if operated for a single period but high return if operated for two periods. But there are at least two key differences: First, while in Diamond-Dybvig model it is uncertain whether an agent is type 1 or type 2 ex-ante, preference shocks generated by hyperbolic discounting is certain. For example, the date-1 utility function of an agent with hyperbolic discounting factor $\beta$ is $u(c_1) + \beta u(c_2)$. So it is a "shock" only for non-sophisticated agents who mispredict their future utility. Second, consistent with much of the literature, we equate welfare with self-0’s utility, so all agents have the same welfare-maximizing allocation, and the financial intermediary’s role is to offer commitment against early liquidation rather than to provide risk sharing among people with different preferences. Given these differences, the preference shocks in this paper are impulsive and “irrational” from a long-term welfare perspective.

We start by showing that autarky and private market usually cannot maximize welfare, even for fully sophisticated consumers, because ex-ante commitment is costly. Then we study a competitive equilibrium similar to Heidhues and Kőszegi (2010). We define a competitive equilibrium as a set of contracts such that each contract earns zero profits, no contract can generate strictly positive profits, and contracts as well as options in contracts that do not affect expectations or behavior are eliminated.

Our model shows that in competitive equilibrium, financial intermediaries provide perfect commitment for sophisticated agents. However, even if depositors are slightly naïve, financial intermediaries offer seemingly attractive return in the long run, but introduces a discontinuous penalty for withdrawing in advance. Consistent with this prediction, time deposits, or certificates of deposit (CD) usually promise a high yield for a fixed period, but charge a non-trivial early withdrawal penalty. We show that this discontinuity has some adverse welfare consequences. Non-sophisticated depositors, even if they are very close to sophisticated and only slightly mispredict their future behavior, would end up failing to resist the temptation of immediate gratification and being charged with early withdrawal penalty.

A commonly articulated justification for early withdrawal penalties in CD contracts is to offer a commitment device for time-inconsistent consumers by preventing them from making impulsive withdrawals. (Laibson, 1997; Ashraf et al., 2003). However,
empirical work shows that early withdrawal from Certificates of Deposit (CD) accounts is at economically significant level. Gilkeson, List and Ruff (1999) found that depositors withdraw a significant amount of their time deposits before maturity—2.4% and 6.4% of the deposit base each year for shortest and longest maturity type, respectively—despite an average negative reinvestment incentive. In fact, few people ask about how much early withdrawal penalty is when buying CDs. Given this, it is not easy to justify early withdrawal penalties as a costly incentive device to discipline savings behavior, and suggests some depositors may be naïve about their time inconsistency.

Another application of our theory is pension funds, like IRAs, 401(k) plans, etc. Early withdrawals from these retirement funds are also considerable: withdrawals for nonretirement purposes by account holders under 60 amount to $60 billion a year, or 40 percent of the $176 billion employees put into such accounts each year and nearly a quarter of the combined $294 billion that workers and employers contribute. Moreover, most early withdrawals from pension funds are made by low income workers, who assumably are also more naïve about their time inconsistency. These facts support our prediction that financial intermediaries fool naïve depositors by introducing penalties they do not expect.

This paper also connects to a number of papers on nonlinear deposit contracts. Lin (1996) shows that when people have random discount factors, the optimal incentive-compatible risk sharing contract has a convex structure, that is, the interest rate is higher when there is less early withdrawal and larger deposit balance. Ambrus and Egorov (2012) show that withdrawal penalties (money burning) is optimal if consumers face a severe and rare negative liquidity shock. This paper offers another justification for early withdrawal penalty in CD contracts. Besides, all of these papers assume people are time-consistent or sophisticated, so our model is more consistent with the unexpected nature of early withdrawals.

The idea that competitive market offers commitment to sophisticated agents and exploitative contract to naïve agents are prevalent in the literature on contracting with time inconsistency. Our results are most similar to Heidhues and Kőszegi (2010) in that contract and welfare is discontinuous at full sophistication, and even slightly naïve consumer would switch away from her preferred repayment and be penalized ex post. DellaVigna and Malmendier (2004) show that a monopolistic firm offers a two-part tariff to partially naïve consumers, in which the per-usage price falls below the firm’s marginal cost in the case of investment goods, and lies above marginal cost in the case of leisure goods. These contract features have adverse effects on consumer welfare only if consumers are naïve. Eliaz and Spiegler (2006) studies a two-period model in which the firm screens agents by the probability they attach to each state ex ante, namely their sophistication. The optimal menu provides a perfect commitment device for relatively sophisticated types, and exploitative contracts which involve speculation with relatively naive types. This paper specializes these models to investment contracts offered by financial intermediaries, and yields specific predictions, such as early withdrawal penalty, that coincides with features of deposit contracts and pension funds in reality.

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We also consider a linear restricted market in which contracts are restricted to have a linear structure and consumers can transfer consumption with more flexibility between two dates at a pre-specified interest rate. We show that because this intervention prevents financial intermediaries from inducing naïve agents to drastically mispredict their future behavior, it is welfare-improving as long as agents are not too naïve. Demand deposit is a natural example of this linear intervention. Dividing money in multiple accounts or tradable CD both require that the contract is arbitrage-free, therefore are also equivalent to restricting contracts to be linear.

We show that a linear deposit contract implies a unique term structure of interest rates in our three-date model. Interestingly, our numerical example shows that when there are more naïve people in the population, short-term interest rates decrease while long-term interest rates increase, suggesting a term premium for naïveté. Thus our paper offers a new explanation for term premium.

Finally, we study the role of transparency of financial intermediaries. If the financial intermediary has to disclose its financial records by the end of each period, it can no longer offer an unrealistically high interest rate over a long period to fool the naïve agents. In equilibrium, the financial intermediary can earn positive profits by exploiting naïve people’s mispredictions. However, if financial intermediaries own capital, they are not affected by this restriction, and would offer the same contract as when they are opaque.

The rest of the paper is organized as follows. Section 2 introduces our basic three-date model with time-inconsistent preferences. Section 3 characterizes competitive equilibrium with financial intermediaries and derives unrestricted nonlinear competitive-equilibrium deposit contract. In Section 4, we discuss linear intervention’s effect on contracts and consumers’ welfare. Section 5 extends the results to general forms of partial naïveté, and discusses the implications for term structures and results under tradable CDs and transparency requirements. Section 6 concludes with some final thoughts. All proofs are in the Appendix.

2. Basic Model

In an economy there are three dates \((t=0,1,2)\) and a single homogeneous good. Each agent is endowed with 1 unit of good at date 0 and the good is to be consumed at date 1 and 2. The good can be stored from one date to the next or can be invested at \(t=0\) in a long-run technology, which returns \(R>1\) units at \(t=2\), and liquidation is costless: one can get 1 unit for each unit liquidated at \(t=1\). \(^2\)

The agent has time-inconsistent preferences. Self 0’s utility is \(u(c_1) + u(c_2)\), where \(c_1 \geq 0\) and \(c_2 \geq 0\) are her consumption in dates 1 and 2, respectively. Self 1 maximizes \(u(c_1) + \beta u(c_2)\), where \(0 < \beta \leq 1\) is the hyperbolic discount factor. The per-date utility function \(u()\) is strictly concave and twice differentiable, with \(u(0) = 0\) and a sufficiently large \(u'(0)\).

Following O’Donoghue and Rabin (2001), in most of our analysis, we assume that

\(^2\)The assumption of costless liquidation is not essential to our main results, because banks are optimizing and there is no liquidation in equilibrium. It may affect consumer’s choices in autarky case (Section 2.2). Our main results also hold if each unit can be liquidated at value \(R' \in (1,R)\) because it is essentially the costless case with each unit equaling to \(1/R'\).
self 0 believes with certainty that self 1 will maximize \( u(c_1) + \hat{\beta} u(c_2) \), where \( \beta \leq \hat{\beta} \leq 1 \). \( \hat{\beta} \) reflects self 0’s beliefs about her time-inconsistency, so that \( \hat{\beta} = \beta \) corresponds to perfect sophistication about future preferences, and \( \hat{\beta} = 1 \) corresponds to complete naïveté about time-inconsistency. In Section 6, we consider a more general form of partial naïveté, and shows that our qualitative results still hold if self 0 attaches significant probability to her time-inconsistency above \( \beta \).

We measure welfare using long-run self-0 preferences following much of the literature on time inconsistency (DellaVigna and Malmendier, 2004; O’Donoghue and Rabin, 2006). Although for simplification we consider a three-date model, in reality time-inconsistency plays out over many short periods, so weighting each date equally would be more reasonable, but the major results would remain qualitatively the same.

### 2.1. The First Best

We first consider the first-best allocation in our model. Note that in optimum there is no liquidation in date 1, so the welfare-maximizing allocation is determined by the following program:

\[
\max_{c_1, c_2} u(c_1) + u(c_2) \quad (1)
\]

s.t. \( c_1 + c_2 / R \leq 1 \)

The first-order condition is:

\[
\frac{u'(c_1^b)}{u'(c_2^b)} = R \quad (2)
\]

Note that since the first-best solution does not depend on degree of time-inconsistency (\( \beta \)) or sophistication (\( \hat{\beta} \)) of the agent, a social planner can maximize welfare by investing on behalf of all agents and offering each agent \( c_1^b \) and \( c_2^b \) in dates 1 and 2, respectively. It would be more interesting to consider agents’ welfare when they invest on their own or through competitive profit-maximizing financial intermediaries, which are studied in the following sections.

### 2.2. The Autarky Case

For comparison, we also derive the autarky outcomes without financial intermediary. Suppose at date 0, all agents anticipate their preference at time 1 is \( u(c_1) + \beta_1 u(c_2) \). An agent is naïve if his preference turns out to be \( u(c_1) + \beta_1 u(c_2) \) with \( \beta_1 < \beta_2 \). An agent is sophisticated if his preference is time-consistent.

First, suppose there is no trade between agents. Agents cannot commit, and they will optimally choose their liquidation at date 1. All agents maximize their date-1 utility \( u(c_1) + \beta_1 u(c_2) \) subject to budget constraint: \( c_1 + c_2 / R = 1 \) . So their actual consumption satisfies:

\[
\frac{u'(c_1)}{u'(c_2)} = \beta_1 R \quad (3)
\]

Next, we consider a market where people can trade claims on future goods. At date 1, the price of date 2 consumption must be \( 1 / R \), otherwise either \((1,0)\) or \((0,R)\) would dominate all other points on the budget line and it cannot be an equilibrium. In
equilibrium, supply equals demand, and everyone maximizes their date-1 utility on the budget line. Therefore agents’ consumption is the same as in the case without trading, that is, the existence of market cannot improve the agents’ welfare.

In Section 3 and 4, we will introduce competitive financial intermediaries into the model to see whether financial intermediaries can help agents improve their welfare. The competitiveness among financial intermediaries guarantee all the surplus goes to economic agents instead of financial intermediaries. When the agent types are known to financial intermediaries, we find that the existence of financial intermediary weakly improves agents’ welfare.\(^3\) Sophisticated people are strictly better off by investing through a financial intermediary than investing on their own, because the financial intermediary can help them commit against taste for immediate gratification, by limiting their choices or charging an early withdrawal penalty. Compared to autarky, naïve people are equally well off when the contract is nonlinear and strictly better off when the contract is linear, so the financial intermediary can also help partially naïve people to overcome their time inconsistency problem under the linear contract. Moreover, by restraining people’s impatience, the financial intermediary holds more money for a longer period and make more investments to the economy.

When there are both naïve agents and sophisticated agents and financial intermediaries cannot differentiate them, the early withdrawal penalty serves as a commitment device for sophisticated people while “exploiting” naïve people’s wrong expectations, and it makes sophisticated agents better off but could make naïve agents worse off compared with autarky.

\section{Competitive Equilibrium}
\subsection{Definition of Competitive Equilibrium}
In a competitive market, agents interact with competitive, risk-neutral and profit-maximizing financial intermediaries. Financial intermediaries face the long-term technology described above. For simplicity, we assume that in date 0, depositors put all their money in the financial intermediary and can sign unrestricted nonlinear contracts regarding repayment schedule. These contracts are exclusive: once an agent signs with a financial intermediary, she cannot interact with other financial intermediaries.

We assume there are finite \(\beta\)'s among people and \(\beta_1 < \beta_2 < \ldots < \beta_k\), and \(\hat{\beta} \in \{\beta_2, \ldots, \beta_k\}\). Financial intermediaries offer a finite menu of repayment options \(C = \{(c_1, c_2)\} \in S\) to each agent at date 0, where \(c_1\) and \(c_2\) are consumptions in date 1 and 2, respectively. Define \((c_1(\cdot), c_2(\cdot)) : \beta, \ldots, \hat{\beta}, \beta \rightarrow R_+\) as an incentive-compatible map, if agents with hyperbolic discounting factor \(\beta\) prefer \((c_1(\beta), c_2(\beta))\) among all repayment options, that is, \(u(c_1(\beta)) + \beta u(c_2(\beta)) \geq u(c_1) + \beta u(c_2)\) for all \((c_1, c_2) \in C\). An agent of type \((\beta, \hat{\beta})\) believes in date 0 she would choose \((c_1(\hat{\beta}), c_2(\hat{\beta}))\) from \(C\), but in reality she chooses \((c_1(\beta), c_2(\beta))\) when confronted with \(C\) in date 1.

\(^3\)If liquidation is costly, that is, \(L < 1\), then both sophisticated and naïve people would be worse off in autarky, and financial intermediaries still weakly improve people’s welfare.
We define a competitive equilibrium as a contract $C$ offered by the financial intermediaries and incentive compatible map $(c_1(\cdot), c_2(\cdot))$ that satisfies the following properties:

1. [Zero-profit] For each financial intermediary $C$ yields zero expected profits.
2. [No profitable deviation] There exists no contract $C'$ with an incentive compatible map $(c'_1(\cdot), c'_2(\cdot))$ such that for some $\hat{\beta}$, $u(c'_1(\hat{\beta}), c'_2(\hat{\beta})) > u(c_1(\hat{\beta}), c_2(\hat{\beta}))$, and $C'$ yields positive profits.
3. [Non-redundancy] For each repayment option $(c_{1j}, c_{2j}) \in C$, there is a corresponding type $(\beta, \hat{\beta})$ such that either $(c_{1j}, c_{2j}) = (c_1(\beta), c_2(\beta))$ or $(c_{1j}, c_{2j}) = (c_1(\hat{\beta}), c_2(\hat{\beta}))$.

The first two conditions are typical for competitive markets, saying that financial intermediaries earn zero profits by offering these contracts, and they can do no better. The last condition says that all repayment options are relevant in that they affect the expectations or behaviors of depositors. Due to the redundancy condition, many options are excluded from the competitive-equilibrium contracts, in particular, non-sophisticated consumers can only change their repayment option by paying a large early withdrawal penalty as discussed below.

### 3.2. Unrestricted Contracts when $\beta$ and $\hat{\beta}$ are known

We start by considering a fully sophisticated depositor with $\hat{\beta} = \beta < 1$. Since a time-consistent depositor would correctly predict her choice in date 1, only her chosen repayment option is relevant in both periods. We can assume that the contract offered by financial intermediaries only include one repayment option that she actually chooses. The financial intermediary’s problem is:

$$
\max_{c_1, c_2} (1 - c_1)R - c_2 \quad (4)
$$

s.t. $u(c_1) + u(c_2) \geq \underline{u}$ \quad [PC]

$PC$ is the participation constraint, where $\underline{u}$ is a consumer’s perceived utility from the perspective of date 0 if she accepts a purported competitive-equilibrium contract. It is clear that $PC$ binds; otherwise the financial intermediary could increase profits by decreasing $c_1$. Competition drives the financial intermediary’s profit to zero through lifting $\underline{u}$. Since in a competitive market $(1 - c_1)R - c_2 = 0$, the financial intermediary is maximizing self 0’s utility subject to the budget constraint.

**Proposition 1** If $\hat{\beta} = \beta$, the competitive-equilibrium deposit contract has a single repayment option satisfying $u'(c_1)/u'(c_2) = R$, and $(1 - c_1)R - c_2 = 0$. The agent gets the same allocation as the first best.

The situation is entirely different for partially naïve agents. Partially naïve ($\hat{\beta} > \beta$) agents mispredict their utility and thus the repayment option they would choose in date 1. The financial intermediary offers them a former “decoy” repayment option $(\hat{c}_1, \hat{c}_2)$ self-0 expects to choose and a latter “chosen” repayment option $(c_1, c_2)$ self-1 actually chooses subject to the following constraints. First, for the consumer to be willing to accept the financial intermediary’s offer, self 0’s utility from the decoy...
option must be at least $u$. This is a version of the standard participation constraint ($PC$), except that self 0 may make her participation decision based on incorrectly forecasted future behavior. Second, if self 0 is to think that she will choose the decoy option, then given her beliefs $\hat{\beta}$ she must think she will prefer it to the other available options. These are the perceived-choice constraints ($PCC$). Third, if an agent with short-term impatience actually chooses the repayment schedule intended for her, she has to prefer it to the other repayment options. This is analogous to standard incentive-compatibility constraints ($IC$) for self 1.

Therefore the financial intermediary solves:

$$
\max_{\hat{c}_1, \hat{c}_2, c_1, c_2} (1-c_1)R - c_2 \quad (5)
\text{s.t. } u(\hat{c}_1) + u(\hat{c}_2) \geq u, \quad [PC]
\text{s.t. } u(\hat{c}_1) + \hat{\beta}u(\hat{c}_2) \geq u(c_1) + \hat{\beta}u(c_2), \quad [PCC]
\text{s.t. } u(c_1) + \beta u(c_2) \geq u(\hat{c}_1) + \beta u(\hat{c}_2) \quad [IC]
$$

As before, $PC$ must bind in equilibrium. In addition, $IC$ also binds because otherwise the financial intermediary could increase profits by lowering $c_1$. Given that $IC$ binds and $\hat{\beta} > \beta$, $PCC$ is equivalent to $c_1 \geq \hat{c}_1$. Intuitively, if self 1 is in reality indifferent between two repayment options, then self 0—who overestimates her future patience—predicts she will prefer the option with more repayment later. We first derive the optimal solution without $PCC$ below, and confirm that $c_1 \geq \hat{c}_1$ is satisfied.

The relaxed problem is:

$$
\max_{\hat{c}_1, \hat{c}_2, c_1, c_2} (1-c_1)R - c_2 \quad (6)
\text{s.t. } u(\hat{c}_1) + u(\hat{c}_2) \geq u, \quad [PC]
\text{s.t. } u(c_1) + \beta u(c_2) = u(\hat{c}_1) + \beta u(\hat{c}_2) \quad [IC]
$$

The optimal solution must have $\hat{c}_1 = 0$, otherwise since $\beta < 1$, the financial intermediary could decrease $u(\hat{c}_1)$ and increase $u(\hat{c}_2)$ by the same amount, keeping $PC$ constraint unchanged and loosing $IC$ constraint, allowing it to lower $c_1$.

**Proposition 2** Suppose $\hat{\beta} > \beta$, the competitive-equilibrium deposit contract has two repayment options, with the consumer expecting to choose $\hat{c}_1 = 0$, $\hat{c}_2 > 0$, and actually choosing $c_1$, $c_2$ satisfying $u'(c_1)/u'(c_2) = \beta R$, and $(1-c_1)R - c_2 = 0$.

This contract offers an option for the consumer to consume little in the short run, but also introduces an option to withdraw in advance for a significant penalty. Since the financial intermediary designs the contract to induce repayment behavior that self 0 does not expect, its goal with the chosen option is to maximize the gains from trade with self 1 keeping the $IC$ constraint satisfied, and it caters fully to self 1’s taste for immediate gratification.

Notice that there is a discontinuity at full sophistication: all non-sophisticated depositors, even near-sophisticated ones, receive discretely different contracts from and discretely lower welfare than sophisticated depositors. This is because agents are
maximizing utility from the “decoy” repayment option, while the financial intermediary’s profit is determined by the “chosen” repayment option. These two options differ as long as the depositors are not fully sophisticated.

By designing an unrestricted nonlinear contract and catering to preferences of self 1, the financial intermediary exaggerates even an arbitrarily small amount of naïveté, and induces the agent to make a non-trivial mistake.

These properties closely resemble some important features of CD contracts in banks. For example, for a one-year CD, the penalty of early withdrawal is often three months’ interests. CDs promise a favorable return as long as money is locked away for a fixed period, but if the depositor withdraws even one dollar before maturity, she would be charged with a substantial penalty, and investing in CD becomes extremely unprofitable. Pension funds also share some similar features with penalty of early withdrawal.

In the autarky case, non-sophisticated depositor’s long-run self-0 utility is

\[ u(c_1) + u(c_2) \]

where \( c_1 \) and \( c_2 \) satisfy

\[ u'(c_1)/u'(c_2) = \beta R \quad \text{and} \quad (1 - c_1)R - c_2 = 0 \ . \]

Obviously, the welfare of non-sophisticated depositor does not change with the presence of financial intermediary because the realized consumptions are identical.

### 3.3. Unrestricted Nonlinear Contracts when \( \beta \) and \( \hat{\beta} \) are unknown

In this section we study competitive equilibria when either \( \hat{\beta} \), or \( \beta \) and \( \hat{\beta} \), are unknown to the financial intermediary. We show that the main results from last section remain: sophisticated and non-sophisticated depositors are identical ex-ante and sign the same contract, but non-sophisticated depositors choose the repayment option with large early withdrawal penalties and have discontinuously lower welfare than sophisticated depositors.

#### 3.3.1. Known \( \hat{\beta} \), Unknown \( \beta \)

Suppose that all agents has \( \hat{\beta} \) at date 0 and is known to the firm, and at date 1 they have \( \beta = \beta_1 < \hat{\beta} \) with probability \( p_1 \), and \( \beta = \beta_2 = \hat{\beta} \) with probability \( p_2 \). Because sophisticated and naïve agents have the same belief at date 0, they accept the same contract. The financial intermediary’s problem is:

\[
\begin{align*}
\max_{c_{11}, c_{12}, c_{21}, c_{22}} & \quad p_1((1-c_{11})R-c_{21}) + p_2((1-c_{12})R-c_{22}) \\
\text{s.t.} & \quad u(c_{12}) + u(c_{22}) \geq u, \quad [PC] \\
& \quad u(c_{11}) + \beta u(c_{21}) \geq u(c_{12}) + \beta u(c_{22}), \quad [IC_1] \\
& \quad u(c_{12}) + \beta u(c_{22}) \geq u(c_{11}) + \beta u(c_{21}), \quad [IC_2]
\end{align*}
\]

Clearly \( PC \) binds, and \( IC_1 \) also binds, otherwise the financial intermediary can decrease \( c_{11} \) without violating the constraints, and increase profits. Then we have the following proposition.

**Proposition 3** Suppose \( \hat{\beta} \) is known, and \( \beta \) takes two values, \( \beta_1 < \hat{\beta} \) and \( \beta_2 = \hat{\beta} \) with probability \( p_1 \) and \( p_2 \), respectively. In competitive equilibrium, the repayment options \( (c_{11}, c_{21}) \) and \( (c_{12}, c_{22}) \) satisfies \( p_1((1-c_{11})R-c_{21}) + p_2((1-c_{12})R-c_{22}) = 0 \),
and

\[
\frac{u'(c_{11})}{u'(c_{21})} = \beta R \quad (8)
\]

\[
\frac{u'(c_{12})}{u'(c_{22})} = R \left( 1 + (1 - \beta) \frac{p_1 u'(c_{12})}{p_2 u'(c_{11})} \right) \quad (9)
\]

Condition (8) is a version of the standard “efficiency-at-the-top” result common in many screening problems. It says that repayment schedule of non-sophisticated agents is efficient from self 1’s perspective—the same result as in the case with known \(\hat{\beta}\) and \(\beta\). By condition (9), sophisticated agents’ repayment schedule is too back-loaded even from the long-term self 0’s perspective. When considering whether to allocate more of the sophisticated agent’s repayment to date 1, the financial intermediary faces a trade-off: on one hand, this adjustment increases sophisticated agent’s expected utility in date 0, and increases financial intermediary’s profits; on the other hand, this adjustment also increases non-sophisticated agent’s utility in date 1, making early withdrawal penalties less preferable, and decreases financial intermediary’s profits. This is similar to the tradeoff in standard screening problems between increasing efficiency of less profitable type and decreasing information rent of paid to the more profitable type. Sophisticated agent’s allocation is distorted from date-0 perspective to keep non-sophisticated agent, the more profitable type, incentive compatible from date-1 perspective.

As before, the two first-order conditions imply discontinuity at full sophistication. Even for arbitrarily close \(\beta_1\) and \(\beta_2\), a non-sophisticated agent gets discontinuously different repayments from a sophisticated agent, and is discontinuously worse off as a result. In other words, the welfare of non-sophisticated agents is reduced with the presence of financial intermediary because in this case \(c_{11}\) and \(c_{21}\) satisfies

\[
\frac{u'(c_{11})}{u'(c_{21})} = \beta R \quad \text{and} \quad (1 - c_{11})R - c_{21} < 0.
\]

Next we consider how welfare changes with the proportion of non-sophisticated agents in the population.

**Proposition 4** (The Cross-Subsidy Effect) Suppose \(\hat{\beta}\) is known, and \(\beta\) takes two values, \(\beta_1 < \hat{\beta}\) and \(\beta_2 = \hat{\beta}\) with probability \(p_1\) and \(p_2\), respectively. The sophisticated type’s welfare in the competitive equilibrium is strictly increasing in \(p_1\).

The intuition for this result is that when types are unknown, financial intermediaries make money on non-sophisticated depositors and lose money on sophisticated depositors, so an increase in \(p_1\) would result in positive profits. Competition drives financial intermediaries to offer more attractive contracts, and since sophisticated consumers correctly anticipate their outcomes, this would make them better off.

### 3.3.2. Unknown \(\beta\) and \(\hat{\beta}\)

As in Heidhues and Kőszegi (2010), we provide a condition under which agents self-select according to \(\hat{\beta}\) in date 0, and then according to \(\beta\) in date 1.

Let \(u_i\) be the perceived utility from competitive-equilibrium contract when \(\hat{\beta} = \beta_i\).
is observable, with probability \( p_i \) she is sophisticated, and with probability \( 1 - p_i \) she is type \( \beta_i \).

**Condition 1:** \( u_i \) is increasing in \( \beta \).

Under this condition agents would self-select into the contract intended for her true \( \hat{\beta} \) because: First, she wouldn’t choose a contract intended for any \( \hat{\beta}' < \hat{\beta} \) since it would give her lower utility (by Condition 1); Second, while she prefers the contract for \( \hat{\beta}' > \hat{\beta} \), she expects to switch away from it in date 1 and get lower utility at last (otherwise the contract designed for \( \hat{\beta} \) is suboptimal compared to more attractive and profitable contract for \( \hat{\beta}' > \hat{\beta} \)).

**Proposition 5** If Condition 1 holds, then in the unique competitive equilibrium with \( \hat{\beta} \) unobserved by financial intermediaries, each agent accepts the same contract as when \( \hat{\beta} \) is observed by financial intermediaries.

Even when financial intermediaries do not know agent’s \( \hat{\beta} \) or \( \beta \), a non-sophisticated depositor would self-select into a contract which she would pay the early withdrawal penalty, and our result about discontinuity of welfare and behavior at full sophistication remains.

**4. Restricted Linear Contracting**

In this section, we consider contracts that are restricted to have a linear structure. In date 0, the financial intermediary specifies some \( \tilde{R} \) and \( T \), and agents choose from all \((c_1, c_2)\) that satisfies \( c_1 + c_2 / \tilde{R} = T \). We show that preventing financial intermediaries from fooling slightly naïve agents into discretely mispredicting their behavior, restricting contracts to be linear raises naïve agents’ welfare.

**4.1. Full Information**

First we consider competitive equilibrium in a restricted linear market when both \( \beta \) and \( \hat{\beta} \) are known to the financial intermediaries. A perfectly sophisticated agent is fully aware of her time inconsistency, so it would be profit-maximizing to offer her a contract with an interest rate of \( \tilde{R} = R / \beta \) which aligns self 1’s interest with the long-term welfare. This contract counteracts self 1’s tendency for immediate gratification and maximizes sophisticated depositors’ welfare.

More interestingly, restricted linear contracts prevents financial intermediaries from setting too high early withdrawal penalties, so a slightly naïve agent only mispredicts her future behavior by a small amount, and achieves nearly optimal welfare.

**Proposition 6** Restricting financial intermediaries’ investment contracts to a linear form keeps fully sophisticated agents equally well off and strictly raises not-too-naïve agents’ welfare compared to unrestricted nonlinear contract. The welfare of naïve agents is increased by the presence of financial intermediary when preferences are observable.

Notice that this proposition holds only for not-too-naïve agents since both linear and unrestricted nonlinear contracts would lead very naïve agents to severely
underestimate their consumption in date 1. But if all agents are not too naïve, the linear intervention offers a Pareto improvement upon unrestricted nonlinear contracts.

4.2. Unknown Types

Then we consider the welfare effects of linear intervention when \( \beta \) is unknown to the financial intermediaries.

**Proposition 7** Suppose \( \hat{\beta} \) is known, and \( \beta \) takes the values \( \beta_1 < \hat{\beta} \) and \( \beta_2 = \hat{\beta} \) with probability \( p_1 \) and \( p_2 = 1 - p_1 \), respectively. Agents strictly prefer the competitive equilibrium in the unrestricted market over that in the restricted market. However, if non-sophisticated agents are sufficiently sophisticated (\( \beta_1 \) is sufficiently close to \( \beta_2 \)), their welfare and the population-weighted sum of two types’ welfare, in greater in the restricted linear market than in unrestricted market.

As in the case of full information, restricted linear contracts prevent non-sophisticated agents from drastically mispredicting their behavior and raise their welfare. But all agents strictly prefer unrestricted market to restricted linear market, and since fully sophisticated agents correctly predict their future behavior, they are made worse off by the linear intervention. This is because linear contracts eliminates discontinuity in welfare and reduces the cross-subsidy from non-sophisticated agents to sophisticated ones.

When types are unknown, restricting contracts to have a linear structure generally does not Pareto-dominate unrestricted nonlinear contracts. Nevertheless, the benefit of this intervention to non-sophisticated agents outweighs the harm to sophisticated depositors. Since this intervention decreases the distortion in repayment schedules to both types, it increases the population-weighted sum of welfare. Furthermore, this intervention has redistributive benefits if non-sophisticated agents are poorer than sophisticated agents in general.

An example of linear intervention is the demand deposit, which allows depositors withdraw any amount at a pre-specified interest rate. But to reduce risks of bank runs, interest rates of demand deposits are usually very low, and depositors’ welfare under demand deposit contracts may not be higher than under restricted deposit contracts. A similar example is enhanced or “flex” CDs introduced in recent years. These CDs offer an option to withdraw early once without penalty. Though enhanced CDs were introduced in order to reduce liquidity or interest rate risks, in some extent they restrict CD contracts to have a linear structure by allowing depositors to withdraw early at the specified interest rate. As shown above, enhanced CDs would increase population-weighted sum of welfare even though they have a slightly lower rate than standard CDs.

Pension fund deposit contracts also have a linear structure. For example, earnings from money withdrawn from the new Roth 401(k) or 403(b) accounts before the age of 59 1/2 are subject to income tax and a 10 percent early withdrawal penalty, and this 10 percent penalty corresponds to the linear contract in our model.

4.3. Term Premium

The restricted linear contract implies a term structure of interest rates in our three-date model. Suppose the one-period and two-period interest rates are \( i_1 \) and \( i_2 \), respectively, then the repayment options under this term structure include all \( (c_1, c_2) \)
that satisfies:

\[ \frac{c_1}{(1+i_1)} + \frac{c_2}{(1+i_2)^2} = 1 \]  

(10)

So a linear contract \( c_1 + c_2 / \tilde{R} = T \) has the term structure:

\[
\begin{cases}
1 + i_1 = T \\
1 + i_2 = (\tilde{R}T)^{1/2}
\end{cases}
\]

(11)

When all agents are sophisticated (\( \hat{\beta} = \beta \)), as shown in Proposition 6, financial intermediary offers \( \tilde{R} = R / \beta \), and \( T = q^h + r^h / \tilde{R} \), where \( (c_1^h, c_2^h) \) is the first-best allocation. This implies \( 1 + i_1 = c_1^h + \beta c_2^h / R \), and \( (1+i_2)^2 = Rc_1^h / \beta + c_2^h \), so when people get more impatient, financial intermediary would offer a lower one-period interest rate and a higher two-period interest rate such that people are committed to consume the first-best allocation.

Next we consider the term structure of investment contract when types are unknown. The assumptions the same as in Section 4.1. Suppose the financial intermediary offers the contract \((i_1, i_2)\), and type-1 and type-2 depositors consume \((c_{11},c_{21})\) and \((c_{12},c_{22})\), respectively. Since every depositor maximizes her self-1 utility among all repayment options, \((c_{11},c_{21})\) is the solution to \( c_{11} / (1+i_1) + c_{21} / (1+i_2)^2 = 1 \) and \( u'(c_{11}) / u'(c_{21}) = \beta_1(1+i_2)^2 / (1+i_1) \), \((c_{12},c_{22})\) is the solution to \( c_{12} / (1+i_1) + c_{22} / (1+i_2)^2 = 1 \) and \( u'(c_{12}) / u'(c_{22}) = \beta_2(1+i_2)^2 / (1+i_1) \), and \( c_{11},c_{21},c_{12},c_{22} \) are all continuous functions of \( i_1 \) and \( i_2 \).

**Proposition 8** Suppose \( \hat{\beta} \) is known, and \( \beta \) takes two values, \( \beta_1 < \hat{\beta} \) and \( \beta_2 = \hat{\beta} \) with probability \( p_1 \) and \( p_2 \), respectively. The competitive equilibrium linear contract has a term structure \((i_1,i_2)\) where \( i_1,i_2 \) are one-period and two-period interest rates, respectively, and is determined by:

\[
(R \frac{\partial c_{11}}{\partial i_1} + \frac{\partial c_{21}}{\partial i_1})(\beta_2 (1+i_2)^2 \frac{\partial c_{12}}{\partial i_2} + \frac{\partial c_{22}}{\partial i_2}) = (R \frac{\partial c_{11}}{\partial i_2} + \frac{\partial c_{21}}{\partial i_2})(\beta_2 (1+i_1)^2 \frac{\partial c_{12}}{\partial i_1} + \frac{\partial c_{22}}{\partial i_1})
\]

(11)

and

\[ p_1((1-c_{11})R-c_{21}) + p_2((1-c_{12})R-c_{22}) = 0 \]

(12)

in which \( c_{11},c_{21},c_{12},c_{22} \) are continuous functions of \( i_1 \) and \( i_2 \) as defined above. In particular, when \( \beta_1 = \beta_2 = \beta \), \( i_1 \) is increasing in \( \beta \) and \( i_2 \) is decreasing in \( \beta \).

Next we study how the term structure changes with the proportion of naïve agents. By the cross-subsidy effect, the financial intermediary earns money on naïve agents to subsidize sophisticated agents. When there are more naïve agents, thus more agents to subsidize the financial intermediary, the financial intermediary can remain budget balanced by exploiting every naïve agent less. So the financial intermediary would increase the long-term interest rate relative to the short-term interest rate to induce naïve agents to consume less at date 0, and mitigate their time inconsistency problem. This indicates a term premium for naïveté: the difference between long-term and short-term yields increases with the share of naïve agents.
The following proposition confirms the above intuition.

**Proposition 9** Suppose \( \hat{\beta} \) is known, \( \beta \) takes two values, \( \beta_1 < \hat{\beta} \) and \( \beta_2 = \hat{\beta} \) with probability \( p_1 \) and \( p_2 \), and the linear contract has a term structure \((i_1, i_2)\) where \( i_1, i_2 \) are one-period and two-period interest rates. Then, in the competitive equilibrium, term premium \( i_2 - i_1 \) is increasing in \( p_1 \) if \(-xu''(x)/u'(x) > 1 - \beta_2\).

5. Extensions and Discussions

5.1. General Consumer Beliefs

In previous sections we use the definition of partial naïveté by O’Donoghue and Rabin that self 0 believes with certainty that self 1’s discount factor is \( \hat{\beta} \), in this section we extend the concept of partial naïveté to a more general form which incorporates partial naïveté \( (\hat{\beta}) \) as special cases, and study contracts under this general specification of consumer beliefs.

Let the cumulative distribution function \( F(\hat{\beta}) \) represent consumers’ beliefs about \( \beta \) in date 0. Suppose financial intermediaries know consumers’ true \( \beta \). This is plausible given that financial intermediaries have a lot of information about depositors and spend a lot on researching their behavior. Suppose consumers expect to choose \((\hat{c}_1(\hat{\beta}), \hat{c}_2(\hat{\beta}))\) in date 0 for each \( \hat{\beta} \). Denote the support of \( F \) by \( F \). First we suppose financial intermediaries know \( F(.) \). Then we can show as in Heidhues and Köszegi (2010)’s Appendix that the contract is competitive equilibrium even if financial intermediaries do not observe consumers’ beliefs. The financial intermediary’s problem is:

\[
\begin{align*}
\max_{c_1, c_2, \hat{c}_1(\hat{\beta}), \hat{c}_2(\hat{\beta})} & \quad (1-c_1)R - c_2 \\
\text{s.t.} & \quad \int \left[ u(\hat{c}_1(\hat{\beta})) + u(\hat{c}_2(\hat{\beta})) \right] dF(\hat{\beta}) \geq u. \quad \text{[PC]} \\
& \quad u(\hat{c}_1(\hat{\beta})) + \hat{\beta}u(\hat{c}_2(\hat{\beta})) \geq u(\hat{c}_1(\hat{\beta}')) + \hat{\beta}u(\hat{c}_2(\hat{\beta}')) \quad \text{for any } \hat{\beta}, \hat{\beta}' \in \overline{F}, \quad \text{[PCC]} \\
& \quad u(c_1) + \beta u(c_2) \geq u(\hat{c}_1(\hat{\beta}')) + \beta u(\hat{c}_2(\hat{\beta}')) \quad \text{for any } \hat{\beta} \in \overline{F} \quad \text{[IC]}
\end{align*}
\]

**Proposition 10** Either when financial intermediaries know consumers’ beliefs or not, in a competitive equilibrium the repayment schedule a consumer with belief \( F(.) \) chooses satisfies:

\[
\frac{u'(c_1)}{u'(c_2)} = \frac{\beta R}{F(\beta)\beta + (1 - F(\beta))} \quad (11)
\]

Notice that \( \beta R \leq u'(c_1)/u'(c_2) \leq R \). Sophisticated consumers have \( F(\beta) = 1 \), thus the repayment schedule satisfies \( u'(c_1)/u'(c_2) = R \) and is welfare-maximizing; Partially naïve \( (\hat{\beta} > \beta) \) consumers have \( F(\beta) = 0 \), so the repayment schedule satisfies \( u'(c_1)/u'(c_2) = \beta R \). Therefore we replicate the results in Section 3. The consumer’s welfare depends solely on \( 1 - F(\beta) \), the probability she attaches to
unrealistically high levels of self-control.

Another example is the definition of partial naïveté in Eliaz and Spiegler (2006). Suppose the agent believes with probability $p$ she is time-consistent ($\beta = 1$) and with probability $1 - p$ her type is $\beta$, and $p$ measures her degree of sophistication. The repayment schedule satisfies $u'(c_1)/u'(c_2) = \beta R / ((1 - p)\beta + p)$ and is continuous in $p$; it approaches sophisticated agents’ repayment schedule as $p \to 0$, and approaches type- $\beta$ partially naïve agents’ repayment schedule as $p \to 1$.

5.2. Trading of CDs

Some CDs, like brokerage CDs, can be traded on a secondary market. So for this type of CDs, if a depositor wants to liquidate before maturity, she can sell the deposits on a secondary market rather than paying the early withdrawal penalty. The following proposition shows that, when there are both naïve and sophisticated people, tradable CDs with unrestricted nonlinear contracts is equivalent to linear deposit contract.

**Proposition 11** Suppose $\hat{\beta}$ is known, and $\beta$ takes the values $\beta_1 < \hat{\beta}$ and $\beta_2 = \hat{\beta}$ with probability $p_1$ and $p_2 = 1 - p_1$, respectively. If investment contracts can be traded on a secondary market at date 1, in competitive equilibrium all the agents get the same allocations as when investment contracts are restricted to be linear.

Intuitively, when agents can trade date-1 consumption for date-2 consumption under a certain price, people’s choice set is a linear set. In equilibrium, supply equals demand, and everyone maximizes their date-1 utility along the linear set, therefore tradable CDs are equivalent to restricted linear deposit contracts.

5.3. Transparency of Financial Intermediaries

So far we have assumed that the financial intermediary is not transparent. In this section we consider a transparent financial intermediary, that is, the financial intermediary has to disclose its amount of investment at date 0 and needs to be able to satisfy all customers’ needs according to the contract. For example, the unrestricted nonlinear contract with naïve agents is not feasible for a transparent financial intermediary, because the expected repayment schedule is out of the financial intermediary’s budget line, and though it would not be realized in equilibrium, the financial intermediary cannot remain solvent ex-ante when there is possibility that all agents choose that expected repayment option.

We reconsider the financial intermediary’s problem when all agents are naïve and the financial intermediary is transparent. Since liquidation of investment is costless, the financial intermediary can satisfy agents’ needs as long as all repayment options lie within the budget line. Since the budget constraint of the “chosen” repayment option is implied by nonnegative profits, we only need to add the budget constraint of the expected “decoy” repayment option, and this is the transparency constraint (TC). The financial intermediary’s problem is:

$$\max_{c_1, c_2} (1-c_1)R - c_2 \quad (12)$$

s.t. $u(\hat{c}_1) + u(\hat{c}_2) \geq u$. \hspace{1cm} [PC]

$u(\hat{c}_1) + \hat{\beta}u(\hat{c}_2) \geq u(c_1) + \hat{\beta}u(c_2)$. \hspace{1cm} [PCC]

$u(c_1) + \beta u(c_2) \geq u(\hat{c}_1) + \beta u(\hat{c}_2)$. \hspace{1cm} [IC]
\[(1 - \hat{c}_1)R - \hat{c}_2 \geq 0 \quad [TC]\]

If \( Ic \) and \( TC \) bind, the “decoy” repayment option is the same as the first-best allocation, while the “chosen” repayment option satisfies:

\[
\frac{u'(c_1)}{u'(c_2)} = \beta R \quad (13)
\]

and

\[
u(c_1) + \beta u(c_2) = u(c_1^b) + \beta u(c_2^b) \quad (14)\]

Note that the financial intermediary makes positive profits, and there is no deviation that can earn higher profits, because ex-ante naïve agents only care about the “decoy” repayment option, but the “decoy” repayment option is constrained by the transparency condition. So the financial intermediary can exploit naïve agents using the actually “chosen” repayment option without influencing their ex-ante expectation. The regulation that requires the financial intermediary to disclose its financial records and remain transparent in fact hurts the naïve people by restricting the financial intermediary’s choices of repayment options.

One way to increase naïve agents’ welfare when the financial intermediary is transparent is to have capital. If the financial intermediary has some capital, it would be able to offer repayment options beyond the budget line, and achieve the same welfare as when the financial intermediary is opaque. For example, suppose all depositors are naïve, if the financial intermediary has sufficient capital \( K \), the financial intermediary will offer \((\hat{c}_1, \hat{c}_2)\) at date 0, where \((\hat{c}_1, \hat{c}_2)\) are as defined in Proposition 2. If everyone chooses the option \((\hat{c}_1, \hat{c}_2)\), the financial intermediary’s budget constraint is:

\[
\hat{c}_2 \leq (1 + K)R \quad (15)
\]

The minimum capital required to achieve the same welfare as an opaque financial intermediary would be:

\[
K_{\text{min}} = \hat{c}_2 / R - 1 \quad (16)
\]

5.4. Saving Amount

In previous analysis, we assume each agent has unit saving. In this section, we consider the case where investors can choose their optimal saving amount. For simplicity, we assume that saving an amount \( c \) reduces utility by \( c \), i.e., self 0 has utility \( u(c_1) + u(c_2) - c \). Therefore, the first best allocation in this case solving the following program:

\[
\max_{c_1, c_2, c} u(c_1) + u(c_2) - c \quad (17)
\]

s.t. \( c_1 + c_2 / R \leq c \)

It is easy to know that the optimal \( c_1^* = I(1/l) \), \( c_2^* = I(1/R) \), \( c^* = c_1^* + c_2^* / R \), where \( I(\cdot) \) is the inverse function of \( u'(\cdot) \).

If individual agents have no choice but to invest through intermediary and the intermediary offer an unrestricted nonlinear contract. When there are only sophisticated depositors with \( \hat{\beta} = \beta < 1 \), the financial intermediary’s problem is:
\[
\max_{c_1, c_2} (c - c_1)R - c_2 \quad (18)
\]
s.t. \(u(c_1) + u(c_2) - c = u\) \quad [PC]

The first order condition is \(u'(c_1) = 1\) and \(u'(c_2) = 1/R\).

When there are only naive depositors with \(\hat{\beta} > \beta\), The problem is:

\[
\max_{\hat{c}_1, \hat{c}_2, c_1, c_2} (c - c_1)R - c_2 \quad (19)
\]
s.t. \(u(\hat{c}_1) + u(\hat{c}_2) - c = u\) \quad [PC]
\[
u(c_1) + \beta u(c_2) = u(\hat{c}_1) + \beta u(\hat{c}_2) \quad [IC]
\]

The first order condition is \(u'(\hat{c}_1) = 1/\beta\) and \(u'(c_2) = 1/R\). This suggests the saving of a naive agent is higher than that of a sophisticated agent.

6. Conclusion

Our model offers a new explanation of early withdrawal penalties in financial intermediaries’ investment contacts when people are time-inconsistent, and identifies simple welfare-improving interventions. For example, a secondary market in which long term deposit contracts can be traded could improve naïve agents’ welfare. However, while these interventions raise social welfare, they would not be accepted by agents who all believe they are rational. So it remains to be investigated whether there are modifications of this intervention that agents would prefer.

For simplification, we consider a model with three dates throughout the analysis. An interesting direction for future research is to extend this model to infinite horizon with overlapping generations and consider the term structure of interest rates in the infinite horizon model.

To isolate the commitment problem of time-inconsistent consumers, we assume that that liquidity need is totally generated by “irrational” time-inconsistent preferences and all agents differ only in their beliefs (or degree of sophistication). Future research would likely consider a more general model combining time inconsistency and real liquidity shocks, as in Amador, Werning and Angeletos (2006), in order to provide a more complete characterization of financial intermediaries’ role of providing risk-sharing and commitment in competitive equilibrium. It would also be interesting to investigate whether bank run is an equilibrium in the generalized model.
REFERENCES


Lin, Ping (1996): “Banking, Incentive Constraints, and Demand Deposit Contracts


Appendix : Proofs

Proof of Proposition 2. In the text we have established that \( \hat{c}_1 = 0 \). Using \( u(0) = 0 \), the two constraints in the relaxed problem combines into:

\[
 u(c_1) + \beta u(c_2) = \beta u \quad (A-1)
\]

So the first-order condition is: \( u'(c_1)/u'(c_2) = \beta \hat{R} \). And \( (1-c_1)R-c_2 = 0 \) comes from the zero profit condition of competitive equilibrium.

Proof of Proposition 3. Since \( PC \) and \( IC_1 \) both bind, we first ignore \( IC_2 \). With Lagrange multipliers \( \lambda \) and \( \mu \), the first order conditions are:

\[
\begin{align*}
-p_1 R + \mu u'(c_{11}) & = 0 \\
-p_1 + \mu \beta u'(c_{11}) & = 0 \\
-p_2 R + \lambda u'(c_{12}) - \mu u'(c_{12}) & = 0 \\
-p_2 + \lambda u'(c_{22}) - \mu \beta u'(c_{22}) & = 0
\end{align*}
\]

\( \lambda = (p_1/\beta u'(c_{11}) + p_2/\beta u'(c_{22}))R > 0 \) and \( \mu = p_1 R/\beta u'(c_{11}) > 0 \). Eliminating \( \lambda \) and \( \mu \), we then get equation (8) and (9) in Proposition 3. Since \( u(c_{11}) + \beta u(c_{21}) = u(c_{12}) + \beta u(c_{22}) \), and \( u'(c_{11})/u'(c_{12}) \leq u'(c_{21})/u'(c_{22}) \), we have \( c_{21} < c_{22} \), thus verify that \( IC_2 \) holds.

Proof of Proposition 4. Let \( u(p_1) \) be the perceived outside option in the competitive equilibrium when the proportion of non-sophisticated depositors is \( p_1 \), this is also the sophisticated depositors’ actual welfare with such a distribution of types. Take any \( p_1 > p_1 \). Since the financial intermediary makes money on non-sophisticated depositors, if the proportion increases to \( p_1 \) and the outside option is \( u(p_1) \), the equilibrium contract with proportion \( p_1 \) makes positive profits and satisfies the participation and incentive constraints. Therefore we must have \( u(p_1) > u(p_1) \).

Proof of Proposition 5. We have argued in the text that when Condition 1 holds, depositors would self-select into the same contracts as when \( \hat{\beta} \) is observed, so it is a competitive equilibrium. Now we prove it is a unique one.

By contradiction suppose there is an equilibrium in which not all \( \hat{\beta} \) types are offered the competitive equilibrium as when \( \hat{\beta} \) is observed. Let \( u_i \) be the perceived utility of \( \hat{\beta}_i \) in this equilibrium. First we show that there exists some \( i \) such that \( u_i < u_i \). Suppose \( u_i \geq u_i \) for all \( i \). Then even if \( \hat{\beta} \) is observable, the financial intermediary can only break even and achieve this by providing each type the equilibrium contract—a contradiction.

Now consider the highest \( i \) such that \( u_i < u_i \). For sufficiently small \( \varepsilon > 0 \), we can
find a contract with perceived outside option \( u_i + \epsilon < u_i \) that attracts type \( \hat{\beta}_i \) and makes positive profits on this type. For \( j > i \), \( u_i + \epsilon < u_j < u_j \), so this contract does not attract type \( \hat{\beta}_j \). Besides, even if it attracts some \( \hat{\beta}_j \) for \( j < i \) (though by self-selection this may not happen), they would select the non-sophisticated repayment option and generates positive profits. So the contract makes positive expected profits.

**Proof of Proposition 6.** The case of the fully sophisticated has already been proved in text. The financial intermediary offers linear contract \((\tilde{R}, T)\), where \( T \) is the discounted value of total repayment at date 1. Suppose fully sophisticated agents are offered \((1/\beta, T')\). The contract chosen by agents at date 0 depends only on \( \hat{\beta} \), denote the optimal contract offered to agents with \( \hat{\beta} \) by \((\tilde{R}(\hat{\beta}), T(\hat{\beta}))\). The agents’ perceived utility is continuous in \( \hat{\beta} \). We have shown that for \( \hat{\beta} = \beta \), the function has a unique maximum at \((1/\beta, T')\), therefore, as \( \hat{\beta} \to \beta \), we must have \((\tilde{R}(\hat{\beta}), T(\hat{\beta})) \to (1/\beta, T')\). Therefore the welfare of the unsophisticated approaches that of the sophisticated as \( \hat{\beta} \to \beta \), while it’s not true with unrestricted nonlinear contract by Proposition 3 and 4.

**Proof of Proposition 7.** First, we show that the perceived utility \( u \) in competitive equilibrium in unrestricted market is higher than that in restricted market. Suppose not, the contract in restricted market satisfies PC, IC, PCC, and breaks even, and is therefore a competitive equilibrium contract. But the competitive equilibrium we derived cannot hold in restricted market, because \( u'(c_{11})/u'(c_{21}) = \beta_1 R \) requires \( \tilde{R} = 1 \), while at this interest rate sophisticated consumers would not choose \((c_{12}, c_{22})\). (They have the same belief ex-ante)

To prove that restricted market raises welfare for sufficiently sophisticated consumers we use a similar argument as proof of Proposition 8. Holding the distribution of consumers fixed, for any \( \tilde{R} \), there is a unique \( T \) such that the contract \((\tilde{R}, T)\) yields zero profits, which in turn induces a perceived utility \( u \) as a function of \( \tilde{R} \). For \( \beta_1 = \beta_2 \), all consumers are sophisticated, so the function has a unique maximum at \((1/\beta_2, T')\). Since the function is continuous in \( \beta_1 \), the same argument in Proposition 6 follows.

Next we consider social welfare. As \( \beta_1 \to \beta_2 \), both types’ repayment schedules approach the welfare-optimal one in long-term restricted market. While in unrestricted market, the repayment schedule of non-sophisticated approaches to \( u'(c_{11})/u'(c_{21}) = \beta_2 R \) and is thus inefficient.

**Proof of Proposition 8.** The financial intermediary’s problem is:

\[
\max_{c_{11}, c_{21}, c_{12}, c_{22}} p_1 ((1-c_{11})R - c_{21}) + p_2 ((1-c_{12}) - c_{22}) \quad (A-3)
\]
First order conditions with respect to \( i_1 \) and \( i_2 \) are:

\[
\begin{align*}
-p_1 R \frac{\partial c_{i1}}{\partial i_1} - p_1 \frac{\partial c_{i1}}{\partial i_1} - p_2 R \frac{\partial c_{i2}}{\partial i_1} - p_2 \frac{\partial c_{i2}}{\partial i_1} + \lambda u'(c_{i1}) \frac{\partial c_{i2}}{\partial i_1} + \lambda u'(c_{i2}) \frac{\partial c_{i2}}{\partial i_1} &= 0 \\
-p_1 R \frac{\partial c_{i2}}{\partial i_2} - p_1 \frac{\partial c_{i2}}{\partial i_2} - p_2 R \frac{\partial c_{i2}}{\partial i_2} - p_2 \frac{\partial c_{i2}}{\partial i_2} + \lambda u'(c_{i1}) \frac{\partial c_{i2}}{\partial i_2} + \lambda u'(c_{i2}) \frac{\partial c_{i2}}{\partial i_2} &= 0
\end{align*}
\]  
(A-4)

Eliminating \( \lambda \), and using \( u'(c_{i1})/u'(c_{i2}) = \beta_2 (1 + i_2)^2 / (1 + i_1) \), we have:

\[
\left( R \frac{\partial c_{i1}}{\partial i_1} + \frac{\partial c_{i1}}{\partial i_1} \right) \left( R \frac{\partial c_{i1}}{\partial i_1} + \frac{\partial c_{i1}}{\partial i_1} \right) = \left( R \frac{\partial c_{i1}}{\partial i_1} + \frac{\partial c_{i1}}{\partial i_1} \right) \left( R \frac{\partial c_{i1}}{\partial i_1} + \frac{\partial c_{i1}}{\partial i_1} \right)
\]  
(A-5)

which is equation (11). And equation (12) is the zero profit condition of competitive equilibrium.

**Proof of Proposition 9** The budget line can be rewritten as \( c_1 + c_2 / \tilde{R} = T \), then \( i_1 = T - 1 \) and \( i_2 = (\tilde{R} T)^{1/2} - 1 \). In period 1, type \( i \) \( (i = 1, 2) \) maximizes his utility \( u(c_{i1}) + \beta u(c_{i2}) \) such that \( c_1 + c_2 / \tilde{R} = T \). So \( u(c_{i1})/u(c_{i2}) = \beta \tilde{R} \) and \( c_{i2} = \tilde{R} (T - c_{i1}) \). Simple exercises show that

\[
\frac{\partial c_{i1}}{\partial T} = \frac{\beta \tilde{R}^2 u''(c_{i2})}{\beta \tilde{R}^2 u''(c_{i2}) + u''(c_{i1})}
\]

and

\[
\frac{\partial c_{i2}}{\partial \tilde{R}} = \frac{\beta (u''(c_{i2}) c_{i2} + u'(c_{i2}))}{\beta \tilde{R}^2 u''(c_{i2}) + u''(c_{i1})}.
\]

Now assume the economy is in the equilibrium, the intermediary slightly increases \( \tilde{R} \) and decreases \( T \) such that the equilibrium \( c_{i1} \) decreases and \( c_{i2} \) increases, and \( u(c_{i1}) + u(c_{i2}) \) keeps constant, that is, \( u(c_{i1}) + u(c_{i2}) = u(c_{i1} - \delta c_{i1}) + u(c_{i2} + \delta c_{i2}) \). Since \( \beta < 1 \), we have \( u(c_{i1}) + \beta u(c_{i2}) > u(c_{i1} - \delta c_{i1}) + \beta u(c_{i2} + \delta c_{i2}) \), suggesting \( c_{i1} - \delta c_{i1} + (c_{i2} + \delta c_{i2}) / \tilde{R} < T \) and \( c_{i1} - \delta c_{i1} + (c_{i2} + \delta c_{i2}) / \tilde{R} < T \). If \( \tilde{R} < R \), \( \sum_{i=1}^{2} p_i (-\delta c_{i1} R + \delta c_{i2}) < 0 \), indicating that the equilibrium \( \tilde{R} \) is larger than \( R \). Otherwise, we can increase the intermediary’s profit without lowering self 0’s utility by increasing \( \tilde{R} \).

By zero-profit condition \( p_1(c_{i1}R + c_{i2}) + p_2(c_{i2}R + c_{i2}) = R \), we have \( R = (p_1 c_{i1} + p_2 c_{i2}) (R - \tilde{R}) + \tilde{R} \tilde{R} \), thus \( \tilde{R} > R \) implies \( T < 1 \).

Now we show that the \( T \) and \( \tilde{R} \) must satisfy \( \partial T / \partial T < 0 \) and \( \partial \tilde{R} / \partial T < 0 \). Because the optimal \( T \) and \( \tilde{R} \) maximize \( u(c_{i1}) + u(c_{i2}) \), we have

\[
u'(c_{i1}) \left( \frac{\partial c_{i1}}{\partial T} + \frac{\partial c_{i1}}{\partial \tilde{R} \partial T} \right) + u'(c_{i2}) \left( \frac{\partial c_{i2}}{\partial T} + \frac{\partial c_{i2}}{\partial \tilde{R} \partial T} \right) = 0.
\]
Thus,
\[
\frac{\partial T}{\partial \tilde{R}} = \frac{\tilde{R}(1 - \beta_2) \frac{\partial c_{12}}{\partial \tilde{R}} - c_{12}}{\tilde{R} - \tilde{R}(1 - \beta_2) \frac{\partial c_{12}}{\partial T}} = \tilde{R} \beta_2 (1 - \beta_2) u'(c_{22}) - (T - c_{12}) u''(c_{12}) - \beta_2^2 \tilde{R} u'(c_{22}) c_{22} < 0
\]

Similarly, we have
\[
\frac{\partial T \tilde{R}}{\partial T} = \frac{-\tilde{R}c_{12} / T - \frac{\partial c_{12}}{\partial T} \tilde{R}(\beta_2 - 1)}{\tilde{R}(\beta_2 - 1) \frac{\partial c_{12}}{\partial T} + c_{22} / (T \tilde{R})}
\]

Since
\[
\frac{\partial c_{12}}{\partial T} = \frac{\beta_2 \tilde{R}^2 c_{12} u''(c_{22}) - u'(c_{12})}{T(\beta_2 \tilde{R}^2 u''(c_{22}) + u''(c_{12}))}
\]

and
\[
\frac{\partial c_{12}}{\partial T} = \frac{-\beta_2 (c_{12} u''(c_{22}) + u'(c_{12}))}{T(\beta_2 \tilde{R}^2 u''(c_{22}) + u''(c_{12}))}
\]

we know that
\[
- \tilde{R}c_{12} / T - \frac{\partial c_{12}}{\partial T} \tilde{R}(\beta_2 - 1) = - \frac{\tilde{R} \beta_2 \tilde{R}^2 c_{12} u''(c_{22}) + u''(c_{12}) c_{12} + u'(c_{12})(1 - \beta_2)}{\beta_2 \tilde{R}^2 u''(c_{22}) + u''(c_{12})} < 0
\]

and
\[
\tilde{R}(\beta_2 - 1) \frac{\partial c_{12}}{\partial \tilde{R}} + c_{22} / (T \tilde{R}) = \frac{c_{22} u''(c_{12}) + \tilde{R} \beta_2 (\tilde{R} - 1) u'(c_{22}) + u''(c_{22}) c_{22}}{T(\beta_2 \tilde{R}^2 u''(c_{22}) + u''(c_{12}))} > 0
\]

if \(-xu'(x) / u'(x) > (1 - \beta_2)\).

As shown in Proposition 6, the optimal \(\tilde{R} = R / \beta_2\) when \(p_1 = 0\). In this equilibrium, \(c_{11} R + c_{21} = c_{12} R + c_{22} = R\). Because \(c_{11} < T < 1\), there always exists a \(R' < R\) such that \(c_{11} R' + c_{21} = R^*\). In other words, for the equilibrium \(\tilde{R}\) and \(T\) given \(R\) and \(p_1 = 0\), there exists a corresponding \(R' < R\) and \(p_1 = 1\) such that the equilibrium \(\tilde{R}\) and \(T\) are the same. This indicates the equilibrium \(\tilde{R}\) when \(p_1 = 1\) should be larger than \(R / \beta_2\). If the optimal \(\tilde{R}\) and \(T\) are continuously change with \(p_1\), they must be monotone. Otherwise, we can find a \(p_1\) such that the perturbation of \(p_1\) would not change optimal \(\tilde{R}\) and \(T\). This is impossible because the zero profit condition would be violated when \(p_1\) changes but \(\tilde{R}\) and \(T\) keep constant. Therefore, when \(p_1\) increases from 0 to zero, the optimal \(\tilde{R}\) is increasing, \(T\) is decreasing and \(T \tilde{R}\) is increasing, that is, the term premium increases.

Proof of Proposition 10. \(PC\) binds because otherwise the financial intermediary
can lower \( c_i \) and \( \hat{c}_1(\hat{\beta}) \) so that \( u(\hat{c}_1(\hat{\beta})) \) decreases equally, keeping \( PCC \) and \( IC \) unchanged and raise profits. For \( \hat{\beta} \leq \beta \), from \( PCC \) and \( IC \) we can get: 
\[
 u(c_1) + u(c_2) \geq u(\hat{c}_1(\hat{\beta})) + u(\hat{c}_2(\hat{\beta})), \forall \hat{\beta} \in \overline{F} .
\]
Hence given \( PC \) it is optimal to set \((\hat{c}_1(\hat{\beta}), \hat{c}_2(\hat{\beta})) = (c_1, c_2)\) for all \( \hat{\beta} \leq \beta \).

Next consider \( \hat{\beta} > \beta \). We ignore \( PCC \) first and verify that the optimal solution satisfies it. As before it’s optimal to set \( \hat{c}_1(\hat{\beta}) = 0 \), otherwise we could decrease \( u(\hat{c}_1(\hat{\beta})) \) by some amount and increase \( u(\hat{c}_2(\hat{\beta})) \) by \( 1/\beta \) times the same amount, keeping \( IC \) fixed and easing \( PC \). With \( \hat{c}_1(\hat{\beta}) = 0 \) for all \( \hat{\beta} > \beta \), it is optimal to set \( \hat{c}_1(\hat{\beta}) = \hat{c}_2 \) such that \( IC \) binds. The financial intermediary’s problem is simplified to:

\[
\begin{align*}
\max_{c_1, c_2} & (1-c_1)R - c_2 \\
\text{s.t.} \quad & F(\beta)(u(c_1) + u(c_2)) + (1-F(\beta))u(\hat{c}_2) = y, \quad [PC] \\
& u(c_1) + \beta u(c_2) = \beta u(\hat{c}_2) \quad [IC]
\end{align*}
\]

Combining the two constraints and eliminate \( u(\hat{c}_2) \), then the first order condition yields equation (11) in Proposition 10.

**Proof of Proposition 11.** First, suppose the financial intermediary only offers one repayment option \((c_{10}, c_{20})\), we prove there exists a date-1 price of date-2 consumption \( p \) that market clears. In date 1, agent \( i \) maximizes:

\[
 u(c_{10} - px_i) + \beta u(c_{10} + x_i), \quad \text{where} \quad x_i \quad \text{is her purchase of date-2 consumption.}
\]

The FOC of naïve agent is:

\[
\frac{u'(c_{10} - px_i)}{u'(c_{20} + x_i)} = \frac{\beta_i}{p} \quad (A-7)
\]

FOC of sophisticated agent is:

\[
\frac{u'(c_{10} - px_2)}{u'(c_{20} + x_2)} = \frac{\beta_2}{p} \quad (A-8)
\]

Market clearing requires: \( p x_1 + p x_2 = 0 \), so there exists a \( p \) within the range \( \left( \frac{\beta_1 u'(c_{20})}{u'(c_{10})}, \frac{\beta_2 u'(c_{20})}{u'(c_{10})} \right) \) so that market clears. An agent maximizes her date-1 utility along the line: \( pc_1 + c_2 = pc_{10} + c_{20} \). Therefore by choosing a repayment option \((c_{10}, c_{20})\), the financial intermediary is also choosing a linear contract that is determined by the point, and to maximize profits the financial intermediary would choose exactly the same linear contract as the equilibrium contract under linear restriction in Section 4.

Next, suppose the financial intermediary offers two repayment options \((c'_{10}, c'_{20})\) and \((c''_{10}, c''_{20})\). We only need to prove that there exists a unique equilibrium \((c_{10}, c_{20}, p)\) in which everybody maximizes her date-1 utility along the line
\[pc_1 + c_2 = pc_{10} + c_{20}.\] Then as above we can show such equilibrium is equivalent to the competitive equilibrium with linear contract. Suppose \(c'_{10} < c''_{10}\) and denote
\[p = -(c'_{20} - c''_{20})/(c'_{10} - c''_{10}).\] By the proof above, there exists a price \(p'\) such that market clears at the repayment option \((c'_{10}, c'_{20})\). If \(p' \leq p\), \((c'_{10}, c'_{20})\) dominates \((c''_{10}, c''_{20})\), and \((c'_{10}, c'_{20}, p')\) is an equilibrium. Similarly, there exists a \(p''\) such that market clears at \((c''_{10}, c''_{20})\), and \((c''_{10}, c''_{20}, p'')\) is an equilibrium if \(p'' \geq p\).

Now we consider the case when \(p' > p\) and \(p'' < p\). By substitution effects, under price \(p\), there is excessive supply of date 2 consumption at the repayment option \((c'_{10}, c'_{20})\), and there is excessive demand for date 2 consumption at the repayment option \((c''_{10}, c''_{20})\). So there exists an equilibrium \((c_{10}, c_{20}, p)\), where \((c_{10}, c_{20})\) is a point on the line segment connecting \((c'_{10}, c'_{20})\) and \((c''_{10}, c''_{20})\). When the financial intermediary offers more than two repayment options, we can also prove in similar ways that such an equilibrium exists.