Asset Substitution and Underinvestment: A Dynamic View

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Abstract

Asset substitution and underinvestment are two of the most discussed agency problems in finance. However, the empirical study by Graham and Harvey (2001) finds little evidence that corporate executives are concerned about the asset substitution and underinvestment problems when they make investment and financing decisions. This paper notices that the two problems are formulated based on one-shot relationships between equity holders and debt holders. The paper examines these two issues in multi-period dynamic arrangements. It is found that if a firm issues debt only once, then the firm has an incentive to increase the firm risk after the debt is in place. If a firm needs to issue debt periodically, however, in order to capture the tax benefit associated with all future debts, the firm may not have an incentive to increase its risk. It is also found that the underinvestment problem can be greatly reduced or even essentially eliminated in a dynamic model because undertaking an investment now benefits equity holders in the future.

JEL classification: G31; G32; G33.

Keywords: Asset substitution; Underinvestment; Growth option.

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1 Introduction

Asset substitution and underinvestment are two of the most discussed agency problems in finance. In a one-period arrangement, Jensen and Meckling (1976) argue that once a debt is in place, the value of the equity is like an option due to the limited liability of the equity holders. Consequently, they conclude that equity holders will have incentives to increase the risk of the firm so as to increase the equity value at the expense of debt holders. Also in a one-period relationship, Myers (1977) argues that equity holders may not undertake certain positive net-present-value projects because they bear the full costs of the projects while sharing the benefits with debt holders. In particular, when the firm value is low, additional firm value-enhancing investments may benefit mostly debt holders whereas equity holders bear all of the costs. Consequently, underinvestment arises.

In an empirical study, Graham and Harvey (2001) survey 392 CFOs about their investment and financing decisions. Overall, they find that support for asset substitution and underinvestment arguments is weak. In particular, they find little support for the idea that short-term debt is used to alleviate the underinvestment problem as well as little evidence that executives issue short-term debt to minimize asset substitution problems. On the potential conflicts between managers and equity holders, Khanna and Poulsen (1995) consider empirically whether financial distress is a result of self-serving managerial decisions. Their results suggest that the potential conflict between managers and shareholders is not a major contributor to the financial distress of firms. Our focus is on the potential conflicts between equity holders and bond holders. Theoretically, we note that the concepts of asset substitution and underinvestment, which are proposed based on static arguments, may not be robust in more general settings. For example, in the principal-agent literature, Fama (1980) and Holmstrom (1999) point out the possibility that the results based on a one-period arrangement may not be robust in a multi-period one with implicit incentives or career concerns.

In this paper we study the asset substitution problem and the underinvestment problem in multi-period arrangements in which a firm issues debt periodically for additional investment needs. In the discussion of asset substitution, we assume that there are no explicit contracts between debt holders and equity holders or that debt holders cannot force equity holders to adopt a specific volatility for the firm’s investment. Equity holders, however, face implicit incentives, that is, if they increase the firm risk after the debt is in place, then debt holders
would price future debts at the increased risk level. As a result, the optimal debt amounts in future periods will be lower, resulting in lower tax shields for the firm.

We formulate the asset substitution problem in a continuous-time framework in which the firm issues debts potentially infinite times (as long as it is solvent). Each debt matures in a finite time period and both the optimal leverage ratio and the optimal debt maturity are jointly determined. In a static model in which a firm issues debt only once, we show that the firm always wants to increase risk after the debt is in place. In a dynamic model in which the firm issues debt multiple times, implicit incentives can greatly reduce the asset substitution problem. This is due to the repetitive nature of the dynamic arrangement. Suppose that after one issue of debt, equity holders decide to raise the volatility of the firm, for example, from 20% to 30%. This has two effects. On the one hand, equity holders can extract value from the current debt holders (the option analogy). On the other hand, once the risk of the firm has been raised, equity holders can no longer issue debt at the lower volatility in the future. Therefore, they cannot optimally issue as much debt as they could otherwise, and as a result, the tax benefit would fall in the future. Because of the second effect, the equity holders’ incentive to increase the risk of the firm is greatly reduced. Even when the debt amount is not adjusted as in Leland and Toft (1996), due to the increased default probability and loss of tax benefit if the risk is raised, the firm does not have any incentive to raise firm risk if the firm value is high enough. In our model with the option to dynamically optimally adjust the capital structure, the cost of raising firm risk is even higher since the option value is lost if default occurs. Only when the firm value is sufficiently low and the equity is deep out of the money, do equity holders have an incentive to raise risk. In this case, the firm faces bankruptcy, and its implicit incentives are very low.

In the discussion of underinvestment, the implicit incentives faced by equity holders are two fold in our dynamic model. First, the investment in the current period reduces the likelihood of the firm going bankrupt, so that the equity holders are more likely to benefit from future investment projects. Second, the current investment increases the asset base that makes future investments more valuable. This discussion is also set in a continuous-time framework in which there are infinite number of time periods, each of which lasts for a finite number of years. Three cases are considered. (1) A firm has a debt in place as well as a growth option that increases the firm value for one period. (2) A firm has a debt
only in the first period but it has a growth option in every period. (3) A firm has both a debt and a growth option in every period. We define the first best solution as the one that maximizes the firm value (equity value plus debt value) and the second best solution as the one that maximizes the equity value. We do not consider explicit incentives in our model. DeMarzo and Fishman (2007) develop a general model of investment decisions in the presence of explicit optimal incentive contracts. They show that the agency problems between principal (investors) and agent (managers) can be alleviated by making additional investment following good firm performance and disinvestment following bad firm performance.¹

Our first case corresponds to the original Myers’ (1977) static model. In this case, the exercise of the growth option makes the firm less likely to be bankrupt so that the debt value strictly increases. Because equity holders bear the full cost of the investment, they will exercise the option only when their benefit equals or exceeds the cost. As a result, equity holders will bypass some positive NPV investments. In the second case, equity holders have implicit incentives associated with all future growth options. We show that the critical firm asset value above which the current option is exercised is always lower than that in the first case because exercising the growth option in the current period increases the present values of all future options, which solely benefit equity holders. In the third case, the level at which the current option is exercised is again lower than that in the static case. Although the firm has a debt in place in every period, exercising the current growth option increases the firm’s asset base, which makes future growth options more valuable, benefitting equity holders.

This paper studies the asset substitution problem and the underinvestment problem in dynamic settings in which implicit incentives exist. It is demonstrated that implicit incentives can greatly reduce these agency problems. Our results suggest that these and other agency problems, as proposed in static arrangements, may not be evident in reality where dynamic arrangements are the norm. Our results offer a potential explanation for the survey results of Graham and Harvey (2001) in which the CFOs do not view asset substitution and underinvestment as major problems.

Barnea, Haugen, and Senbet (1980), Green and Talmor (1986), and others study asset substitution in one-period models. It is suggested that shorter maturity debt can be used to reduce the agency costs associated with asset substitution and underinvestment problems.

¹For more models of agency conflicts and optimal contracting, see DeMarzo and Fishman (2007) and references therein.
In a continuous-time framework, Leland (1998) allows equity holders’ decision to choose a high or a low risk level once a debt is in place and studies the impact of equity holders’ ex post flexibility to choose risk on the firm’s optimal capital structure. Leland finds that when the agency costs increase, the firm chooses a strategy with higher average risk, the optimal leverage ratio is lower, the debt maturity is shorter, and the yield spread of debt rises. Titman and Tsyplakov (2007) and Childs, Mauer, and Ott (2005) study the impact of debt issuance on the firm’s underinvestment behavior. In Titman and Tsyplakov, when a firm can issue debt dynamically, it increases the debt amount when the firm value increases, but, by assumption, it cannot as easily reduce the debt amount when the firm value decreases. As a result, the agency costs may increase if the firm can issue debt dynamically as opposed to the case in which the firm can issue debt only once. The magnitude of the agency costs can be high in this model. In Childs, Mauer, and Ott, the firm can both increase and decrease the debt amount easily, so the agency costs decrease if the firm can issue debt dynamically.


In summary, the previous work on asset substitution does not capture the notion that although increasing the risk of the firm benefits the equity holders in the current period, it will hurt them in the future. The reason is that all future debts of the firm will be priced at the raised risk level or the firm cannot issue as much debt as it could, so that the firm will benefit less from future tax savings. The previous work on underinvestment does not capture the idea that a firm may have many growth options or investment opportunities and that although exercising the growth option in the current period may benefit the debt holders, it will benefit the equity holders in the future.

The rest of this paper is organized as follow. Section 2 presents a dynamic capital structure model in which the asset substitution problem is studied. Section 3 addresses the underinvestment problem in a dynamic model in which the investment decision rules of a leveraged firm are solved. Section 4 concludes the paper.
2 Examining the Debt-Induced Asset Substitution Problem within a Dynamic Model of Optimal Capital Structure

2.1 Model Assumptions

Assumption 1 Financial markets are dynamically complete, and trading takes place continuously. Therefore, there exists an equivalent martingale measure or a risk-neutral measure, \( Q \), under which discounted price processes are martingales.

Assumption 2 The unlevered asset value process under \( Q \) follows geometrical Brownian motion,

\[
\frac{dV_t}{V_t} = (r - \delta)dt + \sigma dW^Q_t, \tag{1}
\]

where \( r \) is the constant riskless interest rate, \( \delta \) a constant payout rate, \( \sigma \) a constant volatility of the asset return and \( W^Q \) is a standard Wiener process under the risk-neutral probability measure \( Q \).

Assumption 3 The firm issues coupon debt with a finite maturity date. Coupon payments are tax deductible at the corporate tax rate \( \theta \) and the full loss offset is assumed.

Let \((C, P, T)\) be the coupon, principal, and maturity of the debt, respectively. Assumption 3 implies that when the firm pays coupon \( C \), it deducts taxes by the amount of \( \theta C \).

Assumption 4 Bankruptcy occurs when the firm’s unlevered asset value falls to an exogenous default boundary \( V_B(t) \), which is specified by

\[
V_B(t) = \rho Pe^{\gamma t}. \tag{2}
\]

where \( P \) is the face value of the bond and \( \rho = e^{-\gamma T} \). We require that the bond in our model be always priced at par. So \( P \) is also the price of the bond when it is issued. In the event of default, \( \alpha \in (0, 1) \) fraction of the firm’s unlevered value is lost to the bankruptcy procedure and the bondholders receive the rest, \((1 - \alpha)V_B(t)\).
The above bankruptcy-triggering mechanism includes some of the most widely used ones as special cases. Black and Cox (1976) also use an exponential bankruptcy-triggering boundary, and a special case with $\gamma = r$. $\gamma = 0$ corresponds to the protected debt case in Leland (1994). Longstaff and Schwartz (1995) consider a constant bankruptcy level (corresponding to $\gamma = 0$), which is not necessarily equal to the principal $P$. Kim, Ramaswamy, and Sundaresan (1993) also assume a constant bankruptcy level. $\gamma = \infty$ corresponds to the model in Merton (1974), where bankruptcy can only occur at debt maturity.

**Assumption 5** The firm rebalances its capital structure every $T$ years. At time zero the firm issues a $T$-year coupon bond. If the firm has not gone bankrupt in $T$ years, it issues another $T$-year coupon bond at time $T$. This process continues indefinitely as long as the firm is solvent.

It is noted that the capital structures in Leland (1994) and Leland and Toft (1996) are static in the sense that as the firm’s asset value evolves, the coupon and the face value of the debt remain the same. This is not optimal because the firm may want to change the debt structure when its asset value changes. When the firm rebalances its capital structure, Assumption 5 allows the firm to scale up (down) the coupon and the face value if the firm’s asset value has increased (decreased). Given this assumption, we show in Subsection 2.2.2 that the optimal initial debt value at each restructuring point is proportional to the prevailing asset value. For example, if the firm’s asset value has doubled since the previous bond issue, the coupon and principal (and market value too) of the new bond should be twice of that of the previous bond.

**Assumption 6** The transactions cost of issuing and serving a bond is $\kappa \in (0, 1)$ fraction of the market value of the bond issued.

While this paper is not about optimal capital structure and optimal maturity, a transactions cost is introduced to exclude the firm from rebalancing too often.

### 2.2 Model Derivation

The dynamic model we develop in this section is similar to that in Ju, Parrino, Poteshman, and Weisbach (2005) and the derivations follow that paper closely. To this end, define the
first passage time $\tau$ as

$$\tau = \min\{t : V_t \leq V_B(t)\}, \quad (3)$$

which is the first time at which the asset value $V_t$ hits the default boundary $V_B(t)$ in some state $\omega \in \Omega$ under $Q$. Let $f(t)$ be the density function of $\tau$. For simplicity we define the following three quantities:

$$F(t) = \int_0^t f(s)ds, \quad G(t) = \int_0^t e^{-rs}f(s)ds, \quad \overline{G}(t) = \int_0^t e^{-(r-\gamma)s}f(s)ds. \quad (4)$$

Note that $F(t)$ is the distribution function of $\tau$. Closed form expressions of $F(t)$, $G(t)$ and $\overline{G}(t)$ are provided in Appendix A.

### 2.2.1 Closed form formulas of securities in the first period

Following Leland and Toft (1996), the initial value of the risky bond is given by:

$$D(V_0; C, P, T) = C \int_0^T e^{-rt}(1 - F(t))dt + \int_0^T e^{-rt}(1 - \alpha)Pe^{-\gamma(T-t)}f(t)dt + P(1 - F(T))e^{-rT} = \frac{C}{r} \left(1 - (1 - F(T))e^{-rT} - G(T)\right) + (1 - \alpha)Pe^{-\gamma T}\overline{G}(T) + P(1 - F(T))e^{-rT}. \quad (5)$$

When the firm pays coupon $C$, it deducts $\theta C$ for taxes. The tax deduction exists as long as the firm value has not reached $V_B(t)$. Therefore the present value of the tax benefits is given by

$$tb(V_0; C, P, T) = \theta C \int_0^T e^{-rt}(1 - F(t))dt = \frac{\theta C}{r} \left(1 - (1 - F(T))e^{-rT} - G(T)\right). \quad (6)$$

Since bankruptcy occurs when the firm value first reaches $V_B(t)$, the present value of bankruptcy costs is given by

$$bc(V_0; C, P, T) = \int_0^T \alpha Pe^{-\gamma(T-t)}e^{-rt}f(t)dt = \alpha Pe^{-\gamma T}\overline{G}(T). \quad (7)$$

Since the debt is issued at par, the transactions cost is given

$$tc(V_0; C, P, T) = \kappa P. \quad (8)$$
2.2.2 Total values of tax benefits, bankruptcy costs and transactions costs

Note that (6), (7) and (8) give the values of tax benefit, bankruptcy cost and transactions cost resulting from the first issue of debt. Even though at any one point there is only one issue of debt outstanding, the optimal capital structure will reflect the benefit and cost of future issues of debt. Therefore we need to find the total tax benefit, total bankruptcy cost, and total transaction cost from all future periods. However, the existence of all future periods depends on the firm not having gone bankrupt in all previous periods. This is a multi-period conditional first passage time problem. Generally, such problems are very hard to solve. Fortunately, because of the repetitive nature of the multi-period game, we are able to transform this difficult problem into a one-period fixed-point problem as we show now.

Suppose the optimal coupon and principal in the first period are $C$ and $P$, respectively. We argue that if the unlevered firm value at $T$ is $V_T$, the optimal coupon and principal in the second period will be $CV_T/V_0$ and $PV_T/V_0$, respectively. That is, the coupon and principal are scaled by $V_T/V_0$. The reason is that, due to the proportional process of (1), the firm at time $T$ is identical to itself at time zero, except that it is $V_T/V_0$ times as large. In fact, the optimal coupon and principal in all future periods scale with the unlevered firm value at the beginning of each period. It follows that the total value of tax benefits, bankruptcy costs and transactions costs all also scale with the unlevered firm value.

We now use the scaling property to obtain the total tax benefit at the beginning of the first period. Let it be denoted by $TB(V_0; C, P, T)$. By the scaling argument, $TB(V_0; C, P, T)V_T/V_0$ is the total tax benefit at the beginning of the second period. However, the total tax benefit $TB(V_0; C, P, T)$ at the beginning of the first period is, of course, the tax benefit of the first period $tb(V_0; C, P, T)$, plus the present value of all potential tax benefits in the future periods. Therefore, we have

$$TB(V_0; C, P, T) = tb(V_0; C, P, T) + e^{-rT}E^Q \left[ \frac{V_T}{V_0} TB(V_0; C, P, T) | \text{No default by } T \right]$$

$$= tb(V_0; C, P, T) + \phi TB(V_0; C, P, T), \tag{9}$$

where

$$\phi = e^{-rT}E^Q \left[ \frac{V_T}{V_0} | \text{No default by } T \right]. \tag{10}$$
A closed form expression for $\phi$ is given in Appendix A. Solving for $TB(V_0; C, P, T)$, we have

$$TB(V_0; C, P, T) = \frac{tb(V_0; C, P, T)}{1 - \phi}. \quad (11)$$

Similarly, total bankruptcy cost $BC(V)$ and the total transactions cost $TC(V)$ are respectively given by

$$BC(V_0; C, P, T) = \frac{bc(V_0; C, P, T)}{1 - \phi}, \quad TC(V_0; C, P, T) = \frac{tc(V_0; C, P, T)}{1 - \phi}. \quad (12)$$

### 2.3 Model Implications for the Asset Substitution Problem

#### 2.3.1 Optimal capital structure and optimal maturity

The total levered firm value equals to the unlevered firm value $V_0$, plus the total tax benefit $TB(V_0; C, P, T)$, less the total bankruptcy cost $BC(V_0; C, P, T)$, and less total transactions cost $TC(V_0; C, P, T)$:

$$TV(V_0; C, P, T) = V_0 + \frac{tb(V_0; C, P, T) - bc(V_0; C, P, T) - tc(V_0; C, P, T)}{1 - \phi}. \quad (13)$$

For a given asset volatility, $\sigma$, and the other structural parameter values of the model, the optimal capital structure and maturity are obtained by maximizing $TV(V_0; C, P, T)$ over $C, P$ and $T$, subject to the constraint that the bond is issued at par. That is, the firm solves the following constrained maximization problem:

$$\max_{C, T, P} TV(V_0; C, T, P) \quad \text{such that } D(V_0; C, T, P) = P. \quad (14)$$

For numerical calculations, we use the following parameter values: $r = 7.5\%$, $\delta = 5.0\%$, $\sigma = 0.2$, $\alpha = 0.5$, $\theta = 0.2$,$^{2}$ $\kappa = 2\%$, and $\rho = 0.8$.\footnote{Even though 35\% is the top corporate tax rate, the effective corporate tax rate for most firms is likely to be lower due to investment credit, loss-carry forward, and personal tax effects.} With these input values, we find that the optimal $(C, P, T) = (2.76, 35.55, 6.05)$.\footnote{Fischer, Heinkel and Zechner (1989) have used transaction costs ranging from 1\% to 10\%. Kane, Marcus and MacDonald (1985) have considered 1\% and 2\% transaction costs while Goldstein, Ju and Leland (2001) have used 1\%.} The optimal maturity of 6.05 years is close to $3$ Even though 35\% is the top corporate tax rate, the effective corporate tax rate for most firms is likely to be lower due to investment credit, loss-carry forward, and personal tax effects.

$^{3}$ Fischer, Heinkel and Zechner (1989) have used transaction costs ranging from 1\% to 10\%. Kane, Marcus and MacDonald (1985) have considered 1\% and 2\% transaction costs while Goldstein, Ju and Leland (2001) have used 1\%.

$^{4}$ Leland and Toft (1996) consider a model with finite-maturity debt which is rolled over. Their endogenously determined bankruptcy level for five year maturity debt is about 0.9 of the face value of total debt outstanding.

$^{5}$ The optimal leverage ratio is 33.52\% which compares favorably with that of a typical large firm which is around 30\%.
the average maturity of all debts. Barclay and Smith (1995) find that during the period of
1974–1992, firms in their sample have 51.7% of their debts due in more than three years.
Stohs and Mauer (1996) find that the mean maturity is 3.38 years for their sample. Because
on average, the debt would have existed for half of the lifetime at any point of time, the
average maturity at issue appears to be between 6 years and 7 years. When $T$ is optimally
chosen, we call this version of the model the **Dynamic Model**.

To discuss the implications of the dynamic model on the asset substitution problem in the
next subsection, we also consider a model where debt is issued only once. To this end, we set
$T = \infty$. For $\rho$ and $\gamma$ in (2), we set $\rho = 0.8$ and $\gamma = 0$. These values imply that a perpetual
bond is issued and the default boundary is a constant which is 0.8 of the debt principal. In
this case the optimal $(C, P) = (3.67, 44.78)$. The corresponding optimal leverage ratio is
42.45%. When $T$ is exogenously fixed at $\infty$, we call this version of the model the **Static
Model**.

Although it is straightforward to have more extensive comparative statics results and
discussions on the optimal capital structure and debt maturity, it is not the goal of this
paper. In the next subsection, we focus our attention to the implications of the dynamic
nature of the model on the problem of debt-induced asset substitution.

### 2.3.2 Volatility sensitivity of the equity value and debt value versus the firm
asset value

A classical problem created by debt in the capital structure is the agency problem between
the equity holders and the debt holders. It has been well recognized that once the debt is
issued, the equity holders may have the incentive to increase the volatility of the firm to
transfer wealth from the debt holders. The basic intuition is that the equity is a call option
on the firm value. Option theory shows that the value of an option is an increasing function
of the volatility of the underlying asset. But this analogy is not complete, because almost
all bonds are not pure discount bonds. We also demonstrate that because the firm needs to
go to the debt market many times in the future, it may not want to increase the volatility of
the firm unless the firm value becomes low enough. The intuition is that if the firm increases
its volatility now, it can not optimally issue as much debt as otherwise it could in the future.
Therefore the dynamic nature of the dynamic model reduces the equity holders’ incentive to
increase the volatility of the firm.
To show the effect of an *increased* asset volatility on the equity value and debt value after the debt is issued, following Leland and Toft (1996), we calculate the partial derivatives of the equity value and bond debt value with respect to the volatility for different asset values. To do so, we need the equity value and debt value as a function of the asset value $V_t$ at any time before debt maturity, $t \in (0, T)$. Note that given $(C, P, T)$, and the firm asset value $V_t$ at $t \in (0, T)$, it is straightforward to obtain the corresponding debt value $D(V_t; C, P, T)$ in (5). The equity value $E(V_t; C, P, T)$ equals to the difference between the total firm value, $TV(V_t, C, P, T)$, and the prevailing debt value, $D(V_t; C, P, T)$:

$$E(V_t; C, P, T) = TV(V_t, C, P, T) - D(V_t; C, P, T) \quad (16)$$

To obtain $TV(V_t, C, P, T)$, we first calculate the tax benefit $tb(V_t; C, P, T)$ in (6), and bankruptcy cost $bc(V_t; C, P, T)$ in (7), resulting from the remaining life of the outstanding debt from $t$ and $T$. Similar to (9), we have

$$TV(V_t; C, P, T) = tb(V_t; C, P, T) - bc(V_t; C, P, T) + e^{-r(T-t)} E_t \left( \frac{V_T}{V_0} TV(V_0; C, P, T) 1(\tau > T) \right)$$

$$= tb(V_t; C, P, T) - bc(V_t; C, P, T) + \frac{V_t}{V_0} TV(V_0; C, P, T) e^{-r(T-t)} E_t \left( \frac{V_T}{V_t} 1(\tau > T) \right). \quad (17)$$

The indicator function $1(\tau > T)$ indicates the condition that no default has occurred by $T$. $e^{-r(T-t)} E_t \left[ V_T/V_t 1(\tau > T) \right]$ is similar to $\phi$ in (10) and can be obtained similarly in closed form. Equation (17) states that the total firm value at $t$ (with asst value $V_t$) equals to the net benefit from the outstanding debt (tax shield less default cost),

6 plus the present value of the total firm value at $T$ which scales with the total firm value at time zero.

With (17), we can obtain $E(V_t; C, P, T)$ from (16). We are now ready to calculate the partial derivatives of the equity value and bond debt value with respect to the volatility for different asset values. Figure 1 plots $\partial D(V_t; C, P, T)/\partial \sigma$ (dotted line) and $\partial E(V_t; C, P, T)/\partial \sigma$ (solid line) for the **Static Model**. Figure 2 plots the same partial derivatives for the **Dynamic Model**. Recall that the (initial) time-to-maturity is $T = 6.05$ years. In Figure 2 the time-to-maturity is the remaining life of the outstanding debt.

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6 No need to subtract the transactions cost because it occurred at time zero.

7 Note that when $\sigma$ is changed, the optimal $C$, $P$, $T$ and $TV(V_0; C, P, T)$ also change. To compute $TV(V_t; C, P, T)$ using (17) if $\sigma$ is changed after the debt $(C, P, T)$ is in place, the new optimal $C$, $P$, $T$ in the third term are different from those in the first two terms. To avoid introducing new variables, we have used the same $C$, $P$, $T$, but we should keep in mind that they are different.
Figure 1 reveals that if no future debt is ever issued, the incentive for the equity holders to increase the volatility after the outstanding debt is issued always exists. However, Figure 2 shows that when the firm needs to optimally issue new debts periodically, the incentive for the firm to increase its volatility is greatly reduced. The reason is that once the firm increases its asset volatility, the debt in the future will be issued based on the higher risk level. Since the optimal debt amounts, or rather the optimal net tax benefits (tax shields less default costs) are negatively related to the risk level, raising the risk level in the first period will reduce the net tax benefits in the future. Therefore increasing the volatility of the firm may not be in the best interest of the equity holders unless the firm value becomes low enough.

As the firm value becomes very high, the partial derivatives of the equity and the debt in the Static Model both approach zero because the debt becomes essentially riskless and the equity holders will not be able to transfer value from the debt holders. But this is a consequence of the static nature of the Static Model because the debt capacity is not adjusted as the firm value changes. In the Dynamic Model as the firm value becomes very high, the debt becomes essentially riskless, therefore its partial derivative approaches zero from below. But the equity holders will be worse off if they increase the risk of the firm. The reason is that, on the one hand, they could not transfer value from the debt holders by reducing their value, but on the other hand, they hurt themselves because they could not optimally issue as much debt as otherwise they could in the future to take advantage of the tax shield. This is also the reason that the partial derivative of the equity decreases as the firm value increases on the right portion of each graph in Figure 2 because as the firm value becomes higher, the reduction in the optimal debt amount to be issued in the future is greater if the risk of the firm is increased.

The firm’s consideration of future issues of debt does not completely eliminate the asset substitution problem. After the outstanding debt is issued, if the asset value becomes close enough to the default boundary and the equity value is very small, the equity holders have the incentive to raise the volatility to increase the expected payoff because the equity behaves like a call option. However, if the asset value is extremely close to the default boundary, even though the incentive for the equity holders to raise asset volatility still exists, it is not as strong as when the asset value is not so close to the default boundary. The reason is that
when the asset value is very close to the default boundary, the equity value (and debt value too) are insensitive to the volatility value because default is almost certain. The incentive for the equity holders to raise the volatility is the greatest when the asset value is close, but not too close to the default boundary. At these values, the probability of avoiding default is small but not too small. With an increased volatility, when default is avoided, the expected payoff for the equity holders becomes larger. However, as the asset value increases further, the incentive for the equity holders to raise volatility becomes weaker because the loss of the future tax benefits becomes a concern.

In sum, the asset substitution problem is greatly reduced in the **Dynamic model**. This result is very intuitive in a dynamic setting. Suppose, for example, that after one issue of debt, the equity holders decide to change the volatility of the firm, say, from 20% to 30%. This has two effects. First, the equity holders can transfer wealth from the current debt holders to themselves (the option analogy). However, once the risk of the firm has been increased, the equity holders can no longer optimal issue debt at 20% volatility in the future. Therefore they could not optimally issue as much debt as they could otherwise, and the tax benefit would fall in the future. Because of the second effect, the incentive for the equity holders to increase the risk of the firm is greatly reduced.

In our model and calculations, we do not allow further increasing the firm risk in future periods. This amounts to the assumption that the equityholders can credibly and freely commit not to raise risk again. If they cannot credibly and/or freely commit, future debtholders will rationally anticipate that the firm may increase the firm risk again and will price their debt at an even higher risk level if the firm has raised risk in the first period. This will result in even lower optimal tax shields. Thus, if the equityholders have to take into account the adverse effects of their action of increasing risk in the first period on future debt issues, they have even less incentive to raise risk in the first period.

Option features in bond indentures have been argued to be effective controlling the risk shifting problem of asset substitution. Green (1984) argues that convertible bonds can help align the interests of bondholders and equityholders, while Hennessy and Tserlukovich (2008) show that in a world with market imperfections such as taxes, default costs, and debt issue costs, callable bonds with premia can dominate convertible bonds and help reduce agency conflicts between bondholders and equityholders. Call and conversion features are certainly
useful controlling the agency conflicts. However, even without them, we have argued that the equityholders may not have any incentive to raise risk because they have to take into account the intertemporal trade-off between gain in the first period by decreasing the value of the existing bonds and the loss of net tax benefits in future periods if they raise firm risk. Our model helps to understand the finding in Graham and Harvey (2001) that senior managers are not concerned about the debt induced risk shifting problem because the bondholders will realize that it is not in the interest of the equityholders for them to raise risk.8

3 Examining the Debt-Induced Underinvestment Problem within a Dynamic Model of Growth Options

To build a multi-period model, define one period as every $T$ years. For the capital structure and growth option combinations we consider three cases:

1. A one-period levered firm with only one growth option in the first period;
2. A one-period levered firm with one growth option every period period;
3. An every-period levered firm with one growth option every period period.

For each case, we will consider the first best and second best policies of exercising the growth options. The first best policy is chosen to maximize the initial firm value. This is equivalent to the policy by the equity holders of an all-equity firm. The second best policy is chosen by the equity holders of a levered firm after the outstanding debt is issued.

3.1 Model Assumptions

**Assumption 7** The value of a firm’s asset-in-place at time $t$, denoted by $V_t$, follows a standard Black-Scholes geometric Brownian motion, given by (1).

**Assumption 8** Each growth option (investment opportunity) occurs at the middle of the prevailing period.

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8In a debtless model where the manager is risk averse and holds employee stock options and restricted company shares, Ju, Leland and Senbet (2014) find that the manager’s chosen risk level is lower than the optimal level for well-diversified risk-neutral shareholders. Any debt-induced incentive to raise risk may be beneficial to the shareholders while the effects on debt holders can be ambiguous since in such a model the increased likelihood of default probability can be offset by the increase in firm asset value.
The assumption that a growth option exists at the middle of one period is for simplicity and can be replaced by the assumption that a growth option can emerge at any time during one period with a distribution, for example, an exponential one, $\beta e^{-\beta t}$.$^9$ However, all our results and intuitions will hold.

**Assumption 9** The cost (size) of each investment is a fraction $f$ of the prevailing asset-in-place value. If the investment is made, the prevailing asset value increases by a factor of $\lambda$ where $\lambda > f$.

This assumption implies a constant return of scale of all investments and the size and the net present value of each investment are proportional to the asset value when the investment is taken. In particular, it means that for every $f$ dollars invested, the cash flows generated by the investment has a present value of $\lambda$ dollars, for a constant return of $(\lambda - f)/f$. In their study of investment and growth, DeMarzo and Fishman (2007) also make a proportional assumption in that any investment in a project rescales its size and all of its cash flows.

Let $I_n$ denote the cost of investment in period $n$, where $n = 1, 2$, etc. This assumption implies that $I_n = fV(n-0.5)T$. If this investment is made, the firm’s asset value at $t = (n-0.5)T$ increases from $V(n-0.5)T$ to $(1+\lambda)V(n-0.5)T$. Therefore, the net present value of the investment at $t = (n-0.5)T$ is $(\lambda - f)V(n-0.5)T$. Mauer and Ott (2000) consider a model where the firm can expand the scale of operation once. For simplicity and tractability, we have assumed that when a growth option is exercised, the added assets have characteristics similar to those of the existing assets. Childs, Mauer and Ott (2005) consider a model where the growth options assets are not perfectly correlated with the existing assets.

**Assumption 10** Zero coupon bonds exist in the capital structure. The bonds are issued at the beginning of each period and mature at the end of the period. The face value of each bond is proportional to the asset value at the beginning of the period.

This assumption is for tractability. It implies that if the face value of the (zero coupon) bond is $P$ in the first period, it is $\frac{V(n-1)T}{V_0}P$ in period $n$.

$^9$In particular, this implies that there is a possibility that no growth option emerges in this period.
3.2 First Best Growth Option Exercising Policy

Since $\lambda - f > 0$, exercising the growth option always increases firm value. Therefore, the first best exercising policy in all three cases is always to exercise the growth option. However, when there is debt in the capital structure, exercising the growth option may not increase the equity value unless the asset value at the exercising point is high enough. To examine the second best policy and the associated (agency) cost is the major focus of this section.

3.3 Case 1: A One-Period Levered Firm with Only One Growth Option in the First Period

Assume that there is one outstanding zero coupon bond maturing at $t = T$ with face value $P$. Since there is no growth option in later periods and more importantly there is no agency conflict after $t = T$, this case is equivalent to a one-period model where there is debt in the capital structure and there is a growth option at the middle of the period. This case is the classic static (one-period) model where the intuition of debt-induced underinvestment is obtained.

3.3.1 Determining the second best exercising policy

Let the critical asset value above which the growth option at $t = 0.5T$ is exercised be denoted by $V^s_1$. At the exercising point $V_{0.5T} = V^s_1$, if the growth option is exercised, the investment increases the firm asset value by $\lambda V^s_1$. However, while the equity holders incur all the investment cost $I_1 = fV_{0.5T} = fV^s_1$, they only receive part of the increase in firm asset value, $\lambda V^s_1$. The critical value, $V^s_1$, is determined by equating the net increase in equity value ($\lambda V^s_1$ less the increase in debt value) to the investment cost ($fV^s_1$).

Let $V^T_e$ be the firm value at $t = T$ from the existing assets, i.e. without exercising the growth option and $V^o_T$ be the firm value with the option exercised at $t = 0.5T$. The increase in maturity date bond value (payoff), given by

$$[P - (P - V^o_T)1(V^o_T < P)] - [P - (P - V^e_T)1(V^e_T < P)] = [(P - V^e_T)1(V^e_T < P) - (P - V^o_T)1(V^o_T < P)],$$

is the difference between the payoffs of two put options and is nonnegative because $V^o_T \geq V^e_T$. The increase in bond value at $t = 0.5T$ is the risk-neutral discounted expected value of (18).
and is positive. The bond holders always benefit from exercising the growth option because they do not pay any investment costs and underinvestment always occurs. The condition that determines the critical exercising point $V_1^S$ can be stated as that at $V_{0.5T} = V_1^S$,

$$\lambda V_1^S - e^{-rT/2}E\left[ (P - V_T^S)1(V_T^S < P) - (P - V_T^O)1(V_T^O < P) \right] = fV_1^S. \quad (19)$$

To obtain concrete results, by assumption 7, between growth option exercising points the firm asset value follows a diffusion process described by (1). By assumption 9, at the exercising points, the firm asset value jumps by a factor of $\lambda$. Starting with $V_{0.5T} = V_1^S$, we have

$$V_T^S = V_1^S \exp \left( (r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q \right), \quad (20)$$

$$V_T^O = (1 + \lambda)V_1^S \exp \left( (r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q \right). \quad (21)$$

Using (20) and (21), it is straightforward to obtain that at $V_{0.5T} = V_1^S = X_1^S V_0$,\(^{10}\)

$$e^{-rT/2}E\left[ (P - V_T^S)1(V_T^S < P) - (P - V_T^O)1(V_T^O < P) \right] = \left( \frac{\lambda - f}{X_1^S V_0} \right) e^{-rT/2} \left[ (1 - d_2^*) - (1 + \lambda) (1 - d_1^*) \right]. \quad (22)$$

where

$$d_1 = \frac{\log(X_1^S V_0/P) + (r - \delta + \sigma^2/2)T/2}{\sigma \sqrt{T/2}}, \quad d_2 = d_1 - \sigma \sqrt{T/2}, \quad (23)$$

$$d_1^* = \frac{\log((1 + \lambda)X_1^S V_0/P) + (r - \delta + \sigma^2/2)T/2}{\sigma \sqrt{T/2}}, \quad d_2^* = d_1^* - \sigma \sqrt{T/2}. \quad (24)$$

Applying (22) to (19), we have that $X_1^S$ is determined by

$$\lambda X_1^S - \left[ (P/V_0) e^{-rT/2}(N(-d_2) - N(-d_2^*)) - X_1^S e^{-\delta T/2}(N(-d_1) - (1 + \lambda) N(-d_1^*)) \right] = fX_1^S. \quad (25)$$

### 3.3.2 Determining the growth option value following first and second best exercising policies

The first best growth option value, denoted by $VO_1^F$, is easily computed as

$$VO_1^F = e^{-rT/2}E[(\lambda - f)V_{0.5T}] = (\lambda - f)V_0 e^{-rT/2}.$$
The second best value of (growth) option, denoted by $VO_1^S$, can similarly be computed as

$$VO_1^S = e^{-rT/2}E[(\lambda - f)V_{0.5T}1(V_{0.5T} > X_1^S V_0)] = (\lambda - f)V_0 e^{-\delta T/2}N(d_1^S),$$

where

$$d_1^S = -\log(X_1^S) + \frac{(r - \delta + \sigma^2/2)T/2}{\sigma \sqrt{T/2}}.$$

The cost of debt-induced underinvestment in Case 1 is obviously given by

$$AC_1 = VO_1^F - VO_1^S = (\lambda - f)V_0 e^{-rT/2}N(-d_1^S).$$

### 3.3.3 Remarks

This case considers a firm with one debt and one growth option. This is the typical case examined in the literature upon which the underinvestment intuition is obtained. The second best exercising policy foregoes some investments when the gains from these investments are not high enough, even though an all-equity firm will always make these investments. The reason is that after the exercising of the option, the scale of the asset value increases. Therefore the asset terminal value at $T$ is also higher, reducing the default probability of debt and increasing its value. Since the shareholders bear the full cost of undertaking the option, they will not exercise the option unless $V_{0.5T} > X_1^S V_0$.

### 3.4 Case 2: A One-Period Levered Firm with One Growth Option Every Period

In the previous subsection we analyzed a firm’s optimal policy of exercising a growth option when it is levered and has a growth option in the first period. The model is effectively a static one with one period. However, the assumption that only the first period has a growth option is too restrictive. In this subsection we extend the setting to the case where a one-period levered firm has a growth option in every period. We will show that because of the existence of future growth options, the firm will optimally exercise the growth option in the first period in some situations where a firm with no future growth options will not exercise. Therefore, the existence of future growth options can alleviate the severity of debt-induced underinvestment.

Note that the firm becomes an all-equity firm after $T$. To determine the optimal exercising policy in the first period, we need to determine the firm value at $t = T$. The value of the
growth option at \( t = (n + 0.5)T \) is given by \((\lambda - f)V_{(n+0.5)T}\). Thus the total present value at \( t = T \) of all future growth options is given by

\[
PVGO_T = (\lambda - f) \sum_{n=1}^{\infty} e^{-(n-0.5)rT} (1 + \lambda)^{n-1} V_T E \left[ \frac{V_{(n+0.5)T}}{(1 + \lambda)^{n-1} V_T} \right].
\]

(26)

Since after each exercise of the growth option (at the middle of each period), the scale of the firm increases by a factor of \( \lambda \), we have

\[
\frac{V_{(n+0.5)T}}{(1 + \lambda)^{n-1} V_T} = \exp \left( (r - \delta - \sigma^2/2)(n - 0.5)T + \sigma W_{(n-0.5)T}^Q \right).
\]

Thus, (26) becomes as

\[
PVGO_T = (\lambda - f) \sum_{n=1}^{\infty} e^{-(n-0.5)rT} (1 + \lambda)^{n-1} V_T e^{(r-\delta)(n-0.5)T} e^{\delta T/2} \frac{e^{-\delta T/2}}{1 - (1 + \lambda)e^{-\delta T}} = g V_T.
\]

(27)

where

\[
g = (\lambda - f) \frac{e^{-0.5\delta T}}{1 - (1 + \lambda)e^{-\delta T}}.
\]

We require that \((1 + \lambda)e^{-\delta T} < 1\). This condition is similar to the condition that the growth rate is smaller than the discount rate in the constant growth dividend discount model. In other words, the firm cannot expand its asset base too quickly.

The firm value at \( t = T \) is the value of asset-in-place, \( V_T \), plus the total present value of future growth options, \((1 + g)V_T\). With existing assets only (not exercising the growth option at \( t = 0.5T \)), the firm value at \( t = T \) is given by

\[
V_T = (1 + g) V_{0.5T} \exp \left( (r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q \right).
\]

(28)

With exercising the growth option, the firm value at \( t = T \) is given by

\[
V_T^o = (1 + g)(1 + \lambda) V_{0.5T} \exp \left( (r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q \right).
\]

(29)

Note that \( g/(1 + g) \) fraction of the firm value at \( t = T \) gives \( PVGO_T \).

### 3.4.1 Determining the second best exercising policy, \( X_2^S \)

Let the second best critical asset value above which the growth option at \( t = 0.5T \) is exercised be denoted by \( V_2^S \). Similar to (22), starting with \( V_{0.5T} = V_2^S = X_2^S V_0 \), the increase in bond
value if the growth option at $t = 0.5T$ is exercised is given by

$$e^{-rT/2} E \left[ (P - V_T^e) 1(V_T^e < P) - (P - V_T^o) 1(V_T^o < P) \right] =$$

$$Pe^{-rT/2} \left( N(-d_4) - N(-d_4^*) \right) - X_2^S V_0 e^{-\delta T/2} \left( N(-d_3) - (1 + \lambda) N(-d_3^*) \right), \quad (30)$$

where

$$d_3 = \frac{\log((1 + g) X_2^S V_0/P) + (r - \delta + \sigma^2/2) T/2}{\sigma \sqrt{T/2}}, \quad d_4 = d_3 - \sigma \sqrt{T/2}, \quad (31)$$

$$d_3^* = \frac{\log((1 + g)(1 + \lambda) X_2^S V_0/P) + (r - \delta + \sigma^2/2) T/2}{\sigma \sqrt{T/2}}, \quad d_4^* = d_3^* - \sigma \sqrt{T/2}. \quad (32)$$

Since exercising the growth option at $t = 0.5T$ affects the size of future growth options, the correct condition for determining the exercising policy should take into account the increase in growth option value. With future growth options, exercising the growth option at $t = 0.5T$ not only increases the value of the existing bond, which is not beneficial to the equity holders, but also increases the value of future growth options, which is beneficial to the equity holders. Therefore, the correct condition that determines the exercising policy should be stated as that

$$\lambda V_2^S - \text{increase in bond value} + \text{increase in future growth option value} = f V_2^S. \quad (33)$$

With $V_{0.5T} = V_2^S = X_2^S V_0$, it is easy to see that

$$e^{-rT/2} E_{0.5T}[PVGO_T|\text{No Exercise at } T/2] = e^{-rT/2} E_{0.5T} \left[ \frac{gV_T^e}{1 + g} \right] = gV_2^S e^{-\delta T/2}, \quad (34)$$

$$e^{-rT/2} E_{0.5T}[PVGO_T|\text{With Exercise at } T/2] = e^{-rT/2} E_{0.5T} \left[ \frac{gV_T^o}{1 + g} \right] = g(1 + \lambda)V_2^S e^{-\delta T/2}, \quad (35)$$

The increase in future option value if the growth option at $t = 0.5T$ is exercised is the difference between (35) and (34) and given by $g\lambda V_2^S e^{-\delta T/2}$. Using (30) and (33), we see that $V_2^S$, or equivalently $X_2^S = V_2^S / V_0$, is determined by

$$(\lambda - f) X_2^S - \left[ (P/V_0) e^{-rT/2} (N(-d_4) - N(-d_4^*)) - X_2^S e^{-\delta T/2} (N(-d_3) - (1 + \lambda) N(-d_3^*)) \right] +$$

$$g\lambda X_2^S e^{-\delta T/2} = 0. \quad (36)$$
3.4.2 Determining the growth option value following first and second best exercising policies

The firm value maximizing policy (as if the firm were an all-equity firm) is to always exercise the growth options. The present value of these growth options at \( t = 0 \) is given by (27) except that \( V_0 \) replaces \( V_T \). Thus, the growth option value of first best exercising policy of Case 2, denote by \( VO^F_2 \), is given by

\[
V_O^F_2 = gV_0.
\]

To determine the second best growth option value of Case 2, denoted by \( VO^S_2 \), we note that the present value from exercising the growth option at \( t = 0.5T \), is given by

\[
e^{-rT/2}E\left[ (\lambda - f)V_{0.5T}1(V_{0.5T} > X^S_2V_0) \right] = (\lambda - f)V_0e^{-\delta T/2}N(d^S_2),
\]

where

\[
d^S_2 = \frac{-\log(X^S_2) + (r - \delta + \sigma^2/2)T/2}{\sigma\sqrt{T/2}}.
\]

To obtain the value of future growth options after \( t = 0.5T \), we note the following. First, assume that the growth option at \( t = 0.5T \) is not exercised. In this case we have

\[
e^{-rT}E[\text{PVGO}_T|\text{no exercise at } T/2] = e^{-rT}E\left[ E_{0.5T} \left[ \frac{gV^e_T}{1 + g} 1(V_{0.5T} < X^S_2V_0) \right] \right] =
\]

\[
e^{-rT}gE\left[ V_{0.5T}e^{(r-\delta)T/2}1(V_{0.5T} < X^S_2V_0) \right] = gV_0(1 - N(d^S_2))e^{-\delta T}.
\]

On the other hand, if the growth option is exercised at \( t = 0.5T \), the asset value at \( t = 0.5T \) jumps from \( V_{0.5T} \) to \( (1 + \lambda)V_{0.5T} \). Thus, we have

\[
e^{-rT}E[\text{PVGO}_T|\text{with exercise at } T/2] = e^{-rT}E\left[ E_{0.5T} \left[ \frac{gV^e_T}{1 + g} 1(V_{0.5T} > X^S_2V_0) \right] \right] =
\]

\[
e^{-rT}g(1 + \lambda)E\left[ V_{0.5T}e^{(r-\delta)T/2}1(V_{0.5T} > X^S_2V_0) \right] = g(1 + \lambda)V_0N(d^S_2)e^{-\delta T}.
\]

The second best value \( t = 0 \) of all future growth options is the sum of (37), (38) and (39), and given by

\[
VO^S_2 = gV_0 \left( e^{-\delta T} + (1 - e^{-\delta T})N(d^S_2) \right) = gV_0 \left( 1 - (1 - e^{-\delta T})N(-d^S_2) \right).
\]

Thus, the agency cost induced by debt is given by

\[
AC^2 = VO^F_2 - VO^S_2 = gV_0(1 - e^{-\delta T})N(-d^S_2).
\]
3.4.3 Remarks

Note that, due to the last term on the left hand side of (36) which is positive, relative to Case 1 where there exists only one growth option, the debt-induced underinvestment is reduced when growth options exist in future periods. The reason is that, while it is still the case that the equity holders receive only part of the direct benefit of exercising the option, $\lambda X_2^S V_0$, they also benefit from more future growth options if they exercise the option at $t = 0.5T$. Thus, compared with the no future option case, they are willing to exercise the growth option in the first period at a lower exercising value.

On the other hand, if the equity holders ignore the increased benefit to them in the future and determine the exercising policy by equating the direct benefit of exercising the growth option at $t = 0.5T$, $\lambda X_2^S V_0 - \Delta d$, where $\Delta d$ is the increase in debt value if the growth option is exercised, to the cost of exercising the option, $f X_2^S V_0$, they will underinvest, relative to the (optimal) exercise policy determined by (36). To see this, suppose that the exercising policy is determined by

\[
(\lambda - f) X_2^S - \left[\frac{P}{V_0} e^{-rT/2} (N(-d_4) - N(-d'_4)) - X_2^S e^{-\delta T/2} (N(-d_3) - (1 + \lambda) N(-d'_3))\right] = 0.
\]

(40)

It is easy to see that the solution to (40) is greater than that to (36) because the term, $g \lambda X_2^S e^{-\delta T/2}$, in (36) is positive.

3.5 Case 3: An Every-Period Levered Firm with One Growth Option Every Period

To determine the second best exercising policy, $V_3^S = X_3^S V_0$, we first determine the present value of all future growth options at $t = T$, $PVGO_T$, given the exercising policy $V_3^S$ in the first period. To this end, we develop a scaling property which facilitates the determination of $PVGO_T$.

3.5.1 The series of option values scales with the initial asset value

Let the exercising policy in period $n$ be $V_n^*$, where $n = 1, 2, \text{etc.}$. By assumption 9, the value (benefit) at $t = (n - 0.5)T$ of the growth option in period $n$ is given by

\[
B_n = (\lambda - f) V_{(n-0.5)T} 1(V_{(n-0.5)T} > V_n^*) = V_{(n-1)T} (\lambda - f) X_n 1(X_n > X_n^*),
\]

(41)
where
\[ X_n = \frac{V_{(n-0.5)T}}{V_{(n-1)T}} = \exp \left( \frac{(r - \sigma^2/2)T}{2} + \sigma B_{0.5T} \right), \quad X_n^* = \frac{V_n^*}{V_{(n-1)T}}. \]

That is, \( X_n \) is the ratio of the asset value at the middle of period \( n \) before exercising the growth option to that at the beginning of the period. Note that the distribution of \( X_n \) is independent of \( n \). That is, the \( X_n \)'s are i.i.d random variables.

From (41), we see that at \( t = T \), the series of values (not PV) of future growth options at \( t = 1.5T, 2.5T, 3.5T, 4.5T, \cdots \), is given by
\[ V_T \left[ (\lambda - f)X_21(X_2 > X_2^*), \frac{V_{2T}}{V_T}(\lambda - f)X_31(X_3 > X_3^*), \frac{V_{3T}}{V_T}(\lambda - f)X_41(X_4 > X_4^*), \cdots \right]. \] (42)

Similarly, at \( t = 2T \), the series is given by
\[ V_{2T} \left[ (\lambda - f)X_31(X_3 > X_3^*), \frac{V_{3T}}{V_{2T}}(\lambda - f)X_41(X_4 > X_4^*), \frac{V_{4T}}{V_{2T}}(\lambda - f)X_51(X_5 > X_5^*), \cdots \right]. \] (43)

Since the distribution of \( V_nT/V_T \) is the same as that of \( V_{(n+1)T}/V_{2T} \) and the \( X_n \)'s are i.i.d., the series is the same at \( t = T \) as it appears at \( t = 2T \) except the scaling factors, \( V_T \) vs. \( V_{2T} \). Therefore, the optimal exercising value \( X_2^* \) in the second period is the same as \( X_3^* \) in the third period. Similarly, \( X_3^* \) is the same as \( X_4^* \). It follows that the \( X_n^* \)'s are the same and independent of \( n \). In fact, they all equal to the exercising policy in the first period, denoted by \( X_S^3 \), because the series also looks the same at \( t = 0 \).

3.5.2 Total present value of future growth options at \( t = T \)

We are ready to use the scaling property to find the present value of all future growth options at \( t = T \), denoted by \( PVGO_T \). According to the scaling property, the present value of future growth options at \( t = 2T \) is \( \frac{V_{2T}}{V_T} PVGO_T \). On the other hand, the value at \( t = T \), \( PVGO_T \), equals to the present value of the growth option at \( t = 1.5T \), denoted by \( Z \), plus the present value of all growth options afterwards. From risk-neutral pricing we have
\[ PVGO_T = Z + e^{-rT}E_T \left[ \frac{V_{2T}}{V_T} PVGO_T \right], \] (44)
which has the simple solution that
\[ PVGO_T = \frac{Z}{1 - \phi}, \] (45)
where \( \phi = e^{-rT}E_T[V_{2T}/V_T] \).
We can obtain $Z$ and $\phi$ by the following steps. It is straightforward to see that

$$Z = e^{-rT/2}V_T E_T[(\lambda - f)X_21(X_2 > X_3^S)] = (\lambda - f)V_T e^{-\delta T/2}N(d_3^S),$$  \hspace{1cm} (46)

where

$$d_3^S = -\log(X_3^S) + (r - \delta + \sigma^2/2)T/2 \sigma\sqrt{T/2}.$$  \hspace{1cm} (47)

To compute $\phi$, we note that

$$\phi = e^{-rT}E_T \left[ \frac{V_{2T}}{V_T} \right] = e^{-rT}E_T \left[ \frac{V_{1.5T}}{V_{1.5T}} \frac{V_{2T}}{V_{1.5T}} 1(V_{1.5T} > V_2^*) + \frac{V_{1.5T}}{V_{1.5T}} \frac{V_{2T}}{V_{1.5T}} 1(V_{1.5T} < V_2^*) \right]$$

$$= e^{-(r+\delta)T/2}E_T \left[ \frac{V_{1.5T}}{V_T} (1 + \lambda) 1(V_{1.5T} > X_3^SV_T) + \frac{V_{1.5T}}{V_T} 1(V_{1.5T} < X_3^SV_T) \right]$$

$$= e^{-\delta T}((1 + \lambda)N(d_3^S) + N(-d_3^S)) = (1 + \lambda - \lambda N(-d_3^S))e^{-\delta T}.$$  \hspace{1cm} (48)

To obtain the second line, we note that if $V_{1.5T} > V_2^*$ (the growth option exercised), the asset value at $t = 1.5T$ jumps to $(1 + \lambda)V_{1.5T}$ and it then follows a diffusion process until $t = 2T$. In this case we have

$$\frac{V_{2T}}{V_{1.5T}} = (1 + \lambda)e^{(r-\delta-\sigma^2/2)T/2+\sigma B_{0.5T}}.$$  

On the other hand, if $V_{1.5T} < V_2^*$ (the growth option not exercised), we have

$$\frac{V_{2T}}{V_{1.5T}} = e^{(r-\delta-\sigma^2/2)T/2+\sigma B_{0.5T}}.$$  

In either case, the expectation operator can be iterated inside to yield that

$$E_{1.5T}[e^{(r-\delta-\sigma^2/2)T/2+\sigma B_{0.5T}}] = e^{(r-\delta)T/2}.$$  

Thus, the second line follows. The third line follows because

$$\frac{V_{1.5T}}{V_T} = e^{(r-\delta-\sigma^2/2)T/2+\sigma B_{0.5T}} \text{ and } E_T \left[ \frac{V_{1.5T}}{V_T} \right] = e^{(r-\delta)T/2}.$$  

Similar to (27), we have

$$PVGO_T = \bar{g}V_T, \quad \bar{g} = (\lambda - f)\frac{e^{-\delta T/2}N(d_3^S)}{1 - (1 + \lambda - \lambda N(-d_3^S))e^{-\delta T}}.$$  \hspace{1cm} (49)

Similar to (28) and (29), given the asset value at $t = 0.5T$, $V_{0.5T}$, the firm value at $t = T$ without (with) exercising the growth option at $t = 0.5T$, $V_T^e$ ($V_T^o$), is respectively given by

$$V_T^e = (1 + \bar{g})V_{0.5T} e^{(r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q},$$  \hspace{1cm} (50)

$$V_T^o = (1 + \bar{g})(1 + \lambda)V_{0.5T} e^{(r - \delta - \sigma^2/2)T/2 + \sigma W_{0.5T}^Q}.$$  \hspace{1cm} (51)
Note that \( \tilde{g}/(1 + \tilde{g}) \) fraction of the firm value at \( t = T \) gives \( PVGOT \).

To determine \( X_3^S \), similar to (33), we have that, at the critical value \( V_3^S = X_3^SV_0 \),

\[
\lambda V_3^S - \text{increase in bond value} + \text{increase in future growth option value} = fV_3^S. \tag{52}
\]

At the critical exercising value, \( V_{0.5T} = V_3^S = X_3^SV_0 \), the increase in bond value is given by

\[
e^{-rT/2}E[(P - V_T^e)1(V_T^e < P) - (P - V_0^p)1(V_0^p < P)] = \]

\[
P e^{-rT/2} (N(-d_6) - N(-d_6^*)) - X_2^S V_0 e^{-\delta T/2} (N(-d_5) - (1 + \lambda)N(-d_5^*)) , \tag{53}
\]

where

\[
d_5 = \frac{\log((1 + \tilde{g})X_3^SV_0/P) + (r - \delta + \sigma^2/2)T/2}{\sigma \sqrt{T/2}}, \quad d_6 = d_5 - \sigma \sqrt{T/2}, \tag{54}
\]

\[
d_6^* = \frac{\log((1 + \tilde{g})(1 + \lambda)X_3^SV_0/P) + (r - \delta + \sigma^2/2)T/2}{\sigma \sqrt{T/2}}, \quad d_6^* = d_5^* - \sigma \sqrt{T/2}. \tag{55}
\]

To compute the increase in growth option value, we note that

\[
e^{-rT/2}E_{0.5T}[PVGOT|\text{No Exercise at }T/2] = e^{-rT/2}E_{0.5T}\left[\frac{\tilde{g}V_T^e}{1 + \tilde{g}}\right] = \tilde{g}V_3^S e^{-\delta T/2}, \tag{56}
\]

\[
e^{-rT/2}E_{0.5T}[PVGOT|\text{With Exercise at }T/2] = e^{-rT/2}E_{0.5T}\left[\frac{\tilde{g}V_0^p}{1 + \tilde{g}}\right] = \tilde{g}(1 + \lambda)V_3^S e^{-\delta T/2}. \tag{57}
\]

The increase in future option value if the growth option at \( t = 0.5T \) is exercised is the difference between (57) and (56) and given by \( \tilde{g}\lambda V_3^S e^{-\delta T/2} \). Plugging (53), (56) and (57) into (52), we see that \( X_3^S \) is determined by

\[
(\lambda - f)X_3^S - [(P/V_0)e^{-rT/2}(N(-d_6) - d(d_6^*)) - X_3^S e^{-\delta T/2}(N(-d_5) - (1 + \lambda)N(-d_5^*))] + \\
\tilde{g}\lambda X_3^S e^{-\delta T/2} = 0. \tag{58}
\]

Note that (58) is very similar to (36) except that \( \tilde{g} \) replaces \( g \).

3.5.3 Remarks

It can be argued that if for some reasons, there is the need for debt financing in the first period, similar reasons may also exist in future periods. Therefore, it is more reasonable to assume that there is debt in every period. Similarly, if a growth option exists in the first period, it is reasonable to assume a growth option may exist in every period.
In a model with debt and a growth option in every period (our current case), debt-induced underinvestment exists in every period. Therefore, compared with Case 2 with debt only in the first period but a growth option in every period, it is intuitive that the critical value in this case is greater than in Case 2, i.e. \( X_3^S > X_2^S \).

On the other hand, even with debt-induced underinvestment in the future, nevertheless there are more growth opportunities (benefits) to the equity holders in the future if they exercise the growth option in the first period. Therefore, compared with Case 1 with debt and growth option only in the first period, the exercising policy in this is lower, i.e. \( X_3^S < X_1^S \).

Similar to our discussion in the last paragraph of Section 3.4.3, if the equity holders mistakenly determine the exercising policy by equating the direct benefit of exercising, \( \lambda X_3^S V_0 - \Delta d \), where \( \Delta d \) is the increase in debt value if the growth option is exercised, to the cost of exercising the option, \( f X_3^S V_0 \), they will underinvest, relative to the (optimal) exercising policy determined by (58). That is, the solution to

\[
(\lambda - f)X_3^S - \left[ (P/V_0)e^{-rT/2}(N(-d_6) - d(d_6^*)) - X_3^S e^{-\delta T/2}(N(-d_5) - (1 + \lambda)N(-d_5^*)) \right] = 0
\]

(59)
is bigger than that to (58). The reason is that (59) ignores the benefit of more future growth options resulting from an enlarged asset base if the growth option at \( t = 0.5T \) is exercised.

4 Numerical Results

We report the numerical results in Table 1. For the calculations we have used the following base values: \( r = 7.5\% \), \( \delta = 5.0\% \), \( V_0 = 100 \), \( \sigma = 0.3 \), and \( f = 0.3 \). For the other parameters, \( \lambda \), \( P \), and \( T \), the values are provided under column 1. Given \( \lambda \) and \( P \), \( T \) is chosen such that the value of \( \varphi = e^{-\delta T/2}/(1 - (1 + \lambda)e^{-\delta T}} \) matches the value under \( \varphi \).

We first examine Case 1. In this static case with one bond and one growth option, debt-induced underinvestment always exists. Column 2 reports the critical asset value above which the equity holders will exercise the growth option at \( t = 0.5T \). It is noted that as the growth option becomes more valuable (higher \( \lambda \)), the critical asset value becomes smaller because the equity holders do not want to forego the investment unless the asset value falls too low. For these low asset values, the net gain, \( (\lambda - f)V_{0.5T} \), from the investment is not enough to offset the increase in bond value if the growth option is exercised, so equity holders
forego the investment for these asset values. Column 4 reports the percentage agency cost, denoted by $100(VO_F^t - VO_S^t)/VO_S^t$, where $VO_F^t$ and $VO_S^t$ are respectively the $t = 0$ value of the growth option $t = 0.5T$ following the first and second exercising policy.\(^{11}\)

Recall that the corresponding values of the first best exercising policy are all zero, i.e., the first best policy is always to make the investment. The time zero (present) value of the growth option at $t = 0.5T$ is reported under column 3. We note that as the growth option becomes more valuable, the percentage agency cost from the second best exercising policy becomes smaller. Given the amount of debt in place, it does not appear to be the case that for larger growth options, debt induces more underinvestment because it is more costly to forego these investments. On the other hand, as expected, the higher the amount of debt in place, the more severe is the underinvestment problem.

Going beyond the static model, in Case 2 we consider a firm which is levered only in the first period but has a growth option every period. Comparing the results in Case 1 with those in Case 2, we note that there is little agency cost from a second best exercising policy. In other words, a firm in this setting almost always exercises the growth option at $t = 0.5T$. The reason is that while exercising the growth option benefits bond holders, it also increases the size of all future growth options. When these options have high returns per dollar invested, there is little concern for underinvestment. Only when the growth options have low returns ($\lambda = 0.4$),\(^{12}\) do underinvestments exist. The most notable conclusion is that with the existence of future growth options, the concern for underinvestment in the first period is greatly reduced or even eliminated.

Of course, the assumption that there is only debt in the first period is restrictive. In Case 3 we consider a firm which has both a debt and a growth option in every period. Comparing with Case 2 reveals that with debt in every period, the underinvestment problem is more severe when it exists. The reason is that each and every debt can induce underinvestment, while in Case 2 the firm becomes an all-equity firm after the first period. Nevertheless, when the growth options have high returns per dollar invested ($\lambda = 0.5$ or $\lambda = 0.6$), equity holders essentially always exercise the growth options in every period. Comparing with the

\(^{11}\)For space consideration, $V_S^t$ is not reported but can be recovered from $V_S^t = V_F^t/(1 + AC_1^F/100)$, where $AC_1^F/100$ converts the percentage number into a decimal.

\(^{12}\)This means that for every 0.3 dollars invested ($f = 0.3$), the firm asset value increases by 0.4 dollars ($\lambda = 0.4$), for a return of 1/3.
static setting of Case 1, we note that even though there is debt in every period, the critical exercising asset value is always lower than that in the static case and the agency cost is either much smaller or zero in our dynamic setting.

In sum, we can conclude that underinvestment in a dynamic arrangement is not a major problem.

5 Conclusion

In this paper, we apply the idea of implicit incentives to analyze the asset substitution and underinvestment problems in dynamic settings. For asset substitution, the agency problem between debt holders and equity holders is that equity holders may raise the risk of the firm once the debt is in place. An implicit incentive is that if equity holders raise risk now, its future debt will be priced at the raised risk level. We show that because of this implicit incentive, a firm’s tendency to raise risk is greatly reduced. For underinvestment, equity holders may bypass positive NPV projects because they bear the full costs of investments but have to share the benefits with debt holders. In a dynamic setting, a firm has future investment opportunities. An implicit incentive that reduces the underinvestment problem is that investing now will increase the firm’s asset base as well as reduce the firm’s probability of default, both of which enhance the present values of future investment opportunities. As a result, the implicit incentive greatly reduces or can even eliminate the underinvestment problem. For a wide range of parameter values, we illustrate that the magnitude of the agency costs associated with underinvestment is small. Our findings are consistent with the survey results of Graham and Harvey (2001), which find little evidence that CFOs are concerned about asset substitution and underinvestment.
Appendix A. Derivation of \( F(t), G(t), \overline{G}(t) \) and \( \phi \)

To obtain \( F(T), G(T) \) and \( \overline{G}(T) \), we need the first-passage time density function \( f(t) \). To this end, define

\[
x(t) = \log\left(\frac{V(t)}{Pe^{-\gamma(T-t)}}\right).
\]

(A1)

One simple application of Ito’s lemma yields

\[
dx = (r - \delta - \gamma - \sigma^2/2)dt + \sigma dw^Q.
\]

Therefore \( x(t) \) is a Brownian motion with drift \( \mu = r - \delta - \gamma - \sigma^2/2 \) and diffusion \( \sigma \), starting at \( x_0 = \log(V/(Pe^{-\gamma T})) \). From Ingersoll (1987) the first-passage time density function \( f(t) \) for \( x(t) \), starting at \( x_0 > 0 \), to cross the origin is given by

\[
f(t) = \frac{x_0}{\sigma t^{3/2}} n\left(\frac{x_0 + \mu t}{\sigma t^{3/2}}\right),
\]

(A2)

where \( n(\cdot) \) is the standard normal density function.

Tedious but straightforward calculations yield that

\[
F(T) = N[h_1(T)] + \left(\frac{V}{Pe^{-\gamma T}}\right)^{-2a} N[h_2(T)],
\]

\[
G(T) = \left(\frac{V}{Pe^{-\gamma T}}\right)^{-a+z} N[q_1(T)] + \left(\frac{V}{Pe^{-\gamma T}}\right)^{-a-z} N[q_2(T)],
\]

\[
\overline{G}(T) = \left(\frac{V}{Pe^{-\gamma T}}\right)^{-a+z} N[\overline{q}_1(T)] + \left(\frac{V}{Pe^{-\gamma T}}\right)^{-a-z} N[\overline{q}_2(T)],
\]

\[
h_1(T) = \frac{(-x_0 - a\sigma^2 T)}{\sigma \sqrt{T}}, \quad h_2(T) = \frac{(-x_0 + a\sigma^2 T)}{\sigma \sqrt{T}},
\]

\[
q_1(T) = \frac{(-x_0 - z\sigma^2 T)}{\sigma \sqrt{T}}, \quad q_2(T) = \frac{(-x_0 + z\sigma^2 T)}{\sigma \sqrt{T}},
\]

\[
\overline{q}_1(T) = \frac{(-x_0 - \bar{z}\sigma^2 T)}{\sigma \sqrt{T}}, \quad \overline{q}_2(T) = \frac{(-x_0 + \bar{z}\sigma^2 T)}{\sigma \sqrt{T}},
\]

\[
a = \frac{(r - \delta - \gamma - \sigma^2/2)}{\sigma^2}, \quad z = \frac{[(a\sigma^2)^2 + 2r\sigma^2]^{1/2}}{\sigma^2}, \quad \bar{z} = \frac{[(a\sigma^2)^2 + 2(r - \gamma)\sigma^2]^{1/2}}{\sigma^2},
\]

where \( N(\cdot) \) is the cumulative standard normal distribution function.

To find \( \phi \), we need the conditional distribution of \( V_T \) (i.e., \( X(T) \)), which is defined in equation (A1) such that the firm has not gone bankrupt at time \( T \). Again from Ingersoll
(1987) we find the following density function for \( X(T) \):

\[
g(x) = \frac{1}{\sigma \sqrt{T}^n} \left( \frac{x - x_0 - \mu T}{\sigma \sqrt{T}} \right) - e^{2x_0^2} \frac{1}{\sigma \sqrt{T}^n} \left( \frac{x + x_0 - \mu T}{\sigma \sqrt{T}} \right). \quad (A3)
\]

Using the above density function, tedious calculations show that \( \phi \) is given by

\[
\phi = e^{-\delta T} \left( N(d_1) - \left( \frac{Pe^{-\gamma T}}{V} \right)^{2(1+\mu/\sigma^2)} N(d_2) \right), \quad (A4)
\]

where

\[
d_1 = \frac{-\log(Pe^{-\gamma T}/V) + (r - \delta - \gamma + \sigma^2/2)T}{\sigma \sqrt{T}},
\]

\[
d_2 = \frac{\log(Pe^{-\gamma T}/V) + (r - \delta - \gamma + \sigma^2/2)T}{\sigma \sqrt{T}}.
\]


Table 1: Comparison of the second best exercising policy and the associated agency cost for the three cases. The first column provides the \((\gamma, P, T, \varphi)\) combinations. \(T\) is chosen to yield the value under \(\varphi\) defined by \(\varphi = e^{-\delta T/2}/(1 - (1 + \gamma)e^{-\delta T}}\). \(V_1^S, V_2^S\) and \(V_3^S\) are respectively the critical asset values for the three cases above which the growth option is exercised at \(t = 0.5T\). \(VO_1^F, VO_2^F\) and \(VO_3^F\) are respectively the growth option values for the three cases following the first best exercising policies, i.e. the growth options are always exercised. \(AC_1^%, AC_2^%\) and \(AC_3^%\) are respectively the percentage agency cost of the second best exercising policy for the three cases. For space consideration, the second best growth option values, \(VO_1^S, VO_2^S,\) and \(VO_3^S\) are not reported, but they can be easily computed. For example, \(VO_1^S\) is given by \(VO_1^S = VO_1^F/(1 + AC_1^%/100)\), where \(AC_1^%/100\) converts the percentage number into a decimal number.
Figure 1: Volatility sensitivity of the **Static Model**. The x-axis denotes the unlevered firm asset value and the y-axis denotes the derivative of the equity value with respect to the asset volatility at various asset value (solid line) and that of the debt value (dotted line).
Figure 2: Volatility sensitivity of the Dynamic Model, $T = 6.05$ years. The x-axis denotes the unlevered firm asset value and the y-axis denotes the derivative of the equity value with respect to the asset volatility at various asset value (solid line) and that of the debt value (dotted line).