A New Measure of Vector Dependence, with an Application to Financial Contagion*

Ivan Medovikov†  Artem Prokhorov‡

April 2015

Abstract

We propose a new nonparametric measure of association between an arbitrary number of random vectors. The measure is based on the empirical copula process for the multivariate marginals, corresponding to the vectors, and is insensitive to the dependence between the within-vector components. It is bounded by the [0, 1] interval, covering the entire range of dependence between vector independence and a vector version of an exact monotonic relationship. We study the properties of the new measure under several well-known copulas and provide a non-parametric estimator of the measure, along with its asymptotic theory, under fairly general assumptions. To illustrate the applicability of the new measure, we use it to assess the degree of interdependence between the financial markets in the North and South America, Europe and Asia, surrounding the financial crisis of 2008. We find strong evidence of previously unknown contagion patterns, with selected regions exhibiting little dependence before and after the crisis and a lot of dependence during the crisis period.

JEL Codes: C13
Key Words: copula, measures of vector dependence, nonparametric statistics, Hoeffding’s Phi-square

*Helpful comments of the participants of IMS/ASS conference in Sydney are gratefully acknowledged. We also thank Axel Bücher for his contribution to the paper.
†Department of Economics, Brock University, St. Catharines, ON, Canada. Tel.: +1 905 688 55 50 ext.6148; email: imedovikov@brocku.ca
‡University of Sydney Business School, Sydney; email: artem.prokhorov@sydney.edu.au
1 Introduction

Measures of multivariate association between \( d \geq 2 \) scalar components of a random vector \( \mathbf{X} = (X_1, \ldots, X_d) \) received substantial attention in the literature. Such measures include Kendall’s \( \tau \), Spearman’s \( \rho \), Pearson’s \( \phi^2 \) and its multivariate extension, various multivariate divergence measures such as relative entropy measures of Joe (1987, 1989) and many others (see, e.g., Joe 1997 for a survey). Most desirable of them are invariant to increasing transformations and therefore capture dependence regardless of specific marginals.

Less attention has been paid to constructing measures of multivariate association between several vectors, invariant to dependence between the within-vector components, even though such measures are key to multivariate modelling in many fields, for example for the study of contagion in financial markets, where we look for a measure of dependence between entire markets, which is robust to co-movements within them. The reason why such measures are of interest in their own right is that mutual independence between elements of vectors is not implied by pairwise independence.

In a recent paper, Gaißer et al. (2010) generalize nonparametric bivariate dependence measure of Hoeffding (see Hoeffding, 1940 and Fisher and Sen, 1994) to the multivariate case. The measure, termed Multivariate Hoeffding’s \( \Phi^2 \), is based on an \( L^2 \)-distance between the copula of \( \mathbf{X} \) and the independence copula. Mutual independence between the components of \( \mathbf{X} \) occurs when \( \Phi^2 = 0 \), and \( \Phi^2 = 1 \) represents co-monotonicity, or increasing deterministic relationship between \( (X_1, \ldots, X_d) \). The estimator for \( \Phi^2 \) is based on the empirical copula, and as Gaißer et al. (2010) note, carries low computation cost even when the dimension \( d \) is large, making it convenient to use in practice.

The main objective of this paper is to generalize the measure of Gaißer et al. (2010) to the case of vectors, that is, to the case when association is to be measured between \( p \geq 2 \) partitions of \( \mathbf{X} = (X_1, \ldots, X_p) \), where now \( X_k \in \mathbb{R}^{d_k}, \ k = 1, \ldots, p \) and \( d_1 + d_2 + \cdots + d_p = d \). Such problems often arise in practical situations such as, for example, in finance when dependence between whole classes of assets such as equities or bonds needs to be measured, or when analyzing degree of interdependence between groups of financial markets (e.g. between Asian, European and American exchanges). Note that such measures cannot be obtained through a combination of existing scalar statistics, since in general not all information about dependence between sets is contained in dependence structures of the sub-sets; for additional details see Section 1.8 in Romano (1986). In a closely related paper, Grothe et al. (2014) work out similar generalizations of Spearman’s \( \rho \) and Kendall’s \( \tau \). However, their measures do not explicitly account for the case of more than two dependent
vectors.

We refer to the proposed measure as Hoeffding’s Vector Phi-squared and denote it by $\Phi^2$. Unlike the scalar statistics, $\Phi^2$ is designed to distinguish dependence between $X_1, \ldots, X_p$ from dependence within individual vectors. In fact, as we show, $\Phi^2$ remains unaffected by any dependence among the components of individual vectors, and when the partitions are independent, we always have $\Phi^2 = 0$, even when the within-vector components are not independent. We provide a nonparametric estimator for the new measure and derive its asymptotic behavior in the iid case. We further conduct a simulation study to investigate the finite-sample properties of $\Phi^2$ under a variety of dependence scenarios. Finally, to illustrate the use of $\Phi^2$ in practice we measure the degree of interdependence between stock market of different regions during the period surrounding the 2008 financial crisis.

The paper is organized as follows. Hoeffding’s Vector $\Phi^2$ is introduced in Section 2 and a nonparametric estimator of $\Phi^2$ is discussed in Section 3, where we also derive the large sample asymptotics of the estimator. In Section 2 we also present results of a few simulations investigating the ability of the new measure to capture between-vectors rather than within-vector dependence, under several well known copulas. Section 4 provides an empirical application of $\Phi^2$ to the analysis of contagion between Asian, European and American equity markets during the financial crisis of 2008. Section 5 contains concluding remarks.

2 Hoeffding’s $\Phi^2$

Let $F$ be the joint cdf of $X = (X_1, \ldots, X_d)$, $X_j \in \mathbb{R}, j = 1, \ldots, d$, and let $F_1, \ldots, F_d$ denote the corresponding univariate marginal cdf’s. Following a result by Sklar [1959], the function $F$ can be represented in terms of the marginals $F_1, \ldots, F_d$ and the copula $C$ as

$$F(x) = C(F_1(x_1), \ldots, F_d(x_d)), \quad (x_1, \ldots, x_d) \in \mathbb{R}^d. \quad (1)$$

The copula $C : [0,1]^d \to [0,1]$ provides a complete, and in the case of continuous marginals, a unique description of the relationship between $X_1, \ldots, X_d$. Many well-established and new measures of association such as Spearman’s $\rho$, Kendall’s $\tau$, mutual information coefficient $\delta$ [Blumentritt and Schmid 2012] are based on $C$ or the corresponding density $c(u) = \partial^d/\partial_1\ldots\partial_d C(u)$, $u \in [0,1]^d$. Schmid et al. [2010] provide a survey of copula-based dependence measures.
2.1 Multivariate Hoeffding’s $\Phi^2$

It is easy to see that mutual independence between the $d$ scalar components of $X$ is characterized by independence copula $C^\perp(u) = \prod_{j=1}^d u_j, u \in [0,1]^d$, which makes it natural to develop measures of association based on the distance $C(u) - C^\perp(u)$. The measure proposed by Gaißer et al. (2010) can be written as follows

$$\Phi^2 = \frac{||C - C^\perp||_2^2}{||M - C^\perp||_2^2},$$

where $|| \cdot ||_2$ denotes the $L_2$-norm, and $||M - C^\perp||_2^2$ is the normalization factor where the function $M(u) = \min(u_1, \ldots, u_d), u \in [0,1]^d$, is the so-called comonotonic copula and represents an almost-sure strictly-increasing functional relationship between all components of $X$. Here, $M$ is the upper Frechet-Hoeffding bound on copulas, that is, for any valid copula $C$, $C(u) \leq M(u), \forall u \in [0,1]^d$.

The statistic of Gaißer et al. (2010) represents a multivariate extension of the bivariate measure of association initially proposed in Hoeffding (1940) (also see Fisher and Sen (1994)) – we refer to it as Multivariate Hoeffding’s $\Phi^2$. For any copula $C$, the measure $\Phi^2$ is constrained to the $[0, 1]$ interval where the case $\Phi^2 = 1$ occurs when there is co-monotonicity between all components of $X$, while $\Phi^2 = 0$ corresponds to mutual independence. It is important to note that the converse is generally not true – it is possible to have $\Phi^2 < 1$ when $X$ are actually deterministically dependent. This is possible because the copula $C$ may be arbitrarily close to $C^\perp$ when it can be expressed as a shuffle of $M$. It is therefore possible for the statistic $\Phi^2$ to be made arbitrarily small, while maintaining a deterministic relationship between all components of $X$. This implication follows from the copula theory rather than from the definition of $\Phi^2$, and is a feature of any measure of association that is based on distances between $C$ and $C^\perp$. For additional details, see Section 3.2.3 in Nelsen (2006).

It is clear that $\Phi^2$ is not well-suited for the measurement of dependence between random vectors since even under mutual independence between vectors, $\Phi^2$ may be non-zero due to dependence within them. Our aim here is to develop a measure of multivariate association that factors out the dependence between the within-vector elements.

2.2 Additional definitions and notation

Before we proceed, we need to introduce some additional notation to assist with the partitioning of $X$ into $p$ sub-vectors $(X_1, \ldots, X_p)$ of dimension $d_1, \ldots, d_p$, where
\[ d_k \geq 1. \] Define \( b_0 = 0 \) and, for any integer \( k \in \{1, \ldots, p\} \), set \( b_k = \sum_{\ell=1}^{k} d_\ell \).

Moreover, for \( k = 1, \ldots, d \) and \( u \in [0,1]^d \), let
\[
\begin{align*}
\mathbf{u}^{[k]} &= (u_{b_{k-1}+1}, \ldots, u_{b_k}) \in \mathbb{R}^{d_k}, \\
\mathbf{u}^{[k]} &= (\mathbf{1}_{1+d_1+\ldots+d_{k-1}}, \mathbf{u}^{(k)}_{,1}, \mathbf{1}_{d_{k+1}+\ldots+d_p}) \in \mathbb{R}^d.
\end{align*}
\]

Here, for \( m \in \mathbb{N} \), \( \mathbf{1}_m \) denotes an \( m \)-dimensional vector with each entry equal to 1.

For \( k = 1, \ldots, p \), let \( C^{(k)} : [0,1]^{d_k} \to [0,1] \), \( C^{(k)}(\mathbf{v}) = C(1_{1+d_1+\ldots+d_{k-1}}, \mathbf{v}, 1_{d_{k+1}+\ldots+d_p}) \) denote the copula of \( X_k \). Also, define a copula \( C^{\Pi} : [0,1]^d \to [0,1] \) through
\[
C^{\Pi}(\mathbf{u}) = \prod_{k=1}^{p} C^{(k)}(\mathbf{u}^{[k]}) = \prod_{k=1}^{p} C(\mathbf{u}^{[k]}).
\]

This copula serves as a vector analogue of the scalar independence copula \( C^\perp \).

### 2.3 Hoeffding’s Vector \( \bar{\Phi}^2 \)

First we note that \( X_1, \ldots, X_p \) are mutually independent if and only if \( C = C^{\Pi} \). In this case, the joint distribution of \( p \) vectors is a product of \( p \) multivariate marginals, corresponding to the vector dimensions. This observation suggests that we can define a measure of dependence between the vectors \( X_1, \ldots, X_p \) by considering a suitable distance between \( C \) and \( C^{\Pi} \).

The new measure we propose is a generalization of the scalar version of \( \Phi^2 \) discussed in the previous section. We follow Gaißer et al. (2010) and use an \( L^2 \)-type distance. The vector version of Hoeffding’s \( \Phi^2 \), which we denote as \( \bar{\Phi}^2 \), is defined as follows
\[
\bar{\Phi}^2 := \bar{\Phi}^2(C, d_1, \ldots, d_p) := \frac{\|C - C^{\Pi}\|_2^2}{\|M - C^{\Pi}\|_2^2},
\]
where as before \( \| \cdot \|_2 \) denotes the \( L_2 \)-norm. Now, we use \( C^{(k)} \), \( k \in \{1, \ldots, p\} \) in the distance and in the normalizing factor. The scalar version is obtained as a special case when \( d_j = 1, j = 1, \ldots, p \).

We use the same bound \( M \) in our measure as in the scalar case above. There may exist a vector equivalent of this bound. For example, we may consider using \( \min(C^{(1)}, \ldots, C^{(p)}) \) instead. However, little is known about the properties of this object, specifically we cannot guarantee that \( C \leq \min(C^{(1)}, \ldots, C^{(p)}) \) for any copula \( C \). What is known is that this would be the right Frechet-Hoeffding bound for
some copula \( C_c(C^{(1)}, \ldots, C^{(p)}) \) but, unless all margins are scalar distributions, the resulting distribution is a \( d \)-variate copula only if \( C_c(u_1, \ldots, u_p) = C_\perp(u_1, \ldots, u_p) \) (see, e.g., Quesada-Molina and Rodriguez-Lallena [1994] [Genest et al.] [1995]). So this would not be the right bound to use in our case.

Next we can state a few properties of the measure. Some of them are less obvious than others so we provide a brief discussion.

(i) **Bounds:** For any copula \( C \) and marginals \( C^{(k)} \), we have that \( \bar{\Phi}^2 \in [0, 1] \). \( \bar{\Phi}^2 = 1 \) when \( C = M \) and \( \bar{\Phi}^2 = 0 \) when \( C = C^\Pi \).

(ii) **Independence of partitions:** If the partitions \( X_1, \ldots, X_p \) are independent then \( \bar{\Phi}^2 = 0 \). The converse is also true.

(iii) **Vector dependence:** Vectors \( X_1, \ldots, X_p \) are associated when \( \bar{\Phi}^2 > 0 \). The converse is also true.

(iv) **Comonotonicity of components:** For the case of two vectors \( p = 2 \), when \( \bar{\Phi}^2 = 1 \), each of the individual components of \( X \) is almost-surely a strictly-increasing function of another. The converse is not true. (There is no equivalent result for countermononicity.)

(v) **Invariance with respect to partitions order:** Given partition sizes \( d_1, \ldots, d_p \), for every permutation \( \pi \) of partitions order \( \{1, \ldots, p\} \), we have that \( \Phi^2(C, d_1, \ldots, d_p) = \Phi^2(C, d_{\pi(1)}, \ldots, d_{\pi(p)}) \), as long as the composition of each partition is maintained. This, and the next property follow from Fubini’s Theorem.

(vi) **Invariance with respect to ordering of components within partitions:** Given partition sizes \( d_1, \ldots, d_p \) and partition composition, \( \Phi^2 \) is invariant with respect to permutations of components within each partition.

(vii) **Invariance with respect to strictly-increasing transformations:** For any \( d \geq 2 \), we have that \( \Phi^2 \) is invariant with respect to strictly-increasing transformations of one or many components of \( X \). This is true since the copula \( C \) is invariant under such transformations.

### 2.4 Numerical Examples

In this section, we use Monte-Carlo simulations to calculate and compare the approximate values of \( \Phi^2 \) and \( \Phi^2 \) for some commonly used multivariate families of copulas, each having a range of dependence parameter values. To keep computational costs low, we set \( p = 2 \), overall dimension to \( d = 4 \), and partition data into two vectors.
of equal size, so that $d_1 = d_2 = 2$ in all cases. We begin with the case when $C$ is the equi-correlated Gaussian copula and plot the corresponding values of $\Phi^2$ and $\Phi^2$ in the top-left panel of Figure 1 against correlation coefficient $\rho$. Both statistics are increasing in $\rho$ as expected, and the values of $\Phi^2$ are substantially smaller than $\Phi^2$ due to factoring out dependence within the partitions. To further illustrate the difference between $\Phi^2$ and $\Phi^2$, the top-right panel of Figure 1 shows the case when $C$ is a product of two equi-correlated bivariate Gaussian copulas, implying independence of partitions. Here, as expected, $\Phi^2$ remains insensitive to dependence within partitions, regardless of the degree of correlation, while the measure $\Phi^2$ is increasing in $\rho$.

The remaining panels in Figure 1 show the two measures for four-dimensional Gumbel, Clayton, Frank and t copulas. In the Gumbel, Clayton and Frank copula case, when the dependence parameter is at the origin, we have mutual independence of all components of $X$, and increasing values away from the origin indicate increasing dependence between all components so that $C$ approaches $M$. For the t-copula, we vary correlation while keeping the degrees of freedom constant at $\tau = 2$. Symmetric tail dependence is therefore always present in this case, even when $\rho = 0$. In all the cases we consider, $\Phi^2$ overestimates $\Phi^2$ by a large margin.

3 Statistical inference for $\Phi^2$

We now consider the formal definition and asymptotic properties of an estimator of $\Phi^2$.

3.1 An estimator of $\Phi^2$

For a sample $X^{(1)}, \ldots, X^{(n)}$, $X^{(i)} = (X^{(i)}_1, \ldots, X^{(i)}_d) \sim F = C(F_1, \ldots, F_d)$ and continuous marginal cdfs $F_1, \ldots, F_d$, let $\hat{U}^{(i)} = (\hat{U}^{(i)}_1, \ldots, \hat{U}^{(i)}_d)$ denote pseudo-observations from the copula $C$ defined through

$$\hat{U}^{(i)}_j = \frac{1}{n} \left( \text{rank of } X^{(i)}_j \text{ among } X^{(1)}_j, \ldots, X^{(n)}_j \right),$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, d$. Let $\hat{C}_n : [0,1]^d \to [0,1]$ denote the associated empirical copula, defined for $\mathbf{u} = (u_1, \ldots, u_d) \in \mathbb{R}^d$ as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(\hat{U}^{(i)} \leq \mathbf{u}) = \frac{1}{n} \sum_{i=1}^n \prod_{j=1}^d \mathbb{I}(\hat{U}^{(i)}_j \leq u_j).$$

7
Figure 1: Approximate values of $\Phi^2$ and $\Phi^2$ for common multivariate copula families obtained using Monte-Carlo integration (data dimensionality $d = 4$ and partition sizes $d_1 = d_2 = 2$).
The empirical copula easily allows to define a sample version of \( \Phi^2 \) through

\[
\hat{\Phi}_n^2 := \Phi^2(\hat{C}_n, d_1, \ldots, d_p) = \frac{\|\hat{C}_n - \hat{C}_n^\Pi\|^2}{\|M - \hat{C}_n^\Pi\|^2},
\]

where \( \hat{C}_n^\Pi(u) = \hat{C}_n(\Pi_{k=1}^p C_{n[k]}). \)

The following lemmas and the subsequent proposition show that the estimator \( \hat{\Phi}_n^2 \) can be calculated directly from the pseudo-observations. In part, the proposition relies on calculations provided in Proposition 10 of Kojadinovic and Holmes (2009).

For notational convenience, for any positive integer \( i \) define \([i] = \{1, 2, \ldots, i\}.\)

**Lemma 1.** For some \( a, b \in [0, 1] \) s.t. \( b > a \), some \( p > 0 \), and integer \( d > 1 \), let \( m(a, b, d, p) \) represent the value of the definite integral

\[
\int_a^b \ldots \int_a^b \min(u)^p du.
\]

Then, we can find \( m(a, b, d, p) \) as

\[
m(a, b, d, p) = d! \frac{b^{d+p} - a^{d+p}}{\prod_{i=1}^d (p + i)} - \sum_{k=1}^{d-1} \frac{\Gamma(d + 1)}{\Gamma(d - k + 1)} \frac{a^p f^k}{\prod_{j=1}^k (p + j)} (b - a)^{d-k}, \tag{3}
\]

where \( \Gamma \) denotes the Gamma function.

**Proof.** See Appendix for all proofs.

**Lemma 2.** For some integer \( d > 1 \), \( a \in [0, 1]^d \), and \( p > 0 \), let \( I(a, p) \) represent the value of the definite integral

\[
I(a, p) = \int_{a_1}^1 \ldots \int_{a_d}^1 \min(u_1, \ldots, u_d)^p du_1 \ldots du_d, \tag{4}
\]

where for \( j \in [d] \), \( a_j \) denotes the \( j \)’th component of \( a \). Then, we can find the value of \( I(a, p) \) as

\[
I(a, p) = \sum_{j=1}^{d-1} \sum_{k=1}^j \sum_{l=1}^{d-k} \left( \prod_{n=1}^{d-k} (1 - a_{[A_{(d,j,k,l,n)]}}) \right) m(a_{[j]}, a_{[j+1]}, k, p) \tag{5}
\]

\[+ m(a_{[d]}, 1, d, p), \]
where for \( j \in [d] \), \( a_{[j]} \) denotes the \( j \)th largest component of \( \mathbf{a} \) (so that \( a_{[1]} \leq a_{[2]} \leq \ldots \leq a_{[d]} \)), and the function \( A : \mathbb{Z}_+^d \to \mathbb{Z}_+ \) is defined below.

For some \( j, k \in \mathbb{Z}_+ \) such that \( k \leq j \), let \( [j_k] \) denote the set of \( k \)-combinations of \( [j] \), and let \( [j_k](s) \) denote its \( s \)th element. For example, we have that \( [2_1] = \{\{1\}, \{2\}\} \), \( [3_2] = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\} \), and \( [3_2](2) = \{1, 3\} \). Further, define a set \( J(d, j, k, s) = [d] \setminus [j_k](s) \), which is a set of components of \( [d] \) not in \( [j_k](s) \), and let \( J(d, j, k, s, n) \) represent its \( n \)th element. Lastly, for positive integers \( d, j, k, s, n \), the function \( A \) is defined as follows:

\[
A(d, j, k, s, n) = \mathbb{I}(J(d, j, k, s)(n) \in [j])(j + 1) + \mathbb{I}(J(d, j, k, s)(n) \not\in [j])J(d, j, k, s)(n),
\]

where \( \mathbb{I}() \) is an indicator function as before.

**Proposition 1.** The numerator and denominator of \( \Phi_n^2(C) \) can be expressed as

\[
\| \hat{C}_n - \hat{C}_n^{\Pi} \|_2^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{j=1}^{d} [1 - \hat{u}_{i,j} \lor \hat{u}_{l,j}] \tag{7}
\]

\[
- \frac{2}{n^{p+1}} \sum_{i=1}^{n} \prod_{k=1}^{p} \sum_{j=1}^{n} \prod_{b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \lor \hat{u}_{l,j}] \tag{8}
\]

\[
+ \frac{1}{n^{2p}} \prod_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} \prod_{b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \lor \hat{u}_{l,j}], \tag{9}
\]

and

\[
\| M - \hat{C}_n^{\Pi} \|_2^2 = \frac{2}{(d + 1)(d + 2)} \tag{10}
\]

\[
- \frac{2}{n^{p}} \sum_{i_1=1}^{n} \ldots \sum_{i_p=1}^{n} I(\{[\hat{u}_{i_{k(1)}}, \hat{u}_{i_{k(2)}}, \ldots, \hat{u}_{i_{k(d)}}] \}, 1) \tag{11}
\]

\[
+ \frac{1}{n^{2p}} \prod_{k=1}^{p} \sum_{i=1}^{n} \sum_{l=1}^{n} \prod_{b_{k-1}+1}^{b_k} [1 - \hat{u}_{i,j} \lor \hat{u}_{l,j}], \tag{12}
\]

where for \( j \in [d] \), \( k(j) : [d] \to [p] \) is a function such that \( b_{k(j)-1} + 1 \leq j \leq b_{k(j)} \), and \( I(\mathbf{a}, p) \) is defined in Lemma 3 and \( m(a, b, d, p) \) is defined in Lemma 4.
3.2 Asymptotic properties of $\Phi_n^2$

Both consistency and a weak convergence result regarding $\Phi_n^2$ can be obtained using functional weak convergence results for the empirical copula process $C_n = \sqrt{n}(\hat{C}_n - C)$ and for the process $C_n^\Pi$, defined for $u \in [0, 1]^d$ through

$$C_n^\Pi(u) = \sqrt{n}(\hat{C}_n^\Pi(u) - C^\Pi(u)).$$

Naturally, both $C_n$ and $C_n^\Pi$ can be considered as elements of the space of real-valued, bounded functions on $[0, 1]^d$, denoted by $\ell_\infty([0, 1]^d)$, equipped with the uniform metric induced by the sup-norm $\|f\|_\infty = \sup_{u \in [0, 1]^d} |f(u)|$. Weak convergence of the empirical copula process $C_n$ has been investigated by various authors under slightly different assumptions (see, e.g., Rüschendorf [1976] [Gaenssler and Stute [1987] Fer-\-manian et al. [2004] Segers [2012] Bücher and Volgushev [2013]). Under appropriate smoothness conditions and under $C = C^\Pi$, Kojadinovic and Holmes [2009, Theorem 3] gave a weak convergence result for $H_n = C_n - C_n^\Pi$. The following result can be regarded as an extension of that result.

**Theorem 1.** Let $C$ be a copula such that, for any $j = 1, \ldots, d$, the $j$th first order partial derivative $\hat{C}_j = \partial C/\partial u_j$ exists and is continuous on the set $\{u \in [0, 1]^d : u_j \in (0, 1)\}$. Let $B_C$ denote a $C$-Brownian bridge on $[0, 1]^d$, i.e., a centered Gaussian process with continuous sample paths and covariance

$$\text{cov}\{B_C(u), B_C(v)\} = C(u \wedge v) - C(u)C(v).$$

Then, $(C_n, C_n^\Pi) \Rightarrow (C_C, C_C^\Pi)$ in $\{\ell_\infty([0, 1]^d)\}^2$, where

$$C_C(u) = B_C(u) - \sum_{j=1}^d \hat{C}_j(u)B_C(1, \ldots, 1, u_j, 1, \ldots, 1),$$

with $\hat{C}_j$ defined as 0 wherever it does not exist, and where

$$C_C^\Pi(u) = \sum_{k=1}^p C_C(u^{[k]}) \prod_{k'\neq k}^p C(u^{[k']}).$$

The preceding theorem also holds under many serial dependence scenarios for time series (e.g., under alpha-mixing), see Bücher and Volgushev [2013]. More precisely, provided the weak limit $\tilde{B}_C$ of the empirical process

$$u \mapsto \sqrt{n} \left\{ \frac{1}{n} \sum_{i=1}^n I(U^{(i)} \leq u) - C(u) \right\}$$

11
exists and has continuous sample paths, the statement of Theorem 1 holds with the C-Brownian bridge \( \mathbb{B}_C \) replaced by \( \tilde{\mathbb{B}}_C \).

**Proposition 2.** Under the assumptions of Theorem 1,

(a) if \( \bar{\Phi}^2 \neq 0 \), then

\[
\sqrt{n}(\hat{\Phi}_n^2 - \bar{\Phi}^2) \sim 2 \int_{[0,1]^d} \left\{ C(u) - C^{\Pi}(u) \right\} \left\{ C_C(u) - C^{\Pi}_C(u) \right\} du \left\| M - C^{\Pi} \right\|_2^2
+ 2 \left\| C - C^{\Pi} \right\|_2^2 \int C^{\Pi}_C(u) \left\{ M(u) - C^{\Pi}(u) \right\} du \left\| M - C^{\Pi} \right\|_2^4.
\]

(b) if \( \bar{\Phi}^2 = 0 \), then

\[
n\hat{\Phi}_n^2 \sim \left\| C_C - C^{\Pi}_C \right\|_2^2 \left\| M - C^{\Pi} \right\|_2^2.
\]

Note that in the case \( \bar{\Phi}^2 \neq 0 \) the limiting distribution of \( \sqrt{n}(\hat{\Phi}_n^2 - \bar{\Phi}^2) \) is Gaussian. This follows from the fact that \( C_C \) is a centered Gaussian process. The limiting variance depends in a complicated way on \( C \) and its partial derivatives. In practice, the true copula is unknown and analytical calculation, even approximate, of these quantities is infeasible and bootstrap distributions will be used instead.

4 **Empirical example: global financial crisis**

4.1 **Literature review**

To illustrate the applicability of the proposed measure, we use it to study linkages between the global equity markets before, during and after the financial crisis of 2008. We are particularly interested in financial contagion, that is in the transmission of shocks sustained by one market to the other markets elsewhere. Following Forbes and Rigobon (2002), we define contagion as a change in cross-market linkages during or after financial shocks and aim to assess global contagion allowing for a partition of markets into regional groups. Specifically, we study linkages between Europe, Asia, North and South America, in the time periods surrounding the global financial crisis of 2008. Our measure is well-suited for this task since it is not affected by the association between markets belonging to the same region, meaning that financial contagion between, say, Germany and the U.K. will have no influence on our estimates of contagion between Europe and, say, Asia.
The issue of contagion continues to receive substantial attention in the literature which is not surprising given the severe economic consequences of financial and currency crises and the resources devoted by policy makers to crisis management. The evidence to date overwhelmingly supports the presence of contagion in financial markets, that is, it shows an increase in cross-market linkages during major financial crises; for an overview see, for example, Forbes and Rigobon (2001). In most cases, contagion is interpreted as an increase in correlation coefficients or other measures of association between cross-country asset returns. For example, in one of earlier attempts at measuring contagion, King and Wadhwani (1990) tested for an increase in correlations between equity returns in the US, UK and Japan and found that correlations rose substantially following a US stock market crisis, interpreting this as contagion. Similarly, Lee and Kim (1993) document the strengthening of co-movements between several stock market indexes in the aftermath of the 1987 stock market crash.

A related strand of literature studies transmission of volatility rather than returns dynamics, using a GARCH framework. For example, Edwards (1998) uses an augmented GARCH model to assess the role of capital control in transmission of shocks following the Mexican peso crisis.

More recently, the copula approach has been adopted to probe for financial contagion that may be nonlinear and nonmonotone in nature. For example, using a copula model Rodriguez (2007) documents increased cross-country tail-dependence during crises in Asia, suggesting that transmission of shocks may be asymmetric and generally has a complicated, nonlinear form. Similarly, Chiang and Wang (2011) study volatility transmission using a measure based on a transition copula and document volatility spillovers from the US to the other G7 countries.

Interestingly, few attempts appear to have been made in the literature at measuring cross-region rather than cross-country contagion. One exception is Bae et al. (2003), who study regional contagion using a multinomial logistic regression model which captures probabilities of co-occurrence of extreme returns in pairs of global regions such as Asia and Europe and find evidence of inter-regional contagion. This approach, however, requires a relatively complicated set up along with fairly strong assumptions about the joint distributions of market returns (specifically, joint normal or multivariate t-distribution), meaning that the evidence of contagion can be equally interpreted as evidence against joint normality of the returns.

While the issue of regional contagion appears to be largely overlooked, regional partitioning can represent an important factor in the propagation of financial shocks from one country to another. For example, some markets can be closely-linked for institutional reasons such as membership in currency or customs union, or common
bailout guarantees such as those issued by the European Central Bank to some Euro-zone members. To this end, the statistic that we propose allows for such partitions, even when they have arbitrary many members.

Our contribution to the empirical literature on financial contagion in this section is two-fold. First, this appears to be the first attempt to estimate contagion using a fully-nonparametric copula-based measure that is sensitive to any form of dependence, including non-linear and non-monotone. Second, our estimates seem to be the first that encompass more than two regions simultaneously and are therefore closer to what could be viewed as regional contagion in global sense.

4.2 Estimates of inter-regional contagion

We restrict our attention to the total of 15 national equity market indexes, which we partition into four regional groups as follows:

- **North America**: S&P 500 Index (US), TSX-S&P Composite Index (Canada), IPC Index (Mexico).
- **South America**: HSBC Chile Index, HSBC Colombia Index, Bovespa Index (Brazil), Merval Index (Argentina).
- **Europe**: FTSE 100 Index (UK), Euronext 100 Index (Netherlands, France, Belgium and Portugal), DAX 100 Index (Germany).
- **Asia**: Hang Seng Index (Hong Kong), Shanghai Composite Index (China), Nikkei 225 Index (Japan), ASX Index (Australia), BSE 500 Index (India).

We use monthly closing values for all of the indexes in our sample for the period January 2004 to January 2014 extracted from the Standard and Poor’s COMPU-STAT database. The series are plotted in Figure 2. There is some homogeneity among indexes within a region but regional profiles differ substantially in spite of the common trough in late 2008.

Most markets in our sample suffered substantial drawdowns at some point during the 2007 - 2009 period, and we next attempt to establish whether this coincided with an increase in cross-market linkages. To get a comprehensive picture of contagion and to better understand how market linkages differ for the four regions we estimate the values of both $\Phi^2$ and $\Phi^2$ for all regional pairs in our sample. The estimates are provided in Tables 1. We estimate the statistics separately for the full sample as well as for the pre-crisis sub-sample which we define to span January 2004 - December 2007, the crisis sub-sample spanning January 2008 - May 2009, and the post-crisis
Figure 2: Normalized monthly closing values of leading global stock market indexes.
sub-sample which begins in April 2009 and ends in January 2014. For any regional pair, the measure $\Phi^2$ captures overall association between all countries belonging to either of the two regions, while $\bar{\Phi}^2$ captures linkages between the regions as a whole while netting out dependence due to co-movements of markets belonging to the same partition.

<table>
<thead>
<tr>
<th>Region pair</th>
<th>Multivariate $\Phi^2$</th>
<th>Vector $\Phi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Full sample, 2004 - 2014</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North America - Europe</td>
<td>0.5118</td>
<td>0.3751</td>
</tr>
<tr>
<td>North America - South America</td>
<td>0.2090</td>
<td>0.1189</td>
</tr>
<tr>
<td>North America - Asia</td>
<td>0.2168</td>
<td>0.1287</td>
</tr>
<tr>
<td>South America - Europe</td>
<td>0.2307</td>
<td>0.1263</td>
</tr>
<tr>
<td>South America - Asia</td>
<td>0.1517</td>
<td>0.0892</td>
</tr>
<tr>
<td>Europe - Asia</td>
<td>0.2594</td>
<td>0.1500</td>
</tr>
<tr>
<td><strong>Pre-crisis, 2004 - 2007</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North America - Europe</td>
<td>0.4110</td>
<td>0.2943</td>
</tr>
<tr>
<td>North America - South America</td>
<td>0.1717</td>
<td>0.1077</td>
</tr>
<tr>
<td>North America - Asia</td>
<td>0.1277</td>
<td>0.0819</td>
</tr>
<tr>
<td>South America - Europe</td>
<td>0.1591</td>
<td>0.0855</td>
</tr>
<tr>
<td>South America - Asia</td>
<td>0.0632</td>
<td>0.0396</td>
</tr>
<tr>
<td>Europe - Asia</td>
<td>0.0911</td>
<td>0.0438</td>
</tr>
<tr>
<td><strong>Crisis, 2007-2009</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North America - Europe</td>
<td>0.4759</td>
<td>0.6006</td>
</tr>
<tr>
<td>North America - South America</td>
<td>0.3096</td>
<td>0.3693</td>
</tr>
<tr>
<td>North America - Asia</td>
<td>0.2282</td>
<td>0.2201</td>
</tr>
<tr>
<td>South America - Europe</td>
<td>0.4331</td>
<td>0.6108</td>
</tr>
<tr>
<td>South America - Asia</td>
<td>0.4005</td>
<td>0.5331</td>
</tr>
<tr>
<td>Europe - Asia</td>
<td>0.3974</td>
<td>0.4891</td>
</tr>
<tr>
<td><strong>Post-crisis, 2009-2014</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>North America - Europe</td>
<td>0.4615</td>
<td>0.3308</td>
</tr>
<tr>
<td>North America - South America</td>
<td>0.1477</td>
<td>0.0747</td>
</tr>
<tr>
<td>North America - Asia</td>
<td>0.1990</td>
<td>0.1189</td>
</tr>
<tr>
<td>South America - Europe</td>
<td>0.1823</td>
<td>0.0954</td>
</tr>
<tr>
<td>South America - Asia</td>
<td>0.1175</td>
<td>0.0667</td>
</tr>
<tr>
<td>Europe - Asia</td>
<td>0.2706</td>
<td>0.1691</td>
</tr>
</tbody>
</table>

Table 1: Measures of global market linkages.
Perhaps unsurprisingly, substantial differences in regional linkages appear to exist; the strongest connection by far appears to be between North America and Europe, followed by links between Europe and Asia, North America and Asia, and South America and Europe. Interestingly, the connection between the Americas is among the weakest, stronger only than the South America - Asia link.

Pre-crisis, global regional links appear to be rather weak for all regional pairings with the exception of North America - Europe, and increase very sharply during the crisis period. The increase in purely-regional links captured by $\Phi^2$ is particularly dramatic, where in some cases such as South America - Asia, regions that were almost entirely-disconnected before crisis become tightly associated, with dependence estimate increasing more than tenfold, from 0.0438 in the pre-crisis period to 0.4891 during crisis.

Overall, our findings suggest that regional linkages increased very sharply during the 2007 - 2009 period, which we interpret as evidence of regional contagion. Interestingly, we find that post crisis, with the exception of North America - Europe pair global regions became once again largely disjoint. This is evident in the case of South America, where $\Phi^2$ dropped to nearly same values as pre-crisis, and in the case of South America - North America – to an even-lower value than in 2004 - 2007.

5 Concluding Remarks

We have proposed a new vector dependence measure and considered its properties. The new measure has important advantages over the other available dependence measures. First, it applies to between-, rather than within-, vector dependence. Second, the number of vectors it can accommodate is not limited to two. We derive its asymptotic distribution, discuss various estimation issues and provide an important example where using our measure we come to conclusions which differ substantively from those available in the literature.

An alternative way of constructing vector dependence measure can be based on linkage functions of , which are generalizations of copulas. However, the properties of the linkage functions are not as well studied as the properties of copulas and so we leave the construction of such measure to future research.
\section*{A Proofs}

**Proof of Lemma 1** Note that for any \( u \in [0, 1]^d \), we can expand \( \min(u)^p \) as

\[
\min(u) = \sum_{j=1}^d \left[ \prod_{k \in [d], k \neq j} \mathbb{I}(u_j \leq u_k) \right] u_j^p,
\]

and for any \( j \in [d] \), express the integral of the summand as

\[
\int_a^b \ldots \int_a^b \left[ \prod_{k \in [d], k \neq j} \mathbb{I}(u_j \leq u_k) \right] u_j^p du_j du^{[-j]}
= \int_a^b \ldots \int_a^b u_j^p du_j du^{[-j]}
= \frac{1}{p+1} \int_a^b \ldots \int_a^b \min(u^{[-j]})^{p+1} du^{[-j]} - \frac{a^{p+1}}{p+1} \int_a^b \ldots \int_a^b 1 du^{[-j]},
\]

where \( u^{[-j]} \in [0, 1]^{d-1} \) denotes sub-vector of \( u \) with its \( j \)'th component removed.

Applying induction, it is tedious but relatively straightforward to show that for any \( d > 1 \), by continuing the expansion for \( d - 1 \) steps, we can write \( \int_a^b \ldots \int_a^b \min(u)^p du \) as the sum containing \( d! \) integrals of the form

\[
\frac{1}{\prod_{j=1}^{d-1}(p+j)} \int_a^b u^{p+d-1} du,
\]

and \( \sum_{k=1}^{d-1} b\{k\} \) constants of the form

\[
-\frac{a^s}{t} \int_a^b \ldots \int_a^b 1 u^{[-j]},
\]

where \( b\{k\} = d!/(d-k)! = \Gamma(d+1)/\Gamma(d-k+1) \) is the falling factorial (Pochhammer function), \( J \) is the subset of elements of \( u \) that have been removed, and \( s \) and \( t \) are some exponents. The resulting sum of single integrals is then given by

\[
\sum_{j=1}^d \frac{1}{\prod_{j=1}^{d-1}(p+j)} \int_a^b u^{p+d-1} du = \frac{d!}{\prod_{j=1}^{d}(p+j)} (b^{p+d} - a^{p+d}), \quad (13)
\]
and the sum of multiple integrals by

\[
\sum_{k=1}^{d-1} \sum_{m=1}^{d[k]} \prod_{j=1}^{k} (p+j) \int_a^b \cdots \int_a^b 1 du_2 \cdots du_{d-k+1}
\]

(14)

\[
= \sum_{k=1}^{d-1} \sum_{m=1}^{d[k]} \prod_{j=1}^{k} (p+j) \prod_{j=2}^{d-k} (b-a)
\]

(15)

\[
= \sum_{k=1}^{d-1} \frac{\Gamma(d+1)}{\Gamma(d-k+1)} \prod_{j=1}^{k} (p+j) (b-a)^{d-k},
\]

(16)

which yields the desired result.

**Proof of Lemma 2** First, note that for any \( a \in [0, 1]^d \) we can change integration order by arranging \( u_1, \ldots, u_d \) according to the values of \( a_1, \ldots, a_d \) as

\[
I(a, p) = \int_{a_1}^1 \cdots \int_{a_d}^1 \min(u_1, \ldots, u_d)^p du_1 \cdots du_d
\]

(17)

\[
= \int_{a_1}^1 \cdots \int_{a_d}^1 \min(u_1, \ldots, u_d)^p du_{[1]} \cdots du_{[d]},
\]

(18)

where for \( j \in [d] \), \( u_{[j]} \) denotes the variable with \( j \)’th lowest integration limit \( a_{[j]} \). Since \( a_{[1]} \leq a_{[2]} \leq \ldots \leq a_{[d]} \), as a first step, we can write the integral as

\[
\int_{a_{[1]}}^1 \cdots \int_{a_{[d]}}^1 \min(u_1, \ldots, u_d)^p du_{[1]} \cdots du_{[d]}
\]

(19)

\[
= \int_{a_{[1]}}^{a_{[2]}} u_{[1]}^p du_{[1]} \int_{a_{[2]}}^1 1 du_{[2]} \cdots \int_{a_{[d]}}^1 1 du_{[d]}
\]

(20)

\[
+ \int_{a_{[2]}}^1 \int_{a_{[2]}}^1 \cdots \int_{a_{[d]}}^1 \min(u_1, \ldots, u_d)^p du_{[1]} \cdots du_{[d]}
\]

(21)

\[
= m(a_{[1]}, a_{[2]}, 1, p) \prod_{j=2}^d (1 - a_{[j]})
\]

(22)

\[
+ \int_{a_{[2]}}^1 \int_{a_{[2]}}^1 \cdots \int_{a_{[d]}}^1 \min(u_1, \ldots, u_d)^p du_{[1]} \cdots du_{[d]},
\]

(23)

where \( m() \) is the function defined in Lemma 1. Continuing with such expansion for \( d - 1 \) more steps, we arrive at the final integral in this series which can also be
expressed in terms of \( m() \) as
\[
\int_{a[d]}^{1} \cdots \int_{a[d]}^{1} \min(u_{[1]}, \ldots, u_{[d]})^p du_{[1]} \cdots du_{[d]} = m(a[d], 1, d, p). \tag{24}
\]

Induction will reveal that for \( 1 \leq n < d \), step \( n \) of the expansion yields a total of \( 2^n - 1 \) new terms of the form \( m(.) \prod (1 - a[i]) \). Since \( 2^n - 1 = \sum_{i=1}^{n} \binom{n}{i} \), the new terms can be collected in \( n \) groups containing \( \binom{n}{i} \) elements each, for \( i = 1, \ldots, n \). Further induction yields combinations of indexes to be used in \( m(.) \) as well as the function \( A(.) \) that generates product indexes in every group.

**Proof of Proposition 1** Expression for the numerator is available directly from Proposition 10 of [Kojadinovic and Holmes (2009)](KojadinovicAndHolmes2009). For the denominator, we have that

\[
\| M \hat{C}_n^p \|_2^2 = \int_{[0,1]^d} M(u)^2 du - 2 \int_{[0,1]^d} M(u) \prod_{k=1}^{p} C_n(u[k]) du \tag{25}
\]
\[
+ \int_{[0,1]^d} \left( \prod_{k=1}^{p} C_n(u[k]) \right)^2 du. \tag{26}
\]

For the first term, using Lemma 2 we have that

\[
\int_{[0,1]^d} M(u)^2 du = I(0_d, 2) = \frac{2}{(d + 1)(d + 2)}. \tag{27}
\]

The last term is the same as that in Proposition 10 of [Kojadinovic and Holmes (2009)](KojadinovicAndHolmes2009). For the middle term, we have that

\[
\int_{[0,1]^d} M(u) \prod_{k=1}^{p} C_n(u[k]) du \tag{28}
\]
\[
= \int_{[0,1]^d} M(u) \prod_{k=1}^{p} \left( \frac{1}{n} \sum_{i=1}^{n} \prod_{j=b_{k-1}+1}^{b_k} \mathbb{I}(\hat{u}_{ij} \leq u_j) \right) du \tag{29}
\]
\[
= \frac{1}{n^p} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \int_{[0,1]^d} M(u) \prod_{k=1}^{p} \prod_{j=b_{k-1}+1}^{b_k} \mathbb{I}(\hat{u}_{i_k,j} \leq u_j) \right) du \tag{30}
\]
\[
= \frac{1}{n^p} \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} I(\hat{u}_{i_1[1]}, \hat{u}_{i_2[2]}, \ldots, \hat{u}_{i_p[d]}], 1), \tag{31}
\]
where for $j \in [d], k[j] = k \in [p]$ such that $b_{k-1} + 1 \leq j \leq b_k$.

**Proof of Theorem 1** The proof is similar to the one of Theorem 3 in Kojadinovic and Holmes (2009). First, define a map $\Psi : \ell^\infty([0,1]^d) \to \{\ell^\infty([0,1]^d)^2\}$ through

$$\Psi(F)(u) = \left(F(u), \prod_{k=1}^p F(u^{[k]})\right), \quad u \in [0,1]^d$$

and note that

$$H_n = \sqrt{n}\{\Psi(\hat{\phi}_n) - \Psi(C)\}.$$

It follows as in Lemma 2 in Kojadinovic and Holmes (2009) that $\Psi$ is Hadamard-differentiable at any function $F$ with derivative $\Psi'_{F} : \ell^\infty([0,1]^d) \to \{\ell^\infty([0,1]^d)^2\}$, defined through

$$\Psi'_F(G)(u) = \left(G(u), \sum_{k=1}^p G(u^{[k]}) \prod_{k' = 1}^p F(u^{[k']})\right), \quad u \in [0,1]^d.$$

Therefore, the assertion of the theorem easily follows from an application of the functional delta method, see Theorem 3.9.6 in Van der Vaart and Wellner (1996), observing that, under the assumptions on the partial derivatives of $C$, $C_n \to C$ in $(\ell^\infty([0,1]^d), \|\cdot\|_\infty)$ from Proposition 3.1 in Segers (2012).

**Proof of Proposition 2** Decompose $\hat{\Phi}_n^2 - \bar{\Phi}^2 = A_{n1} + A_{n2}$ where

$$A_{n1} = \frac{1}{\| M - C^\Pi \|_2^2} \left\{ \| \hat{C}_n - \hat{C}_n^\Pi \|_2^2 - \| C - C^\Pi \|_2^2 \right\}$$

$$A_{n2} = \| \hat{C}_n - \hat{C}_n^\Pi \|_2^2 \left\{ 1 - \frac{1}{\| M - C^\Pi \|_2^2} - \frac{1}{\| M - C^\Pi \|_2^2} \right\}$$

Some simple algebra reveals that the expression $A_{n1}$ can be written as

$$\| M - C^\Pi \|_2^2 A_{n1}$$

$$= \int (\hat{C}_n - \hat{C}_n^\Pi)^2 - (C - C^\Pi)^2 \, d\lambda$$

$$= \int (\hat{C}_n - \hat{C}_n^\Pi - C + C^\Pi)^2 \, d\lambda + 2 \int (C - C^\Pi)(\hat{C}_n - \hat{C}_n^\Pi - C + C^\Pi) \, d\lambda$$

$$= \frac{1}{n} \| C_n - C^\Pi \|_2^2 + \frac{2}{\sqrt{n}} \int (C - C^\Pi)(C_n - C^\Pi) \, d\lambda$$

21
Regarding $A_{n2}$, we have

$$\frac{A_{n2}}{\|\hat{C}_n - \hat{C}^{\Pi}_n\|^2_2} = \frac{\|M - C^{\Pi}\|^2_2 - \|M - \hat{C}^{\Pi}_n\|^2_2}{\|M - \hat{C}^{\Pi}_n\|^2_2 \|M - C^{\Pi}\|^2_2},$$

and the numerator on the right-hand side can be further simplified to

$$\|M - C^{\Pi}\|^2_2 - \|M - \hat{C}^{\Pi}_n\|^2_2 = \frac{1}{\sqrt{n}} \int C^{\Pi}_n (2M - C^{\Pi} - \hat{C}^{\Pi}_n) d\lambda.$$

Assembling terms, we have

$$\hat{\Phi}_n^2 - \Phi^2 = \frac{1}{\|M - C^{\Pi}\|^2_2} \left\{ \frac{1}{n} \|C_n - C^{\Pi}_n\|^2_2 + \frac{2}{\sqrt{n}} \int (C - C^{\Pi})(C_n - C^{\Pi}_n) d\lambda \right\}$$

$$+ \frac{\|\hat{C}_n - \hat{C}^{\Pi}_n\|^2_2}{\|M - \hat{C}^{\Pi}_n\|^2_2 \|M - C^{\Pi}\|^2_2} \left\{ \frac{1}{\sqrt{n}} \int C^{\Pi}_n (2M - C^{\Pi} - \hat{C}^{\Pi}_n) d\lambda \right\}.$$

This readily implies the assertion for $\Phi^2 \neq 0$, i.e., for $C \neq C^{\Pi}$, by Theorem 1. Moreover, if $\Phi^2 = 0$ and hence $C = C^{\Pi}$, then

$$n\|\hat{C}_n - \hat{C}^{\Pi}_n\|^2_2 = \|C_n - C^{\Pi}_n\|^2_2 = O_P(1),$$

which implies the second assertion.
References


23


URL http://books.google.com/books?hl=en&amp;lr=&amp;id=B3ONT5rBv0wC&amp;oi=fnd&amp;pg=PP7&amp;dq=An+Introduction+to+Copulas&amp;ots=4x7tfURwu7&amp;sig=lIE2srS3sXVHh936N-nVAmlzXA


URL http://linkinghub.elsevier.com/retrieve/pii/S0927539806000582


URL http://www.springerlink.com/index/10.1007/978-3-642-12465-5

URL http://dx.doi.org/10.3150/11-BEJ387

