Precautionary Monetary Policy at the Zero Lower Bound*

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Abstract
Long-run real rates of interest have been following over time in many developed countries, increasing the probability of episodes in which the nominal interest rate hits the zero lower bound (ZLB). Debates on optimal policy in this context, however, have been based on analytical frameworks that neglect precautionary behavior by the central bank. I address this issue by showing how to account for the policy kink generated by the ZLB and its transition probability in order to design a precautionary optimal monetary policy when the ZLB is occasionally binding. I show that it prescribes strong policy gradualism, whose degree increases with the long-run probability of hitting the zero lower bound. In exercises, I show that the precautionary optimal policy welfare-dominates standard optimal policies, whose derivations take into consideration the ZLB constraint on the nominal interest rate, but do not account for the effect of the ZLB kink in the expected welfare metric.

Keywords: Precautionary behavior, Gradualism, Optimal Policy, Zero Lower Bound, Trend Inflation
Jel Codes: E31, E43, E52

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1 Introduction

Long-run real rates of interest have been following over time in many developed countries (see e.g. Laubach and Williams (2015), Bauer and Rudebusch (2016) and Yi and Zhang (2016)). Given constant inflation targets, this process increases the likelihood that episodes of hitting the zero lower bound (ZLB) become more likely over time. This possibility has sparked debates about optimal policy and optimal inflation target in this context. The literature, however, has focused mainly on analytical settings that do not take into account the possibility of precautionary behavior by the central bank (e.g. Eggertsson and Woodford (2003a,b) and Nakov (2008)). However, it is well known in the literature that precautionary behavior has quantitatively important effects when generated by kinks, such as the one around the ZLB on nominal interest rates (see e.g. Zeldes (1989) and Carroll and Kimball (2001)). Therefore, precautionary-driven motives should not be neglected in any analysis on optimal policy in the context of the zero lower bound.

In this paper, I address this issue by showing how to account for the kink in order to design precautionary optimal monetary policy when the ZLB is occasionally binding. To illustrate the approach, I work upon Damjanovic et al. (2008) to derive optimal policies under unconditionally commitment in an economy subject to persistent demand and technology shocks. For this task, I first derive how the transition probability of hitting the ZLB endogenously evolves as a function of the shocks distribution. Following, I use the Alves (2014) trend-inflation welfare-based (TITeB) loss function to derive what I call the Precautionary Optimal Policy under unconditionally commitment. In that paper, I show that this loss function is the appropriate metric for policy evaluation at different levels of trend inflation, as it fully internalizes the distortions created by larger inflation rates. Therefore, I also assess the issue of whether it is worthy to increase the inflation target in order to mitigate the probability of hitting the ZLB. Finally, I perform welfare exercises, comparing the Precautionary Optimal Rule to the Standard Optimal Rule and a Taylor Rule. I call the Standard Optimal Rule the one obtained under occasionally binding ZLB episodes, with Kuhn-Tucker restrictions, but without accounting for the kink or the probability of hitting the ZLB prior to optimization. In this sense, Eggertsson and Woodford (2003a,b) and Nakov (2008) derive Standard Optimal Rules.

I first find that the targeting rule describing the Precautionary Optimal Policy is a convex combination between terms related to inflation and output-gap stabilization, and components leading to precautionary behavior. In addition, the long-run probability of hitting the ZLB works as the weight on the latter components. To the best of my knowledge, targeting rules describing Standard Optimal Policies only have terms related to stabilization.

The Precautionary Optimal Policy also gives the rationale to justify ad-hoc approaches that impose interest rate smoothing or monetary policy gradualism in the vicinity of the ZLB (see e.g. Nakata and Schmidt (2018)). Even not having any quadratic interest rate term in the loss function, the precautionary targeting rule suggests that monetary policy should respond to a long history of lagged nominal interest rates. The optimal degree of policy gradualism, or precautionary behavior, increases as the long-run probability of hitting the zero lower bound rises. In this case, policy responses to shocks become more sluggish, especially near the ZLB. As an illustration of the induced precautionary behavior, the central bank will not decrease the nominal rate as much right after moderate negative demand shocks hit the economy. This creates more room for future effective monetary policy changes. On the other hand, the central bank will take even longer to increase the rate, when compared to what the standard optimal policy prescribes, after the shocks have dissipated. Hence, forward guidance is always optimized under the Precautionary Optimal Rule.

I also characterize the conditions at which the precautionary optimal policy can be approximated by price level targeting. A necessary condition, but not sufficient, is the long-run level of nominal

\[ 1 \]In this paper, I follow an analytical approach to deal with the transition probability, under particular distributions and assumptions. Lansing (2018), on the other hand, uses an numerical approach to obtain the transition probability under weaker assumptions.
interest rates to be low. At higher rates, optimal policy is more entangled and do not resemble price level targeting.

When comparing welfare-based performances of policy type, under occasionally binding ZLB episodes, I find that the precautionary optimal policy dominates both the standard optimal policy and Taylor Rule for every level of real interest rates and trend inflation. Finally, by using a welfare-based loss function that fully accounts for the distortionary effects of larger levels of trend inflation, the results show that it is not worthy to increase the inflation target in order to mitigate the probability of hitting the zero lower bound on interest rates.

The remainder of the paper is organized as follows. The model is described in Section 2. Key results on the probability of hitting the ZLB are derived in Section 3. The design of the Precautionary Optimal Policy is shown in Section 4. The calibration strategy is described in Section 5. Policy evaluation at different levels of long-run real interest rates and trend inflation is discussed in Section 6.1, while Section 6.2 assesses how optimal policies perform after negative demand shocks. Section 7 summarizes the paper’s conclusions.

2 The model

For simplicity, I follow Woodford (2003, chap. 4) to describe the standard new-Keynesian model with Calvo (1983) price setting and flexible wages. The economy consists of a representative infinite-lived household that consumes an aggregate bundle and supplies differentiated labor to a continuum of differentiated firms indexed by $z \in (0, 1)$, which produce and sell goods in a monopolistic competition environment.

2.1 Households

Household’s workers supply $h_t(z)$ hours of labor to each firm $z$, at nominal wage $W_t(z) = P_t w_t(z)$, where $P_t$ is the consumption price index and $w_t(z)$ is the real wage. Disutility over hours worked in each firm is $v_t(z) \equiv \chi h_t(z)^{1+\nu} / (1 + \nu)$, where $\nu^{-1}$ is the Frisch elasticity of labor supply. The household’s aggregate disutility function is $v_t \equiv \int_0^1 v_t(z) \, dz$. Consumption $c_t(z)$ over all differentiated goods is aggregated into a bundle $C_t$, as in Dixit and Stiglitz (1977), and provides utility $u_t \equiv \epsilon_t C_t^{1-\sigma} / (1 - \sigma)$, where $\sigma^{-1}$ is the intertemporal elasticity of substitution and $\epsilon_t$ is a preference shock. Aggregation and expenditure minimization relations are described by $C_t^{\theta+1} = \int_0^1 c_t(z)^{\theta+1} \, dz$, $P_t^{1-\theta} = \int_0^1 p_t(z)^{-\theta} \, dz$, and $P_t C_t = \int_0^1 p_t(z) c_t(z) \, dz$, where $\theta > 1$ is the elasticity of substitution between goods.

The budget constraint is $P_t C_t + E_t q_{t+1} B_{t+1} \leq B_t + P_t \int_0^1 w_t(z) h_t(z) \, dz + d_t$, under complete financial markets, where $B_t$ is the state-contingent value of the portfolio of financial securities held at the beginning of period $t$, $d_t$ denotes nominal dividend income, and $q_{t+1}$ is the stochastic discount factor from $(t+1)$ to $t$. The household chooses the sequence of $C_t, h_t(z)$ and $B_{t+1}$ to maximize its welfare measure $W_t \equiv \max E_t \sum_{\tau=t}^{\infty} \beta^{T-\tau} (u_t - v_t)$, subject to the budget constraint and a standard no-Ponzi condition, where $\beta$ denotes the subject discount factor. In equilibrium, optimal labor supply satisfies $w_t(z) = u_t'(z) / u_t''$, where $u_t' \equiv \partial u_t / \partial C_t$ is the marginal utility to consumption and $u_t'(z) \equiv \partial v_t(z) / \partial h_t(z)$ is the marginal disutility to hours. The optimal consumption plan and dynamics of the stochastic discount factor are described by $1 = \beta E_t \left( \frac{w_t}{u_t'} \frac{l_t}{\Pi_{t+1}^t} \right)$ and $q_t = \beta^t \frac{w_t}{u_t'} \frac{1}{\Pi_t}$, where $\Pi_t = 1 + \pi_t$ and $I_t = 1 + i_t$ are the gross inflation and interest rates at period $t$, which satisfies $I_t = 1 / E_t q_{t+1}$, and $i_t$ is the riskless one-period nominal interest rate. As usual, the gross real interest rate is defined as $R_t \equiv i_t / E_t \Pi_{t+1}$.

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2Equilibrium is defined as the equations describing the first order conditions, a transversality condition $\lim_{T \to \infty} E_T q_{t,T} B_T = 0$, where $q_{t,T} \equiv \Pi_{t+1}^{T+1} q_T$, and the market clearing conditions. 
I also consider the auxiliary variable $I_t$, defined as the gross nominal interest rate that would prevail if the gross nominal interest rate could be allowed to be smaller than unity at the particular period $t$.\footnote{Note that it is not the same as stating that $I_t$ is the nominal interest rate that would prevail if the ZLB was never a constraint.} In regular times, the ZLB constraint affects the equilibrium at period $t$ by means of the non-linear relation $I_t = \max(1, I_t)$. For the remaining of the paper, I refer to $I_t$ as the unbounded nominal interest rate.

2.2 Firms

Firm $z \in (0, 1)$ produces differentiated goods using the technology $y_t(z) = \mathcal{A}_t h_t(z)^\varepsilon$, where $\mathcal{A}_t$ is the aggregate technology shock and $\varepsilon \in (0, 1)$. The aggregate output $Y_t$ is implicitly defined by $P_t Y_t = \int_0^1 p_t(z) y_t(z) \, dz$. Using the market clearing condition $y_t(z) = c_t(z), \forall z$, the definition implies that the firm’s demand function is $y_t(z) = Y_t (p_t(z)/P_t)^{-\theta}$, where $Y_t = C_t$.

With probability $(1 - \alpha)$, the firm optimally readjusts its price to $p_t(z) = p_t^*$. With probability $\alpha$, the firm sets its price according to $p_t(z) = p_{t-1} (z) \Pi_{t}^{\text{ind}}$, where $\Pi_{t}^{\text{ind}} \equiv \Pi_{t-1}^{\text{ind}}$ and $\gamma_z \equiv (0, 1)$. When optimally readjusting at period $t$, the price $p_t^*$ maximizes the expected discounted flow of nominal profits $\mathcal{P}_t(z) = p_t(z) y_t(z) - P_t w_t(z) h_t(z) + E_t q_{t+1} \mathcal{P}_{t+1}(z)$, given the demand function and the price setting structure. At this moment, the firm’s real marginal cost is $mc_t^* = (1/\mu) X_t^{(\omega + \sigma)} (p_t^*/P_t)^{-\theta \omega}$, where $\omega \equiv (1 + \nu)/\varepsilon - 1$ is a composite parameter, $\mu \equiv \theta/(\theta - 1) > 1$ is the static markup parameter, $X_t \equiv Y_t/Y_t^*$ is the gross output gap, and $Y_t^*$ is the natural (flexible prices) output, defined in Session 2.4.

Following e.g. Ascari and Sbordone (2013, Section 3) and Ascari (2004, online Appendix), the firm’s first order condition can be conveniently written, in equilibrium, as $(p_t^*/P_t)^{1+\theta \omega} = \frac{N_t}{D_t}$. The numerator $N_t$ and the denominator $D_t$ functions can be written in recursive forms, avoiding infinite sums:

\begin{align*}
N_t &= (X_t)^{(\omega + \sigma)} + E_t n_{t+1} N_{t+1} \quad ; \quad n_t = \alpha q_t \mathcal{G}_t \Pi_t \left( \frac{\Pi_{t-1}^{\text{ind}}}{\Pi_{t}^{\text{ind}}} \right)^{\theta(1+\omega)} \\
D_t &= 1 + E_t d_{t+1} D_{t+1} \quad ; \quad d_t = \alpha q_t \mathcal{G}_t \Pi_t \left( \frac{\Pi_{t-1}^{\text{ind}}}{\Pi_{t}^{\text{ind}}} \right)^{(\theta-1)}
\end{align*}

(1)

where $\mathcal{G}_t \equiv Y_t/Y_{t-1}$ denotes the gross output growth rate. The price setting structure implies the the dynamics: $1 = (1 - \alpha) \left( \frac{p_t^*/P_t}{1} \right)^{-(\theta-1)} + \alpha \left( \frac{p_t^*/P_t}{1} \right)^{(\theta-1)}$.

2.3 Aggregates

Following, I present a set of equations describing the evolution of the aggregate disutility $u_t \equiv \int_0^1 u_t(z) \, dz$ to labor and the aggregate hours worked $h_t \equiv \int_0^1 h_t(z) \, dz$. For that, let $\mathcal{P}_t^{-\theta(1+\omega)} \equiv \int^1_0 (p_t(z)/P_t)^{-\theta(1+\omega)} \, dz$ and $\mathcal{P}_{ht}^{-\theta(1+\omega)} \equiv \int^1_0 (p_t(z)/P_t)^{-\theta(1+\omega)} \, dz$ denote two distinct measures of aggregate relative prices, where $\tilde{\omega} \equiv \frac{1}{\varepsilon} - 1$. Using the Calvo (1983) price setting structure, I am able to derive the laws of motion of $\mathcal{P}_t$ and $\mathcal{P}_{ht}$\footnote{The way I derive the law of motion of $\mathcal{P}_t$ and $\mathcal{P}_{ht}$ is very similar to how e.g. Alves (2014), Schmitt-Grohe and Uribe (2007) and Yun (2005) derive relevant price dispersion variables for aggregate output, employment, resource constraints and aggregate disutility in their models.}. The result is general and independent of any level of...
trend inflation. The following system describes the evolution of \( v_t, h_t, \mathcal{P}_t \) and \( \mathcal{P}_{ht} \):

\[
\begin{align*}
v_t &= \left( \frac{Y_t}{\bar{A}_t} \right)^{(1+\omega)} \mathcal{P}_t^{-(1+\omega)} ; \\
h_t &= \left( \frac{Y_t}{\bar{A}_t} \right)^{(1+\tilde{\omega})} \mathcal{P}_{ht}^{-(1+\tilde{\omega})} + \theta \left( 1 - \alpha \right) \left( \frac{\tilde{p}_{t+1}^n}{\bar{P}} \right)^{-(1+\omega)} + \alpha \left( \frac{\Pi^n_{t+1}}{\bar{P}} \right)^{\omega(1+\omega)} \mathcal{P}_{t-1}^{-(1+\omega)} \\
\mathcal{P}_t^{-(1+\omega)} &= (1 - \alpha) \left( \frac{\tilde{p}_{t+1}^n}{\bar{P}} \right)^{-(1+\omega)} + \alpha \left( \frac{\Pi^n_{t+1}}{\bar{P}} \right)^{\omega(1+\omega)} \mathcal{P}_{t-1}^{-(1+\omega)} \\
\mathcal{P}_{ht}^{-(1+\tilde{\omega})} &= (1 - \alpha) \left( \frac{\tilde{p}_{t+1}^n}{\bar{P}} \right)^{-(1+\tilde{\omega})} + \alpha \left( \frac{\Pi^n_{t+1}}{\bar{P}} \right)^{\omega(1+\tilde{\omega})} \mathcal{P}_{ht-1}^{-(1+\tilde{\omega})}
\end{align*}
\]

where \( \tilde{p}_{t+1}^n = \frac{\tilde{p}_{t+1}^n}{\bar{P}} \) is the optimal resetting relative price.

### 2.4 Equilibrium with flexible prices

In the abstract equilibrium with flexible prices (with superscript \( n \), after natural equilibrium), i.e. with \( \alpha = 0 \), the natural output \( Y_{t}^n \) evolves according to \( Y_{t}^{n(\omega+\sigma)} = \frac{\varepsilon}{\chi \sigma} \zeta_t \mathcal{A}_t^{(1+\omega)} \), the ZLB is never a constraint and the (gross) natural real interest rate \( R_{t}^n \) satisfies

\[
\frac{1}{R_{t}^n} = \beta E_t \left( \left( \frac{\varepsilon_{t+1}}{\varepsilon_t} \right)^{\theta_1} \left( \frac{\mathcal{A}_t}{\mathcal{A}_{t+1}} \right)^{\theta_2} \right) ; \quad \theta_1 \equiv \frac{\omega}{(\omega+\sigma)} ; \quad \theta_2 \equiv \frac{\sigma(1+\omega)}{(\omega+\sigma)}
\]

The fact that the ZLB is never a constraint in this abstract equilibrium ensures that monetary policy is able to keep inflation always at its (gross) target \( \bar{\Pi} \geq 1 \). Therefore, I assume that \( \Pi^n_{t} = \bar{\Pi} \) \( \forall t \) in the equilibrium with flexible prices. For that, the central bank sets the nominal interest rate at \( I_{t}^n = I_{t}^n = \bar{\Pi} R_{t}^n \), the unbounded nominal interest rate under flexible prices.

### 2.5 The log-linearized model

For any variable \( \mathcal{M}_t, \tilde{\mathcal{M}}_t \equiv \log \left( \frac{\mathcal{M}_t}{\bar{\mathcal{M}}} \right) \) represents its log-deviation from its steady state level \( \bar{\mathcal{M}} \) with non-zero trend inflation (Trend StSt).

In the equilibrium with flexible prices, real interest rate \( \hat{r}_{t}^n \) and output \( \hat{y}_{t}^n \) evolve according to the following equations:

\[
\hat{r}_{t}^n = E_t \left[ \sigma \left( \hat{y}_{t+1}^n - \hat{y}_{t}^n \right) - \left( \hat{\varepsilon}_{t+1} - \hat{\varepsilon}_t \right) \right] ; \quad \hat{y}_{t}^n = \frac{1}{(\omega+\sigma)} \left[ (1 + \omega) \mathcal{A}_t + \hat{\varepsilon}_t \right] \tag{2}
\]

Under sticky prices, i.e. for \( \alpha > 0 \), the dynamics of the (log-deviation) output gap \( \hat{x}_t = \hat{y}_t - \hat{y}_{t}^n \) are described by the IS curve \( \hat{x}_t = E_t \hat{x}_{t+1} + \frac{1}{\sigma} E_t \left( \hat{\eta}_{t} + \hat{\varepsilon}_t \right) \).

The Generalized New Keynesian Phillips Curve (GNKPC) under trend inflation, as coined by Ascarì and Sbordone (2013), is obtained by log-linearizing the firm’s first order conditions and the price setting structure about the Trend StSt. As in Alves (2014), I describe the GNKPC system in terms of the output gap as the only demand variable:

\[
\begin{align*}
(\hat{\pi}_t - \hat{\pi}_{t}^{ind}) &= \beta E_t \left( \hat{\pi}_{t+1} - \hat{\pi}_{t}^{ind} \right) + \bar{\kappa} \hat{x}_t + \left( \bar{\nu} - 1 \right) \bar{\kappa} \omega \beta E_t \hat{\omega}_{t+1} + \hat{\mu}_t \\
\hat{\omega}_t &= \alpha \bar{\nu} \beta E_t \hat{\omega}_{t+1} + \theta (1 + \omega) \left( \hat{\pi}_t - \hat{\pi}_{t}^{ind} \right) + (1 - \alpha \bar{\nu} \beta) (\omega + \sigma) \hat{x}_t + (1 - \sigma) (\hat{x}_t - \hat{x}_{t-1}) \\
\hat{\mu}_t &= \alpha \bar{\nu} \beta E_t \hat{\mu}_{t+1} + \left( \bar{\nu} - 1 \right) \beta E_t \hat{\xi}_{t+1} \\
\hat{\xi}_t &= \bar{\kappa} \omega \left( \frac{1 + \omega}{(\omega+\sigma)} \right) \left[ (1 - \sigma) \left( \mathcal{A}_t - \mathcal{A}_{t-1} \right) + (\hat{\varepsilon}_t - \hat{\varepsilon}_{t-1}) \right]
\end{align*}
\]

where \( \hat{\pi}_{t}^{ind} = \gamma \hat{\pi}_{t-1} \) is the indexation term, \( \hat{\omega}_t \) is an ancillary variable with no obvious interpre-
tation,⁵ $\dot{\xi}_t$ is an aggregate shock term that collects the effects of the technology shock $\hat{A}_t$ and the utility shock $\hat{e}_t$, and $\hat{u}_t$ is the endogenous trend inflation cost-push shock, which ultimately depends only on the technology and preference shocks. As for the composite parameters, $\dot{\vartheta} \equiv \Pi(1+\vartheta_\omega)(1-\gamma_\varepsilon)$ is a positive transformation of the level $\tilde{\pi}$ of trend inflation and $\tilde{\alpha} \equiv \alpha \Pi(\delta_{\omega}^{-1}(1-\gamma_\varepsilon))$ is the effective degree of price stickiness.⁶ Since $\tilde{\alpha}$ and $\dot{\vartheta}$ increase as trend inflation rises, the trend inflation cost-push shock $\hat{u}_t$ amplifies, by means of $(\dot{\vartheta} - 1)$ and the coefficient $\tilde{\alpha} \dot{\vartheta}$ on $E_t \hat{u}_{t+1}$, the effect of the aggregate shock $\dot{\xi}_t$ and transmits it through the inflation dynamics. The remaining composite parameters are 

$$\dot{\kappa} \equiv \frac{(1-\dot{\vartheta})(1-\tilde{\alpha} \dot{\vartheta})}{\tilde{\alpha}} \omega \sigma_{\omega} \quad ; \quad \dot{\kappa}_w \equiv \frac{(1-\dot{\vartheta})(1-\tilde{\alpha})}{\tilde{\alpha}} \quad ; \quad \dot{\omega} \equiv \frac{(1+\vartheta \varepsilon)^{\vartheta_\omega}}{\varepsilon} - 1$$  (4)

As well documented in the literature on trend inflation, the GNKPC becomes flatter ($\dot{\kappa}$ decreases) and more forward looking ($(\dot{\vartheta} - 1) \dot{\kappa}_w \beta$ and $\tilde{\alpha} \dot{\vartheta}$ increases) with trend inflation.⁷ The effect of $\hat{\omega}_t$ on the inflation dynamics is to make it even more forward looking. This is due to the fact that the coefficients $((\dot{\vartheta} - 1))$ on $E_t \hat{\omega}_{t+1}$, in the first equation, and $\tilde{\alpha} \dot{\vartheta}$ on $E_t \hat{\omega}_{t+1}$, in the second equation, increase as trend inflation rises.

As for the aggregates, where $\dot{\vartheta} \equiv \Pi(1+\vartheta_\omega)(1-\gamma_\varepsilon)$ and $\dot{\omega} \equiv \frac{1}{\varepsilon} - 1$, we have:

$$\dot{\tilde{u}}_t = (1 + \dot{\omega}) \left( \hat{y}_t - \hat{A}_t - \theta \hat{P}_t \right) \quad ; \quad \dot{\tilde{h}}_t = (1 + \dot{\omega}) \left( \hat{y}_t - \hat{A}_t - \theta \hat{P}_{ht} \right)$$

$$\dot{\hat{P}}_t = \tilde{\alpha} \dot{\vartheta} \hat{P}_{t-1} - \frac{(\dot{\vartheta} - 1)\tilde{\alpha}}{(1-\dot{\alpha})} \left( \dot{\hat{\pi}}_t - \dot{\hat{\pi}}_{t \text{ ind}} \right) \quad ; \quad \dot{\hat{P}}_{ht} = \tilde{\alpha} \dot{\vartheta} \hat{P}_{ht-1} - \frac{(\dot{\vartheta} - 1)\tilde{\alpha}}{(1-\dot{\alpha})} \left( \dot{\hat{\pi}}_t - \dot{\hat{\pi}}_{t \text{ ind}} \right)$$

### 3 Probability of hitting the ZLB

For more details for the results shown in this session, see Appendix B. I assume that preference (demand) and technology shocks follow AR(1) processes, i.e. $\epsilon_t = \epsilon_{t-1}, \epsilon_{t} \approx \mathcal{N}(0, \sigma_{\epsilon_{t}}^2)$, and $\hat{A}_t = \hat{A}_{t-1}, \hat{A}_{t} \approx \mathcal{N}(0, \sigma_{\hat{A}_{t}}^2)$, where $\epsilon_{u,t}$ and $\epsilon_{a,t}$ are unity-averaged independent white noise disturbance terms. Let us assume that $\epsilon_{u,t}, \epsilon_{a,t} \approx \mathcal{N}(0, \sigma_{\epsilon_{u,t}}^2)$ is independent of $\epsilon_{a,t}, \epsilon_{a,t} \approx \mathcal{N}(0, \sigma_{\epsilon_{a,t}}^2)$, where $\sigma_{\epsilon_{u,t}}^2$ and $\sigma_{\epsilon_{a,t}}^2$ are dispersion parameters.

For each monetary policy framework, given the information set $\mathcal{F}_t$ at period $t$, I define the transition probability $p_{o,t}$ of hitting the ZLB at period $t + 1$ as $p_{o,t} \equiv \mathbb{P}(T_{t+1} = 1 | \mathcal{F}_t)$, where again $T_t$ is the unbounded nominal interest rate under flexible prices.

Let us first consider the unbounded equilibrium with flexible prices, described in Section 2.4. Using the Euler equation and the marginal utility definition, note that $(T_{t+1})^{-1} = \beta E_{t+1} \left( \frac{\epsilon_{t+2}}{\epsilon_{t+1}} \frac{V_{t+2}}{V_{t+1}} \right) \left( 1 - t \right)$, where $T_t$ is again the unbounded nominal interest rate under flexible prices, for an economy in which the inflation rate is kept at $\tilde{\Pi}$. Therefore, the probability $p_{o,t}$ is defined as follows:

$$p_{o,t} \equiv \mathbb{P}(T_{t+1} = 1 | \mathcal{F}_t)$$

The first step in determining $p_{o,t}$ is to compute the expectation term in the right-hand side of $(T_{t+1})^{-1}$, and rewrite $\frac{1}{T_{t+1}}$ and its expected value as follows (see Appendix B for more details):

$$\frac{1}{T_{t+1}} = \mathcal{E}_{o,t}(\epsilon_{t+1}) \left( \frac{1}{T_{t+1}} \right) \quad ; \quad E_{t+1} \left( \frac{1}{T_{t+1}} \right) = \exp \left( \frac{\sigma_{\epsilon_{u,t}}^2}{2} \right) \mathcal{E}_{o,t}$$

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⁵In the literature on trend inflation, there are two usual ways to describe trend inflation Phillips curves: (i) with ancillary variables (e.g. Ascari and Ropele (2007)); and (ii) with infinite sums (e.g. Cogley and Sbordone (2008) and Cobion and Gorodnichenko (2011)).

⁶The composite parameters $\dot{\alpha}$ and $\dot{\vartheta}$ are bounded by $\max (\dot{\alpha}, \dot{\vartheta}) < 1$ to guarantee the existence of an equilibrium with trend inflation.

⁷As Ascari and Ropele (2007) show, the GNKPC reduces to the usual form when the level of trend inflation is zero. In this case, the ancillary variable $\hat{\omega}_t$ become irrelevant and the trend inflation cost-push shock $\hat{u}_t$ vanishes to zero.
where
\[
\mathcal{E}_{o,t} = \beta \frac{\pi}{\rho(1-\rho_2)} \exp\left(\frac{\theta_1}{\rho(1-\rho_2)}\right) \quad \text{and} \quad \mathcal{I}_{o,t} = \frac{\omega}{\rho(1-\rho_2)} + \frac{\sigma(1+\omega)}{\rho(1-\rho_2)}
\]

Therefore, it is easy to derive how \( p_{o,t} \) evolves:
\[
p_{o,t} = \mathbb{P}\left(T_{t+1} \leq 1 | \mathcal{I}_{t}\right) = \mathbb{P}\left(\epsilon_{ua,t} < \exp\left(-\frac{s_{ua}^2}{2}\right) E_t \left(\frac{1}{T_{t+1}}\right) | \mathcal{I}_{t}\right)
\]

where \( \epsilon_{ua,t} \equiv \frac{(\epsilon_{ua})^{(1-\rho_2)/2}}{(\epsilon_{ua})^{(1-\rho_1)/2}} \). This result implies that
\[
p_{o,t} = \mathbb{F}_{ua}\left[\exp\left(-\frac{s_{ua}^2}{2}\right) E_t \left(\frac{1}{T_{t+1}}\right)\right]
\]

where \( \mathbb{F}_{ua}(x) \equiv \mathbb{P}(\epsilon_{ua,t} \leq x) \) is the cdf of the aggregate disturbance \( \epsilon_{ua,t} \iid LN(0, s_{ua}^2) \).

For notation simplicity, it is convenient to consider the equivalent result \( p_{o,t} = \mathbb{F}_{ua}(\mathcal{E}_{o,t}) \). Let us define the hazard rate of hitting the ZLB as \( h_{o,t} \equiv \frac{f_{o,t}}{s_{ua}^2} \), where \( f_{o,t} = f_{ua}(\mathcal{E}_{o,t}) \) and \( f_{ua} \) is the density function of \( \epsilon_{ua,t} \). That is, considering the properties of log-normal densities, the probability \( p_{o,t} \) and the hazard rate \( h_{o,t} \) of hitting the ZLB at period \( t + 1 \) under the equilibrium with flexible prices evolve according to:
\[
p_{o,t} = \frac{1}{2} \left[1 + \text{erf}\left(\frac{\log(\mathcal{E}_{o,t})}{\sqrt{2s_{ua}^2}}\right)\right] ; \quad h_{o,t} = \frac{f_{o,t}}{s_{ua}^2} ; \quad f_{o,t} = \frac{1}{\mathcal{F}_{o,t} \sqrt{2\pi s_{ua}^2}} \exp\left(-\frac{1}{2} \left(\frac{\log(\mathcal{E}_{o,t})}{s_{ua}^2}\right)^2\right)
\]

At the steady state, with \( \bar{\epsilon} = 1 \) and \( \bar{A} = 1 \), I obtain:
\[
\bar{p}_o = \mathbb{F}_{ua}(\bar{\mathcal{E}}_o) ; \quad \bar{h}_o = \frac{\bar{f}_o}{(1-\bar{p}_o)} ; \quad \bar{f}_o = f_{ua}(\bar{\mathcal{E}}_o) ; \quad \bar{\mathcal{E}}_o = \frac{\beta}{\pi} \exp\left(\frac{1}{2} s_{ua}^2\right)
\]

A linear approximation of \( p_{o,t} \) about the trend inflation steady state is what I call the natural ZLB Probability curve:
\[
p_{o,t} \approx \bar{p}_o - \phi_{ua} \left[\rho_a (1-\rho_a) \theta_1 \bar{A} - \rho_a (1-\rho_a) \theta_2 \bar{A}\right]
\]

where \( \phi_{ua} \equiv \frac{\bar{f}_o \bar{\mathcal{E}}_o}{\bar{p}_o} \) is the shock-elasticity of ZLB probability:
\[
\phi_{ua} = \frac{1}{\sqrt{2\pi s_{ua}^2}} \exp\left(-\frac{1}{2} \left(\frac{\bar{\epsilon} - \frac{1}{2} s_{ua}^2}{s_{ua}^2}\right)^2\right)
\]

where \( \bar{\epsilon} \equiv \log\left(\frac{\bar{p}_o}{\bar{p}_s}\right) \).

Under sticky prices and a particular monetary policy framework, there is no closed-form solution for the probability \( p_{o,t} \) of hitting the ZLB as a function of structural disturbances, as it depends on the way monetary policy is implemented and how it affects the joint distribution of the expected path of the endogenous variables. Notwithstanding, it is easy to conclude that \( \bar{p}_s = \bar{p}_o \) once we realize that the distortive contribution of non-zero levels of trend inflation is offset in the steady-state value of \( Y_{t+2}/Y_{t+1} \). Therefore, any first order approximation of the transition probability about the steady state has the following format, in which \( \mathcal{O}(\cdot) \) is the order operator:
\[
p_{o,t} \approx \bar{p}_o + \mathcal{O}(1)
\]
As I show in the next section, this result suffices for deriving the optimal precautionary policy rule. Nevertheless, I can derive a closed-form proxy \( p_{p,t} \) for the probability of hitting the ZLB, assuming a perturbation about the equilibrium with flexible prices. As detailed in Appendix B, I highlight that \( p_{p,t} \) can only be thought as a proxy for \( p_{o,t} \), as my assumptions might be strong. Again, my results presented in the next section only depend on \( p_{o} = \bar{p}^{o} \), and are independent of the dynamics of \( p_{o,t} \). That is, any gap between \( p_{p,t} \) and \( p_{o,t} \) will not affect the equilibrium and the optimal results I present in the next section.

That being said, as I show in Appendix B, the proxy \( p_{p,t} \) evolves according to the following result:

\[
p_{p,t} = \bar{P}_{ua} \left( \exp \left( -\frac{\bar{s}_{ua}^{2}}{2} \right) E_{t} \left( \frac{1}{T_{t+1}} \right) \right) \tag{9}
\]

Using similar definitions and conclusions as those used for deriving \( p_{o}^{t} \), the probability proxy \( p_{p,t} \) and its hazard rate \( h_{p,t} \) of hitting the ZLB at period \( t+1 \) under the equilibrium with sticky prices evolve according to:

\[
p_{p,t} = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(\xi_{p,t})}{\sqrt{2\pi s_{ua}^{2}}} \right) \right] \; ; \; h_{p,t} = \frac{I_{p,t}}{(1-p_{p,t})} \; ; \; f_{p,t} = \frac{1}{\xi_{p,t} \sqrt{2\pi s_{ua}^{2}}} \exp \left( -\frac{1}{2} \left( \frac{\log(\xi_{p,t})}{\sqrt{2\pi s_{ua}^{2}}} \right)^{2} \right) \tag{10}
\]

where \( \xi_{p,t} \equiv \exp \left( -\frac{\bar{s}_{ua}^{2}}{2} \right) E_{t} \left( \frac{1}{T_{t+1}} \right) \) and \( \bar{\xi}_{p} = \bar{\xi}_{ua}^{o} \). The linear approximation of \( p_{p,t} \) about the trend inflation steady state is:

\[
p_{p,t} \approx \bar{p}_{o}^{n} - \phi_{ua} E_{t} \frac{\hat{\pi}}{\hat{I}_{t+1}} \tag{11}
\]

Recall that \( I_{t} \) is the unbounded nominal interest rate under sticky prices. It implies that \( I_{t} \leq I_{t} \). And so, \( p_{p,t} \) can achieve large values.

4 Optimal monetary policy

In Alves (2014), I derive a trend-inflation welfare based (TIWeB) loss function, which implies the following second order log-approximation of the (negative) welfare function \( W_{t} = -\frac{1}{2} \nu E_{t} \sum_{t=0}^{\infty} \beta^{t} \tilde{L}_{t+1} + \tilde{f}_{t}^{W} \), where \( \tilde{L}_{t} \equiv \left( \hat{\pi}_{t} - \hat{\pi}_{t}^{ind} + \hat{\phi}_{u} \right)^{2} + \bar{\nu} \left( \hat{x}_{t} - \hat{\varphi}_{x} \right)^{2} \) is the trend inflation welfare-based (TIWeB) loss function, \( \tilde{f}_{t}^{W} \) stands for terms independent of policy at period \( t \), \( \hat{\phi}_{u} \) and \( \hat{\varphi}_{x} \) are constants that depend on the inefficiency parameters \( \Phi_{\theta} \equiv (\bar{\theta} - 1) \) and \( \Phi_{\bar{y}} \equiv 1 - \bar{u}_{Y} / \bar{u}_{Y} \), and \( \bar{\nu} \) corrects for the aggregate reduction in the welfare when trend inflation increases. Those composite parameters are defined as follows:

\[
\hat{\varphi}_{x} \equiv \frac{(1-\bar{\alpha})}{(1-\alpha\beta)(1+\theta\omega)} \bar{\Phi}_{\theta} \; ; \; \hat{\varphi}_{x} \equiv \frac{1}{(\omega+\sigma)} \bar{\Phi}_{\bar{y}} \; ; \; \bar{\nu} \equiv \frac{(\omega+\sigma)}{\bar{\nu}} \bar{\nu}^{-1-\sigma} \; ; \; \bar{\nu} \equiv \frac{(1-\bar{\alpha})}{(1-\alpha\beta)} \bar{\nu} \tag{12}
\]

Here, I expand trend inflation optimal policies results I obtained in Alves (2014) by internalizing the influence of \( p_{o,t} \) when deriving precautionary optimal policies rules under unconditionally commitment. The unconditionally approach has its roots on Damjanovic et al. (2008), who show that the unconditionally commitment optimal policy (under zero trend inflation) welfare-dominates the Woodford (2003) timeless perspective targeting rule and is time-consistent. That is, there is no inconsistency arising from first-order conditions obtained at first periods of optimization steps. Therefore, unconditionally, the monetary authority has no incentive to deviate from the optimal policy rule.

I assume that the welfare-concerned central bank implements inflation targeting by internalizing the ZLB Probability curve (8) to minimize the unconditional expectation of the Lagrangian problem formed by the discounted sum of the TIWeB loss function, subject to the IS curve, the GNKPC (3)
and the ZLB constraint $I_t \geq 1$, which is log-linearized as $\dot{i}_t + \ddot{i}_t \geq 0$, for $\dot{i} \equiv \log (\bar{I})$. Inflation targeting is implemented by keeping the unconditional mean of the inflation rate at the central target $\bar{\pi}$, i.e. $E\Pi = \bar{\Pi}$, which is log-linearized as $E\bar{\pi}_t = 0$.

When dealing with occasionally binding ZLB constraints, we must properly evaluate the TIWeB loss function and the first order conditions for states in which $\dot{i}_t + i_t \geq 0$ is binding, with probability $p_{o.t}$, and those in which the restriction is loose, with probability $(1 - p_{o.t})$. When the restriction binds, I impose $\dot{i}_t = -i_t$ into the IS curve, which is the only one affected by the restriction. The remaining equations are not affected. Analogously, the only loss function quadratic term affected by the restriction is $\bar{\chi} (\ddot{x}_t - \ddot{\phi}_x)^2$. When building the Lagrangian form, the simplest approach is to directly impose the restricted IS curve $\dot{x}_t = \hat{\gamma}_t + \frac{1}{\sigma} \ddot{x}_t + \frac{1}{\sigma} \ddot{c}_t$, where $\hat{\gamma}_t = E_t (\dot{x}_{t+1} + \frac{1}{2} \ddot{\pi}_{t+1})$, into $\bar{\chi} (\ddot{x}_t - \ddot{\phi}_x)^2$ when the restriction binds.

In addition, the whole Lagrangian problem must be of order $O(2)$, for this is the order to which the welfare function is log-linearized. Since log-linearized equations are used as restrictions, Lagrangian multipliers must be of order $O(1)$. This order issue is relevant when considering the first order approximation of $p_{o.t} \approx \bar{p}_o + O(1)$ into the problem. The issue arises when multiplying this approximation by the second order components from the loss function, as we must disregard all $O(3)$ terms from the resulting multiplication. In Alves (2014), I show that the distortion parameters $\hat{\phi}_x$ and $\hat{\phi}_x$ must be of order $O(1)$ in order for the trend inflation welfare-based loss function to be properly used with log-linearized equations when deriving optimal policy rules. With the same logic, I assume that $i_t$ is of order $O(1)$. This assumption is reasonable once we realize that precautionary rules become more relevant for low levels of the long-run nominal interest rate $i \equiv \log (\bar{I})$.

The following proposition summarizes the results on the optimal precautionary policy under unconditionally commitment:

**Proposition 1** When a welfare-concerned central bank targets $\bar{\pi}$ as the inflation target, follows the recommendations of the TIWeB loss function, and recognizes its role in influencing occasionally binding episodes of hitting the zero-lower bound (ZLB) on nominal interest rates, the optimal precautionary policy under unconditionally commitment is described by the following targeting rule, when the ZLB constraint is not binding:

$$0 = (\hat{\pi}_t - \hat{\pi}^{ind}_t) + (1 - \bar{p}_o) \frac{1}{c_t} \bar{\chi} [\dddot{x}_t - \beta \dddot{x}_{t-1} - (\varepsilon_2 - \varepsilon_1) \dddot{z}_t] + \bar{p}_o \bar{\chi} \left( \frac{1}{\sigma} \dddot{\gamma}_1, + \frac{1}{\sigma} \dddot{\gamma}_2 \right)$$ (13)

where $\dddot{z}_t$, $\dddot{\gamma}_1$, and $\dddot{\gamma}_2$ are ancillary variables, whose dynamics are described by

$$\dddot{z}_t = \frac{\alpha_1}{c_1} \dddot{x}_t - \frac{\alpha_1}{c_1} \dddot{x}_{t-1} - \frac{\beta}{c_1} (1 - \alpha_1 \beta \bar{\theta}) \dddot{x}_{t-2}$$
$$\dddot{\gamma}_1 = \gamma_x E_t \dddot{\gamma}_1, t+1 + \left( \dddot{x}_{t-1} + \frac{1}{\sigma} \dddot{x}_{t-1} \right)$$
$$\dddot{\gamma}_2 = \frac{\alpha_1}{c_1} \dddot{\gamma}_2, t+1 + \left( \dddot{x}_{t-1} + \frac{1}{\sigma} \dddot{x}_{t-1} \right) - \frac{1}{c_1} \left[ (1 + \alpha \bar{\theta}) \beta + \bar{\theta} (\varepsilon_2 - \varepsilon_1) \right] \left( \dddot{x}_{t-2} + \frac{1}{\sigma} \dddot{x}_{t-2} \right) + \frac{1}{c_1} \bar{\theta} \beta^2 \left( \dddot{x}_{t-3} + \frac{1}{\sigma} \dddot{x}_{t-3} \right)$$ (14)

and the composite parameters are defined as follows:

$$\varepsilon_1 \equiv 1 - (\bar{\theta} - 1) \frac{\beta \bar{\pi}}{\bar{\pi}} (1 - \sigma) ; \varepsilon_2 \equiv 1 + (\bar{\theta} - 1) \frac{\beta \bar{\pi}}{\bar{\pi}} (\omega + \sigma)$$
$$\varepsilon_3 \equiv \theta \pi c_1 + (1 - \alpha_1 \beta \bar{\theta}) ; \varepsilon_4 \equiv c_1 - (1 - \alpha_1 \beta \bar{\theta}) c_2$$ (15)

The proof is shown in Appendix C.

Of course, $\dot{\pi}_t \rightarrow -\dot{i}_t$ when the ZLB constraint binds. Therefore, the full targeting rule must be understood as the one to be pursued in between occasionally binding episodes.
In order to tell the contributions of trend inflation and precautionary behavior induced by internalizing \( p_{o,t} \), when compared to the benchmark targeting rule derived by Damjanovic et al. (2008), who considered a steady state equilibrium with zero trend inflation. The authors derive the rule

\[
0 = (\hat{\pi}_t - \hat{\pi}^\text{ind}_t) + \frac{\lambda}{\kappa} (\hat{x}_t - \beta \hat{x}_{t-1}), \quad \text{where } \lambda \text{ and } \kappa \text{ are the values of } \lambda^* \text{ and } \kappa^*, \text{ evaluated at } \overline{\pi} = 0.
\]

In Alves (2014), after accounting for the effects of trend inflation in welfare, I find that the optimal rule prescribes a more (output-gap) history dependent policy rule as trend inflation rises. In that case, the optimal rule is

\[
0 = (\hat{\pi}_t - \hat{\pi}^\text{ind}_t) + \frac{1}{c_1 \kappa} [\hat{x}_t - \beta \hat{x}_{t-1} - (c_2 - c_1) \hat{x}_{1,t-1}]
\]

(16)

Expanding that result to internalize the transition probability \( p_{o,t} \) when deriving optimal policies under commitment and occasionally binding ZLB episodes, I conclude that the monetary authority faces a larger trade-off when adjusting inflation to the history of output gaps. She must imbed a precautionary behavior, adjusting inflation to a weighted average between the history of output gaps and components depicting the long-past instance of nominal interest rates and output-gaps. In addition, the weight of this convex combination is the steady state level of the transition probability \( \pi_o \).

Note that under low steady level of the (gross) nominal interest rate \( \bar{I} = \bar{\Pi}/\beta \), the parameters \( \pi_o \) and \( \phi_{ua} \) fast increase, which makes the precautionary targeting rule (13) even more directly dependent on the history of output-gaps and nominal interest rates. For larger steady state levels of nominal interest rate, on the other hand, \( \pi_o \) shrinks towards zero and the rule starts to resemble the one depicted in (16), derived for the case in which the ZLB is never binding.

Under optimality, the central bank consciously and directly adopts precautionary behavior in normal times in order not to cut nominal interest rates so fast after negative demand shocks and taking longer to increase the rate after the shock has dissipated. Due to the particular format of the targeting rule, the intuition behind this property is easy to understand. Suppose that the economy is hit by a moderate negative demand shock. Under precautionary behavior, monetary policy look further into the history of output gap and nominal interest rates. Consequently, monetary policy does not strongly react on impact, for it also looks into periods in which the output gap was not yet hit by the shock and the nominal interest rate was not low. This slowdown in reducing the rate makes more room for monetary policy to avoid hitting the ZLB, especially if the shock is perceived as weakly persistent.

After the negative demand shock has dissipated, monetary policy continues to look longer in the past and still take in consideration that the output gap has fallen in the past and nominal interest rates were low. Therefore, nominal rates take longer to return. I highlight that this duration depends on \( \pi_o \). Hence, forward guidance is always optimized under this policy rule.

Finally, the presence of \( h_{-1} \) in the targeting rule is a novelty in the literature of optimal policy prescriptions. It naturally arises as \( \pi_o \) rises and it serves to smooth optimal changes on nominal interest rates. As a consequence, the rate does not fall (rise) as fast under negative (positive) demand shocks when compared to responses under standard optimal policy prescriptions such as in Damjanovic et al. (2008), Woodford (2003), and Nakov (2008).

I highlight the fact that those precautionary terms would not arise in the targeting rule when not directly considering the transition probability \( p_{o,t} \) for deriving optimal policies under commitment and occasionally binding ZLB episodes. In light of the previous comments, this conclusion is immediate once it would resemble the case in which \( \pi_o \) is close to zero. For instance, Eggertsson and Woodford (2003a,b) and Nakov (2008) derive optimal policies from the timeless perspective (with zero trend inflation), only imposing the ZLB constraint, without assessing the effects of the transition probability into the loss function. As a consequence, their targeting rules for normal times has the standard format, in which inflation has only to adjust against changes in output gap in normal times.\(^8\) In general equilibrium, nonetheless, policy precautionary behavior partially (indirectly) arises as the

\(^8\) Optimal policies from the timeless perspective has \( (\hat{x}_t - \hat{x}_{t-1}) \) instead of \( (\hat{x}_t - \beta \hat{x}_{t-1}) \) as its last term.
nominal interest rate is recommended to remain at the lower bound even after dissipation of the shock that brought it to zero in the first place.

The Precautionary Optimal Policy (13) I derive here, on the other hand, prescribes a stronger precautionary behavior. For comparison, consider the targeting rule that would be obtained in my exercises should I have disregarded the effects of the transition probabilities in the loss function. In that case, the Standard Optimal Policy under occasionally binding ZLB constraint would have the same format, in normal times, as the pure targeting rule under trend inflation (16). Based on previous comments, the Precautionary Optimal Policy prescribes the rate not to fall as much as what the Standard Optimal Policy prescribes after a moderate negative demand shock, and prescribes the rate to remain at zero for even longer.

Finally, recall that \( \mathbf{p}_o \) and \( \phi_{ua} \) are direct functions of the steady level of the (gross) nominal interest rate \( \bar{I} = \bar{\Pi}/\beta \), which increases with larger levels of (gross) trend inflation \( \bar{\Pi} \) or long-run (gross) real interest rate \( \bar{R} = 1/\beta \). Therefore, the effects of rising \( \bar{\Pi} \) or \( \bar{R} \) are perfectly substitutes on what regards \( \mathbf{p}_o \) and \( \phi_{ua} \). Both margins are not substitutes, however, as their effects on welfare and policy prescription are very different. Hence, rising the trend inflation (inflation target) level is not a perfect remedy to offset the effects of a declining trend on the long-run real interest rate. Actually, rising levels of trend inflation lead to more welfare distortions and instability, as shown in the literature of trend inflation, despite being successful in minimizing the probability of hitting the ZLB.

In the next part of the analysis, I assess whether price-level targeting can be optimal.

### 4.1 Price-level targeting

I now assess the optimality of price level targeting under occasionally binding ZLB episodes. In particular, the following proposition shows that it can be approximately optimal when certain conditions hold. For that, we need both the long-run inflation and real interest rates to be close to zero, and the coefficient \( \bar{k} \) of output-gap in the GNKPC, the intertemporal elasticity of substitution \( \sigma^{-1} \) and the degree of price indexation \( \gamma_{\pi} \) to be small. Addressing the case under larger values of trend inflation \( \bar{\pi} \), even under low real interest rates, is more entangled. The targeting rules would not resemble price level targeting anymore.

**Proposition 2** When \( \bar{\pi} \) is close to zero, \( \beta \) is close to unity, and both the composite parameter \( \bar{k}/\sigma \) and the degree of price indexation \( \gamma_{\pi} \) are sufficiently small, the optimal precautionary policy under unconditionally commitment (13) can be reasonable approximated by a price-level targeting rule:

\[
0 \approx (\hat{p}_t - \hat{p}^{ind}_t) + (1 - \mathbf{p}_o) \frac{\bar{X}}{\bar{k}} \hat{x}_t + \mathbf{p}_o \frac{\bar{X}}{\bar{k}} (\hat{x}_{t-1} + \frac{1}{\sigma} \hat{i}_{t-1})
\]

where \( \hat{p}_t \) and \( \hat{p}^{ind}_t = \gamma_{\pi} \hat{p}_{t-1} \) stand for log-deviations of the aggregate price level and indexed price from their targets.

**Proof.** If \( \bar{\pi} \) is close to zero, \( \mathbf{p}_o \) fast increase and so the precautionary targeting rule depends more intensively on the history of nominal interest rates:

\[
0 = (\hat{x}_t - \hat{x}^{ind}_t) + (1 - \mathbf{p}_o) \frac{\bar{X}}{\bar{k}} (\hat{x}_t - \beta \hat{x}_{t-1}) + \mathbf{p}_o \frac{\bar{X}}{\bar{k}} \left( \frac{1}{\sigma} \hat{\delta}_{1,t} + \frac{1}{\sigma} \hat{\delta}_{2,t} \right)
\]

\[
\hat{\delta}_{1,t} = \gamma_{\pi} E_t \hat{\delta}_{1,t+1} + \left( \hat{x}_t + \frac{1}{\sigma} \hat{i}_t \right)
\]

\[
\hat{\delta}_{2,t} = \alpha \beta \hat{\delta}_{2,t-1} + \left( \hat{x}_{t-1} + \frac{1}{\sigma} \hat{i}_{t-1} \right) - (1 + \alpha) \beta \left( \hat{x}_{t-2} + \frac{1}{\sigma} \hat{i}_{t-2} \right) + \alpha \beta^2 \left( \hat{x}_{t-3} + \frac{1}{\sigma} \hat{i}_{t-3} \right)
\]
Using the lag $L(\cdot)$ operator, note that we can rewrite the rule as follows:

$$
0 = (\hat{x}_t - \hat{x}_t^{\text{ind}}) + (1 - \bar{p}_o) \frac{\bar{\sigma}}{\bar{\kappa}} (1 - \beta L) \hat{x}_t + \bar{p}_s \bar{X} \left( \frac{1}{\bar{\sigma}} \hat{\delta}_{1,t} + \frac{1}{\bar{\kappa}} \hat{\delta}_{2,t} \right) 
$$

$$
\hat{\delta}_{1,t} = \gamma \pi \sigma_i \hat{\delta}_{1,t+1} + (\hat{x}_t + \frac{1}{\bar{\kappa}} \hat{u}_t) 
$$

$$
(1 - \alpha \beta L) \hat{\delta}_{2,t} = (1 - \alpha \beta L) (1 - \beta L) (\hat{x}_{t-1} + \frac{1}{\bar{\kappa}} \hat{u}_{t-1})
$$

Consider the case in which the (gross) real interest rate $\hat{R} = 1/\beta$ has been reducing over time, i.e. $\beta$ is getting closer and closer to unity. In this case, $(1 - L)$ is a good approximation for $(1 - \beta L)$. Moreover, empirical microevidence strongly suggests that there is none or very small degree of price stickiness in the US, i.e. $\gamma \pi$ is very small, which induces $\hat{\delta}_{1,t}$ to be close to zero. Since $\hat{\pi}_t = (1 - L) \hat{p}_t$ and $\hat{x}_t^{\text{ind}} = (1 - L) \hat{p}_t^{\text{ind}}$, for $\hat{p}_t^{\text{ind}} = \gamma \pi \hat{p}_{t-1}$, the targeting rule is reasonably approximated by the following expression when $\hat{R}$ is small:

$$
0 \approx (1 - L) (\hat{p}_t - \hat{p}_t^{\text{ind}}) + (1 - \bar{p}_o) \frac{\bar{\sigma}}{\bar{\kappa}} (1 - L) \hat{x}_t + \bar{p}_s \frac{\bar{\sigma}}{\bar{\kappa}} \left[ \frac{\bar{\sigma}}{\bar{\kappa}} + (1 - L) L \right] (\hat{x}_t + \frac{1}{\bar{\kappa}} \hat{u}_t)
$$

Since macroevidence suggests that $\sigma \gg \bar{\kappa}$, i.e. $\frac{\bar{\sigma}}{\bar{\kappa}}$ is very small, the second term dominates the expression inside brackets, and so it is reasonably approximated by

$$
0 \approx (1 - L) \left[ (\hat{p}_t - \hat{p}_t^{\text{ind}}) + (1 - \bar{p}_o) \frac{\bar{\sigma}}{\bar{\kappa}} \hat{x}_t + \bar{p}_s \frac{\bar{\sigma}}{\bar{\kappa}} L (\hat{x}_t + \frac{1}{\bar{\kappa}} \hat{u}_t) \right]
$$

Therefore, we can rewrite it using its simpler version

$$
0 \approx (\hat{p}_t - \hat{p}_t^{\text{ind}}) + (1 - \bar{p}_o) \frac{\bar{\sigma}}{\bar{\kappa}} \hat{x}_t + \bar{p}_s \frac{\bar{\sigma}}{\bar{\kappa}} L (\hat{x}_t + \frac{1}{\bar{\kappa}} \hat{u}_t)
$$

### 5 Calibration

The calibration is described as follows. As in Cooley and Prescott (1995), I set the elasticity to hours at the production function at $\varepsilon = 0.64$. As in Coibion et al. (2012), I set the elasticity of substitution at $\phi = 7$, which implies a steady state price markup of $\mu = 1.17$.\(^9\) Recall that the (log-deviation) technology shock evolves according to $\tilde{\lambda}_t = \rho_a \tilde{\lambda}_{t-1} + \tilde{\epsilon}_{a,t}$, where $\tilde{\epsilon}_{a,t} \overset{\text{iid}}{\sim} N(0, s_a^2)$. Using the central estimate obtained by Smets and Wouters (2007) for the larger sample, I set the autoregressive coefficient of the technology shock at $\rho_a = 0.95$ and the shock’s standard deviation at $s_a = 0.0045$. The remaining parameters were based on central estimates obtained by Smets and Wouters (2007), for the Great Moderation. I set the reciprocal of the intertemporal elasticity of substitution at $\sigma = 1.47$. As for the elasticity $\nu$ of the disutility from hours $h_t(z)$, i.e. the reciprocal of the Frisch elasticity, I use $\nu = 2.30$. Note that this value is consistent with micro evidence, as reported by Chetty et al. (2011).\(^10\) I set the degree of price stickiness at $\alpha = 0.73$, while the price indexation parameter is fixed at $\gamma \pi = 0.21$. In addition, I set the disutility nuisance parameter at $\chi = 1$.

Recall that the (log-deviation) demand shock evolves according to $\tilde{\epsilon}_t = \rho_a \tilde{\epsilon}_{t-1} + \tilde{\epsilon}_{u,t}$, where $\tilde{\epsilon}_{u,t} \overset{\text{iid}}{\sim} N(0, s_u^2)$, and that I do not assume consumption habit persistence in my model. Therefore, $\rho_a$ will play a similar role as the degree of habit persistence in this model. Therefore, based on the authors’ estimated habit persistence parameter, I set the persistence of the demand shock $\rho = 0.68$. In order to adjust the implied dynamics implied by this assumption, I estimate $s_u$ using quarterly US data

\(^9\)For instance, Ravenna and Walsh (2008, 2011) set the steady state price markup to 1.2.

\(^10\)The authors conduct meta analyses of existing micro evidence. Their point estimate of the Frisch elasticity of intensive margin is $(1/\nu) = 0.54$. 
from the *Great Moderation* period 1985:Q1-2005:Q4. For that, I fix $\beta = 0.995$ (consistent with annual real interest rate $\bar{r} = 2\%$) and (annual) $\bar{\pi} = 3.05\%$ (consistent with the sample average of the CPI inflation rate).

For estimation, I considered the following observed variables: $(i)$ inflation rate $\hat{\pi}_t$ is the (log) BLS CPI inflation rate (US city average, all urban consumers), demeaned from its sample average; $(ii)$ output $\hat{y}_t$ is the (log) BLS GDP, detrended by its linear trend; and $(iii)$ nominal interest rate $\hat{i}_t$ is the (log) quarterly average of the Federal Funds Rate, demeaned from its sample average.

Since I observe the nominal interest rate in the estimation, I assume that monetary policy followed a simplified Trend Inflation Taylor rule, based on Coibion and Gorodnichenko (2011):

$$\hat{i}_t = \phi_i \hat{i}_{t-1} + (1 - \phi_i) [\phi_{\pi} \hat{E}_t \hat{\pi}_{t+1} + \phi_{gy} (\hat{y}_t - \hat{y}_{t-1})] + \hat{\epsilon}_{i,t}$$  \hspace{1cm} (18)

where $\hat{\epsilon}_{i,t} \overset{iid}{\sim} N(0, \sigma_i^2)$ is the monetary policy shock, $\phi_i$ is the policy smoothing parameter, and $\phi_{\pi}$ and $\phi_{gy}$ are response parameters consistent with stability and determinacy in equilibria with rational expectations in a equilibrium with positive trend inflation. The authors find that reacting to the observed output growth has two major advantages over responding to output gap: $(i)$ it has more stabilizing properties when the trend inflation is not zero; and $(ii)$ it is empirically more relevant. Based on Coibion and Gorodnichenko (2011) central estimations, I keep $\phi_i = 0.92$ and estimate the response parameters so that the estimated model adjusts to a possible misscalibration and absence of additional shocks. Since I aim at inferring $\sigma_u$ and $\sigma_a$, this strategy is quite reasonable.

Using Bayesian MCMC estimation, with flat priors and 200000 draws, table 1 reports posterior means and 95% credible intervals for $\phi_{\pi}$, $\phi_{gy}$, $\sigma_i$, and $\sigma_u$. For simulations shown in Section 6, I set $\sigma_u$ and $\sigma_a$ at the posterior means. Since all exercises are focused on evaluating optimal policy rules, I did not consider outcomes under monetary policy shocks.

<table>
<thead>
<tr>
<th>Table 1: Posterior Distributions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Post Mean</strong></td>
</tr>
<tr>
<td>$\phi_{\pi}$</td>
</tr>
<tr>
<td>$\phi_{gy}$</td>
</tr>
<tr>
<td>$\sigma_i$</td>
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<tr>
<td>$\sigma_u$</td>
</tr>
</tbody>
</table>

Using the calibration set, note that the steady-state levels of the ZLB probability $\bar{p}_e$ and probability-elasticity of shocks $\phi_{ua}$ increase very fast as the steady state level of the annual nominal interest rate $\bar{i}$ falls towards the ZLB, as depicted in Figure 1. I highlight the fact that $\bar{p}_e$ is a reference level, for the expected frequency $E_p_{\phi_{ua}}$ according to which the ZLB binds is highly policy-dependent. In Section 6, I show simulations under different monetary policy frameworks.

---

11I have also estimated an extended version which included $\phi_{x_x}$ as an additional term in the Trend Inflation Taylor Rule. The results, however, pointed at a zero posterior mode and mean for $\phi_x$. 

13
For instance, the observed frequency at which the effective annualized Federal Funds rate is below 0.15%, which I call the Effective Lower Bound (ELB), from 2006Q1 to 2016Q4 (after the Great Moderation period) was 41%. During this period, the average Fed Funds rate was 1.19%. In this context, even though \( p_0 = 32\% \), the Standard Optimal policy delivers \( E{p}_{e,t} \) between 43% and 44% depending on the combination of \( \beta \) and \( \bar{\pi} \) such that \( \bar{\pi} = 1.19 \). In a similar exercise, for the larger sample 1985Q1 to 2016Q4, the average Fed Funds rate was 3.73% and the rate was below the ELB at 14% of the time. In this context, \( p_0 = 7.5\% \) and \( E{p}_{e,t} \) ranges between 14% and 16%, depending on the combination of \( \beta \) and \( \bar{\pi} \) such that \( \bar{\pi} = 3.73 \).

6 Simulations

This section studies the welfare gains and dynamics implied by trend inflation optimal policies under unconditionally commitment. I perform simulations using Ocibin, by Guerrieri and Iacoviello (2015), to account for the occasionally biding ZLB constraint on the nominal interest rate. Due to ZLB restrictions, there is no closed form solution to compute the model’s unconditional moments. Therefore, in order to infer them, I simulate artificial equilibria with 10,000 periods simultaneously using fixed sequences of exogenous demand and technology shocks, based on the distribution detailed in the last section.

In the first exercise, I compute welfare gains from using the TITeB precautionary optimal policy under unconditionally commitment (PrOP) over the TITeB standard optimal policy under unconditionally commitment (StOP), obtained by extending the Nakov (2008) analysis to a trend inflation economy, and estimated TTrend Inflation Taylor Rule (TayR). I do not compare with the Trend Inflation optimal policy under discretion, which I derive in Alves (2014), for it is only compatible with stability and determinacy at very small levels of trend inflation (see Alves (2014) for more details). In the second exercise, I compare impulse responses to negative demand shocks obtained under different policy frameworks.

6.1 Policy evaluation

As for studying the welfare gains, I follow Schmitt-Grohe and Uribe (2007) and Alves (2014) by computing welfare cost rates, in terms of consumption equivalence results, of each optimal monetary policy framework. The analysis is done in terms of assessing the gains from commitment as trend inflation rises from 0 percent to 2 percent.\(^{12}\) paralleling the exercises done by Ascari and Ropele

\(^{12}\)If nominal interest rates were allowed to be negative, optimal monetary policy under unconditionally commitment would fully stabilize the economy under zero trend inflation, and the model would not be disturbed by exogenous

Figure 1: ZLB Probability and Probability-Elasticity
I assess the gains from using the PrOP optimal policy under unconditionally commitment against the alternative StOP optimal policy under unconditionally commitment, considering the welfare cost rate \( \lambda \) of adopting each specific policy framework. In order to simplify the evaluation, I consider the TIWeB loss function to compute the unconditional expected value of the second order log-approximation of the welfare function, under occasionally biding ZLB restrictions, 

\[
E_W t \approx \bar{W} - \frac{1}{2 (1-\beta)} E \tilde{\lambda}_t, \quad \text{where } E \tilde{\lambda}_t = Var(\tilde{\pi}_t - \tilde{\pi}_t^{ind}) + \tilde{\lambda} Var(\tilde{x}_t) + \left[ E (\tilde{\pi}_t - \tilde{\pi}_t^{ind})^2 + \tilde{\lambda} E \tilde{x}_t^2 \right].
\]

Note that the term inside brackets might be relevant as occasionally biding ZLB restrictions induce non-zero values for \( E (\tilde{\pi}_t) \) and \( E \tilde{x}_t \).

The welfare cost rate \( \lambda \) is interpreted as a tax rate that must be applied to the steady state output level \( \bar{Y}_0 \) under the equilibrium with flexible prices (\( \bar{\pi} = 0 \)) in order to the representative household to be indifferent between this equilibrium and a stochastic one with non-zero trend inflation and occasionally binding ZLB constraints over the nominal interest rate:

\[
\frac{1}{(1-\beta)} \left[ u \left( (1-\lambda) \bar{Y}_0 - \bar{v}^0 \right) \right] = E W_t
\]

Tables 2 and 3 report welfare cost rates \( \lambda \), for different optimal policy frameworks and different levels of (annual) real interest rates, \( \bar{r} = 2\% \) and \( \bar{r} = 1\% \), as trend inflation rises from \( \bar{\pi} = 0\% \) to \( \bar{\pi} = 2\% \). The compared policy structures are TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP) and TI Taylor Rule (TayR). For benchmark purposes, the tables also show the outcome in the fictitious economy where the ZLB constraint is not at play.

The tables also compare steady state levels \( \bar{p}_s \) of the probability of hitting the ZLB with average probabilities \( E p_{s,t} \) obtained under different policy rules. Two lessons are learned from the tables: (i) if the ZLB constraint occasionally binds, relative gains from precautionary (PrOP) optimal commitment over standard (StOP) optimal commitment increase as trend inflation rises and the real interest rate falls; (ii) in the fictitious economy where the ZLB constraint is not at play, even though I obtain the expected result that the StOP optimal policy always dominates, the losses from adopting the PrOP optimal policy are negligible; (iii) the PrOP optimal policy delivers larger probabilities of hitting the ZLB, as it finds it optimal to remaining longer at the ZLB even after large negative shocks have dissipated (see Section 6.2); (iv) even though the Taylor Rule delivers much smaller probabilities of hitting the ZLB, its implied losses are much larger than those of both optimal policies.
Table 2 - Gains from Precautionary Optimal Policy at $\bar{r} = 2\%$ ($\beta = 0.995$)

| A) Under ZLB constraints | \begin{tabular}{cccccc}
| Steady States \ $\bar{r}=2\%$ | PrOP Rates (%) | StOP Rates (%) | TayR Rates (%) |
| $\pi$ | $i$ | $p_o$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ |
| 0 & 2 & 21.8 & 0.05 & 32.1 & 0.06 & 31.4 & 0.40 & 0.8 |
| 1 & 3 & 12.1 & 0.54 & 19.5 & 0.55 & 18.3 & 0.84 & 0.0 |
| 2 & 4 & 6.0 & 2.49 & 12.2 & 2.54 & 11.7 & 2.76 & 0.0 |
| B) No ZLB constraints | \begin{tabular}{cccccc}
| Steady States \ $\bar{r}=2\%$ | PrOP Rates (%) | StOP Rates (%) | TayR Rates (%) |
| $\pi$ | $i$ | $p_o$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ |
| 0 & 2 & 21.8 & 0.00 & 26.9 & 0.00 & 28.0 & 0.35 & 0.9 |
| 1 & 3 & 12.1 & 0.48 & 18.9 & 0.48 & 19.7 & 0.84 & 0.0 |
| 2 & 4 & 6.0 & 2.40 & 12.6 & 2.40 & 13.1 & 2.76 & 0.0 |

Note: TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP), TI Taylor Rule (TayR), welfare loss ($\lambda$), trend inflation ($\pi$), steady state annual real interest rate ($i$), steady state annual nominal interest rate ($\bar{r}$), steady state probability of hitting the policy rate ZLB constraint ($p_o$), expected policy-based probability of hitting the policy rate ZLB constraint ($\hat{E}p_{o,t}$).

Table 3 - Gains from Precautionary Optimal Policy at $\bar{r} = 1\%$ ($\beta = 0.9975$)

| A) Under ZLB constraints | \begin{tabular}{cccccc}
| Steady States \ $\bar{r}=1\%$ | PrOP Rates (%) | StOP Rates (%) | TayR Rates (%) |
| $\pi$ | $i$ | $p_o$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ |
| 0 & 1 & 34.8 & 0.12 & 52.3 & 0.13 & 49.9 & 0.96 & 8.9 |
| 1 & 2 & 21.7 & 0.59 & 31.2 & 0.59 & 25.0 & 0.91 & 0.8 |
| 2 & 3 & 12.1 & 2.56 & 18.9 & 2.65 & 19.0 & 2.78 & 0.0 |
| B) No ZLB constraints | \begin{tabular}{cccccc}
| Steady States \ $\bar{r}=1\%$ | PrOP Rates (%) | StOP Rates (%) | TayR Rates (%) |
| $\pi$ | $i$ | $p_o$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ | $\lambda$ | $\hat{E}p_{o,t}$ |
| 0 & 1 & 34.8 & 0.00 & 37.2 & 0.00 & 38.1 & 0.35 & 11.3 |
| 1 & 2 & 21.7 & 0.48 & 26.8 & 0.48 & 28.0 & 0.84 & 0.9 |
| 2 & 3 & 12.1 & 2.42 & 18.9 & 2.42 & 19.7 & 2.78 & 0.0 |

Note: TIWeB precautionary unconditional commitment (PrOP), TIWeB standard unconditional commitment (StOP), TI Taylor Rule (TayR), welfare loss ($\lambda$), trend inflation ($\pi$), steady state annual real interest rate ($i$), steady state annual nominal interest rate ($\bar{r}$), steady state probability of hitting the policy rate ZLB constraint ($p_o$), expected policy-based probability of hitting the policy rate ZLB constraint ($\hat{E}p_{o,t}$).
6.2 Impulse Responses

In order to clearly illustrate the role of a precautionary optimal policy under unconditionally commitment, Figures 2 and 3 depict responses after a one-period ($t = 2$) negative demand innovation shocks, with amplitudes varying from $\epsilon_{u,t} = -(0.5)s_u$ to $\epsilon_{u,t} = -(5.0)s_u$, where again $s_u$ is the estimated standard deviation of the demand shock. In all simulations, I consider $\bar{r} = 1\%$ and trend inflation fixed at $\bar{\pi} = 2\%$. Figures 4 and 5 depict the responses after two-periods negative demand shocks.

At those levels, there are distinct responses differences under the precautionary and standard optimal policy rules. In each exercise, I compare the responses obtained under the Precautionary Optimal Policy (13), Standard Optimal Policy, estimated Trend Inflation Taylor Rule (18) and under the Equilibrium with Flexible Prices without ZLB constraints. In this equilibrium, I assume that the nominal interest rate adjusts in order to keep the nominal interest rate constant at $\bar{\pi} = 2\%$, given the path of the real interest rate, $i_t = R^n_t$.

The figures depict the responses of output $\hat{Y}_t$, annualized inflation rate $\pi_t$, annualized nominal interest rate $i_t$ and the expected probability of hitting the ZLB in the next period $E_t p_{o,t+1}$. Six lessons are learned from the responses: (i) output losses and inflation falls are smaller under precautionary (PrOP) optimal commitment over standard (StOP) optimal commitment and Taylor Rule; (ii) the PrOP policy delays the reduction in the nominal interest rate as the shock hits, making more room for policy effectiveness, and delays even further the nominal rate return to normal levels after the shock dissipates; (iii) in line with the conclusions obtained in the analyses from the last section, the PrOP optimal policy deliver larger probabilities of hitting the ZLB, as it finds it optimal to remaining longer at the ZLB even after the negative shocks have dissipated; (iv) when not bounded, the interest rate response under the PrOP policy tends to mimic that of the nominal interest rate under the equilibrium with flexible prices; (v) even though the Taylor Rule generates very low probability of hitting the ZLB, it generates costs in terms of larger declines in output and inflation when compared to the PrOP optimal policy; (vi) under large enough negative demand shocks, the Taylor Rule starts to dominate the StOP optimal policy.

The second and third lessons characterizes the precautionary nature of PrOP policies. In this exercise, the resulting optimal forward guidance structure depends on the size of the negative shock. For small shocks, which prevents the ZLB to actually bind, both optimal policy frameworks deliver very similar results in terms of output and inflation. Indeed, they virtually succeed in bringing price stability. For larger negative shocks, the differences become very clear, as the precautionary nature of PrOP dominates.
Figure 2: Responses to a one-period negative demand shocks of 0.5 (A) and 1.0 (B) Std. Dev.

Note: $\bar{\tau} = 1$, (A) $\epsilon_{u,t} = -(0.5)\sigma_u$, (B) $\epsilon_{u,t} = -(1.0)\sigma_u$. Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\tau} = 2$ (black dotted)
Figure 3: Responses to a one-period negative demand shock of 1.5 (A) and 5.0 (B) Std. Dev.

Note: $\bar{r} = 1$, (A) $\epsilon_{u,t} = -(1.5)\sigma_u$, (B) $\epsilon_{u,t} = -(5.0)\sigma_u$. Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and $\bar{\pi} = 2$ (black dotted)
Figure 4: Responses to a two-period negative demand shock of 0.5 (A) and 1.5 (B) Std. Dev.

Note: \( \bar{\epsilon} = 1 \), (A) \( \epsilon_{u,t} = -(0.5)s_{u} \), (B) \( \epsilon_{u,t} = -(1.0)s_{u} \). Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and \( \bar{\pi} = 2 \) (black dotted).
Figure 5: Responses to a two-period negative demand shock of 5.0 Std. Dev.

Note: \( \bar{r} = 1, \epsilon_{u,t} = -(5.0)\delta_u \). Stars show when shocks hit. Taylor Rule (black circles), Standard commitment (red dash-dotted), Precautionary commitment (blue line), Equilibrium with Flexible Prices with no ZLB Constraints and \( \bar{\pi} = 2 \) (black dotted)

7 Conclusions

I derive the Precautionary Optimal Policy under unconditionally commitment and occasionally binding zero lower bound constraint on the nominal interest rate, for a standard New Keynesian model with continuously distributed demand and technology shocks. I depart from the literature by directly addressing the kink generated by the ZLB constraint into the expected discounted flow of the Trend Inflation Welfare-based loss function.

Therefore, I show how the central bank internalizes the probability of hitting the zero lower bound prior to optimization in order to adopt a precautionary behavior. I show that the Precautionary Optimal Policy welfare-dominates Standard Optimal Policies that do not account for the kink on the expected loss function, even when derived for occasionally binding ZLB episodes.

Finally, I show that optimal precautionary behavior induces policy gradualism, whose degree optimally increases with the long-run probability of hitting the zero lower bound. Therefore, the response of monetary policy to shocks becomes more sluggish as the long-run nominal rates declines. That is, optimal precautionary responses to negative demand shocks induces a slower reduction in the nominal rate as the shock hits, making more room implementing monetary policy in the future. As the negative shock dissipates, the rate takes longer to return to normal values than what the standard optimal policy prescribes. That is, the Precautionary Optimal Policy prescribes longer forward guidance periods.

References


## A Steady state levels

Tables 2 and 3 define the structural and composite parameters. Table 4 describes the steady state levels under trend inflation.

<table>
<thead>
<tr>
<th>$\sigma \equiv$ reciprocal of intertemp elast substit</th>
<th>$\gamma_\pi \equiv$ coeff lag inf on index rule</th>
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</thead>
<tbody>
<tr>
<td>$\nu \equiv$ reciprocal of the Frisch elasticity</td>
<td>$\varepsilon \equiv$ labor elasticity prod function</td>
</tr>
<tr>
<td>$\chi \equiv$ scale parameter on labor disutility</td>
<td>$\alpha \equiv$ Calvo degree of price rigidity</td>
</tr>
<tr>
<td>$\theta \equiv$ elasticity of substit between goods</td>
<td>$\bar{\pi} \equiv$ level of trend inflation</td>
</tr>
</tbody>
</table>
\[ I \equiv \frac{\theta}{1-\gamma_s} \]
\[ \omega \equiv \frac{1+\omega}{\omega} - 1 \]
\[ \hat{\omega} \equiv \frac{1}{\epsilon} - 1 \]
\[ \delta \equiv \frac{1}{1-\gamma_n} \]
\[ \bar{\alpha} \equiv \alpha (\Pi)^{(\theta-1)(1-\gamma_s)} \]
\[ \beta \equiv (\Pi)^{(1+\theta)(1-\gamma_s)} \]

\[ \tilde{\gamma} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} \]
\[ \frac{\kappa}{\bar{\kappa}} = \left( \tilde{\gamma} \right)^{1+\theta_n} \]
\[ \bar{\kappa} \equiv \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} \tilde{\gamma} \]
\[ \frac{\nu}{\bar{\nu}} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} \tilde{\gamma} \]

\[ \bar{\gamma} \equiv \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} \tilde{\gamma} \]

**Table 3: Composite parameters**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu \equiv \frac{\theta}{1-\gamma_s} )</td>
<td>( \frac{1-(\alpha)(1-\alpha+\beta)\omega}{(1+\theta_n)} )</td>
</tr>
<tr>
<td>( \bar{\mu} \equiv \frac{(1-\alpha)(1-\alpha+\beta)\omega}{(1+\theta_n)} )</td>
<td>( \frac{1-(\alpha)(1-\alpha+\beta)\omega}{(1+\theta_n)} )</td>
</tr>
</tbody>
</table>

**Table 4: Steady state levels**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{I} = \beta^{-1}(\Pi) )</td>
<td>( \beta^{-1}(\Pi) )</td>
</tr>
<tr>
<td>( \tilde{\gamma} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} )</td>
<td>( \tilde{\gamma} )</td>
</tr>
<tr>
<td>( \tilde{\gamma} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} )</td>
<td>( \tilde{\gamma} )</td>
</tr>
<tr>
<td>( \tilde{\gamma} = \left( \frac{1-\alpha}{\alpha} \right)^{\frac{1}{\gamma_n}} )</td>
<td>( \tilde{\gamma} )</td>
</tr>
</tbody>
</table>

**B Deriving the probability of hitting the ZLB**

For the analyses carried out for now on, I assume that preference (demand) and technology shocks follow AR(1) processes, i.e. \( \epsilon_t \equiv \epsilon_{t-1} + \epsilon_{u,t} \) and \( \epsilon_t \equiv \epsilon_{t-1} + \epsilon_{a,t} \), where \( \epsilon_{u,t} \) and \( \epsilon_{a,t} \) are unity-averaged independent white noise disturbance terms.

Assume that \( \epsilon_{u,t} \overset{iid}{\sim} LN(0, \sigma_u^2) \) is independent of \( \epsilon_{a,t} \overset{iid}{\sim} LN(0, \sigma_a^2) \), where \( \sigma_u^2 \) and \( \sigma_a^2 \) are dispersion parameters. It implies that, for any parameters \( \varphi_u \) and \( \varphi_a \), a random variable defined as \( \epsilon_t \overset{iid}{\sim} LN(0, \varphi_u^2 \sigma_u^2 + \varphi_a^2 \sigma_a^2) \), whose expected value is \( E \epsilon_t = \exp \left( \frac{\varphi_u^2 \sigma_u^2 + \varphi_a^2 \sigma_a^2}{2} \right) \). Moreover, recall that the cdf and pdf of any log-normal distributed random variable \( z \sim LN(\mu, \sigma^2) \) is \( F(z) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(z)-\mu}{\sqrt{2\sigma^2}} \right) \right] \) and \( f(z) = \frac{1}{2 \sqrt{2\pi}\sigma^2} \exp \left( -\frac{1}{2} \left( \frac{\log(z)-\mu}{\sigma^2} \right)^2 \right) \).

Let us first consider the equilibrium with flexible prices, described in Section 2.4. Using the Euler equation and the marginal utility definition, note that \( \left( I_t \right)^{-1} = \beta E_t(1-\beta)(\epsilon_{u,t+1} \epsilon_{a,t+1}^2)^{-\sigma} \), where \( I_t^* \) is again the unbounded nominal interest rate under flexible prices, for an economy in which the inflation rate is kept at \( \Pi \). Therefore, the probability \( p_{t+1}^a \) satisfies:

\[ 1 - p_{t+1}^a = P \left( \frac{1}{I_t} < 1 \right) \]

Therefore, the first step in determining \( p_{t+1}^a \) is to compute the expectations term of the last result:
It means that

\[
\frac{1}{T_{t+1}} = \frac{\beta}{\prod} \exp \left( \frac{1}{2} \frac{\sigma^2}{\sigma^2_{ua}} \right) \frac{(A_{t+1})^{(1 - \rho_a)} \theta_2}{(e_{t+1})^{(1 - \rho_a)} \theta_1}
\]

where

\[
\sigma^2_{ua} = \sigma^2 + \sigma^2_{ua} ; \quad \theta_1 = \frac{\omega}{(\omega + \sigma)} ; \quad \theta_2 = \frac{\sigma(1 + \omega)}{(\omega + \sigma)}
\]

Since \( e_{t+1} = \epsilon_{t+1}^a \epsilon_{t+1} + A_{t+1} \), we can rewrite \( \frac{1}{T_{t+1}} \) and its expected value as follows:

\[
\frac{1}{T_{t+1}} = \mathcal{E}_{\epsilon_{t+1}} \left( (\epsilon_{t+1})^{(1 - \rho_a)} \theta_2 \right) ; \quad E_t \left( \frac{1}{T_{t+1}} \right) = \exp \left( \frac{\sigma^2_{ua}}{2} \right) \mathcal{E}_{\epsilon_{t+1}}
\]

where

\[
\mathcal{E}_{\epsilon_{t+1}} = \frac{\beta}{\prod} \exp \left( \frac{1}{2} \frac{\sigma^2}{\sigma^2_{ua}} \right) \left( \epsilon_{t+1} \right)^{-(1 - \rho_a) \theta_1} \left( A_{t+1} \right)^{\rho_a (1 - \rho_a) \theta_2} ; \quad \sigma^2_{ua} = (1 - \rho_a)^2 \sigma_1^2 \sigma_2^2 + (1 - \rho_a)^2 \sigma_2^2 \sigma_2^2
\]

Therefore, the probability \( p^n_{\epsilon_{t+1}} \) can be determined as follows:

\[
p^n_{\epsilon_{t+1}} = 1 - \mathbb{P} \left( \mathcal{E}_{\epsilon_{t+1}} < \theta \right) = 1 - \mathbb{P} \left( \left( \epsilon_{t+1} \right)^{-(1 - \rho_a) \theta_1} \left( A_{t+1} \right)^{\rho_a (1 - \rho_a) \theta_2} > \mathcal{E}_{\epsilon_{t+1}} \theta \right) = \mathbb{P} \left( \left( \epsilon_{t+1} \right)^{-(1 - \rho_a) \theta_1} \left( A_{t+1} \right)^{\rho_a (1 - \rho_a) \theta_2} \leq \mathcal{E}_{\epsilon_{t+1}} \theta \right)
\]

This result implies that \( p^n_{\epsilon_{t+1}} = \mathbb{P}_{ua} \left( \mathcal{E}_{\epsilon_{t+1}} \right) \), where \( \mathcal{E}_{\epsilon_{t+1}} \) is an aggregate shock defined below, whereas \( \mathbb{P}_{ua} (\epsilon_{t+1}) \equiv \mathbb{P} \left( \epsilon_{t+1} \leq \epsilon \right) \) is the cdf of the aggregate disturbance \( \epsilon_{t+1} \overset{iid}{\sim} LN(0, \sigma^2_{ua}) \). It is also convenient to define the hazard rate of hitting the ZLB \( h^n_{\epsilon_{t+1}} = \frac{f_{ua} \left( \mathcal{E}_{\epsilon_{t+1}} \right)}{1 - \mathbb{P}_{ua} \left( \mathcal{E}_{\epsilon_{t+1}} \right)} \), where \( f_{ua} (\epsilon_{t+1}) \) is the density function of \( \epsilon_{t+1} \):

\[
\mathbb{P}_{ua} \left( \epsilon_{t+1} \right) = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(\epsilon_{t+1})}{\sqrt{2} \sigma_{ua}} \right) \right] ; \quad f_{ua} (\epsilon_{t+1}) = \frac{1}{\sigma_{ua} \sqrt{2\pi}} \exp \left( \frac{1}{2} \frac{(\log(\epsilon_{t+1}))^2}{\sigma_{ua}^2} \right)
\]

That is, under the unbounded equilibrium with flexible prices, the probability \( p^n_{\epsilon_{t+1}} \) and hazard rate \( h^n_{\epsilon_{t+1}} \) of hitting the ZLB at period \( t + 1 \) are:

\[
p^n_{\epsilon_{t+1}} = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(\epsilon_{t+1})}{\sqrt{2} \sigma_{ua}} \right) \right] ; \quad h^n_{\epsilon_{t+1}} = \frac{f_{\epsilon_{t+1}}}{1 - \mathbb{P}_{ua} \left( \mathcal{E}_{\epsilon_{t+1}} \right)} ; \quad f_{\epsilon_{t+1}} = \frac{1}{\mathcal{E}_{\epsilon_{t+1} \sqrt{2\pi}} \sigma_{ua}} \exp \left( \frac{1}{2} \frac{(\log(\epsilon_{t+1}))^2}{\sigma_{ua}^2} \right)
\]

At the steady state, with \( \bar{c} = 1 \) and \( \bar{A} = 1 \), I obtain:

\[
p^n = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(\bar{c})}{\sqrt{2} \sigma_{ua}} \right) \right] = \frac{1}{2} \left[ 1 + \text{erf} \left( \frac{\log(\bar{c})}{\sqrt{2} \sigma_{ua}} \right) \right]
\]

and

\[
f^n = \frac{1}{\bar{c} \sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{(\log(\bar{c}))^2}{\sigma_{ua}^2} \right) = \frac{1}{\bar{c} \sqrt{2\pi}} \exp \left( -\frac{1}{2} \frac{(\log(\bar{c}))^2}{\sigma_{ua}^2} \right)
\]
where
\[ \overline{E}_o^n = \frac{\beta}{\Pi} \exp \left( \frac{1}{2} \sigma_{ua}^2 \right) ; \overline{e}_{\text{ua}} = 1 \]

And so, we obtain:
\[ \bar{p}_o^n = \frac{1}{2} \left[ 1 + \operatorname{erf} \left( \frac{-1}{\sqrt{2}} \left( \frac{1-\frac{1}{2} \sigma_{ua}^2}{\sigma_{ua}^2} \right) \right) \right] ; \bar{E}_{n,t} = \frac{\bar{p}_o^n}{(1-\bar{p}_o^n)} ; \bar{I}_0^n = \frac{\bar{I}}{\sqrt{2\pi \sigma_{ua}^2} \exp \left( \frac{1}{2} \left( \frac{1-\frac{1}{2} \sigma_{ua}^2}{\sigma_{ua}^2} \right)^2 + \frac{1}{2} \sigma_{ua}^2 \right)} \]

where \( \bar{I} = \bar{\Pi}/\beta \) and \( \bar{t} \equiv \log (\bar{I}) \).

Based on those results, it is easy to rewrite \( p_{o,t}^n \) as follows:
\[ p_{o,t}^n = \Pi_{ua} \left[ \exp \left( -\frac{\sigma_{ua}^2}{2} \right) E_t \left( \frac{1}{\bar{I}_{t+1}} \right) \right] \]

Under sticky prices and a particular monetary policy framework, there is no closed-form solution for \( p_{o,t} \) as a function of structural disturbances, as it depends on the joint distribution of the expected path of the endogenous variables and the exogenous shocks. For that reason, I derive below an approximated result based on a perturbation component about the unbounded equilibrium with flexible prices.

In this context, let us now consider an exogenous perturbation \( \psi_t \) on the unbounded nominal interest rate under flexible prices, bringing \( I_{t+1}^n \) to \( I_{p,t+1} = \psi_t I_{t+1}^n \). In what follows, I assume that the perturbation \( \psi_t \) belongs to the information set \( \mathcal{I}_t \) at period \( t \) and that its steady state level is unity, i.e. \( \bar{\psi} = 1 \).

For now, \( I_{p,t+1} \) cannot be understood as \( I_{t+1} \). Nonetheless, we can compute the proxy probability \( p_{p,t} \equiv \mathbb{P}(I_{p,t+1} \leq 1 \mid \mathcal{I}_t) \). For now, \( p_{p,t} \) cannot be understood as \( p_{o,t} \). That being said, I can derive the following result:
\[ p_{p,t} = \mathbb{P}(I_{p,t+1} \leq 1 \mid \mathcal{I}_t) = \mathbb{P}(\psi_t I_{t+1}^n \leq 1 \mid \mathcal{I}_t) \]
\[ = \mathbb{P} \left( \frac{(\epsilon_{ua,t+1})(1-\rho_{e})}{(\epsilon_{ua,t+1})(1-\rho_{e})} \leq \mathcal{E}_{p,t}^n \mid \mathcal{I}_t \right) = \Pi_{ua} \left( \mathcal{E}_{p,t}^n \right) \]

where \( \mathcal{E}_{p,t}^n \equiv \frac{\mathcal{E}_{p,t}}{\psi_t} \). Note now that my assumptions imply that \( T_p = T^n \) and that
\[ \psi_t I_{t+1}^n = \frac{1}{T_{p,t+1}} \cdot \psi_t E_t \left( \frac{1}{I_{p,t+1}} \right) = E_t \left( \frac{1}{T_{p,t+1}} \right) \cdot \psi_t = \exp \left( \frac{\sigma_{ua}^2}{2} \right) E_t \left( \frac{1}{T_{p,t+1}} \right) \mathcal{E}_{p,t}^n \]

And so
\[ \mathcal{E}_{p,t} = \exp \left( -\frac{\sigma_{ua}^2}{2} \right) E_t \left( \frac{1}{I_{p,t+1}} \right) ; \mathcal{E}_{p,t}^n = \Pi_{ua} \left( \exp \left( -\frac{\sigma_{ua}^2}{2} \right) E_t \left( \frac{1}{I_{p,t+1}} \right) \right) \]

If we were to approximate \( I_{t+1} \) for \( I_{p,t+1} \), I highlight again that \( p_{p,t} \) would still be a proxy for \( p_{o,t} \). Finally, since \( \bar{\psi} = 1 \), I obtain:
\[ \overline{E}_p = \frac{\overline{E}_o}{\bar{\psi}} = \overline{E}_o \]

C Deriving the optimal precautionary optimal policy

The monetary authority solves the problem described below, written in the Lagrangian form with unconditional expectations applied to the discounted flow of the TIWeB Loss function, where
\[ \hat{\theta}_t = E_t \left( \hat{x}_{t+1} + \frac{1}{\sigma} \hat{\pi}_{t+1} \right) \] and \( p_{e,t} \approx \hat{p}_e + O(1) \). In this problem, let \( \Lambda \) denote the Lagrangian multiplier for the inflation targeting restriction \( E \hat{\pi}_t = 0 \) and \( \lambda_t^\pi \) denote the Kuhn-Tucker multiplier for the ZLB inequality \( \hat{\delta}_t + \hat{i}_t \geq 0 \).

\[
\min_{\{i_t, \hat{\theta}_t, \hat{x}_t, \hat{\pi}_t, \hat{\beta}_t\}} \min_{\{\hat{x}_t, \hat{\pi}_t, \hat{\beta}_t\}} \frac{1}{(1-\beta)} E \left[ \left( \hat{\pi}_t - \hat{\pi}_t^{ind} + \phi_x \right)^2 + (1 - p_{e,t}) \bar{X} \left( \hat{x}_t - \phi_x \right)^2 + p_{e,t} \bar{X} \left( \hat{\theta}_t + \frac{1}{\sigma} i_t + \frac{1}{\sigma} \hat{\pi}_t^{\pi} - \phi_x \right)^2 \right]
\]

\[
+ \frac{1}{(1-\beta)} E \Lambda (\hat{\pi}_t - 0) + \frac{1}{(1-\beta)} E \lambda_t^\pi \left[ \hat{\theta}_t - \hat{x}_{t+1} - \frac{1}{\sigma} \hat{\pi}_{t+1} \right] + \frac{1}{(1-\beta)} E \lambda_t^{\pi \pi} \left[ \hat{\pi}_t^{ind} - \gamma \hat{\pi}_{t-1} \right] + \frac{1}{(1-\beta)} E \lambda_t^{\pi \beta} \left[ \hat{\pi}_t - \hat{\beta} \left( \hat{\pi}_{t+1} - \hat{\pi}_t^{ind} \right) \right] - (1 - \alpha) \hat{\beta} \hat{\pi}_{t+1} \right] + \frac{1}{(1-\beta)} E \Lambda (\hat{\pi}_t - \hat{\pi}_t^{ind}) - \theta (1 + \omega) \left( \hat{\pi}_t - \hat{\pi}_t^{ind} \right) \left( \hat{\alpha} \hat{\beta} \hat{\pi}_{t+1} \right) - \hat{x}_t - (1 - \sigma) (\hat{x}_t - \hat{x}_{t-1}^-) \right] + \frac{1}{(1-\beta)} E (1 - p_{e,t}) \lambda_t^\pi \left[ \hat{x}_t - \hat{\theta}_t + \frac{1}{\sigma} i_t \right]
\]

constant and exogenous terms

In short, the Damjanovic et al. (2008) approach to find optimal policies under unconditionally commitment can be understood as follows. Consider a generic variable \( \hat{\pi}_t \) and a generic Lagrangian multiplier \( \lambda_t^\pi \). Using properties of unconditional expectations on stationary time series, we can substitute \( E \lambda_t^{\pi \pi} \hat{\pi}_t \) for \( E \lambda_t^\pi \hat{\pi}_t \). Likewise, we substitute \( E \lambda_t^{\pi \beta} \hat{\pi}_t \) for \( E \lambda_t^\pi \hat{\pi}_t \). With this strategy, we simplify the Lagrangian problem to set all relevant variables to current time, leaving Lagrangian multipliers at appropriate lags or leads:

\[
\min_{\{i_t, \hat{\theta}_t, \hat{x}_t, \hat{\pi}_t, \hat{\beta}_t\}} \min_{\{\hat{x}_t, \hat{\pi}_t, \hat{\beta}_t\}} \frac{1}{(1-\beta)} E \left[ \left( \hat{\pi}_t - \hat{\pi}_t^{ind} + \phi_x \right)^2 + (1 - p_{e,t}) \bar{X} \left( \hat{x}_t - \phi_x \right)^2 + p_{e,t} \bar{X} \left( \hat{\theta}_t + \frac{1}{\sigma} i_t + \frac{1}{\sigma} \hat{\pi}_t^{\pi} - \phi_x \right)^2 \right]
\]

\[
+ \frac{1}{(1-\beta)} E \Lambda \hat{\pi}_t + \frac{1}{(1-\beta)} E \lambda_t^\pi \left[ \hat{x}_t + \hat{i}_t \right] + \frac{1}{(1-\beta)} E \left[ \lambda_t^\pi \hat{\theta}_t - \lambda_t^{\pi \beta} \hat{x}_t - \frac{1}{\sigma} \lambda_t^{\pi \beta} \hat{\pi}_t \right] + \frac{1}{(1-\beta)} E \lambda_t^{\pi \pi} \left[ \hat{\pi}_t^{ind} - \gamma \lambda_t^{\pi \beta} \hat{\pi}_{t-1} \right] + \frac{1}{(1-\beta)} E \lambda_t^{\pi \beta} \left[ \hat{\pi}_t - \hat{\beta} \left( \hat{\pi}_{t+1} - \hat{\pi}_t^{ind} \right) \right] - \hat{x}_t - (1 - \sigma) (\hat{x}_t - \hat{x}_{t-1}^-) \right] + \frac{1}{(1-\beta)} E \lambda_t^\pi \left[ \lambda_t^{\pi \pi} \hat{x}_t - \hat{\alpha} \hat{\beta} \lambda_t^{\pi \beta} \hat{\pi}_{t+1} \right] \left[ \hat{x}_t - \hat{\pi}_t^{ind} \right] - \lambda_t^{\pi \pi} \hat{x}_t - (1 - \sigma) \left( \lambda_t^{\pi \pi} \hat{x}_t - \lambda_t^{\pi \beta} \hat{x}_{t+1} \right) \right] + \frac{1}{(1-\beta)} E (1 - p_{e,t}) \left[ \lambda_t^\pi \hat{x}_t - \lambda_t^{\pi \pi} \hat{x}_t + \frac{1}{\sigma} \lambda_t^{\pi \pi} \hat{x}_t \right]
\]

constant and exogenous terms

In addition, the whole Lagrangian problem must be of order \( O(2) \), for this is the order to which the welfare function is log-approximated. Since log-linearized equations are used as restrictions, Lagrangian multipliers must be of order \( O(1) \). This order issue is relevant when adding the ZLB Probability curve (8), i.e. first order approximation \( p_{e,t} \approx \hat{p}_e + O(1) \), into the problem. The issue
arises when multiplying this approximation by the second order components from the loss function. We must disregard all $\mathcal{O}(3)$ terms from the resulting multiplication, and so the whole Lagrangian problem is equivalent to the one in which $\bar{\pi}_i$ substitutes $\bar{p}_{\sigma t}$ everywhere. In Alves (2014), I show that the distortion parameters $\bar{\phi}_x$ and $\bar{\phi}_x$ must be of order $\mathcal{O}(1)$ in order for the trend inflation welfare-based loss function to be properly used with log-linearized equations when deriving optimal policy rules. With the same logic, I assume that $\bar{\phi}_L$ is of order $\mathcal{O}(1)$. This assumption is reasonable once we consider that the problem is relevant when the long-run nominal interest rate is low enough.

Therefore, I rewrite the Lagrangian problem as follows:

\[
\min_{\{\pi_t, \pi_t, \pi_t\}} \mathcal{L} E \left[ \frac{1}{2} (\pi_t - \bar{\pi}_t) \left( \frac{\pi_t}{\pi_t} \right)^2 + \frac{1}{2} (\bar{\pi}_t) + \frac{1}{2} (1 - \bar{\pi}_t) \bar{\pi}_t \left( \frac{\pi_t}{\pi_t} \right)^2 \right]
\]

\[
+ \mathcal{L} \left[ \lambda_t^\pi - \phi_x - (1 - \beta L) \lambda_t^\pi + \theta (1 + \omega) \lambda_t^\pi \right] \frac{\pi_t}{\pi_t}
\]

\[
+ \mathcal{L} \left[ \phi_x + \Lambda - \frac{1}{\sigma} \lambda_{t-1}^\pi - \gamma_x \lambda_{t-1}^\pi + (1 - \beta L) \lambda_t^\pi - \theta (1 + \omega) \lambda_t^\pi \right] \frac{\pi_t}{\pi_t}
\]

\[
+ \mathcal{L} \left[ \bar{\pi}_t \bar{\pi}_t \frac{1}{\sigma} \frac{\pi_t}{\pi_t} \right] \frac{\pi_t}{\pi_t}
\]

\[
+ \mathcal{L} \left[ \left[ (1 - \alpha \bar{\pi}_t) \omega + \sigma + (1 - \sigma) (1 - L^{-1}) \right] \lambda_t^\pi \right] \frac{\pi_t}{\pi_t}
\]

\[
+ \mathcal{L} (1 - \bar{\pi}_t) \lambda_t^\pi - \lambda_{t-1}^\pi = \kappa \lambda_{t-1}^\pi \x_t
\]

\[
+ \mathcal{L} \left[ \frac{1}{\sigma} (1 - \bar{\pi}_t) \lambda_t^\pi + \lambda_t^\pi \right] \x_t
\]

\[
+ \text{constant and exogenous terms}
\]

In normal times, $i_t + \bar{\pi}_t > 0$, which requires that $\lambda_t^\pi = 0$. Using the lag operator $L(\cdot)$, the remaining first-order conditions are:

**For $i_t$:**

\[
0 = \frac{1}{\sigma} (1 - \bar{\pi}_t) \lambda_t^\pi
\]

Or:

\[
\lambda_t^\pi = 0
\]

**For $\pi_t$:**

\[
\frac{\pi_t}{\pi_t} = \bar{p}_t \bar{\pi}_t \bar{\pi}_t \frac{1}{\sigma} \frac{\pi_t}{\pi_t} + \bar{p}_t \bar{\pi}_t \left( \frac{1}{\sigma} - \phi_x \right) + \lambda_t^\pi - (1 - \bar{\pi}_t) \lambda_t^\pi
\]

Or:

\[
\lambda_t^\pi = \bar{p}_t \bar{\pi}_t \left( \frac{1}{\sigma} - \phi_x \right)
\]

**For $\pi_t^\pi$:**

\[
0 = - (\bar{\pi}_t - \pi_t^\pi) + \phi_x - \lambda_t^\pi - \lambda_{t-1}^\pi + \theta (1 + \omega) \lambda_t^\pi
\]

Or:

\[
\lambda_t^\pi = (\bar{\pi}_t - \pi_t^\pi) + (1 - \beta L) \lambda_t^\pi - \theta (1 + \omega) \lambda_t^\pi + \phi_x
\]

**For $\pi_t$:**

\[
0 = - (\bar{\pi}_t - \pi_t^\pi) + \phi_x - \lambda_t^\pi - \lambda_{t-1}^\pi - \theta (1 + \omega) \lambda_t^\pi + \phi_x + \Lambda
\]

Or:

\[
\lambda_t^\pi = \kappa \pi_t^\pi \left( \frac{\bar{\pi}_t - \pi_t^\pi}{\bar{\pi}_t - \pi_t^\pi} \right)
\]

**For $\pi_t$:**

\[
0 = (\bar{\pi}_t - \pi_t^\pi) - \lambda_t^\pi - \gamma_x \lambda_{t+1}^\pi - \lambda_t^\pi - \lambda_{t-1}^\pi - \theta (1 + \omega) \lambda_t^\pi + \phi_x + \Lambda
\]

Or:

\[
0 = (1 - \gamma_x L^{-1}) \left[ (1 - \beta L) - \theta (1 + \omega) \kappa \pi_t^\pi \left( \frac{\bar{\pi}_t - \pi_t^\pi}{\bar{\pi}_t - \pi_t^\pi} \right) \right] \lambda_t^\pi
\]

\[
+ (1 - \gamma_x L^{-1}) \left( \pi_t - \pi_t^\pi \right) + \bar{p}_t \bar{\pi}_t \frac{1}{\sigma} \left( \bar{\pi}_t - \pi_t^\pi \right)
\]

\[
+ (1 - \gamma_x) \phi_x + \lambda + \bar{p}_t \bar{\pi}_t \frac{1}{\sigma} \left( \bar{\pi}_t - \phi_x \right)
\]
For \( \tilde{x}_t \): \( 0 = (1 - \tilde{p}_e) \tilde{X} \tilde{x}_t - (1 - \tilde{c}_x) \tilde{X} \tilde{\phi}_x - \lambda_{t-1}^\sigma - \tilde{r} \lambda_t^\sigma - (1 - \tilde{\alpha} \tilde{\beta}) (\omega + \sigma) \lambda_t^\omega - (1 - \sigma) \lambda_t^\sigma + (1 - \tilde{p}_x) \lambda_t^\sigma \)

Or: \( 0 = -\tilde{r} \left\{ 1 + \left[ \left( 1 - \tilde{\alpha} \tilde{\beta} \right) (\omega + \sigma) + (1 - \sigma) (1 - L^{-1}) \right] \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) \right\} \lambda_t^\sigma + (1 - \tilde{p}_x) \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) \) \( \tilde{x}_t + \frac{1}{\sigma} \tilde{r}_t 

Finally, I replace out \( \lambda_t^\sigma \) from the last two first-order conditions and obtain the optimal condition for normal times:

\[
\tilde{c}_1 \equiv 1 - (\tilde{\alpha} - 1) \frac{\tilde{X} \tilde{\phi}_x}{\tilde{X}} (1 - \sigma) ; \quad \tilde{c}_2 \equiv 1 + (\tilde{\alpha} - 1) \frac{\tilde{X} \tilde{\phi}_x}{\tilde{X}} (\omega + \sigma) ; \quad \tilde{c}_3 \equiv \tilde{r} \tilde{c}_1 + (1 - \tilde{\alpha} \tilde{\beta}) \tilde{c}_4 \quad \text{and} \quad \tilde{c}_4 \equiv \tilde{c}_1 - (1 - \tilde{\alpha} \tilde{\beta} \tilde{\beta}) \tilde{c}_2
\]

Therefore, I can simplify the targeting rule for normal times as follows:

\[
\Lambda = \frac{(1 - \gamma_n) \left[ (1 - \beta) - (1 + \omega) \tilde{r} \tilde{c}_1 + (1 - \tilde{\alpha} \tilde{\beta}) \tilde{c}_4 \right]}{(1 - \gamma_n) \left[ (1 - \beta) - (1 + \omega) \tilde{r} \tilde{c}_1 + (1 - \tilde{\alpha} \tilde{\beta}) \tilde{c}_4 \right] + (1 - \tilde{p}_e) \tilde{X} \tilde{\phi}_x - \tilde{p}_e \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) + (1 - \tilde{p}_x) \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) - (1 - \tilde{p}_x) \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) \}
\]

Note now that, using the IS curve, we obtain:

\[
\hat{\tilde{c}}_t + \frac{1}{\sigma} \hat{\tilde{r}}_t = \tilde{E}_t \hat{\tilde{c}}_{t+1} + \frac{1}{\sigma} \tilde{E}_t \hat{\tilde{r}}_{t+1} + \frac{1}{\sigma} \hat{\tilde{r}}_t = \hat{\tilde{x}}_t + \frac{1}{\sigma} \hat{\tilde{r}}_t
\]

The inflation targeting condition \( \dot{\hat{\tilde{p}}}_t = 0 \) implies then that \( \dot{\hat{\tilde{p}}}_t = \dot{\tilde{E}} \hat{\tilde{t}}_t = \dot{\tilde{E}} \hat{\tilde{r}}_t = 0 \). Therefore, plugging this result into the last equation allows me to pin down the inflation targeting Lagrangian multiplier \( \Lambda \) as a function of the remaining composite parameters:

\[
\Lambda = \frac{(1 - \gamma_n) \left[ (1 - \beta) - (1 + \omega) \tilde{r} \tilde{c}_1 + (1 - \tilde{\alpha} \tilde{\beta}) \tilde{c}_4 \right]}{(1 - \gamma_n) \left[ (1 - \beta) - (1 + \omega) \tilde{r} \tilde{c}_1 + (1 - \tilde{\alpha} \tilde{\beta}) \tilde{c}_4 \right] + (1 - \tilde{p}_e) \tilde{X} \tilde{\phi}_x - \tilde{p}_e \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) + (1 - \tilde{p}_x) \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) - (1 - \tilde{p}_x) \tilde{X} \tilde{\phi}_x + \tilde{p}_x \tilde{X} \left( \frac{1}{\sigma} \tilde{\phi}_x \right) \}
\]
where

\[ \dot{\chi}_t \equiv \left[ \frac{\gamma}{\alpha^2} - \left( 1 - \alpha \sigma_\alpha \right) L \right] \dot{x}_{t-1} = \frac{\gamma}{\alpha^2} \dot{\chi}_{t-1} + \frac{\gamma}{\alpha^2} \dot{x}_{t-1} + \alpha \left( 1 - \alpha \sigma_\alpha \right) \dot{x}_{t-2} \]

\[ \hat{\chi}_{1,t} \equiv \frac{(\dot{x}_{t-1} + \frac{1}{\alpha} \dot{u}_{t-1})}{(1 - \gamma u_{t-1})} = \gamma E_t \hat{\chi}_{t+1} + (\dot{x}_{t-1} + \frac{1}{\alpha} \dot{u}_{t-1}) \]

\[ \hat{\chi}_{2,t} \equiv \frac{1}{\alpha^2} \frac{\left( 1 - \beta \left( 1 - \alpha \sigma_\alpha \right) \right)}{(1 - \gamma u_{t-2})} \left( \dot{x}_{t-1} + \frac{1}{\alpha} \dot{u}_{t-1} \right) \]

\[ = \frac{\gamma}{\alpha} \hat{\chi}_{2,t-1} + \frac{1}{\alpha} \left( \dot{x}_{t-1} + \frac{1}{\alpha} \dot{u}_{t-1} \right) - \frac{1}{\alpha} \left[ \left( 1 + \alpha \sigma_\alpha \right) \beta + \kappa \theta (\epsilon_2 - \epsilon_1) \right] \left( \dot{x}_{t-2} + \frac{1}{\alpha} \dot{u}_{t-2} \right) + \frac{1}{\alpha} \alpha \sigma_\alpha \beta^2 \left( \dot{x}_{t-3} + \frac{1}{\alpha} \dot{u}_{t-3} \right) \]