Sovereign Debt Maturity Structure of a Small Open Economy: Short or Long?

by Sergii Kiiashko∗†

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Abstract

I study the sovereign debt maturity structure of a small open economy in a model with opportunity to default by the government, stochastic interest rates, and risk of self-fulfilling debt crisis. If default premiums are perfectly foreseen and roll-over crisis is impossible, the optimal debt policy is to issue only one-period debt. Short-term debt disciplines the future governments not to over borrow compared to ex ante optimal allocations because, otherwise, the sovereign has an incentive to dilute the value of long-term debt ex post. Longer maturity structure hedges the government against unpredictable swings in interest rates, smooths consumption over states of the world and minimizes probability of default due to self-fulfilling debt crisis. I find that uncertainty regarding future interest rates is a weak incentive to issue long-term debt if government lacks commitment and can dilute the value of long-term debt. However, adding risk of self-fulfilling debt crisis significantly increases share of long-term debt.

1 Introduction

There are two major motives to use maturity of sovereign debt: the incentive motive and hedging motive. The first motive implies that if there is a lack of commitment then maturity can discipline future governments to pursue ex ante optimal policies. The second motive implies that in absence of state-contingent bonds different maturities of debt can be used to complete the markets. However, when studied in isolation these two motives can lead to very different implications of the optimal maturity structure. According to the literature1, a small-open economy which lacks commitment to repay its debt should issue short-term debt to minimize the debt dilution problem. On the other hand, long-term debt can be used to hedge economy against exogenous shocks in interest rates and to smooth consumption over different states of the world. In addition, long-term debt minimizes

∗Kyiv School of Economics and National Bank of Ukraine
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1see, for example, Aguiar et al. (2018), Arellano and Ramanayanan (2012), Fernandez and Martin (2015) etc.
the probability of a roll-over crisis. The question is how a small open economy should structure the maturity of its debt if both problems are present and, particularly, what problem is quantitatively more important: the problem of lack of commitment or the problem of lack of insurance?

This question is answered in a recent paper by Debortoli et al. (2017) but for a closed economy with endogenous risk-free interest rates and no default. They find that the maturity of government debt should structure its maturity so that maturity minimizes the costs of lack of commitment, i.e., the future governments do not have an incentive to distort risk-free interest rates ex post, while problem of lack of insurance is quantitatively much less important. Following Lucas and Stokey (1983) we know that to resolve the problem of lack of commitment a government issues approximately flat maturity structure. However, consumption smoothing in absence of state-contingent bonds require very large and tilted positions of short-term and long-term debt, for example, around -10000% of GDP of short-term debt and 5000% of GDP of long-term debt. Not surprisingly that given such large positions of debt a government has a huge incentive to distort risk-free interest rates. In other words, marginal deviations from flat maturity structure (or whatever maturity structure that minimizes the problem of lack of commitment) have extremely small benefit of hedging compared to costs of distortion of interest rates ex post.

In contrast to a model with endogenous risk-free interest rates and no default, a small open-economy model (with exogenous risk-free interest rates but endogenous default risk) does not require such large and tilted positions of debt to achieve full hedging. A three-period example in this paper shows that while the problem of lack of commitment can be resolved with only one-period debt, the hedging problem can be resolved using 50% of one-period debt and 50% of long-term debt. Therefore, it is not clear whether the conclusion of Debortoli et al (2017) that minimizing the costs of commitment is quantitatively more important than the costs of insurance applies to a small-open economy model as well.

I consider a classical small-open economy model with exogenous risk-free interest rates and opportunity to default on sovereign debt. Economy is endowed with a deterministic but not constant stream of output. Sovereign can borrow from foreign risk-neutral lenders issuing bonds with different maturities. The default is modeled similar to Aguiar et al. (2018). In the beginning of a period, a sovereign has an option to default on its debt and to receive an outside option. The value of outside option accounts for potential costs of default, it is stochastic and drawn from a continuous distribution. The latter is very important for the incentive motive: a marginal increase in government debt marginally increases default risk, thus, maturity matters for the debt-dilution problem. However, in contrast to Aguiar et al. (2018) I assume that the value of outside option can be driven from two different distributions which correspond to two different states of the world, one of which is associated with higher default risk. I call it the “crisis” state of the world and the other state is the “normal” state of the world. The government learns the state of the world at date

\[ \text{see Buera and Nicolini (2004) for numerical examples of Angeletos (2002) model} \]
1 but does not know it initially. In addition, I assume a possibility of self-fulfilling debt crisis: the sovereign can opt to default if financial markets are closed. Thus, I introduce the hedging motive for using maturity.

There are several contributions of this paper. First, I develop an algorithm of numerical solution of the model using bicubic interpolation approach. The benefit of bicubic interpolation compared to discretization approach is that instead of a step function the price function remains continuously differentiable with respect to both state variables (short-term debt and long-term debt). Therefore, to solve for the optimal solution we can use first-order conditions instead of iteration on a grid that increases preciseness and saves time significantly.

Second, I solve the model numerically and demonstrate that in an environment with lack of commitment the sovereign issues long-term debt to minimize the risk of self-fulfilling crisis. However, an incentive to use maturity structure to hedge against uncertainty in future interest rates is not sufficient to issue significant share of long-term debt.

Finally, I demonstrate that standard numerical methods that approximate continuous distribution with a discrete distribution can lead to biased results exaggerating the role of long-term debt. To show it I solve a three-period version of Aguiar et al. (2018) model. The analytical solution of this model implies that the government issues only short-term debt. However, if continuous distribution is approximated with a discrete, then numerical solution implies issuance of positive and significant amount of long-term debt contradicting the analytical solution. As the number of possible values of discrete distribution goes up the bias decreases, however, in practice distributions are approximated with a small number of points\(^3\) which questions the accuracy of the numerical solutions.

The structure of the paper is as follows. Section 2 describes the model. In Section 3 I consider three benchmark models which can be solved numerically: a model with state-contingent bonds, a model with one state of the world, and a model with a discrete distribution. Section 4 explains the numerical algorithm and provides numerical solution to the problem. Section 5 discusses limitations of the standard numerical approaches and potential bias in the solution. The last section concludes.

## 2 A Three-Period Model

Consider a small open-economy. A sovereign makes all decisions on a behalf of a representative household who values private consumption. It is a three-period version of Aguiar et al. (2018) model with two extensions. First, the risk-premium varies over states of the world for the same level of debt and its maturity structure. Second, I allow for a self-fulfilling debt crisis.

Every period economy is endowed with \(y_t\) units of a single tradable good, \(t = 0, 1, 2\). The stream of endowment \(\{y_t\}_{t=0}^2\) is deterministic, however, \(y_t\) can be different in different periods. The

\(^3\)For example, Arellano and Ramanayanan (2012) discretize continuous distribution with only 6 possible values
sovereign can reallocate consumption across time periods only through borrowing and lending on international financial markets. Particularly, I assume that \( y_0 < y_1 = y_2 = y \) so that the sovereign has an incentive to issue some stock of debt at date 0 to smooth consumption over time.

The key friction of the model is that the sovereign lacks commitment and can default on its debt. I model default as in Aguiar et al. (2018), i.e., at period 1 and 2 the sovereign has an outside option which can be achieved upon default. The sovereign’s utility function is

\[
u(c_0) + \beta \mathbb{E} \max \left\{ \nu(c_1) + \beta \mathbb{E} \max \left\{ \nu(c_2), V_{2}^{def} \right\}, V_{1}^{def} \right\}
\]

where \( \beta \in (0, 1) \) is a discount factor, \( c_t \) is consumption at period \( t \), \( \nu() \) is a strictly concave, continuously differentiable function, and \( V_t^{def} \) is the value of outside option if sovereign defaults at date \( t \).

The international lenders are risk-neutral and risk-free interest rate is \( R = \frac{1}{\beta} \). Sovereign can issue discount bonds with different maturities. Let denote by \( b_{t+k}^{t} \) the amount of discount bonds issued at period \( t \) maturing at period \( t+k \) and let \( q_{t+k}^{t} \) be the corresponding bond price. Suppose that the initial debt is zero. Thus, at date 0 government issues \( b_1^0 \) of a one-period (or short-term) debt and \( b_2^0 \) of two-period (or long-term) debt. The total market value of debt issued at \( t = 0 \) is \( q_1^0 b_1^0 + q_2^0 b_2^0 \). At the intermediate date the sovereign pays \( b_1^0 \) and issues \( q_1^1 (b_1^1 - b_2^0) \). At the beginning of the last period, conditional on no default, the sovereign pays \( b_2^1 \) to the international lenders.

Similarly to Aguiar et al. (2018), I assume that the outside option is stochastic and continuously distributed. Particularly, let \( V_t^{def} \) is drawn from distribution \( F_t \). However, in contrast to Aguiar et al. (2018), I assume that \( F_t \) is not deterministic. More specifically, suppose that \( F_2 \in \{ F_2^N, F_2^C \} \), where ”\( N \)” stands for a “normal” state of the world which corresponds to relatively lower default risk, and ”\( C \)” stands for a “crisis”, riskier state of the world. At date 0 the government does not know the distribution of \( V_2^{def} \) but the sovereign learns the distribution at period 1. Without loss of generality, I assume that \( F_1 \) is deterministic and known at \( t = 0 \). Important features of the outside option are summarized in Assumption 1.

**Assumption 1. Outside Option:**

(i) \( V_t^{def} \in [v_t^{min}(F_t), v_t^{max}(F_t)] \);

(ii) \( v_2^{max}(F_2) \leq u(y), v_1^{max} \leq \frac{u(y)}{1+\beta} \);

(iii) \( f(v^{min}(F_1)) = f(v^{max}(F_1)) = 0 \);

(iv) \( \text{Prob}_0(F_2 = F_2^C) = p; \)

(v) \( F_2^C(v) \leq F_2^N(v) \forall v \in \mathbb{R} \).

First assumption implies that any \( F_t \) has a bounded support. Assumption (ii) states that an upper bound of outside option is always below the value achieved by consuming endowment.
only ensuring that it is never optimal to choose the outside option if there is no debt. The third assumption states that the density of the boundaries equals zero. This assumption is made to avoid kinks in value, pricing and policy functions. The fourth assumption implies that at period 0 the probability of crisis distribution of outside option is \( p \) and, hence, the probability of normal distribution is \( 1 - p \). The last assumption implies that default is weakly more likely in the crisis state of the world.

Such structure of outside option is isomorphic to assuming that default is costly, i.e., in case of default the sovereign does not repay its debt back, but there is an endowment loss associated with foreign lenders sanctions or other type of punishment. For example, a popular assumption is that upon default economy loses a fraction \( \chi \) of its output and, hence, the outside option at \( t = 2 \) equals to \( V_{2}^{def} = u(y \cdot (1 - \chi)) \). Assuming that \( \chi \) is continuously distributed and has a bounded support is equivalent to the assumption that \( V_{2}^{def} \) is continuously distributed with a bounded support. In addition to default risk due to higher value of outside option, there is a possibility of self-fulfilling debt crisis as in Cole and Kehoe (2000).

**Bellman Equations and Break-Even Conditions**

Suppose that the sovereign does not default at the beginning of period 2, then its value is simply \( V_{2}(b_{2}^{1}) = u(y - b_{2}^{1}) \) where \( s = \{C, N\} \) is the state of the world at date 2 known at \( t = 1 \).

At date 1, conditional on no default and no roll-over crisis, the sovereign chooses consumption \( c_{1} \) and the stock of debt to be paid in the next period \( b_{1}^{2} \) to solve

\[
V_{1}(b_{0}^{1}, b_{0}^{2}, s) = \max_{c_{1}, b_{1}^{2}} u(c_{1}) + \beta \mathbb{E} \max\{V_{2}(b_{1}^{2}), V_{2}^{def}(s)\} \tag{2}
\]

subject to

\[
c_{1} = y - b_{0}^{1} + q_{1}^{2}(b_{1}^{2}, s) \cdot (b_{1}^{2} - b_{0}^{2}) \tag{3}
\]

where

\[
\mathbb{E} \max\{V_{2}(b_{1}^{2}), V_{2}^{def}(s)\} = \int_{v_{2}^{\max}(s)}^{v_{2}^{\min}(s)} \max\{u(y - b_{1}^{2}), v\} \, dF_{2}^{s}(v) \tag{4}
\]

is the expected sovereign’s value in the last period which incorporates an opportunity to choose an outside option with stochastic value. As the lenders are risk-neutral, bond prices depend on exogenous risk-free interest rate and default risk:

\[
q_{1}^{2}(b_{1}^{2}, s) = \frac{1}{R} \cdot \text{Prob}(V_{2}(b_{1}^{2}) > V_{2}^{def} | F_{2} = F_{2}^{s}) = \frac{1}{R} \cdot F_{2}^{s}(u(y - b_{1}^{2})) \tag{5}
\]
In an event of roll-over crisis when the sovereign cannot issue new debt but is obligated to repay maturing debt, the sovereign’s problem is

\[
V_1(b_0^1, b_0^2, s) = \max_{c_1, b_1^2} u(c_1) + \beta \mathbb{E} \max\{V_2(b_1^2), V_2^{def}(s)\}
\]

s.t. (3) and \( b_1^2 \leq b_0^2 \)

At date 0 the sovereign makes consumption and debt decisions to solve

\[
V_0 = \max_{c_0, b_0^1, b_0^2} u(c_0) + \beta W_1(b_0^1, b_0^2)
\]

s.t. \( c_0 = y_0 + q_0^1(b_0^1, b_0^2) \cdot b_0^1 + q_0^2(b_0^1, b_0^2) \cdot b_0^2 \)

where

\[
W_1(b_0^1, b_0^2) = \sum_s p_s \cdot \left\{ V_1(b_0^1, b_0^2, s) \cdot \left( 1 - \pi_1(b_0^1, b_0^2, s) \right) + \lambda \int \frac{V_1(b_0^1, b_0^2, s)}{V_1(b_0^1, b_0^2, s)} v F_1(v) + \int \frac{v^{max}}{V_1(b_0^1, b_0^2, s)} v F_1(v) \right\}
\]

is the expected value of the sovereign next period. It consists of four important elements. The first term in the brackets reflects the sovereign’s value \( V_1(b_0^1, b_0^2, s) \) if there is no default. The probability of no default is \( 1 - \pi_1 \) and discussed below. If \( V_1(b_0^1, b_0^2, s) < V_1^{def} < V_1(b_0^1, b_0^2, s) \) then with conditional probability \( \lambda \) there is a chance of roll-over crisis. The second term in the brackets corresponds to the expected value of outside option in case of the crisis. The last element in the brackets is the expected value of outside option if it exceeds the value of paying debt even if there is no roll-over crisis. Finally, \( p_s \) corresponds to probability of crisis or normal states of the world realized in the beginning of period 1.

The probability of paying debt consists of two parts: the sovereign does not default if its value is greater than the value of outside option and self-fulfilling debt crisis does not occur:

\[
1 - \pi_1(b_0^1, b_0^2, s) = F_1(\frac{V_1(b_0^1, b_0^2, s)}{V_1(b_0^1, b_0^2, s)}) - \lambda \cdot [F_1(\frac{V_1(b_0^1, b_0^2, s)}{V_1(b_0^1, b_0^2, s)}) - F_1(\frac{V_1(b_0^1, b_0^2, s)}{V_1(b_0^1, b_0^2, s)})]
\]

The price of a short-term bond incorporates risk-free interest rate and probability of default in period 1. The price of long-term debt depends also on the long-term default risk:

\[
q_0^1(b_0^1, b_0^2) = \frac{1}{R} \mathbb{E} \left( 1 - \pi_1(b_0^1, b_0^2, s) \right) = \frac{1}{R} \sum_s p_s \left( 1 - \pi_1(b_0^1, b_0^2, s) \right)
\]

\[
q_0^2(b_0^1, b_0^2) = \frac{1}{R} \cdot \mathbb{E} \left[ (1 - \pi_1(b_0^1, b_0^2, s)) \cdot q_1^2(b_1^2, s) \right] =
\]
\[ \frac{1}{R^2} \sum_s p_s \cdot \left[ \left( 1 - \pi_1(b_1^1, b_0^2, s) \right) \cdot F_s^2 \left( u(y - b_1^2(b_1^1, b_0^2, s)) \right) \right] \quad (12) \]

where \( b_1^2(b_1^1, b_0^2, s) \) is the optimal debt policy at \( t = 1 \) as a function of outstanding debt and state of the world.

The key object of the analysis is the maturity structure \((b_1^1, b_0^2)\). I will first proceed with the benchmark models for which we can derive analytical solutions to discuss the so called “incentive” and “hedging” motives of maturity.

3 Benchmark Models

In this section I consider three benchmark models which allow me to analytically derive the optimal maturity structure of sovereign debt when studying debt-dilution problem, uncertainty about default risk, and self-fulfilling debt crisis in isolation. A sovereign issues only short-term debt to resolve the first problem and longer-term debt to resolve the latter two problems.

3.1 A Model with Incentive Motive

Consider problem (7) without uncertainty about the distribution of the value of outside option in the last period and without self-fulfilling debt crisis in the intermediate period. More specifically, assume that \( F_2^C = F_2^N = F_2 \) and \( \lambda = 0 \). The model is a simplified version of Aguiar et al. (2018).

Proposition 1 states that the optimal debt policy is to issue one-period debt only if default risk in period 2 is positive.

**Proposition 1. Incentive Motive.** Suppose that \( F_2^C = F_2^N = F_2 \) and \( \lambda = 0 \). If default risk in the last period is positive then \( b_0^2 = 0 \).

The proof of Proposition 1 is in Appendix. The problem is characterized by considering a planning problem in which government can commit to debt policies in the future, however, cannot promise to pay debt if sequential default decision is optimal. The planning problem provides an upper bound for (7) which can be achieved only if sovereign issues no long-term debt.

The key friction is time-consistency problem. When making decision about \( b_1^1 \) in period 1, the sovereign faces a trade-off between a marginal decrease in the expected value in the last period and utility from extra consumption today due to marginal increase in the market value of newly issued debt \((b_1^1 - b_0^1)\). However, from government in period 0 perspective, marginal increase in \( b_1^1 \) increases the market value of the whole stock of long-term debt \( b_0^2 \). Thus, relative to the government in period 0, the government in period 1 does not internalize how an increase in default risk affects
the value of long-term debt issued in the initial period. In period 0, though, the government can enforce the future government to conduct ex-ante optimal policies by issuing only short-term debt.

3.2 A Model with Uncertainty about Default Risk

To understand the role of maturity structure when sovereign faces uncertainty about default risk, assume again that there are two states of the world, but instead of a continuous distribution consider the following discrete distribution. Suppose in each state of the world there are only two possible values of outside option in the last period and they are the same for all states of the world: 

\[ v_{2 \text{max}}^{(C)} = v_{2 \text{max}}^{(N)} = v^{\text{max}} \equiv u(y) \] 

and 

\[ v_{2 \text{min}}^{(C)} = v_{2 \text{min}}^{(N)} = v^{\text{min}} \equiv \lim_{c \to 0} u(c). \] 

Thus, \( v_{2 \text{max}} \) and \( v_{2 \text{min}} \) imply that the default is either absolutely cost less and, thus, sovereign defaults on any positive debt in period 2, or default is infinitely costly and any feasible amount of debt will be repaid. However, the states of the world are different in the probability of the outcomes. Let \( p_s \) be the probability of \( V_{2 \text{def}} = v_{2 \text{max}} \) if state of the world is \( s \). Then \( p^C > p^N \) meaning that there is a higher chance of default at the crisis state of the world. Without loss of generality, let assume that 

\[ V_{1 \text{def}} = \lim_{c \to 0} u(c) + \beta u(c), \] 

i.e., it is never optimal to default if sovereign can repay outstanding debt.

Similarly to section 3.1, consider the modified commitment problem in which the sovereign can commit to future consumption decisions but cannot commit to repay debt if default is optimal:

\[
\max_{c_0, c_1(s), c_2(s)} \quad u(c_0) + \beta \lambda \left( u(c_1(C)) + \beta (1 - p^C) \cdot u(c_2(C)) + \beta p^C \max \{ u(c_2(C)); v^{\text{max}} \} \right) \\
+ \beta (1 - \lambda) \left( u(c_1(N)) + \beta (1 - p^N) \cdot u(c_2(N)) + \beta p^N \max \{ u(c_2(N)); v^{\text{max}} \} \right) \\
\text{s.t. } c_0 + \frac{\lambda}{R} \left( c_1(C) + \frac{1}{R} \cdot \frac{1 - p^C}{c_2(C)} \right) + \frac{1 - \lambda}{R} \left( c_1(N) + \frac{1 - p^N}{R} \cdot c_2(N) \right) = \\
\leq y_0 + \frac{y}{R} + \left( \lambda (1 - p^C) + (1 - \lambda) (1 - p^N) \right) \cdot \frac{y}{R^2} + \\
+ \frac{\lambda p^C}{R^2} \mathbb{1}_{\{c_2(C) \geq y\}} \cdot (y - c_2(C)) + \frac{(1 - \lambda) p^N}{R^2} \mathbb{1}_{\{c_2(N) \geq y\}} \cdot (y - c_2(N)) \]

where \( \mathbb{1}_{\{c \geq y\}} \) is the indicator function that equals 1 if \( c \geq y \) and 0 otherwise.

Lemma A.1 in Appendix proves that if \( y_0 < y = y \) the government always opts to be in the crisis region. The reason is that by escaping crisis region in any state of the world \( s \), i.e., by setting \( c_2(s) = y \), an increase in present value of income is compensated by an increase in present value of consumption. Therefore, there is no benefit of decreasing default risk. Instead, the government distorts consumption in different states of the world that strictly reduces its value. The best
allocation is to smooth consumption over time and states of the world.

**Lemma 1. Optimal Allocation of MCP with discrete distribution.** \( c_0 = c_1(s) = c_2(s) = c \).

Proposition 2 states that the planner’s allocation can be implemented with flat maturity structure.

**Proposition 2. Hedging Motive of Maturity Structure.** Suppose that \( F_s^2 \) is a discrete distribution with two outcomes: \( v_{2_s}^{\text{max}} = u(y) \) and \( v_{2_s}^{\text{min}} = \lim_{c \to 0} u(c) \) \( \forall s \), but different probabilities of those outcomes, and \( V_{1_s}^{\text{def}} = \lim_{c \to 0} u(c) + \beta u(c) \). Then the optimal maturity structure is unique and flat:

\[
b_1^1 = b_0^2
\]

If marginal increase in \( b \) does not change the default risk, i.e., \( f(v) = 0 \) for \( v \in (v_{\text{min}}; v_{\text{max}}) \), then the optimality conditions for both the planner and the Markov government are identical and does not depend on maturity structure of debt. In other words, maturity structure is not used as a disciplining device since the intermediate period government has no incentive to dilute the value of outstanding long-term debt. However, the maturity still enters the intermediate period budget constraints and the perfect consumption smoothing is possible only if it is structured properly. The budget constraint at \( t = 1 \) can be rewritten as follows:

\[
c_1(s) = y_1 - b_0^1 + \frac{1 - p^s}{R} \left( y - c_2(s) - b_0^2 \right)
\]

The consumption smoothing \( (c_1(s) = c_2(s) = c) \) then implies that \( b_0^1 = \frac{1 - p^s}{R} \left( (y - c) - b_0^2 \right) = y - c \) for any \( p^s \in [0, 1] \) leading to the conclusion that \( b_0^1 = b_0^2 = y - c \).

Thus, the flat maturity structure allows to hedge the sovereign against any changes in interest rates. Notice that this result can be extended to a model with multiple states of the world because under flat maturity structure the sovereign does not roll over any short-term debt in the intermediate period so the interest rate is basically irrelevant.

### 3.3 A Model with Self-Fulfilling Debt Crisis

Finally, consider a model with self-fulfilling debt crisis only. Suppose that \( F_1 \) is discrete with two outcomes: \( v_1^{\text{max}} = (1 + \beta)u(y) \) and \( v_1^{\text{min}} < v_1^{\text{max}} \). The probability of the first outcome is \( p_1 \). If outstanding debt is positive, the government always defaults if \( V_1^{\text{def}} = v_1^{\text{max}} \). In addition, if \( V_1^{\text{def}} = v_1^{\text{max}} \),
$v_{1}^{min}$ and intermediate government value $V_1(b_0^1, b_0^2) \geq v_{1}^{min}$ but the value without opportunity to roll-over short-term debt $V_1(b_0^1, b_0^2) < v_{1}^{min}$ there is additional probability of default due to self-fulfilling debt crisis. Without loss of generality suppose that default never occurs in the last period, i.e., $V_2^{def} = \lim_{c \to 0} u(c)$.

The problem of the sovereign in the initial period is

$$V_0 = \max_{c_0, b_0^1, b_0^2} u(c) + \beta \left( \pi_1^{min} v_{1}^{min} + \pi_1^{max} v_{1}^{max} + (1 - \pi_1^{min} - \pi_1^{max}) V_1(b_0^1, b_0^2) \right)$$

$$\text{s.t. } c = y_0 + \frac{1}{R}(1 - \pi_1^{min} - \pi_1^{max}) \cdot \left( b_0^1 + \frac{1}{R} b_0^2 \right)$$

where $\pi_1^{min}$ and $\pi_1^{max}$ denote the probability that sovereign will choose the outside option with value $v_{1}^{min}$ and $v_{1}^{max}$ correspondingly:

$$\pi_1^{min} = (1 - p_1) \mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{min}\}} + (1 - p_1) \lambda \mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{min} \leq V_1(b_0^1, b_0^2)\}}$$

$$\pi_1^{max} = p_1 \mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{max}\}} + p_1 \lambda \mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{max} \leq V_1(b_0^1, b_0^2)\}}$$

The second term of $\pi_1^{min}$ and $\pi_1^{max}$ correspond to the probability of self-fulfilling debt crisis. Even if default decision is not optimal, there is a chance of default if $V_1(b_0^1, b_0^2) < V_1^{def}$. Lemma 2 states that $V_0$ is decreasing in $\lambda$ and strictly decreasing if a self-fulfilling debt crisis can occur with positive probability.

**Lemma 2.** Consider problem (14). Then $\frac{\partial V_0}{\partial \lambda} \leq 0$ with $\frac{\partial V_0}{\partial \lambda} < 0$ if $\mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{min} \leq V_1(b_0^1, b_0^2)\}} = 1$ or $\mathbb{I}_{\{V_1(b_0^1, b_0^2) < v_{1}^{max} \leq V_1(b_0^1, b_0^2)\}} = 1$.

To understand the intuition, fix $b_0^1$ and $b_0^2$. By decreasing risk of self-fulfilling debt crisis government can increase both present consumption because the price of issued discount bonds go up, and the expected value of the government next period as it increases probability of $V_1(b_0^1, b_0^2)$ and decreases probability of receiving outside option $V_1^{def} < V_1(b_0^1, b_0^2)$. The next proposition establishes that issuing long-term debt only is always a (weakly) optimal decision.

**Proposition 3. Self-Fulfilling Debt Crisis.** Consider problem (14). Then $b_0^1 \leq \frac{1}{R} b_0^2$ is an optimal strategy.

Observe that conditional on no roll-over crisis in period 1, maturity structure of debt is irrelevant for the sovereign in the intermediate period and $V_1^{def}(b_0^1, b_0^2)$ stays constant as long as $b_0^1 + \frac{1}{R} b_0^2$ is constant. However, maturity structure is relevant for sovereign’s value in case of self-fulfilling debt.
crisis. Given assumption that default never occurs in the last period, the sovereign tends to smooth consumption over time. If the stock of short-term debt is large, however, consumption smoothing is impossible if government cannot roll-over its short-term debt. If the stock of short-term debt is low enough, then \( V_1(b_0^1, b_0^2) = V_1(b_0^1, b_0^2) \) so that adverse effect of possible self-fulfilling debt crisis is completely eliminated.

**Discussion**

To conclude, the implications for the optimal maturity structure depend on the assumptions of the model. To resolve time-consistency problem in isolation the optimal debt policy is to issue only one-period debt. However, in order to insure itself against default premium risk and minimize possibility of self-fulfilling debt crisis, the sovereign issues longer debt. This raises question about optimal maturity structure in environment where all three features are present.

4 Optimal Maturity Structure: Numerical Exercises

In order to better understand the optimal maturity structure in a general model I consider a number of numerical exercises. In this section I explain the choice of parameters and functional forms and numerical algorithm which is different from what is common in the literature.

4.1 Functional Forms and Parameters

I assume CRRA per-period utility function:

\[
    u(c) = \frac{c^{1-\gamma}}{1-\gamma}
\]

where \( \gamma \) is the relative risk aversion coefficient and I set \( \gamma = 2 \). Output at period 1 and 2 is normalized to one: \( y_1 = y_2 = y = 1 \). For simplicity, \( \beta = R = 1 \). At date 0, the probabilities of a crisis state and a normal state of the world are equal, i.e., \( \lambda = 0.5 \).

The distribution density function of the value of outside option at period is a polynomial of order 4 which satisfies:

\[
    f_s^2(v) = \begin{cases} 
    a_2(s) \cdot (v - v_{2, \min}^2(s))^2 \cdot (v - v_{2, \max}^2(s))^2 & \text{if } v \in (v_{2, \min}^2(s), v_{2, \max}^2(s)) \\
    0 & \text{otherwise}
    \end{cases}
\]

(16)

where \( a(s) \) satisfies \( \int_{v_{2, \min}^2(s)}^{v_{2, \max}^2(s)} a(s) \cdot (v - v_{2, \min}^2(s))^2 \cdot (v - v_{2, \max}^2(s))^2 \, dv = 1 \).

I implicitly assume that default is equivalent to loosing a fraction of output. I set \( v_{2, \max}^2(C) = v_{2, \max}^2(N) = u(y) \) meaning that for any positive debt at the beginning of date 2 the probability of
default is positive as well. In the benchmark case, I set $v_2^{min}(C) = u(0.75y)$ and $v_2^{max}(N) = u(0.4y)$. The probability and cumulative density functions are presented on Figure 1.

![Figure 1. Density Functions of Outside Value Option at $t = 2$](image)

In addition to default at date 2, I allow the government to default at period 1 as well. Analogously, $V_1^{def} \sim F_1$ where $F_1$ has a bounded support $[v_1^{min}, v_1^{max}]$, $v_1^{min} = (1 + \beta)v_2^{min}(N)$, $v_1^{max} = (1 + \beta)v_2^{max}(N)$, and $f_1 = a_1 \cdot (v - v_1^{min})^2 \cdot (v - v_1^{max})^2$ such that $\int_{v_1^{min}}^{v_1^{max}} a_1 \cdot (v - v_1^{min})^2 \cdot (v - v_1^{max})^2 dv = 1$.

### 4.2 Numerical Algorithm

The key distinction between the computational approach of this paper and other approaches used in the literature is that I do not discretize the continuous distributions $F_1$ and $F_2$. In addition, instead of finding the maximum value on the grid I approximate pricing and value functions with bicubic splines and solve the first-order necessary optimality conditions. One benefit of bicubic interpolation is that an approximated function and its first-order derivatives are continuous functions.

First, I find functions which approximate $V_1(b_1^0, b_2^0, s)$, and $q_2^2(b_1^0, b_2^0, s)$. To approximate any function $g(x, y)$ on $[0, 1] \times [0, 1]$ with a multivariate polynomial function of order three $p(x, y)$ we
find a vector of 16 coefficients \( \alpha = \{a_{ij}\}_{i=0,...,3, j=0,...,3} \) for which the values of \( g \) and its derivatives \( g_x, g_y, \) and \( g_{xy} \) equal to \( p, p_x, p_y, \) and \( p_{xy} \) at the four corners \( (0, 0), (0, 1), (1, 0) \) and \( (1, 1) \). Then

\[
g(x \in [0, 1], y \in [0, 1]) \approx p(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} \cdot x^i \cdot y^j
\]  

(17)

In addition, such approach allows to get an approximate for \( \frac{\partial g(x,y)}{\partial x} \) and \( \frac{\partial g(x,y)}{\partial y} \) as follows

\[
\frac{\partial}{\partial x} g(x \in [0, 1], y \in [0, 1]) \approx p_x(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} \cdot ix^{i-1} \cdot y^j
\]  

(18)

\[
\frac{\partial}{\partial y} g(x \in [0, 1], y \in [0, 1]) \approx p_y(x, y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} \cdot x^i \cdot jy^{j-1}
\]  

(19)

Let \([b_0^{1min}, b_0^{1max}]\) and \([b_0^{2min}, b_0^{2max}]\) be the range of possible values of short-term debt and long-term debt. Let \(N_s\) and \(N_i\) be the number of nodes for short-term debt and long-term debt respectively. Then for each node \((i \in [1, N_s], j \in [1, N_j], s \in \{C, N\})\) such that

\[
\left(b_0^1\right)_i = b_0^{1min} + (b_0^{1max} - b_0^{1min}) \cdot \frac{i - 1}{N_s}
\]

\[
\left(b_0^2\right)_j = b_0^{2min} + (b_0^{2max} - b_0^{2min}) \cdot \frac{j - 1}{N_i}
\]

we find the values of \(V_1((b_0^1)_i, (b_0^2)_j, s)\) and \(q_1^2((b_0^1)_i, (b_0^2)_j, s)^4\). In addition, I find the derivatives \(\frac{\partial}{\partial b_0^1} V_1((b_0^1)_i, (b_0^2)_j, s)\), \(\frac{\partial}{\partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s)\), and \(\frac{\partial^2}{\partial b_0^1 \partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s)\) (and analogously the derivatives of \(q_1^2((b_0^1)_i, (b_0^2)_j, s)\)) by finding the values at the points \((b_0^1)_i + \epsilon, (b_0^2)_j, s\), \((b_0^1)_i - \epsilon, (b_0^2)_j, s\), \((b_0^1)_i, (b_0^2)_j + \epsilon, s\), \((b_0^1)_i, (b_0^2)_j - \epsilon, s\), \((b_0^1)_i + \epsilon, (b_0^2)_j + \epsilon, s\), and \((b_0^1)_i - \epsilon, (b_0^2)_j - \epsilon, s\) where \(\epsilon > 0\) is a very small number so that

\[
\frac{\partial}{\partial b_0^1} V_1((b_0^1)_i, (b_0^2)_j, s) = \frac{V_1((b_0^1)_i + \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i - \epsilon, (b_0^2)_j, s)}{2\epsilon}
\]

\[
\frac{\partial}{\partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s) = \frac{V_1((b_0^1)_i, (b_0^2)_j + \epsilon, s) - V_1((b_0^1)_i, (b_0^2)_j - \epsilon, s)}{2\epsilon}
\]

\[
\frac{\partial^2}{\partial b_0^1 \partial b_0^2} V_1((b_0^1)_i, (b_0^2)_j, s) = \frac{V_1((b_0^1)_i + \epsilon, (b_0^2)_j + \epsilon, s) - V_1((b_0^1)_i + \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i, (b_0^2)_j + \epsilon, s) + V_1((b_0^1)_i, (b_0^2)_j, s)}{2\epsilon^2}
\]

\(^4\)technically, \(q_1^2\) is a function of \(b_0^1\) and \(s\) but \(b_0^1\) is a function of \(b_0^1, b_0^2,\) and \(s\)
\[
2V_1((b_0^1)_i, (b_0^2)_j, s) - V_1((b_0^1)_i - \epsilon, (b_0^2)_j, s) - V_1((b_0^1)_i, (b_0^2)_j - \epsilon, s) + V_1((b_0^1)_i - \epsilon, (b_0^2)_j - \epsilon, s) \\
\frac{2\epsilon^2}{2}.
\]

Then for each square \([(b_0^1)_i, (b_0^1)_{i+1}] \times [(b_0^2)_j, (b_0^2)_{j+1}]\) and \(s \in \{C, N\}\) I find the vectors of coefficients \(\alpha_{ij}s(V_1)\) and \(\alpha_{ij}s(q_1^2)\) to approximate the value and policy functions and their derivatives analogously to (17) - (19).

To find \(V(b_0^1, b_0^2, s)\) and \(q_1^2(b_0^1, b_0^2, s)\) I solve the system of equations consisting of the budget constraint at period 1 (3) and the first-order optimality condition (20) is different from (17) - (19).

\[
-\beta \cdot \frac{\partial \mathbb{E}V_2(b^2, s)}{\partial b} = u'(c_1) \cdot \frac{1}{R} \left[ F^s \left( u(y - b_1^2) \right) + \frac{\partial}{\partial b} F^s \left( u(y - b_1^2) \right) \cdot \left( b_1^2 - b_0^2 \right) \right] \tag{20}
\]

where

\[
\frac{\partial \mathbb{E}V_2(b, s)}{\partial b} = -\frac{\partial u(y - b)}{\partial b} \cdot F^s(u(y - b))
\]

and find \(b_1^2(b_0^1, b_0^2, s)\) and \(c_1^2(b_0^1, b_0^2, s)\). Then the value function is just

\[
V_1(b_0^1, b_0^2, s) = u(c_1^2(b_0^1, b_0^2, s)) + \beta \mathbb{E}V_2(b_1^2(b_0^1, b_0^2, s), s)
\]

where

\[
\mathbb{E}V_2(b, s) = \int_{u(y-b)}^{v_{max}(s)} v \cdot dF^s(v) + u(y - b) \cdot F^s(u(y - b))
\]

The first component can be easily computed given that \(F^s\) is a polynomial. The pricing function is

\[
q_1^2(b_0^1, b_0^2, s) = \frac{1}{R} \cdot F^s(u(y - b_1^2(b_0^1, b_0^2, s)))
\]

The value of the sovereign in the intermediate period in case of roll-over crisis is

\[
V_1(b_0^1, b_0^2, s) = \begin{cases} 
V_1(b_0^1, b_0^2, s) & \text{if } b_1^2(b_0^1, b_0^2, s) \geq b_0^2 \\
 u(y - b_0^1) + \beta \mathbb{E}V_2(b_0^2, s) & \text{otherwise}
\end{cases}
\]

The final step is to find the optimal maturity structure \((b_0^1, b_0^2)\) by solving the system of equations consisting of the budget constraint at period 0 and first-order optimality conditions with respect to short-term debt and long-term debt:

\footnote{optimality condition (20) is different from (17) - (19) because we use \(b_1^2\) and \(b_0^2\) instead of \(b_1^2(s)\) and \(b_0^2(s)\).}
\[
c_0 = y_0 + q_0^1(b_{01}, b_{02}) \cdot b_{10}^1 + q_0^2(b_{01}, b_{02}) \cdot b_{02}
\]

\[
- \frac{\partial}{\partial b_{01}} \beta W_1(b_{01}, b_{02}) = u'(c_0) \cdot \left( q_0^1(b_{01}, b_{02}) + \frac{\partial}{\partial b_{01}} q_0^1(b_{01}, b_{02}) \cdot b_{10}^1 + \frac{\partial}{\partial b_{02}} q_0^2(b_{01}, b_{02}) \cdot b_{02} \right)
\]

\[
- \frac{\partial}{\partial b_{02}} \beta W_1(b_{01}, b_{02}) = u'(c_0) \cdot \left( q_0^2(b_{01}, b_{02}) + \frac{\partial}{\partial b_{01}} q_0^1(b_{01}, b_{02}) \cdot b_{10}^1 + \frac{\partial}{\partial b_{02}} q_0^2(b_{01}, b_{02}) \cdot b_{02} \right)
\]

The left-hand sides of the optimality conditions show a marginal decrease in discounted expected value tomorrow \(\beta W_1(b_{01}, b_{02})\) if the sovereign increases issuance of debt by one unit. See Lemma A.2 for derivation of the derivatives. The right-hand side of the optimality conditions shows how an increase in debt by a unit increases the sovereign’s utility in the initial period. Notice that it takes into account how such action affects the market value of already issued debt.

### 4.3 Optimal Maturity Structure

As discussed in Section 3, the sovereign issues only short-term debt to resolve time-consistency problem. In this section, I consider two numerical exercises to understand how sovereign’s decision to issue only short-term debt is affected by adding incentives to issue long-term debt: uncertainty regarding future interest rates and possibility of self-fulfilling debt crisis.

#### 4.3.1 Incentive Motive vs Hedging Motive

In the first numerical exercise, I study the optimal maturity structure in an environment with lack of commitment and uncertainty about future interest rates. More specifically, I consider problem (7) and set \(\lambda = 0\). Thus, there is no incentive to use maturity to eliminate self-fulfilling debt crisis as it never occurs, however, sovereign might be willing to use long-term debt to smooth consumption between different states of the world in which risk premium can be higher or lower.

Figure 2 displays the results of the model without self-fulfilling debt crisis. The left top panel shows the equilibrium ex ante default risk at period 1 and for each state of the world at period 2. For the parameter of the model, the default risk at \(t = 1\) and in the normal state of the world at \(t = 2\) are negligibly small. The default risk in the crisis state of the world is substantial. The right top panel shows equilibrium consumption at period 1 in different states of the world. It is clear that the sovereign does not prefer to smooth consumption between the states of the world.

Figure 2. Maturity Structure with Default Risk Uncertainty
The left bottom panel shows the optimal maturity structure of sovereign debt. The maturity structure is mostly short term as the fraction of long-term is negligibly small. The right bottom panel displays the share of long-term debt. Even though the share increases as borrowing in the initial period goes up, it’s share is small with slightly above 1% for \( y_0 = 0.8 \) which corresponds to approximately 2% default risk in the crisis state of the world next period.

4.3.2 Incentive Motive vs Self-Fulfilling Debt Crisis

In the second exercise I abstract away from hedging motive by assuming that there is no uncertainty regarding future default risk, but add the possibility of self-fulfilling debt crisis. Specifically, I assume that \( F_2^C = F_2^N \) and set \( \lambda = 0.3 \).

The results of this model are presented in Figure 3. Analogically to Figure 2, I plot default risk and consumption in both periods, maturity structure and share of long-term debt as a function of \( y_0 \). As income in the initial period decreases, the sovereign borrows more increasing default risk in both periods. The sovereign still prefers to issue mostly short-term debt, however, the fraction of long-term is significantly higher than in the first exercise. For different values of initial income the share of long-term debt ranges within 13-16%. Interestingly, the share seems to be almost uncorrelated with the level of default risk.

Figure 3. Maturity Structure with Self-Fulfilling Debt Crisis
Figure 4 displays results for different probabilities of sun-spot equilibrium $\lambda$. As $\lambda$ goes up, keeping other parameters and control variables constant, likelihood of self-fulfilling debt crisis increases. As seen from the left bottom panel, the sovereign issues more long-term debt and less short-term debt for a higher $\lambda$. The share of long-term debt increases from 0 for $\lambda = 0$ to almost 30% for $\lambda = 1$.

Figure 4. Maturity Structure with Self-Fulfilling Debt Crisis
Interestingly, default risk in the first period remains approximately constant for different values of $\lambda$. This is possible because increase in fraction of long-term debt increase the sovereign’s value in case of roll-over crisis decreasing the range of values of the outside option at which the self-fulfilling debt crisis can occur. The probability of default in the last period increases due to higher share of long-term debt.

To conclude, in the presence of lack of commitment, uncertainty regarding future interest rates is a weak incentive to issue long-term debt. Long-term debt allows to smooth consumption between states of the world only if the distribution of the value of outside option is discrete so that marginal increase in debt does not affect the default probability. However, if the distribution is continuous and marginal increase in debt leads to a marginal increase in default risk, then the cost due to lack of commitment outweighs the benefit of hedging. The reason is that default risk increases in both states of the world and there is even higher incentive to dilute the value of long-term debt in the crisis state of the world. Thus, the positive effect of smoothing consumption between states of the world is minimized.

If self-fulfilling debt crisis is possible though marginal increase in long-term debt leads to a marginal increase in long-term interest rates. However, it also marginally decreases the probability of roll-over crisis next period, decreasing short-term interest rates. Therefore, issuance of long-term debt is justified.
5 Limitations of Discretization Approach

The literature studying maturity structure of sovereign debt often discretize the continuous distribution as, for example, Arellano and Ramanarayanan (2012). They approximate continuous distribution with quadrature procedure (Tauchen and Hussey, 1991) using only six-state Markov chain. In this section I demonstrate that the numerical algorithm matters for the solution of the maturity structure and discretization can lead to a biased results.

Consider a model discussed in Section 3.1 but with no default in period 1. From Proposition 1 we know that if equilibrium default risk is positive the sovereign should issue only one-period bonds and does not issue two-period bonds at all. Figure 5 shows the solution to such an example using the algorithm explained in Section 4.2.

![Figure 5. Solution to a single-state model using bicubic approximation approach](image)

We see that the numerical solution coincides with the analytical solution and is very accurate. The share of long-term debt does not exceed $10^{-8}$ which corresponds to computational error. However, as we will see, discretization approach does not lead to such results.

Suppose that $f$ has a functional form as in (16) and $V_2^{def} \in [v^{min}, v^{max}]$ where $v^{max} = u(y)$ and $v^{min} = u(0.5y)$. Let construct discrete distribution $\tilde{f}$ with $N_f$ points such that
\[ x_i = v^{\text{min}} + \left( \frac{v^{\text{max}} - v^{\text{min}}}{N_f} \right) \cdot \left( \frac{1}{2} + (i - 1) \right), \quad i = 1, \ldots, N_f \]

\[ \tilde{f}_i = \frac{f(x_i)}{\sum_{i=1}^{N_f} f(x_i)}, \quad i = 1, \ldots, N_f \]

and solve the problem finding the maximum on the fine grid. Figure 6 shows the approximated distribution for \( N_f = 5 \).

Figure 6. Discrete distribution, \( N_f = 5 \)

Figure 7 displays the optimal maturity structure and equilibrium default risk at period 2. We can see that the government issues more short-term debt. However, it also issues positive and substantial amount of long-term debt. For example, if \( y_0 = 0.8 \) then the sovereign issues more than 0.06 of long-term debt while short-term debt amounts to less than 0.08. Obviously, it contradicts the analytical solution to the problem that implies issuance of short-term debt only.

Figure 7. Solution to a single-state model using discretization approach for \( N_f = 5 \)
In addition, notice that the total debt issued and default risk presented on Figure 7 does not correspond to the results displayed on Figure 3. On Figure 5 default risk increases smoothly and reaches approximately 3% at $y_0 = 0$. If we consider the discretization solution, then default risk “jumps” to about 5% at $y_0 = 0.85$. As a result, not only the maturity structure is wrong but also the total level of debt.

Obviously, as $N_f$ goes up and thus the approximated discrete distribution because closer to the original continuous distribution, the bias decreases. Let increase $N_f$ to 25. Figure 8 shows the discrete distribution.

Figure 8. Discrete distribution, $N_f = 25$
The optimal maturity structure and equilibrium default risk are presented on Figure 9. We can see issuance of short-term debt and default risk schedule are much closer to Figure 9 compared to the results presented on Figure 7.

Figure 9. Solution to a single-state model using discretization approach for $N_f = 25$

Nevertheless, the solution to the problem does not imply that the issuance of long-term debt is zero or, at least, significantly close to zero. For example, it exceeds 30% for $y_0 = 0.87$. Therefore, precise solution to the optimal maturity structure problem requires that continuous probability distributions are approximated with a large number of nodes. Otherwise, the results can be biased and indicate exaggerated share of long-term debt.

Appendix A. Proofs

Proof of Proposition 1.

The proof is in three steps. First, I set up and solve the modified commitment problem as in Kiiashko (2018). I show that the sovereign who makes all decisions sequentially cannot choose a strictly better allocation than that corresponding to the solution of the modified commitment problem. Finally, I show that the best allocation can be achieved only if the sovereign issues no long-term debt.

Consider the planning problem: a sovereign can commit to future consumption but makes
strategic default decisions sequentially

\[
\hat{V}_0 = \max_{c_0, c_1, c_2} u(c_0) + \beta \mathbb{E} \max_{V_1} \left\{ u(c_1) + \beta \mathbb{E} \max_{V_2^{\text{def}}} \left\{ u(c_2), V_2^{\text{def}} \right\} \right\}
\]

\[
s.t. (y_0 - c_0) + \frac{1}{R} F_1(\hat{V}_1) \cdot (y - c_1) + \frac{1}{R^2} F_1(\hat{V}_1) F_2(u(c_2)) \cdot (y - c_2) \geq 0
\]

Notice that

\[
\mathbb{E} \max_{\{V_1, V_1^{\text{def}}\}} = \int_{v_1^{\min}}^{v_1^{\max}} \max_{V_1} \left\{ \hat{V}_1, V_1^{\text{def}} \right\} dF_1(v) + \int_{F_1(\hat{V}_1)}^{v_1^{\max}} \max_{V_1} \left\{ \hat{V}_1, V_1^{\text{def}} \right\} dF_1(v) = F_1(\hat{V}_1) \cdot \hat{V}_1 + \int_{F_1(\hat{V}_1)}^{v_1^{\max}} vdF_1(v) = F_1(\hat{V}_1) \cdot u(c_1) + \int_{F_1(\hat{V}_1)}^{v_1^{\max}} vdF_1(v) + \beta F_1(\hat{V}_1) \cdot \mathbb{E} \max \left\{ u(c_2), V_2^{\text{def}} \right\}
\]

where \( \mathbb{E} \max \left\{ u(c_2), V_2^{\text{def}} \right\} \) is

\[
\mathbb{E} \max \left\{ u(c_2), V_2^{\text{def}} \right\} = \int_{v_2^{\min}}^{v_2^{\max}} \max_{V_2^{\text{def}}} \left\{ u(c_2), V_2^{\text{def}} \right\} dF_2(v) + \int_{F_2(u(c_2))}^{v_2^{\max}} \max_{V_2^{\text{def}}} \left\{ u(c_2), V_2^{\text{def}} \right\} dF_2(v) = F_2(u(c_2)) \cdot u(c_2) + \int_{F_2(u(c_2))}^{v_2^{\max}} vdF_2(v)
\]

Next observe that for \( i = 1, 2 \)

\[
\frac{\partial \mathbb{E} \max \left\{ \hat{V}_1, V_1^{\text{def}} \right\}}{\partial c_i} = F_1(\hat{V}_1) \cdot \frac{\partial \hat{V}_1}{\partial c_i} + \frac{\partial F_1(\hat{V}_1)}{\partial c_i} \cdot \hat{V}_1 - \frac{\partial F_1(\hat{V}_1)}{\partial c_i} \cdot \hat{V}_1 = F_1(\hat{V}_1) \cdot \frac{\partial \hat{V}_1}{\partial c_i}
\]

so that

\[
\frac{\partial \mathbb{E} \max \left\{ \hat{V}_1, V_1^{\text{def}} \right\}}{\partial c_1} = F_1(\hat{V}_1) \cdot u'(c_1)
\]

\[
\frac{\partial \mathbb{E} \max \left\{ \hat{V}_1, V_1^{\text{def}} \right\}}{\partial c_2} = \beta F_1(\hat{V}_1) \cdot \frac{\partial \mathbb{E} \max \left\{ u(c_2), V_2^{\text{def}} \right\}}{\partial c_2} = \beta F_1(\hat{V}_1) \cdot \left( F_2(u(c_2)) \cdot u'(c_2) + \frac{\partial F_2(u(c_2))}{\partial c_2} \cdot u(c_2) - \frac{\partial F_2(u(c_2))}{\partial c_2} \cdot u(c_2) \right) = \beta F_1(\hat{V}_1) F_2(u(c_2)) u'(c_2)
\]
Recall that $\beta = \frac{1}{R}$. Optimal allocation $(\hat{c}_0, \hat{c}_1, \hat{c}_2)$ satisfies the optimality conditions

$$u'(c_0) = \hat{\mu}$$

$$\beta F_1(\hat{V}_1)u'(c_1) = \hat{\mu}\beta \left( F_1(\hat{V}_1) - \frac{\partial F_1(\hat{V}_1)}{\partial c_1} \cdot ((y - c_1) + \beta F_2(u(c_2)) \cdot (y - c_2)) \right)$$  \hspace{1cm} (23)

$$\beta^2 F_1(\hat{V}_1)F_2(u(c_2))u'(c_2) = \hat{\mu}\beta \frac{\partial F_1(\hat{V}_1)}{\partial c_2} \cdot ((y - c_1) + \beta F_2(u(c_2)) \cdot (y - c_2)) +$$

$$+ \hat{\mu}\beta^2 F_1(\hat{V}_1) \cdot \left( F_2(u(c_2)) - \frac{\partial F_2(u(c_2))}{\partial c_2} \cdot (y - c_2) \right)$$  \hspace{1cm} (24)

where $\hat{\mu}$ is the Lagrange multiplier on (22). The ratio of the left-hand sides of (23) and (24) is $\frac{\partial \hat{V}_1}{\partial c_1} / \frac{\partial \hat{V}_1}{\partial c_2}$. Note that

$$\frac{\partial F_1(\hat{V}_1)}{\partial c_1} \frac{\partial \hat{V}_1}{\partial c_1} = \frac{f_1(\hat{V}_1) \frac{\partial \hat{V}_1}{\partial c_1}}{f_1(\hat{V}_1) \frac{\partial \hat{V}_1}{\partial c_2}} = \frac{\frac{\partial \hat{V}_1}{\partial c_1}}{\frac{\partial \hat{V}_1}{\partial c_2}}.$$

Therefore, combining (23) and (24) yields

$$\frac{u'(c_2)}{u'(c_1)} = F_2(u(c_2)) - \frac{\partial F_2(u(c_2))}{\partial c_2} \cdot (y - c_2)$$  \hspace{1cm} (25)

Now consider the problem without commitment. First of all, note that in a problem without commitment sovereign cannot choose an allocation which yields strictly higher value than (21). Suppose by contradiction that there is a better allocation $(c^*_0, c^*_1, c^*_2)$ with corresponding debt policy $(b^*_0, b^*_1, b^*_2)$. This allocation must satisfy budget constraints in period 0 and 1. Combining those budget constraints we conclude that the allocation also satisfies budget constraint of the modified commitment problem (22). Therefore, the allocation is feasible for the planner, implying $\hat{V}_0 > V_0$.

In the period 1 the sovereign chooses consumption today and tomorrow to maximize

$$V_1(b_0, b_2) = \max_{c_1, c_2} u(c_1) + \beta \mathbb{E} \max \left\{ u(c_2), V_2^{\text{def}} \right\}$$

s.t. $(y - b_0 - c_1) + \frac{1}{R} F_2(u(c_2)) \cdot (y - b_0^2 - c_2) \geq 0$

The optimality condition for the sovereign in the intermediate period is

$$\frac{u'(c_2)}{u'(c_1)} = F_2(u(c_2)) - \frac{\partial F_2(u(c_2))}{\partial c_2} \cdot (y - b_0^2 - c_2)$$  \hspace{1cm} (26)

Note that (25) and (26) are identical only if $b_0^2 = 0$.  

\[\square\]
Lemma A.1. Consider the modified commitment problem (13). If \( y_0 < y \) then \( c_2(s) < y \) \( \forall s \).

Proof.

Basically, the planner has 4 options:

(i) \( c_2(C) \geq y, c_2(N) \geq y \);
(ii) \( c_2(C) < y, c_2(N) \geq y \);
(iii) \( c_2(C) \geq y, c_2(N) < y \);
(iv) \( c_2(C) < y, c_2(N) < y \).

They correspond to no default risk, default risk in the crisis state of the world and no default risk in the normal state of the world, default risk in the normal state of the world and no default risk in the crisis state of the world, or default risk in both states of the world.

First, notice that if the government decides to avoid default risk in state \( s \) it sets \( c_2(s) = y_2 \). If \( c_2(s) > y_2 \) the government can be better off by decreasing \( c_2(s) \) and by smoothing consumption, i.e., increasing consumption in other periods.

For each case (i)-(iv) the first-order necessary conditions imply that \( c_0 = c_1(s) = c \) \( \forall s \). In addition, if the government does not restrict \( c_2(s) \) to be equal to \( y_2 \) then \( c_2(s) = c \):

\[
u'(c_0) = \mu
\]

\[
\beta p_0(F = F^s)u'(c_1(s)) = p_0(F = F^s)\frac{\mu}{R} \Rightarrow u'(c_1(s)) = \mu
\]

\[
\beta^2 p_0(F = F^s) \left(1 - p^s + p^s \cdot \mathbb{I}_{c_2(s) \geq y} \right) u'(c_2(s)) = \mu \frac{p_0(F = F^s) \left(1 - p^s + p^s \cdot \mathbb{I}_{c_2(s) \geq y} \right)}{R^2}
\]

\[
\Rightarrow u'(c_2(s)) = \mu
\]

where \( \mu \) is the Lagrange multiplier and \( p^s = p^C \) if \( s = \{C\} \) and \( p^s = p^N \) otherwise.

Notice that the cases (i)-(iv) have exactly the same objective functions (because \( u(c_2(s)) = v^{max}_2 \)) and budget constraints (because if \( \mathbb{I}_{c_2(s) \geq y} > 0 \) then \( c_2(s) = y \)). The only difference is that the cases (i)-(iii) impose additional restrictions on \( c_2(s) \) as the government chooses to avoid default risk: (i) implies \( c_2(C) = c_2(N) = y \), (ii) imposes \( c_2(N) = y \), and (iii) imposes \( c_2(C) = y \). Thus, the government is strictly better off by choosing \( c_2(C) = c_2(N) = c < y \).

\( \blacksquare \)
Proof of Lemma 1.

Result follows from the first-order necessary conditions (see proof of Lemma A.1 for details).

Proof of Proposition 2.

Analogously to Lemma A.1, one can show that the MCP is an upper bound for the MPCE problem. Consider the problem of the intermediate period government with the state \((b_1^0, b_2^0, s)\)

\[
V_1(b_1^0, b_2^0, s) = \max_{c_1, c_2} u(c_1) + \beta(1 - p^s) \cdot u(c_2(s)) + \beta p^s \max \{u(c_2(s)); v^{\text{max}}\}
\]

s.t. \(y - c_1 - b_1^0 + \frac{1}{R} \left(1 - p^s + p^s \mathbb{1}_{(c_2(s) \geq y)}\right)(y - c_2(s) - b_2^0) = 0\)

First, notice that for any \(b_1^0 > 0\) and \(b_2^0 > 0\) if the government decides to avoid default risk by choosing \(c_2(s) = y\), it reduces its budget set because then \(y - c_2(s) - b_2^0 < 0\). Therefore, if debt is positive the equilibrium default risk is also positive. The optimal consumption is given by the first-order necessary conditions:

\[c_1(s) = c_2(s) = y - \frac{b_1^0 + \frac{1-p^s}{R} b_2^0}{1 + \frac{1-p^s}{R}}\]

Recall that Lemma 1 implies that the first-best allocation requires perfect consumption smoothing \(c_0 = c_1(s) = c_2(s) \forall s\). Therefore, the Markov government can implement the planner’s allocation if

\[\frac{b_1^0 + \frac{1-p^s}{R} b_2^0}{1 + \frac{1-p^s}{R}} = \text{const}\]

for any \(p^s \in [0, 1]\). The latter is true if and only if \(b_1^0 = b_2^0\).

Proof of Lemma 2.

Let \(J(c, b_1^0, b_2^0, \lambda)\) be the objective function of (14) as a function of control variables and probability of self-fulfilling debt crisis. Let \(V_0\) be the value at \(\lambda \in (0, 1]\) and \(V'_0\) the value at \(0 \leq \lambda' < \lambda\), and \((b_1^{1*}, b_2^{1*})\) be optimal solutions to (14) if risk of self-fulfilling debt crisis is \(\lambda\). Denote by \(c(b_1^{1*}, b_2^{1*}, \lambda)\) consumption that satisfies the budget constraint (15).

Note that \(J(c(\lambda'), b_1^{1*}, b_2^{1*}, \lambda') \geq J(c(\lambda), b_1^{1*}, b_2^{1*}, \lambda)\) with strict inequality if \(\{V_1(b_1^0, b_2^0) < v^{\text{min}} \leq V_1(b_1^0, b_2^0)\} = \)
1 or \([1_{\{V_1(b_0^1, b_0^2) \leq V_1^{\max} \leq V_1(b_0^1, b_0^2)\}}] = 1\). In addition, observe that

\[
V_0' \geq J(c(\lambda'), b_0^1, b_0^2, \lambda'), \quad V_0 = J(c(\lambda), b_0^1, b_0^2, \lambda)
\]

\[
\Rightarrow V_0' \geq V_0 \Rightarrow \frac{\partial V_0}{\partial \lambda} \leq 0
\]

\[\blacksquare\]

Lemma A.2.

\[
\frac{\partial W_1(b_0^1, b_0^2)}{\partial b_0^0} = \sum_s p^s \left\{ \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} \cdot (1 - \pi_1(b_0^1, b_0^2, C)) - \lambda f_1(V_1(b_0^1, b_0^2, s))(\frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0}) \left( V_1(b_0^1, b_0^2, s) - V_1(b_0^1, b_0^2, s) \right) \right\}
\]

(27)

where \(W_1(b_0^1, b_0^2)\) is defined by (9).

Proof.

\[
\frac{\partial (1 - \pi_1(b_0^1, b_0^2, s))}{\partial b_0^0} = f_1(V_1(b_0^1, b_0^2, s)) \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} - \lambda f_1(V_1(b_0^1, b_0^2, s)) \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} + \lambda f_1(V_1(b_0^1, b_0^2, s)) \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0}
\]

(28)

\[
\frac{\partial W_1(b_0^1, b_0^2)}{\partial b_0^0} = \sum_s p^s \left\{ V_1(b_0^1, b_0^2, s) \cdot \frac{\partial (1 - \pi_1(b_0^1, b_0^2, s))}{\partial b_0^0} + \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} \cdot (1 - \pi_1(b_0^1, b_0^2, C)) + \lambda \left( V_1(b_0^1, b_0^2, s) \cdot f_1(V_1(b_0^1, b_0^2, s)) \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} \right) - \right.
\]

\[
\left. - V_1(b_0^1, b_0^2, s) \cdot f_1(V_1(b_0^1, b_0^2, s)) \frac{\partial V_1(b_0^1, b_0^2, s)}{\partial b_0^0} \right\}
\]

(29)

Plug (28) to (29) results in (27).

\[\blacksquare\]

References


