Intrinsic Bubbles in Stock Prices Under Persistent Dividend Growth Rates

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Abstract

We extend the constant discount factor model with intrinsic bubbles developed in Froot and Obstfeld (1991) to account for serial correlation in dividend growth rates. We derive an exact analytical expression for both the present value stock price and an intrinsic bubble component when dividend growth rates evolve as a Gaussian first-order autoregressive process. We estimate the model with two sets of annual U.S. stock prices and dividends data, namely the DJIA and the S&P 500 series, over the last century. Hypotheses tests reject an AR(0) process for dividend growth rates in favor of an AR(1) process for both data series. Likelihood ratio tests also favor the AR(1)-based model developed here for price-dividends ratios to the AR(0)-based model considered in Froot and Obstfeld (1991). Hypotheses tests also reject the absence of a bubble component in both series. This inference is robust to whether or not the parameters governing the intrinsic bubbles process are restricted to values implied by our model or freely estimated. Incorporating the bubble component into our model provides a significant improvement in fit to observed P/D ratios and stock prices as compared to the present value stock prices alone.

JEL Classification: C12, G10, G12

Keywords: Stock Prices, Price-Dividend Ratios, Present-Value Model, Intrinsic Bubbles, Closed-form solutions

1 Introduction

Rationalization of observed stock prices is a task of great interest in financial economics. Several comprehensive surveys of the literature are available. See, for example, Hansen (2014), for a recent effort. It is well known that traditional asset pricing models do not capture variation in stock prices very well. Examples of such models include the popular constant discount factor present value model under a rational expectations framework. Deviations in stock prices from those predicted by simple present value models have proven empirically significant and persistent over time (LeRoy & Porter, 2008).
One of several approaches to rationalize these deviations in stock prices is the rational bubbles theory (Blanchard, 1979; Blanchard & Watson, 1982; Diba & Grossman, 1983, 1988a, and 1988b).

Within this framework, Froot & Obstfeld (1991) develop a specific type of rational bubble that they call a rational “intrinsic” bubble. Here, movements in stock prices are exclusively driven by economic fundamentals alone (i.e., dividends) and not from any extraneous factors, as are common in stock price bubbles literature. Specifically, they assume a random walk process for log dividends and derive the present value stock price within a constant discount factor present value model of stock prices. Another solution to the present value model exists that violates the transversality condition. This solution for stock prices consists of the present value stock price plus a bubble component. In the intrinsic bubbles setup of Froot & Obstfeld (1991), this bubble component is driven by fundamentals alone, which are exogenous dividends.

A random walk model of log dividends used in Froot & Obstfeld (1991) is deficient. It fails to capture observed autocorrelation in dividend growth rates. The Lintner (1956) model of corporate dividends payout assumes that firms have a target payout ratio in mind that is a fraction of current earnings. However, firms are assumed to only make partial adjustments every period. This results in a smoothing of dividends paid over time, resulting in autocorrelation in their growth rates as observed in the data. Lansing (2010) notes that, in order to generate observed persistence in price-dividend ratios, the present value-type asset pricing model requires a persistent process for dividend growth rates.

In this paper, we extend the framework of Froot & Obstfeld (1991) to account for this autocorrelation. We derive an exact analytical expression for both the present value stock price and an intrinsic bubble component when dividend growth rates evolve as a Gaussian first-order autoregressive process. Our solution for the present value stock price is an adaptation of the one provided in Burnside (1998) for the consumption-based asset pricing model under a Gaussian AR(1) process for dividend growth rates. The analytical form of the intrinsic bubble in our work is an augmented version of the one provided in Froot & Obstfeld (1991). Their framework is a special case of the one considered here.

Within a random walk framework for log dividends, Bidarkota & Dupoyet (2007) extend the intrinsic bubbles model of Froot & Obstfeld (1991) to account for observed leptokurtosis and negative skewness in dividend growth rate by modeling the innovations to the random walk as random variables drawn from a non-normal, fat-tailed probability distribution. While it would be ideal to incorporate both persistence in dividend growth rates, as they are being done here, and non-normality, as considered in Bidarkota & Dupoyet (2007), a solution to the present value model and characterization of the intrinsic bubble component poses a significant challenge under such a setting.

Recent literature has studied bubbles to rationalize price movements of alternative assets and markets, such as real estate in the U.S. (Nneji et al., 2013 and Hu & Oxley, 2018) and China (Yu, 2011), Chinese stock markets (Chang & Cai, 2016), and cryptocurrency (Cheah & Fry, 2015; Corbet et al., 2018). Another strand of research has focused on developing new econometric methods to statistically test for the existence of speculative bubbles (Phillips et al., 2011 and 2015; Whitehouse, 2019; Homm & Breitung, 2011; Breitung & Kruse, 2013; and Yuhn et al., 2015).

We estimate the model developed here with two sets of annual U.S. stock price and dividends data, namely the DJIA and the S&P 500 series, over the last century. Hypothe-
ses tests reject an AR(0) process for dividend growth rates in favor of an AR(1) process for both data series. Likelihood ratio tests also favor the AR(1)-based model developed here for price-dividends ratios over the AR(0)-based model considered in Froot and Obstfeld (1991). Hypotheses tests also reject the absence of a bubble component in both series. This inference is robust to whether or not the parameters governing the intrinsic bubbles process are restricted to values implied by our model or freely estimated. Incorporating the bubble component into our model provides a significant improvement in fit to observed P/D ratios and stock prices as compared to the present value stock prices alone. Lansing (2010) has completed work similar to our analytical solutions and approach by using calibration techniques to match the moments of price-dividend data, while we instead use econometric methods.

We organize this paper as the following. In Section 2, we introduce the present value model for stock prices in which we describe the fundamental value stock price and a bubble solution that violates the transversality condition. In Section 3, we derive closed-form solutions to the model with a fundamental stock price component and an intrinsic bubbles component under the assumption that dividends growth evolves as an AR(1) process. In Section 4, we introduce the data and econometric specifications, and provide a series of empirical results and inferences. We summarize our main findings in the concluding section.

2 Present Value Model

The present value model with a constant discount rate is given by:

\[ P_t = e^{-r}E_t [D_t + P_{t+1}] . \] (1)

Here, \( P_t \) is the real price of a share at the beginning of period \( t \), \( D_t \) is the real dividend per share paid out over period \( t \), \( r \) is the non-stochastic and constant discount rate, \( E_t \) is the mathematical expectation conditioned on information available at the start of period \( t \). It is often useful to think about this pricing equation as arising from a Lucas (1978)-type asset pricing model under risk neutrality.

On forward iteration, the present value equation yields:

\[ P_t = \sum_{s=t}^{\infty} e^{-r(s-t+1)}E_t (D_s) + \lim_{s \to \infty} e^{-rs}E_t (P_s) . \] (2)

One solution to stock prices in the above equation, denoted \( P_{pv}^t \), is obtained by imposing the transversality condition:

\[ \lim_{s \to \infty} e^{-rs}E_t (P_s) = 0. \] (3)

Imposing the transversality condition on Equation (2) gives:

\[ P_{pv}^t = \sum_{s=t}^{\infty} e^{-r(s-t+1)}E_t (D_s) . \] (4)

Thus, this equation provides the fundamental value of the stock price. One specifies an exogenous stochastic process for dividends and evaluates \( P_{pv}^t \).
There exist other solutions to the present value model given in Equation (1) that do not satisfy the transversality condition in Equation (3). For instance, let \( B_t \) be any sequence of random variables that satisfy:
\[
B_t = e^{-r}E_t \{ B_{t+1} \}.
\]
(5)

One can easily show that \( (P_t^{PV} + B_t) \) satisfies Equation (1) but violates Equation (3) for all \( B_t \neq 0 \).

If \( B_t \) is constructed as a function of the fundamentals alone, i.e., as a function of the dividends \( D_t \) alone in the present value model of Equation (1), it is termed an intrinsic rational bubble by Froot and Obstfeld (1991). Intrinsic bubbles turn out to be a non-linear function of dividends. Their exact functional form depends on the assumed stochastic process for the dividends.

3 Solution to the Model

In this section, we obtain an exact analytical solution for the present value stock price \( P_t^{PV} \) when the dividend growth rate follows a first-order autoregressive process. We also derive conditions under which a posited functional form for \( B_t \) satisfies all the conditions for a rational intrinsic bubble.

3.1 The Present Value Stock Price under AR(1) Dividends Growth Rate Process

Let \( x_t \equiv \ln (D_t) - \ln (D_{t-1}) \) denote the dividend growth rate. We assume that \( x_t \) stochastically evolves as a first-order autoregressive process:
\[
x_t - \mu = \rho (x_{t-1} - \mu) + \xi_t, \quad |\rho| < 1, \quad \xi_t \sim \text{iid N}(0, \sigma^2)
\]
(6)

One can now derive the present value stock price by evaluating the right hand side of Equation (4). Following up on the results in Burnside (1998), Appendix A shows that the present value stock price is given by:
\[
P_t^{PV} = D_t \sum_{s=t}^{\infty} \exp \{ -r(s - t + 1) + b_{s-t} (x_t - \mu) + a_{s-t} \}
\]
(7)

where
\[
a_{s-t} = (s - t)\mu + \frac{\sigma^2}{2(1 - \rho)^2} \left[ (s - t) - 2\frac{\rho}{1 - \rho} (1 - \rho^{s-t}) + \rho^2 \frac{1 - \rho^{s-t}}{1 - \rho^2} \right]
\]
(8)

and
\[
b_{s-t} = \frac{\rho}{1 - \rho} \left( 1 - \rho^{s-t} \right).
\]
(9)

The following theorem provides conditions for the infinite summation in Equation (7) to converge, and hence for the price–dividend ratio to be finite.
**Theorem 1.** The series in Equation (7) converges if
\[ R \equiv \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1 - \rho^2)} \right\} < 1. \] (10)

**Proof:** See Appendix B.

The next theorem derives an expression for the mean of the fundamental stock price-dividend ratio, i.e., the unconditional expectation of \( \frac{P_{pv}}{D_t} \). It also provides conditions under which this mean is finite.

**Theorem 2:** The mean of the price dividend ratio \( \frac{P_{pv}}{D_t} \) is given by:
\[ E \left( \frac{P_{pv}}{D_t} \right) = \sum_{i=0}^{\infty} \exp \left\{ -r(i + 1) + a_i + \frac{b_i^2 \sigma^2}{2(1 - \rho^2)} \right\} \] (11)

where
\[ a_i = i\mu + \frac{\sigma^2}{2(1 - \rho^2)} \left[ i - 2\rho \frac{1 - \rho}{1 - \rho^2} (1 - \rho^i) + \rho^2 \frac{1 - \rho^{2i}}{1 - \rho^2} \right] \] (12)
and
\[ b_i = \frac{\rho}{1 - \rho} \left[ 1 - \rho^i \right] \] (13)

It is finite if \( R \equiv \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1 - \rho^2)} \right\} < 1. \)

**Proof:** See Appendix C

### 3.2 Intrinsic Rational Bubbles

Let us postulate that intrinsic rational bubbles take the form:
\[ B (D_t) = cD_t^\lambda \exp \{hx_t\}. \] (14)

Here, \( \lambda > 0 \) for the bubble to grow with an increase in dividends, \( c > 0 \) to ensure non-negativity of stock prices, and \( h \) is a constant.

Appendix D shows that the functional form for the intrinsic bubble in Equation (14) satisfies Equation (5) defining a bubble, provided that \( \lambda \) and \( h \) are chosen to satisfy:
\[ r = (\lambda + h)(1 - \rho)\mu + (\lambda + h)^2 \sigma^2 / 2 \] (15)
and
\[ h = (\lambda + h)\rho. \] (16)

If the dividend growth rate stream follows an AR(0) process, then the solution for the present value stock price is easily obtained by setting \( \rho = 0 \) in the equations above. One can readily show that the expression obtained for the present value stock price in this case is identical to the one given in Froot and Obstfeld (1991). From Equation (16) \( h = 0 \) when \( \rho = 0 \). Therefore, the bubble component of the stock price given in Equation (14) reduces to \( B (D_t) = cD_t^\lambda \), exactly the expression in Froot and Obstfeld (1991). In this case the conditions needed for convergence of the fundamental stock price-dividend ratio as well as the conditions for \( B(D_t) \) to be a rational intrinsic bubble are also identical to those in Froot and Obstfeld (1991).
4 Empirical Assessment of the Model

4.1 Data Description

For empirical assessment of the model, we employ two aggregate stock price indices: the Dow Jones Industrial Average (DJIA) and the S&P 500. We retrieved annual DJIA index data, including average, yearly closing values and dividends for a sample period of 1920-2017 from two data sources. For the 1920-2005 period, we use the 2006 Value Line publication, A Long-Term Perspective: Dow Jones Industrial Average, 1920–2005 and for the 2006-2017 period, we use Standard & Poor’s Compustat database. Additionally, we retrieved monthly S&P 500 index data, including closing values and dividends for a sample period of 1900-2018 from Robert Shiller’s Irrational Exuberance (2000) publication. Each series used in the Shiller dataset is of January values. Although S&P 500 data spanning from 1871 are available, to follow Froot & Obstfeld (1991), we begin the series in 1900.

Table 1 provides summary statistics on real dividend growth rates and P/D ratios for both stock indices. As established in extant literature, dividend growth rates and P/D ratios both indicate strong and statistically significant leptokurtosis, negative skewness for the former and positive for the latter, with normality being strongly rejected for both series. The two series also exhibit strong first-order autocorrelation. Figures 1 and 2 plot real stock prices, real dividends and their growth rates, and price-dividend ratios for DJIA and S&P 500, respectively. Our objective in this paper is to attempt to rationalize movements in price-dividend ratios, and hence movements in the stock prices themselves, through movements in dividend growth rates which we take here to be exogenous.

4.2 Econometric Specification

Empirical evaluation of our model requires specification of an exogenous stochastic process for dividend growth rates. This is taken to be the AR(1) process given in Equation (6):

\[ x_t - \mu = \rho (x_{t-1} - \mu) + \xi_t, \quad |\rho| < 1, \quad \xi_t \sim iid N(0, \sigma^2_{\xi}) . \]  

(17)

Assumption of a normal distribution for \( \xi_t \) is inconsistent with its strong rejection reported in Table 1 and discussed in Section 4.1. A non-normal, fat-tailed probability distribution that explicitly accounts for leptokurtosis and negative skewness in the dividend growth rate was considered in Bidarkota and Dupoyet (2007). However, their analysis ignored persistence in dividend growth rates and considered a random walk process, instead. While it would be ideal to incorporate both persistence in dividend growth rates, as they are being done here, and non-normality, as considered in Bidarkota and Dupoyet (2007), a solution to the present value model and characterization of the intrinsic bubble component pose a significant challenge under such a setting. Burnside (1998) provides a solution to the consumption-based asset pricing model under a Gaussian AR(1) process for dividend growth rates. However, he neither considers intrinsic bubbles in his work nor undertakes an empirical assessment of the model.

Considering the discussion following Equation (5), one can write the complete solution to the present value model as:

\[ P_t = P_t^{pv} + B_t. \]  

(18)
Now using Equations (7)-(9) and (14)-(16), one obtains:

\[ P_t = \kappa_t D_t + c D_t^\lambda \exp\{h x_t\}. \]  

(19)

Here

\[ \kappa_t = \sum_{s=t}^{\infty} \exp\{-r(s-t+1) + b_{s-t}(x_t - \mu) + a_{s-t}\} \]  

(20)

from Equation (7), and \(a_{s-t}\) and \(b_{s-t}\) are given in Equations (8) and (9), respectively. Dividing Equation (19) by \(D_t\), we can write:

\[ \frac{P_t}{D_t} = \kappa_t + c D_t^{\lambda-1} \exp\{h x_t\}. \]  

(21)

We follow standard practice in the literature by augmenting the stock price-dividend ratio obtained by solving the present value equation with a regression residual \(\eta_t\) when fitting the model to the data. As noted by Hamilton (1986), the regression residual captures omitted variables such as time-varying real interest rates, risk premia, and changes in tax laws. Consequently, one obtains the following econometric model for the stock price-dividend ratio:

\[ \frac{P_t}{D_t} = b_0 \kappa_t + b_1 D_t^{\lambda-1} \exp\{h x_t\} + \eta_t, \quad \eta_t \sim \text{iid N}(0, \sigma^2_\eta). \]  

(22)

where \(b_0, b_1, \lambda, \) and \(h > 0\). The error term \(\eta_t\) is assumed to be independent of the innovations \(\xi_t\) to the dividend growth rate in Equation (17), at all leads and lags. \(\kappa_t\) can be thought of as a time-varying dividends multiplier.

Thus, our econometric specification, motivated by the present value model, is made up of Equations (17) and (22), for the dividend growth rates and price-dividend ratios respectively, subject to restrictions on the parameters governing the intrinsic bubble process given in Equations (15) and (16). These restrictions can now be stated as follows:

\[ r = (\lambda + h)(1 - \rho)\mu + (\lambda + h)^2 \sigma^2_\xi / 2 \]  

(23)

and

\[ h = (\lambda + h)\rho. \]  

(16)

### 4.3 Model Estimates for Dividend Growth Rates

Table 2 reports maximum likelihood estimates of the AR(1) model for dividend growth rates given in Equation (17) in the top panel. The estimates for \(\mu, \sigma^2, \) and \(\rho\) are close to the empirical mean, variance, and first-order autocorrelation coefficient of raw dividend growth rates reported in Table 1.

Froot & Obstfeld (1991) consider a random walk process for dividends, i.e. an AR(0) process for their growth rates \(x_t:\)

\[ x_t = \mu + \nu_t, \quad \nu_t \sim \text{iid N}(0, \sigma^2_\nu). \]  

(24)

Estimates of the benchmark AR(0) process for dividend growth rate are reported in Panel B of Table 2.

A test of the benchmark AR(0) versus AR(1) process for dividend growth rates can
be conducted by testing for $\rho = 0$. The likelihood ratio (LR) test for such a hypothesis is reported in the last column of Table 2. The test rejects AR(0) for both series at better than the 10 percent significance level. This provides empirical justification for considering extension of the work reported in Froot & Obstfeld (1991).

Figure 3 plots the unconditional distributions of the AR(1) and AR(0) models, along with the kernel density of the dividend growth rates. Figure 3a suggests that the AR(0) model renders a better fit to the DJIA kernel density, while Figure 3b indicates that the AR(1) model provides a better fit for the S&P 500 kernel density. However, neither appear to provide a significant improvement in fit over the other. This is not surprising, given that the maximum likelihood estimates for $\mu$ and $\sigma^2$ for both models are close to the empirical mean and variance of raw dividend growth rates reported in Table 1.

4.4 Present Value Stock Prices

In order to calculate present value stock prices implied by our model, we need a value for the constant discount rate, $r$. As in Froot & Obstfeld (1991) and Bidarkota & Dupoyet (2007), we choose $r$ equal to 8.6%. Using maximum likelihood parameter estimates from Table 2, we verify that the convergence condition given in Equation (10), required for finiteness of the present value stock price given in Equation (7), is satisfied. The time-varying present value stock price to dividends ratio, or the dividends multiplier, $\kappa_t$ is estimated by evaluating the expression on the right hand side of Equation (20), with appropriate truncation. Figure 4 plots these values for the two data series. Their mean values are reported in the last column of the top panel in Table 3. We note that these values are considerably below their empirical counterparts reported in Table 1. However, these mean $\kappa_t$ values are larger than the constant $\kappa$ estimates of about 14, reported in Froot & Obstfeld (1991) and Bidarkota & Dupoyet (2007) for the Gaussian AR(0) dividend growth rate.

4.5 Intrinsic Bubble Parameter Estimates

Implied values of the parameters $\lambda$ and $h$, governing the intrinsic bubble process given in Equation (14), are obtained by solving Equations (23) and (16).

For the benchmark AR(0) process for dividend growth rates in Equation (24), the form for the intrinsic bubbles term in Equation (14) reduces to:

$$B(D_t) = cD_t^\lambda$$

with $\lambda > 0$ for the bubble to grow with increasing dividends, and $c > 0$ to ensure non-negativity of stock prices, as for the AR(1) dividend growth rate process. The constant $h$ that appears in the AR(1) case is now equal to zero. Parameter restrictions governing the intrinsic bubble process given in Equations (15) and (16) now reduce to:

$$r = \lambda(\mu + \lambda)\sigma^2_v. \quad (26)$$

Table 3 reports values for the intrinsic bubble parameters for both AR(1) and AR(0) process for dividend growth rates. In Panel A, for the AR(1) process, the solution yields values of 1.772 and 2.148 for $\lambda$ for the two data series. By contrast, Panel B reports values of $\lambda = 2.062$ and 2.609 for the AR(0) process. For comparison, Froot & Obstfeld (1991) obtain an estimate of $\lambda = 2.74$ while Bidarkota & Dupoyet (2007) obtain $\lambda = 2.50$ for...
the Gaussian AR(0) process. This is not surprising given the expression for the intrinsic bubble in the AR(1) case given by Equation (14):

\[ B(D_t) = cD_t^\lambda \exp \{h x_t \} \]

which can be re-expressed as:

\[ B(D_t) = cD_t^{(\lambda+h)} D_t^{(-h)}. \] (27)

In Panel A, for the AR(1) process, the solution yields values of 0.557 and 0.73 for parameter h for the two data series.

### 4.6 Price-Dividend Ratio Regression

#### 4.6.1 Models

We now proceed with estimation of the econometric model for price-dividends ratios given in Equation (22). We estimate several versions of this model with the two data series. The first two rows of Table 4 list the two main models of interest at this point. The model in Equation (22), along with the restrictions specified by Equations (16) and (23), is the primary model of stock prices developed in this paper, with an AR(1) process driving the dividend growth rate and comprising of intrinsic bubbles. This is referred to as the Implied AR(1) Model. When estimating this model, the dividends growth, \( x_t \), follows an AR(1) process whose parameter values are the estimates reported in Panel A of Table 2. The values of the bubble component parameters \( \lambda \) and \( h \) are set equal to the implied parameter values reported in Panel A of Table 3. Implied AR(0) Model is the version of the above model, where the dividend growth rates follow an AR(0) process instead, given by Equation (24). The econometric model for price-dividends ratio is now:

\[ \frac{P_t}{D_t} = b_0 \kappa + b_1 D_t^{\lambda-1} + \upsilon_t, \quad \upsilon_t \sim iid N \left( 0, \sigma_\upsilon^2 \right). \] (28)

where \( \kappa = \sum_{s=t}^{\infty} \exp \left\{ -r(s-t+1) + a_{s-t} \right\} \), and \( a_{s-t} \) is given in Equation (8) with \( \rho = 0 \). This is the model estimated by Froot & Obstfeld (1991). When estimating this model, the dividends growth, \( x_t \), follows an AR(0) process whose parameter values are the estimates reported in Panel B of Table 2. As stated in the discussion in Section 3.2, in this instance, \( h = 0 \). The value of the bubble component parameter \( \lambda \) is now set equal to the implied parameter values reported in Panel B of Table 3. For each of these models, we estimate an unrestricted version which is the model described above. This is referred to as Sub-Model A. We also estimate three restricted versions, referred to as Sub-Models B, C, and D. The restrictions describing these three versions of the models are specified in the last three rows of Table 4. Sub-Model B is a semi-restricted model with \( b_0=1 \). Sub-Model C is a semi-restricted model with \( b_1=0 \). Sub-Model D is a restricted model with \( b_1=0 \) and \( b_0=1 \). As stated earlier, our primary model of interest is the Implied AR(1) Model developed in this paper in the present value context with an intrinsic bubble component. Implied AR(0) Model is the one estimated by Froot & Obstfeld (1991). Within these two models, Sub-Model A is the unrestricted version with an intrinsic bubble component whereas Sub-Model D is the most restricted version with no bubble component and the mean price-dividend ratio equal to the one dictated by the present value stock price.
4.6.2 Estimates

Tables 5a–5b present maximum likelihood estimates of the models, Implied AR(1) and AR(0) Model, described above. Each table presents estimates of all four Sub-Models A-D.

For the unrestricted Sub-Model A of the Implied AR(1) Model, we obtain estimates of \( b_0 = 0.67 \) (0.79) and \( b_1 = 0.29 \) (0.63) for the DJIA (S&P 500) series as reported in Table 5a. Estimated variance of the model error is now just over half that of the series reported in Table 1. Minimum AIC criterion selects Sub-Model B (Sub-Model A) of Implied AR(1) Model as best for the DJIA (S&P 500) series among the four Sub-Models, suggesting the importance of the bubble component in rationalizing movements in both series.

For the unrestricted Sub-Model A of the Implied AR(0) Model, we obtain estimates of \( b_0 = 0.92 \) (1.21) and \( b_1 = 0.05 \) (0.11) for the DJIA (S&P 500) series as reported in Table 5b. Thus, the estimated slope coefficients on the bubble component are now lower than those for the AR(1) model. The slope coefficients on the fundamental component imply that the estimated fundamental present values (i.e., the product of \( b_0 \) and \( \kappa \)) are very similar to the DJIA (S&P 500) constant, theoretical price-dividend ratios \( \kappa \) of 17.102 (15.02) reported in the bottom panel of Table 3, but are also much lower than empirically observed mean price-dividend ratios of 28.30 (29.79) reported in Table 1. For comparison, we note that the estimated fundamental present values and slope coefficients on the bubble component obtained here are higher for the former and lower for the latter than the estimates of roughly 14 and at least 0.26, respectively as reported in Froot & Obstfeld (1991) and Bidarkota & Dupoyet (2007) for the Gaussian AR(0) dividend growth rate. Minimum AIC criterion once again selects Sub-Model B (Sub-Model A) of the Implied AR(0) Model as best for the DJIA (S&P 500) series among the four Sub-Models.

Among all the models estimated in Tables 5a and 5b, minimum AIC criterion selects Sub-Model B of Implied AR(0) Model (Sub-Model A of Implied AR(1) Model) as best for the DJIA (S&P 500) series. Thus, the broader stock market index favors the AR(1) model developed here, whereas the blue chip index favors the Froot and Obstfeld (1991) model by this criterion.

Figures 5 and 6 plot the observed price-dividend ratios and prices, along with the fitted values of the fundamental and bubble components from Sub-Model A of both the Implied AR(1) and AR(0) Models, respectively. The contribution of the fundamental present value component alone in accounting for variation in observed ratios and prices is vividly insufficient in both figures. Taking the bubble component into account provides a much better fit to the P/D ratios and stock prices.

Figures 7 and 8 compare the performance of the fundamental present value component alone and fundamental plus bubble components, respectively, between Sub-Model A of the Implied AR(1) and AR(0) Models. Visually, the empirical performance of the two models is indistinguishable from one another.

In summary, while there is support for the AR(1) model developed here, particularly for the S&P 500 series, as compared to the AR(0) model of Froot and Obstfeld (1991), both models provide very similar performance when judged in terms of implied fluctuations in the P/D ratios and stock prices.

4.6.3 Tests of Hypotheses

A test of the present value model can be conducted by testing for the null hypothesis that \( b_0 = 1 \) and \( b_1 = 0 \) for Sub-Model A of the Implied AR(1) Model described in Equation
above. The alternative hypothesis of $b_1 > 0$ and/or $b_0 \neq 1$ is a rejection of the present value model. The null hypothesis of no intrinsic bubbles implies that $b_1 = 0$ in Equation (22). $b_1 > 0$ implies rejection of the absence of bubbles. These null and alternative hypotheses are identical in the case of Sub-Model A of the Implied AR(0) Model estimated by Froot & Obstfeld (1991).

Tables 6a-6b report likelihood ratio (LR) tests for various hypotheses of interest. In Table 6a, under an AR(1) process for dividend growth rates, LR tests for the two hypotheses, $b_1 = 0$ and joint hypothesis $b_0 = 1$ and $b_1 = 0$ are unequivocally rejected. Thus, we can reject the null hypothesis of no bubbles for this model. LR test of $b_0 = 1$ is rejected for the DJIA, but not for the S&P 500.

For comparison, in Table 6b under an AR(0) process for dividend growth rates, we find from Sub-Model A of the Implied AR(0) Model that, as in Table 6a, LR tests for the single hypothesis, $b_1 = 0$ and joint hypothesis, $b_0 = 1$ and $b_1 = 0$ are rejected. Like the Implied AR(1) Model, the no bubbles hypothesis is rejected for this model as well. However, in contrast to the results in Table 6a, the null hypothesis of $b_0 = 1$ is rejected for the S&P 500, but not for the DJIA.

To provide support of optimal model selection, Table 6c shows the LR test results between the two models. The null hypothesis of Implied AR(0) as the optimal $\hat{f}_{it}$ is unarguably rejected. In other words, there is evidence that favors the Implied AR(1) over the Implied AR(0) model. This is a critical result which lends support for our model extension from previous work.

4.6.4 Free Models - A Purely Econometric Specification

We also estimate an alternative version of the above two models, referred to as Free AR(1) Model and Free AR(0) Model, respectively. In these models, the values of the bubble component parameters are no longer set equal to the implied parameter values reported in the respective panels of Table 3, but are instead estimated freely along with the rest of the regression parameters of the econometric model.

Thus, the most general model is the Free AR(1) Model. We also have the Free AR(0) Model. Among these two models, here too, we estimate an unrestricted version, which is referred to as Sub-Model A, and three restricted versions, referred to as Sub-Models B, C, and D. The restrictions describing these three versions of the models are identical to those described earlier.

Tables 5c–5d (not shown) present maximum likelihood estimates of these two models, Free AR(1) Model and Free AR(0) Model, described above. Each table presents estimates of all four Sub-Models A-D. In Table 5c, under the AR(1) process for dividends growth and free estimation of the nonlinear bubble components, the $\lambda$ parameter estimates are 1.713 and 1.741 for the two data series. The estimate for $\lambda$ is similar to the DJIA implied parameter value in Panel A of Table 3. However, the S&P 500 $\lambda$ estimate of 1.741 is quite lower than the implied parameter value of 2.148 reported there. The parameter $h$ has been constrained to 0 in order to comply with the non-negativity assumption in our theoretical model.

Figures 9 and 10 plot observed P/D ratios and prices, along with fitted values of the fundamental and bubble components from Sub-model A of the Free AR(1) and AR(0) Model, respectively. As seen earlier in Figures 5-6 for the Implied AR(1) and AR(0) Mod-

\footnote{Complete details for ‘Free’ models are available (click ‘Full version’) at https://www.faisalawwal.com/research}
els, the fundamental component alone does not track observed index P/D ratios and prices adequately. Adding the bubble component allows the model to track the observed data remarkably better. Figure 11 (not shown) compares the fit of both the fundamental and bubble components to the observed ratios and prices of Sub-model A between Free AR(1) and AR(0) Models. Visually, the empirical performance of both models is indistinguishable from one another. Table 6c (details for ‘Free’ models not shown) confirms this indistinguishance as the LR test does not reject the hypothesis that Free AR(0) Model is the optimal model. Put another way, Free AR(1) Model is no better for predictive performance than that of the Free AR(0) Model.

For the sake of completeness, we visually and quantitatively compare Implied & Free AR(1) Models and Implied & Free AR(0) Models in Figures 12 and 13 (not shown), respectively. Again, visually the models are not considerably different from each other. Table 6d (not shown) confirms this observation for the DJIA results as the LR tests do not reject the null Implied AR(1) and AR(0) Models, respectively; however, for the S&P 500 results in Table 6e (not shown), we can easily reject Implied AR(1) and AR(0) Models. In other words, under both AR(1) and AR(0) process-driven specifications, we found no evidence that the theoretical implied values of $\lambda$ and $h$ in Table 3 are not accurate estimates of the degree of non-linearity in the price-dividend data for DJIA, but we can statistically significantly reject the S&P 500 implied parameter values.

### 4.7 Summary of Results and Model Comparison

The nonlinear price-dividend ratio regression results reported in Tables 5 and 6, and discussed in subsection 4.6 above, indicate that we can conclusively reject the null hypothesis of the absence of intrinsic bubbles across all four models, Implied and Free, AR(1) and AR(0), considered here. This inference is robust, regardless of whether or not we restrict $b_0$ to 1. It aligns with the conclusions of Froot and Obstfeld (1991) and Bidarkota and Dupoyet (2007). Further substantiating the regression results, Figures 5-6 and 9-10 clearly demonstrate that the fundamental and bubble components together track the observed stock indices more closely than the fundamental present value component alone.

We conduct additional model analysis to determine a performance leader. When using implied parameter values, the nonlinear model in which dividend growth rates follow an AR(1) process outperforms a model driven by an AR(0) process. This shows the improvement in the performance of the extension considered here when compared to the model in Froot and Obstfeld (1991).

We could not find distinct differences between each model visually as Figures 7-8 show. However, LR tests reported in Tables 6a-6c allow us to infer quantitatively that Implied AR(1) Model is statistically superior to Implied AR(0) Model. This demonstrates the usefulness of the extension to the work in Froot and Obstfeld (1991) considered here.

### 5 Conclusions

We extend the constant discount factor model with intrinsic bubbles developed in Froot & Obstfeld (1991) to account for serial correlation in the dividend growth rate. We derive an exact analytical expression for both the present value stock price and an intrinsic bubble component when dividend growth rates evolve as a Gaussian first-order autoregressive process. We estimate the model with two sets of annual U.S. stock price
and dividends data, namely the DJIA and the S&P 500 series, over the last century. We compare the results with the benchmark specification under which dividend growth rates follow an AR(0) process, as in Froot & Obstfeld (1991).

Hypotheses tests reject an AR(0) process for dividend growth rates in favor of an AR(1) process for both data series. Likelihood ratio tests also favor the AR(1)-based model developed here for price-dividends ratios to the AR(0)-based model considered in Froot and Obstfeld (1991). Information-based model selection criteria favor the AR(1) model developed here, particularly for the S&P 500 series, as compared to the AR(0) model of Froot and Obstfeld (1991). Nonetheless, the implied P/D ratios and stock prices from the two models are visually indistinguishable from one another.

Hypotheses tests also reject the absence of a bubble component in both series. This inference is robust to whether or not the parameters governing the intrinsic bubbles process are restricted to values implied by our model or freely estimated. Incorporating the bubble component into our model provides a significant improvement in fit to observed P/D ratios and stock prices as compared to the present value stock prices alone.

6 References


7 Appendix

7.1 Appendix A

Derivation of the Fundamental Stock Price

In this appendix we derive the expression for the fundamental stock price $P_{t}^{ps}$ given in Equation (7). From Equation (2), we have:

$$P_{t}^{ps} = \sum_{s=t}^{\infty} \text{exp}\{-r(s-t+1)\}E_{t}(D_{s}) \cdot (7.1.1)$$

For $s = t + 1, t + 2, \ldots$

$$D_{s} = D_{t+s-t} = \text{exp}\{d_{t+s-t}\} = \text{exp}\{x_{t+s-t} + x_{t+s-t-1} + x_{t+s-t-1} + \ldots + x_{t+1} + d_{t}\}.$$ Using Equation (15) from Burnside (1998),

$$\sum_{j=1}^{s-t} x_{t+j} = (s-t)\mu + \frac{\rho}{1-\rho} \{1 - \rho^{s-t}\} (x_{t} - \mu) + \frac{(1 - \rho)\{1 - \rho^{s-t}\}}{1 - \rho} \xi_{t+1} + (1 - \rho^{s-t}) \xi_{t+2} + \ldots + (1 - \rho)\xi_{t+s-t}. \cdot (7.1.2)$$

Therefore,

$$E_{t}(D_{s}) = D_{t} \text{exp}\left\{\frac{\sigma^{2}}{2(1-\rho)^{2}} \left\{ (s-t)\mu + \frac{\rho}{1-\rho} \{1 - \rho^{s-t}\} (x_{t} - \mu) + \frac{(1 - \rho)\{1 - \rho^{s-t}\}}{1 - \rho} \right\} \right\}. \cdot (7.1.3)$$

This simplifies to:

$$E_{t}(D_{s}) = D_{t} \text{exp}\left\{\frac{1}{2(1-\rho)^{2}} \left\{ (s-t)\mu + \frac{\rho}{1-\rho} \{1 - \rho^{s-t}\} (x_{t} - \mu) + \frac{(1 - \rho)\{1 - \rho^{s-t}\}}{1 - \rho} \right\} \right\}. \cdot (7.1.4)$$

which can be expressed as:

$$E_{t}(D_{s}) = D_{t} \text{exp}\{b_{s-t} (x_{t} - \mu) + a_{s-t}\}. \cdot (7.1.5)$$

with

$$a_{s-t} = (s-t)\mu + \frac{\sigma^{2}}{2(1-\rho)^{2}} \left\{ (s-t) - 2 \frac{\rho}{1-\rho} (1 - \rho^{s-t}) + \rho^{2} \frac{1 - \rho^{(s-t)}}{1 - \rho^{2}} \right\}. \cdot (7.1.6)$$

and

$$b_{s-t} = \frac{\rho}{1-\rho} \{1 - \rho^{s-t}\}. \cdot (7.1.7)$$

Equation (7.1.1) can be expressed as $P_{t}^{ps} = \text{exp}\{-r\}D_{t} + \sum_{s=t+1}^{\infty} \text{exp}\{-r(s-t+1)\}E_{t}(D_{s})$. Substituting for $E_{t}(D_{s})$ yields:

$$P_{t}^{ps} = \text{exp}\{-r\}D_{t} + D_{t} \sum_{s=t+1}^{\infty} \text{exp}\{-r(s-t+1) + b_{s-t} (x_{t} - \mu) + a_{s-t}\}$$

which can be rewritten as:

$$P_{t}^{ps} = D_{t} \sum_{s=t}^{\infty} \text{exp}\{-r(s-t+1) + b_{s-t} (x_{t} - \mu) + a_{s-t}\}. \cdot (7.1.8)$$

7.2 Appendix B

Proof of Theorem 1: Convergence of the Price-Dividend Ratio

From Equation (7),

\[ P^p_t = D_t \sum_{s=t}^{\infty} \exp \left\{ -r(s-t+1) + b_{s-t} (x_t - \mu) + a_{s-t} \right\} \tag{7.2.1} \]

where

\[ a_{s-t} = (s-t)\mu + \frac{\sigma^2}{2(1-\rho)^2} \left[ (s-t) - 2 \frac{\rho}{1-\rho} (1 - \rho^{s-t}) + \rho^2 \frac{1-\rho^{2(s-t)}}{1-\rho^2} \right]. \tag{7.2.2} \]

and

\[ b_{s-t} = \frac{\rho}{1-\rho} \{1 - \rho^{s-t}\}. \tag{7.2.3} \]

Denoting \( i = s - t \), \( v_t = \frac{P^p_t}{D_t} = \sum_{i=0}^{\infty} \exp \{ -r(i+1) \} \exp \{ a_i + b_i (x_t - \mu) \} \) \tag{7.2.4}

where

\[ b_i = \frac{\rho}{1-\rho} \{1 - \rho^i\} \] and \( a_i = i\mu + \frac{\sigma^2}{2(1-\rho)^2} \left[ i - 2 \frac{\rho}{1-\rho} (1 - \rho^i) + \rho^2 \frac{1-\rho^{2i}}{1-\rho^2} \right] \]

Let

\[ v_t \equiv \sum_{i=0}^{\infty} z_i \tag{7.2.5} \]

Then

\[ \frac{z_{i+1}}{z_i} = \frac{\exp \{ -r(i+1+1) \} \exp \{ a_{i+1} + b_{i+1} (x_t - \mu) \} \exp \{ -r(i+1) \} \exp \{ a_i + b_i (x_t - \mu) \} }{\exp \{ -r(i+1+1) \} \exp \{ a_{i+1} + b_{i+1} (x_t - \mu) \} \exp \{ -r(i+1) \} \exp \{ a_i + b_i (x_t - \mu) \} } \]

\[ = \exp \{ -r \} \exp \{ (a_{i+1} - a_i) + (b_{i+1} - b_i) (x_t - \mu) \} \]

\[ b_{i+1} - b_i = \frac{\rho}{1-\rho} \left\{ 1 - \rho^{i+1} - 1 + \rho^i \right\} = \rho^{i+1} \]

\[ a_{i+1} - a_i = \mu + \frac{\sigma^2}{2(1-\rho)^2} \left[ 1 - 2 \frac{\rho}{1-\rho} (1 - \rho^{i+1} - 1 + \rho^i) + \frac{\rho^2}{1-\rho^2} \left\{ 1 - \rho^{2(i+1)} - 1 + \rho^{2i} \right\} \right] \]

\[ = \mu + \frac{\sigma^2}{2(1-\rho)^2} \left[ 1 + \rho^{i+1} \left\{ \rho^{i+1} - 2 \right\} \right] \tag{7.2.7} \]

Therefore,

\[ \frac{z_{i+1}}{z_i} = \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1-\rho^2)} \left[ 1 + \rho^{i+1} \left( \rho^{i+1} - 2 \right) \right] + \rho^{i+1} (x_t - \mu) \right\}. \]

Now \( \frac{z_{i+1}}{z_i} \). With \( |\rho| < 1 \), \( \lim_{i \to \infty} \frac{z_{i+1}}{z_i} = \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1-\rho^2)} \right\}. \]

Therefore, from Burnside’s (1998) proof of Theorem 1, \( \sum_{i=0}^{\infty} z_i \) converges if

\[ R \equiv \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1-\rho^2)} \right\} < 1 \tag{7.2.8} \]
### 7.3 Appendix C

**Derivation of Equation (11): Mean of the Price-Dividend Ratio**

From Equation (7), we have:

\[
v_t \equiv \frac{P^\text{pv}}{D_t} = \sum_{i=0}^{\infty} \exp\{-r(i+1)\} \exp\{a_i + b_i (x_t - \mu)\}
\] (7.3.1)

Therefore,

\[
E(v_t) = \sum_{i=0}^{\infty} \exp\{-r(i+1)\} E[\exp\{b_i (x_t - \mu)\}].
\]

We have from Equation (6),

\[
x_t - \mu = \rho (x_{t-1} - \mu) + \xi_t, \quad |\rho| < 1, \quad \xi_t \sim \text{iid } N\left(0, \sigma^2\right). \tag{7.3.2}
\]

Therefore, \(x_t - \mu \sim N\left(0, \frac{\sigma^2}{1-\rho^2}\right)\) which implies \(b_i (x_t - \mu) \sim N\left(0, \frac{b_i^2 \sigma^2}{2(1-\rho^2)}\right)\). Therefore

\[
E[\exp\{b_i (x_t - \mu)\}] = \exp\left\{\frac{b_i^2 \sigma^2}{2(1-\rho^2)}\right\}.
\]

Therefore,

\[
E(v_t) = \sum_{i=0}^{\infty} \exp\left\{-r(i+1) + a_i + \frac{b_i^2 \sigma^2}{2(1-\rho^2)}\right\}. \tag{7.3.3}
\]

**Proof of Convergence of Mean of the Price-Dividend Ratio**

Let

\[
E(v_t) \equiv \sum_{i=0}^{\infty} Z_i. \tag{7.3.4}
\]

\[
\frac{Z_{i+1}}{Z_i} = \frac{\exp\{-r(i+1) + a_{i+1} + \frac{\sigma^2}{2(1-\rho^2)} b_{i+1}^2\}}{\exp\{-r(i+1) + a_i + \frac{\sigma^2}{2(1-\rho^2)} b_i^2\}} = \exp\left\{-r + (a_{i+1} - a_i) + \frac{\sigma^2}{2(1-\rho^2)} (b_{i+1}^2 - b_i^2)\right\}.
\]

Following from the proof on convergence of the fundamental stock price-dividend ratio \(P_t^\text{pv}/D_t\), it suffices to show that \(\lim_{i \to \infty} \frac{\sigma^2}{2(1-\rho^2)} (b_{i+1}^2 - b_i^2) = 0\) for a version of Theorem 2 in Burnside (1998) to hold.

We have from Appendix B, \((b_{i+1}^2 - b_i^2) = \frac{\rho^2}{(1-\rho)^2} \left[(1 + \rho)^i - (1 - \rho)^i\right]^2\) which implies \(b_{i+1}^2 - b_i^2 = \frac{\rho^2}{(1-\rho)^2} \left[2 - (1 + \rho)\rho^i\right]\). Therefore, \(\lim_{i \to \infty} (b_{i+1}^2 - b_i^2) = 0\).

Now, we have \(\frac{Z_{i+1}}{Z_i} = \frac{Z_{i+1}}{Z_i}. \) Therefore \(\lim_{i \to \infty} \frac{Z_{i+1}}{Z_i} = \exp\left\{-r + \lim_{i \to \infty} (a_{i+1} - a_i)\right\}\). From Equation (7.2.7) we have:

\[
a_{i+1} - a_i = \mu + \frac{\sigma^2}{2(1-\rho^2)} \left[1 - 2 \frac{\rho}{1-\rho} (1 - \rho^{i+1} - 1 + \rho^i) + \frac{\rho^2}{1-\rho^2} (1 - \rho^{2(i+1)} - 1 + \rho^{2i})\right]
\]

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\[ a_{i+1} - a_i = \mu + \frac{\sigma^2}{2(1-\rho)^2} \{ 1 + \rho^{i+1} \{ \rho^{i+1} - 2 \} \}. \]
Therefore \( \lim_{i \to \infty} (a_{i+1} - a_i) = \mu + \frac{\sigma^2}{2(1-\rho)^2} \) since \( |\rho| < 1 \). Therefore, \( \lim_{i \to \infty} \left| \frac{z_{i+1}}{z_i} \right| = \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1-\rho)^2} \right\} \). Therefore, from Burnside’s (1998) proof of Theorem 1, \( \sum_{i=0}^{\infty} z_i \) converges if \( R \equiv \exp \left\{ -r + \mu + \frac{\sigma^2}{2(1-\rho)^2} \right\} < 1 \)

### 7.4 Appendix D

**Intrinsic Bubbles under AR(1) Process for Dividend Growth Rates**

Equation (4) implies \( B_t = e^{-r}E_t \{ B_{t+1} \} \). Let \( B(D_t) = cD^\lambda_t \exp \{ hx_t \} \) where \( \lambda \) and \( h \) are constants to be determined. Therefore,

\[ E_t \{ B_{t+1} \} = E_t \{ cD^\lambda_t \exp \{ hx_{t+1} \} \} \]

\[ = E_t \{ cD^\lambda_t \exp \{ \lambda(1-\rho)\mu + \lambda p x_t + \lambda \xi_{t+1} \} \exp \{ h \{ (1-\rho)\mu + \rho x_t + \xi_{t+1} \} \} \}

Using the moment generating function of normal random variables, we obtain:

\[ E_t \{ B_{t+1} \} = cD^\lambda_t \exp \left\{ (\lambda + h)(1-\rho)\mu + (\lambda + h)\rho x_t + (\lambda + h)^2 \sigma^2/2 \right\} \]

Therefore, the r.h.s. of Equation (4) becomes:

\[ cD^\lambda_t \exp \{ (\lambda + h)\rho x_t \} \exp \{ -r + (\lambda + h)(1-\rho)\mu + (\lambda + h)^2 \sigma^2/2 \} \].

The l.h.s. of Equation (4) is \( B_t = B(D_t) = cD^\lambda_t \exp \{ hx_t \} \). Therefore, for Equation (4) to hold, we must have:

\[ r = (\lambda + h)(1-\rho)\mu + (\lambda + h)^2 \sigma^2/2 \]  \hspace{1cm} (7.4.3)

and

\[ h = (\lambda + h)\rho. \]  \hspace{1cm} (7.4.4)

### 8 Tables and Figures

**Table 1: Summary Statistics**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Index</th>
<th>Sample Size n</th>
<th>Mean</th>
<th>Variance</th>
<th>Median</th>
<th>Min</th>
<th>Max</th>
<th>FOAC</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Test for Normality</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dividend Growth Rate</td>
<td>DJIA</td>
<td>97</td>
<td>0.021</td>
<td>(0.015)</td>
<td>0.020</td>
<td>0.029</td>
<td>-0.545</td>
<td>0.502</td>
<td>-0.587**</td>
<td>7.256***</td>
<td>78.768***</td>
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<td></td>
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<td></td>
<td></td>
<td>(0.003)</td>
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</tr>
<tr>
<td></td>
<td>S&amp;P 500</td>
<td>118</td>
<td>0.017</td>
<td>(0.010)</td>
<td>0.012</td>
<td>0.024</td>
<td>-0.422</td>
<td>0.397</td>
<td>-0.795**</td>
<td>7.267***</td>
<td>101.952***</td>
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<td>(0.002)</td>
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<td></td>
<td>(17.400)</td>
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</tr>
<tr>
<td></td>
<td>S&amp;P 500</td>
<td>118</td>
<td>29.790</td>
<td>(1.440)</td>
<td>24.019</td>
<td>25.320</td>
<td>10.459</td>
<td>85.296</td>
<td>1.519***</td>
<td>5.003***</td>
<td>65.097***</td>
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<td></td>
<td></td>
<td>(31.700)</td>
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</table>

Notes: Numbers in parentheses for mean and variance are their standard errors. FOAC is the first-order autocorrelation coefficient. Numbers in parentheses for skewness and kurtosis are the p-values. The null hypotheses are no skewness and no excess kurtosis, respectively. Test for normality gives the Jarque-Bera test statistic and its p-value in parentheses. *p<0.1; **p<0.05; ***p<0.01.
Table 2: Dividend Growth Rate Process Estimates

\[ x_t - \mu = \rho (x_{t-1} - \mu) + \xi_t, \quad |\rho| < 1, \quad \xi_t \sim \text{iid } N(0, \sigma^2) \] (6)

<table>
<thead>
<tr>
<th>Panel A: ( x_t ) follows AR(1) process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Index</td>
</tr>
<tr>
<td>DJIA</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
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<table>
<thead>
<tr>
<th>Panel B: ( x_t ) follows AR(0) process</th>
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</thead>
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</tr>
<tr>
<td>DJIA</td>
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<tr>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
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<tr>
<td></td>
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</tbody>
</table>

Notes: Maximum likelihood estimates of Eq (6) for the dividend growth rate process are reported in Panel A. Maximum likelihood estimates of a restricted model with \( \rho = 0 \) are reported in Panel B. Numbers in parentheses for the parameter estimates are their standard errors. LR Test in the last column gives the likelihood ratio (LR) test statistic. P-values from \( \chi^2 \) distribution with appropriate df are in parentheses. *p<0.1; **p<0.05; ***p<0.01.

Table 3: Implied Parameter Values

<table>
<thead>
<tr>
<th>Panel A: ( x_t ) follows AR(1) process</th>
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</tr>
<tr>
<td>DJIA</td>
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<td>S&amp;P 500</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: ( x_t ) follows AR(0) process</th>
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</thead>
<tbody>
<tr>
<td>Index</td>
</tr>
<tr>
<td>DJIA</td>
</tr>
<tr>
<td>S&amp;P 500</td>
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</table>
Table 4: Description of various regression specifications for nonlinear P/D ratio models in Tables 5-6

<table>
<thead>
<tr>
<th>Model</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Implied AR(1) Model</strong></td>
<td>This is the nonlinear price-dividend regression specification in which we set ( \lambda ) and ( h ) equal to the implied parameter values in Table 3 and in which dividends growth, ( x_t ), follows an AR(1) process.</td>
</tr>
<tr>
<td><strong>Implied AR(0) Model</strong></td>
<td>This is the nonlinear price-dividend regression specification in which we set ( \lambda ) and ( h ) equal to the implied parameter values in Table 3 and in which dividends growth, ( x_t ), follows an AR(0) process.</td>
</tr>
<tr>
<td><strong>Free AR(1) Model</strong></td>
<td>This is the nonlinear price-dividend regression specification in which we estimate ( \lambda ) and ( h ) freely and in which dividends growth, ( x_t ), follows an AR(1) process.</td>
</tr>
<tr>
<td><strong>Free AR(0) Model</strong></td>
<td>This is the nonlinear price-dividend regression specification in which we estimate ( \lambda ) and ( h ) freely and in which dividends growth, ( x_t ), follows an AR(0) process.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sub-Model</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Unrestricted model</td>
</tr>
<tr>
<td>B</td>
<td>Semi-restricted model (( b_0 = 1 ))</td>
</tr>
<tr>
<td>C</td>
<td>Semi-restricted model (( b_1 = 0 ))</td>
</tr>
<tr>
<td>D</td>
<td>Restricted model (( b_1 = 0 ) and ( b_0 = 1 ))</td>
</tr>
</tbody>
</table>
Table 5a: Maximum likelihood estimation of Implied AR(1) Model

\[ \frac{P_t}{D_t} = b_0 \kappa_t + b_1 D_t^{\lambda_t-1} \exp\{hx_t\} + \eta_t, \quad \eta_t \sim iid N(0, \sigma^2_\eta) \]

<table>
<thead>
<tr>
<th>Sub-Model</th>
<th>Index</th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(\sigma^2_\eta)</th>
<th>(\log L)</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Implied AR(1): A</td>
<td>DJIA</td>
<td>0.669***</td>
<td>0.292***</td>
<td>76.309***</td>
<td>-347.874</td>
<td>701.749</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.144)</td>
<td>(0.040)</td>
<td>(0.627)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S&amp;P 500</td>
<td>0.791***</td>
<td>0.629***</td>
<td>133.267***</td>
<td>-456.085</td>
<td>918.170</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.135)</td>
<td>(0.065)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied AR(1): B</td>
<td>DJIA</td>
<td>1.000</td>
<td>0.206***</td>
<td>80.461***</td>
<td>-350.444</td>
<td>700.888</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.015)</td>
<td>(0.644)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>S&amp;P 500</td>
<td>1.000</td>
<td>0.543***</td>
<td>135.902***</td>
<td>-457.260</td>
<td>918.521</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.033)</td>
<td>(0.759)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| DJIA            | 1.644***  | 117.543***| -368.860  | 741.721          |
|                 | (0.064)   | (0.778)   |           |                 |
|                 | (0.092)   | (1.006)   |           |                 |
| DJIA            | 1.000     | 240.842***| -403.617  | 809.235          |
|                 |           |           | (1.114)   |                 |
| Implied AR(1): D | S&P 500   | 1.000     | 443.647***| -527.036        | 1056.072 |
|                 |           |           | (1.371)   |                 |

Notes: Implied AR(1) Model is the nonlinear price-dividend regression specification in which we set \(\lambda\) and \(h\) equal to the implied parameter values in Table 3 and in which dividends growth, \(x_t\), follows an AR(1) process. Sub-models A-D: Sub-model A estimates \(b_0\) and \(b_1\) freely; Sub-model B restricts \(b_0 = 1\); Sub-model C restricts \(b_1 = 0\); and Sub-model D restricts both \(b_0 = 1\) and \(b_1 = 0\). Numbers in parentheses for the parameter estimates are their standard errors. *\(p<0.1\); **\(p<0.05\); ***\(p<0.01\).
Table 5b: Maximum likelihood estimation of Implied AR(0) Model

\[
\frac{P_t}{D_t} = b_0 + b_1 D_t^{\lambda-1} + \nu_t, \quad \nu_t \sim \text{iid } N(0, \sigma^2_\nu)
\]

<table>
<thead>
<tr>
<th>Sub-Model</th>
<th>Index</th>
<th>(b_0)</th>
<th>(b_1)</th>
<th>(\sigma^2_\nu)</th>
<th>log L</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>DJIA</td>
<td>0.923***</td>
<td>0.047***</td>
<td>75.024***</td>
<td>-347.049</td>
<td>700.098</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.111)</td>
<td>(0.006)</td>
<td>(0.622)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied AR(0): A</td>
<td>S&amp;P 500</td>
<td>1.208***</td>
<td>0.105***</td>
<td>139.435***</td>
<td>-458.750</td>
<td>923.501</td>
</tr>
<tr>
<td></td>
<td>(0.112)</td>
<td>(0.011)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>DJIA</td>
<td>1.000</td>
<td>0.043***</td>
<td>75.405***</td>
<td>-347.297</td>
<td>698.593</td>
</tr>
<tr>
<td></td>
<td>(restricted to 1)</td>
<td></td>
<td>(0.003)</td>
<td>(0.623)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied AR(0): B</td>
<td>S&amp;P 500</td>
<td>1.000</td>
<td>0.121***</td>
<td>143.575***</td>
<td>-460.482</td>
<td>924.963</td>
</tr>
<tr>
<td></td>
<td>(restricted to 1)</td>
<td></td>
<td>(0.008)</td>
<td>(0.780)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>DJIA</td>
<td>1.655***</td>
<td></td>
<td>118.990***</td>
<td>-369.421</td>
<td>742.841</td>
</tr>
<tr>
<td></td>
<td>(0.065)</td>
<td></td>
<td></td>
<td>(0.783)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Implied AR(0): C</td>
<td>S&amp;P 500</td>
<td>1.984***</td>
<td></td>
<td>240.170***</td>
<td>-490.834</td>
<td>985.667</td>
</tr>
<tr>
<td></td>
<td>(0.095)</td>
<td></td>
<td></td>
<td>(1.009)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>DJIA</td>
<td>1.000</td>
<td></td>
<td>244.412***</td>
<td>-404.331</td>
<td>810.663</td>
</tr>
<tr>
<td></td>
<td>(restricted to 1)</td>
<td></td>
<td></td>
<td>(1.122)</td>
<td></td>
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</tr>
<tr>
<td>Implied AR(0): D</td>
<td>S&amp;P 500</td>
<td>1.000</td>
<td></td>
<td>458.418***</td>
<td>-528.976</td>
<td>1059.951</td>
</tr>
<tr>
<td></td>
<td>(restricted to 1)</td>
<td></td>
<td></td>
<td>(1.394)</td>
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</tr>
</tbody>
</table>

Notes: Implied AR(0) Model is the nonlinear price-dividend regression specification in which we set \(\lambda\) and \(h\) equal to the implied parameter values in Table 3 and in which dividends growth, \(x_t\) follows an AR(0) process. Sub-model A estimates \(b_0\) and \(b_1\) freely; Sub-model B model restricts \(b_0 = 1\); Sub-model C restricts \(b_1 = 0\); and Sub-model D restricts both \(b_0 = 1\) and \(b_1 = 0\). Numbers in parentheses for the parameter estimates are their standard errors. *p<0.1; **p<0.05; ***p<0.01.
<table>
<thead>
<tr>
<th>Index</th>
<th>Sub-Model</th>
<th>LR Test (Null $H_0: b_0=1$)</th>
<th>LR Test (Null $H_0: b_1=0$)</th>
<th>LR Test (Null $H_0: b_0=1$ and $b_1=0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DJIA</td>
<td>Implied AR(1): Sub-model A vs Sub-model B</td>
<td>5.140** (0.023)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model C vs Sub-model D</td>
<td>69.514*** (0.000)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model A vs Sub-model C</td>
<td></td>
<td>41.972*** (0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model B vs Sub-model D</td>
<td></td>
<td>106.350*** (0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model A vs Sub-model D</td>
<td></td>
<td>111.490*** (0.000)</td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>Implied AR(1): Sub-model A vs Sub-model B</td>
<td>2.351 (0.125)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model C vs Sub-model D</td>
<td>72.949*** (0.000)</td>
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</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model A vs Sub-model C</td>
<td></td>
<td>68.953*** (0.000)</td>
<td></td>
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<tr>
<td></td>
<td>Implied AR(1): Sub-model B vs Sub-model D</td>
<td></td>
<td>139.550*** (0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(1): Sub-model A vs Sub-model D</td>
<td></td>
<td>141.900*** (0.000)</td>
<td></td>
</tr>
</tbody>
</table>

Notes: See Table 4 for full model descriptions. Sub-model A estimates $b_0$ and $b_1$ freely; Sub-model B restricts $b_0 = 1$; Sub-model C restricts $b_1 = 0$; and Sub-model D restricts both $b_0 = 1$ and $b_1 = 0$. LR Tests give the likelihood ratio (LR) test statistic. P-values from $\chi^2$ distribution with appropriate df are in parentheses. *$p<0.1$; **$p<0.05$; ***$p<0.01$. 
## Table 6b: Sub-Model Comparisons: AR(0) Model

<table>
<thead>
<tr>
<th>Index</th>
<th>Sub-Model</th>
<th>LR Test (Null $H_0: b_0=1$)</th>
<th>LR Test (Null $H_0: b_1=0$)</th>
<th>LR Test (Null $H_0: b_0=1$ and $b_1=0$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Implied AR(0): Sub-model A vs Sub-model B</td>
<td>0.495</td>
<td>(0.482)</td>
<td></td>
</tr>
<tr>
<td>DJIA</td>
<td>Implied AR(0): Sub-model C vs Sub-model D</td>
<td>69.821***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model A vs Sub-model C</td>
<td>44.744***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model B vs Sub-model D</td>
<td>114.070***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model A vs Sub-model D</td>
<td>114.560***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model A vs Sub-model B</td>
<td>3.463*</td>
<td>(0.063)</td>
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</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model C vs Sub-model D</td>
<td>76.284***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>Implied AR(0): Sub-model A vs Sub-model C</td>
<td>64.166***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model B vs Sub-model D</td>
<td>136.990***</td>
<td>(0.000)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Implied AR(0): Sub-model A vs Sub-model D</td>
<td>140.450***</td>
<td>(0.000)</td>
<td></td>
</tr>
</tbody>
</table>

Notes: See Table 4 for full model descriptions. Sub-model A estimates $b_0$ and $b_1$ freely; Sub-model B restricts $b_0 = 1$; Sub-model C restricts $b_1 = 0$; and Sub-model D restricts both $b_0 = 1$ and $b_1 = 0$. LR Tests give the likelihood ratio (LR) test statistic. P-values from $\chi^2$ distribution with appropriate df are in parentheses. *$p<0.1$; **$p<0.05$; ***$p<0.01$. 
Table 6c: Full Model Comparisons: Implied AR(1) and AR(0) Models

<table>
<thead>
<tr>
<th>Index</th>
<th>Sub-Model</th>
<th>LR Test (Null H₀: Implied AR(0) optimal fit)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sub-model A: Implied AR(1) vs.</td>
<td>1.651*** (0.000)</td>
</tr>
<tr>
<td></td>
<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model B: Implied AR(1) vs.</td>
<td>6.295*** (0.000)</td>
</tr>
<tr>
<td></td>
<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model C: Implied AR(1) vs.</td>
<td>1.121*** (0.000)</td>
</tr>
<tr>
<td>DJIA</td>
<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model D: Implied AR(1) vs.</td>
<td>1.428*** (0.000)</td>
</tr>
<tr>
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<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model A: Implied AR(1) vs.</td>
<td>5.331*** (0.000)</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model B: Implied AR(1) vs.</td>
<td>6.443*** (0.000)</td>
</tr>
<tr>
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<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model C: Implied AR(1) vs.</td>
<td>0.545*** (0.000)</td>
</tr>
<tr>
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<td>Implied AR(0)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Sub-model D: Implied AR(1) vs.</td>
<td>3.879*** (0.000)</td>
</tr>
<tr>
<td></td>
<td>Implied AR(0)</td>
<td></td>
</tr>
</tbody>
</table>

Notes: See Table 4 for full model and sub-model descriptions. LR Tests give the likelihood ratio (LR) test statistic. P-values from \( \chi^2 \) distribution with appropriate df are in parentheses. *p<0.1; **p<0.05; ***p<0.01.
Figure 1: Plots of Dow Jones Industrial Average Data
Figure 2: Plots of S&P 500 Data

(a) Real Dividends (S&P 500)

(b) P/E Ratios (S&P 500)

(c) Real Stock Prices (S&P 500)

(d) Growth Rates of Real Dividends (S&P 500)
Figure 3: Probability distributions of real dividend growth rates

(a)

(b)
Figure 4: Plots of Time-varying $\kappa$ and Constant $\kappa$
Figure 5: Implied AR(1) Model Results

(a) Implied AR(1) Model Results (DJIA)

(b) Implied AR(1) Model Results (S&P 500)

(c) Implied AR(1) Model Results (S&P 500)

(d) Implied AR(1) Model Results (S&P 500)
Figure 6: Implied AR(0) Model Results

Implied AR(0) Model Results (DJIA)

(a)

Implied AR(0) Model Results (S&P 500)

(c)

Implied AR(0) Model Results (DJIA)

(b)

Implied AR(0) Model Results (S&P 500)

(d)
Figure 7: Comparison of fundamental components [Implied AR(1) vs. AR(0)]
Figure 8: Comparison of both fundamental and bubble components [Implied AR(1) vs. AR(0)]
Figure 9: Free AR(1) Model Results
Figure 10: Free AR(0) Model Results

(a) Free AR(0) Model Results (DJIA)

(b) Free AR(0) Model Results (S&P 500)

(c) Free AR(0) Model Results (S&P 500)