Ambiguous Leverage Cycles*

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Abstract

Financial crises often originate in debt markets, where collateral constraints and opacity of asset values are the norm. In such ambiguous contexts, beliefs formation is essential in explaining dynamics. We introduce ambiguity attitudes, which include both ambiguity aversion and ambiguity seeking (see Baillon et. al. [3] and Abdellaoui et. al.[1]), in a small open economy model where borrowers investing in risky assets face occasionally binding collateral constraints. We use GMM estimation with latent value functions to estimate the ambiguity attitudes process. The latter imply that borrowers endogenously act optimistically in booms (form right skewed beliefs) and pessimistically in recessions (form left skewed beliefs). By simulating a crisis scenario in our model we show that optimism in booms is responsible for higher asset price and leverage growth and pessimism in recessions is responsible for sharper deleveraging and asset price bursts. Analytically and numerically (using global methods) we show that our ambiguity attitudes coupled with the collateral constraints help to explain the frequency of crises, as well as asset price and debt cycle facts. At last, introducing an intermediation channel (credit supply) we show that it contributes to the severity of the crisis and to the debt pro-cyclicality, while preserving the role of ambiguity.

JEL: E0, E5, G01

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1 Introduction

Most financial crisis originate in debt markets and asset price as well as leverage cycles have important effects on the real economy. Opacity and collateral constraints are the two most notable features of debt markets and both can be a source of instability (See Holmstrom[39]). First, collateral constraints expose debt markets to the fluctuations in collateral values and the anticipatory effects associated to their endogenous changes trigger large reversal in debt and asset positions. Second, agents trading in debt markets hold doubts about the fundamental value of the collateral. In this context ambiguity attitudes and endogenous beliefs formation are crucial in determining the dynamic of asset values and debt, also since the latter is tied to the first through the collateral constraint. The surge in asset prices and leverage observed prior to most financial crises and their collapse observed following it have often been linked to a combination of institutional factors, captured by collateral constraints, and endogenous beliefs formation1. Optimism in booms, generated by assigning higher subjective beliefs to gains than to losses, can explain the surge in asset demand, prices and, through the collateral channel, in debt. Pessimism in recessions produces the opposite chain of events2. Despite the joint relevance of those elements in explaining the unfolding of financial crises, as well as the dynamic of asset prices and leverage over the business cycle, they are absent from the literature.

We fill this gap by assessing the role of ambiguity attitudes in a small open economy model where borrowers, investing in risky assets, are subject to occasionally binding collateral constraints that tie the scarcity or availability of debt to asset valuations. The latter is then affected by ambiguity attitudes, which render beliefs formation endogenous. Indeed the borrower, endowed with a sequence of subjective beliefs upon which he holds different amount of confidence, optimally chooses the degree of entropy, namely the distance between subjective and objective probability distributions, subject to bounds on it. The confidence in subjective beliefs are captured by an ambiguity parameter. Given the optimal entropy or likelihood ratio (LR hereafter), which affects also the value of risky assets through the stochastic discount factor (SDF here-after), the borrower solves optimal portfolio and leverage decisions.

1 See also Barberis[5].
2 See Barberis[5] for discussion on the role of over-confidence and under-confidence in particular for asset prices and leverage also at around the 2007-2008 financial crisis.
Importantly we depart from the standard ambiguity averse framework\(^3\) and consider all variations of ambiguity attitudes, which include ambiguity seeking\(^4\). While the latter have been neglected in macro, there is extensive experimental and survey evidence that they are pervasive: through this paper we show that in a macro model of leverage they are also crucial in accounting for the dynamic of asset prices and leverage. We capture the span of ambiguity attitudes through extended multiplier preferences (see Baillon, Bleichrodt, Huang and van Loon\(^3\) for theoretical foundations). To provide sound empirical ground we determine the mapping between the ambiguity attitudes and the states of the economy through structural estimation of the model. Specifically, we estimate the ambiguity parameter of our extended multiplier preferences through non-linear method of moments applied to our model-based combined Euler equation, in debt and risky asset\(^5\). To this purpose we develop a novel estimation procedure for models with collateral constraints and uncertainty. The estimated value for the ambiguity parameter results being negative, capturing ambiguity seeking, in good states, namely when the current realization of the value function is above its mean, and above zero in bad states. Those attitudes endogenously result in optimism or right-skewed beliefs in booms and pessimism in recessions\(^6\). This structure of the beliefs coupled with the anticipatory effects, which are typically associated with occasionally binding collateral constraints\(^7\), have important implications for asset price, debt capacity and leverage dynamic. Consider a boom. Borrowers endogenously tend to act optimistically and increase their demand of risky assets. This boosts asset prices and through anticipatory effects also the demand of debt, which in turn endogenously relaxes the constraint. This is also consistent with the fact that in booms the evaluation of optimistic agents drives the debt capacity. The opposite is true in the loss domain, defined as shocks that bring the value function below its mean.

\(^{3}\) See pioneering work by Hansen and Sargent\(^{[32]}\), \(^{[33]}\) and Maccheroni, Marinacci and Rustichini\(^{[53]}\).

\(^{4}\) Ambiguity seeking is strongly supported in experimental evidence. See Dimmock et. al.\(^{[21]}\) and \(^{[22]}\), Baillon et. al.\(^{[3]}\) and Trautmann and van Kuilen\(^{[62]}\) among others.

\(^{5}\) For this we use the procedure developed in Chen, Favilukis, and Ludvigson\(^{[16]}\), where one step involves the estimation of a latent unobservable variable given by the continuation value ratio.

\(^{6}\) Our macro estimates are well in line with experimental evidence. Abdellaoui et. al.\(^{[1]}\) provide foundations for S-shaped preferences with changing ambiguity attitudes and show through experimental evidence that pessimism (left-skewed beliefs) prevails in face of losses, while optimism prevails in face of gains. Further experimental evidence by Boiney\(^{[12]}\) and Kraus and Litzenberg\(^{[47]}\) has associated ambiguity seeking (aversion) with right (left) skewed beliefs. On another front, survey evidence by Rozszypal and Schlafmann\(^{[57]}\), shows that low-income households hold pessimistic beliefs about the future, while the opposite is true for high-income households.

\(^{7}\) Mendoza\(^{[54]}\) shows that the occasionally binding nature of the collateral constraints gives a role to anticipatory effects. As agents expect the constraint to bind in the future, they off-loads risky assets and debt in anticipation.
With the above model we obtain a series of analytical and numerical results related to asset prices and debt dynamic. Analytically we discuss implications for asset prices and the Sharpe ratio. For the first, we show that the conditional LR heightens asset price growth in booms and depresses it in recessions. Second, the kink in the stochastic discount factor induced by the shift from optimism to pessimism helps to move the model-based Sharpe ratio closer to the Hansen and Jagannathan[28] bounds.

Next, we solve our model numerically by employing global non-linear methods with occasionally binding constraints. The policy functions and a simulated crisis event, which allow us to discuss the economic intuition behind our model, show that optimism increases the build-up of leverage in booms, while pessimism steepens the recessionary consequence of the crisis. In both cases the comparison is done relatively to a model featuring solely collateral constraints, but no deviations between subjective and objective beliefs. Ambiguity attitudes play a crucial role in this result. In booms optimism boosts collateral values, hence, by relaxing the constraint, it facilitates the build-up of leverage, asset demand and the asset price boom. In recessions pessimism materializes, which drives the transmission channel in the opposite direction. To subject our model and belief formations process to further rounds of empirical validation, we estimate all parameters by minimizing the distance between some targeted model-based moments and their empirical counterparts using data for the US economy over the sample 1980-2016, namely the sample of both a rapid growth in leverage and then a sudden collapse in debt positions. Under the optimized calibration, the model can match asset price volatilities and equity premia (both the long run and the dynamic pattern), returns, Sharpe ratios, volatilities of debt and its pro-cyclicality. The comparison with the model featuring solely the collateral constraint shows that our model performs better in the data matching. To explain asset price facts borrowers’ ambiguity attitudes over the tails are crucial.

The rest of the paper is structured as follows. Section 2 compares the paper to the literature. Section 3 describes the model and the ambiguity attitudes specification. Section 4 presents the estimation procedure and results. Section 5 investigates analytical results. Section 6 discusses quantitative findings (the solution method is detailed in the appendix). Section 7 concludes.

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8 We employ policy function iterations based on a Tauchen and Haussey[61] discretization of the state space and by accommodating different regimes (portions of the state space) with binding or non-binding constraints.

9 It is well documented by Jorda, Schularick and Taylor[43] at aggregate level and using historical data. But it is also well document for consumer debt, see for instance Fieldhouse, Livshits, MacGee[51] among others.
pendices, Tables and figures follow.

2 Comparison with Past Literature

Following the 2007 financial crisis which was triggered by panics in various debt markets (for structured products, for short-term bank funding and in repo markets, see Gorton and Metrick[26]) there has been a growing interest in understanding the determinants and the dynamics of the leverage cycle and the role of the underlying externalities (pecuniary and demand) for the real economy. Most recent literature tends to assess the dynamic of debt over the business cycle through models with occasionally binding constraints. Papers on this topic include Geanakoplos[25], Lorenzoni[52], Mendoza[54], which among many others examine both positive and normative issues related to the leverage cycle. Papers focusing on the positive aspects show that anticipatory effects produced by occasionally binding constraints are crucial in generating sharp reversals in debt markets and in establishing the link between the tightening of the constraint and the unfolding of financial crisis.

None of the past papers however assesses the joint role of financial frictions, in the form of collateral constraints, and belief formation, while both play a crucial role in determining the asset price and leverage cycle in normal times and in explaining endogenously the unfolding of crises even in face of small shocks. One exception is Boz and Mendoza[13] which introduces learning on asset valuation in a model with occasionally binding collateral constraints. Contrary to them our beliefs are endogenously formed based on ambiguity attitudes toward model mis-specification. Moreover none of the past papers conducts a quantitative analysis aimed at assisting the quantitative relevance of those elements in jointly matching asset price and debt facts and cyclical moments.

The relevance of ambiguity and of the beliefs formation process is crucial in debt markets in which opacity is the norm (see Holmstrom[39]). Indeed, contrary to equity markets in which buyers of the asset wish to exert monitoring and control on the investment activity, participants in debt markets usually trade under the ignorance of the fundamental value of collateral. For this reason in debt markets a collateral guarantee is part of the contractible set-up. This indeed serves the purpose of overcoming the pervasive asymmetric information. However even if the information asymmetry underlying the specific debt relation is solved through the contracts, doubts remain about the fundamental value of the asset, implying that optimism or pessimism of subjective beliefs affect the
agents’ saving and investment problem, hence the dynamic of asset prices and leverage. Despite the realism and importance of the connection between ambiguity and debt dynamic, this nexus has not been studied so far.

Since we choose to model endogenous beliefs formation through ambiguity attitudes our model is also connected to the literature on ambiguity aversion (see Hansen and Sargent[32], [33] and Maccheroni, Marinacci and Rustichini[53]). In this context some papers also assess the role of ambiguity aversion for asset prices or for portfolio allocation. For instance Barillas, Hansen and Sargent[7] show that ambiguity aversion is akin to risk-sensitive preferences a’ la Tallarini[60] and as such it helps the model’s Sharpe ratio to get closer to the Hansen and Jagannathan[28]\textsuperscript{10}. Epstein and Schneider[?] also analyze the properties of asset prices focusing on More recently in a production economy Bianchi, Ilut and Schneider[11] have assessed the role of ambiguity aversion for firms debt policies and stock prices. We depart from this literature in two important ways. First, we model ambiguity attitudes that encompass both ambiguity aversion and ambiguity seeking behavior. Ambiguity seeking is well documented in experimental evidence (see Dimmock et. al.[21] and [22], Baillon et. al.[3] and Trautmann and van Kuilen[62], Roca, Hogarth and Maule[56] among others). We introduce the whole span of ambiguity attitudes through the extended multiplier preferences, which have been founded theoretically by Baillon, Bleichrodt, Huang and van Loon[3]. We confirm the existence and significance of ambiguity attitudes through time-series estimation or our model. Furthermore, it is only by accounting jointly for ambiguity aversion and ambiguity seeking that our model is able to match numerically the volatilities, the persistence and the cyclical behavior of asset prices and debt.

At last our paper contributes to the literature on the estimation of SDF with behavioral elements. The closer contribution to ours is Chen, Favilukis, and Ludvigson[16]. A series of papers have developed procedures for SDF estimation. We review most of them in the section describing our model estimation. An important aspect we contribute to this literature is the development of an estimation procedure for a model which jointly accounts for collateral constraints and for ambiguity attitudes. Our estimation uncovers the state-contingent nature of ambiguity attitudes, namely optimistic in booms and pessimistic in recessions, while not previously noted in the literature.

\textsuperscript{10}On a different line of research Benigno and Nistico’[8] show how ambiguity averse preferences can be used to explain the home bias in international portfolio allocations due to the need to hedge against long run risk.
3 A Model of Ambiguous Leverage Cycle

Our baseline model economy is an otherwise standard framework with borrowers facing occasionally binding collateral constraints. One of the novel ingredients stems from the interaction between ambiguity attitudes and debt capacity. Debt supply is fully elastic with an exogenous debt rate as normally employed in most recent literature on the leverage cycle. Collateral in this economy is provided by the value of the risky asset funded through debt. To this framework we add ambiguity attitudes, which includes both ambiguity aversion and seeking. The latter is modelled through the extended multiplier preferences, for which Baillon et. al. [3] and Abdellaoui et. al.[1] have provided experimental evidence and theoretical foundation. The underlying logic is similar to the one pioneered and proposed by the game-theoretic approach a’ la Hansen and Sargent[33] in which agents are assumed to have fears of model mis-specification and play a two-stage game with a malevolent agent (nature) that amplifies deviations from the true probability model and helps the borrower to explore the fragility of a decision rule with respect to various perturbations of the objective shock distribution. Hansen and Sargent[33] focus on ambiguity averse attitudes. Under this case the game of interaction between the agents and nature results in the latter inducing more pessimistic beliefs with the goal of testing agents’ ability to make robust decisions. Agents therefore optimally attempt to minimize the distortion induced by nature, by enforcing a positive penalty parameter under the case of maxmin preferences.

While ambiguity aversion has been the norm in macro and finance models, a crucial departure introduced by our framework is to consider the whole span of ambiguity attitudes, namely ambiguity aversion and ambiguity seeking. Extensive experimental studies (reviewed above) finds support for both. To include the whole span of ambiguity attitudes we employ the extended multiplier preferences discussed in Baillon et. al. [3] and Abdellaoui et. al.[1], as explained above. Moreover through empirical analysis we uncover the state-contingent nature of the ambiguity attitudes, whereby aversion prevails in recessions and ambiguity seeking prevails in booms.

Importantly the contingent reason for considering this extended set-up for ambiguity attitudes is that, as our analysis below shows through several steps, this is crucial for explaining the facts

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11 This model economy corresponds to a limiting case in which lenders are risk-neutral. Alternatively the model can be interpreted as a small open economy with debt supplied from the rest of the world.
we focus on, namely the patterns observed around the unfolding and development of debt crises as well as the full array of asset price and debt statistics. At this stage it is also useful to mention that within the structure of the zero-sum game the economic interpretation of the ambiguity seeking attitudes is just similar to the one described above for the ambiguity averse attitudes. This implies that under positive realizations of income nature induces optimistic beliefs with the intent again of testing robustness of agents’ decisional process. Given this interpretation, such beliefs formation process is also akin to the one considered in Brunnermeier and Parker[14] in which a small optimistic bias in beliefs typically leads to first-order gains in anticipatory utility.12

Below, we show that ambiguity aversion results endogenously in left-skewed or pessimistic beliefs, relatively to rational expectation, namely relatively to the case in which objective and subjective beliefs coincide. On the other side ambiguity seeking results in right-skewed or optimistic beliefs. Importantly the changing nature of the ambiguity attitudes contributes to the occasionally binding nature of the collateral constraint. As agents become optimist their demand for risky assets contributes to boost collateral values and to expand debt capacity. The opposite is true with pessimism.

3.1 Beliefs Formation and Preferences

The source of uncertainty in the model is a shock to aggregate income $y_t$, which is our exogenous state and follows a finite-space stationary Markov process. We define the state space as $S_t$, the realization of the state at time $t$ as $s_t$ and its history as $s^t = \{s_0, s_1, \ldots, s_t\}$ with associated probability $\pi(s^t)$. The initial condition of the shock is known and defined with $s_{-1}$.

Borrowers are endowed with the approximated model $\pi(s^t)$ over the history $s^t$ but they also consider alternative probability measures, indicated by $\tilde{\pi}(s^t)$, which deviate from $\pi(s^t)$.13 Borrowers can have different degrees of trust in their own subjective beliefs, so that act as ambiguity averse when they fear deviations from the approximated model and they act as ambiguity seeking when they hold high confidence in their beliefs. Following the relevant literature, we introduce the

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12 Some connections between the economic interpretation of ambiguity seeking attitudes under loss domain can also be traced with the news averse preferences introduced by Koszegi and Rabin[45] and examined more recently in asset price context by Page[55]. In this case as well agents prefer not to receive news fearing the bad ones.

13 The alternative probability measure $\tilde{\pi}$ is absolutely continuous with respect $\pi$. This means that events that receive positive probability under the alternative model, also receive positive probability under the approximating model.
measurable function \( M(s') = \hat{\pi}(s')/\pi(s') \), which we define as the likelihood ratio. We can also define the conditional likelihood ratio as, \( m(s_{t+1}|s') = \hat{\pi}(s_{t+1}|s')/\pi(s_{t+1}|s') \). For ease of notation since now onward we use the following notation convention: \( M_t = M(s') \), \( M_{t+1} = M(s^{t+1}) \) and \( m_{t+1} = m(s_{t+1}|s') \), where the sub-index refers to the next period state. The above definition of \( M_t \) allows us to represent the subjective expectation of a random variable \( x_t \) in terms of the approximating probability models:

\[
\tilde{E}_t[x_t] = E_t[M_t x_t] \quad (1)
\]

where \( E_t \) is the subjective expectation operator conditional to information at time \( t \) for the probability \( \pi(s') \), while \( \tilde{E}_t \) is the expectation operator conditional to information at time \( t \) for the probability \( \hat{\pi}(s') \). The function \( M_t \) follows a martingale process and as such it satisfies the following condition \( E[M_{t+1}] = M_t \). We can decompose \( M_t \) as follows

\[
m_{t+1} = \frac{M_{t+1}}{M_t} \quad \text{for} \quad M_t > 0 \quad (2)
\]

and \( m_{t+1} = 1 \) for \( M_t = 0 \). These incremental deviations satisfy condition \( E_t[m_{t+1}] = 1 \). Moreover, the discrepancy between the approximated and the subjective models is measured by the conditional entropy, defined as follows:

\[
\varepsilon(m_{t+1}) = E_t \{ m_{t+1} \log m_{t+1} \} \quad (3)
\]

where \( \varepsilon(m_{t+1}) \) is a positive-valued, convex function of \( \pi(s') \) and is uniquely minimized when \( m_{t+1} = 1 \), which is the condition characterizing the case with no ambiguity attitudes.

Given the probabilistic specifications above, we now introduce the following extended multiplier preferences:

\[
V(c_t) = \begin{cases} 
\min_{\{m_{t+1}, M_t\}_{t=0}^\infty} \sum_{t=0}^\infty E_0 \{ \beta^t \pi_t M_t u(c_t) + \beta \theta_t \varepsilon(m_{t+1}) \} & \text{if } \theta_t > 0 \\
\sum_{t=0}^\infty E_0 \{ \beta^t \pi_t u(c_t) \} & \text{if } \theta_t = 0 \\
\max_{\{m_{t+1}, M_t\}_{t=0}^\infty} \sum_{t=0}^\infty E_0 \{ \beta^t \pi_t M_t u(c_t) + \beta \theta_t \varepsilon(m_{t+1}) \} & \text{if } \theta_t < 0
\end{cases} \quad (4)
\]

\[
U(c_t) = \frac{c_t^{1-\gamma} - 1}{1 - \gamma} \quad (5)
\]

where \( \theta_t \in \mathbb{R} \) is a process capturing the degree of doubts about the prevailing model, which include ambiguity aversion and ambiguity seeking. The second, not explored in macro so far,
capture the idea that under some states agents have high confidence in their subjective beliefs and value deviations of entropy from a reference level, which captures the case with no ambiguity. Alternatively one can interpret the ambiguity seeking behavior as news aversion\textsuperscript{14}. The multiplier nature of the preferences stem from the inclusion of a bound constraint on entropy, whose multiplier is given by $\theta_t$. The theoretical foundation for their extension to include ambiguity seeking behavior is due to Baillon, Bleichrodt, Huang and van Loon\cite{3}. Note that the specification of the state-contingent nature of $\theta_t$, namely under which states it turns positive or negative, will be obtained through conditional GMM estimation of the model implied equations. This allows us to give sound theoretical ground to the ambiguity process. For the time being we can anticipate that $\theta_t$ turns negative, therefore capturing ambiguity seeking, in good states and positive, therefore capturing ambiguity aversion, in bad states. This, through the solution of the optimal likelihood ratio, will produce endogenously optimistic (right-skewed) beliefs in good-states and vice-versa in bad states. For this reason, as we note further below, our estimated process is well in line with most experimental studies, which find an association between the booms (recessions) and optimistic (pessimistic) behavior.

The rest of the model follows a standard leverage cycle model with risky assets that serve as collateral (see e.g. Mendoza\cite{54}). The representative agent holds an infinitely lived asset $x_t$, which pays a stochastic dividend $d_t$ every period and is available in fixed unit supply. The asset can be traded across borrowers at the price $q_t$. In order to reduce the dimension of the state space, we assume that the dividend is a fraction $\alpha$ of the income realization. Therefore, we indicate with $(1-\alpha)y_t$ the labor income and with $d_t = \alpha y_t$ the financial income. Agents can borrow using one-period non-state-contingent bonds that pay an exogenous real interest rate $R$. The budget constraint of the representative agents can be expressed as following:

$$c_t + q_t x_t + \frac{b_t}{R} = (1-\alpha)y_t + x_{t-1}[q_t + d_t] + b_{t-1}$$

(6)

where $c_t$ indicates consumption and $b_t$ the bond holdings. The agents' ability to borrow is restricted to a fraction $\phi$ of the value of asset holding:

$$-\frac{b_t}{R} \leq \phi q_t x_t$$

(7)

\textsuperscript{14}See Koszegi and Rabin\cite{45} and more recently of Pagel\cite{55} for models with news aversion.
The collateral constraint depends on the current period price of the asset in order to reproduce fire-sales driven amplification dynamics, which for this simple model would not be produced with a different formulation of the constraint.\textsuperscript{15}

### 3.2 Recursive Formulation

Following Hansen and Sargent \[33\], we rely on the recursive formulation of the problem, which allows us to re-write everything only in terms of \(m_{t+1}\). The recursive formulation shall of course be adapted to capture the changing nature of the ambiguity attitudes.

We now partition the state space \(S_t\) in the two blocks, given by the endogenous and the exogenous states, \(S_t = \{B_t, y_t\}\), where \(B_t\) is the aggregate bond holdings and \(y_t\) the income realization. Note that the aggregate asset holdings is not a state variable because it is in fixed supply. Moreover, the problem is also characterized by the two individual state variables \((b_t, x_t)\).

For the recursive formulation we employ a prime and sub-index to indicate variables at time \(t+1\) and no index for variables at time \(t\). The borrowers’ recursive optimization problem reads as follows. Conditional on \(\theta_t > 0\), the recursive two-stage optimization reads as follows:

\[
V(b, x, S) = \max_{c, x', b'} \min_{m'} \max \{u(c) + \beta \mathbb{E}_S [m' V(b', x', S) + \theta m' \log m'] + \\
+ \lambda \left[ y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right] + \\
+ \mu \left[ \phi q(S)x' + \frac{b'}{R} \right] + \beta \theta \psi [1 - \mathbb{E}_S m'] \}
\]

(8)

Conditional on \(\theta_t < 0\), the recursive two-stage optimization reads as follows:

\[
V(b, x, S) = \max_{c, x', b'} \min_{m'} \max \{u(c) + \beta \mathbb{E}_S [m' V(b', x', S) - \theta m' \log m'] + \\
+ \lambda \left[ y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right] + \\
+ \mu \left[ \phi q(S)x' + \frac{b'}{R} \right] + \beta \theta \psi [1 - \mathbb{E}_S m'] \}
\]

(9)

where the aggregate states follow the law of motion \(S' = \Gamma(S)\). In the above problem \(\lambda\) and \(\mu\) are the multipliers associated to the budget and collateral constraints respectively, while the term \(\beta \theta \psi\) is the multiplier attached to the constraint \(\mathbb{E}_S[m'] = 1\).

\textsuperscript{15}Moreover, Bianchi and Mendoza (2015) provide a micro-founded derivation of this constraint, based on a limited enforcement problem.
The above optimization problems are solved sequentially. First an inner optimization and then an outer optimization problem are derived sequentially. In the first stage agents choose the optimal incremental probability distortion for given saving and portfolio choices. In the second stage, for given optimal likelihood ratio, they solve the consumption/saving problem and choose the optimal amount of leverage. Intuitively, the problem is modelled as a game of strategic interactions between the maximizing agents, who face Knightian uncertainty\textsuperscript{16}, and a malevolent agent that draws the distribution (see Hansen and Sargent\cite{33} who proposed this reading).

3.2.1 The Inner Optimization Problem

Through the inner optimization problem the borrowers choose the optimal entropy or conditional likelihood ratio, namely the optimal deviation between his own subjective beliefs and the objective probability distribution.

The first order condition with respect to $m^{'},$ which is functionally equivalent under the two cases, is given by:

$$V(b^{'}, x^{'}, S^{'}) + \theta(\log m^{'}) + \theta \psi = 0$$

(10)

Rearranging terms, we obtain:

$$1 + \log m^{' } = - \frac{V(b^{'}, x^{'}, S^{'})}{\theta} + \psi$$

(11)

$$m^{' } = \exp \left\{ \frac{-V(b^{'}, x^{'}, S^{'})}{\theta} \right\} \exp \{\psi - 1\}$$

Finally, imposing the constraint over probability deviation $m^{'},$ and defining $\sigma = -\frac{1}{\theta}$ we derive the optimality condition for the conditional likelihood ratio:

$$m^{' } = \exp \left\{ \frac{\sigma V(b^{'}, x^{'}, S^{'})}{\theta} \right\} \mathbb{E}[\exp \{\sigma V(b^{'}, x^{'}, S^{'})\}]$$

(12)

Equation 12 also defines the state-contingent incremental probability deviation from the rational expectation case. The magnitude and the direction of this deviation depends on the agents’ value function and the value for the inverse of $\sigma$. We will return on the role of the optimal conditional likelihood ratio later on in section 3.2.3.

\textsuperscript{16}Knight\cite{44} advanced the distinction between risk, namely the known probability of tail events, and uncertainty, namely the case in which such probabilities are not known. Ambiguity usually refers to cases of uncertainty where the state space is well defined, but objective probabilities are not available.
3.2.2 The Outer Optimization Problem

For given optimal LR $m'$ the borrower solves an outer optimization problem in consumption, risky assets and debt. Upon substituting the optimal LR into the value function, the maximization problem reduces to find the optimal allocations of consumption, bond holding and asset holdings. The resulting recursive problem is:

$$V(b, x, S) = \max_{c, x', b'} \left( u(c) + \frac{\beta}{\sigma} \log \left[ \mathbb{E}_S \exp \left\{ \sigma V(b', x', S') \right\} \right] + \lambda \left[ y + q(S)((x + d) + b - q(S)x' - c - \frac{b'}{R}) \right] + \mu \left[ \phi q(S)x + \frac{b'}{R} \right] \right)$$

We will now derive and list all the competitive equilibrium conditions. Since now we return to the notation with $t$ and $t + 1$ indices as this is needed for our analytical derivations in section 5.

The borrowers' first order condition with respect to bond holding and risky assets reads as follows:

$$u_c(c_t) = \beta \mathbb{E}_t \{ m_{t+1}u_c(c_{t+1}) \} + \mu_t$$

$$q_t = \frac{\beta \mathbb{E}_t \{ m_{t+1}u_c(c_{t+1})[q_{t+1} + \alpha y_{t+1}] \}}{u_c(c_t) - \phi t} \tag{15}$$

where $u_c$ indicates the derivative of the utility function with respect to consumption. Equation 14 is the Euler equation for bonds and displays the typical feature of models with occasionally binding collateral constraint. In particular, when the constraint binds there is a wedge between the current marginal utility of consumption and the expected future marginal utility, given by the shadow value of relaxing the collateral constraint. Equation 15 is the asset price condition.

Note that ambiguity attitudes, hence beliefs, affect asset prices since $m_{t+1}$ enters the optimality conditions for risky assets, 15, and they affect the tightness of the debt limit as $m_{t+1}$ enters the optimality conditions for risky assets 14. In other words the optimal $m_{t+1}$ affects the stochastic discount factor and through this it affects the pricing of all assets in the economy.

The model characterization is completed with the complementarity slackness condition associated to the collateral constraint:

$$\mu_t \left[ \frac{b_{t+1}}{R} + \phi q_t \right] = 0 \tag{16}$$
and with the goods and stock markets clearing conditions:

\[ c_t + \frac{b_{t+1}}{R} = y_t + b_t \]  \hspace{1cm} (17)

\[ x_t = 1 \]  \hspace{1cm} (18)

**Definition 1. (Competitive Equilibrium)** A competitive equilibrium is an allocation \( \{c_t, b_t\}_{t=0}^{\infty} \), probability deviations \( \{m_{t+1}\}_{t=0}^{\infty} \) and prices \( \{q_t\}_{t=0}^{\infty} \) such that:

- given the allocation and prices, the probability distortions solve the inner optimization problem;
- given the probability distortions and prices, the allocation solves the outer optimization problem;
- the allocation is feasible, satisfying 17 and 18.

### 3.3 Pessimism and Optimism

To determine under which states the lagrange multiplier, \( \theta_t \), turns positive or negative we will estimate our model implied Euler equations through GMM in the next section. In the meantime it is useful to discuss how the ambiguity averse or ambiguity seeking attitudes affect the endogenous formation of beliefs, as captured by the optimal likelihood ratio. For simplicity of exposition we report the optimal condition for variable \( m_{t+1} \):

\[ m_{t+1} = \frac{\exp \{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}}{\mathbb{E}_t[\exp \{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}]} \]  \hspace{1cm} (19)

The conditional deviation affects how agents assign different subjective probabilities (with respect to the objective ones) to future events, which can be characterized by high and low utility. In particular, if \( m_{t+1} > 1 \) agents assign an higher subjective probability, while if \( m_{t+1} < 1 \) the opposite holds. Given this, the sign of the parameter \( \sigma_t \) affects how these conditions are linked to positive or negative future state realizations.\(^{17}\) The following lemma summarizes this consideration and defines optimism and pessimism in the agents’ attitude.

\(^{17}\)Concerning the size of the distortion, we can say that a large absolute value of \( \theta \) increases the probability distortion in all future states, meaning that \( m' \) is close to unity. At the contrary, a small absolute value of \( \theta \), implies that the decisions are far from the rational expectation setting.
Lemma 1. When $\theta_t < 0$ then $m_{t+1} > 1$ in good states and $m_{t+1} < 1$ in bad states. Hence, beliefs endogenously emerge as right-skewed and agents act with optimism. When $\theta_t > 0$ the opposite is true.

Proof. First we define good states as those in which the current state value function is above its expected value. When $\theta_t < 0$, then $\sigma_t > 0$, in good states $\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\} > \mathbb{E}_t[\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}]$, namely the risk-adjusted value function for the good states is larger than the average one. Based on the above equation, this implies that $m_{t+1} > 1$. The opposite is true in bad states. When $\theta_t > 0$, then $\sigma_t < 0$, this implies that in good states $\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\} < \mathbb{E}_t[\exp\{\sigma_t V(b_{t+1}, x_{t+1}, S_{t+1})\}]$, namely the risk-adjusted value function for the good states is lower than the average one and $m_{t+1} < 1$. The opposite is true in bad states.

3.3.1 Optimal Beliefs Formation

To gain some intuition we discuss a particular case with only two income states, which we define as high, with a sup-index $h$, and low, with a sup-index $l$. We also consider only three periods which we label as $t = 0$, $t = 1$ and $t = 2$. By assumption the high state is high enough that the collateral constraint is slack, while the opposite is true for the low state. This facilitates the computation of the expectation operators. The states have a binomial probability structure such that state $h$ realizes with probability $\pi$, while the state $l$ with its complement $1 - \pi$. Equipped with these assumptions we can characterize the dynamic between time 0 and time 1. In this case the likelihood ratio can be specified as follows:

$$m_1 = \frac{\exp\{\sigma_0 V_1\}}{\pi \exp\{\sigma_0 V_1^h\} + (1 - \pi) \exp\{\sigma_0 V_1^l\}}$$

(20)

where $V_1^h > \mathbb{E}_0\{V_1\}$ and $V_1^l < \mathbb{E}_0\{V_1\}$. Note that depending on the time zero realization of the state we have two different values of the inverse of the penalty parameter, $\sigma_0$. To fix ideas imagine that the income realization at time zero is the low state, $l$. Given our Lemma 1 we have that $\sigma_0^l < 0$. The latter implies that $\exp\{\sigma_0^l V_1^h\} < \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$ and $\exp\{\sigma_0^l V_1^l\} > \mathbb{E}_0\{\exp\{\sigma_0^l V_1\}\}$. Therefore, the marginal likelihood ratio are $m_1^h < 1$ and $m_1^l > 1$. As a consequence, we can define the following subjective probabilities as:

$$\omega^h = \pi m_1^h < \pi \quad \omega^l = (1 - \pi)m_1^l > (1 - \pi)$$

(21)

15
As we can see, agents assign a higher (lower) subjective probability - with respect to the objective probability - to the future negative (positive) events, typical of a pessimistic attitude. The opposite is true when $\sigma_o^l < 0$. In this case $\exp\{\sigma_o^h V_1^l\} > \mathbb{E}_0 \{\exp \{\sigma_o^h V_1\}\}$ and $\exp\{\sigma_o^l V_1^l\} < \mathbb{E}_0 \{\exp \{\sigma_o^l V_1\}\}$ producing $m'^h_l > 1$ and $m'^l_l < 1$. Therefore, agents assign higher (lower) subjective probability to the future positive (negative) events, showing an optimism attitude:

$$\omega^h = \pi m'^h_l > \pi \quad \omega^l = (1 - \pi)m'^l_l < (1 - \pi)$$  \hfill (22)

The interesting feature of this state-contingent behaviour concerns its connections with asset prices, the value of collateral and leverage. Further below we explain this in more details through analytical derivations and quantitative analysis. Intuitively, optimism explains why asset price booms and leverage build-ups are steeper in booms and relatively to the model with no beliefs formation. To fix ideas consider the case with a negative $\theta_t$ and that the borrower experiences a good state today and expects a good state tomorrow. Asset price would grow even in the case with no ambiguity attitudes, however under our extended multiplier preferences, borrowers form today subjective beliefs that induce an LR of $m_{t+1}^h > 1$. As this scales up the SDF in the Euler conditions for debt and for the risky assets, it induces higher demand for both. This is why we label this case as optimism. Consider now the opposite case, namely $\theta_t$ lower than zero. According to Lemma 1 now the optimal LR is left skewed, namely lower than one in good states and larger than one in bad states. In other words the borrower becomes pessimistic. If the bad state materializes in this case, asset prices will fall according to equation 15 and they would do so more sharply than under when $m_{t+1} = 1$ across all states of nature. Hence we shall conclude that pessimism explains why asset price bursts and de-leverages are sharper in recessions and relatively to the case with no ambiguity attitudes. Appendix F considers a more extended version of the three periods model and also shows analytically that our ambiguity attitudes interacting with the collateral constraint induces higher debt levels in booms.

As we explain further below in the quantitative section the heightened dynamic of asset prices and leverage (relatively to the case with no ambiguity attitudes) also allows the model to match high volatility of asset prices, equity premia, Sharpe ratios and debt. The main intuition is the following. To obtain enhanced volatility of asset prices and leverage it is typical to assume that agents assign large weight to the tails (see the literature on long run risks, Bansal, Kiku and Yaron[4]). In our
case this feature emerges endogenously as a result of the inner optimization problem, as under
the ambiguity seeking scenario the subjective probability mass is shifted to the upper tail and
under the ambiguity adverse scenario the subjective probability mass is shifted on the lower tail
(also compared to the case with no ambiguity). Furthermore, the asset price literature noted (see
Cochrane[17] and Hansen and Jagannathan[28]) that in standard consumption-based models the
Sharpe ratio gets closer to the empirical values only when assuming implausibly large values of
the risk-aversion parameter. Alternatively one needs to assume kinked preference or risk-sensitive
preferences a’ la Epstein and Zin[23]. Indeed around the kink the marginal utility tends to infinity
and this raises the volatility of asset prices when averaging across states. Our extended multiplier
preferences act similarly to the risk-sensitive preferences. In the numerical simulations we will
indeed show that the model simulated Sharpe ratio is very close to its empirical counterpart. At
last, an important fact of the leverage cycle concerns debt pro-cyclicality. The combination of the
occasionally binding collateral constraint and of the state-contingent movements in the subjective
beliefs induce the pro-cyclicality in our model. Under optimistic behavior, asset price growth in
good states is higher because of the above described channels. This in turn, by increasing the value
of collateral, relaxes the constraint, making leverage to accumulate more pro-cyclically than under
the case with no ambiguity. The opposite is true under the pessimist behavior. We will return on
this point more extensively later on.

4 Estimation of the Model Implied SDF

To provide empirical ground to ambiguity attitudes within the context of our model and to uncover
how the value of $\theta_t$ changes according to the prevailing state we estimate the model implied Euler
equations. Once equipped with the process for $\theta_t$ we will solve the model analytically to uncover
the main economic channels at work and numerically to assess the quantitative relevance.

We devise a novel estimation method apt to a model with collateral constraints and extended
multiplier preferences. The method is based on adapting the minimum distance estimation condi-
tional on latent variables to our modelling environment. In a nut-shell we derive a moment
condition by using the combined non-linear expression for the Euler equations 14 and 15. As we
show in Appendix A, the latter depends on the value function. We therefore follow the approach in
Chen, Favilukis, and Ludvigson[16], who condition the Euler moment condition to the estimation of the value function. A crucial difference between our method and theirs is that their value function has an unknown functional form, which is estimated semi-nonparametrically, while ours can be derived analytically.

More specifically, the estimation procedure (whose detailed derivations are contained in Appendix A) can be described as follows. First, one shall re-write the value function in terms of an ambiguity factor. For this, we adapt the steps used in the recursive preference literature to the case of our extended multiplier preferences (see Appendix A.8.1). Next, the implied SDF is derived (see Appendix A.8.2) and the value function is estimated (see Appendix A.8.3). Next, substituting the derived SDF into the combined Euler equations for debt and risky assets, 14 and 15, delivers the final moment condition (see Appendix A.8.4). Finally, as it is common for GMM estimation, we condition on a set of instruments, $z_t$. The resulting moment condition reads as follows:

$$
E_t \left\{ \beta \left( \frac{c_t}{c_{t+1}} \right)^{(1-\sigma_t)} \left( \frac{\exp(V_{t+1}/c_t)}{\exp(V_{t+1}/c_{t+1})} \right)^\sigma_t \left( R_{t+1}^s - \phi R_{t+1} \right) + \phi - 1 \right\} z_t = 0
$$

where $R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t}$ is the cum-dividend return on risky asset and $R_{t+1}$ is the risk-free interest rate, which is time-varying in the data. Note that the expression for the SDF can be decomposed into two factors, $\Lambda_{t,t+1}^1 = \beta \left( \frac{c_t}{c_{t+1}} \right)$ and $\Lambda_{t,t+1}^2 = \left( \frac{\exp(V_{t+1}/c_t)}{\exp(V_{t+1}/c_{t+1})} \right)^\sigma_t$, where the second captures the role of ambiguity attitudes. Equation 23 is estimated fully non-linearly with GMM methods\(^{18}\). Note that tight restrictions are placed on asset returns and consumption data since our moment condition embodies both financial frictions and ambiguity attitudes. For the estimation we fix the loan to value ratio at $\phi = 0.6$ and, given that $\theta_t = -\frac{1}{\sigma_t}$, we estimate the preference parameters, $\beta$ and $\theta_t$.

Regarding the data, we use real per capita expenditures on non-durables and services\(^{19}\) as a measure of aggregate consumption. For $R$ we use the three-month T-bill rate\(^{20}\), while $R^s$ is proxied

\(^{18}\)Optimal GMM parameters minimize a quadratic loss function over the weighted distance between population and sample moments, by a two-step GMM.

\(^{19}\)Source NIPA Tables: https://www.bea.gov/iTable/index_nipa.cfm.

\(^{20}\)Source is the CRSP Indices database: http://www.crsp.com/products/research-products/crsp-historical-
through the Standard & Poor 500 equity return\textsuperscript{21}. The choice of the instruments follows the literature on time-series estimation of the Euler equations\textsuperscript{22}. They are grouped into internal variables, namely consumption and interest rates two quarters lagged, and external variables, namely the excess market return, consumption growth, the value and size spreads, the long-short yield spread and the dividend-price ratio (see also Yogo\textsuperscript{63}). A constant is additionally included in order to restrict model errors to have zero mean. Finally, the model’s over-identifying restrictions are tested through the J-test (test of over-identifying restrictions, Hansen \textsuperscript{27})\textsuperscript{23}.

Table 1 presents the results. The estimated values of \( \theta_t \) are conditioned to the logarithm of the continuation value ratio, defined as \( \tilde{v}_t = \log \left( \frac{v_t}{c_t} \right) \). Consistently we previous definition good states are those for which the latent value function is higher than its mean and viceversa for bad states. Column 3 shows results conditioned upon the relation \( \tilde{v}_t \geq \mathbb{E}\{\tilde{v}_t\} \), while column 4 reports the results for the complementary condition. We find that a negative value (-4.28) prevails over good states, namely those for which \( \tilde{v}_t \geq \mathbb{E}\{\tilde{v}_t\} \), and that a positive value (4.00) prevails in bad states, namely those for which \( \tilde{v}_t < \mathbb{E}\{\tilde{v}_t\} \). This gives clear indication on the state-contingent nature of the ambiguity attitudes, being averse to entropy deviations in bad states and opportunistic toward them in good states. According to Lemma 1 above we know that \( \theta_t < 0 \), which prevails in good states, implies that agents act optimistically. Similarly a \( \theta_t > 0 \), which prevails in bad states, speaks in favour of pessimism.

To further test our result above we ran unconditional estimation over two different historical periods. We choose the first to be Great Moderation sample (1985:Q1-2007:Q2), which captures the boom phase preceding the 2007-2008 financial crisis. The sub-sample representing the recessionary states is the period following the crisis, namely the (2007:Q3-2016:Q4). Estimations, reported in the last two rows, confirm the same state-contingent nature uncovered in the conditional estimates. Finally note that for each sample reported the \( J \) test fails to reject model in equation 23 at conventional significance levels.

Next, given the estimated preference parameters we investigate cyclical properties of the pricing indexes.

\textsuperscript{22} See Stock et al. [59] for a survey on the relevance of instruments choice in a GMM setting.
\textsuperscript{23} This is a specification test of the model itself and it verifies whether the moment conditions are enough close to zero at some level of statistical confidence, if the model is true and the population moment restrictions satisfied.
kernel, namely the empirical SDF. Among other things this also gives indications on the cyclical  
properties of the asset price. To this purpose we decompose the SDF in the two components  
highlighted above, \( \Lambda_{t,t+1}^1 \) and \( \Lambda_{t,t+1}^2 \), where the latter captures the role of ambiguity attitudes. For  
this exercise we use the sample 1980-2016, which among other things is consistent with the one used  
later on for the data-model moments comparison. The empirical moments of the SDF are listed in  
Table 2, which shows that the volatility of \( \Lambda_{t,t+1} \) is explained almost in full by the ambiguity factor  
\( \Lambda_{t,t+1}^2 \). The latter also accounts for a lower correlation with respect to both consumption and risky  
returns.

Table 1: Estimated Moments of the Pricing Kernel.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Estimated parameters</th>
<th>( \beta )</th>
<th>( \theta )</th>
<th>( \theta(\hat{v}_t \geq E\hat{v}_t) )</th>
<th>( \theta(\hat{v}_t &lt; E\hat{v}_t) )</th>
<th>( J - test )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980-2016</td>
<td></td>
<td>0.836</td>
<td>-4.278</td>
<td>4.00</td>
<td>7.014</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.016)</td>
<td>(0.053)</td>
<td>(0.068)</td>
<td>(0.857)</td>
<td></td>
</tr>
<tr>
<td>1985:Q1-2007:Q2</td>
<td></td>
<td>0.814</td>
<td>-4.261</td>
<td>4.51</td>
<td>4.51</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.014)</td>
<td>(0.039)</td>
<td>(0.985)</td>
<td>(0.985)</td>
<td></td>
</tr>
<tr>
<td>2007:Q3-2016:Q4</td>
<td></td>
<td>0.852</td>
<td>5.499</td>
<td>7.318</td>
<td>7.318</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.015)</td>
<td>(0.019)</td>
<td>(0.885)</td>
<td>(0.885)</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Estimated Moments of the Pricing Kernel.

<table>
<thead>
<tr>
<th>Moments</th>
<th>( \Lambda_{t,t+1} )</th>
<th>( \Lambda_{t,t+1}^1 )</th>
<th>( \Lambda_{t,t+1}^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean SDF</td>
<td>0.806</td>
<td>0.833</td>
<td>0.967</td>
</tr>
<tr>
<td>Standard deviation SDF</td>
<td>8.263</td>
<td>0.332</td>
<td>9.874</td>
</tr>
<tr>
<td>( Corr(SDF, \Delta c_t) )</td>
<td>-0.105</td>
<td>-0.999</td>
<td>-0.063</td>
</tr>
<tr>
<td>( Corr(SDF, R_{t+1}^s) )</td>
<td>-0.081</td>
<td>-0.332</td>
<td>-0.067</td>
</tr>
</tbody>
</table>

Note: \( \Lambda_{t,t+1}^1 = (\frac{c_t}{c_{t-1}})^{1-\sigma} \) and \( \Lambda_{t,t+1}^2 = \left[ \exp\left(\frac{\hat{v}_{t+1}}{\sigma} \right) \right]^{\sigma} \).
4.1 Process for the Penalty Parameter

Given the above estimation results the process for $\theta_t$ reads as follows:

$$\theta_t = \begin{cases} 
\theta^- & \text{if } V_t \geq EV_t \\
\theta^+ & \text{if } V_t < EV_t 
\end{cases}$$

(24)

where $\theta^- = -1.35$ and $\theta^+ = 1.35$. We will use this process structure and calibration since now on.

5 Analytical Results

In this section we derive analytical expressions for asset price, premia and Sharpe ratio and show their dependence on the optimal LR and the shadow price of debt, $\mu_t$. The analytical derivations will allow us to gain first economic intuition on the combined role of occasionally binding constraints and ambiguity attitudes for asset prices and leverage.

5.1 The Impact of Ambiguity on Asset Prices

Proposition 1. (Asset price recursion) The recursive formula for the asset price over the infinite horizon in our model reads as follows:

$$q_t = \lim_{T \to \infty} E_t \left\{ \sum_{i=1}^{T} d_{t+i} \prod_{j=1}^{i} K_{t+j-1, t+j} \right\}$$

(25)

where $K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1-\mu_t'}$ with $\Lambda_{t,t+1} = \beta \frac{u(c_{t+1})}{u(c_t)} m_{t+1}$ and $\mu_t' = \frac{\mu_t}{u(c_t)}$.

Proof. See Appendix 6.1.

The asset price clearly depends upon the optimal LR, $m_{t+1}$, and the shadow price of debt, $\mu_t$. Consider first good states. In this case endogenous beliefs are right skewed toward the upper tails according to Lemma 1, hence both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ are higher than when $m_{t+1} = 1$ for all positive states. In good states the asset price grows, due to increase asset demand, but it does so more under optimist beliefs. Similarly in bad states endogenous beliefs are left-skewed toward the lower tails, hence both $\Lambda_{t,t+1}$ and $K_{t,t+1}$ are higher than in the case with no ambiguity for all negative states. Asset price falls, but they do more so with pessimism. This is the sense in which ambiguity attitudes contribute to the heightened dynamic of the asset price boom and bust cycles. The asset
price also depends upon the shadow price of debt, which proxies the margin or the down-payment requested to borrowers. When the constraint is binding margins are positive and increasing, in line with empirical observations (see Geanakoplos[25]). The higher margins paid by borrowers or the higher collateral value of the asset is reflected in higher asset prices. This also contributes to heightened asset price dynamics.

**Proposition 2. (Equity Premium)** The return for the risky asset reads as follows:

$$\mathbb{E}_t\{R^s_{t+1}\} = \frac{R(1 - \text{cov}(\Lambda_{t,t+1}, R^s_{t+1}) - \phi\mu'_t)}{1 - \mu'_t}$$  (26)

while the premium of its return over debt return reads as follows:

$$\Psi_t = \frac{1 - \text{cov}(\Lambda_{t,t+1}, R^s_{t+1}) - \phi\mu'_t}{1 - \mu'_t}.$$  (27)

where $\Lambda_{t,t+1} = \beta \frac{\mu^c(c_{t+1})}{u(c_t)} m_{t+1}$ and $\mu'_t = \frac{\mu^d}{u(c_t)}$.

**Proof.** See Appendix 6.3.

The above proposition also shows unequivocally the dependence of the premia over the beliefs as captured by $m_{t+1}$ and the shadow price of debt. While the exact dynamic of the equity premium depends on the solution of the full-model and upon its general equilibrium effects, we can draw some general conclusions on the dependence of the equity premium upon the beliefs and the shadow price of debt.

First, a negative covariance between the SDF and the risky asset returns implies that borrowers are less hedged. This results in a higher return required to hold the risky asset. The opposite is true for positive covariances. While we cannot say with certainty whether the $\text{cov}(\Lambda_{t,t+1}, R^s_{t+1})$ \footnote{This indeed depends on whether $\mathbb{E}_t(\Lambda_{t,t+1}, R^s_{t+1}) > \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R^s_{t+1})$ or $\mathbb{E}_t(\Lambda_{t,t+1}, R^s_{t+1}) < \mathbb{E}_t(\Lambda_{t,t+1})\mathbb{E}_t(R^s_{t+1})$.}, we know by the Cauchy–Schwarz inequality that $\text{cov}(\Lambda_{t,t+1}, R^s_{t+1}) \leq \sqrt{\text{Var}(\Lambda_{t,t+1})\text{Var}(R^s_{t+1})}$. Therefore anything that either increases the variance of $\Lambda_{t,t+1}$ or $R^s_{t+1}$ will increase their covariance, whether in the positive or the negative domain. Endogenous beliefs formation by inducing fluctuations in $m_{t+1}$ tend to increase the variance of the stochastic discount factor which is given by $\text{Var}(\Lambda_{t,t+1}) = \text{Var}(\beta \frac{\mu^c(c_{t+1})}{u(c_t)} m_{t+1})$.

Second, the premium also depends upon the shadow price of debt. Taking as given again the covariance between the SDF and the risky return, one can compute the following derivative:
\[ \frac{\partial \psi_t}{\partial \mu_t} = \frac{(1 - \phi) - \text{cov}(\Lambda_t, R_{t+1}^s)}{(1 - \mu_t)^2}. \]

If the \( \text{cov}(\Lambda_t, R_{t+1}^s) \) is negative the derivative is certainly negative\(^{25}\)

In other words when there are low hedging opportunities a tightening of the constraint implies that borrowers require higher premia to hold the risky asset. The asset already conveys poor insurance opportunities, a tightening of the constraint by reducing the asset collateral value, reduces its demand. Hence borrowers are willing to hold only at higher premia. Endogenous beliefs also play an indirect role in this dependence. Indeed as explained above fluctuations in beliefs generally raise the absolute value of the covariance. Hence, consider again the case of a negative covariance. In this case fluctuations in beliefs impair even more the hedging abilities of the risky assets and this in turn increases the premium that borrowers ask in face of a tightening of the borrowing limit.

**Proposition 3. (Sharpe Ratio)** The Sharpe ratio in our model reads as follows:

\[ SR = \frac{E_t\{z_{t+1}\}}{\sigma_z} = \frac{\sigma_{\Lambda_t}^2}{\lambda^{2}} - 2\mu_t \left( \phi - 1 \right) \frac{\sigma_{\Lambda_t}^2}{\lambda^{2}} - \frac{\mu_t^2 (\phi - 1)^2}{\lambda^{2}} \] \( (28) \)

where \( z_{t+1} = R_{t+1}^s - R \) is the asset excess return, \( \Lambda \) is the long run value for the SDF, \( \sigma_{\Lambda_t}^2 \) is the volatility of the SDF and \( \sigma_z^2 \) is the volatility of the excess return.

**Proof.** See Appendix 6.2.

The presence of endogenous beliefs raises the Sharpe ratio and brings it close to the empirical values as we show in Table 4. Matching the empirical values of the Sharpe ratios is typically hard for models with asset pricing and/or financial frictions. The reason being that typically an increase in the excess returns of the risky assets is accompanied by an increase in its volatility. Analytically it is easy to see why the Sharpe ratio raises in our model. First fluctuations in \( m_{t+1} \) raise fluctuations in the stochastic discount factor, \( \Lambda_t^s \), hence in its variance. This in turn raises the Sharpe ratio. Second, fluctuations in \( \theta_t \) enhance fluctuation in beliefs, \( m_{t+1} \). Third, the kinked nature of the value function steepens fluctuations in \( m_{t+1} \) and the SDF also since marginal utilities tend to infinity around the kink. In turn any increase in the variance of \( m_{t+1} \) raises the variance of \( \Lambda_t^s \) and the Sharpe ratio. Intuitively in presence of uncertainty or ambiguity agents require a premium which goes beyond the one needed to cover risk\(^{26}\), as measured by the volatility of the excess return. If agents knew the objective probability distribution, they would need to be compensated only for

\(^{25}\)If the \( \text{cov}(\Lambda_t, R_{t+1}^s) > 0 \), then whether \( \frac{\partial \psi_t}{\partial \mu_t} \) is positive or negative depends upon whether the \( \text{cov}(\Lambda_t, R_{t+1}^s) > (1 - \phi) \) or not.

\(^{26}\)Here we refer to the distinction between uncertainty and risk introduce by Knight\([44]\).
bearing tail risk. As the tail itself is uncertain, borrowers require a higher premia.

In past literature it was noted that the model implied Sharpe ratio can match the empirical counterpart by assuming implausibly large values for the risk-aversion parameter (see Cochrane[17], chapter 13). In the numerical simulations below we show that this is not the case for our model.

At last, note also that the Sharpe ratio depends negatively upon the shadow value of debt. When the constraint binds borrowers start to de-leverage and to reduce the demand of risky asset. As a result this reduces the expected excess returns relatively to the return on debt. This is compatible with the pro-cyclical nature of the returns on risky assets observed in the data.

6 Quantitative Results

In this section we solve the model numerically employing a global solution method, namely policy function iterations with occasionally binding constraints. We provide details on the solution method in Appendix C below. We group our results in three. First, we search for the optimal model calibration. To do so we choose some target moments in the data and we search for the set of parameters that minimizes the distance between the targets and the model-implied moments. This gives further empirical validation of our model. Second, under the optimal calibration we verify if the model can match several volatilities and correlations for asset prices, returns, equity premia and leverage. We show that in fact the model does it well. At last, under this optimal calibration we examine policy functions and we conduct a crisis event exercise. Our main result is that with ambiguity the model produces steeper asset prices and leverage increases in booms, which are then followed by sharper de-leverage and crises in recessions.

6.1 Calibration Strategy

This section describes the calibration strategy. We divide the set of structural parameters in three groups. The first group includes parameters which are calibrated using external information. Those are the risk free rate, the loan-to-value ratio, the fraction of financial wealth over total wealth. The second group includes parameters calibrated using a matching moments routine. Those are $\theta$, the absolute risk aversion coefficient, the discount factor and the volatility of the income process. The third group includes parameters which are calibrated with the estimation of the income process,
Table 3: Values for the calibrated parameters.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Meaning</th>
<th>Strategy</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R )</td>
<td>Risk-free rate</td>
<td>3month T-bill rate</td>
<td>1.0114</td>
</tr>
<tr>
<td>( \phi )</td>
<td>Loan-to-value ratio</td>
<td>Crises Probability</td>
<td>0.15</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>Share of dividend</td>
<td>Fraction of financial wealth</td>
<td>0.10</td>
</tr>
<tr>
<td>( \theta^+(V_t &lt; EV_t) )</td>
<td>Pessimism</td>
<td>Matching Moments</td>
<td>1.35</td>
</tr>
<tr>
<td>( \theta^-(V_t \geq EV_t) )</td>
<td>Optimism</td>
<td>Matching Moments</td>
<td>-1.35</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>Risk aversion</td>
<td>Matching Moments</td>
<td>2.075</td>
</tr>
<tr>
<td>( \beta )</td>
<td>Discount factor</td>
<td>Matching Moments</td>
<td>0.930</td>
</tr>
<tr>
<td>( \sigma_y )</td>
<td>Income Volatility</td>
<td>Matching Moments</td>
<td>0.0415</td>
</tr>
<tr>
<td>( \rho_y )</td>
<td>Income Persistence</td>
<td>Estimation</td>
<td>0.634</td>
</tr>
</tbody>
</table>

more specifically the autocorrelation of the income process. Table 3 summarizes the results of the calibration procedure.

In order to calibrate the second group of parameters, we choose to match six empirical moments (the matching is shown in Table 4, where also other moments are displayed), namely the volatility of debt \( \sigma^b \), the autocorrelation of debt, \( \rho^b \), the correlation between debt and consumption \( \text{Corr}(\Delta b^t, \Delta c^t) \), the expected return on risky asset \( \mathbb{E}_t(R^s_t) \), the volatility of return on risky asset \( \sigma^R \), the correlation between return on risky asset and consumption growth \( \text{Corr}(R^s_t, \Delta c_t) \). To compute the empirical equivalent we focus on the data sample 1980:Q1-2016:Q4, which captures a period of both of large debt growth and subsequent de-leverage. More details on the data sources are in Appendix D. We do not include the equity premium among our targets because the risk free rate is exogenous in the model, but we show later on that our model can match it well. Note that while the income shock correlation is directly estimated in the data, its volatility is instead calibrated. It is indeed well known from past literature that estimated values exhibit large measurement errors (see Heaton and Lucas [38] and Deaton [20]).

The matching moment routine starts from the following grids: \( \sigma^y \in [0.02, 0.07] \) for the states of the income shock, \( \beta = [0.92, 0.98] \), \( \gamma = [1, 2.2] \), and finally \( \theta_t \in \{-5, 5\}, 100 \)\(^{27}\). In the grid for \( \theta_t \) we introduce the value 100, in order to check if the model with no ambiguity produces theoretical

\(^{27}\)For each parameter we check that the optimal values do not hit the bounds of the grid.
moments which perform better than our model with waves of optimism and pessimism. Moreover, the grid is defined between 5 and -5 because out of these bounds the difference between the model with and without ambiguity becomes negligible.

It is interesting to note that the estimation of the full model through moments matching equally delivers the same type of state-contingent process for the parameter $\theta_t$ as the one we uncovered with our GMM estimation above. The estimated values are naturally different between the two estimation methods, since in the GMM case the regression is based on one equation summarizing only borrowers’ first order conditions, while in the second case the estimation involves the full set of model equations. But the fact that the two estimations deliver the same type of state-contingent process is important.

### 6.2 Empirical Moments Matching

In this section we evaluate the model’s ability to match the empirical moments under the optimal calibration determined above. We also compare the theoretical moments of our model with ambiguity attitudes (labelled AA since now on) with those of the equivalent model with rational expectation (labelled RE since now on). The following Table 4 summarizes the main results:

The upper panel of Table 4 shows the matched moments (according to the criteria set in the previous section), while in the lower panel other relevant moments are shown. The overall message is that our model fits well the empirical moments. First, it is better capable of matching empirical debt and risky asset return volatilities, relatively to the RE model. This is so despite both models exhibit amplification induced by the occasionally binding collateral constraint. This shows that endogenous beliefs are also needed to explain asset and debt markets dynamics. The equity premium as well as its cyclical properties are also well captured and again the presence of ambiguity attitudes seem to improve even above the benchmark model featuring solely the collateral constraint. As explained in Cochrane [17] the ability to match contemporaneously the long run equity premia and asset returns and their cyclical properties is related to the agents’ attitude toward events on the tails. In our model borrowers are endogenously optimistic, hence risk-takers, on the upper tail, while they are pessimistic, hence risk-sensitive, on the lower tail. This additional effect, stemming from the endogenous waves of confidence, improves the ability of the model to match the equity premium and its cyclical properties. In terms of matching the Sharpe ratio and the
Table 4: Empirical and model based moments for selected asset price and debt variables.

<table>
<thead>
<tr>
<th>Moments</th>
<th>Mnemonics</th>
<th>Empirical</th>
<th>Model AA</th>
<th>Model RE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility debt</td>
<td>$\sigma^b$</td>
<td>12.52</td>
<td>12.37</td>
<td>7.24</td>
</tr>
<tr>
<td>Persistence debt</td>
<td>$\rho^b$</td>
<td>0.846</td>
<td>0.539</td>
<td>0.331</td>
</tr>
<tr>
<td>Cyclicality debt</td>
<td>$corr(\Delta b_t, \Delta c_t)$</td>
<td>0.668</td>
<td>0.378</td>
<td>0.821</td>
</tr>
<tr>
<td>Exp risky return</td>
<td>$E_t(R_t^s)$</td>
<td>9.38</td>
<td>8.19</td>
<td>7.38</td>
</tr>
<tr>
<td>Volatility risky returns</td>
<td>$\sigma^{R_t^s}$</td>
<td>16.21</td>
<td>17.46</td>
<td>12.40</td>
</tr>
<tr>
<td>Cyclicality risky returns</td>
<td>$corr(\Delta R_t^s, \Delta c_t)$</td>
<td>0.474</td>
<td>0.989</td>
<td>0.989</td>
</tr>
</tbody>
</table>

Other Relevant Moments

<table>
<thead>
<tr>
<th>Moments</th>
<th>Mnemonics</th>
<th>Empirical</th>
<th>Model AA</th>
<th>Model RE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity premium</td>
<td>$E_t(R_t^s - R)$</td>
<td>8.25</td>
<td>7.05</td>
<td>6.24</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>$\frac{E_t(R_t^s - R)}{\sigma^R_{t+1}}$</td>
<td>0.522</td>
<td>0.404</td>
<td>0.503</td>
</tr>
<tr>
<td>SDF</td>
<td>$E_t(\Lambda_{t,t+1})$</td>
<td>0.806</td>
<td>0.940</td>
<td>0.939</td>
</tr>
<tr>
<td>Volatility SDF</td>
<td>$\sigma^{\Lambda_{t,t+1}}$</td>
<td>8.263</td>
<td>15.10</td>
<td>12.987</td>
</tr>
<tr>
<td>Cyclicality SDF</td>
<td>$corr(\Lambda_{t,t+1}, \Delta c_t)$</td>
<td>-0.105</td>
<td>-0.976</td>
<td>-0.988</td>
</tr>
<tr>
<td>Corr SDF with risky returns</td>
<td>$corr(\Lambda_{t,t+1}, R_t^s)$</td>
<td>-0.081</td>
<td>-0.967</td>
<td>-0.98</td>
</tr>
<tr>
<td>Prob(crisis)$^2$</td>
<td></td>
<td>4</td>
<td>3.16</td>
<td>4.51</td>
</tr>
</tbody>
</table>

1. In the data this refers to the SDF estimated in section 2.
2. We do not calculate the empirical frequency of the financial crises but we follow Bianchi and Mendoza [10], who derive an average of 4 crises every 100 years in the developed countries; a value of 4% perfectly fits this result.
3. Column 2 and 3 compare theoretical moments under ambiguity versus rational expectation.
empirical SDF, both models seem to perform similarly and with acceptable performance, thereby showing that the kink induced by the occasionally binding collateral constraint contributes alone to this result. At last, both model match the pro-cyclicality of leverage which is well documented in the data. Leverage indeed increases in booms due to a combination of exuberance and lax debt constraints and declines in recessions due to a combination of pessimism and increasing margins, namely borrowers’ down-payments. Here neither our model nor the RE model seem to match the empirical value with precision, as the first underestimates, while the second overestimates.

At last note, that the model reasonably matches the empirically probability of the crisis. For the empirical counterpart of such a probability we rely on the value presented in Bianchi and Mendoza [10].

6.3 Excess Returns Predictability

Before turning to the implications of this model for the unfolding of crises, it is instructive to conclude our assessment of its empirical validity by examining also the implied excess returns predictability. In asset pricing this is an important test on whether the model ingredients are able to account for the sources of risk that drive expected returns. A number of empirical observations (Fama and French [24]) established predictability of risk premia through current or past price-dividend ratios: the pro-cyclical movements in stock prices generate a large countercyclical variation in expected risk premia. In macro so far the introduction of habits in consumption proved successful in providing a theory for return predictability (Campbell, Cochrane [19]). Here, along the same logic, we evaluate what is the role of our behavioral ingredient, given by ambiguity attitudes, in the debate.

Reminding that we first estimate and then define the ambiguity parameter conditioning on the demeaned value function $(\tilde{v}_t - E(\tilde{v}_t))$, we assess in Figure 1 the model’s implied price-dividend and risk premia determination in terms of our ambiguity attitudes. Note that the plot reports results for 3 realizations of the income process. More importantly when the level of the value function positively departs from its mean we’re in an region where agents displays ambiguity-seeking behaviors, while ambiguity aversion prevails for negative deviations. Then, compatibly with our analytical results in Proposition 1 we conclude that ambiguity attitudes map crucially into the price-dividend ratio generating asset price build-ups and low equity returns under ambiguity seeking; while asset price bursts and high returns under ambiguity aversion.
Figure 1: **Price-consumption ratio and stock returns as function of the distance between the value function and its expected value.**

From these results the intuition that since the type of beliefs deviation depends on the realizations of the demeaned value function and we showed that the latter maps into the price-dividend ratio, then measuring return predictability over the price-dividend ratio (or the dividend yields) would account for the influence of ambiguity attitudes. How they intervene in the ability of the model to forecast future risk premia is evaluated through a return predictability regression (see also Fama and French [24] and Cochrane[18]) of future excess returns, at various horizons $k$, on the dividend yield, $\frac{d_t}{q_t}$, using both historical and model’s simulated data. The regression\(^\text{28}\) we consider is the following:

\[
E_t\{R_{t+k}^e\} - E_t\{R_{t+k}\} = a + b \frac{d_t}{q_t} + \varepsilon_{t+k}
\]

(29)

Table 5 reports the results, comparing empirical evidence and model’s performance. For the former, the estimated coefficients are positive: high dividend yields reliably anticipate periods

\(^\text{28}\) Estimation is done through overlapping OLS regression, standard errors are computed based on Hansen and Hodrick[31]. Data for this estimation are the price-dividend ratio for the S&P stock returns from the Shiller Database and the 3-months T-bill rate from CRSP Indices Database.
Table 5: Empirical regression of excess returns over dividend-price ratios: historical data 1960-2016 in the upper block of the Table and model-equivalent (using simulated series) in the lower block of the Table.

Excess returns regressions, \( E_t\{R^k_{t+k}\} - E_t\{R_{t+k}\} = a + b \frac{d}{q} + \varepsilon_{t+k} \)

<table>
<thead>
<tr>
<th>Historical data_ 1960-2016</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon (years)</td>
<td>( \frac{d}{q} )</td>
<td>t(( \frac{d}{q} ))</td>
<td>( R^2 )</td>
</tr>
<tr>
<td>1</td>
<td>1.06</td>
<td>0.60</td>
<td>0.01</td>
</tr>
<tr>
<td>5</td>
<td>11.44</td>
<td>1.94</td>
<td>0.06</td>
</tr>
<tr>
<td>7</td>
<td>26.04</td>
<td>4.87</td>
<td>0.17</td>
</tr>
<tr>
<td>10</td>
<td>54.87</td>
<td>5.90</td>
<td>0.28</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Simulated data</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizon (years)</td>
<td>( \frac{d}{q} )</td>
<td>t(( \frac{d}{q} ))</td>
<td>( R^2 )</td>
</tr>
<tr>
<td>1</td>
<td>18.57</td>
<td>8.42</td>
<td>0.58</td>
</tr>
<tr>
<td>5</td>
<td>26.03</td>
<td>11.0</td>
<td>0.54</td>
</tr>
<tr>
<td>7</td>
<td>30.66</td>
<td>7.81</td>
<td>0.54</td>
</tr>
<tr>
<td>10</td>
<td>35.16</td>
<td>5.27</td>
<td>0.52</td>
</tr>
</tbody>
</table>

of high returns. Predictability however proves to be poor in the short-run, but increases with the forecasting horizon, as widely stated in the related literature. Model’s results, in the second panel of the table, are instead much more significant and informative at all horizons. Moreover, the comparison between the model with ambiguity attitudes (labelled AA as usual) and the model without (labelled RE as usual) highlights the role of ambiguity, which by affecting the price-dividend ratio (as in Figure 1) is responsible for substantially higher excess returns predictability.

6.4 Policy Function and Crisis Event

We have argued that our leverage cycle model with state-contingent waves of confidence has a sound empirical ground. The estimated SDF implied by our model shows that the role of ambiguity attitudes is significant and sizeable. Under the estimated ambiguity parameter and the empirically optimal calibration, we also showed that our model can account well for several asset price and leverage moments. This second exercise serves a cross-check of the model empirical validity.

Given the above, we proceed describing the dynamic properties of our model in comparison to the RE benchmark and by focusing in particular on the leverage and asset price cycles and on the unfolding of a crisis. We do so in two steps. First, we plot policy functions of debt and asset
prices. Next, we simulate a crisis event.

### 6.4.1 Borrowing and Asset Pricing decisions

Figure 2 below shows the decision rules for debt and asset prices with respect to past debt holdings across the model with ambiguity attitudes, labelled AA (red line) and the model with no ambiguity, labelled RE (blue line). Note that the full set of policy functions can be found in Appendix E below. We interpret the results distinguishing between positive (+5% from income trend; left panels) and negative realizations of the shock (−5% from trend; right panels) in order to appreciate the non-linearity arising by the changing ambiguity attitudes over the different states of the economy. Moreover, in each panel the kink separates the constrained from the unconstrained region and it represents the point at which the collateral constraint is marginally binding in each economy. Finally, the intersection between the 45 degree line and the policy function defines the stationary levels of debt. Several considerations emerge. First, both economies are able to produce the V-shaped bond holdings decision rules, which are a typical feature of models with high deleveraging and financial crises (see e.g. Bianchi [9], Bianchi and Mendoza [10]). To the right of the kink the policy functions are upward-sloping, corresponding to the unconstrained values of debt, while to the left they are downward-sloping identifying the constrained region where next-period bond holdings decrease in current bond holding. The kinked policy functions for asset prices follow accordingly: they increase with wealth and more steeply in the constrained region.

Second, the policy function for the AA model moves away from the one under RE both in the scale of the dynamics in each region and in the position of the kinks. In particular, given a negative state of the economy, higher previous-period debt induces a binding constraint earlier, increasing the probability of lying in the financial amplification region. The opposite holds for booms, where optimism boosts the collateral values, which in turn relaxes the constraint and facilitates the build-up of leverage. Thus, given the shifted location of the binding and slack regions, debt choices under AA, when constrained, associate a sharper or a more damped contraction in debt whether the economy is in booms or in busts. This nonlinearity reflects optimistic and pessimistic attitudes toward future realizations and generates amplification dynamics in the leverage cycles. We will visualize the size of this result below. At last, focusing on the asset price panels the comparison between the two models turns to be quite interesting. Asset prices in the AA model lie always
Figure 2: Policy functions for debt and asset prices conditional on the positive (left panels) and negative shock realizations (right panels). In each panel the threshold identifies the binding (left) versus non-binding region for the debt constraint (right). Current debt, shown on the y-axis, decreases (de-leverage) as we move upward.
above the RE benchmark in booms and always below in busts, which is coherent with the ability of the AA model to associate to a given initial debt position more debt and less debt, respectively for the two income states. Next we compute how large would be the extent of a de-leverage when the steady state of the economy is perturbed by a one-period 5% fall in income. This exercise offers a clear visualization of the enhanced financial amplification dynamics produced by AA, keeping the parallel with the RE model.

Figure 3 reproduces the following experiment. Assume that the two economies lie in equilibrium in A and B, respectively. Then, at the time of the shock the new negative realization of income forces a sharp upward adjustment of the bond decision rules and the temporary equilibria jump to C and D. The arrows define a drop in bond holdings which results to be much more pronounced for the model under the AA model. Interestingly, the AA model generates a drop of -33.9%, which exceeds the RE equivalent by about 10 points. This speaks about the model’s quantitative relevance in producing amplified leverage cycles.
Figure 4: Crisis event exercise. We define a crisis event following Bianchi and Mendoza[10].

6.4.2 Financial Crises

The crisis event displayed in Figure 4 proves the model’s ability to generate financial crises and studies relevant macro dynamics around it. More in detail, the event analysis is realized using model-simulated data for the two economies, AA and RE, and defining as crises the events in which the collateral constraint binds and the current account is at least two standard deviations above the trend. Then, we construct seven-periods event windows centred on the crisis to analyse pre- and post-crisis patterns:

The most interesting aspect emerging from the comparison between the two economies lies in the ability of the model with AA to account for stronger build-up of leverage prior to the crisis (around +3%) and sharper de-leveraging at the crisis (around -7%). Again the role of the state contingent confidence is important in understanding this dynamic. In booms optimism boosts collateral values, relaxing the constraint and facilitates the build-up of leverage. In recessions the
opposite is true. Pessimism induces assets’ fire sales, this generates sharper declines of the collateral values forcing borrowers to de-leverage earlier and more severely. Accordingly, looking at asset prices, consumption and equity returns helps understanding the results around debt decisions. Indeed, all of them display more severe dynamics under ambiguity aversion. The asset price collapses, for instance, playing an important role in explaining the more pronounced decline in debt under the AA model, reflecting a strong Fisherian deflation mechanism. Moreover, consumption falls 2 percentage points more and the risky return results to drive the enhanced pre- and post-crisis debt patterns, falling more sharply in booms and increasing when the crisis occurs.

6.5 Intermediation Sector and Intermediation Shocks

Lack of transparency and ambiguity play an important role in crises developments as we showed so far, but by no means instability stemming from the intermediation sector, hence originating in the credit supply, has a major role too. This is particularly true within the context of the 2007-2008 financial crisis. While including all possible sources of intermediation disincentives is beyond the scope of this paper, we nevertheless wish to assess the role of the intermediation channel. This is important as one should test whether the information channels described so far persist even when the supply side of credit is inserted in the model. In fact, we find that not only the role of ambiguity attitudes is preserved, but in most cases is amplified and the interaction with the intermediation channels is compelling.

We introduce intermediation by assigning the role of debt monitoring to a bank. This is actually realistic since atomistic lenders do not monitor or screen debtors individually, but largely assign this function to an intermediary. In this context the collateral constraint results from the bank design of a debt contract that is incentive compatible, meaning that it reduces the incentives of the borrower to divert resources and default. We formalize this type of contracts and show how the collateral constraint emerges from such incentive compatibility constraint in Appendix 14. Within this context an intermediation shock, which suddenly tightens the supply of credit, affects the parameter governing the loan-to-value ratio, \( \phi \), which itself governs the strength of the incentive problems. Intuitively the shock can be interpreted in two ways, both affecting the contractual agreement in a similar vein. It could capture financial innovation in the form of derivatives and/or asset back securities issuance, which being pervasive prior to the crisis, allowed banks to off-load
credit risk and reduced the tightness of the debt contract. A sudden freeze of the asset backed market liquidity due for instance to the sub-prime shock would have then induced a sudden fall in \( \phi \).

A second interpretation, linked to the first, is that higher availability of liquidity\(^{29}\) prior to the crisis had lessened banks’ monitoring incentives, something which resulted in higher loan-to-value ratios, \( \phi \). After the crisis occurs, the squeeze in liquidity, hence banks’ funding, could suddenly tightens the loan-to-value ratio. Both interpretations, which are realistic particularly in the context of the recent financial crisis, have the effect of producing a sudden tightening of credit supply. Within this context we subject our model to an intermediation shock to \( \phi \) and assess its role as well as its interaction with ambiguity attitudes. We do so by analyzing again policy functions, crisis events and second moments of the model.

Before proceeding to the assessment of the quantitative results, a few words are needed regarding the calibration of the shock. We define a high and a low level of the loan-to-value ratio, respectively \( \phi_l = 0.22 \) and \( \phi_h = 0.28 \), calibrated in order to match the empirical volatility of debt. The shock then follows a two-state regime-switching Markov process, with a transition matrix calibrated to replicate the empirical probability and duration of the crises events, as in Bianchi and Mendoza [10]. More in detail, the probability to remain in a high state, \( \pi_{hh} \) is set equal to 0.955 in order to match a frequency of crises close to 4%, while the transition probability from a low to high state \( \pi_{lh} \) is equal to one, implying a one year duration of the crises. The remaining transition probabilities are set as complements of the previous ones, i.e \( \pi_{hl} = 1 - \pi_{hh} \) and \( \pi_{ll} = 1 - \pi_{lh} \).

We start in this case from a crisis event, since this makes immediately visible the role of the credit supply for the crisis development on top of the role of ambiguity attitudes. Figure 5 compares the crisis event in the model with ambiguity attitudes and with rational expectations. The crisis event is defined as before, but now it is triggered by a combination of income and intermediation shocks. Specifically, we simulate the model in response to both shocks, we then observe that the crisis originates exactly when both shocks turn negative. The Figure shows two interesting facts. First, the role of ambiguity attitudes remains. It is still true that beliefs formation by affecting the value of collateral through endogenous skewed beliefs induce sharper crises than under the case with no ambiguity. Second and interestingly, this time the drop in the crisis is even larger.

\(^{29}\)This again could be due either to the possibility of raising additional bank liabilities through asset backed securities or through the ample availability of liquidity in interbank and repos markets prior to the 2007-2008 crisis.
Figure 5: Crisis event exercise with income and intermediation shocks. We define a crisis event following Bianchi and Mendoza[10].

This is reasonable since now both credit demand side and supply side components are operative. Intuitively the steepness of the crises now depends on two channels. As before the positive skewed beliefs, valid prior to the crisis, induced higher demand for leverage and the negative skewed beliefs, materializing after the crisis, induce de-leveraging. On top of this the progressive reduction of $\phi$ facilitates debt supply prior to the crisis and produces a credit crunch after the crisis.

To examine more in details the intermediation channel we examine the policy functions for debt and asset prices. Figure 6 below shows the policy functions conditional to positive realizations of the income shock for asset prices and debt by comparing various scenarios. In the first column we compare the model with ambiguity attitudes for two values of $\phi$. This case allows us to isolate only the contribution of credit supply. As before the kink represents the turn in which the constraint shifts from binding to non-binding. The comparison shows that a low $\phi$, namely tight credit due to high monitoring standards or low availability of liquidity, has two effects. On the one side,
it enlarges the constrained region. On the other side, it reduces leverage, and this effect can be beneficial in the medium to long run. The second and the third columns compare the models with and without ambiguity attitudes, respectively for low levels of $\phi$ (second column) and high levels of $\phi$ (third column). Two interesting observations emerge. First, as before under the model with ambiguity attitudes asset prices are higher and debt displays the previously underlined nonlinear dynamics over constrained and unconstrained regions. This as before is due to the nature of the positive skewed beliefs that emerge under positive income shocks. Second, the comparison between a high and a low level of $\phi$ shows that the qualitative pattern of the policy functions remains unaltered, albeit the constrained region is expanded under the low loan to value ratio. In other words, the forces operating through the ambiguity channel remain active even when introducing supply side elements. The dominant effect of the latter is more evident in terms of changes in the size of the constrained region.

To fully complete the assessment of the policy functions Figure 7 shows the results for the policy functions conditional on negative income realizations. The message is largely symmetric to the one described above.

At last, we ask whether the introduction of the intermediation shock can improve upon the moment matching and if so along which dimension. Table 6 below shows again the comparison of a selected numbers of second moments between the data, the model with and without ambiguity attitudes. This time the comparison is done by simulating the model also in response to the intermediation shock. The addition of the intermediation shock preserves most of the previous moments and improves in terms of data matching along other dimensions. The Table highlights primarily moments that change with the introduction of the intermediation shock. The most noteworthy result is that the introduction of credit supply fluctuations increases debt pro-cyclicality, which as discussed before, is an important stylized fact. The reason is intuitive. The double occurrence of the negative income and credit supply shock tightens leverage much more sharply. Equally the double-coincidence of the positive income and credit supply realizations heightens the build-up of leverage. Those movements on the tails help to increase average pro-cyclicality. The volatility of debt is also somewhat higher, mostly so in the model with ambiguity attitudes, and is closer to the data value. This again might be due to the contribution of the tails. On the other
Figure 6: Policy functions for debt and asset prices conditional on the positive income shock realizations. Left column shows comparison of the model with ambiguity attitudes under low and high loan-to-value ratios. The second and the third columns compare the model with ambiguity attitudes and with rational expectations respectively under low and high levels of loan to value ratios, $\phi$. 
Figure 7: Policy functions for debt and asset prices conditional on the negative income shock realizations. Left column shows comparison of the model with ambiguity attitudes under low and high loan-to-value ratios. The second and the third columns compare the model with ambiguity attitudes and with rational expectations respectively under low and high levels of loan to value ratios, $\phi$. 
Table 6: **Selected empirical and model based moments in the model with the intermediation channel.**

<table>
<thead>
<tr>
<th>Moments</th>
<th>Mnemonics</th>
<th>Empirical</th>
<th>Model AA</th>
<th>Model RE</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Matched Moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility debt</td>
<td>$\sigma^b$</td>
<td>12.52</td>
<td>11.55</td>
<td>9.78</td>
</tr>
<tr>
<td>Persistence debt</td>
<td>$\rho^b$</td>
<td>0.846</td>
<td>0.432</td>
<td>0.385</td>
</tr>
<tr>
<td>Cyclicality debt</td>
<td>$\text{Corr}(\Delta b_t, \Delta c_t)$</td>
<td>0.668</td>
<td>0.792</td>
<td>0.795</td>
</tr>
<tr>
<td>Exp risky return</td>
<td>$E_t(R^*_t)$</td>
<td>9.38</td>
<td>8.67</td>
<td>7.88</td>
</tr>
<tr>
<td>Volatility risky returns</td>
<td>$\sigma^{R^*_t}$</td>
<td>16.21</td>
<td>23.45</td>
<td>19.40</td>
</tr>
<tr>
<td>Cyclicality risky returns</td>
<td>$\text{Corr}(\Delta R^*_t, \Delta c_t)$</td>
<td>0.474</td>
<td>0.983</td>
<td>0.992</td>
</tr>
<tr>
<td>Equity premium</td>
<td>$E_t(R^*_t - R)$</td>
<td>8.255</td>
<td>7.013</td>
<td>7.050</td>
</tr>
<tr>
<td>Prob(crisis) %</td>
<td>-</td>
<td>4.0</td>
<td>4.06</td>
<td>5.53</td>
</tr>
</tbody>
</table>

1. We do not calculate the empirical frequency of the financial crises but we follow Bianchi and Mendoza [10], who derive an average of 4 crises every 100 years in the developed countries; a value of 4% perfectly fits this result.

2. Column 2 and 3 compare theoretical moments under ambiguity attitudes versus rational expectations.

side, it shall be mentioned that the introduction of the intermediation shock worsens the volatility of risky returns, which now goes above the one detected in the data. This effect is possibly due to the fact that our model does not account for loss absorption capacity of the intermediation sector in terms of equity capital and/or liquidity buffers. Those elements would indeed limit the extent of fire sales in risky assets when credit supply tightens, hence they would reduce fluctuations in asset prices.

To sum up the main contribution of the intermediation channel in our model is that of modifying the size of the constrained versus the unconstrained region, that of contributing to explain the severity of a financial crisis and that of contributing to explain debt pro-cyclicality.

## 7 Conclusions

Financial crisis are most often triggered by endogenous instability in debt markets. The latter are typically characterized by collateral constraints and opacity in asset values. Under lack of transparency the beliefs formation process acquires an important role since eventually it affects the value of collateral and with it the debt capacity. The narrative of most crises depict sharp increases in debt and asset prices prior to them and sharp reversal afterwards.

We therefore introduce in a model in which borrowers fund risky assets through debt and are subject to occasionally binding collateral constraints, belief formation, driven by ambiguity
attitudes that endogenously induce optimism in booms and pessimism in recessions. In booms optimistic borrowers demand more risky assets, which results in higher asset price growth (compared to the case with only collateral constraints), and lever up more. In recessions pessimistic borrowers de-leverage sharply and off load risky assets. This beliefs formation process coupled with the occasionally binding nature of the collateral constraint is a crucial element in explaining the combined amplified dynamic of asset prices and leverage as well as the whole span of their long run and short run statistics. Importantly we assess the empirical validation of our model both through GMM estimation of the Euler equation and through data-model moment matching.
References


8 Appendix A. GMM Estimation of the Ambiguity Parameter

In this section we detail the derivations needed to achieve the moment condition that is the object of our estimation. Further below we also provide a description of the dataset used in the estimation.

8.1 General Approach

We use a GMM estimation procedure based on the moment condition obtained from the combined Euler equation for debt and risky assets and is a variant of the techniques developed for asset pricing models with recursive preferences, pioneered by Epstein and Zin [23] and Kreps and Porteus [48]. Hence the starting point is to reformulate our value function, capturing multiplier preferences, in terms of an ambiguity term. The latter is achieved by mapping the multiplier preferences to a special case of the recursive preferences. This can be done by assuming a logarithmic continuation value, a logarithmic utility function and an exponential ambiguity adjustment factor, $\vartheta$ which accounts for confidence waves. Indeed we depart from the well-known equivalence between multiplier and recursive preferences by embedding state-contingent ambiguity attitudes. We start by reporting the value function derived after substituting the solution of the inner problem, presented in section 3.2.1:

$$V_t = u(c_t) - \beta \theta_t \log \left[ E_t \left\{ e^{\left( -\frac{V_{t+1}}{\theta_t} \right)} \right\} \right]$$

(30)

The above equation embeds a logarithmic ambiguity-adjusted component $\vartheta_{V_t+1}$, which maps future continuation values into current realizations. Indeed we can rewrite 31 as follows:

$$V_t = u(c_t) + \beta h^{-1} E_t \{ h(V_{t+1}) \}$$

$$= u(c_t) + \beta Q_t(V_{t+1})$$

(31)

The equivalence between specifications under recursive and multiplier preferences is achieved by assuming the following functional form $h(V)$ (see Hansen et al. [30]):

$$h(V_{t+1}) = \left( -\frac{V_{t+1}}{\theta_t} \right).$$

The latter implies that the exponential ambiguity adjustment component reads as follows:

$$Q_t(V_{t+1}) = h^{-1} E_t \{ h(V_{t+1}) \} = -\theta_t \log \left[ E_t \left\{ e^{\left( -\frac{V_{t+1}}{\theta_t} \right)} \right\} \right]$$

(32)
8.2 Pricing Kernel-SDF

The next step to obtain our moment condition is to derive an expression for the stochastic discount factor as function of the $Q_t(V_{t+1})$. To this purpose, we shall derive expressions for the marginal utilities in period $t$ and $t+1$. Given the needed functional forms detailed above, namely a logarithmic utility function $u(c_t) = \log(c_t)$, the marginal utility of consumption simplifies to $MC_t = c_t^{-1}$. The marginal utility of next-period continuation value is instead derived as follows:

$$\mu_{t+1} = \beta \exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)$$

Using the above expressions for the marginal utility we can derive the SDF as function of the $Q_t$ factor:

$$\Lambda_{t,t+1} = \frac{MV_{t+1}MC_{t+1}}{MC_t} = \beta \left(\frac{c_{t+1}}{c_t}\right)^{-1} \exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)$$

where $m_{t+1} = \exp\left(-\frac{1}{\theta_t}(V_{t+1} - Q_t(V_{t+1}))\right)$ is the optimal likelihood ratio. Equation 34 shows that the SDF has a two-factor structure. The first factor is the standard fundamental consumption growth, while the second is the added ambiguity factor, which is conditioned to the distance between the future value function and its certainty equivalent (the future insurance premium). Under no uncertainty this premium vanishes.  

8.3 Estimation of the Continuation Value Ratio

Since estimation requires strictly stationary variables, we shall re-scale the value function by consumption (see Hansen, Heaton, and Li [29] (HHL henceforth). We take log deviations from the log of consumption, $\tilde{c}_t = \log(c_t)$, where the tilde indicates logarithms:

$$\tilde{v}_t = \beta Q_t(\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1})$$

We define $\tilde{v}_t$ as the log value of continuation value ratio, $\frac{V_{t+1}}{c_t}$. Next using 32 into 35 we obtain:

$$\tilde{v}_t = -\beta \theta_t \log(\mathbb{E}_t \{e^{\sigma_t(\tilde{v}_{t+1} + \Delta \tilde{c}_{t+1})}\})$$

Indeed the continuation value would be perfectly predictable $\left(\exp\left(-\frac{V_{t+1}}{\theta_t}\right) = \mathbb{E}_t \exp\left(-\frac{V_{t+1}}{\theta_t}\right), m_{t+1}^* = 1\right)$ with zero adjustment ($Q_t(V_{t+1}) = V_{t+1}$).
where \( \sigma_t = -1/\theta_t \), and it is negative when \( \theta_t > 0 \) and positive when \( \theta_t < 0 \). We rely on this expression when we guess a process for \( \hat{v}_t \), which we then estimate. Indeed, since the continuation value ratio is a function of state variables governing the dynamic behaviour of consumption growth, we start by assuming that the latter is a function of state, \( \xi_t \), which in turn evolves according to the following first-order Markov process:

\[
\eta_{t+1} = \tilde{c}_{t+1} - \tilde{c}_t = \mu + H \xi_t + A \varepsilon_{t+1} \\
\xi_{t+1} = F \xi_t + B \varepsilon_{t+1}
\]  

(37)  
(38)

where \( \varepsilon_{t+1} \) is a (2x1) i.i.d. vector with zero mean and covariance matrix \( I \). \( A \) and \( B \) are (2x1) vectors. The exogenous states, \( \varepsilon_{t+1} \), which could capture income shocks, have both a direct impact on consumption and an indirect one through the endogenous state, \( \xi_t \). The latter can indeed capture endogenous movements in wealth which affect consumption one period later. The estimated value of the endogenous states, \( \hat{\xi}_t \), is obtained through Kalman filtering consumption data. The value function depends upon the estimated endogenous states, \( \hat{\xi}_t \), and consumption growth, \( g_{t+1} \). Since the latter also depends upon the endogenous states, we can guess the continuation value ratio as follows:

\[ \tilde{v}_t = \mu + U_v \hat{\xi}_t \]  

(39)

where \( U_v \hat{\xi}_t \) is the discounted sum of expected future growth rates of consumption. After some derivations we can write \( U_v \) and \( \mu_v \) as follows:

\[ U_v = \beta (I - \beta \mathcal{F})^{-1} H \]  

(40)

\[ \mu_v = \frac{\beta}{1 - \beta} \left( \mu + \frac{\sigma_t}{2} |A + U_v B|^2 \right) \]

where the term \( A + U_v B \) captures the dependence between the the continuation value and the exogenous shocks.

### 8.4 SDF and the Euler Equation

Next, given the estimated \( \hat{v}_t \) from 39. Substituting 36 into 34 delivers:

\[
\Lambda_{t,t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{-1} \left( \frac{\exp \left( \frac{V_{t+1}}{c_{t+1}} \right) \frac{c_{t+1}}{c_t}}{\exp \left( Q_t \left( \frac{V_{t+1}}{c_{t+1}} + \Delta \hat{c}_{t+1} \right) \right)} \right)^\sigma
\]  

(41)
Note that equation 41 is equivalent to the SDF obtained under Epstein and Zin [23] preferences and given the assumption of unitary EIS. At last, we substitute the above SDF into the combined Euler for debt and risky asset 14 and 15, which results in:

\[ \mathbb{E}_t \left\{ \beta \left( \frac{c_{t+1}}{c_t} \right)^{(1-\sigma)} \left( \frac{\exp \left( \frac{\nu_{t+1}}{c_{t+1}} \right)}{\beta \exp \left( \frac{\nu_t}{c_t} \right)} \right)^{\sigma} (R_{t+1}^* - \phi R_{t+1}) + \phi - 1 \right\} = 0 \] (42)

where \( R_{t+1}^* = \frac{d_{t+1} + q_{t+1}}{q_t} \) and for the estimation we shall write the debt rate as time-varying.

9 Appendix B. Analytical Derivations

This appendix derives analytical expressions for asset prices and returns.

9.1 Asset Price

From the borrowers’ optimality condition on risky assets:

\[ q_t = \beta \mathbb{E}_t \left\{ \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1}(q_{t+1} + d_{t+1}) \right\} + \phi \mu'_t q_t \] (43)

\[ = \beta \mathbb{E}_t \{ \Lambda_{t,t+1}(d_{t+1} + q_{t+1}) \} + \phi \mu'_t q_t \]

using the definitions for \( \Lambda_{t,t+1} = \beta \frac{u_c(c_{t+1})}{u_c(c_t)} m_{t+1} \) and \( \mu'_t = \frac{\mu_t}{u_c(c_t)} \). Then denoting \( K_{t,t+1} = \frac{\Lambda_{t,t+1}}{1 - \phi \mu'_t} \), we derive the following expression for the asset price:

\[ q_t = \mathbb{E}_t \{ K_{t,t+1}(d_{t+1} + q_{t+1}) \} \] (44)

Proceeding by forward recursion:

\[ q_t = \mathbb{E}_t \{ K_{t,t+1}(d_{t+1} + K_{t+1,t+2}(d_{t+2} + q_{t+2})) \} \] (45)

\[ = \mathbb{E}_t \{ K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2}) \} + \mathbb{E}_t \{ K_{t,t+1} K_{t+1,t+2} K_{t+2,t+3}(d_{t+3} + q_{t+3}) \} \]

\[ = \mathbb{E}_t \{ K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2} + K_{t+1,t+2}K_{t+2,t+3}d_{t+3}) \} + \]

\[ + \mathbb{E}_t \{ K_{t,t+1} K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}(d_{t+4} + q_{t+4}) \} \]

\[ = \mathbb{E}_t \{ K_{t,t+1}(d_{t+1} + K_{t+1,t+2}d_{t+2} + \]

\[ + K_{t+1,t+2}K_{t+2,t+3}d_{t+3} + K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}d_{t+4}) \} + \]

\[ + \mathbb{E}_t \{ K_{t,t+1} K_{t+1,t+2}K_{t+2,t+3}K_{t+3,t+4}q_{t+4} \} \]
At the final recursion step, the solution for the asset price:

\[
q_t = \mathbb{E}_t \left\{ \sum_{i=1}^{T} dt_i \prod_{j=1}^{i} K_{t+j-1,t+j} \right\} + \mathbb{E}_t \left\{ \prod_{i=0}^{T} K_{t+i,t+i+1} q_{t+T} \right\}
\]  

(46)

Taking the limit for \( T \to \infty \) of the above condition delivers equation 25 in Proposition 1.

9.2 The Sharpe Ratio and the Hansen and Jagannathan[28] Bounds

Writing down the two borrowers’ optimal conditions for the risk-free and risky assets, respectively:

\[
1 = \mathbb{E}_t \{ \Lambda_{t,t+1} R \} + \mu'_t
\]  

(47)

\[
1 = \mathbb{E}_t \{ \Lambda_{t,t+1} R^s \} + \phi \mu'_t
\]  

(48)

where \( \mu'_t = \frac{\mu_t}{\sigma_\tau} \), \( \Lambda_{t,t+1} = \beta \frac{\mu_t \sigma_\tau}{\sigma_\tau} m_{t+1} \) and \( R^s_{t+1} = \frac{\sigma_\tau + d_{t+1}}{\sigma_t} \). In order to derive the excess return between the risky asset and the risk-free asset, we subtract 47 from 48, obtaining:

\[
0 = \mathbb{E}_t \{ \Lambda_{t,t+1} (R^s_{t+1} - R) \} + \mu'_t (\phi - 1).
\]  

(49)

Then, we define the excess return as \( z_{t+1} = R^s_{t+1} - R \). Assuming a linear general form for the stochastic discount factor \( \Lambda_{t,t+1} \):

\[
\Lambda^*_{t,t+1} = \tilde{\Lambda}^* + \beta^m (z_{t+1} - E_t z_{t+1})
\]  

(50)

it must satisfy the following condition:

\[
0 = \mathbb{E}_t \{ \Lambda^*_{t,t+1} z_{t+1} \} + \mu'_t (\phi - 1);
\]  

(51)

which, once expanded, gives:

\[
0 = \mathbb{E}_t \{ \Lambda^*_{t,t+1} \} \mathbb{E}_t \{ z_{t+1} \} + \text{cov}(\Lambda^*_{t,t+1}, z_{t+1}) + \mu'_t (\phi - 1) \]

(52)

\[
= \mathbb{E}_t \{ \Lambda^*_{t,t+1} \} \mathbb{E}_t \{ z_{t+1} \} + \mathbb{E}_t \{ (z_{t+1} - \bar{z})(\Lambda^*_{t,t+1} - \bar{\Lambda}^*) \} + \mu'_t (\phi - 1)
\]

\[
= \mathbb{E}_t \{ \Lambda^*_{t,t+1} \} \mathbb{E}_t \{ z_{t+1} \} + \mathbb{E}_t \{ (z_{t+1} - \bar{z})(z_{t+1} - \bar{z}) \tilde{\beta}^m \} + \mu'_t (\phi - 1)
\]

\[
= \mathbb{E}_t \{ \Lambda^*_{t,t+1} \} \mathbb{E}_t \{ z_{t+1} \} + \sigma_2^m \tilde{\beta}^m + \mu'_t (\phi - 1).
\]

Hence:

\[
\tilde{\beta}^m = - (\sigma_2^m)^{-1} \mathbb{E}_t \{ \Lambda^*_{t,t+1} \} \mathbb{E}_t \{ z_{t+1} \} - (\sigma_2^m)^{-1} \mu'_t (\phi - 1)
\]  

(53)
The variance of the stochastic discount factor is then obtained:

\[ Var(\Lambda_{t+1}^*) = Var((z_{t+1} - E_t\{z_{t+1}\})\tilde{\beta}) \]

\[ = \beta^2 \sigma_z^2 \tilde{\beta} \]

\[ = (-\sigma_z^2)^{-1}\tilde{\Lambda}^*E\{z_{t+1}\} - (\sigma_z^2)^{-1}\mu_t(\phi - 1)) \]

\[ = (\sigma_z^2)^{-1}(\tilde{\Lambda})^2(E_t\{z_{t+1}\})^2 + 
+ 2\mu_t^2(\phi - 1)((\sigma_z^2)^{-1}\tilde{\Lambda}^*E\{z_{t+1}\} + ((\sigma_z^2)^{-1}(\mu'_t)^2(\phi - 1)^2). \]

Hence:

\[ \frac{\sigma_{\tilde{\Lambda}^*}^2}{\Lambda^{x^2}} = \frac{(E_t\{z_{t+1}\})^2}{\sigma_z^2} - 2\frac{\mu_t^2(\phi - 1)E_t\{z_{t+1}\}}{\sigma_z^2} - \frac{\mu_t^2(\phi - 1)^2}{\Lambda^{x^2}}. \]

The Sharpe Ratio (SR hereafter) on stock asset returns over bonds results to be:

\[ \frac{E_t\{\sigma_{\tilde{\Lambda}^*}^2\}}{\Lambda^{x^2}} = \frac{\sigma_{\tilde{\Lambda}^*}^2}{\Lambda^{x^2}} - 2\frac{\mu_t^2(\phi - 1)E_t\{z_{t+1}\}}{\sigma_z^2} - \frac{\mu_t^2(\phi - 1)^2}{\Lambda^{x^2}} \]

Thus, the SR depends on the variance of the adjusted for distorted beliefs stochastic discount factor and on \( \mu'_t \).

### 9.3 The Risk Premium

Expanding the borrower’s FOC for the risky asset and plugging in it the derivation for \( E_t\{\Lambda_{t+1}\} \) and the definition \( R_{t+1}^s = \frac{q_{t+1} + d_{t+1}}{q_t} \) we get:

\[ 1 = E_t \left\{ \Lambda_{t+1} \frac{q_{t+1} + d_{t+1}}{q_t} \right\} + \phi \mu_t' \]

\[ = E_t \{\Lambda_{t+1}E_t \left\{ \frac{q_{t+1} + d_{t+1}}{q_t} \right\} + cov(\Lambda_{t+1}, \frac{q_{t+1} + d_{t+1}}{q_t}) + \phi \mu_t' \]

\[ = \left( \frac{1 - \mu_t'}{R} \right)E_t \{R_{t+1}^s\} + cov(\Lambda_{t+1}, R_{t+1}^s) + \phi \mu_t'. \]

The return on risky assets is obtained:

\[ E_t \{R_{t+1}^s\} = \frac{R(1 - cov(\Lambda_{t+1}, R_{t+1}^s) - \phi \mu_t')}{1 - \mu_t} \]

Dividing by the risk-free return rate, the premium between the return on the risky asset and the risk-free rate can be derived:

\[ \Psi_t = \frac{1 - cov(\Lambda_{t+1}, R_{t+1}^s) - \phi \mu_t'}{1 - \mu_t}. \]
10 Appendix C. Numerical Method

Our numerical method extends the algorithm of Jeanne and Korinek [41] to persistent shocks and state-contingent ambiguity attitudes. The method, following the endogenous grid points approach of Carroll [15], performs backwards time iteration on the agent’s optimality conditions. We derive the set of policy functions \( \{c(b, s), b'(b, s), q(b, s), \mu(b, s), V(b, s)\} \) that solve competitive equilibrium characterized by the system:

\[
c(b, s)^{-\gamma} = \beta R \mathbb{E} \{m(b', s)c(b', s')\} + \mu(b, s) \\
qu(b, s) = \beta \frac{\mathbb{E} \{m(b', s)c(b', s')\}}{c(b, s)^{-\gamma} - \phi \mu(b, s)} \\
\mu(b, s) \left[ \frac{b'(b, s)}{R} + \phi q(b, s) \right] = 0 \\
c(b, s) + \frac{b'(b, s)}{R} = y + b \\
V(b, s) = \frac{c(b, s)^{1-\gamma} - 1}{1 - \gamma} + \frac{\beta}{\sigma} \ln \mathbb{E} \{\exp \{\sigma V(b', y')\}\}
\]

where \( m(b, s) \) is the expectation distortion increment. The solution method works over the following steps:

1. We set a grid \( G_b = \{b_1, b_2, \ldots, b_H\} \) for the next-period bond holding \( b' \); and a grid \( G_s = \{s_1, s_2, \ldots, s_N\} \) for the shock state space \( s = \{y, \sigma\} \). The income process \( y \), is discretized with Tauchen and Hussey [61] method, while the grid for the inverse of the penalty parameter \( \sigma \) (recall that \( \theta \) is the inverse of \( \sigma \)) follows a simple two-state rule:\footnote{We use 800 grids point for bonds and 45 grid points for the exogenous shocks; we implement linear interpolation in order to approximate the policy functions outside the grids}

\[
\sigma = \begin{cases} 
\sigma^- & \text{if } V < \mathbb{E} \{V\} \\
\sigma^+ & \text{if } V \geq \mathbb{E} \{V\}
\end{cases}
\]

2. In iteration step \( k \), we start with a set of policy functions \( c_k(b, s), q_k(b, s), \mu_k(b, s) \) and \( V_k(b, s) \).

For each \( b' \in G_b \) and \( s' \in G_s \):

a) we derive the expectation distortion increment:

\[
m_k(b', s') = \frac{\exp \{\sigma V_k(b', s')\}}{\mathbb{E} \{\exp \{\sigma V_k(b', s')\}\}}
\]
and then, the distorted expectations in the Euler equation for bonds and for the risky assets (equations (1) and (2)).

b) we solve the system of optimality conditions under the assumption that the collateral constraint is slack:

\[ \mu^u(b', s) = 0 \]  

(67)

As a result, \( c^u(b', s) \), \( q^u(b', s) \), \( \mu^u(b', s) \), \( V^u(b', s) \) and \( b^u(b', s) \) are the policy functions for the unconstrained region;

c) in the same way, we solve the system for the constrained region of the state space, where the following condition holds:

\[ q^c(b', s) = -\frac{b'/R}{\phi} \]  

(68)

\( c^c(b', s) \), \( q^c(b', s) \), \( \mu^c(b', s) \), \( V^c(b', s) \) and \( b^c(b', s) \) are the respective policy functions.

d) we derive the next period bond holding threshold \( \bar{b}' \) such that the borrowing constraint is marginally binding. For each \( s \in G_s \) it satisfies the following condition:

\[ \bar{b}'(\bar{b}', s) \]  

(69)

When this point is out of the grid we use linear interpolation. Given this value, we can derive for each policy function the frontier between the binding and non-binding region:

\[ x^u(\bar{b}'(\bar{b}', s)) \]  

for \( x = \{c, b, q, \mu, V\} \).

3. In order to construct the step \( k + 1 \) policy function, \( x_{k+1}(b, s) \), we interpolate on the pairs \( (x^c(\bar{b}'(\bar{b}', s))) \) in the constraint region, and on the pairs \( (x^u(\bar{b}'(\bar{b}', s))) \) in the unconstrained region. As a result we find: \( c_{k+1}(b, s) \), \( q_{k+1}(b, s) \), \( \mu_{k+1}(b, s) \) and \( V_{k+1}(b, s) \)

4. We evaluate convergence. If

\[ \sup \|x_{k+1} - x_k\| < \epsilon \]  

for \( x = c, q, \mu, V \)  

(70)

we find the competitive equilibrium. Otherwise, we set \( x_k(b, s) = (1 - \delta)x_{k+1}(b, s) + \delta x_k(b, s) \) and continue the iterations from point 2. We use a value of \( \delta \) close to 1.
11 Appendix D. Data Description for Empirical Moments

In this section we describe the data employed for the computation of the empirical moments used for model matching. We compute several moments for asset prices, returns and debt data. Data are from the US. The used sample spans 1980:Q1 to 2016:Q4, since this corresponds to the period of rapid debt growth. The dataset is composed as follows: debt is given by private non-financial sector, by all sectors (BIS: (http://www.bis.org/publ/qtrpdf/r_qt1403g.pdf)); consumption is given by Personal Consumption Expenditure (NIPA Tables\textsuperscript{32}), GDP (NIPA Tables); the risk-free rate is the 3-month T-bill rate (CRSP Indices database\textsuperscript{33}); risky returns are proxied by the S&P500 equity return with dividends (Shiller Database\textsuperscript{34}). All variables are deflated by CPI index. Note that HP-filtered series are computed as deviations from a long-term trend. Therefore, we work with a much larger smoothing parameter (\(\lambda = 400,000\)) than the one employed in the business cycle literature, to pick up the higher expected duration of the credit cycle (see Borio and Lowe (2002), http://www.bis.org/publ/bcbs187.pdf).

12 Appendix E. Policy Functions

Figure 8 shows the policy functions \(c_t(b,y)\), \(q_t(b,y)\), \(b_{t+1}(b,y)\) and \(\mu_t(b,y)\) for a medium income shock realization. It proves that our model, even with state contingent ambiguity attitudes, is able to reproduce all the salient characteristics of the financial crises models (see Jeanne and Korinek [41]). Indeed, in the binding region the next period bond holding is downward sloping and the policy functions for consumption and asset price display a higher inclination than in the unconstrained region. The latter feature implies that in the constrained region variables respond very strongly to changes in the current wealth, as the financial amplification theory states.

13 Appendix F. Three Period Model

In this section we outline an extended version of the three period model with occasionally binding collateral constraints and with ambiguity attitudes. The goal is to show the combined effect of

\textsuperscript{32} See https://www.bea.gov/iTable/index_nipa.cfm.
Figure 8: **Full set of policy functions from the model under the baseline calibration.**
those two elements on debt growth. The economy we consider is populated by a continuum of agents, who live for three periods: \( t = 0, 1, 2 \). Preferences are given by the following specification:

\[
U = u(c_0) + \mathbb{E}_{S} [\beta u(c_1) + \beta^2 u(c_2)]
\]

(71)

where \( u(c) = \frac{1}{2} [\bar{c} - c]^2 \). In period 0 we can assume a linear utility function \( u(c_0) = c_0 \) in order to simplify the analysis. We also assume that \( \beta R = 1 \). The endowment structure is characterized as follows. Agents receive endowment income in period 1 and 2, but none in period 0. In period 1 the endowment is stochastic depending on the realization of the state \( s \in S \). We assume that \( S = \{s_1, s_2, \ldots s_N\} \) is a monotone increasing sequence. The realization of the endowment are affected monotonically from the realization of \( s \), so that for example \( y^{s_n} > y^{s_{n-1}} \). The probability that a state \( s \) occurs is given by \( \pi_s \). Similarly to the main text we assume that the dividend is lead by the same source of volatility. This allows us to simplify the state space. Therefore, in each period a fraction \( (1 - \alpha)y_t \) is the labor income, and the fraction \( d_t = \alpha y_t \) is the dividends’ income. The budget constraints for each period reads as follows:

\[
c_0 + q_0 x_0 + \frac{b_0}{R} = 0
\]

(72)

\[
c_1^s + q_1^s x_1 + \frac{b_1^s}{R} = (1 - \alpha)y_1^s + x_0(q_1^s + \alpha y_1^s) + b_0
\]

(73)

\[
c_2^s = (1 - \alpha)y_2 + x_1 \alpha y_2 + b_1^s
\]

(74)

Note that the sup-index \( s \) in period 1 indicates that uncertainty materializes in this period.

We have assumed that \( b_{-1} = b_2 = 0, x_{-1} = x_2 = 0, q_2 = 0 \) and \( d_{-1} = 0 \). In period 1 the collateral constraint limits the amount of debt:

\[-\frac{b_1^s}{R} \leq \phi q_1^s x_1^s\]

(75)

The agents expectation formation process is derived as in the main text. Since uncertainty refers to period 1 income, agents form expectation in period 0. Their optimal likelihood ratio in period 0 is given by:

\[
m_1^s = \frac{\exp\{\sigma_0 V_1^s\}}{\mathbb{E}_0 \{\exp\{\sigma_0 V_1^s\}\}}
\]

(76)

where the value function recursion is defined as following\(^{35}\): \( V_1^s = u(c_1^s) + \beta u(c_2^s) \). The relation

\(^{35}\)This simplified representation is obtained under the assumption that there is no uncertainty in period 2.
that links the level of \( m_1^s \) to the state of the economy is:

\[
\text{if } \quad V_1^s < \mathbb{E}_0 \{ V_1^s \} \quad \text{then} \quad m_1^s > 1 \quad (77)
\]

Given the above optimization problems the \textit{decentralized equilibrium} is characterized as follows. The bonds’ Euler equations between periods 0 and 1 and between periods 1 and 2, read as follows:

\[
1 = \beta R \mathbb{E}_0 \{ m_1^s u_c(c_1^s) \} \quad (78)
\]

\[
u_c(c_1^s) = \beta Ru_c(c_2^s) + \mu_1^s \quad (79)
\]

The Euler conditions on the risky asset between periods 0 and 1 and between periods 1 and 2 read as follows:

\[
q_0 = \beta \mathbb{E}_0 \{ m_1^s u_c(c_1^s)[q_1^s + \alpha y_1^s] \} \quad (80)
\]

\[
q_1^s = \beta \frac{u_c(c_2^s) \alpha y_2}{u_c(c_1^s) - \phi \mu_1^s} \quad (81)
\]

The complementarity slackness condition is:

\[
\mu_1^s \left[ \frac{b_1^s}{R} + \phi q_1^s \right] = 0 \quad (82)
\]

Finally, the decentralized equilibrium is closed with a condition on expectations, equation 6, and the following market clearing conditions:

\[
c_0 + q_0 + \frac{b_0}{R} = 0 \quad (83)
\]

\[
c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \quad (84)
\]

\[
c_2^s = y_2 + b_1^s \quad (85)
\]

where we have imposed the stock market clearing condition \( x_t = 1 \).

### 13.1 Time 1 Continuation Equilibrium

We now proceed to the model solution by backward induction. We start from period the last period and since there is no uncertainty between time 1 and time 2 we can solve for the two periods simultaneously. We start from characterizing the continuation value under the \textit{unconstrained region}. 


The system of equilibrium conditions for the unconstrained region (the sup-index $U$ will be used since now on to indicate the solution for this region) is (we can use $\beta = R^{-1}$ and $\mu_1 = 0$):

$$ u_c(c_1^s) = u_c(c_2^s) \quad c_1^s = c_2^s = c^{U,s} \tag{86} $$

$$ q_1^s = \beta \frac{u_c(c_2^s)}{u_c(c_1^s)} \alpha y_2 \tag{87} $$

$$ c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \tag{88} $$

$$ c_2^s = y_2 + b_1^s \tag{89} $$

Given the above the consumption function depends on lifetime wealth and reads as follows:

$$ c^{U,s} = \frac{1}{1+\beta} \left( y_1^s + b_0 + \frac{y_2}{R} \right) \tag{90} $$

Using the budget constraint and the consumption function one can derive the optimal level of debt:

$$ b_1^U(s) = \frac{\beta}{1+\beta} \left( y_1(s) + b_0 - \frac{y_2}{R} \right) \tag{91} $$

Finally, the equilibrium asset price condition, which depends on the value of the dividend in the last period, reads as follows:

$$ q_1 = \beta \alpha y_2 \tag{92} $$

In the constrained region ($\mu_t > 0$, the sup-index $C$ is used since now onward to indicate equilibrium values for this region), the system of equilibrium conditions reads as follows:

$$ \mu_1^s = u_c(c_1^s) - u_c(c_2^s) \quad c_1^s < c_2^s \tag{93} $$

$$ q_1^s = \beta \frac{u_c(c_2^s)}{u_c(c_1^s)} \phi \mu_1^s \alpha y_2 \tag{94} $$

$$ c_1^s + \frac{b_1^s}{R} = y_1^s + b_0 \tag{95} $$

$$ c_2^s = y_2 + b_1^s \tag{96} $$

$$ \frac{b_1^s}{R} = -\phi q_1^s \tag{97} $$
13.2 Time Zero Equilibrium

To characterize the time 0 equilibrium we first partition the state space into two blocks, $S^C$ and $S^U$, where the constraint is binding and slack respectively. Assuming that the $u(c_0) = c_0$ we have:

$$1 = \sum_{s \in S^U} \pi_s m_1^{U,s} u_c^{U,s}(b_0; y_1, y_2) + \sum_{s \in S^C} \pi_s m_1^{C,s} u_c^{C,s}(b_0; y_1, y_2)$$  (98)

$$q_0 = \beta \left\{ \sum_{s \in S^U} \pi_s m_1^{U,s} u_c^{U,s}(b_0; y_1, y_2) \left[ q_1^U + y_1^s \right] + \sum_{s \in S^C} \pi_s m_1^{C,s} u_c^{C,s}(b_0; y_1, y_2) \left[ q_1^C + y_1^s \right] \right\}$$  (99)

$$c_0 = \frac{b_0}{R} - q_0$$  (100)

where $c_1^i, b_1^i, q_1^i$ are the solutions of the time 1 continuation equilibrium.

13.3 The Expectation Distortion under a Binomial State Space

Our goal is to assess the role of ambiguity attitudes on debt growth. To this purpose we shall derive a closed form solution for policy functions. To do that we assume a simple binomial structure for the state space. Hence we assume that the state space is comprised of two states, which we label high, with sup-index $h$, occurring with probability $\pi$, and low, with sup-index $l$, occurring with probability $(1 - \pi)$. The exogenous state space therefore reads as follows $S = \{h, l\}$. We assume that the in state $h$ the income realization is high enough that the collateral constraint is slack. Similarly we assume that in state $l$, the income realization is low enough that the collateral constraint binds. Given this structure for the objective probability, the expectation distortions are given by:

$$m_1^h = \frac{\exp \{\sigma_0 V_1^{h} \}}{\pi \exp \{\sigma_0 V_1^{h} \} + (1 - \pi) \exp \{\sigma_0 V_1^{l} \}}$$  (101)

where the value function has the following form, $V_1^i = u(c_1^i) + \beta u(c_2^i)$. Given the assumptions on the state space, it follows that:

$$V_1^h > \mathbb{E}_0 \{V_1^s\} \quad \text{and} \quad V_1^l < \mathbb{E}_0 \{V_1^s\}$$  (102)

Equation 101 and jointly imply that, if $\theta_0 > 0$, hence $\sigma_0 = -\frac{1}{\theta_0} < 0$, the following holds:

$$\exp \sigma_0 V_1^h < \mathbb{E}_0 \{\exp \sigma_0 V_1^s\} \quad \Rightarrow \quad m_1^h < 1$$  (103)

$$\exp \sigma_0 V_1^l > \mathbb{E}_0 \{\exp \sigma_0 V_1^s\} \quad \Rightarrow \quad m_1^l > 1$$  (104)
Intuitively the above implies that agents assign an higher subjective probability (with respect to the objective) to the bad history and a lower probability to the good history. We can call this behaviour pessimism. Similarly if \( \theta_0 < 0 \), then \( \sigma_0 = -\frac{1}{\theta_0} > 0 \), we have that:

\[
\exp \sigma_0 V^h_1 > \mathbb{E}_0 \{ \exp \sigma_0 V^s_1 \} \quad \Rightarrow \quad m^h_1 > 1
\]

\[
\exp \sigma_0 V^l_1 < \mathbb{E}_0 \{ \exp \sigma_0 V^s_1 \} \quad \Rightarrow \quad m^l_1 < 1
\]

Note that in this second case agents assign an higher subjective probability to the good history and a lower probability to the bad history, depicting borrowers’ optimistic behavior.

We shall now solve the equilibrium and derive the implied debt policy functions under the above beliefs’ structure. We start by characterizing the equilibrium at time zero, given by the optimal decisions \((b_0, c_0, q_0)\). We distinguish the solution with optimistic beliefs from the one with pessimistic beliefs. We also compare the two solutions to the case with no ambiguity attitudes.

The debt policy function is best characterized by the following relation:

\[
b_0 = -R[c_0 + q_0]
\]

Next to characterize the time 0 policy function for consumption we rely on the Euler equation between period 0 and period 1:

\[
u_c(c_0) = \pi m^h_1 u_c(c^h_1) + (1 - \pi) m^l_1 u_c(c^l_1)
\]

We can reformulate the above equation in terms of the subjective weights of the ambiguity averse agent:

\[
u_c(c_0) = \psi^h u_c(c^h_1) + (1 - \psi) \psi^l u_c(c^l_1)
\]

where \( \psi^h = \pi m^h_1 \) and \( \psi^l = (1 - \pi) m^l_1 \). Given the model structure (incomplete financial markets, hence lack of insurance to equalize consumption), the events structure and the condition on the collateral constraint, we can conclude that:

\[
c^h_1 > c^l_1 \quad \Rightarrow \quad u_c(c^h_1) < u_c(c^l_1)
\]

Next, recall that in the optimism case beliefs imply:

\[
\psi^h = \pi m^h_1 > \pi
\]

\[
\psi^l = (1 - \pi) m^l_1 < (1 - \pi)
\]
This implies that agents assign a higher weight, with respect to the case with no ambiguity, to the component $u_c(c_t^1)$. Hence, the marginal utility of consumption in $t = 0$ is lower (than in absence of ambiguity considerations) and the consumption is higher:

$$c_0^o > c_0^{RE}$$  \hfill (113)

where $c_0^o$ indicates consumption under optimist behavior, while $c_0^{RE}$ indicates consumption under no ambiguity. Intuitively agents assign higher weight to good future states, hence they prefer to postpone consumption and to invest in the risky asset. This in turn will raise asset price, since the demand of asset has increased. As investment takes place through leverage, they will also leverage more.

In the pessimism case the borrower assigns the following weights:

$$\psi^h = \pi m_1^h < \pi$$  \hfill (114)

$$\psi^l = (1 - \pi)m_1^l > (1 - \pi)$$  \hfill (115)

This implies:

$$c_0^u < c_0^{RE}$$  \hfill (116)

where $c_0^u$ indicates consumption under pessimistic behavior. In this case the agent expects more likely the bad state to take place in the future. The agent will then anticipate consumption and invest less in the risky asset. They will in turn leverage less. We can generalize this relation with the following condition:

$$c_0^o > c_0^{RE} > c_0^u$$  \hfill (117)

This also implies that:

$$b_0^o > b_0^{RE} > b_0^u$$  \hfill (118)

14 Intermediation Channel

In this section we provide micro-foundations for a delegated monitoring problem in which the collateral constraint emerges as resulting from an incentive-compatible debt contract enforced through a bank. The micro-foundations follows Bianchi and Mendoza[10]. Debt contract are signed by a bank that must enforce debtor incentives. Between periods borrowers can divert revenues for an
amount $\tilde{d}$. At the end of the period the diversion is no longer possible and payment is enforced. Banks can monitor financial diversion due to special relationship lending abilities. If the bank detects the diversion asset can be seized up to a percentage $\phi$. As common in dynamic economies we assume that the contract is done under no memory, so that in the next periods borrowers can re-enter debt agreement even if they defaulted in the previous period. This assumption allows us to preserve the Markov structure of the contracting/intermediation problem.

We shall show that the collateral constraint can emerge as resulting from an incentive compatibility constraint imposed by the bank through the debt design. Specifically the collateral constraint can be derived as an implication of incentive-compatibility constraints on borrowers if limited enforcement prevents banks from redeploying more than a fraction $\phi$ of the value of the assets owned by a defaulting borrower. Define $V^R$ and $V^D$ respectively the value of repayment and default and define as $V$ the continuation value.

If the borrower defaults the diverted resources enter his budget constraint and the recursive problem reads as follows (for notational convenience we skip the beliefs constraints for the purpose of this derivation):

$$V^D(b, x, S) = \max_{c, x', b'} \left\{ u(c) + \beta \mathbb{E}S + \right.$$}

$$+ \lambda \left[ y + q(S)(x + \alpha y) + \tilde{d} + b - q(S)x' - c - \frac{b'}{R} \right] +$$

$$+ \mu \left[ \phi q(S)x' + \frac{b'}{R} \right]$$

On the other side if the borrower repays his value function reads as:

$$V^D(b, x, S) = \max_{c, x', b'} \left\{ u(c) + \beta \mathbb{E}S + \right.$$}

$$+ \lambda \left[ y + q(S)(x + \alpha y) + b - q(S)x' - c - \frac{b'}{R} \right] +$$

$$+ \mu \left[ \phi q(S)x' + \frac{b'}{R} \right]$$

The comparison of the two easily shows that the households repay if and only if $\tilde{d} < \phi q(S)x'$.

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36 We assume zero monitoring costs for simplicity. Extending it to the case with positive monitoring costs is rather straightforward.