Inflation Puzzles in the New Keynesian Model: The Implications of Anchored Expectations *

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Abstract

In this paper we introduce bounded rationality into an otherwise standard New Keynesian Model. Agents are assumed to behave as econometricians, using time-series models to forecast inflation and the output gap similar to that of Stock and Watson (2007). The agent’s perceived optimal forecast rules are defined by the Kalman filter. In a unique equilibrium, the values of the two Kalman gain parameters are pinned down by the observed autocorrelation of inflation and output gap changes. This methodology can be applied directly to U.S. data. We show that if agents perpetually update their estimates of the Kalman gains using a moving window of recent data, the identified Kalman gain for inflation exhibits a downward drift during the so-called “Great Moderation” period. A low Kalman gain implies a low weight on recent inflation in the agent’s forecast rule. This helps anchor inflation near the central bank’s target rate when the output gap falls sharply during the Great Recession. In the longer term, however, the recession leads to a downward revision of agents’ inflation forecast, which generates a moderate – but highly persistent – decline in inflation. Thus, the model can help account for both the “missing disinflation” in the immediate wake of the recession as well as the “missing inflation” in recent years. Forecasts with the model suggest that inflation will undershoot the central bank’s target rate for several years after the output gap has fully recovered. Consequently, the model predicts that monetary policy will remain accommodative and contribute to a positive output gap in the future.

Keywords: Inflation Expectations, Phillips Curve, Bounded Rationality

JEL Classification: E31, E37

*The views in this paper are our own and not necessarily those of The Federal Reserve Bank of San Francisco, The Federal Reserve System or Danmarks Nationalbank

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1 Introduction

Since the outbreak of the Great Recession, inflation dynamics in the U.S. and the eurozone have sparked a debate over the New Keynesian Phillips Curve (NKPC) (e.g. Hall, 2011). From the perspective of the NKPC, inflation rates did not decline as much as expected in the aftermath of the recession. The absence of a persistent decline in inflation was dubbed the “missing disinflation puzzle” (Coibion and Gorodnichenko, 2015). Subsequently, inflation rates have not recovered as fast as expected. Since mid-2012, PCE inflation has been persistently below the Fed’s target of 2 pct., which has given birth to the so-called “missing inflation puzzle”. In this paper, we introduce a form of boundedly-rational expectations into an otherwise standard New Keynesian model.1 Agents are assumed to behave as econometricians, using time-series models for inflation and the output gap similar to that of Stock and Watson (2007). The agent’s perceived optimal forecast rules are defined by the Kalman filter. We show that the model has a unique equilibrium where the perceived optimal values of the two Kalman gain parameters are pinned down by the observed autocorrelations of inflation changes and output gap changes. By computing the values of the two autocorrelation coefficients, the agent can identify the two “signal-to-noise ratios” in the inflation and output gap data. The model’s methodology for identifying the two signal-to-noise ratios can be applied directly to U.S. data. We show that if the agent perpetually updates estimates of the two signal-to-noise ratios using moving windows of recent data, the identified signal-to-noise ratio for inflation will exhibit a downward drift during the so-called “Great Moderation” period from 1984-2007. A lower signal-to-noise ratio implies a lower weight on recent inflation in the agent’s forecast rule, which is consistent with the idea of “anchored” inflation expectations. Anchored expectations in the NKPC imply that inflation is less sensitive to changes in the output gap. Consequently, when the output gap drops sharply in 2008, the initial response of inflation is muted. However, the recession gradually leads to a moderate downward revision of agent’s inflation expectations, which generates a highly persistent decline in inflation in the longer term. Thus, the model can help account for both the “missing disinflation” in the immediate wake of the recession as well as the “missing inflation” since 2012. Model forecasts suggest that inflation will undershoot the central bank’s target rate for several years after the output gap has fully recovered. Thus, according to the model, monetary policy will remain accommodative and contribute to a positive output gap in the future.

Standard New Keynesian models with rational expectations tend to produce two counterfactual predictions. First, they generate large and persistent declines in inflation in response to deep contractions in the output gap such as the Great Recession.2 Second, they predict

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1The paper builds on Lansing (2007), who introduces bounded rationality into the NKPC with an exogenous output gap. We adopt the same type of boundedly rational expectations, but develop a fully-articulated general equilibrium model.

2See, for instance, Auroba and Schorfheide (2016) or Christiano et. al. (2015)
that the recovery of inflation and the output gap closely mirrors the exogenous shock process. Thus, as soon as the shock stops operating, the output gap will be closed and inflation will be back at the central bank’s target rate. However, as shown in Figure 1, actual U.S. inflation has behaved markedly different. First, there was no persistent disinflation in the aftermath of the recession. PCE Inflation dropped sharply in 2008:q4 when energy prices collapsed, but almost fully recovered within two quarters. Since mid-2012, however, PCE inflation has been persistently below the Fed’s target of 2 pct. In the context of a standard NKPC this decline is surprising considering the simultaneous recovery of the output gap. Figure 1 shows that inflation expectations did not respond much to the Great Recession. After a moderate decline in 2008, 1-year inflation expectations from the Michigan Survey of Consumers recovered between 2009 and 2012. Since 2012, however, they have declined. Similarly, 10-year expectations from the SPF began to decline in 2012. 1-year expectations from the Survey of Professional Forecasters did not fully recover after the initial decline in 2008, but have converged to a level which is below its pre-recession trend.

A growing empirical literature has tried to resolve the inflation puzzles that arise in the New Keynesian Model. According to Coibion and Gorodnichenko (2015), the missing disinflation is explained by a rise in household inflation expectations from 2009 to 2011, which reflected the increase in oil prices over this time period. In Bobeica and Jarocinski (2017), the inflation puzzles disappear in a vector autoregression which accounts for both domestic and global variables. Closely related to our work is a recent paper by Ball and Mazumder (2011), who argue that the Great Recession provides new evidence against the NKPC with rational expectations. According to these authors, a backward-looking Phillips Curve with a time-varying slope can match U.S. inflation during the Great Recession. Moreover, they find strong evidence of expectations anchoring during the Great Moderation. According to Bernanke (2010), well-anchored inflation expectations made the risk of deflation in the wake of the Great Recession insignificant. In our model, anchored expectations are equivalent to a low perceived signal-to-noise ratio, which in turn implies a low weight on recent inflation in the agents forecast rule. We argue that inflation forecasts that are based on the unobserved components model of Stock and Watson (2007) may be a good proxy for real world inflation expectations. Since their influential paper, the unobserved components model has become a popular tool among economists to forecast inflation. Several recent papers, including Arouba and Schorfheide (2016), use the unobserved components model to generate inflation forecasts. Moreover, we show that model-based inflation expectations track well with survey-based inflation expectations since the late 1970’s.

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3 Coibion and Gorodnichenko (2015) shows that household’s inflation expectations followed oil prices closely over this period.

4 According to these authors, the NKPC fits the data poorly, because it predicts that the output gap has a negative effect on the expected change in inflation.

5 Other papers reach the same conclusion. For instance, Edge et. al. (2007) find that an unobserved components model using real-time data describes economists long-run productivity growth forecasts extremely
In the New Keynesian literature, a series of recent papers have argued that the missing disinflation puzzle can be resolved by extending the standard NK model with various types of financial frictions. For instance, in a model with a working capital channel, Christiano et. al. (2015) shows that a fall in productivity growth and a rise in the costs of working capital can account for the small drop in inflation during the recession. In Del Negro et. al. (2015), the missing disinflation dissapears if the NKPC is sufficiently flat.

Other recent papers deviate from the rational expectations assumption. In the context of an adaptive learning model, Evans et. al. (2017) argue that the U.S. is stuck in a distinct stagnation steady state characterized by pessimistic expectations and a binding ZLB. Lansing (2017), on the other hand, shows that a model with endogenous switching between two local rational expectations equilibria and a time-varying natural rate of interest can produce highly negative output gaps and a binding ZLB, reminiscent of the U.S. Great Recession. While both of these papers are closely related to our work, we focus more closely on the effects of "anchored" inflation expectations.

The remainder of the paper is organized as follows. Section 2 describes the standard New Keynesian model. Section 3 derives the unique solution under rational expectations. In Section 4, we define the concept of a “consistent expectations” equilibrium and prove the uniqueness of such an equilibrium. In Section 5 we apply the methodology of the CE model directly to U.S. data and reassess the inflation puzzles that arise under rational expectations. Section 6 concludes.

Figure 1: Key Macroeconomic Variables, 2005:q1-2017:q2
2 The New Keynesian Model

The starting point for the analysis is a standard New Keynesian model. The model consists of three main elements: A New Keynesian Phillips Curve (NKPC), an IS curve and a Taylor rule for monetary policy. Throughout the paper, model variables are expressed in terms of log-deviations from steady state. We use the notation \( x_t = \ln \left( \frac{X_t}{X^*} \right) \) where \( X^* \) is the steady state value of a variable, \( X_t \).

The NKPC can be derived from Calvo’s (1983) model of sticky prices. It links inflation to expected inflation and the output gap:

\[
\pi_t = \beta \bar{E}_t \pi_{t+1} + \kappa y_t + u_t, \quad \beta \in [0, 1), \quad \kappa > 0, \quad u_t \sim N \left( 0, \sigma_u^2 \right),
\]

where \( \pi_t \) is the deviation of the inflation rate from the central bank’s target.\(^6\) \( \beta \) is the representative agent’s subjective time discount factor, \( y_t \) is the output gap and \( u_t \) is an \( iid \) cost-push shock. The symbol \( \bar{E}_t \) represents the agent’s subjective expectation conditioned on information available at time \( t \). Under rational expectations, \( \bar{E}_t \) corresponds to the mathematical expectations operator, \( E_t \).

The IS curve links the output gap to the expected future output gap and the real interest rate:

\[
y_t = \bar{E}_t y_{t+1} - \alpha \left( R_t - \bar{E}_t \pi_{t+1} \right) + v_t, \quad \alpha > 0, \quad v_t \sim N \left( 0, \sigma_v^2 \right),
\]

where \( R_t \) is the log deviation of the gross nominal interest rate, \( \alpha \) is the inverse of the coefficient of relative risk aversion and \( v_t \) is an \( iid \) demand shock that is uncorrelated with the cost-push shock.

Monetary policy is characterized by a Taylor-type rule, where the central bank responds to forecasts of inflation and the output gap.

\[
R_t = \mu_\pi \bar{E}_t \pi_{t+1} + \mu_y \bar{E}_t y_{t+1}, \quad \mu_\pi > 0, \quad \mu_y > 0
\]

where \( \mu_\pi \) and \( \mu_y \) are the Taylor coefficients on the central bank’s forecasts of the inflation gap and the output gap, respectively. We assume that the Taylor principle is satisfied, i.e. \( \mu_\pi > 1 \).

3 Rational Expectations

Under rational expectations, the inflation rate and the output gap are uniquely pinned down by the shocks, \( v_t \) and \( u_t \). The unique rational expectations solution is given by:

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\(^6\)This holds under the assumption that price adjustments are costless at the steady state rate. Thus we abstract from changes in the functional form of the NKPC that arise when the Calvo pricing equation is log-linearized around a non-zero inflation rate, as shown by Ascari (2004) and Sahuc (2006).
\[ \pi_t^{re} = \kappa v_t + u_t \]  \hspace{1cm} (4)

\[ y_t^{re} = v_t \]  \hspace{1cm} (5)

\[ R_t^{re} = 0 \]  \hspace{1cm} (6)

(4)-(5) imply that the one-period ahead rational forecasts of inflation and the output gap are always zero:

\[ E_t \pi_{t+1}^{re} = 0 \]  \hspace{1cm} (7)

\[ E_t y_{t+1}^{re} = 0 \]  \hspace{1cm} (8)

where we have replaced \( \bar{E}_t \) with \( E_t \). Moreover, from (4)-(6) we obtain the following unconditional moments:

\[ \text{Var} (\pi_t^{re}) = \kappa^2 \sigma_v^2 + \sigma_u^2 \]

\[ \text{Var} (y_t^{re}) = \sigma_v^2 \]

\[ \text{Var} (R_t^{re}) = 0 \]

\[ \text{Corr} (\pi_t^{re}, \pi_{t-1}^{re}) = 0 \]

\[ \text{Corr} (y_t^{re}, y_{t-1}^{re}) = 0 \]

\[ \text{Corr} (R_t^{re}, R_{t-1}^{re}) = 0 \]

These expressions show that, under rational expectations, the variables inherit their stochastic properties solely from the white-noise shocks. Moreover, they exhibit no first order autocorrelation. The latter conflicts sharply with U.S. data. From 1984 to 2007 the first order autocorrelation of quarterly PCE inflation, the CBO output gap and the Federal Funds rate were, respectively, \( 0.46 \), \( 0.93 \) and \( 0.97 \).

The literature has proposed numerous ways to overcome the problem of weak persistence in New Keynesian models.\(^7\) A straightforward solution would be to model the exogenous shocks,
and \( u_t \), as AR(1) processes. In the following, however, we will instead introduce a form of boundedly rational expectations which generates enough endogenous persistence to match the moments in U.S. data.\(^8\)

4 Consistent Expectations

We introduce bounded rationality by assuming that the representative agent behaves as an econometrician, using time series models to forecast inflation and the output gap. Specifically, the agent is assumed to employ an unobserved components model which allows for both permanent and temporary shocks – along the lines of Stock and Watson (2007). The perceived law of motion for inflation is given by:

\[
\begin{bmatrix}
\pi_t \\
\bar{\pi}_t
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\pi_{t-1} \\
\bar{\pi}_{t-1}
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\zeta_t \\
\eta_t
\end{bmatrix},
\]

\( \zeta_t \sim N \left(0, \sigma^2_{\zeta} \right), \) \( \eta_t \sim N \left(0, \sigma^2_{\eta} \right), \) \( \text{Cov} (\zeta_t, \eta_t) = 0, \) \( (9) \)

where \( \pi_t \) is the unobservable inflation trend, \( \zeta_t \) is a transitory shock that pushes \( \pi_t \) away from trend, and \( \eta_t \) is permanent shock (uncorrelated with \( \zeta_t \)) that shifts the trend over time. The specification implies that the subjective forecast \( \tilde{E}_t \pi_{t+1} \) equals the Kalman filter estimate of \( \pi_t \). Some technical points are worth noting. First, although the perceived law of motion \( (9) \) allows for permanent deviations from steady state, the equilibrium inflation process (to be defined below) remains stationary around the steady state inflation rate. Moreover, we abstract from “long-horizon expectations” that arise in the NKPC when forward-looking agents employ subjective forecasts of future inflation, as discussed by Preston (2005). The perceived law of motion \( (9) \) implies \( \tilde{E}_t \pi_{t+j} = \tilde{E}_t \pi_{t+1} \) for all future horizons \( j = 2, 3, 4... \) Equation \( (1) \) can therefore be viewed as a log-linear approximation of a more-complicated NKPC that explicitly incorporates long-horizon inflation expectations.

The representative agent uses a similar time series model to forecast the output gap:

\[
\begin{bmatrix}
y_t \\
\bar{y}_t
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
y_{t-1} \\
\bar{y}_{t-1}
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
\chi_t \\
\varphi_t
\end{bmatrix},
\]

\( \chi_t \sim N \left(0, \sigma^2_{\chi} \right), \) \( \varphi_t \sim N \left(0, \sigma^2_{\varphi} \right), \) \( \text{Cov} (\chi_t, \varphi_t) = 0, \) \( (10) \)

where \( \bar{y}_t \) is the perceived long-run output gap, \( \chi_t \) is a transitory shock and \( \varphi_t \) is permanent shock (uncorrelated with \( \chi_t \)). Agents do not need to believe that the output gap literally has a unit root but that such a specification is a local approximation that is convenient for forecasting.\(^9\)

\(^8\)Later, we extend the rational version of the model with a persistent demand shock which generates the inflation puzzles described in the introduction.

\(^9\)The CBO output gap was negative from 2008-2017, so any mean reversion to zero is obviously very slow. The unobserved components model captures this feature with a unit root.
As originally shown by Muth (1960), the perceived laws of motion (9) and (10) imply the following error-correction forecasting rules for inflation and the output gap, respectively:

\[
\tilde{E}_{t+1} = \tilde{E}_t + \lambda_e \left( \pi_t - \tilde{E}_t \pi_t \right), \quad 0 < \lambda_e \leq 1,
\]

and

\[
\tilde{E}_{t+1} = \tilde{E}_t + \lambda_y \left( y_t - \tilde{E}_t y_t \right), \quad 0 < \lambda_y \leq 1,
\]

where \( \pi_t - \tilde{E}_t \pi_t \) and \( y_t - \tilde{E}_t y_t \) are the forecast errors in period \( t \). We assume that the agent’s subjective forecast makes use of the contemporaneous realizations \( \pi_t \) and \( y_t \). This setup avoids the introduction of an extra lag of variables that might be viewed as artificially influencing the resulting dynamics.\(^{10}\) Equations (11) and (12) imply that the agent’s forecasts at time \( t \) are exponentially-weighted moving averages of the current and past observed inflation rates and output gaps, respectively.

The agent’s perceived optimal choices of the weights, \( \lambda_e \) and \( \lambda_y \), in equations (11) and (12), respectively, are determined by the Kalman filter, where the objective is to minimize the mean squared forecast errors \( E \left( \pi_{t+1} - \tilde{E}_{t+1} \pi_{t+1} \right)^2 \) and \( E \left( y_{t+1} - \tilde{E}_{t+1} y_{t+1} \right)^2 \). In steady-state, the unique solution for the perceived optimal gain parameter for inflation is:

\[
\lambda_e = \frac{-\phi_e + \sqrt{\phi_e^2 + 4\phi_e}}{2}, \tag{13}
\]

where \( \phi_e = \sigma_{\pi}^2 / \sigma_{\zeta}^2 \) is the perceived signal-to-noise ratio for inflation.\(^{11}\) As \( \phi_e \to \infty \), the gain parameter approaches 1. From the agent’s perspective, the shocks themselves \( \zeta_t \) and \( \eta_t \) are unobservable but the shock variances \( \sigma_{\zeta}^2 \) and \( \sigma_{\eta}^2 \) can be inferred from the moments of inflation changes \( \Delta \pi_t \), which are observable. Similarly, the unique solution for the perceived optimal gain parameter for the output gap is:

\[
\lambda_y = \frac{-\phi_y + \sqrt{\phi_y^2 + 4\phi_y}}{2}, \tag{14}
\]

\(^{10}\)A lagged information assumption is often used in learning models to avoid simultaneity in the determination of the actual and expected values of the forecast variable. In the continuous time limit, the distinction between contemporaneous and lagged information disappears.

\(^{11}\)For details of the derivation of (13), see Nerlove (1967, pp. 141-143). His results are expressed as a formula for \( 1 - \lambda \).
where $\phi_y = \frac{\sigma_\phi^2}{\sigma_\chi^2}$ is the perceived signal-to-noise ratio for the output gap.

**Proposition 1.** If the representative agent’s perceived laws of motion are given by equations (9) and (10), respectively, then the perceived optimal value of the Kalman gain parameter $\lambda_\pi$ is uniquely pinned down by the autocorrelation of observed inflation changes, $\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})$, while the perceived optimal value of the Kalman gain parameter $\lambda_y$ is uniquely pinned down by the autocorrelation of observed output gap changes, $\text{Corr} (\Delta y_t, \Delta y_{t-1})$.

**Proof:** From (9), we have $\Delta \pi_t = \eta_t + \zeta_t - \zeta_{t-1}$. Since $\eta_t$ and $\zeta_t$ are perceived to be independent, we have $\text{Cov} (\Delta \pi_t, \Delta \pi_{t-1}) = -\sigma_\zeta^2$ and $\text{Var} (\Delta \pi_t) = \sigma_\eta^2 + 2\sigma_\zeta^2$. Combining these two expressions and solving for the signal-to-noise ratio yields

$$\phi_\pi = \frac{-1}{\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})} - 2, \quad (15)$$

where $\phi = \frac{\sigma_\eta^2}{\sigma_\chi^2}$ and $\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1}) = \text{Cov} (\Delta \pi_t, \Delta \pi_{t-1}) / \text{Var} (\Delta \pi_t)$. The above expression shows that $\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})$ uniquely pins down $\phi_\pi$ which, in turn, uniquely pins down $\lambda_\pi$ from equation (13). Similarly, we can prove that the Kalman gain parameter, $\lambda_y$, is pinned down by the autocorrelation of changes in the output gap:

$$\phi_y = \frac{-1}{\text{Corr} (\Delta y_t, \Delta y_{t-1})} - 2, \quad (16)$$

The model (1)-(3) and the forecast rules (11)-(12) can be written on the following matrix form which defines the actual law of motion of the economy:  

$$Z_t = AZ_{t-1} + BU_t \quad (17)$$

where $Z_t = \begin{bmatrix} \pi_t & y_t & R_t & \bar{E}_{t+1} & \bar{E}_{t+1} \end{bmatrix}'$ and $U_t = \begin{bmatrix} u_t & v_t \end{bmatrix}'$. The variance-covariance matrix $V$ of the left-side variables in (17) can be computed using the formula:

$$\text{vec} (V) = [I - A \otimes A]^{-1} \text{vec} \left( \Omega B B' \right), \quad (18)$$

where $\Omega$ is the variance-covariance matrix of the fundamental shocks $u_t$ and $v_t$. Using (18) we can compute the autocorrelation coefficients of $\Delta \pi_t$ and $\Delta y_t$. These coefficients pin down the perceived signal-to-noise ratios $\phi_\pi$ and $\phi_y$ in (15) and (16), which – in turn – uniquely pin down the optimal Kalman gains, $\lambda_\pi$ and $\lambda_y$, in (13) and (14).

**4.1 Defining the Consistent Expectations Equilibrium**

This section defines the concept of a “consistent expectations equilibrium” along the lines of Hommes and Sorger (1998).

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12The derivations can be found in Appendix A.1
Definition 1. A consistent expectations equilibrium is defined as the law of motion (17), and associated Kalman gain parameters, $\lambda_\pi$ and $\lambda_y$, such that $\lambda_\pi$ and $\lambda_y$ are the fixed points of the multidimensional nonlinear maps $\lambda_\pi = T_\pi (\lambda_\pi, \lambda_y)$ and $\lambda_y = T_y (\lambda_\pi, \lambda_y)$, where

$$
T_\pi (\lambda_\pi, \lambda_y) = \frac{-\phi_\pi (\lambda_\pi, \lambda_y) + \sqrt{\phi_\pi (\lambda_\pi, \lambda_y)^2 + 4\phi_\pi (\lambda_\pi, \lambda_y)}}{2},
$$

(19)

$$
\phi_\pi (\lambda_\pi, \lambda_y) = \frac{-1}{\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})} - 2 = \frac{-\text{Var} (\Delta \pi_t)}{\text{Cov} (\Delta \pi_t, \Delta \pi_{t-1})} - 2,
$$

and

$$
T_y (\lambda_\pi, \lambda_y) = \frac{-\phi_y (\lambda_\pi, \lambda_y) + \sqrt{\phi_y (\lambda_\pi, \lambda_y)^2 + 4\phi_y (\lambda_\pi, \lambda_y)}}{2},
$$

(20)

$$
\phi_y (\lambda_\pi, \lambda_y) = \frac{-1}{\text{Corr} (\Delta y_t, \Delta y_{t-1})} - 2 = \frac{-\text{Var} (\Delta y_t)}{\text{Cov} (\Delta y_t, \Delta y_{t-1})} - 2,
$$

with the unconditional moments, $\text{Var} (\Delta \pi_t)$, $\text{Var} (\Delta y_t)$, $\text{Cov} (\Delta \pi_t, \Delta \pi_{t-1})$ and $\text{Cov} (\Delta y_t, \Delta y_{t-1})$, computed from the actual law of motion (17), using equation (18).

4.2 Numerical Solution for the Equilibrium

The complexity of the nonlinear maps (19) and (20) necessitates a numerical solution for the equilibrium. To accomplish this, the model is calibrated using a set of baseline parameter values that are standard in the literature. Table 1 reports the baseline calibration.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.995</td>
<td>Discount factor</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>0.01</td>
<td>Output gap coefficient in NKPC</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.5</td>
<td>Interest rate coefficient in IS curve</td>
</tr>
<tr>
<td>$\mu_\pi$</td>
<td>2</td>
<td>Policy response to inflation forecast</td>
</tr>
<tr>
<td>$\mu_y$</td>
<td>0.2</td>
<td>Policy response to output gap forecast</td>
</tr>
<tr>
<td>$\sigma_u$</td>
<td>0.001</td>
<td>Std. dev. of cost push shock</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.001</td>
<td>Std. dev. of aggregate demand shock</td>
</tr>
</tbody>
</table>

We choose the discount factor $\beta = 0.995$, which corresponds to an annual real interest rate of 2 pct. $\kappa = 0.01$ corresponds to a relatively flat NKPC, which is motivated by empirical evidence\textsuperscript{13}. The coefficient on the interest rate in the IS curve, $\alpha = 0.5$, corresponds to a coefficient of relative risk aversion of $1/\alpha = 2$, which is standard in the literature. The Taylor rule coefficients, $\mu_\pi = 2$ and $\mu_y = 0.2$, are close to the estimates in Smets & Wouters (2007).

\textsuperscript{13}See, for instance, Mavroeidis et. al. (2014)
The standard deviations of the structural shocks, \( \sigma_u \) and \( \sigma_v \), are chosen so that the standard deviations of inflation and the output gap are reasonably close to those observed in US data for the period 1984:Q1 to 2007:Q4. We also examine the sensitivity of the results to alternative parameter values.

Given the parameter values in Table 1, we can solve numerically for the equilibrium. An equilibrium \((\lambda^*_\pi, \lambda^*_y)\) requires that the following two conditions are satisfied:

\[
\begin{align*}
    f_\pi (\lambda^*_\pi, \lambda^*_y) &= \lambda^*_\pi - \frac{-\phi_\pi (\lambda^*_\pi, \lambda^*_y) + \sqrt{\phi_\pi (\lambda^*_\pi, \lambda^*_y)^2 + 4\phi_\pi (\lambda^*_\pi, \lambda^*_y)}}{2} = 0 \\
    f_y (\lambda^*_\pi, \lambda^*_y) &= \lambda^*_y - \frac{-\phi_y (\lambda^*_\pi, \lambda^*_y) + \sqrt{\phi_y (\lambda^*_\pi, \lambda^*_y)^2 + 4\phi_y (\lambda^*_\pi, \lambda^*_y)}}{2} = 0
\end{align*}
\]

Figure 2 plots these conditions in \((\lambda_\pi, \lambda_y)\)-space. The figure shows that a unique fixed point occurs at \((\lambda^*_\pi, \lambda^*_y) = (0.5003, 0.8600)\). This corresponds to \(\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1}) = -0.3999\) and \(\text{Corr} (\Delta y_t, \Delta y_{t-1}) = -0.1373\).

Table 2 shows how the equilibrium changes with parameter values. The values of \(\lambda^*_\pi\) and \(\lambda^*_y\) increase with the values of \(\beta\) and \(\kappa\), but decreases with the value of \(\mu_y\). Roughly speaking, parameter changes that increase the persistence in the model have the effect of increasing the perceived signal-to-noise ratios, \(\phi_\pi\) and \(\phi_y\), and hence \(\lambda^*_\pi\) and \(\lambda^*_y\). The intuition behind these effects of parameter changes are straightforward. From the agent’s perspective, inflation and the output gap are comprised of persistent signal components, \(\pi_t\) and \(\overline{y}_t\), respectively, and transitory noise components, \(\zeta_t\) and \(\chi_t\), respectively. If a parameter shift causes the observed inflation rate or output gap to become more persistent, then the agent’s inferred value of the signal-to-noise ratio, \(\phi_\pi\) or \(\phi_y\), will increase.
### Table 2: Sensitivity Analysis

<table>
<thead>
<tr>
<th>Result</th>
<th>Baseline</th>
<th>$\beta = 0.9975$</th>
<th>$\kappa = 0.03$</th>
<th>$\mu_y = 0.1$</th>
<th>$\sigma_u^2/\sigma_v^2 = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi^*_\pi$</td>
<td>0.50</td>
<td>0.53</td>
<td>0.62</td>
<td>0.61</td>
<td>1.02</td>
</tr>
<tr>
<td>$\lambda^*_\pi$</td>
<td>0.50</td>
<td>0.51</td>
<td>0.54</td>
<td>0.53</td>
<td>0.62</td>
</tr>
<tr>
<td>$\phi^*_y$</td>
<td>5.28</td>
<td>5.62</td>
<td>5.78</td>
<td>12.84</td>
<td>1.23</td>
</tr>
<tr>
<td>$\lambda^*_y$</td>
<td>0.86</td>
<td>0.87</td>
<td>0.87</td>
<td>0.93</td>
<td>0.83</td>
</tr>
<tr>
<td>$\text{Corr} (\pi_t, \pi_{t-1})$</td>
<td>0.79</td>
<td>0.80</td>
<td>0.58</td>
<td>0.67</td>
<td>0.80</td>
</tr>
<tr>
<td>$\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})$</td>
<td>-0.40</td>
<td>-0.39</td>
<td>-0.38</td>
<td>-0.38</td>
<td>-0.33</td>
</tr>
<tr>
<td>$\text{Corr} (y_t, y_{t-1})$</td>
<td>0.95</td>
<td>0.95</td>
<td>0.86</td>
<td>0.90</td>
<td>0.92</td>
</tr>
<tr>
<td>$\text{Corr} (\Delta y_t, \Delta y_{t-1})$</td>
<td>-0.14</td>
<td>-0.13</td>
<td>-0.13</td>
<td>-0.07</td>
<td>-0.17</td>
</tr>
<tr>
<td>$\text{Corr} (R_t, R_{t-1})$</td>
<td>0.87</td>
<td>0.87</td>
<td>0.74</td>
<td>0.78</td>
<td>0.82</td>
</tr>
</tbody>
</table>

Notes: Baseline values are reported in Table 1. Changes in $\sigma_v^2/\sigma_u^2$ are accomplished by adjusting $\sigma_v^2$ while maintaining $\sigma_u^2 = (0.001)^2$. Autocorrelation coefficients in the rational model are always zero.

#### 4.3 Real-Time Learning

In the previous section, the equilibrium Kalman gains, $\lambda^*_\pi$ and $\lambda^*_y$, were computed using the population autocorrelation coefficients, $\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1})$ and $\text{Corr} (\Delta y_t, \Delta y_{t-1})$. This implied that the Kalman gains were fixed over time. However, in a real-time learning environment, agents will only have knowledge of the sample autocorrelations, which, in turn, are influenced by the Kalman gains. In the following we will assume that agent’s use the sample autocorrelations to compute the Kalman gains and investigate the convergence properties of the model. The learning algorithm is summarized in Appendix A.2. We run a series of 10,000 period simulations, each generating a unique path of $(\lambda^*_\pi, \lambda^*_y)$, for the first 500 periods. Figure 3 shows the evolution of the Kalman gains. The end-of-simulation values are clustered in the range of the theoretical equilibrium values. However, initial sample variation in the shocks, $u_t$ and $v_t$, influences the estimated autocorrelation coefficients and results in sizable differences in the end-of-simulation Kalman gains. For instance, the full-sample (10,000 period) autocorrelations of inflation changes range between -0.4441 and -0.3164, and the corresponding end-of-simulation Kalman gains are between 0.4298 and 0.6435. Over the 10 simulations shown in Figure 3, the average autocorrelation coefficients are -0.4187 for inflation, and -0.1450 for the output gap, which are close to the theoretical values of $\text{Corr} (\Delta \pi_t, \Delta \pi_{t-1}) = -0.3999$ and $\text{Corr} (\Delta y_t, \Delta y_{t-1}) = -0.1373$, respectively.
5 Applying the Model’s Methodology to U.S. Data

5.1 Endogenous Volatility and Persistence under Consistent Expectations

This section compares different measures of volatility and persistence in U.S. data with the corresponding moments generated by the consistent expectations (CE) model, respectively. We consider three different specifications of the CE model. In the constant gains model, the Kalman gains are computed using the population autocorrelation coefficients. In the variable gains model, the Kalman gains are computed using rolling windows of recent data.\footnote{The learning algorithm is described in Appendix A.2} We assume that agents use either 10-year windows (40 quarters) or 20-year windows (80 quarters) of past observations. Table 3 shows the results. First, under rational expectations, the standard deviations of inflation and the output gap are simply given by the standard deviations of the supply and demand shocks, respectively.\footnote{Note that the standard deviations of the shocks are chosen so that the CE model can roughly match the standard deviations in the data. Thus, the rational model can by construction not generate the volatility observed in the data.} Consistent expectations generates endogenous volatility. Particularly, the variable gains version of the model is more volatile than the constant gains model, especially when the moving window of past observations is relatively short. Moreover, consistent expectations adds a substantial degree of endogenous persistence into the model. Under rational expectations, none of the variables exhibit first order autocorrelation.

Figure 3: Convergence Properties under Real-time Learning
In the CE model, the first order autocorrelation coefficients are positive and generally match well with the corresponding moments in the data. For instance, the autocorrelation of the interest rate matches very well with the data despite the absence of a smoothing term in the Taylor rule.

<table>
<thead>
<tr>
<th>Statistic</th>
<th>U.S. Data 84-07</th>
<th>Model simulations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>CPI</td>
<td>PCE</td>
</tr>
<tr>
<td>Std. Dev. ($\pi_t$)</td>
<td>1.72</td>
<td>1.23</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$Corr$ ($\pi_t, \pi_{t-1}$)</td>
<td>0.09</td>
<td>0.46</td>
</tr>
<tr>
<td>$Corr$ ($\Delta \pi_t, \Delta \pi_{t-1}$)</td>
<td>-0.55</td>
<td>-0.42</td>
</tr>
<tr>
<td>CBO output gap</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Std. Dev. ($y_t$)</td>
<td>1.25</td>
<td>0.10</td>
</tr>
<tr>
<td>$Corr$ ($y_t, y_{t-1}$)</td>
<td>0.93</td>
<td>0.00</td>
</tr>
<tr>
<td>$Corr$ ($\Delta y_t, \Delta y_{t-1}$)</td>
<td>0.19</td>
<td>-0.50</td>
</tr>
<tr>
<td>FFR</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Std. Dev. ($R_t$)</td>
<td>2.38</td>
<td>0.00</td>
</tr>
<tr>
<td>$Corr$ ($R_t, R_{t-1}$)</td>
<td>0.97</td>
<td>0.00</td>
</tr>
<tr>
<td>$Corr$ ($\Delta R_t, \Delta R_{t-1}$)</td>
<td>0.51</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Notes: $T_s =$ length of rolling sample (in quarters) for computing the perceived signal-to-noise ratios $\phi_\pi$ and $\phi_y$. Parameter values are reported in Table 1.

5.2 Inflation Puzzles under Rational Expectations

Numerous papers have demonstrated that standard NK models with rational expectations have large difficulties accounting for inflation dynamics during and after the Great Recession. In the following we will briefly illustrate the resulting inflation “puzzles” in the context of the rational expectations version of our model.

We adopt a standard approach in the New Keynesian literature and assume that the Great Recession was caused by a highly persistent adverse demand shock. For this purpose, we momentarily replace the I(1) demand shock, $v_t$, in the IS-curve (2) with an AR(1) process, $v_t$:

$$y_t = \tilde{E}_t y_{t+1} - \alpha \left( R_t - \tilde{E}_t \pi_{t+1} \right) + v_t, \quad \alpha > 0, \quad (21)$$

where $v_t$ follows the first order autoregressive process:

$$v_t = \rho_v v_{t-1} + \psi_t, \quad 0 < \rho_v < 1, \psi_t \sim N \left( 0, \sigma^2 \right), \quad (22)$$

---

16Note that several extensions could help reconcile the rational model with the data, for instance, a smoothing term in the Taylor rule.

17See, for instance, Auroba and Schorfheide (2016). The shock can be interpreted as a real rate shock when $\alpha = 1$. 

13
We use $\rho_v = 0.96$, which is standard in the literature\textsuperscript{18}. Moreover, we replace the Taylor rule (3) with the ZLB condition:

$$R_t = \max \left\{ -\log(R\Pi), \mu_x \tilde{E}_t \pi_{t+1} + \mu_y \tilde{E}_t y_{t+1} \right\} \quad \alpha > 0, \quad (23)$$

Where $R$ is the steady state gross nominal interest rate and $\Pi$ is the central bank’s gross inflation target. We assume a net inflation target of 2 pct. annually, i.e. $4 \log(\Pi) = 0.02$. To keep the exercise as simple as possible we assume that agent’s never expect the ZLB to bind.\textsuperscript{19} Under this assumption, the (near) rational forecasts are given by:\textsuperscript{20}

$$E_t \pi^{re}_{t+1} = \frac{\kappa \rho_v}{(1 - \beta \rho_v + \frac{\kappa \rho_v (\mu_x - 1)}{1 - \rho_v + \alpha \mu_y \rho_v}) (1 - \rho_v + \alpha \mu_y \rho_v)} v_t \quad (24)$$

$$E_t y^{re}_{t+1} = \frac{\rho_v}{(1 + \frac{1}{1 - \beta \rho_v + \frac{\kappa \rho_v (\mu_x - 1)}{1 - \rho_v + \alpha \mu_y \rho_v}}) (1 - \rho_v + \alpha \mu_y \rho_v)} v_t \quad (25)$$

Thus, the rational model consists of the NKPC (1), the IS curve (21), the shock process (22), the ZLB condition (23) and the two forecast rules (24) and (25).

To replicate the path of the output gap during the recession, we use the CBO output gap as forcing variable from 2008:q1 to 2017:q2. Then we backtrack the sequence of innovations, $\psi_t$, needed to match the output gap data. Specifically, taking $y_t$ as given and treating $\psi_t$ as endogenous, we solve the non-linear system (1) and (21)-(25) every period. Figure 4 shows the results. The shock process, $v_t$, implies that the model matches the CBO output gap from Figure 1 by construction. On impact the recession induces a sharp drop in inflation of around 4 percentage points. This decline is almost fully driven by an equivalent drop in inflation expectations. Following the contraction from 2008 to 2009, the model predicts that inflation will gradually recover. However, inflation is negative for approximately 5 years. These inflation dynamics match poorly with the data. While inflation declined sharply in 2008:q4, it was only negative for two quarters. Moreover, as noted by Coibion and Gorodnichenko (2015), these fluctuations were largely explained by the collapse and subsequent recovery of oil prices. Furthermore, survey-based measures of expectations did not respond much to the recession. However, since mid-2012, inflation has persistently declined. Moreover, this decline was accompanied by a decline in several survey-based measures of inflation expectations, including 1-year expectations from the Michigan Survey of Consumers. Thus, while the output gap recovered, inflation declined. From the perspective of the rational expectations model, these dynamics are puzzling.

\textsuperscript{18}A high autocorrelation coefficient is necessary to match the autocorrelation in output gap data under rational expectations

\textsuperscript{19}This assumption implies that expectations are not fully rational, and it implies that expectations are more stable at the ZLB. Alternatively, we could use the Occbin Toolkit (Guerrieri and Iacoviello, 2014) to solve endogenously for the expected duration of the ZLB episode.

\textsuperscript{20}It is easy to check that these forecasts collapse to (7) and (8) when $\rho_v = 0$
5.3 Inflation Dynamics under Consistent Expectations

In this section we apply the methodology of the CE model directly to U.S. data and reassess the inflation puzzles that arise under rational expectations.

5.3.1 Inflation Expectations During The Great Moderation

Figure 5 shows what happens if the methodology of the consistent expectations model is applied directly to U.S. inflation data. Specifically, we assume that the agent continuously update the estimated signal-to-noise ratio and the associated inflation Kalman gain using a 20-year rolling window of past observations. Thus, the autocorrelation of inflation changes is computed directly from a rolling window of recent data. The signal-to-noise ratio and the associated Kalman gain are computed from the learning-versions of equations (15) and (13), respectively, while the resulting model-implied inflation expectations are computed from the learning-version of the forecast rule (11).\footnote{The learning algorithm is described in Appendix A.2}

Figure 5 shows that the estimated signal-to-noise ratio for inflation exhibits a downward drift during the Great Moderation. This development is consistent with the idea of inflation expectations “anchoring”. The decline in the signal-to-noise ratio implies a lower Kalman gain in the agent’s forecast rule (11). The resulting model-based inflation expectations track well with survey-based measures of expectations from the Michigan Survey of Consumers and the Survey of Professional Forecasters over the period.\footnote{Note, that the survey-based expectations are 1-year expectations, while the model-based expectations are 1-year expectations over the period.} Thus, survey expectations appear to be
5.3.2 Reassessment of Inflation Puzzles

The historical decline in the model-implied signal-to-noise-ratio depicted in Figure 5 is associated with a lower weight on recent inflation in the agent’s forecast rule (11). A lower $\lambda_\pi$ implies that inflation is less sensitive to changes in the output gap. This can be seen by inserting the forecast rule (11) into the NKPC (1), and taking the derivative with respect to $y_t$:

$$\pi_t = \beta \left[ \hat{E}_{t-1} \pi_t + \lambda_\pi \left( \pi_t - \hat{E}_{t-1} \pi_t \right) \right] + \kappa y_t + u_t$$

$$\Leftrightarrow \pi_t = \frac{1}{1 - \beta \lambda_\pi} \left[ (1 - \lambda_\pi) \beta \hat{E}_{t-1} \pi_t + \kappa y_t + u_t \right]$$

$$\Rightarrow \frac{\partial \pi_t}{\partial y_t} = \frac{\kappa}{1 - \beta \lambda_\pi}$$ \hspace{1cm} (26)

In this section we use the CE model to reassess the inflation puzzles that arise under rational expectations. We continue to assume that agent’s use rolling windows of recent data. To make the exercise as realistic as possible, we assume that, prior to the contraction starting annualized quarterly expectations.

Figure 5: Inflation Expectations and the Great Moderation

well described as a moving average of current and past observed inflation rates (as assumed in the forecast rule (11)).
in 2008:q1, agents use a 20-year window of actual U.S. inflation data. After 2008:q1, inflation is endogenously determined in the model and model-generated inflation data will begin to enter the rolling sample window.\textsuperscript{23} Similarly to the exercise under rational expectations, we use the CBO output gap as forcing variable from 2008:q1 to 2017:q2 and backtrack the sequence of demand shocks, $v_t$, needed to match the data. Specifically, taking $y_t$ as given, we compute the response of inflation and inflation expectations directly from the NKPC (1) and the learning-version of the forecast rule (11). Then we solve for $E_t y_{t+1}$ and $R_t$ using the learning-version of (12) and the ZLB condition (23). Finally, we use the IS-curve (2) to backtrack the demand shock, $v_t$.

Figure 6 shows the implied model-based forecasts of PCE inflation, inflation expectations and the Federal Funds Rate from 2008:q1 to 2017:q2. The figure also plots the sequence of demand shocks needed to replicate the CBO output gap from Figure 1.\textsuperscript{24} The inflation forecast tracks PCE inflation very accurately. On impact, the response of inflation is muted. This is because expectations are “anchored” prior to the Great Recession due to the historical decline in the estimated signal-to-noise ratio for inflation (as shown in Figure 5). Model-based inflation expectations track well with expectations from the Michigan Survey of Consumers and the Survey of Professional Forecasters. Similarly to the survey data, model-based expectations are remarkably stable. Unlike in the rational expectations model, expectations do not drop sharply on impact and then mean-revert. Instead, they gradually decline over the period since they are computed as a moving average of current and past observed inflation rates. On average, model-implied expectations decline slightly more than in survey data, but survey-expectations are generally within the confidence bands of the model. Thus, from the perspective of the CE model, inflation dynamics are associated with neither “missing disinflation” in the wake of the recession, nor “missing inflation” since 2012. If anything, it’s puzzling that inflation has not declined to lower levels in recent years given the depth and duration of the downturn. The model-implied interest rate tracks the Federal Funds Rate quite closely over the period. It is close to zero for approximately 5 years - almost as long as in the data. Figure 7 repeats the exercise under the assumption that agent’s use a shorter (10-year) moving window to compute the Kalman gain for PCE inflation. The model continues to track inflation and inflation expectations closely over the period.

5.3.3 The Slope of the NKPC

The response of inflation to changes in the output gap depends crucially on the slope of the NKPC (which is clear from (26)). A flat Phillips Curve reduces the forecasted fall in inflation when the output gap drops sharply during the Great Recession. Recent empirical papers

\textsuperscript{23}Similarly, we assume that agents use a 20-year moving window of output gap data to compute the Kalman gain for the output gap.

\textsuperscript{24}The adverse demand shocks peak at around 5 standard deviations on average, which tracks well with output gap data.
Figure 6: Model-implied Inflation Forecast 2008-2017

Figure 7: Robustness: Moving Window and Price Index
supports the assumption of a relatively flat Phillips Curve. Figure 8 shows how sensitive the inflation forecasts are to changes in the slope parameter, $\kappa$. Clearly, the robustness of the results depend crucially on the the length of the moving window. This is because each sample window implies a different value of the inflation Kalman gain. Clearly, when agent’s use a 10-year moving window of PCE inflation data, expectations are more firmly “anchored” prior to the recession than when agent’ use a 20-year moving window of data. In the latter case, a relatively steep NKPC ($\kappa = 0.05$) implies that inflation eventually declines to around -10%. This is clearly at odds with the data. However, it is important to note, that inflation would decline even more under rational expectations.

5.3.4 Inflation Forecast

For how long will the PCE inflation rate continue to undershoot the Fed’s target of 2%? This question can be adressed within the context of the CE model. Given the path of inflation and the output gap in the data up to 2017:q2, we solve for the future path of the economy implied by the variable gain model. The model-based forecasts for PCE inflation with a 20-year sample window are shown in Figure 9. According to the model, the median inflation rate will undershoot the central bank’s target for several years to come. Consequently, monetary policy will remain accomodative and contribute to a positive output gap in the future. The economy is projected to return to its long run equilibrium around 2021. Obviously, the duration of the undershooting episode depends crucially on the initial values of inflation and the output gap (given by the data values in 2017:q2). Moreover, it depends on which price index we consider and the assumed length of the moving window, since each specification implies different values of the Kalman gains for inflation and the output gap. Figure 10 plots the model-implied forecasts of inflation and the output gap for alternative specifications (PCE inflation with a 10-year sample window, CPI inflation with a 10-year window, and CPI inflation with a 20-year window, respectively). Across these specifications, inflation will undershoot the central bank’s

\[ \text{See, for instance, Ball and Mazumder (2011)} \]

\[ \text{With rational expectations and } \kappa = 0.05, \text{ the inflation rate declines to roughly -11\% in 2009 and remains negative until 2017.} \]
target for roughly 2 to 6 years into the future, and the output gap will be positive for an equivalent period of time.\textsuperscript{27}

\textsuperscript{27}Positive cash-push shocks, $v_t > 0$, such as an increase in oil prices, could substantially shorten the duration of inflation undershooting
Figure 10: Inflation Forecasts 2017-2024: Robustness
6 Concluding Remarks

Since the outbreak of the Great Recession, U.S. inflation has been associated with two “puzzles”. First, the absence of persistent disinflation in the immediate wake of the recession was dubbed the “missing disinflation puzzle”. Subsequently, unexpectedly low inflation rates gave birth to the so-called “missing inflation puzzle”. These are puzzles judging by the predictions of a standard New Keynesian Phillips Curve with rational expectations. In this paper we introduced a form of boundedly rational expectations into an otherwise standard New Keynesian model. In the model, agents use time-series models to forecast inflation and the output gap. The time series models allow for both permanent and transitory shocks – similar to that of Stock and Watson (2007). The perceived optimal forecast rules are defined by the Kalman filter. In a unique “consistent expectations” equilibrium, the values of the Kalman gains for inflation and the output gap are pinned down by the observed autocorrelation of inflation and output gap changes, respectively. We showed that if agents continuously update their estimate of the Kalman gain using a moving window of recent data, the identified Kalman gain for inflation exhibit a downward drift during the so-called Great Moderation period (Figure 5). A low Kalman gain implies a low weight on recent inflation in the agent’s forecast rule, which is consistent with the idea of “well-anchored” expectations. This helps anchor inflation near the central bank’s target rate when the output gap falls sharply during the Great Recession. The recession, however, gradually generates a moderate – but highly persistent – decline in inflation expectations, which lowers the inflation rate in the longer term. Thus, the model can help account for both the “missing deflation” in the aftermath of the recession as well as the “missing inflation” since 2012. Forecasts with the model suggest that inflation will undershoot the central bank’s target rate for several years to come. Consequently, the model predicts that monetary policy will remain accommodative and contribute to a positive output gap in the future.
Appendix

A.1 Actual Law of Motion under Consistent Expectations

The model is given by the following equations

\[ \pi_t = \beta \tilde{E}_{t+1} \pi_{t+1} + \kappa y_t + u_t \]  
(27)

\[ y_t = \tilde{E}_{t+1} y_{t+1} - \alpha \left( \pi_t - \tilde{E}_{t+1} \pi_{t+1} \right) + v_t \]  
(28)

\[ R_t = \mu_x \tilde{E}_{t+1} \pi_{t+1} + \mu_y \tilde{E}_{t+1} y_t \]  
(29)

\[ \tilde{E}_{t+1} \pi_{t+1} = \tilde{E}_{t} \pi_t + \lambda_x \left( \pi_t - \tilde{E}_{t} \pi_{t} \right) \]  
(30)

\[ \tilde{E}_{t+1} y_t = \tilde{E}_{t} y_{t+1} + \lambda_y \left( y_t - \tilde{E}_{t} y_{t} \right) \]  
(31)

which we wish to write on the form:

\[ Z_t = AZ_{t-1} + BU_t \]

where \( Z_t = \begin{bmatrix} \pi_t & y_t & R_t & \tilde{E}_{t+1} \pi_{t+1} & \tilde{E}_{t+1} y_t \end{bmatrix} \) and \( U_t = \begin{bmatrix} u_t & v_t \end{bmatrix} \)

Insert (30) into (27) and (29)-(31) into (28) to obtain:

\[ \pi_t = \frac{1}{1 - \beta \lambda_x} \left\{ \beta (1 - \lambda_x) \tilde{E}_{t} \pi_t + \kappa y_t + u_t \right\} \]

\[ y_t = \frac{1}{1 - \lambda_y + \alpha \mu_y \lambda_y} \left\{ (1 - \lambda_y) \tilde{E}_{t} y_t - \alpha \left( \mu_x - \mu_x \lambda_x - (1 - \lambda_x) \right) \tilde{E}_{t} \pi_t 
+ \mu_y (1 - \lambda_y) \tilde{E}_{t} y_{t} - (1 - \mu_x) \lambda_x \pi_t \right\} \]

Combine these two expressions to derive the following equations for \( \pi_t \) and \( y_t \):

\[ \pi_t = \frac{1}{1 - \beta \lambda_x} \left\{ \beta + \frac{\kappa \alpha (1 - \mu_x)}{1 - \lambda_y + \alpha \mu_y \lambda_y} \left( 1 - \lambda_x \right) \tilde{E}_{t} \pi_t 
+ \frac{(1 - \alpha \mu_y) \kappa (1 - \lambda_y)}{1 - \lambda_y + \alpha \mu_y \lambda_y} \tilde{E}_{t} y_t + \frac{\kappa}{1 - \lambda_y + \alpha \mu_y \lambda_y} v_t + u_t \right\} \]

and
\[
y_t = \frac{1}{1 - \lambda_y + \alpha \mu_y \lambda_y - \alpha (1 - \mu_y) \lambda_y \kappa} \left\{ (1 - \alpha \mu_y) (1 - \lambda_y) \bar{E}_{t-1} y_t + \alpha (1 - \mu_y) \lambda_y \bar{E}_{t-1} \kappa \right\}
\]

These can then be inserted into the forecast rules to obtain:

\[
\bar{E}_{t+1} = \frac{1 - \lambda_y}{1 - \beta \lambda_y - \kappa \alpha (1 - \mu_y) \lambda_y} \bar{E}_{t-1} + \frac{\lambda_y}{1 - \lambda_y + \alpha \mu_y \lambda_y} \left[ \alpha (1 - \lambda_y) \kappa (1 - \lambda_y) \bar{E}_{t-1} y_t + \frac{\kappa}{1 - \lambda_y + \alpha \mu_y \lambda_y} v_t + u_t \right]
\]

and

\[
\bar{E}_{t+1} = (1 - \lambda_y) \left( \frac{1 - \alpha (1 - \mu_y) \lambda_y \kappa}{1 - \beta \lambda_y - \kappa \alpha (1 - \mu_y) \lambda_y} \right) \bar{E}_{t-1} y_t + \frac{\lambda_y}{1 - \lambda_y + \alpha \mu_y \lambda_y} \left[ \alpha (1 - \lambda_y) \kappa (1 - \lambda_y) \bar{E}_{t-1} y_t + \frac{\kappa}{1 - \lambda_y + \alpha \mu_y \lambda_y} v_t + u_t \right]
\]

Finally, the forecasts (34) and (35) can be inserted directly into the Taylor rule (29) to obtain a long and complicated expression for the interest rate, \( R_t \). The equations (32)-(35) and the implied expression for \( R_t \) constitutes the actual law of motion under consistent expectations.

### A.2 Learning Algorithm

Real-time learning is discussed in Section 4 of the text. The learning algorithm is described by the following system of nonlinear equations

\[
\pi_t = \beta \bar{E}_{t+1} \pi_t + \kappa y_t + u_t
\]

\[
y_t = \bar{E}_{t} y_t - \alpha \left( R_t - \bar{E}_{t} \pi_t \right) + v_t
\]

\[
R_t = \mu \bar{E}_{t} \pi_t + \mu_y \bar{E}_{t} y_t + u_t
\]

\[
\bar{E}_{t+1} = \bar{E}_{t} \pi_t + \lambda_{t+1} \left( \pi_t - \bar{E}_{t} \pi_t \right)
\]

\[
\bar{E}_{t+1} = \bar{E}_{t} \pi_t + \lambda_{t+1} \left( y_t - \bar{E}_{t} y_t \right)
\]
\begin{align*}
Ave_{\pi,t} &= \left[ \frac{t}{t+1} \right] Ave_{\pi,t-1} + \left[ \frac{1}{t+1} \right] \Delta \pi_t, \quad (38) \\
n_{\pi,t} &= n_{\pi,t-1} + \left[ \frac{t}{t+1} \right] (\Delta \pi_t - Ave_{\pi,t-1})^2 \quad (39) \\
m_{\pi,t} &= m_{\pi,t-1} + (\Delta \pi_t - Ave_{\pi,t-1}) \left[ \Delta \pi_{t-1} - \frac{\Delta \pi_t}{(t+1)} - \frac{(t^2 + 3t + 1) Ave_{\pi,t-1}}{(t+1)^2} \right] \quad (40) \\
Ave_{y,t} &= \left[ \frac{t}{t+1} \right] Ave_{y,t-1} + \left[ \frac{1}{t+1} \right] \Delta y_t, \quad (41) \\
n_{y,t} &= n_{y,t-1} + \left[ \frac{t}{t+1} \right] (\Delta y_t - Ave_{y,t-1})^2 \quad (42) \\
m_{y,t} &= m_{y,t-1} + (\Delta y_t - Ave_{y,t-1}) \left[ \Delta y_{t-1} - \frac{\Delta y_t}{(t+1)} - \frac{(t^2 + 3t + 1) Ave_{y,t-1}}{(t+1)^2} \right] \quad (43) \\
\phi_{\pi,t} &= -\frac{n_{\pi,t}}{m_{\pi,t}} - 2, \quad (44) \\
\lambda_{\pi,t} &= \frac{\lambda_{\pi,t-1} + \phi_{\pi,t}}{1 + \lambda_{\pi,t-1} + \phi_{\pi,t}} \quad (45) \\
\phi_{y,t} &= -\frac{n_{y,t}}{m_{y,t}} - 2, \quad (46) \\
\lambda_{y,t} &= \frac{\lambda_{y,t-1} + \phi_{y,t}}{1 + \lambda_{y,t-1} + \phi_{y,t}} \quad (47) \\
\end{align*}

Equations (36) and (37) are the forecast rules when the Kalman gains are evolving over time. Equations (38)-(40) and (41)-(43) are used to recursively estimate the autocorrelation of inflation and output gap changes, respectively, using all past data.\textsuperscript{28} Equations (44) and (46) are the full-sample estimate of the signal-to-noise ratios. Equations (45) and (47) are the out-of-steady state versions of the Kalman gain formulas (13) and (14).\textsuperscript{29}

To obtain the “variable-gain” version of the model that is discussed in Section 5 of the text, equations (38) through (43) are modified to compute the autocorrelation of inflation and output gap changes over a rolling sample period rather than over the full sample period. Both the real-time learning algorithm and the variable-gain model employ a “projection facility,” which sets \( \phi_{i,t} = \phi_{i,t-1} \) for \( i = \{ \pi, y \} \) whenever the sample autocorrelations of \( \Delta \pi_t \) and \( \Delta y_t \) yield the result that \( \phi_{i,t} < 0 \).

\textsuperscript{28}These formulas are adapted from Hommes and Sorger (1998, pp. 320-321).
\textsuperscript{29}For the derivation of equation (A.7), see McCulloch (2005).
References


