New Results on Betting Strategies and Market Selection

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Abstract

We consider a repeated betting market populated by two agents who wage on a binary event according to generic betting strategies. We show that complete knowledge of the betting strategies is not necessary to assess their survival or dominance. Rather, the asymptotic wealth of agents can be generally inferred from the odds they believe fair and how much they would bet when the odds are equal to the ones the other agent believes fair. As an example, we apply our conditions to the case of CRRA betting.

Keywords: Bounded rationality, Betting Strategies, Market Selection, Recurrent processes

JEL Classification: C60, D53, G11, G12

1 Introduction

In this paper we study a repeated market for parimutuel betting where two agents wage on the outcomes of a binary event. Agents follow generic betting functions that depend only upon the bet inverse odd ratio. In this frame we provide sufficient and, a part from hairline cases, necessary conditions for assessing the long-run selection outcomes of the model. That is, checking the conditions one can understand whether one agent of the two is able to get all the wealth in the long-run, whether both manage to maintain a positive wealth share or whether who gets everything depends on the sequence of events realized. The conditions rely only on the model’s parameters and can be easily computed once one knows
a particular quantity: the fraction of wealth bet by an agent when the bet inverse odd ratio is equal to that the other agent believes fair. Our analysis relates and extends several contributions dealing with betting and market selection. Beygelzimer et al. (2012) and Kets et al. (2014) report that when agents bet according to Kelly betting (Kelly, 1956), then only the agent who maximizes a given function gets all the wealth, eventually. These results match those of Blume and Easley (1992) and Blume and Easley (2009) derived in a different but equivalent setting. Indeed Blume and Easley show that when agents use Kelly betting (or, more generally, are expected utility maximizers with rational price expectations) then only the agent with the most accurate beliefs survives in the long-run. Kets et al. (2014) considers also fractional Kelly and CRRA betting, however they cannot provide analytical results and have to rely on numerical simulations to understand the selection outcomes. Bottazzi and Giachini (2016) partially fill this gap providing sufficient and, apart from hairline cases, necessary conditions for establishing the long-run market selection in the case of fractional Kelly traders.\footnote{As showed by Bottazzi and Giachini (2017), the same conditions can be recovered looking at those required for the existence of a non-degenerate price (or wealth) invariant distribution.} Considering general rules that depend only on the bet inverse odd ratio, we generalize those previous results and we show that a key feature needed to evaluate long-run selection outcomes is how much an agent would like to wage when the inverse odd ratio of the market matches the one the other agent believes fair. Moreover the conditions we provide are global, in the sense that they apply no matter which is the initial wealth distribution. Thus, we extend and complement the contributions of Bottazzi and Dindo (2013) and Bottazzi and Dindo (2014) which perform local analyses. In the end we use our conditions to study the long-run selection outcomes of a market populated by myopic CRRA bettors under different combinations of beliefs, risk preferences, success probabilities.

2 Model

Consider two agents who make a sequence of bets against each other on binary events. The outcome of the event \( s_t \in \{0, 1\} \) is an independent Bernoulli trial with success probability \( \pi^* \): \( s_t = 1 \) means that the event occurs while \( s_t = 0 \) that it does not. In each round, agent \( i \in \{1, 2\} \) has to choose the fraction of wealth to be wagered \( b_i^t \) and the side of the bet \( \sigma_i^t \in \{0, 1\} \), where 1 means betting on the occurrence of the event while 0 betting against it. We assume the amount bet is redistributed among the winners according to the parimutuel procedure, that is, proportionally to how much they have bet, without any house-take. Let \( p_t \) be the prevailing inverse odd ratio at round \( t \) for the occurrence of the event. Thus if \( s_t = 1 \) the agent betting on the occurrence of the event receives \( 1/p_t \) time the...
amount bet while if \( s_t = 0 \) the agent betting against the occurrence of the event receives \( 1/(1 - p_t) \) times the amount bet. Agents’ betting strategies are based on prevailing odds and they try to maximize their gain by increasing their bet when they perceive favorable opportunities. However we rule out the possibility that they bet all their wealth.\(^2\)

Formally, and partially following Kets et al. (2014), we assume that for each agent \( i \) there exists a “fair” inverse odd \( \bar{p}_i \in (0, 1) \) and a continuous function \( b_i \in [0, 1) \) such that \( \sigma_i^t = 1 \) if \( p_t < \bar{p}_i \), \( \sigma_i^t = 0 \) if \( p_t > \bar{p}_i \), \( b_i(\bar{p}_i) = 0 \) and \( b_i(p_t) \) is strictly decreasing if \( p_t < \bar{p}_i \) and strictly increasing if \( p_t > \bar{p}_i \).

Without loss of generality we set \( \bar{p}_1 < \bar{p}_2 \). Thus, if \( w_{i,t-1} \) is the wealth of agent \( i \in \{1, 2\} \) before the event at time \( t \) is realized, the prevailing inverse odd \( p_t \) is set by the equation\(^3\)

\[
 w_{i,t-1} b_i(p_t) = w_{i,t-1} b^2_i(p_t)
\]

being always \( \sigma_1^t = 0 \) and \( \sigma_2^t = 1 \). The amount of wealth that is not bet is invested in a risk-less asset that pays no interest. Hence, after the event at round \( t \) is realized, the wealth of agents is updated according to

\[
 w_i^t = (1 - b_i(p_t)) w_{i,t-1} + \delta_{s_t,i-1} w_{i,t-1} b_i(p_t) \left( \frac{\delta_{i,1}}{1 - p_t} + \frac{\delta_{i,2}}{p_t} \right)
\]

where \( \delta_{a,b} \) is the Kronecker delta. Since the house takes no fees, the aggregate wealth is constant and we set \( w_t = w_1^t + w_2^t = 1 \) such that \( p_t \in [\bar{p}_1, \bar{p}_2] \) and \( p_t = \bar{p}_i \) if and only if \( w_i^t = 1 \).

## 3 Long-run Selection

The dynamics of wealth described by (1) can lead to two different outcomes: either a single agent accrues all the wealth and dominates the market or both agents will indefinitely survive, each with a positive, and fluctuating, fraction of wealth. In general, the fate of an agent could depend on the specific sequence of realizations of the random variable \( s_t \). Let \( \sigma = \{s_1, s_2, \ldots\} \) denote a realization of the Bernoulli process and let \( w_i^t(\sigma) \) be the associated sequence of agent \( i \)'s wealth. The long-term outcomes of the model are formalized in the following.

**Definition 3.1.** Agent \( i \) (asymptotically) dominates on \( \sigma \) if \( \lim_{t \to \infty} w_i^t(\sigma) = 1 \). Agent \( i \) (asymptotically) survives on \( \sigma \) if \( \lim \sup_{t \to \infty} w_i^t(\sigma) > 0 \).

Agent \( i \) (asymptotically) dominates if \( \lim_{t \to \infty} w_i^t(\sigma) = 1 \) for almost all \( \sigma \). Agent \( i \) (asymptotically) survives if \( \lim \sup_{t \to \infty} w_i^t > 0 \) for almost all \( \sigma \).

\(^2\)This would lead them to wealth zero almost surely

\(^3\)Relaxing the assumption of continuity and monotonicity on betting strategies can easily lead to indeterminacy and non-uniqueness of prevailing odds
Notice that dominance implies survival. If one agent dominates, the other cannot survive and we say that it vanishes. At the same time, if one agent survives the other cannot dominate.

Kets et al. (2014) show that if \( \pi^* < \bar{p}_1 \) agent 1 dominates, while if \( \pi^* > \bar{p}_2 \) it is agent 2 to get all the wealth. Hence we focus on the case \( \bar{p}_1 < \pi^* < \bar{p}_2 \). We will show that in order to decide survival or dominance of agents, it is not generically necessary to know all the details of the investment strategies, but simply the Bernoulli probability \( \pi^* \), the odds considered fair by the two agents, \( \bar{p}_i \), and two positive numbers, \( b_1(\bar{p}_2) \) and \( b_2(\bar{p}_1) \), representing the fraction of wealth one agent invests if the odds are equal to those the other agent would consider fair.

Proposition 3.1. Consider the quantities

\[
\mu_1 = \pi^* \log \frac{\bar{p}_1}{\bar{p}_1} + (1 - \bar{p}_1) \frac{b_2(\bar{p}_1)}{\bar{p}_1} + (1 - \pi^*) \log(1 - b_2(\bar{p}_1))
\]  
and

\[
\mu_2 = -\pi^* \log(1 - b_1(\bar{p}_2)) - (1 - \pi^*) \log \frac{1 - \bar{p}_2 + \bar{p}_2 b_1(\bar{p}_2)}{1 - \bar{p}_2}.
\]

If agent investment strategies satisfy the requirements of Section 2, then

i) if \( \mu_1 > 0 \) and \( \mu_2 > 0 \), agent 2 dominates and \( \lim_{t \to \infty} p_t = \bar{p}_2 \) almost surely;

ii) if \( \mu_1 < 0 \) and \( \mu_2 < 0 \), agent 1 dominates and \( \lim_{t \to \infty} p_t = \bar{p}_1 \) almost surely;

iii) if \( \mu_2 < 0 \) and \( \mu_1 > 0 \) both agents survive;

iv) if \( \mu_2 > 0 \) and \( \mu_1 < 0 \) then either agent 1 dominates or agent 2 dominates depending on the realization of the Bernoulli process.

Proof. Consider the process \( \{ z_t = \log w_t^2 / w_t^1 \} \) and notice that \( \mu_1 = \lim_{x \to -\infty} E[z_{t+1}-z_t| z_t = z] \) and \( \mu_2 = \lim_{x \to +\infty} E[z_{t+1}-z_t| z_t = z] \). Define the (conditional) increment \( g(p, s) = z_{t+1} - z_t \) when \( p_t = p \) and \( s_{t+1} = s \). From (1) remembering that, by hypothesis, \( b_1(p) \) and \( b_2(p) \) cannot be both zero for the same \( p \) and are monotonic, it is immediate to see that

\[
\log \frac{1 - b_2(\bar{p}_1)}{1 + b_1(\bar{p}_2)\bar{p}_2/(1 - \bar{p}_2)} < g(p, 0) < 0 < g(p, 1) < \log \frac{1 + b_2(\bar{p}_1)(1 - \bar{p}_1)/\bar{p}_1}{1 - b_1(\bar{p}_2)}.
\]

Thus the increments \( g \) are finite and bounded and Theorems 2.2, 3.1 and 3.2 of Bottazzi and Dindo (2015) can be applied to the process \( \{ z_t \} \). If \( \mu_1 > 0 \) and \( \mu_2 > 0 \)

\footnote{The definition of survival and dominance in Kets et al. (2014) are weaker than the one adopted here. Given the relative simplicity of the considered process, however, their conclusions are still valid under Definition 3.1.}
Figure 1: Agents’ betting strategies. In both panels we set \( \bar{p}^1 = 0.3 \) and \( \bar{p}^2 = 0.75 \), in the first plot we have \( \gamma^1 = \gamma^2 = 2 \) while in the second it is \( \gamma^1 = \gamma^2 = 0.5 \).

then \( \lim_{t \to \infty} z_t = +\infty \), whence \( i \). If \( \mu^1 < 0 \) and \( \mu^2 < 0 \) then \( \lim_{t \to \infty} z_t = -\infty \), whence \( ii \). If \( \mu^2 < 0 \) and \( \mu^1 > 0 \) then there exists a finite interval \( A \) such that \( z_t \in A \) almost surely for any \( t \), whence \( iii \). If \( \mu^2 > 0 \) and \( \mu^1 < 0 \) then on any Bernoulli sequence or \( \lim_{t \to \infty} z_t = +\infty \) or \( \lim_{t \to \infty} z_t = -\infty \), whence \( iv \).

4 Example with CRRA Bettors

The betting strategies introduced in Section 2 are flexible enough to accommodate several behavioral prescriptions. As illustrative example, we consider the case in which agents bet to maximize the expected utility of wealth using a power utility function, with Constant Relative Risk Aversion (CRRA). Call \( \gamma^i > 0 \) the relative risk aversion coefficient of agent \( i \) and \( \pi^i \) the subjective probability (belief) that agent \( i \) assigns to the realization of the event, which is precisely the inverse odd that agent \( i \) would consider fair. Assuming \( \pi^1 < \pi^2 \), for \( p_t \in [\pi^1, \pi^2] \) agent 1 bets against the occurrence of the event a fraction of wealth \( b^1 \) which maximizes \( \pi^1(1 - b^1)^{1-\gamma^1} + (1 - \pi^1)(1 - \pi^1 b_t/(1 - p_t))^{1-\gamma^1} \) to obtain

\[
b^1(p_t) = \frac{(p_t(1 - \pi^1))^{\frac{\gamma^1}{1+\gamma^1}} - (\pi^1(1 - p_t))^{\frac{\gamma^1}{1+\gamma^1}}}{(p_t(1 - \pi^1))^{\frac{\gamma^1}{1+\gamma^1}} + p_t (\pi^1)^{\frac{\gamma^1}{1+\gamma^1}} (1 - p_t)^{\frac{1-\gamma^1}{1+\gamma^1}}}.
\] (4)

Conversely agent 2 bets in favor of the realization of the event a fraction of wealth \( b^2 \) which maximizes \( \pi^2(1 + b^2(1 - p_t)/p_t)^{1-\gamma^2} + (1 - \pi^2)(1 - b^2(1 - \pi^2))^{1-\gamma^2} \) to obtain

\[
b^2(p_t) = \frac{(\pi^2(1 - p_t))^{\frac{\gamma^2}{1+\gamma^2}} - (p_t(1 - \pi^2))^{\frac{\gamma^2}{1+\gamma^2}}}{(\pi^2(1 - p_t))^{\frac{\gamma^2}{1+\gamma^2}} + (1 - p_t) (1 - \pi^2)^{\frac{1-\gamma^2}{1+\gamma^2}} (p_t)^{\frac{1-\gamma^2}{1+\gamma^2}}}.
\] (5)
Figure 2: Dominance, survival and vanishing for different combinations of fair inverse odd ratios. 1D: agent 1 dominates and agent 2 vanishes, 2D: agent 2 dominates and agent 1 vanishes, 1,2S: both agents survive, 1D2D: either agent 1 dominates and agent 2 vanishes or agent 2 dominates and agent 1 vanishes. White areas with $\bar{p}_2 > \bar{p}_1$ indicate those combinations of fair inverse odd ratios such that monotonicity is violated over $[\bar{p}_1, \bar{p}_2]$ for at least one betting strategies. Panel a: $\pi^* = 0.45, \gamma^1 = \gamma^2 = 2$; panel b: $\pi^* = 0.45, \gamma^1 = \gamma^2 = 0.5$; panel c: $\pi^* = 0.45, \gamma^1 = 2, \gamma^2 = 0.5$; panel d: $\pi^* = 0.75, \gamma^1 = 0.5, \gamma^2 = 2$. 
The positive risk aversion implies that agents never invest the totality of their wealth. Figure 1 provides two examples of how agents’ betting strategies vary depending on the inverse odd ratio. In the effective price support, betting strategies are always continuous while, for some combinations of beliefs and risk aversion, they may be non-monotonic. Figure 2 reports the long-run selection outcomes inferred using the conditions from Proposition 3.1 when all assumptions are satisfied. Depending on agents’ risk aversion and beliefs any case of Proposition 3.1 may generically occur.

5 Conclusion

In this paper we consider a market for bets where two agents repeatedly wage on an uncertain event with two possible outcomes using generic betting strategies. We find sufficient conditions for having the dominance of one of the two agents, the survival of both or the history dependent case in which, depending on the sequence of events, either one agent or the other eventually accrues all the wealth. Such conditions rely on the sign of two quantities which depend upon the model’s parameters. In particular they are function of the fraction waged by one agent when the bet inverse odd ratio is fair for the other agent.

References


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