Dynamic and Stochastic Search Equilibrium*

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Abstract

Wage posting models are an appealing way of studying wage growth and dispersion, as firms offer high employment values to retain and poach more workers from other companies. In this paper, I study the business cycle properties of these types of models with random search. When abstracting from firm entry and exit, I show that productivity shocks do not generate fluctuations in labor market quantities (such as unemployment) because fear of competition for workers makes wages absorb most of these shocks. Even though separation rate shocks can generate large unemployment fluctuations, they generate a positive correlation between unemployment and hires and, depending on the recruiting cost function, these shocks may also generate a positive correlation between unemployment and vacancies. I show that introducing wage rigidity into the model increases the responses of labor market quantities to aggregate productivity shocks, but the size of these responses depends on how large (relative to productivity) the flow opportunity cost of employment is. To assess these models quantitatively, I propose a new algorithm that finds the steady state and computes transitional dynamics in seconds. Hence, integrating wage posting models with random search to larger models becomes possible (and easy) with this new algorithm.

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1 Introduction

Wage posting models have emerged as an appealing way of studying wage dispersion and growth, as firms post high employment values to prevent their employees from taking other job offers and to poach more workers from other companies. Since the wage posting model of Burdett and Mortensen (1998), significant progress has been achieved in this literature. Moscarini and Postel-Vinay (2013 and 2016) proved the existence and uniqueness of a rank-preserving equilibrium in the context of a model with random search in which firms post and commit to employment values. Menzio and Shi (2011) propose a model with directed search and random match quality in which, in equilibrium, agents’ decision rules do not depend on the distribution of employment. This equilibrium result simplifies the simulation of the model. Coles and Mortensen (2011) develop a model in which firms cannot commit to wage contracts, but the effect of offering a high wage on firms’ reputation works as a commitment device. They also assume a hiring cost function per current workers, which implies that the equilibrium path depends only on the level of employment and not on its distribution.

In spite of the significant theoretical progress in this literature, quantifying the cyclical properties of these types of models has remained a challenging task.¹ In this paper, I study the business cycle properties of wage posting models with random search by proposing a new algorithm that computes the steady state and transitional dynamics in seconds. The goal is to study a class of models in which the distribution of employment and wages matter for agents’ decision rules and the aggregate equilibrium path. These models do not only provide a richer description of the economy, but also enable us to study in deeper details distributional dynamics facts documented, for example, by Haltiwanger, Hyatt and McEntarfer (2015) and Moscarini and Postel-Vinay (2012) in an economy with a well defined distribution. I present a model close to Moscarini and Postel-Vinay (2013) [MPV13], but I abstract from entry and exit and I introduce capital and a strictly concave utility function for households. There is a continuum of firms that produced a homogeneous good that is sold in a competitive market to the household that can be used for consumption or capital accumulation. A priori the only difference among firms is their permanent log-TFP level. Firms open new vacancies and post and commit to employment values to all of their workers to retain and poach more workers.

Moscarini and Postel-Vinay (2016) [MPV16] also proposed an algorithm for solving these types of models, which I discuss below. Even though I am able to reproduce the simulated business cycle moments of MPV16 with my algorithm, my conclusions are significantly different. First, I show that aggregate productivity shocks (alone) do not generate meaningful responses in labor

¹For example, Moscarini and Postel-Vinay (2016) said “In a series of articles (...) we explore, both theoretically and empirically, the business cycle implications of the wage posting paradigm. Progress in this direction has been stunted by technical difficulties in finding equilibrium where the law of one price fails.” p. 136.
market quantities such as unemployment and vacancies because fear of competition for workers makes wages absorb most of these shocks. Under some specific (but widely used) assumptions, I show that aggregate productivity shocks do not generate any responses in aggregate labor market quantities (Proposition 1). Second, while shocks to the separation rate generate large responses in labor market quantities, they also generate a positive correlation between unemployment and hires, which is inconsistent with the data. Also, if firms face a vacancy cost function, separation rate shocks also generate a positive correlation between unemployment and vacancies. Third, the cyclical behavior of wages is inconsistent with the data. The simulated wage volatility is much larger than in the data, and the model predicts a poor (almost non-existent) wage correlation with other labor market variables. These results comes from the fact that firms commit to employment values and not to wages. When firms offer a higher employment value to attract more workers and the economy is hit by a positive shock, current wages go down. Since workers were promised an employment value for today \((W_t)\), and today they are promised a higher employment value for the next period \((W_{t+1})\) in response to a positive shock; current wages can go down in order to keep the current value of employment constant. Hence, in response to a positive aggregate shock, wages have a big drop on impact, increase in the subsequent period, and slowly return to their steady state value. I conclude that wage posting models with random search have more difficulties accounting for the cyclical behavior of the labor market than the standard DMP model (Shimer, 2005).

To address the lack of meaningful responses to productivity shocks and the unrealistic wage responses in wage posting models, I propose two modifications to the baseline model. First, I assume that firms post wages instead of employment values to retains and attract more workers. In other words, I assume that firms post non-contingent wages. I show that this extension does not overcome the lack of labor market responses to productivity shocks (Proposition 2), but it delivers wage responses that are more in line with the data. Second, I assume that firms face a quadratic cost of wage adjustment \(a la\) Rotemberg (1982). This extension makes the model display large responses in labor market quantities to productivity shocks. I also find that the magnitude of these responses depends on the average size, relative to productivity, of the Flow Opportunity Cost of Employment (FOCE), which is consistent with Hagedorn and Manovskii (2008) and Ljungqvist and Sargent (2017).\(^2\) When the value of FOCE is large and wages are sluggish, a 1% increase in productivity implies large proportional changes in the value of a new worker, which makes hiring decision more sensitive to aggregate shocks.

To assess quantitatively the performance of this model, I propose a new algorithm that computes transitional dynamics in seconds and may be of interest in its own right. This new method consists

\(^2\)FOCE is the forgone value of unemployment benefits plus the forgone value of non-working activities in terms of consumption.
of three steps: (1) calibrating/imposing a distribution of employment value offer in steady state
(2) computing the deterministic steady state and (3) taking a first order approximation of the
model around that point, as proposed by Reiter (2009). MPV16 also proposed an algorithm for
computing the equilibrium of this type of models, which is based on the full non-linear solution.
Even though my method is an approximation of the dynamics around the deterministic steady
state, I am able to reproduce the results of MPV16. While I see the MPV16 algorithm as a useful
method that enables researchers to answer particular questions that my algorithm cannot, the
method proposed in this paper has some particular (and powerful) advances over MPV16. My
method does not suffer from the curse of dimensionality. I can include as many shocks and state
variables as I want without increasing significantly the computational burden, allowing researcher
to study other frictions and sources of aggregate fluctuations. Hence, one can easily integrate wage
posting models with random search in an even more general framework (such as a medium scale
New-Keynesian model). Also, because of the nature of my algorithm, stochastic simulations and
Impulse Response Functions are an easy and useful exercise to implement.

The rest of this paper is organized as follows: Section 2 presents the model. Section 3 defines
and characterizes the equilibrium of the model. I describe the computational method in Section
4, present the calibration of my model in Section 5, and present the results of the baseline model
in Section 6. I extend the baseline model to include wage rigidity in Section 7, and Section 8
concludes.

2 Theoretical Framework

The model presented in this section is a generalization of the wage posting model presented in
MPV13. The main differences to their model is that I abstract from entry and exit, and that I
introduce capital and a strictly concave utility function for households.\(^3\)

2.1 Model Overview

There are two types of agents in this economy: households and firms. There is a representative
household in the economy made up of a continuum of workers that supplies capital and labor
to firms and owns all firms in the economy. The household derives utility from consumption
and leisure and discounts future utility at rate \(\beta\). Capital is supplied in a perfectly competitive
market at the capital rental rate \(r_t\) and depreciates at rate \(\delta_k\), while labor supply is subject to

\(^3\)I do not think that abstracting from firm entry and exit changes the conclusions of this paper. On the one
hand, MPV2013 only allow for firm entry and exit at the bottom of the distribution. Hence, those firms have low
impact on aggregate variable as they are the smallest firms. On the other hand, I show that I am able to replicate
quantitatively the results of MPV2016.
search frictions. I assume complete consumption insurance, which implies that workers seek to maximize income for the household. A worker can be employed or unemployed at each point in time. Unemployed workers receive unemployment compensation $b_t$ and are matched with a firm with probability $q_t$. Employed workers are separated from their job with exogenous probability $\delta_{nt}$, in which case they must spend at least one period in unemployment before they can be matched with another firm. Employed workers can search on the job. An employed worker is matched with another firm with probability $\bar{\iota}_t \cdot q_t$, where $\bar{\iota}_t$ is the search intensity of employed workers relative to unemployed workers. However, employed workers only change jobs if they find a firm that offers an equal or better employment value.

There is a continuum of firms indexed by $j$ with mass normalized to 1. All firms produce a homogeneous good that is sold in a competitive market to the household and can be used for consumption or capital accumulation. A priori the only difference among firms is their permanent log-TFP level, which is denoted by $a_j$ and is distributed across firms according to a continuous pdf $f$ over the interval $[a, \infty)$. Without loss of generality, I assume that $a_j$ is increasing in $j$ ($a_x \geq a_y$ for all $x \geq y$), and to save on notation $f_j = f(a_j)$. Firms produce with capital $k_{jt}$ and labor $n_{jt}$, and firms’ output is denoted by $y_{jt} = e^{a_j + a_t} k_{jt} n_{jt}^{1-\alpha}$, where $a_t$ stands for aggregate log-TFP, which is common to all firms and follows an AR(1) process. At the beginning of each period, firms rent capital, open new vacancies ($v_{jt}$) and post a (net) employment value for the next period ($W_{jt+1} \geq 0$). A vacancy is matched with a worker with probability $\bar{q}_t$. If a vacancy is matched with an unemployed worker, the vacancy is filled with probability 1. However, if a vacancy is matched with an employed worker at firm $y_j$, the vacancy is filled only if firm $W_{jt+1} \geq W_{yt+1}$. As is standard, new workers (filled vacancies) become productive in the subsequent period.

The total number of matches in the economy $m_t(v_t, s_t)$ is an increasing function in the total number of vacancies ($v_t$) and the total number of job searchers ($s = u + (1 - \delta_{nt})\bar{\iota}_t n_t$), where $u_t = 1 - n_t$ is the number of unemployed workers. Following the literature, $m_t(v_t, s_t)$ is assumed to be homogeneous of degree 1. Hence, $q_t = m_t(\theta_t, 1)$ and $\bar{q}_t = m_t(1, \theta_t^{-1})$ where $\theta_t = v_t / s_t$ is labor market tightness.

The timing of the model each period is as follows: (1) aggregate shocks are realized; (2) firms rent capital, open new vacancies and employment offers; (3) production takes place, and factors are paid; (4) the household makes a consumption decision; (5) a fraction $\delta_{nt}$ of employed workers loose their job, and a fraction $q_t$ of unemployed workers finds a new job; (6) a fraction $(1 - \delta_{nt})\bar{\iota}_t q_t G_{jt}$ of employed workers leaves firm $j$ to join another firm, where $G_{jt}$ is the probability of firm $j$’s employees being matched with a firm that offers a higher employment value.

\footnote{In other words, value $W_{jt+1}$ is net of the unemployment value.}
2.2 Household

There is a representative household made up of a continuum of members with mass normalized to 1. The household is the owner of all firms in the economy, and it supplies capital and labor to firms. Capital is supplied in a perfectly competitive market at the rental rate $r_t$, while labor supply is subject to search frictions. I assume complete consumption insurance, which implies that workers seek to maximize income for the household. Consumption and savings decision are made at the household level to maximize the life-time utility function

$$U(k_t, n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \Psi \tilde{n}_t^{1+\eta} + \beta E_t [U(k_{t+1}, n_{t+1})]$$

subject to the budget constraint (2) and the aggregation of labor (3):

$$c_t + k_{t+1} \leq (r_t + 1 - \delta_k)k_t + \int w_j n_j t dj + \int \pi_j t dj + b \cdot u_t - T_t$$

$$\tilde{n}_t = \left( \int n_{jt}^{1+\xi} dj \right)^{\frac{1}{1+\xi}}$$

where $c$ is consumption, $k$ is capital, $w_j$ is the wage paid by firm $j$, and $\pi_j$ stands for firm $j$’s profits. $u = \int_0^1 (1 - n_j) dj$ is the total number of unemployed workers, and $b$ is unemployment compensation, which is financed by lump sum taxes ($T = b \cdot u$). Parameter $\xi$ in (3) governs the elasticity of substitution between $n_x$ and $n_y$ for all $x \neq y$.\(^5\) Hence, the optimality condition for consumption:

$$c_t^{-\sigma} = \beta E_t [(1 - \delta_k + r_{t+1})c_{t+1}^{-\sigma}]$$

A worker can be employed or unemployed at each point in time. Unemployed workers receive unemployment compensation $b_t$ and are matched with a firm with probability $q_t$. Meanwhile, employed workers are separated from their job with exogenous probability $\delta_{nt}$, in which case they have to spend at least one period in unemployment before they can be matched with another firm. I assume that employed workers can search on the job and are matched with another firm with probability $\bar{i}_t q_t$. However, employed workers only change jobs if they find a firm that offers an

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\(^5\)We can interpret this parameter as follows: firms and workers are located uniformly on a circle, and firms hire workers who are closer to them. As firms increase in size, they have to attract workers who are farther away, which implies that workers have to spend more time commuting, which reduces their utility level.
equal or better employment value. Hence, the net value of employment at firm \( j \) is given by:

\[
W_{jt} = w_{jt} - z_{jt} + E_t\{Q_t[(1 - \delta_{nt})(1 - \tilde{v}_t q_t G_{jt})W_{jt+1}]
+ (1 - \delta_{nt})\tilde{v}_t q_t \int_{W_{jt+1}}^{\infty} W f_W^v dW - q_t \int_{0}^{\infty} W f_W^v dW\}\]  

(5)

where employment offers are distributed according to a continues pdf \( f_W^v \) over the interval \( W \in [0, \infty) \), \( z_{jt} \) is the flow-opportunity cost of employment for firm \( j \), and \( G_{jt} \) is the probability of receiving a better employment value than \( W_{jt+1} \):

\[
z_{jt} = b_t + \Psi_{\tilde{n}_t} \frac{n_{jt}^{\eta - \xi}}{c_t} \eta_{jt}^\xi \]  

(6)

\[
G_{jt} = \int_{W_{jt+1}}^{\infty} f_W^v dW \]  

(7)

2.3 Firms

There is a continuum of firms indexed by \( j \) and with mass normalized to 1. \( A \text{ priori} \), the only difference among firms is their permanent log-TFP level, which is denoted by \( a_j \) and is distributed across firms according to a continues p.d.f. \( f \) over \( [a, \infty] \). Without loss of generality, I assume that \( a_x \geq a_y \) for all \( x \geq y \), and to save on notation \( f_j = f(a_j) \).

Firms produce with capital and labor, and their output can be used for consumption or for capital accumulation. At the beginning of each period, firms rent capital in a perfectly competitive market at rate \( r_t \), open \( v_{jt} \) new vacancies, and post and commit to an employment value \( (W_{jt+1}) \), based on which workers decide to accept employment at firm \( j \). As is standard, each firm is subjected to an equal treatment constraint and has to pay the same wage to all of its employees. Hence, a worker employed at firm \( j \) will move to firm \( y \) if and only if \( W_{yt+1} \geq W_{jt+1} \). Since negative employment offers are never accepted, \( W_{jt+1} \geq 0 \) for all \( j \). As a consequence, unemployed workers and continuing workers that are not contacted by any other firm always accept an employment offer. Vacancies are filled with probability \( \tilde{q}_{jt} \), which will be defined below. I assume that firms face either a vacancy cost function \( \kappa v_{jt+1}^{1+x} \) or a hiring cost function \( \kappa (\tilde{q}_{jt} v_{jt+1}^{1+x}) \). These functions can be written as: \( \kappa (\tilde{q}_{jt} v_{jt})^{1+x} \tilde{I}_h \), where \( I_h \) is an indicator function equal to 1 if firms face a hiring cost
function and 0 otherwise. Hence, the problem for firm $j$ is given by:

$$\Pi_{jt}(n_{jt}, \bar{W}) = \max_{v_{jt}, k_{jt}, W_{jt+1}} \pi_{jt} + E_t \left[ Q_t \Pi_{jt+1}(n_{jt+1}, W_{jt+1}) \right]$$

s.t.

$$\pi_{jt} = y_{jt} - w_{jt}n_{jt} - \tau_{jt} k_{jt} - \kappa \left( \tilde{q}_{jt} v_{jt} \right)^{1+\chi}$$

$$y_{jt} = e^{a_j + a_t k_{jt} n_{jt}^{-\alpha}}$$

$$n_{jt+1} = (1 - \delta_n) (1 - \tilde{\tau}_t q_t G_{jt}) n_{jt} + \tilde{q}_j v_{jt}$$

$$W_{jt} = w_{jt} - z_{jt} + E_t \left\{ Q_t \left[ (1 - \delta_n) (1 - \tilde{\tau}_t q_t G_{jt}) W_{jt+1} \right. \right.$$ 

$$+ (1 - \delta_{nt}) \tilde{\tau}_t q_t \int_{W_{jt+1}}^{\infty} W f_{Wt}^n dW - q_t \int_0^{W_{jt+1}} W f_{Wt}^n dW \}$$

$$G_{jt} = \int_{W_{jt+1}}^{\infty} f_{Wt}^n dW$$

$$\tilde{q}_j = \frac{q_t}{s_t} \left( u_t + \tilde{\tau}_t (1 - \delta_{nt}) \int_0^{W_{jt+1}} n_{Wt} f_{Wt}^n dW \right)$$

$$W_{jt} \geq W$$

where $f_{Wt}^n$ and $f_{Wt}^v$ denote the density functions of wage offers and employment. Letting $f(a(W))$ denote the density of firms offering an employment value equal to $W$, and $v(W)$ and $n(W)$ the vacancy and employment decisions of those firms:

$$f_{Wt}^v = \frac{v(W)_t}{v_t} f(a(W))$$

$$f_{Wt}^n = \frac{n(W)_t}{s_t} f(a(W))$$

Hence, the optimality conditions for capital, vacancies and employment values are given by:

$$r_t = \alpha e^{a_j + a_t} \left( \frac{k_{jt}}{n_{jt}} \right)^{\alpha-1}$$

$$\kappa \left( \tilde{q}_{jt}^h v_{jt} \right)^\chi = E_t \left[ Q_t \tilde{q}_{jt}^{1-h} J_{jt+1} \right]$$

$$E_t [Q_t h_{jt}] \geq E_t \left\{ Q_t J_{jt+1} (1 - \delta_{nt}) \tilde{\tau}_t \left[ q_t f_{jt}^v n_{jt} + (1 - I_h) \cdot \tilde{q}_t f_{jt}^n v_{jt} \right] \right.$$

$$\right.$$
where $p_{jt}$ is labor productivity, $h_{jt}$ is hires, and $J_{jt}$ is the value of a filled vacancy:

$$p_{jt} = (1 - \alpha)e^{a_j + a_t} \left( \frac{k_{jt}}{n_{jt}} \right)^{\alpha}$$  \hspace{1cm} (21)

$$h_{jt} = \tilde{q}_{jt} v_{jt}$$  \hspace{1cm} (22)

$$J_{jt} = p_{jt} - w_{jt} + E_t [Q_t(1 - \delta_{nt})(1 - \bar{i}_t q_t G_{jt})J_{jt+1}]$$  \hspace{1cm} (23)

And to save on notation:

$$f^{v}_{jt} = \frac{v (W_{jt+1})}{v_t} f (a (W_{jt+1}))$$  \hspace{1cm} (24)

$$f^{n}_{jt} = \frac{n (W_{jt+1})}{s_t} f (a (W_{jt+1}))$$  \hspace{1cm} (25)

### 2.4 Aggregate Resource

Notice that total income in this economy is used for consumption, capital accumulation and for vacancy posting costs. Hence, the aggregate resource constrain is given by:

$$y_t = c_t + k_{t+1} - (1 - \delta)k_t + \kappa \int \frac{(\tilde{q}_{jt} v_{jt})^{1+\chi}}{1+\chi} dj$$  \hspace{1cm} (26)

where it is straight forward to define aggregate production as: $y_t = \int y_{jt} f_j dj$

### 3 Equilibrium

**Definition 1. Competitive Search Equilibrium.** A competitive search equilibrium is a sequence of prices $\{r_t, w_t\}$, quantities $\{y_t, c_t, k_t, u_t, n_t\}$, probabilities $\{q_t, \tilde{q}_t\}$, and functions $\{v_{jt}, W_{jt+1}, J_{jt}, n_{jt+1}\}$ on productivity $a_j$, firm size $n_{jt}$ and $W_{jt}$, such that given exogenous variables $\{a_t, \delta_{nt}, \bar{i}_t\}$, an initial stock of capital and initial distributions of employment and employment values:

(i) The household optimizes, taken as given prices and exogenous shocks. Consumption satisfies the optimality condition (4). (ii) Taking as given the exogenous variables, $\{r_t\}$, and all other firms strategies (i.e. employment, wage, and vacancies), firms optimize. Functions $\{v_{jt}, W_{jt+1}, J_{jt}, n_{jt+1}\}$ solve equations (11), (19), (20), (23) and prices satisfy equations (5) and (18). (iii) Probabilities evolve according to $q_t = m(\theta_t, 1)$ and $\tilde{q}_t = m(1, \theta_t^{-1})$. (iv) Markets clear: the aggregate resource constrain holds.

It is worth noticing that a competitive search equilibrium does not establish any properties regarding functions $\{v_{jt}, W_{jt+1}, J_{jt}, n_{jt+1}\}$. For example, this definitions allows an equilibrium in which firms’ productivity $a_j$ is not perfectly correlated with its employment size or employment
value offer, which makes the problem intractable. However, MPV2013 defined and proved the existence of a particular class of equilibrium: A rank-preserving equilibrium, on which I will focus throughout this paper.

**Definition 2. Rank-preserving competitive search equilibrium:** A rank-preserving competitive search equilibrium is a competitive search equilibrium in which function $n_{jt+1}$, $W_{jt+1}$ are increasing in $a_j$ and $n_{jt}$.

Intuitively, a rank-preserving equilibrium establishes that most productive firms are larger and offer higher employment values at all times. With this definitions in hand, I am now ready to present an important theoretical result, which argues that under some particular assumptions (but widely used in the literature), aggregate productivity shocks do not generate real responses in aggregate labor quantities such as unemployment or vacancies.

**Proposition 1.** Suppose that $\{r_t^{(1)}, w_t^{(1)}, y_t^{(1)}, c_t^{(1)}, k_t^{(1)}, u_t^{(1)}, n_t^{(1)}, v_t^{(1)}, q_t^{(1)}, \bar{q}_t^{(1)}, v_{jt}^{(1)}, W_{jt+1}^{(1)}, J_{jt}^{(1)}, n_{jt+1}^{(1)}\}$ is a rank-preserving competitive search equilibrium given a sequence of exogenous shocks $\{a_t^{(1)}, \delta_{nt}^{(1)}, \bar{r}_t^{(1)}\}$. Now, suppose that $\{r_t^{(2)}, w_t^{(2)}, y_t^{(2)}, c_t^{(2)}, k_t^{(2)}, u_t^{(2)}, n_t^{(2)}, v_t^{(2)}, q_t^{(2)}, \bar{q}_t^{(2)}, v_{jt}^{(2)}, W_{jt+1}^{(2)}, J_{jt}^{(2)}, n_{jt+1}^{(2)}\}$ is a rank-preserving competitive search equilibrium given a sequence of exogenous shocks $\{a_t^{(2)}, \delta_{nt}^{(1)}, \bar{r}_t^{(1)}\}$ i.e. the aggregate productivity is different in both equilibriums while the remaining exogenous variables are the same. If $\sigma = 0$, and aggregate shocks are small.

Then

- $u_t^{(1)} = u_t^{(2)}, n_t^{(1)} = n_t^{(2)}, v_t^{(1)} = v_t^{(2)}, q_t^{(1)} = q_t^{(2)}, \bar{q}_t^{(1)} = \bar{q}_t^{(2)}$ for all $t$.

- $J_{jt}^{(1)} = J_{jt}^{(2)}, v_{jt}^{(1)} = v_{jt}^{(2)}, n_{jt+1}^{(1)} = n_{jt+1}^{(2)}$ for all $t$ and for all $j > 0$.

*Proof. See Appendix A.1.*

What this proposition is stating is that aggregate labor market quantities (such as unemployment and vacancies) do not respond to aggregate productivity shocks if the utility function is linear in consumption and aggregate shocks are small, regardless of the recruiting cost function. This Proposition implies that employment value offers ($W_{jt+1}$) completely absorb these shocks. In Proposition 1, the assumption of small aggregate shocks is important to guarantee that there is no firm entry and exit and that the new equilibrium is also rank-preserving.

This result may look surprising given that MPV2016 argue that these class of models are able to generate large fluctuations in labor market quantities in response to productivity shocks and that I am able to replicate their results qualitatively. As I will show in the numerical section of this paper, labor market fluctuations in MPV2016 are mainly driven by fluctuations in the separation
rate \( \delta_{nt} \). Notice that in Proposition 1, the separation rate is the same in both equilibriums as well as the search intensity for job stayers \( \bar{\delta}_{it} \).

In MPV2016, the results are not driven exclusively by fluctuations in the separation rate because, in one of the extensions, they allow for firm entry and exit, which I abstracted from. However, allowing for firm entry and exit has little impact on labor market fluctuations MPV2016 because, in these class of models, firms only enter and exit at the bottom of the distribution where firms are small and have a small participation in total employment.

Proposition 1 relies on the assumption that the household utility function is linear in consumption, which is an assumption usually made in the literature. Given that consumption reacts to TFP innovations, assuming that \( \sigma_c > 0 \) implies that aggregate labor market quantities respond to productivity shocks. This is because changes in consumption induce changes in the stochastic discount factor and, as a consequence, in the optimality conditions for vacancies or hiring.\(^7\) However, in the numerical part of this paper, I show that assuming a non-linear utility function generates little volatility in labor market quantities.

The rest of this section characterizes some of the properties of a rank preserving equilibrium in steady state. This characterization will turn out to be extremely useful for the computation of the equilibrium.

**Lemma 1.** In steady state, assuming that firms face a vacancy cost function, if the offer of employment value is distributed according to a decreasing p.d.f. in equilibrium, and \( \chi > \frac{1}{4} \); the FOCs are necessary and sufficient and the equilibrium is Rank-preserving.

*Proof.* See Appendix A.2

Now, Lemma 1 does not provide a necessary property for the distribution of employment value offer in equilibrium. Lemma 1 establishes a sufficient condition for the distribution of employment value offers to be an equilibrium distribution. Hence, this Lemma does not rule out the possibility that this distribution has at least some segments that are upward sloping.

**Lemma 2.** In steady state, assuming that firms face a hiring cost function, the FOCs are necessary and sufficient if \( \left( 1 - J_j \frac{f_{yy}}{f_y} \right) > \frac{1}{\chi} \) for all \( W_j \). Additionally, if \(-J_j \frac{f_{yy}}{f_y} \geq 1\) the equilibrium is rank-preserving.

*Proof.* See Appendix A.3

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\(^6\)In their paper, the only source of aggregate uncertainty is aggregate productivity. They do not consider exogenous changes to the search intensity of employed workers, and the separation rate is a function of aggregate productivity.

\(^7\)Even though changes in consumption induce changes in the flow opportunity cost of employment \( (z_{jt}) \), it is straightforward to prove that those variations alone do not generate labor market fluctuations as long as aggregate shocks are small.
Corollary 1. If the offer of employment value is distributed according to: $f_{jt}^v = \frac{1}{\sigma_v} e^{-\frac{W_{jt}}{\sigma_v}}$ in steady state, $\chi > 0.5$ and firms face a hiring cost function, The FOCs are necessary and sufficient and the equilibrium is rank-preserving.

**Proof.** See Appendix A.4

4 Computation

In this section, I propose a computational algorithm for this class of models, which is easy and fast to implement. On my desktop with Intel Core i7 3.4 GHz processor and Matlab R2016a, it takes about 5 seconds to compute the steady state and the transitional dynamics. The main idea is based on the Reiter method (2009), which solves heterogeneous agents models by numerically approximating the equilibrium dynamics around the deterministic steady state. The main steps of this algorithm are:

1. Calibrate the distribution of employment value offers in steady state.
2. Compute the deterministic steady state.
3. Take a first order approximation around the deterministic steady state.\(^8\)

The main technical challenge is to compute the steady state, which is a complicated task for a given productivity distribution. Notice that if you impose a distribution for $a_j$, you have to find an equilibrium employment value, employment and vacancy functions ($W_{jt+1}$, $n_{jt+1}$, and $v_{jt}$) to compute the distributions of employment value offers ($f_{Wt}^v$) and employment ($f_{Wt}^n$). However, in equilibrium, $W_{jt}$, $n_{jt}$ and $v_{jt}$ are a function of $f_{Wt}^v$ and $f_{Wt}^n$. Hence, in order to find the equilibrium, you would have to iterate on multiple infinite dimensional elements.

What I show below is that by calibrating the equilibrium distribution of employment value offers, and therefore finding the implicit distribution of productivity that generates $f_{Wt}^v$, makes the computation of the steady state a simple task. Given that I would be calibrating an endogenous element, Lemmas 1 and 2 play an important role, as they provide equilibrium properties of this distribution in any rank-preserving equilibrium.

4.1 Calibrating the distribution of wage offers

Following Lemmas 1 and 2, to guarantee that the FOCs are necessary and sufficient, I will only consider wage offer distributions with a strictly decreasing and strictly convex pdf. For simplicity,\(^8\) even though, I work with a first order approximation in this paper. It is straightforward to take an $n^{th}$ approximation.
I assume that wage offers are distributed according to an exponential distribution with associated standard deviation equal to $\sigma_v$:

$$f_{Wt}^v = \frac{1}{\sigma_v} e^{-\frac{W}{\sigma_v}}; \quad W \in [0, \infty) \quad (27)$$

Given that this, as well as other equilibrium elements, is an infinite dimensional element, I will find the exact solution for $GP$ points and approximate all other points by interpolating. In particular, I will find the exact solution for $W = \{W_1, W_2...W_{GP}\}$ and will approximate the solution for all other points.

4.2 Computing the deterministic steady state

Given a distribution of employment value offers, it is not difficult to compute other elements in the steady state. In this section, I preset the computation of the steady state for a given set of targets. However, it is straightforward to change this procedure to match other moments. The goal of this section is to show that the computation of the steady state is significantly simplified by calibrating $f_{Wt}^v$ in steady state.

First, notice that by calibrating the distribution of employment value offers, $G_j$ becomes an easy element to compute. Also notice that in steady state $r = 1/\beta - (1-\delta_k)$. Hence, given a value for $q$, $\tilde{q}$, and $u$ in steady state, which I target in my calibration:

$$s = u + (1-\delta_n)\tilde{i}n \quad (28)$$
$$\theta = \frac{q}{\tilde{q}} \quad (29)$$
$$v = \theta s \quad (30)$$

where $n = 1 - u$. The value of a filled vacancy is given by:

$$J_j = \frac{[1 - (1-\delta_n)(1-\tilde{i}qG)]}{2^{1-i_h}(1-\delta_n)iqf_j^v} \quad (31)$$

Then, we can find the job filling rate by iterating on:

$$\tilde{q}^{(i+1)}(W) = \frac{\tilde{q}}{s} \left( u + \tilde{i}(1-\delta_n) \int_{0}^{W} \left( \frac{n}{\Omega^{(i)}} \right) \frac{v f_x^v}{[1 - (1-\delta_n)(1-\tilde{i}qG_x)]} \tilde{q}_x^{(i)} dx \right) \quad (32)$$
$$\Omega^{(i)} = \int_{0}^{\infty} \frac{v f_x^v}{[1 - (1-\delta_n)(1-\tilde{i}qG_x)]} \tilde{q}_x^{(i)} dx \quad (33)$$

where $\Omega^{(i)}$ converges to $n$, and term $\frac{n}{\tilde{q}}$ in (32) guarantees that total employment is equal to $n$ in equilibrium. Then, we have to calibrate $\kappa$ to make sure that aggregate vacancies and employment
add up to \( v \) and \( n \). To this end, notice that:

\[
v_j = \frac{1}{\tilde{q}^{1-h}_j} \left( \frac{\beta \tilde{q}^{1-h}_j J_j}{\kappa} \right)^{\frac{1}{\chi}}
\]

\[
n_j = \frac{\tilde{q}_j v_j}{1 - (1 - \delta_n)(1 - iqG_j)}
\]

Hence, combining equations (34) and (35) and integrating over \( j \):

\[
\kappa = \left( \frac{1}{n} \int_0^\infty \frac{\tilde{q}^{1-h}_j \left( \beta \tilde{q}^{1-h}_j J_j \right)^{\frac{1}{\chi}}}{1 - (1 - \delta_n)(1 - iqG_j)} f_j dj \right)^x
\]

(36)

where \( f_j \) can be computed from the definition of \( f_j^v \) and equation (34):

\[
f_j = \frac{\tilde{q}^{-h}_j \left( \beta \tilde{q}^{1-h}_j J_j \right)^{\frac{1}{\chi}}}{\tilde{q}^{-h}_j \left( \beta \tilde{q}^{1-h}_j J_j \right)^{\frac{1}{\chi}}} \frac{1}{1} \Omega
\]

(37)

\[
\Omega = \int_0^\infty \frac{v f_j^v}{\tilde{q}_x^{-h} \left( \beta \tilde{q}_x^{1-h} J_x \right)^{\frac{1}{\chi}}} dx
\]

(38)

Finally, notice that for a given \( \Psi \) and \( b \)

\[
z_j = b + \Psi c^\sigma \tilde{n}^{\nu - \epsilon} n_j^\ell
\]

(39)

\[
w_j = [1 - \beta(1 - \delta_n)(1 - \tilde{i}qG_j)] W_j + z_j - \beta \tilde{q} \int W_j f_j^w dW + \beta q \int_0^\infty W f_j^w dW
\]

(40)

\[
p_j = [1 - \beta(1 - \delta_n)(1 - \tilde{i}qG_j)] W_j + w_j
\]

(41)

\[
a_j = \log \left( \left( \frac{p_j}{1 - \alpha} \right)^{1-\alpha} \left( \frac{r}{\alpha} \right)^\alpha \right)
\]

(42)

\[
k_j = (e^{a_j})^{\frac{1}{1-\alpha}} \left( \frac{\alpha}{r} \right)^{\frac{1}{1-\alpha}} n_j
\]

(43)

\[
c = Y - \delta_k K - \int \kappa v_j^{1+\xi} \frac{1}{1+\xi} dj
\]

(44)

where \( \tilde{n} = \left( \int n_j^{1+\xi} dj \right)^{\frac{1}{1+\xi}} \), \( Y = \int y_j dj \) and \( k = \int k_j dj \). Hence, I calibrate \( \Psi \) and \( b \) (by iterating on these equations) to target an average ratio \( \frac{z_j}{p_j} \) across firms and a ratio \( \frac{b}{(Y/n)} \).
4.3 Approximation Around the Deterministic Steady State

Notice that the equilibrium of the economy is described by a system of non-linear equations that can be written as:

$$E_t F(\vec{X}_{t+1}, \vec{X}_t, \vec{\epsilon}_{t+1}, \vec{\epsilon}_t) = 0$$

(45)

where $\vec{X}_t$ and $\vec{\epsilon}_t$ are the vectors of endogenous and exogenous variables of the model and are given by:

$$\vec{X}_t = \begin{bmatrix} \text{vec}(n_{jt}), \text{vec}(W_{jt}), K_t, \text{vec}(w_{jt}), u_t, r_t, Q_t, q_t, \tilde{q}_t, C_t \end{bmatrix}$$

(46)

$$\vec{\epsilon}_t = \begin{bmatrix} a_t, \delta_{nt}, \tilde{i}_t \end{bmatrix}$$

(47)

Hence, following Reiter (2009), the system of equations 45 can be linearized numerically around the deterministic steady to get:

$$E_t F_{\vec{X}_{t+1}} \partial \vec{X}_{t+1} + F_{\vec{X}_t} \partial \vec{X}_t + F_{\vec{\epsilon}_{t+1}} \partial \vec{\epsilon}_{t+1} + F_{\vec{\epsilon}_t} \partial \vec{\epsilon}_t = 0$$

(48)

where $F_x$ is the partial derivative of $F$ with respect to $x$. Hence, this system of linear equations can be solve using a standard method. In this paper, I use the method proposed by Klein (2000).

5 Calibration

I calibrate my model to a monthly frequency. I set the discount factor to $\beta = 1.04^{-1}$ per year, the inverse of the Frisch elasticity ($\eta$) is set to 1.5, and the coefficient of relative risk aversion ($\sigma$) is set to 1. The capital depreciation rate is set to 10% per year. The output elasticity of labor ($\alpha$) is set to 0.33.

I set the exogenous separation rate in steady state to target an unemployment rate of 5.5% and a job finding rate ($q$) equal to 0.27, which implies that $\delta_n = 0.016$. The search intensity of employed workers is calibrated to match a fraction of job changers equal to 2% in steady state (Fallick & Fleischman, 2004). I calibrate $\kappa$ such that the vacancy contact rate is equal to $\tilde{q} = 0.9$, which also implies that in equilibrium $v = \frac{q}{\tilde{q}} s$. I assume a Cobb-Douglas matching function of the from: $m = \bar{m}s^l v^{1-l}$. I set $l$ to 0.5, and $\bar{m}$ is given by $\bar{m} = q\theta^{-l}$.

I calibrate $\epsilon$ to match the standard deviation of log wages in steady state to 0.53, which is consistent with the CPS microdata. The curvature of the hiring or vacancy cost function is set to 1 in order to guarantee that the equilibrium is rank preserving (Lemmas 1 and 2). The standard deviation of employment value offers $\sigma_v$ is set such that the average labor productivity across firms
is equal to 1. Based on Chodorow-Reich and Karabarbounis (2016), the unemployment benefit is calibrated such that it represents 6% of total output per worker, and $\Psi$ is calibrated to target an average ratio $\frac{z}{p_j}$ of 0.71 across firms.

For simplicity, I assume the following exogenous processes for the aggregate productivity ($a$), the separation rate ($\delta_n$) and the search intensity of job stayers ($i$):

$$ a_t = \rho_a a_{t-1} + \varepsilon_a $$  \hspace{1cm} (49)

$$ \log(i_t) = (1 - \rho_i) \log(i_{t-1}) + \varepsilon^{\hat{i}}_t $$  \hspace{1cm} (50)

$$ \log(1 - \delta_{nt}) = (1 - \rho_{\delta_n}) \log(1 - \delta_{n}) + \rho_{\delta_n} \log(1 - \delta_{nt-1}) + \lambda a_t + \varepsilon^{\delta_n} + \zeta \varepsilon^a $$  \hspace{1cm} (51)

where $\varepsilon_a$, $\varepsilon^{\hat{i}}$ and $\varepsilon^{\delta_n}$ are uncorrelated i.i.d. shocks that are normally distributed with standard deviation $\varsigma_a$, $\varsigma^{\hat{i}}$ and $\varsigma^{\delta_n}$ respectively. The persistence of aggregate TFP is set equal to $0.95^{1/3}$ and the standard deviation is calibrated to match the standard deviation of HP-detrended, utilization adjusted TFP. Similarly, the persistence and standard deviation of the separation rate are calibrated to target the empirical behavior of the Employment-to-Unemployment transition rate ($EUR$). Table 1 presents the model parameters.

**Comparison with MPV16** I also present the results of my model for a calibration that matches the moments targeted by MPV16 in the main text, in which they abstracted from firm entry and exit. The parameter values for that calibration can be found under column MPV16 in Table 1. MPV16 assumed a hiring cost function, linear utility in consumption and a zero flow opportunity cost of employment.\footnote{In terms of my models, this assumption translate into $\Psi = b = 0$.} Also, they abstracted from exogenous shocks the separation rate and assumed that this rate was negatively correlated with productivity.

**Steady State Distribution** Figure 1 plots the distribution of employment values and wages in steady state for each one of these models. Since we impose a decreasing density for employment value offer, it is not surprising that the distribution of wages over new employees is downward sloping. However, notice that the distributions over other groups of workers are not decreasing. In particular the distributions over all workers is hump-shaped and, relative to that distribution, job changers are more concentrated at high paying firms since they only accept better paying offers.

### 6 Results

In this section, I present the quantitative predictions of the model, which are judged against quarterly U.S. data from 1994 to 2015. I use data for output, labor productivity, unemployment,
Table 1: Parameter Values

### Externally Calibrated

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value Posting Model</th>
<th>Wage Posting Model</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vacancy Hiring MPV16</td>
<td>Vacancy Hiring</td>
<td></td>
</tr>
<tr>
<td>σ</td>
<td>1.00</td>
<td>1.00</td>
<td>Intertemporal elasticity of substitution</td>
</tr>
<tr>
<td>η</td>
<td>1.50</td>
<td>1.50</td>
<td>Inverse of Frisch elasticity</td>
</tr>
<tr>
<td>α</td>
<td>0.33</td>
<td>0.33</td>
<td>Output Elasticity of labor</td>
</tr>
<tr>
<td>ρ</td>
<td>0.951/3</td>
<td>0.951/3</td>
<td>Persistence of productivity shocks</td>
</tr>
<tr>
<td>β</td>
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<td>0.996</td>
<td>Discount factor.</td>
</tr>
<tr>
<td>δk</td>
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<td>0.0087</td>
<td>Capital depreciation rate.</td>
</tr>
<tr>
<td>χ</td>
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<td>1.00</td>
<td>Convexity of cost function convexity.</td>
</tr>
</tbody>
</table>

### Internally Calibrated

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value Posting Model</th>
<th>Wage Posting Model</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Vacancy Hiring MPV16</td>
<td>Vacancy Hiring</td>
<td></td>
</tr>
<tr>
<td>δh</td>
<td>0.015</td>
<td>0.015</td>
<td>Exogenous separation rate.</td>
</tr>
<tr>
<td>Ψ</td>
<td>0.087</td>
<td>0.309</td>
<td>Desutility of labor parameter.</td>
</tr>
<tr>
<td>b</td>
<td>0.071</td>
<td>0.073</td>
<td>Unemployment benefits.</td>
</tr>
<tr>
<td>l</td>
<td>0.50</td>
<td>0.50</td>
<td>Matching function parameter.</td>
</tr>
<tr>
<td>i</td>
<td>0.337</td>
<td>0.337</td>
<td>Search intensity of employed workers.</td>
</tr>
<tr>
<td>ε</td>
<td>0.445</td>
<td>0.445</td>
<td>Elasticity of substitution between jobs.</td>
</tr>
<tr>
<td>κ</td>
<td>0.973</td>
<td>0.973</td>
<td>Constant of hiring cost function</td>
</tr>
<tr>
<td>σW</td>
<td>0.38</td>
<td>0.38</td>
<td>Std of W_j offers in steady state.</td>
</tr>
<tr>
<td>φ</td>
<td>-</td>
<td>100</td>
<td>Rotemberg cost</td>
</tr>
</tbody>
</table>

Note: This table summarizes the parameterization of the model. Details are reported in section 5. Model “Vacancy” refers to my baseline model in which firms face a vacancy cost function ($I_h = 0$). Model “Hiring” refer to my baseline model in which firms face a hiring cost function ($I_h = 1$). Model MPV16 is my baseline model in which firms face a hiring cost function and calibrated to match the same moments as in MPV16.

Vacancies, hires, employment transition rates and wages. Since the model generates artificial monthly series, I take the quarterly average of these simulated data.

Output is real output in the non farm business sector, labor productivity is measure as real output per hour in the non farm business sector, and aggregate TFP is measured by the utilization adjusted TFP. Unemployment is total number of unemployed workers. Vacancies are measured by the composite help-wanted index computed by Barnichon (2010). Using the CPS microdata, I construct monthly series for transitions rates from Employment to Unemployment ($EUR_r$), Unemployment to Employment ($EUR_r$) and Employment to Employment ($ERR_r$). Based on the CPS, total hires ($H$) is constructed as the sum of flows from Unemployment-to-Employment and Employment-to-Employment. Also, to asses the transitional dynamics generated by this
model, I construct average hourly log-wages for all workers (\( w^a \)), new hires from unemployment (\( w^u \)) and new hires from other jobs (\( w^c \)) controlling for individual characteristics.\(^{10}\) I aggregate these monthly series to a quarterly frequency by taking a simple average of the quarter’s months and seasonally adjust these series using the X-13 filter. Following the literature, I detrend all series in logs using the HP filter with a smoothing parameter equal to 10\(^5\). Figures 2 plots these series in log level (solid black line) along with their HP trends (dash black lines).

### 6.1 Business Cycle Moments

Table 2 compares the volatility and persistence generated by this model versus the data. On the one hand, we can see that both models generate a modest but still significant volatility in unemployment, vacancies, total hires and employment transition rates. However, the wage volatility generated by all models is around 10 times larger than in the data. On the other hand, except for wages, the model generates persistence that is in line with the data. In the case of wages, the model predict a low (almost zero) persistence. In the data, the quarterly autocorrelation for wages is between 0.7 and 0.9.

Table 3 presents the correlation matrix generated by these models. The hiring cost model is able to generate a negative correlation between unemployment and vacancies as in the data, but when firms face a vacancy cost function this correlation becomes positive. Even though this makes

\(^{10}\)From now on, I will refer to new hires from unemployment as new employees and new hires from other jobs as job changers. Details about wages can be found in Appendix ??.
Figure 2: Data
<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>h</th>
<th>$UE_r$</th>
<th>$EU_r$</th>
<th>$EE_r$</th>
<th>$w^a$</th>
<th>$w^v$</th>
<th>$w^c$</th>
<th>y</th>
<th>p</th>
<th>a</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.20</td>
<td>0.19</td>
<td>0.04</td>
<td>0.14</td>
<td>0.11</td>
<td>0.08</td>
<td>0.02</td>
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<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>Vacancy</td>
<td>0.10</td>
<td>0.05</td>
<td>0.06</td>
<td>0.09</td>
<td>0.09</td>
<td>0.09</td>
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<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
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<tr>
<td>Hiring</td>
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<td>0.05</td>
<td>0.17</td>
<td>0.09</td>
<td>0.02</td>
<td>0.24</td>
<td>0.37</td>
<td>0.27</td>
<td>0.02</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>MPV16</td>
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<td>0.13</td>
<td>0.00</td>
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<td>0.15</td>
<td>1.11</td>
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<td>1.12</td>
<td>0.02</td>
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</tr>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>h</th>
<th>$UE_r$</th>
<th>$EU_r$</th>
<th>$EE_r$</th>
<th>$w^a$</th>
<th>$w^v$</th>
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<th>p</th>
<th>a</th>
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</thead>
<tbody>
<tr>
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<td>0.84</td>
<td>0.91</td>
<td>0.76</td>
<td>0.71</td>
<td>0.97</td>
<td>0.92</td>
<td>0.92</td>
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<tr>
<td>Vacancy</td>
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<td>0.87</td>
<td>0.94</td>
<td>0.89</td>
<td>0.90</td>
<td>0.81</td>
<td>-0.04</td>
<td>-0.01</td>
<td>-0.02</td>
<td>0.96</td>
<td>0.96</td>
<td>0.95</td>
</tr>
<tr>
<td>Hiring</td>
<td>0.93</td>
<td>0.89</td>
<td>0.85</td>
<td>0.91</td>
<td>0.90</td>
<td>0.97</td>
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<td>-0.03</td>
<td>-0.05</td>
<td>0.95</td>
<td>0.94</td>
<td>0.95</td>
</tr>
<tr>
<td>MPV16</td>
<td>0.89</td>
<td>0.76</td>
<td>0.89</td>
<td>1.00</td>
<td>0.92</td>
<td>0.91</td>
<td>-0.02</td>
<td>-0.03</td>
<td>-0.02</td>
<td>0.96</td>
<td>0.92</td>
<td>0.96</td>
</tr>
</tbody>
</table>

Note: Statistics for the U.S. economy are based on: $u$: Unemployment level. $v$: Help-wanted index (Barnichon, 2010). $h$: hires from unemployment and other jobs. $UE_r$: Unemployment to Employment transition rate. $EU_r$: Employment to Unemployment transition rate. $w^a$: Average wage in the economy. $w^v$: Average wage for new employees. $w^c$: Average wage for job changers. $y$: Real output in the nonfarm business sector. $p$: Real output per-hour in the non-farm business sector. $a$: Utilization adjusted TFP. All series are seasonally adjusted, logged, and detrended via the HP filter with a smoothing parameter of 100,000. Vacancy refers to a calibrated model in which firms face a vacancy cost function. Hiring refers to a calibrated model in which firms face a hiring cost function. MPV16 refers to a model calibrated to target the same moments as in Moscarini and Postel-Vinay (2016).
the former model look better, notice that both models generate a positive correlation between unemployment and total hires, which is inconsistent with the data. Empirically, a decline in unemployment is associated with an increase in total hiring. Finally, notice the low correlation between wages and labor market variables and the low correlation between TFP and labor market quantities. We know by Proposition 1 that TFP shocks alone do not generate fluctuations in the labor market quantities when the utility function is linear in consumption. Even though these results assume a strictly concave utility function in consumption, based on the discussion in the previous section, we didn’t expected a large correlation between labor market quantities and aggregate productivity. This exercise confirms that intuition.

Given that Tables 2 and 3 assume that the economy faces TFP and separation rates shocks at the same time. Tables ?? and ?? in Appendix ?? reproduce these tables assuming that the economy is shocked by only one type of innovation. In the case of MPV16, I just assume that there is no correlation between the separation rate and aggregate TFP. According to this results, we can see that almost all of the variation in labor market quantities is due to shocks to the separation rate, as suspected given Proposition 1.

6.2 Impulse Response Functions

To better understand the business cycle moments presented in the previous section, I study the transitional dynamics of the model in this section by presenting the Impulse Response Functions (IRFs) generated by this model. Figures 3, and 4 presents the IRF to a 1% increase in aggregate productivity and a 1% decline in the separation rate.

Regarding TFP shocks, we can see that labor market quantities do not significantly react, as expected. Wages have a big drop on impact, increase in the second period and slowly return to the steady state. To understand this initial decline in wages, notice that firms committed to an employment value equal to \( W_{jt} \). Hence, when a higher employment value is promised for the next period \( W_{jt+1} \), firms can lower current wages to keep the current value of employment constant. For this reason, the high volatility of wages shown in Table 2 is not surprising. Also, this IRFs illustrate why aggregate TFP is poorly correlated with labor market variables.

Since TFP shocks generate little volatility in labor market quantities as illustrated in Figure 3, most of the volatility presented in Table 2 is driven by shocks to the separation rate. Figures 4 show that unemployment significantly react to changes in \( \delta_n \).
Table 3: Correlation over Business Cycle

<table>
<thead>
<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>h</th>
<th>$UE_r$</th>
<th>$EU_r$</th>
<th>$EE_r$</th>
<th>$w^u$</th>
<th>$w^w$</th>
<th>$w^x$</th>
<th>$y$</th>
<th>$p$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Data</td>
<td>1</td>
<td>-0.93</td>
<td>-0.37</td>
<td>-0.98</td>
<td>0.90</td>
<td>-0.85</td>
<td>-0.10</td>
<td>-0.22</td>
<td>-0.19</td>
<td>-0.76</td>
<td>0.36</td>
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<tr>
<td>Vacancy</td>
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<td>0.51</td>
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<td>-0.48</td>
<td>0.63</td>
<td>0.66</td>
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<td>-0.01</td>
<td>0.01</td>
<td>-0.01</td>
<td>0.27</td>
<td>0.00</td>
</tr>
<tr>
<td>Hiring</td>
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Figure 3: Impulse Response Functions to a 1% Increase in Aggregate TFP

Note: This figure plots the IRFs to a 1% increase in aggregate productivity. Solid black lines are the IRFs for a model in which firms face a vacancy cost function. Dash lines are the IRFs for a model in which firms face a hiring cost function. Dotted lines represent the IRF for a calibrated model as in MPV16.

Figure 4: Impulse Response Functions to a 1% Increase in the Separation Rate

Note: This figure plots the IRFs to a 1% increase in the separation rates. Solid black lines are the IRFs for a model in which firms face a vacancy cost function. Dash lines are the IRFs for a model in which firms face a hiring cost function. Dotted lines represent the IRF for a calibrated model as in MPV16.
7 Wage Posting and Wage Rigidity

I showed in the previous section that the baseline model generates large wage responses to aggregate shocks on impact. These large responses seem unfeasible since wages for job stayers look sticky in the data (e.g. Kahn, 1997; Barattieri, Basu & Gottschalk, 2014). Given that firms commit to a value of employment, once a higher employment value is offer in the future, current wages decrease in order to keep the current employment value constant for job stayers. Similarly, based on the results presented in the previous section, the baseline model does not generate significant (if any) movements in labor market quantities in response to aggregate productivity shocks, which was a results that was expected given Proposition 1. These results are similar to those in Shimer (2005), who argues that the standard DMP model generates volatility in the labor market in response to productivity shocks and that separation rate shocks tend to generate a positive correlation between unemployment and vacancies.

Given that the Shimer puzzle gave rise to a large body of the literature studying the amplifying effects of wage rigidity (e.g. Hall, 2005; Hall & Milgrom, 2008; Gertler & Trigari, 2009), I modify the baseline model to (1) limit the wage responses on impact to aggregate shocks and (2) assess the amplifying effects of wage rigidity in this framework. To this end, I first propose a modification to the baseline model in which firms post wages instead of employment values to retain and poach workers from other firms. Since workers only care about employment values, firms will target a value of employment when posting a wage. However, as I will point out, this extension to the model does not increase the volatility in the labor market. This new assumption will only limit the impact response of wages to aggregate shocks. Then, I introduce wage rigidity by assuming that firms face quadratic cost of wage adjustment a la Rotemberg (1982).

7.1 Model with Wage Posting and Wage Rigidity

The household problem is the same. However, firm posts wages instead of employment values to attract workers, whose decisions will continue to depend on the value of employment. In addition, I assume that firms face quadratic cost of wage adjustment. Hence, we can re-write the firm’s
problem as follows:

\[
\Pi(n_{jt}, \bar{w}) = \max_{v_{jt}, k_{jt}, w_{jt+1}} \pi_{jt} + E_t [Q_t \Pi_{jt+1}(n_{jt+1}, w_{jt+1})]
\]  

s.t.

\[
\pi_{jt} = y_{jt} - w_{jt}n_{jt} - r_t k_{jt} - \kappa \left( \frac{\bar{q}_{jt} v_{jt}}{1 + \chi} + Y_t \phi \left( \frac{w_{jt}}{w_{jt-1}} - 1 \right)^2 \right)
\]  

\[
y_{jt} = e^{a_j + \alpha} k^{\alpha} n_{jt}^{1-\alpha}
\]  

\[
n_{jt+1} = (1 - \delta_t) (1 - \tilde{q}_t g_{jt}) n_{jt} + q_{jt} v_{jt}
\]  

\[
W_{jt} = w_{jt} - z_{jt} + E_t \{Q_t[(1 - \delta_t)(1 - \tilde{q}_t g_{jt}) W_{jt+1}]
\]

\[
+ (1 - \delta_t) \tilde{q}_t q_t \int_{W_{jt+1}}^{\infty} W f_{W_t}^v dW - q_t \int_0^W W f_{W_t}^v dW\}
\]  

\[
G_{jt} = \int_{W_{jt+1}}^{\infty} f_{W_t}^v dW
\]  

\[
\bar{q}_{jt} = \frac{q_{jt}}{s_t} \left( u_t + \tilde{q}_t (1 - \delta_t) \int_{W_{jt+1}}^{\infty} n_{W_t} f_{W_t}^n dW \right)
\]  

\[
w_{jt} \geq \bar{w}
\]  

as before, \(f_{W_t}^v\) and \(f_{W_t}^n\) denote the density functions of employment value offers and employment. Notice that there is a mapping between wage and employment offers given by (56). Hence, when firms post a wage \(w_{jt+1}\), they are implicitly posting a value of employment \(W_{jt}\) given by (56). Hence, given that a worker only move to jobs that offer higher employment values, the relevant distributions continue to be over employment values and not over wages. It can be shown that the optimality conditions for capital and vacancies (or hires) do not change, but the optimality condition for wage offers is given by:

\[
E_t [Q_t n_{jt+1}] + Y_t \phi \left( \frac{w_{jt+1}}{w_{jt}} - 1 \right) \frac{1}{w_{jt}} - Y_t \phi \left( \frac{w_{jt+2}}{w_{jt+1}} - 1 \right) \frac{w_{jt+2}}{w_{jt+1}^2}
\]

\[
\geq E_t \{Q_t J_{jt+1}(1 - \delta_t) \tilde{q}_t [q_t f_{jt}^v n_{jt} + (1 - I_h) \cdot \tilde{q}_t f_{jt}^n v_{jt}]\}
\]  

When firms commit to a higher wage in the future, they retain a larger fraction of their workforce and poach more workers (right hand side of equation (60)), but their payroll will increase by the size of their new workforce. In contrast, when firms post employment values, the cost of offering a higher employment value is proportional to the increase (and not the level) of the firm’s workforce.

**Definition 3. Competitive Search Equilibrium with Wage Posting.** A competitive search equilibrium with Wage Posting is a sequence of prices \(\{r_t, w_t\}\), quantities \(\{y_t, c_t, k_t, u_t, n_t\}\),
probabilities \( \{q_t, \tilde{q}_t\} \), and functions \( \{v_{jt}, W_{jt+1}, J_{jt}, n_{jt+1}\} \) on productivity \( a_j \), firm size \( n_{jt} \) and \( W_{jt} \), such that given exogenous variables \( \{a_t, \delta_{nt}, \tilde{t}_t\} \), an initial stock of capital and initial distributions of employment and employment values: (i) The household optimizes, taken as given prices and exogenous shocks. Consumption satisfies the optimality condition (4). (ii) Taking as given the exogenous variables, \( \{r_t\} \), and all other firms strategies (i.e. employment, wage, and vacancies), firms optimize. Functions \( \{v_{jt}, W_{jt+1}, J_{jt}, n_{jt+1}\} \) solve equations (11), (19), (23) and (60) and prices satisfy equations (5) and (18). (iii) Probabilities evolve according to \( q_t = m(\theta_t, 1) \) and \( \tilde{q}_t = m(1, \theta_t^{-1}) \). (iv) Markets clear: the aggregate resource constraint holds.

Given this definition, I next show Proposition 1 still applies when firms post wages instead of employment values:

**Proposition 2.** Suppose that \( \{r^{(1)}_t, w^{(1)}_t, y^{(1)}_t, k^{(1)}_t, u^{(1)}_t, n^{(1)}_t, v^{(1)}_t, q^{(1)}_t, \tilde{q}^{(1)}_t, v^{(1)}_{jt}, W^{(1)}_{jt+1}, J^{(1)}_{jt}, n^{(1)}_{jt+1}\} \) is a rank-preserving competitive search equilibrium with wage posting given a sequence of exogenous shocks \( \{a^{(1)}_t, \delta^{(1)}_{nt}, \tilde{t}^{(1)}_t\} \). Now, suppose that \( \{v^{(2)}_t, W^{(2)}_{jt+1}, J^{(2)}_{jt}, n^{(2)}_{jt+1}\} \) is a rank-preserving competitive search equilibrium with wage posting given a sequence of exogenous shocks \( \{a^{(2)}_t, \delta^{(1)}_{nt}, \tilde{t}^{(1)}_t\} \) i.e. the aggregate productivity is different in both equilibriums while the remaining exogenous variables are the same. If \( \sigma = 0, \phi = 0 \) and aggregate shocks are small. Then

- \( u^{(1)}_t = u^{(2)}_t, n^{(1)}_t = n^{(2)}_t, v^{(1)}_t = v^{(2)}_t, q^{(1)}_t = q^{(2)}_t, \tilde{q}^{(1)}_t = \tilde{q}^{(2)}_t \) for all \( t \).
- \( J^{(1)}_{jt} = J^{(2)}_{jt}, v^{(1)}_{jt} = v^{(2)}_{jt}, n^{(1)}_{jt+1} = n^{(2)}_{jt+1} \) for all \( t \) and for all \( j > 0 \).

**Proof.** See Appendix A.5. \( \square \)

In other words, the irrelevance of aggregate productivity shocks for fluctuations in labor market quantities applies for an equilibrium with wage posting and value posting. In both cases, increasing competition for workers makes wages absorb aggregate productivity shocks completely as long as firms do not face quadratic costs of wage adjustment.

### 7.2 Calibration and Results

To calibrate this new version of the model, I target the same moments as before. The only parameter that remains to be specified is \( \phi \) which governs the degree of wage rigidity. I calibrate this parameter to target the standard deviation of the average wage in the economy as a fraction of the standard deviation of output. The parameter values for this new model can be found in Table 1 under columns wage posting. For simplicity, I only present the results for productivity shocks. Table 4 presents the business cycle moments generated by the vacancy and hiring cost models when firms face quadratic costs of wage adjustment, and Figure 5 plots the IRFs to a 1%
increase in TFP. Even though, the volatility of labor market quantities continues to be small in this calibration, TFP shocks are now able to generate meaningful responses in the labor market. Also, the wage responses are smoother and more consistent with the observed persistence in the data.

To close this section, I show that the magnitude of the responses in labor market quantities depends on the size (relative to productivity) of the FOCE. To illustrate this, Figure 6 plots the Impulse Response Functions generated by a model in which firms face a hiring cost function and quadratic costs of wage adjustment for different values of $\int \frac{\partial z_j}{\partial p_j}$. The solid black lines, which correspond to the baseline case, assume that $\int \frac{\partial z_j}{\partial p_j} = 0.72$, while the dashed lines and dotted lines impose a value equal to 0.81 and 0.91 in steady state. This Figure shows that larger values of FOCE, conditional on wage rigidity, generate large responses in the labor market to TFP shocks. As explained by Hagedorn and Manovskii (2008) and Ljungqvist and Sargent (2017), large values of FOCE generate proportional large changes in the match surplus in response to 1% increase in productivity, making hiring decisions more sensitive to aggregate innovations.

Note: This figure plots the IRFs to a 1% increase in aggregate productivity when firms face quadratic costs of wage adjustment. Solid black lines are the IRFs for a model in which firms face a vacancy cost function. Dash lines are the IRFs for a model in which firms face a hiring cost function.
Table 4: Business Cycle Moments. Model with Wage Rigidity

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<tr>
<th></th>
<th>u</th>
<th>v</th>
<th>h</th>
<th>UEr</th>
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<td>-0.54</td>
<td>-0.89</td>
<td>-0.85</td>
<td>-0.77</td>
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</table>

Note: Statistics for the U.S. economy are based on: u: Unemployment level. v: Help-wanted index (Barnichon, 2010). h: hires from unemployment and other jobs. UEr: Unemployment to Employment transition rate. EUr: Employment to Unemployment transition rate. w^a: Average wage in the economy. w^v: Average wage for new employees. w^c: Average wage for job changers. y: Real output in the nonfarm business sector. p: Real output per-hour in the non-farm business sector. a: Utilization adjusted TFP. All series are seasonally adjusted, logged, and detrended via the HP filter with a smoothing parameter of 100,000. Vacancy refers to a calibrated model in which firms face a vacancy cost function and quadratic costs of wage adjustment. Hiring refers to a calibrated model in which firms face a hiring cost function and quadratic costs of wage adjustment.
Figure 6: Role of FOCE

Note: This figure plots the IRFs to a 1% increase in aggregate productivity when firms face quadratic costs of wage adjustment and a hiring cost function. Solid, dashed and dotted lines are the IRFs for a model in which the average value of $\frac{z_i}{p_j}$ is 0.71, 0.81 and 0.91 in steady state respectively.
8 Conclusion

Wage posting models are an appealing way of studying wage dispersion and wage growth. However, quantifying the cyclical properties of these types of models has remained a challenging task.

In this paper, I study the business cycle properties of wage posting models with random search in which the distribution of employment matters for agents decision rules and the equilibrium path. I present a model close to Moscarini and Postel-Vinay (2013), but I abstract from firm entry and exit, and I introduce capital and a strictly concave utility function for the household.

To evaluate this model quantitatively, I propose a new algorithm that computes the steady state and transitional dynamics in a few seconds. This new method may be of interest of its own right. Based on this method, integrating wage posting models with random search to larger models (like a medium scale New-Keynesian model) becomes possible and easy.

Even though I am able to reproduce the results of Moscarini and Postel-Vinay (2016), who also proposed an algorithm for solving these types of models, my conclusions are significantly different. First, I show that wage posting models generate insignificant fluctuations in labor market quantities in response to productivity shocks because fear of competition for workers makes wages absorb most of these shocks. Second, wage posting models display unrealistic wage responses. After a positive aggregate innovation, wages drop significantly on impact, increase in the subsequent period, and then return slowly to the steady state. Third, even though separation rate shocks generate large fluctuations in the labor market, these shock also generate a positive correlation between unemployment and hires that is inconsistent with the data. In addition, if firms face a vacancy cost function, wage posting models generate a positive correlation between unemployment and vacancies in response to separation rate shocks.

To address the lack of volatility in response to TFP shocks and the unrealistic behavior of wages generated by wage posting models, I make two modifications to the baseline model. First, I assume that firms post wages instead of values of employment. Second, I assume that firms face quadratic costs of wage adjustment. I show that these modifications improve significantly the model predictions.

The algorithm presented in this paper has powerful advantages that make it very useful. This method does not suffer from the curse of dimensionality, it takes only a few seconds, and it makes it possible and easy to integrate wage posting models to more general frameworks.

References


31


A Proofs

Before going into the details of these proofs, the following conditions and Lemmas would be useful. First, notice that the following conditions always hold:

\[ q_t = \left( \frac{v_t}{s_t} \right) \tilde{q}_t \]  
(61)

\[ \tilde{q}_t f^n_{jt} v_{jt} = q_t f^v_{jt} n_{jt} \]  
(62)

\[ f^n_{jt} = \left( \frac{v_t}{s_t} \right) \left( \frac{n_{jt}}{v_{jt}} \right) f^v_{jt} \]  
(63)

Then, the following Lemmas will be used to proof the Lemmas presented in the main text.

**Lemma 3.** Regardless of the recruiting cost function, in steady state \( \frac{\partial (n_j)}{\partial W_j} \) and \( \frac{\partial (n_j)}{\partial W_j} \) are positive.

*Proof.* Notice that in steady state the following conditions hold

\[ \left( \frac{n_j}{v_j} \right) = \frac{\tilde{q}_j}{[1 - (1 - \delta_n)(1 - iqG_j)]} \]  
(64)

\[ \left( \frac{n_j}{h_j} \right) = \frac{1}{[1 - (1 - \delta_n)(1 - iqG_j)]} \]  
(65)

Therefore,

\[ \frac{\partial (n_j)}{\partial W_j} = (1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{h_j} \right) + \tilde{q}_j(1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{h_j} \right)^2 \]  
(66)

\[ \frac{\partial (n_j)}{\partial W_j} = (1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{v_j} \right) \left( \frac{n_j}{h_j} \right) + \tilde{q}_j(1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{h_j} \right)^2 \]  
(67)

\[ \frac{\partial (n_j)}{\partial W_j} = (1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{h_j} \right) \left( \frac{n_j}{v_j} \right) + \tilde{q}_j \left( \frac{n_j}{h_j} \right) \]  
(68)

\[ \frac{\partial (n_j)}{\partial W_j} = 2(1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{v_j} \right) \left( \frac{n_j}{h_j} \right) > 0 \]  
(69)

Also

\[ \frac{\partial (n_j)}{\partial W_j} = \tilde{q}_j(1 - \delta_n)\tilde{q}f^v_j \left( \frac{n_j}{h_j} \right)^2 > 0 \]  
(70)
Lemma 4. Regardless of the recruiting cost function, in steady state:

\[
\frac{\partial f_j^n}{\partial W_j} = f_j^{n'} = \left(\frac{v}{s}\right) \left(\frac{n_j}{v_j}\right) \left[ \frac{2(1 - \delta_n)\bar{iq}}{1 - (1 - \delta_n)(1 - iqG_j)} \right] f_j^{v^2} + f_j^{v'}
\]

(71)

Proof. First, note that:

\[
f_j^n = \left(\frac{v}{s}\right) \left(\frac{n_j}{v_j}\right) f_j^v
\]

(72)

\[
\frac{\partial f_j^n}{\partial W_j} = \left(\frac{v}{s}\right) \left[ \frac{\partial}{\partial W_j} \left(\frac{n_j}{v_j}\right) f_j^v + \left(\frac{n_j}{v_j}\right) f_j^{v'} \right]
\]

(73)

\[
\frac{\partial f_j^n}{\partial W_j} = \left(\frac{v}{s}\right) \left(\frac{n_j}{v_j}\right) \left[ \frac{2(1 - \delta_n)\bar{iq}}{1 - (1 - \delta_n)(1 - iqG_j)} \right] f_j^{v^2} + f_j^{v'}
\]

(74)

\[
\frac{\partial f_j^n}{\partial W_j} = \left(\frac{v}{s}\right) \left(\frac{n_j}{v_j}\right) \left[ \frac{2(1 - \delta_n)\bar{iq}}{1 - (1 - \delta_n)(1 - iqG_j)} \right] f_j^{v^2} + f_j^{v'}
\]

(75)

Lemma 5. Regardless of the recruiting cost function, in steady state:

\[
\frac{\partial p_j}{\partial W_j} = -J_j \beta(1 - \delta_s)\bar{iq} f_j^v + \left[ \frac{\partial J_j}{\partial W_j} + 1 \right] \left[ 1 - \beta(1 - \delta_s)(1 - \bar{iq}G_j) \right] + \epsilon \bar{\Pi} \frac{n_j^{\eta - \epsilon}}{c} n_j^{\eta - 1} \frac{\partial n_j}{\partial W_j}
\]

(76)

Proof. From the definition of \( J_j \), we have that in steady state

\[
J_j = p_j - w_j + \beta(1 - \delta_n)(1 - \bar{iq}G_j)J_j
\]

(77)

Hence, solving for \( p_j \) and taking the derivative with respect to \( W_j \):

\[
p_j = J_j \left[ 1 - \beta(1 - \delta_s)(1 - \bar{iq}G_j) \right] + w_j
\]

(78)

\[
\frac{\partial p_j}{\partial W_j} = -J_j \beta(1 - \delta_s)\bar{iq} f_j^v + \frac{\partial J_j}{\partial W_j} \left[ 1 - \beta(1 - \delta_s)\bar{iq}G_j \right] + \frac{\partial w_j}{\partial W_j}
\]

(79)

From the value of employment (5), solving the \( w_j \) in steady state and taking the derivative
with respect to $W_j$:

$$w_j = W_j [1 - \beta(1 - \delta_n)(1 - \bar{q}G_j)] + z_j + \beta(1 - \delta_n)\bar{q} \int_{W_j}^{\infty} W f'_{W}dW - \beta q \int_{0}^{\infty} W f'_{W}dW$$

(80)

$$\frac{\partial w_j}{\partial W_j} = [1 - \beta(1 - \delta_n)\bar{q}G_j] - \beta(1 - \delta_n)\bar{q}W_j f''_j + \frac{\partial z_j}{\partial W_j}$$

(81)

$$\frac{\partial w_j}{\partial W_j} = [1 - \beta(1 - \delta_n)\bar{q}G_j] + \frac{\partial z_j}{\partial W_j}$$

(82)

Finally, from the definition of $z_j$:

$$\frac{\partial z_j}{\partial W_j} = \epsilon \Psi \frac{\bar{n}^{\eta - c}}{c} n_j^{s-1} \frac{\partial n_j}{\partial W_j}$$

(83)

Hence, combining (82), (83) and (76):

$$\frac{\partial p_j}{\partial W_j} = -J_j \beta(1 - \delta_n)\bar{q}f_j'' + \left[ \frac{\partial J_j}{\partial W_j} + 1 \right] [1 - \beta(1 - \delta_n)(1 - \bar{q}G_j)] + \epsilon \Psi \frac{\bar{n}^{\eta - c}}{c} n_j^{s-1} \frac{\partial n_j}{\partial W_j}$$

(84)

\[\square\]

A.1 Proof of Proposition 1

**Proposition 1.** Suppose that $\{r_t^{(1)}, u_t^{(1)}, y_t^{(1)}, c_t^{(1)}, k_t^{(1)}, u_t^{(1)}, n_t^{(1)}, v_t^{(1)}, q_t^{(1)}, \check{q}_t^{(1)}, v_{jt}^{(1)}, W_{jt+1}^{(1)}, J_t^{(1)}, n_{jt+1}^{(1)}\}$ is a rank-preserving competitive search equilibrium given a sequence of exogenous shocks $\{a_t^{(1)}, \delta_{nt}^{(1)}, i_t^{(1)}\}$. Now, suppose that $\{r_t^{(2)}, u_t^{(2)}, y_t^{(2)}, c_t^{(2)}, k_t^{(2)}, u_t^{(2)}, n_t^{(2)}, v_t^{(2)}, q_t^{(2)}, \check{q}_t^{(2)}, v_{jt}^{(2)}, W_{jt+1}^{(2)}, J_t^{(2)}, n_{jt+1}^{(2)}\}$ is a rank-preserving competitive search equilibrium given a sequence of exogenous shocks $\{a_t^{(2)}, \delta_{nt}^{(1)}, i_t^{(1)}\}$ i.e. the aggregate productivity is different in both equilibriums while the remaining exogenous variables are the same. If $\sigma = 0$. Then

- $u_t^{(1)} = u_t^{(2)}, n_t^{(1)} = n_t^{(2)}, v_t^{(1)} = v_t^{(2)}, q_t^{(1)} = q_t^{(2)}, \check{q}_t^{(1)} = \check{q}_t^{(2)}$ for all $t$.

- $J_t^{(1)} = J_t^{(2)}, v_{jt}^{(1)} = v_{jt}^{(2)}, n_{jt+1}^{(1)} = n_{jt+1}^{(2)}$ for all $t$ and for all $j > 0$.

**Proof.** Given that $\sigma_c = 0$, the stochastic discount factor is always equal to $\beta$. Now, by definition, $W_{xt}\{1\} \geq W_{yt}\{1\}$ for all $t$ and all $x \geq y$. Also, notice that I can re-write firms productivity as follows:

$$p_{jt}^{(2)} = (1 + x_j)p_{jt}^{(1)}$$

(85)

Also, by assumption, the sequence for $G_{jt}$, $q_{jt}$ and $z_{jt}$ are the same in both equilibriums. Those variables depend on vacancies and employment, which are the same in both equilibriums. Now,
omitting the superscripts for those variables that are the same in both equilibriums:

\[ J_{jt} = p_{jt}^{(2)} - w_{jt}^{(2)} + \beta E \left[ (1 - \delta_{nt})(1 - \bar{q}_t G_{jt}) \right] J_{jt+1} \]  

(86)

\[ J_{jt} = p_{jt}^{(1)} - \left( w_{jt}^{(2)} - x_t p_{jt}^{(1)} \right) + \beta E \left[ (1 - \delta_{nt})(1 - \bar{q}_t G_{jt}) \right] J_{jt+1} \]  

(87)

Which is only true if wages in the second equilibrium are equal to:

\[ w_{jt}^{(2)} = w_{jt}^{(1)} + x_t p_{jt}^{(1)} \]  

(88)

Now, to prove that the second equilibrium is rank preserving, and as consequence the sequence for \( G_{jt}, \tilde{q}_{jt} \) are the same in both equilibriums, notice that:

\begin{align*}
(W_{xt}^{(2)} - W_{yt}^{(2)}) &= \left( w_{xt}^{(2)} - z_{xt} - w_{yt}^{(2)} + z_{yt} \right) + \beta E \left[ (1 - \delta_{nt})(1 - \bar{q}_t G_{yt}) \left( W_{xt+1}^{(2)} - W_{yt+1}^{(2)} \right) \right] \\
&+ \beta E \left[ (1 - \delta_{nt}) \bar{q}_t \int_{W_{yt+1}}^{W_{xt+1}} \left( W_{xt+1}^{(2)} - W \right) f_{W_t}^{(2)} dW \right] \\
&= E_t \left[ \beta \bar{q}_t^{1-I_h} J_{jt+1} \right] \\
&= E_t \left[ (J_{jt+1} - \delta_{nt} \bar{q}_t) q_t f_{jt}^{n(2)} n_{jt} + (1 - I_h) \cdot \bar{q}_t f_{jt}^{n(2)} v_{jt} \right]
\end{align*}

(89)

Hence, the second equilibrium is rank preserving if (89) is positive for all \( x \geq y \), which is true if:

\begin{align*}
\left( w_{xt}^{(2)} - z_{xt} - w_{yt}^{(2)} + z_{yt} \right) &> 0 \quad \text{(90)} \\
\left( w_{xt}^{(1)} - z_{xt} - w_{yt}^{(1)} + z_{yt} \right) &> -x_t \left( p_{xt}^{(1)} - p_{yt}^{(1)} \right) \quad \text{(91)} \\
- \left( w_{xt}^{(1)} - z_{xt} - w_{yt}^{(1)} + z_{yt} \right) &> x_t \left( p_{xt}^{(1)} - p_{yt}^{(1)} \right) \quad \text{(92)}
\end{align*}

Since, the first equilibrium is rank preserving, the left hand side of equation (92) is always negative. As a consequence, if aggregate shocks are not very large and negative, the second equilibrium is rank preserving. To show that the second equilibrium is in fact an equilibrium, notice that:

\[ \kappa \left( \tilde{q}_j^h v_{jt} \right)^x = E_t \left[ \beta \tilde{q}_j^{1-I_h} J_{jt+1} \right] \]  

(93)

\[ E_t \left[ h_{jt} \right] \leq E_t \left\{ J_{jt+1} (1 - \delta_{nt}) \bar{q}_t q_t f_{jt}^{n(2)} n_{jt} + (1 - I_h) \cdot \bar{q}_t f_{jt}^{n(2)} v_{jt} \right\} \]  

(94)

Hence, given that the second equilibrium preserves the ranking of firms, \( f_{jt}^{n(2)} = f_{jt}^{n(1)} \) and \( f_{jt}^{n(2)} = f_{jt}^{n(1)} \). Finally, even though \( W_{0t} \) will continue to be zero in both equilibrium, and those firm will change their employment and vacancies decisions, the mass of those firms is 0. As a
In steady state, assuming that firms face a vacancy cost function, if the offer of employment value distributes according to a decreasing p.d.f. in equilibrium, and $\chi > \frac{1}{4}$; the FOCs are necessary and sufficient and the equilibrium is Rank-preserving.

**Proof.** Notice that the second order conditions are given by:

$$\Pi v_j, v_j : -\chi \kappa v_{jt}^{\chi - 1} + E_t [Q_t \tilde{q}_{jt} J_{jt+1}]$$  \hspace{2cm} (95)

$$\Pi W_j, W_j : -E_t [Q_t (1 - \delta_{nt}) \tilde{v}_j \tilde{q}_{jt} v_{jt}] - E_t [Q_t (1 - \delta_{nt}) \tilde{v}_j (q t f_{jt}^* n_{jt} + \tilde{q} f_{jt}^* v_{jt})]$$

$$+ E_t [Q_t J_{jt+1} (1 - \delta_{nt}) \tilde{v}_j (q t f_{jt}^* n_{jt} + \tilde{q} f_{jt}^* v_{jt})]$$  \hspace{2cm} (96)

$$\Pi v_j, W_j : -E_t [Q_t \tilde{q}_{jt}] + E_t [Q_t J_{jt+1} (1 - \delta_{nt}) \tilde{v}_j \tilde{q} f_{jt}^*]$$  \hspace{2cm} (97)

Hence, in steady state

$$\Pi v_j, v_j = -\chi \kappa v_{jt}^{\chi - 1} + \beta \tilde{q}_j J_j$$  \hspace{2cm} (98)

$$\Pi W_j, W_j = -\beta (1 - \delta_n) \tilde{q} f_{jt}^* v_j - \beta (1 - \delta_n) \tilde{v}_j (q f_{jt}^* n_j + \tilde{q} f_{jt}^* v_j)$$

$$+ \beta J_j (1 - \delta_n) \tilde{v}_j (q f_{jt}^* n_j + \tilde{q} f_{jt}^* v_j)$$  \hspace{2cm} (99)

$$\Pi v_j, W_j = -\beta \tilde{q}_j + \beta J_j (1 - \delta_n) \tilde{v} f_{jt}^*$$  \hspace{2cm} (100)

Notice that the second order condition with respect to $W_j$ can be simplified using (63), (65) and (71) and the first order condition for $W_j$ as follows:

$$\Pi W_j, W_j = -3 \beta (1 - \delta_n) \tilde{q} f_{jt}^* n_j + \beta J_j (1 - \delta_n) \tilde{v} n_j \left[ q f_{jt}^* \tilde{v} + \tilde{q} f_{jt}^* \left( \frac{v_j}{n_j} \right) \right]$$  \hspace{2cm} (101)

$$= -3 \beta (1 - \delta_n) \tilde{q} f_{jt}^* n_j + \beta 2 J_j (1 - \delta_n) \tilde{v} q n_j f_{jt}^* \left[ \frac{f_{jt}^*}{f_j} + \frac{(1 - \delta_n) \tilde{q}}{1 - (1 - \delta_n)(1 - \tilde{q} G_j)} \right] f_j^*$$

$$= -3 \beta (1 - \delta_n) \tilde{q} f_{jt}^* n_j + \beta h_j \left[ \frac{f_{jt}^*}{f_j} + \frac{(1 - \delta_n) \tilde{q}}{1 - (1 - \delta_n)(1 - \tilde{q} G_j)} \right] f_j^*$$  \hspace{2cm} (102)

$$= n_j \left( -3 \beta (1 - \delta_n) \tilde{q} f_{jt}^* + \beta [1 - (1 - \delta_n)(1 - \tilde{q} G_j)] \frac{f_{jt}^*}{f_j} + \beta (1 - \delta_n) \tilde{q} f_{jt}^* \right)$$  \hspace{2cm} (103)

$$= n_j \left( -2 \beta (1 - \delta_n) \tilde{q} f_{jt}^* + \beta [1 - (1 - \delta_n)(1 - \tilde{q} G_j)] \frac{f_{jt}^*}{f_j} \right)$$  \hspace{2cm} (104)

$$= n_j \left( -2 \beta (1 - \delta_n) \tilde{q} f_{jt}^* + \beta \left( \frac{h_j}{n_j} \right) \left( \frac{f_{jt}^*}{f_j} \right) \right)$$  \hspace{2cm} (105)
Also notice that, in steady state, the first order condition with respect to \( W_j \) could be re-written as follows:

\[
h_j = 2J_j(1 - \delta_n)\bar{q}_j f_j^v \]
\[
\frac{\dot{q}_j}{2} = J_j(1 - \delta_n)\bar{q}_j f_j^v \tag{107}
\]

which implies that the cross derivative could be re-written as:

\[
\Pi_{v_j,W_j} = -\beta \frac{\dot{q}_j}{2} \tag{109}
\]

As a consequence, using the first order conditions for vacancies and value offers:

\[
\Pi_{v_j,v_j} \Pi_{W_j,W_j} - \Pi_{v_j,W_j}^2 = \chi \nu_j \nu_j^{-1} n_j \left[ 2\beta(1 - \delta_n)\bar{q}_j f_j^v - \beta \left( \frac{h_j}{n_j} \right) \left( \frac{f_j^{v'}}{f_j^v} \right) \right] - \beta^2 \frac{\bar{q}_j^2}{4} \tag{110}
\]

\[
\Pi_{v_j,v_j} \Pi_{W_j,W_j} - \Pi_{v_j,W_j}^2 = \chi \nu_j \nu_j^{-1} n_j \left[ 2\beta(1 - \delta_n)\bar{q}_j f_j^v - \beta \left( \frac{h_j}{n_j} \right) \left( \frac{f_j^{v'}}{f_j^v} \right) \right] - \beta^2 \frac{\bar{q}_j^2}{4} \tag{111}
\]

\[
= 2\chi \beta^2 J_j \bar{q}_j \left( \frac{n_j}{v_j} \right) (1 - \delta_n)\bar{q}_j f_j^v - \chi \beta^2 J_j \bar{q}_j \left( \frac{n_j}{v_j} \right) \left( \frac{h_j}{n_j} \right) \left( \frac{f_j^{v'}}{f_j^v} \right) - \beta^2 \frac{\bar{q}_j^2}{4} \tag{112}
\]

\[
= 2\chi \beta^2 J_j \bar{q}_j \left( \frac{n_j}{v_j} \right) \frac{h_j}{2J_j n_j} - \chi \beta^2 J_j \bar{q}_j \bar{q}_j \left( \frac{f_j^{v'}}{f_j^v} \right) - \beta^2 \frac{\bar{q}_j^2}{4} \tag{113}
\]

\[
= \chi \beta^2 \bar{q}_j^2 - \chi \beta^2 J_j \bar{q}_j \frac{f_j^{v'}}{f_j^v} - \beta^2 \frac{\bar{q}_j^2}{4} \tag{114}
\]

Hence, notice that \( \Pi_{v_j,v_j} \) is always negative, \( \Pi_{W_j,W_j} \) is negative as long as \( f_j^{v'} < 0 \) and:

\[
\Pi_{v_j,v_j} \Pi_{W_j,W_j} - \Pi_{v_j,W_j}^2 > 0 \quad if \quad \left( \chi - \frac{1}{4} \right) - \chi J_j \left( \frac{f_j^{v'}}{f_j^v} \right) > 0 \tag{115}
\]

\[
\left( \chi - \frac{1}{4} \right) - \chi J_j \left( \frac{f_j^{v'}}{f_j^v} \right) > 0 \tag{116}
\]

Which is true if \( f_j^{v'} < 0 \) and \( \chi > 1/4 \). Therefore, the FOC are necessary and sufficient. Now, to prove that the equilibrium is rank-preserving, First notice that

\[
J_j = \left( \frac{h_j}{n_j} \right) \frac{1}{2(1 - \delta_n)\bar{q}_j f_j^v} \tag{117}
\]

\[
J_j = \frac{[1 - (1 - \delta_n)(1 - \bar{q}G_j)]}{2(1 - \delta_n)\bar{q}_j f_j^v} \tag{118}
\]

\[
\frac{\partial J_j}{\partial W_j} = -\frac{1}{2} \left[ (1 - \delta_n)\bar{q}_j f_j^v \right] \tag{119}
\]
Then, from the first order conditions for vacancies:

\[
v_j = \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} (\tilde{q}_j J_j)^{\frac{1}{\chi}}
\]  

(120)

\[
\frac{\partial v_j}{\partial W_j} = \frac{1}{\chi} \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} (\tilde{q}_j J_j)^{\frac{1}{\chi} - 1} \left(1 - \delta_s\tilde{q} f_j J_j + \tilde{q}_j \frac{\partial J_j}{\partial W_j}\right)
\]

(121)

Then, substituting (119) into (121)

\[
\frac{\partial v_j}{\partial W_j} = \frac{1}{\chi} \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} (\tilde{q}_j J_j)^{\frac{1}{\chi} - 1} \left(\frac{\tilde{q}_j}{2} + \tilde{q}_j \frac{\partial J_j}{\partial W_j}\right)
\]

(122)

\[
\frac{\partial v_j}{\partial W_j} = -\frac{1}{\chi} \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} (\tilde{q}_j J_j)^{\frac{1}{\chi} - 1} \left[1 - (1 - \delta_s)(1 - \tilde{i}q G_j)\right] f_j^\prime \frac{f_j}{1 - \tilde{i}q (1 - \delta_s)}
\]

(123)

Hence, \(\frac{\partial v_j}{\partial W_j} > 0\) given that \(f_j^\prime\) is assumed to be negative. Now, we can compute \(\frac{\partial n_j}{\partial W_j}\) as follows:

\[
n_j = \left(\frac{n_j}{v_j}\right) v_j
\]

(124)

\[
\frac{\partial n_j}{\partial W_j} = \frac{\partial \left(\frac{n_j}{v_j}\right)}{\partial W_j} v_j + \left(\frac{n_j}{v_j}\right) \frac{\partial v_j}{\partial W_j} > 0
\]

(125)

From Lemma 3, we know that \(\frac{\partial \left(\frac{n_j}{v_j}\right)}{\partial W_j} > 0\). As a consequence, if \(f_j^\prime < 0\), \(\frac{\partial n_j}{\partial W_j} > 0\), and we conclude that employment values are an increasing function of firm size since \(\frac{\partial n_j}{\partial W_j} > 0\). Now, it remains to show that employment values are an increasing function of firms productivity, in order to proof that the equilibrium is rank-preserving. To see this, using Lemma 5 and equation (119) we get:

\[
\frac{\partial p_j}{\partial W_j} = -J_j \beta (1 - \delta_s)\tilde{i}q f_j^v + \left[\frac{\partial J_j}{\partial W_j} + 1 \right] \left[1 - \beta (1 - \delta_s)(1 - \tilde{i}q G_j)\right] + \epsilon \Psi \tilde{n}^{\eta - \epsilon} c n_j^{-1} \frac{\partial n_j}{\partial W_j}
\]

(126)

\[
\frac{\partial p_j}{\partial W_j} = -\frac{1}{2} \left[1 - (1 - \delta_n)(1 - \tilde{i}q G_j)\right] f_j^v \left[1 - \beta (1 - \delta_n)(1 - \tilde{i}q G_j)\right]
\]

(127)

\[
\frac{\partial p_j}{\partial W_j} = -J_j \beta (1 - \delta_s)\tilde{i}q f_j^v + \left[1 - \beta (1 - \delta_s)(1 - \tilde{i}q G_j)\right] \frac{f_j^v}{2} - \frac{1}{\tilde{i}q (1 - \delta_n)} \left[1 - \beta (1 - \delta_s)(1 - \tilde{i}q G_j)\right] f_j^v \frac{f_j^v}{f_j^v} + \epsilon \Psi \tilde{n}^{\eta - \epsilon} c n_j^{-1} \frac{\partial n_j}{\partial W_j}
\]

(128)
Hence, using the first order condition for offer values in steady state:

\[
\frac{\partial p_j}{\partial W_j} = -\beta \left[ 1 - (1 - \delta_n)(1 - \bar{iq}G_j) \right] \frac{[1 - \beta(1 - \delta_s)(1 - \bar{iq}G_j)]}{(1 - \delta_n)\bar{iq}} f_j^{\nu'} f_j^{v} + \epsilon \Psi \frac{\bar{n}_j^{\eta - \epsilon}}{c} \frac{n_j^{\epsilon - 1}}{\partial W_j} \tag{129}
\]

\[
\frac{\partial p_j}{\partial W_j} = \frac{1 - \beta}{2} \left[ 1 - (1 - \delta_n)(1 - \bar{iq}G_j) \right] 
- \left[ 1 - \beta(1 - \delta_s)(1 - \bar{iq}G_j) \right] \frac{f_j^{\nu'}}{f_j^{v}} \tag{130}
\]

\[
\frac{\partial p_j}{\partial W_j} = \frac{1 - \beta}{2} \left[ 1 - (1 - \delta_n)(1 - \bar{iq}G_j) \right] 
- \left[ 1 - \beta(1 - \delta_s)(1 - \bar{iq}G_j) \right] \frac{f_j^{\nu'}}{f_j^{v}} + \epsilon \Psi \frac{\bar{n}_j^{\eta - \epsilon}}{c} \frac{n_j^{\epsilon - 1}}{\partial W_j} \tag{131}
\]

Hence, assuming that \( f_j^{\nu'} < 0 \), implies that employment offers are an increasing function in \( a_j \) since \( \frac{\partial p_j}{\partial W_j} > 0 \) and \( \frac{\partial p_j}{\partial a_j} > 0 \).

\[\square\]

### A.3 Proof of Lemma 2

**Lemma 2.** In steady state, assuming that firms face a hiring cost function, if the offer of employment value distributes according to a decreasing p.d.f. in equilibrium. The FOCs are necessary and sufficient if \( (1 - J_j f_j^{\nu'}) > \frac{1}{\chi} \) for all \( W_j \). Additionally, if \( -J_j f_j^{\nu'} > 1 \) for all \( W_j \), the equilibrium is rank-preserving.

**Proof.** Using equation (62), notice that the second order conditions are given by:

\[
\Pi_{h_j,h_j} = -\chi \kappa h_j^{\chi - 1} \tag{132}
\]

\[
\Pi_{W_j,W_j} = -E \left[ Q_t(1 - \delta_nt)\bar{iq}f_j^{v}n_j + J_tJ_{jt+1}f_j^{v'}n_j \right] + E \left[ Q_tJ_{jt+1}f_j^{v'}n_j^{\epsilon - 1} \right] \tag{133}
\]

\[
\Pi_{h_j,W_j} = -E \left[ Q_t \right] \tag{134}
\]

Hence, in steady state:

\[
\Pi_{h_j,h_j} = -\chi \kappa h_j^{\chi - 1} \tag{135}
\]

\[
\Pi_{W_j,W_j} = -\beta(1 - \delta_n)\bar{iq}f_j^{v}n_j + J_j(1 - \delta_n)\bar{iq}f_j^{v'}n_j \tag{136}
\]

\[
\Pi_{h_j,W_j} = -\beta \tag{137}
\]

The second order conditions with respect to \( h_j \) is always negative. On the other hand, from (136):

\[
\Pi_{W_j,W_j} = -\beta(1 - \delta_n)\bar{iq}f_j^{v}n_j + J_j(1 - \delta_n)\bar{iq}f_j^{v'}n_j \tag{138}
\]

\[
\Pi_{W_j,W_j} = \beta(1 - \delta_n)\bar{iq}f_j^{v}n_j \left[ -1 + J_j \frac{f_j^{v'}}{f_j^{v}} \right] \tag{139}
\]

40
Hence, $\Pi_{W_j, W_j}$ is negative in steady state if $- \frac{J_j f_j'}{f_j^v} > -1$, which is satisfied given the assumption that $- \frac{J_j f_j'}{f_j^v} > 1$. Now, using the FOCs and SOCs in steady state:

$$\Pi_{h_j, h_j} \Pi_{W_j, W_j} - \Pi_{h_j, W_j}^2 = -\chi \kappa \beta (1 - \delta_n) i q f_j^n n_j \left[-1 + \frac{J_j f_j'}{f_j^v} \right] - \beta^2$$

(140)

$$= -\chi \frac{\beta J_j}{h_j} \frac{h_j}{J_j} \left[-1 + \frac{J_j f_j'}{f_j^v} \right] - \beta^2$$

(141)

$$= -\chi \beta^2 \left[-1 + \frac{J_j f_j'}{f_j^v} \right] - \beta^2$$

(142)

Hence, $\left(\Pi_{h_j, h_j} \Pi_{W_j, W_j} - \Pi_{h_j, W_j}^2\right) > 0$ in steady state if $\left(1 - \frac{J_j f_j'}{f_j^v}\right) > \frac{1}{\chi}$. Now, to show verify that the equilibrium is rank-preserving, first notice that in steady state:

$$J_j = \frac{\left[1 - (1 - \delta_n)(1 - i q G_j)\right]}{(1 - \delta_n) i q f_j^n}$$

(143)

$$\frac{\partial J_j}{\partial W_j} = -1 - \frac{J_j f_j'}{f_j^v}$$

(144)

Notice that $\frac{\partial J_j}{\partial W_j}$ is positive by assumption ($- \frac{J_j f_j'}{f_j^v} > 1$). Therefore, in steady state:

$$h_j = \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} J_j^{\frac{1}{\kappa}}$$

(145)

$$\frac{\partial h_j}{\partial W_j} = \frac{1}{\chi} \left(\frac{\beta}{\kappa}\right)^{\frac{1}{\chi}} J_j^{\frac{1}{\kappa} - 1} \frac{\partial J_j}{\partial W_j} > 0$$

(146)

To see that larger firms offer higher employment values notice that:

$$n_j = \left(\frac{n_j}{h_j}\right) h_j$$

(147)

$$\frac{\partial n_j}{\partial W_j} = \frac{\partial}{\partial W_j} \left(\frac{n_j}{h_j}\right) h_j + \left(\frac{n_j}{h_j}\right) \frac{\partial h_j}{\partial W_j} > 0$$

(148)

which follows from equation (148) and Lemma 3. Finally, to see that employment values are
increasing in firms productivity, substitute (144) in (76):

\[
\frac{\partial p_j}{\partial W_j} = -J_j \beta (1 - \delta_s) \tilde{q} f_j - J_j \frac{f_j}{f_j} [1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)] + \epsilon \Psi \frac{\tilde{n}^{\eta - \epsilon}}{c} n_j^{\epsilon - 1} \frac{\partial n_j}{\partial W_j} (149)
\]

\[
= -\left[1 - (1 - \delta_n) (1 - \tilde{q} G_j)\right] \frac{1}{(1 - \delta_n) \tilde{q} f_j} \beta (1 - \delta_s) \tilde{q} f_j - J_j \frac{f_j}{f_j} [1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)] + \epsilon \Psi \frac{\tilde{n}^{\eta - \epsilon}}{c} n_j^{\epsilon - 1} \frac{\partial n_j}{\partial W_j} (150)
\]

\[
= -\left[1 - (1 - \delta_n) (1 - \tilde{q} G_j)\right] - J_j \frac{f_j}{f_j} [1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)] + \epsilon \Psi \frac{\tilde{n}^{\eta - \epsilon}}{c} n_j^{\epsilon - 1} \frac{\partial n_j}{\partial W_j} (151)
\]

As a consequence, \( \frac{\partial p_j}{\partial W_j} > 0 \), for any value of \( \epsilon \), if:

\[
- J_j \frac{f_j}{f_j} [1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)] - [1 - (1 - \delta_n) (1 - \tilde{q} G_j)] \beta > 0 (152)
\]

\[
- J_j \frac{f_j}{f_j} [1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)] > [1 - (1 - \delta_n) (1 - \tilde{q} G_j)] \beta (153)
\]

\[
- J_j \frac{f_j}{f_j} > \frac{[1 - (1 - \delta_n) (1 - \tilde{q} G_j)]}{[1 - \beta (1 - \delta_s) (1 - \tilde{q} G_j)]} \beta < 1 \times < 1 (154)
\]

Hence, \( \frac{\partial p_j}{\partial W_j} > 0 \) if \( - J_j \frac{f_j}{f_j} > 1 \).

\[\square\]

### A.4 Proof Corollary 1

**Corollary 1.** If the offer of employment value distributes according to: \( f_j^v = \frac{1}{\sigma_w} e^{-\frac{w_j}{\sigma_w}} \) in steady state, \( \chi > 0.5 \), and firms face a hiring cost function, The FOCs are necessary and sufficient and the equilibrium is rank-preserving.

**Proof.** Notice that:

\[
f_j^v = \frac{1}{\sigma_w} e^{-\frac{w_j}{\sigma_w}} (155)
\]

\[
f_j^{v'} = \frac{1}{\sigma_w} f_j^v (156)
\]

\[
G_j = \sigma_w f_j^v (157)
\]
Hence, in steady state:

\[ J_j = \frac{[1 - (1 - \delta_n)(1 - \bar{q} G_j)]}{(1 - \delta_n)iq f_j} \]  

(158)

\[ J_j = \frac{\delta_n}{(1 - \delta_n)iq f_j} + \sigma_W \]  

(159)

\[-J_j \frac{f_j''}{f_j} = \frac{\delta_n}{(1 - \delta_n)iq f_j \sigma_W} + 1 \]  

(160)

\[-J_j \frac{f_j''}{f_j} > 1 \]  

(161)

Given that \( f_j'' < 0 \), \(-J_j \frac{f_j''}{f_j} \) is increasing in \( W_j \):

\[-J_j \frac{f_j''}{f_j} = \frac{\delta_n}{(1 - \delta_n)iq f_j} e^{-\frac{w}{\sigma_W} \sigma_W} \]  

(162)

\[-J_j \frac{f_j''}{f_j} \geq \frac{\delta_n}{(1 - \delta_n)iq} + 1 \]  

(163)

\[-J_j \frac{f_j''}{f_j} > 1 \]  

(164)

\[ 1 - J_j \frac{f_j''}{f_j} > 2 \]  

(165)

\[ 1 - J_j \frac{f_j''}{f_j} > \frac{1}{\chi} \]  

(166)

Given equations (161) and (166), the conditions stated in Lemma 2 as satisfied.

\( \square \)

A.5 Proof of Proposition 2

Proposition 2. Suppose that \( \{r_t^{(1)}, u_t^{(1)}, y_t^{(1)}, c_t^{(1)}, k_t^{(1)}, n_t^{(1)}, v_t^{(1)}, q_t^{(1)}, \tilde{q}_t^{(1)}, v_{jt}^{(1)}, W_{jt+1}^{(1)}, J_{jt}^{(1)}, n_{jt+1}^{(1)}\} \) is a rank-preserving competitive search equilibrium with wage posting given a sequence of exogenous shocks \( \{a_t^{(1)}, \delta_n^{(1)}, \bar{\tilde{q}}_t^{(1)}\} \). Now, suppose that \( \{r_t^{(2)}, w_t^{(2)}, y_t^{(2)}, c_t^{(2)}, k_t^{(2)}, u_t^{(2)}, n_t^{(2)}, v_t^{(2)}, q_t^{(2)}, \tilde{q}_t^{(2)}, v_{jt}^{(2)}, W_{jt+1}^{(2)}, J_{jt}^{(2)}, n_{jt+1}^{(2)}\} \) is a rank-preserving competitive search equilibrium with wage posting given a sequence of exogenous shocks \( \{a_t^{(2)}, \delta_n^{(1)}, \bar{\tilde{q}}_t^{(1)}\} \) i.e. the aggregate productivity is different in both equilibriums while the remaining exogenous variables are the same. If \( \sigma = 0 \), \( \phi = 0 \) and aggregate shocks are small. Then

- \( u_t^{(1)} = u_t^{(2)}, n_t^{(1)} = n_t^{(2)}, v_t^{(1)} = v_t^{(2)}, q_t^{(1)} = q_t^{(2)}, \tilde{q}_t^{(1)} = \tilde{q}_t^{(2)} \) for all \( t \).

- \( J_{jt}^{(1)} = J_{jt}^{(2)}, v_{jt}^{(1)} = v_{jt}^{(2)}, n_{jt+1}^{(1)} = n_{jt+1}^{(2)} \) for all \( t \) and for all \( j > 0 \).
Proof. This proof follow the proof of Proposition 1. Just notice that the only difference is the first order condition with respect to $w_{jt+1}$, which doesn’t make any change to this proof. □