Optimal Public Debt with Life Cycle Motives*

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Abstract

Public debt can be optimal in standard incomplete market models with infinitely lived agents, since capital crowd-out induces a higher interest rate that encourages agents to hold more savings which provides insurance against idiosyncratic labor risk. Although public debt is optimal due to households’ savings patterns, this class of economies abstracts from the potentially important life cycle savings pattern. This paper incorporates a life cycle into the incomplete markets model and finds that optimal policy changes from public debt equal to 24% of output in the infinitely lived agent model to public savings equal to 59% of output in the life cycle model. Even though a higher level of public debt similarly encourages life cycle agents to hold more savings during their lifetimes, the act of accumulating this savings mitigates the welfare benefit from public debt. Moreover, public savings improves life cycle agents’ welfare by encouraging a flatter lifetime allocation of consumption and leisure. Abstracting from the life cycle when computing optimal policy reduces average welfare by at least 0.5% of expected lifetime consumption. Our results demonstrate that studying optimal debt policy in an infinitely lived agent model, which abstracts from the realism of a life cycle in order to render models more computationally tractable, is not without loss of generality.

Keywords: Government Debt; Life Cycle; Heterogeneous Agents; Incomplete Markets
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1 Introduction

Motivated by the prevalence of government borrowing across advanced economies, previous work demonstrates that government debt can be optimal in a standard incomplete markets model with infinitely lived agents. For example, in their seminal work, Aiyagari and McGrattan (1998) find that a large quantity of public debt is optimal when such an economic environment is calibrated to the U.S. economy. Public debt is optimal because it crowds out the stock of productive capital and leads to a higher interest rate that encourages households to save more. As a result, households are better self-insured against idiosyncratic labor earnings risk and, therefore, are less likely to be liquidity constrained. While household savings behavior is central to public debt being optimal, previous work largely examines optimal policy in economies inhabited by infinitely lived agents. Such economic environments abstract from empirically relevant life cycle characteristics that influence savings decisions and that can, therefore, affect optimal debt policy.

This paper characterizes the effect of a life cycle on optimal public debt and inspects the mechanisms through which a life cycle affects optimal policy. In order to determine the effect of the life cycle, we contrast optimal policy in two model economies: (i) the standard incomplete markets model with infinitely lived agents, and (ii) a life cycle model. We find that the optimal policies are strikingly different between the two models. In the infinitely lived agent model, it is optimal for the government to be a net borrower with public debt equal to 24 percent of output. In contrast, in the life cycle model, we find that it is optimal for the government to be a net saver, not a net borrower, with public savings equal to 59 percent of output.

Our results demonstrate that studying optimal policy in an infinitely lived agent model, which abstracts from the realism of a life cycle in order to render models more computationally tractable, is not without loss of generality. Not only is the optimal policy quite different when one ignores life cycle features, but the welfare consequences of ignoring them are economically significant. In the life cycle model, we find that if a government is a net borrower (as is optimal in the infinitely lived agent model) instead of being a net saver (as is optimal in the life cycle model), then an average life cycle agent would be worse off by 0.5 percent of expected lifetime consumption.

Two competing mechanisms generate the different optimal policies between the two models. The first, and quantitatively dominant, mechanism is the existence of a particular progression of individual savings, consumption, and labor through the life cycle that is absent in the infinitely lived agent model. In the infinitely lived
agent model, public debt induces a higher interest rate that encourages agents to save. As a result, agents will live in an economy that has more private savings on average, which improves agents’ self-insurance against idiosyncratic labor earnings risk. Although public debt also increases the average amount of savings in the life cycle model, agents enter the economy with little or no wealth and must accumulate savings over their lifetimes. Thus, public debt has a smaller welfare benefit in the life cycle model because agents only experience improved self-insurance after they have accumulated savings. Moreover, in the life cycle model, public debt can increase welfare by leading agents to allocate their consumption differently over their lifetime. In particular, agents tend to increase their consumption over most of their lifetime since the interest rate is sufficiently large relative to their effective discounting of future utility, from both impatience and mortality risk. Abstracting from changes to the level of total lifetime consumption, a level of public debt (or public savings) that results in agents allocating consumption more equally throughout their lifetimes maximizes expected lifetime utility. Thus, public savings can improve welfare in the life cycle model by inducing a lower interest rate that reduces the increase in consumption over the lifetime.

Income inequality is a competing, but quantitatively smaller mechanism, that reduces the divergence between the two models’ optimal policies. Underlying this mechanism are three relationships: (i) increasing or decreasing public debt moves the interest rate and wage in opposite directions, (ii) income inequality is due to inequality in both asset and labor income, and (iii) generally, asset income inequality increases with the interest rate while labor income inequality increases with the wage. Put together, these three relationships imply that optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other. Comparing the two models, the infinitely lived agent model features a larger ratio of asset income inequality to labor income inequality compared to the life cycle model. Accordingly, in the infinitely lived agent model, a reduction in public debt improves welfare by lowering the return to saving and decreasing asset income inequality. Conversely, in the life cycle model, a decrease in public savings improves welfare by lowering the return to labor and decreasing labor income inequality. De-

1Over most of the lifetime mortality rates are low enough compared to interest rates that consumption rises. However, towards the end of the lifetime the increase in mortality rates implies that consumption falls.

2Asset income inequality is smaller in the life cycle model due to a finite lifetime. As agents live longer, there are more opportunities for agents to receive a long string of positive or negative labor productivity shocks, which generate greater dispersion in savings between lucky and unlucky agents. In the life cycle model, the finite lifetime means there are fewer opportunities for labor productivity shocks to propagate into the wealth distribution and create more variance in wealth.
spite its countervailing effect, we find that the inequality channel is quantitatively smaller than the effect on optimal policy from the particular progression of consumption, labor, and savings through the life cycle. Thus, public savings is optimal in the life cycle model and public debt is optimal in the infinitely lived agent model.

This paper is related to an established literature that uses the standard incomplete market model with infinitely lived agents, originally developed in Bewley (1986), İmrohoroğlu (1989), Huggett (1993) Aiyagari (1994) and others, to study the optimal level of steady state government debt. In contrast to this paper, previous work has mostly utilized infinitely lived agent models and finds that public debt is optimal. Aiyagari and McGrattan (1998) is the seminal contribution to the study of optimal debt in the standard incomplete market model, and finds that public debt is optimal in an economy calibrated to resemble the U.S. Floden (2001) finds that increasing government debt can provide welfare benefits if transfers are below optimal levels. Similarly, Dyrda and Pedroni (2016) find that it is optimal for the government to be a net borrower. However, they find that optimizing both taxes and debt at the same time leads to a smaller level of optimal debt than do previous studies. Relative to these papers, we focus on how optimal policy changes when one considers a life cycle model as opposed to an infinitely lived agent model, and find that including life cycle features has large effects on optimal policy.\footnote{Using infinitely lived agent models, Desbonnet and Weitzenblum (2012), Açıkgöz (2015), Dyrda and Pedroni (2016), Röhrs and Winter (2017) find quantitatively large welfare costs of transitioning between steady states after a change in public debt. Moreover, Heinemann and Wulff (2017) demonstrate that debt-financed government stimulus after an aggregate shock can be welfare improving. We do not consider these transitional costs and instead focus on steady state comparisons to more sharply highlight the effect of the life cycle on optimal debt policy.}

Using variants of incomplete market models, Röhrs and Winter (2017) and Vogel (2014) also find that it can be optimal for the government to be a net saver. In both papers, the government’s desire for redistribution partially explains the optimality of public savings, as public savings leads to a lower interest rate and therefore redistributes welfare from wealth-rich agents to wealth-poor agents.\footnote{This motive to redistribute is enhanced in both of these papers since the models are calibrated to match the upper tail of the U.S. wealth distribution, which leads to a small mass of wealth-rich agents and a larger mass of wealth-poor agents.} This paper finds that the redistribution motive affects optimal policy. Yet, we find that the redistribution motive pushes optimal policy in opposing directions, toward less public saving in the life cycle model and toward less public debt in the infinitely lived agent model.\footnote{Specifically, we find that the ratio of asset income inequality relative to lifetime labor income inequality increases with the length of the lifetime (see Dávila et al. (2012) for discussion). Thus, in the infinitely lived agent model, there is a stronger desire for the government to reduce lifetime interest income inequality which they can accomplish through public savings which lowers the interest rate.} How-
ever, we find that the existence of the accumulation phase and the desire for public policy to induce agents to more equally allocate their consumption over their lifetime are the quantitatively dominant, leading to the optimality of public savings in the life cycle model and the optimality of public debt in the infinitely lived agent model.\textsuperscript{6}

This paper is also related to a strand of literature that examines the effects of life cycle features on optimal fiscal policy but generally focuses on taxation instead of government debt. For example, Garriga (2001), Erosa and Gervais (2002) and Conesa et al. (2009) show that introducing a life cycle creates a motive for positive capital taxation, in contrast to the seminal findings of Judd (1985) and Chamley (1986) that optimal capital taxation is zero in the long-run of a class of infinitely lived agent models.\textsuperscript{7} With a life cycle, if age-dependent taxation is not feasible then a positive capital tax may be optimal since it can mimic an age-dependent tax on labor income. Instead of focusing on optimal taxation in a life cycle model, this paper quantifies the effects of life cycle features on optimal government debt.\textsuperscript{8} We find that introducing life cycle features changes optimal policy from public debt to public savings because agents must accumulate savings at the beginning of their lifetimes, not because the government would like to mimic age-dependent policy.

Finally, this paper is related to Dávila, Hong, Krusell, and Ríos-Rull (2012), whose work defines constrained efficiency in a standard incomplete markets model with infinitely lived agents. Constrained efficient allocations account for the effect of individual behavior on market clearing prices, while satisfying individuals’ constraints. The authors show that the price system in the standard incomplete market model does not efficiently allocate resources across agents, and welfare improving equilibrium prices could be attained if agents were to systematically deviate from individually optimal savings and consumption decisions. While this paper does not characterize constrained efficient allocations, this paper’s Ramsey allocation improves welfare for similar reasons: since it understands the relationship between public debt and prices, in contrast, the life cycle model, there is more desire for the government to reduce lifetime labor income inequality, which it can accomplish by increasing public debt and thereby lowering the wage.\textsuperscript{6}

In characterizing optimal public debt, this paper additionally abstracts from aggregate uncertainty (i.e., Barro (1979), Lucas and Stokey (1983), Aiyagari, Marcet, Sargent, and Seppälä (2002), Shin (2006)), political economy distortions (i.e., Alesina and Tabellini (1990), Battaglini and Coate (2008), and Song, Storesletten, and Zilibotti (2012)) and international capital flows (i.e., Azzimonti, de Francisco, and Quadrini (2014)).

\textsuperscript{7}In addition, Aiyagari (1995) and İmrohoroglu (1998) demonstrate that incomplete markets can overturn the zero capital tax result with uninsurable earnings shocks and sufficiently tight borrowing constraints.

\textsuperscript{8}Instead of isolating the effects of life cycle features on optimal debt, Garriga (2001) allows the government to choose sequences for taxes (capital, labor and consumption) as well as government debt. In contrast, our paper explicitly measures how including life cycle features alters optimal debt policy while holding other fiscal instruments constant.
the government can implement a welfare improving allocation that individual agents
cannot attain through private markets. As a result of this common mechanism, both
of our papers find that a higher capital stock improves welfare. However, Dávila
et al. (2012) obtains this result through matching top wealth inequality in an infinitely
lived agent model, while our paper does so through adding life cycle features. In
the life cycle model, the accumulation phase mitigates the welfare benefits of public
debt and the benefit of public savings inducing agents to more equally allocate their
consumption over their lifetime leads to the optimality of public savings.

The remainder of this paper is organized as follows. Section 2 illustrates the un-
derlying mechanisms by which optimal government policy interacts with life cycle
and infinitely lived agent model features. Section 3 describes the life cycle and in-
finity lived agent model environments and defines equilibrium. Section 4 explains
the calibration strategy, Section 5 presents quantitative results and Section 6 performs
robustness exercises. Section 7 concludes.

2 Illustration of the Mechanisms

In this section, we illustrate the mechanisms that lead the government to an optimal
public debt or savings policy. We discuss why optimal government policy may differ
in the life cycle and infinitely lived agent models.

2.1 Life Cycle Phases

In order to highlight how the life cycle may impact optimal debt policy, it will be use-
ful to describe classify agents‘ behavior over their life cycle in three different phases.
Agents enter the economy with little or no wealth and begin the accumulation phase,
which is characterized by the accumulation of wealth for precautionary motives and to
finance post-retirement consumption. While accumulating a stock of savings, agents
tend to work more and consume less.

Once wealth provides sufficient insurance against labor productivity shocks, these
agents have entered the stationary phase. This phase is characterized by savings, hours
and consumption that remain constant on average.

9The stationary level of average savings is related to the “target savings level” in Carroll (1992, 1997).
Given the primitives of the economy, an agent faces a tradeoff between consumption levels and con-
sumption smoothing. The agent targets a level of savings that provides sufficient insurance while
maximizing expected consumption.

10However, underlying constant averages for the cohort are individual agents who respond to shocks
by choosing different allocations, thereby moving about various states within a non-degenerate dis-
Finally, agents enter the deaccumulation phase as they approach the end of their lives. In order to smooth consumption in the final periods of their lives, agents attempt to deaccumulate assets so that they are not forced to consume a large quantity immediately preceding death. Furthermore, with few periods of life remaining, agents no longer want to hold as much savings for precautionary reasons. Thus, the average level of savings and labor supply decreases, while consumption increases slightly.

In comparison, infinitely lived agents only experience the equivalent of a stationary phase. On average, infinitely lived agents’ consumption, hours and savings allocations remain constant.

2.2 Welfare Channels and Life Cycle Features

We identify four main channels through which public debt policy affects welfare: the direct effect, the insurance channel, the inequality channel and the age-allocation channel. We heuristically characterize how these channels affect optimal policy, and how these channels’ effects can differ in the life cycle and infinitely lived agent models.

Direct Effect: The direct effect is the partial equilibrium change in the productive capital stock, aggregate consumption and aggregate output with respect to a change in public debt, when holding constant the aggregate labor supply and aggregate private savings. Mechanically, increased public debt crowds out (e.g., decreases) productive capital, thereby generating less output and decreasing aggregate consumption. Generally, decreased aggregate consumption reduces welfare, which causes this mechanism to push optimal policy toward public savings. Absent any general equilibrium effects, this mechanism should operate similarly in both the life cycle and infinitely lived agent economies.

Indirect Effects: While the direct effect is a partial equilibrium effect of policy on aggregate resources, the remaining channels affect welfare in general equilibrium, that is, by impacting market clearing prices. In particular, decreasing public savings or increasing public debt will crowd out productive capital and lead to an increase in the market clearing interest rate and reduction in the market clearing wage rate.
An increase in the interest rate encourages agents to save. The higher level of savings improves welfare because agents are less likely to face binding liquidity constraints and are, therefore, better insured against labor earnings risk. We refer to this channel as the *insurance channel*.

The insurance channel’s welfare benefit varies substantially across the life cycle and infinitely lived agent models. In the infinitely lived agent model, agents exist in a perpetual stationary phase. This implies that public debt causes agents to live in an economy in which they have more private savings on average. Thus, increased public debt improves insurance for the average agent because he lives with more *ex ante* savings. In the life cycle model, in contrast, agents enter the economy with little or no wealth and immediately begin the accumulation phase.\(^{12}\) While increased public debt may encourage agents to save more over their lifetime, agents need to accumulate this savings in the first place which reduces the benefit from public debt.\(^{13}\)

Second, changes in the interest rate lead agents to choose to allocate their consumption differently over their lifetimes. In a standard consumption-savings problem, abstracting from changes to the level of total lifetime consumption, public debt or public savings policy that results in agents allocating consumption more equally throughout their lifetimes maximizes expected lifetime utility.\(^{14}\) The lower interest rate associated with public savings will lead agents to prefer using more resources for consumption today as opposed to saving them for consumption at a later age. Thus, if consumption tends to increase over the lifetime then the lower interest rate associated with public savings will induce a flatter consumption profile and therefore increase welfare. Similarly, if consumption tends to decline over the lifetime then public savings and the associated higher interest rate would lead to steeper fall in consumption and reduce welfare. We refer to this channel as the *age-allocation channel*. The age-allocation channel only exists in the life cycle model since there is no meaningful concept of age in the infinitely lived agent model.

\(^{12}\)If life cycle features were introduced in a dynastic model, instead of a life cycle model, where old agents bequeath wealth to agents entering the economy, then the accumulation phase may be more responsive to public policy. Consistent with Fuster, İmrohoğlu, and İmrohoğlu (2008), the optimal policy differences with the infinitely lived agent model could be smaller since agents would receive some initial wealth through bequests.

\(^{13}\)Savings accumulation mitigates the welfare benefit from the insurance for two reasons. First, life cycle agents only realize the insurance benefit from precautionary savings after accumulating that savings. Second, although the higher interest rate associated with public debt may encourage agents to accumulate a higher level of savings by the time they enter the stationary phase, agents will need to work more and consume less during the accumulation phase in order to reach a higher level of stationary savings.

\(^{14}\)Similarly, changes in the interest rate can change how agents choose to allocate their leisure and labor over their lifetime and a smooth allocation of labor/leisure will maximize utility. However, for simplicity, we choose to focus on describing the intuition with the allocation of consumption.
The final indirect channel describes the welfare effect of income inequality arising from price changes. Income inequality is composed of both asset and labor income inequality and the amount of inequality from each source increases with each source’s return. Since changing public debt has opposite effects on the wage and interest rate, optimal policy trades-off decreasing income inequality from one income source with increasing income inequality from the other. Therefore, the optimal tradeoff depends on the relative amount of inequality that arises from each source of income. We refer to this channel as the *inequality channel*.

Since the relative inequality deriving from labor income and asset income varies across the two models, so too will the optimal policy tradeoff. As demonstrated in Dávila, Hong, Krusell, and Ríos-Rull (2012), inequality depends on agents’ lifespan. As agents live longer, lifetime labor income inequality increases because there is a greater chance that agents receive a long string of either positive or negative labor productivity shocks. However, asset income inequality will also develop because agents reduce (increase) their wealth in response to a string of negative (positive) shocks. Generally, as each agent’s lifespan increases, asset income inequality increases more than labor income inequality. Accordingly, the ratio of asset income inequality to labor income inequality is larger in the infinitely lived agent model, and smaller in the life cycle model. Therefore, the inequality channel pushes optimal policy in the life cycle model toward more public debt (less public savings) and pushes optimal policy in the infinitely lived agent model toward more public savings (less public debt).

Overall, given the competing mechanisms that may affect welfare, it is a quantitative issue whether public debt or public savings is optimal in either model. In particular, higher public debt lowers welfare through the direct effect and raises welfare through the insurance channel, while the effect through age-allocation channel is uncertain. Furthermore, the inequality channel’s effect differs across models, pushing the life cycle model’s optimal policy toward public debt and the infinitely lived agent model’s optimal policy toward public savings. Likewise, the insurance channel is weaker in the life cycle model than in the infinitely lived agent model because the existence of the accumulation phase, while the infinitely lived agent model does not have an age-allocation effect. Thus, it is unclear whether introducing the life cycle will cause optimal policy to move towards more public debt or towards public savings. In the next section, we turn to a quantitative model in order to determine the relative strength of all these effects.
3 Economic Environment

In this section, we present both the life cycle model and the infinitely lived agent model. Given that there are many common features across models, we will first focus on the life cycle model in detail before providing an overview of the infinitely lived agent model.

3.1 Life Cycle Model

3.1.1 Production

We assume there exist a large number of firms that sell a single consumption good in a perfectly competitive product market, purchase inputs from perfectly competitive factor markets and each operate an identical constant returns to scale production technology, \( Y = ZF(K, L) \). These assumptions on primitives admit a representative firm that chooses capital \( K \) and labor \( L \) inputs in order to maximize profits, given an interest rate \( r \), a wage rate \( w \), a level of total factor productivity \( Z \) and capital depreciation rate \( \delta \in (0, 1) \).

3.1.2 Consumers

Demographics: Let time be discrete and let each model period represent a year. Each period, the economy is inhabited by \( J \) overlapping generations of individuals. We index agents’ age in the model by \( j = 1, \ldots, J \), where \( j = 1 \) corresponds to age 21 in the data and \( J \) is an exogenously set maximum age (set to age 100 in the data). Before age \( J \) all living agents face mortality risk. Conditional on living to age \( j \), agents have a probability \( \psi_j \) of living to age \( j + 1 \), with a terminal age probability given by \( \psi_J = 0 \). Each period a new cohort is born and the size of each successive newly born cohort grows at a constant rate \( g_n > 0 \).

Agents who die before age \( J \) may hold savings when they die since mortality is uncertain. These savings are treated as accidental bequests and are equally divided across each living agent in the form of a lump-sum transfer, denoted \( Tr \).

Preferences: Agents enjoy lifetime paths of consumption and labor, denoted \( \{c_j, h_j\}_{j=1}^J \), according to the following preferences:

\[
E_1 \sum_{j=1}^J \beta^{j-1} \psi_j \left[ u(c_j) - v(h_j, s'_j) \right]
\]
where \( \beta \) is the time discount factor. Expectations are taken with respect to the stochastic processes governing labor productivity. Furthermore, \( u(c) \) and \( v(h) \) are instantaneous utility functions over consumption and labor hours, respectively, satisfying standard conditions. Lastly, \( s'_j \) is a retirement decision that is described immediately below.

**Retirement:** Agents choose their retirement age, which is denoted by \( J_{\text{ret}} \). A retired agent cannot sell labor hours and the retirement decision is irreversible. Agents choose their retirement age in the interval \( j \in [J_{\text{ret}}, J_{\text{ret}}] \) and are forced to retire after age \( J_{\text{ret}} \). Let \( s'_j \equiv 1(j < J_{\text{ret}}) \) denote an indicator variable that equals one when an agent chooses to continue working and zero upon retirement.

**Labor Earnings:** Agents are endowed with one unit of time per period, which they split between leisure and market labor. During each period of working life, an agent’s labor earnings are \( we_jh_j \), where \( w \) is the wage rate per efficiency unit of labor, \( e_j \) is the agent’s idiosyncratic labor productivity drawn at age \( j \) and \( h_j \) is the time the agent chooses to work at age \( j \).

Following Kaplan (2012), we assume that labor productivity shocks can be decomposed into four sources:

\[
\log(e_j) = \kappa + \theta_j + v_j + \epsilon_j
\]

where (i) \( \kappa \sim \mathcal{N}(0, \sigma_\kappa^2) \) is an individual-specific fixed effect that is drawn at birth, (ii) \( \{\theta_j\}_{j=1}^J \) is an age-specific fixed effect, (iii) \( v_j \) is a persistent shock that follows an autoregressive process given by \( v_{j+1} = \rho v_j + \eta_{j+1} \) with \( \eta \sim \mathcal{N}(0, \sigma_v^2) \) and \( \eta_1 = 0 \), and (iv) \( \epsilon_j \sim \mathcal{N}(0, \sigma_\epsilon^2) \) is a per-period transitory shock.

For notational compactness, we denote the relevant state as a vector \( \varepsilon_j = (\kappa, \theta_j, v_j, \epsilon_j) \) that contains each element necessary for computing contemporaneous labor earnings, \( e_j \equiv e(\varepsilon_j) \), and forming expectations about future labor earnings. Denote the Markov process governing the process for \( \varepsilon \) by \( \pi_j(\varepsilon_{j+1}|\varepsilon_j) \) for each \( \varepsilon_j \), \( \varepsilon_{j+1} \) and for each \( j = 1, \ldots, J_{\text{ret}} \).

**Insurance:** Agents have access to a single asset, a non-contingent one-period bond denoted \( a_j \) with a market determined rate of return of \( r \). Agents may take on a net debt position, in which case they are subject to a borrowing constraint that requires their debt position be bounded below by \( a \in \mathbb{R} \). Agents are endowed with zero initial wealth, such that \( a_1 = 0 \) for each agent.
3.1.3 Government Policy

The government (i) consumes an exogenous amount of resources $G$, (ii) collects linear Social Security taxes $\tau_{ss}$ on all pre-tax labor income below an amount $\bar{m}$, (iii) distributes lump-sum Social Security payments $b_{ss}$ to retired agents, (iv) distributes accidental bequests as lump-sum transfers $Tr$, and (v) collects income taxes from each individual.

Social Security: The model’s Social Security system consists of taxes and payments. The social security payroll tax is given by $\tau_{ss}$ with a per-period cap denoted by $\bar{m}$. We assume that half of the social security contributions are paid by the employee and half by the employer. Therefore, the consumer pays a payroll tax given by: 

\[
(\frac{1}{2}) \tau_{ss} \min\{we, \bar{m}\}.
\]

Social security payments are computed using the averaged indexed monthly earnings (AIME) that summarizes an agents lifetime labor earnings. Following Huggett and Parra (2010) and Kitao (2014), the AIME is denoted by $\{m_j\}_{j=1}^J$ and is given by:

\[
m_{j+1} = \begin{cases} 
\frac{1}{j} \left( \min\{we, h_j, \bar{m}\} + (j - 1)m_j \right) & \text{for } j \leq 35 \\
\max \left\{ m_j, \frac{1}{j} \left( \min\{we, h_j, \bar{m}\} + (j - 1)m_j \right) \right\} & \text{for } j \in (35, J_{ret}) \\
m_j & \text{for } j \geq J_{ret} 
\end{cases}
\]

The AIME is a state variable for determining future benefits. Benefits consists of a base payment and an adjusted final payment. The base payment, denoted by $b_{ss}^{base}(m_{J_{ret}})$, is computed as a piecewise-linear function over the individual’s average labor earnings at retirement $m_{J_{ret}}$:

\[
b_{ss}^{base}(m_{J_{ret}}) = \begin{cases} 
\tau_1 m_{J_{ret}} & \text{for } m_{J_{ret}} \in [0, b_1^{ss}) \\
\tau_1 b_1^{ss} + \tau_2 (m_{J_{ret}} - b_1^{ss}) & \text{for } m_{J_{ret}} \in [b_1^{ss}, b_2^{ss}) \\
\tau_1 b_1^{ss} + \tau_2 b_2^{ss} + \tau_3 (m_{J_{ret}} - b_1^{ss} - b_2^{ss}) & \text{for } m_{J_{ret}} \in [b_2^{ss}, b_3^{ss}) \\
\tau_1 b_1^{ss} + \tau_2 b_2^{ss} + \tau_3 b_3^{ss} & \text{for } m_{J_{ret}} \geq b_3^{ss} 
\end{cases}
\]

Lastly, the final payment requires an adjustment that penalizes early retirement and
credits delayed retirement. The adjustment is given by:

\[
b_{ss}(m_{j_{ret}}) = \begin{cases} 
(1 - D_1(J_{nra} - j_{ret}))b_{base}^{ss}(m_{j_{ret}}) & \text{for } J_{ret} \leq j_{ret} < J_{nra} \\
(1 + D_2(j_{ret} - J_{nra}))b_{base}^{ss}(m_{j_{ret}}) & \text{for } J_{nra} \leq j_{ret} \leq j_{ret} 
\end{cases}
\]

where \(D_i(\cdot)\) are functions governing the benefits penalty or credit, \(j_{ret}\) is the earliest age agents can retire, \(J_{nra}\) is the "normal retirement age" and \(j_{ret}\) is the latest age an agent can retire.

**Net Government Transfers**: Taxable income is defined as labor income and capital income net of social security contributions from an employer:

\[
y(h,a,\varepsilon,s) \equiv swe(\varepsilon)h + r(a + Tr) - s\frac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \bar{m}\}
\]

The government taxes each individual’s taxable income according to an increasing and concave function, \(\Upsilon(y(h,a,\varepsilon,s))\).

Define the function \(T(\cdot)\) as the government’s net transfers of income taxes, social security payments and social security payroll taxes to working age agents (if \(s = 1\)) and retired agents (if \(s = 0\)). Net transfers are given by:

\[
T(h,a,\varepsilon,m,s) = (1 - s)b_{ss}(m) - s\frac{\tau_{ss}}{2}\min\{we(\varepsilon)h, \bar{m}\} - \Upsilon(y(h,a,\varepsilon,s))
\]

**Public Savings and Budget Balance**: Each period, the government has a debt balance \(B\) and saves or borrows (denoted \(B'\)) at the market interest rate \(r\). If the government borrows then \(B' < 0\) and the government repays \(rB'\) next period. If the government saves then \(B' > 0\) and the government collects asset income \(rB'\) next period. The resulting government budget constraint is:

\[
G + B' - B = rB + Y_y \tag{1}
\]

where \(Y_y\) is aggregate revenues from income taxation and \(G\) is an unproductive level of government expenditures.\(^\text{15}\) The model’s Social Security system is self-financing and therefore does not appear in the governmental budget constraint.

\(^{15}\)Two recent papers, Röhrs and Winter (2017) and Chaterjee, Gibson, and Rioja (2016) have relaxed the standard Ramsey assumption that government expenditures are unproductive. Both papers show that public savings is optimal with productive government expenditures, intuitively because there is an additional benefit to aggregate output.
3.1.4 Consumer’s Problem

The agent’s state variables consist of asset holdings $a$, labor productivity shocks $\varepsilon \equiv (\kappa, \theta, v, \epsilon)$, Social Security contribution (AIME) variable $m$ and retirement status $s$. For age $j \in \{1, \ldots, J\}$, the agent’s recursive problem is:

$$V_j(a, \varepsilon, m, s) = \max_{c,a',h,s'} \left[ u(c) - v(h, s') \right] + \beta \psi_j \sum_{\varepsilon'} \pi_j(\varepsilon' | \varepsilon) V_{j+1}(a', \varepsilon', m', s')$$  \hspace{1cm} (2)

s.t. \hspace{0.5cm} $c + a' \leq s'w(\varepsilon)h + (1 + r)(a + Tr) + T(h, a, \varepsilon, m, s')$

$$a' \geq a, \hspace{1cm} s' \in \{1(j < \bar{J}_{ret}), 1(j \leq \bar{J}_{ret}) \cdot s\}$$

The indicator function $\mathbb{1}(j < \bar{J}_{ret})$ equals one when an agent is too young to retire and equals zero thereafter. Additionally $\mathbb{1}(j \leq \bar{J}_{ret})$ equals zero for all ages after an agent must retire and equals one beforehand. Therefore the agent’s recursive problem nests all three stages of life: working life, near-retirement and retirement.\footnote{During an agent’s working life (ages $j < \bar{J}_{ret}$) the agent’s choice set for retirement is $s' \in \{1, 1\}$ and therefore the agent must continue working. Near retirement (ages $\bar{J}_{ret} \leq j \leq \bar{J}_{ret}$), the agent’s choice set is $s' \in \{0, 1\}$ and the agent may retire by choosing $s' = 0$. Lastly, if an agent has retired either because he chose retirement at a previous date ($s = 0$) or because of mandatory retirement ($j > \bar{J}_{ret}$), then the choice set is $\{0, 0\}$ and $s' = s = 0$.}

3.1.5 Recursive Competitive Equilibrium

Agents are heterogeneous with respect to their age $j \in J \equiv \{1, \ldots, J\}$, wealth $a \in A$, labor productivity $\varepsilon \in E$, average lifetime earnings $m \in X$ and retirement status $s \in R \equiv \{0, 1\}$. Let $S \equiv A \times E \times X \times R$ be the state space and $B(S)$ be the Borel $\sigma$-algebra on $S$. Let $\mathcal{M}$ be the set of probability measures on $(S, B(S))$. Then $(S, B(S), \lambda_j)$ is a probability space in which $\lambda_j(S) \in \mathcal{M}$ is a probability measure defined on subsets of the state space, $S \in B(S)$, that describes the distribution of individual states across age-$j$ agents. Denote the fraction of the population that is age $j \in J$ by $\mu_j$. For each set $S \in B(S)$, $\mu_j \lambda_j(S)$ is the fraction of age $j \in J$ and type $S \in S$ agents in the economy. We can now define a recursive competitive equilibrium of the economy.

**Definition (Equilibrium):** Given a government policy $(G, B, B', Y, \tau_{ss}, b_{ss})$, a stationary recursive competitive equilibrium is (i) an allocation for consumers described by policy functions $\{c_j, a_j', h_j, s_j'\}_{j=1}^J$ and consumer value function $\{V_j\}_{j=1}^J$, (ii) an allocation for the representative firm $(K, L)$, (iii) prices $(w, r)$, (iv) accidental bequests $Tr$, and (v)
distributions over agents’ state vector at each age \( \{\lambda_j\}_{j=1}^J \) that satisfy:

(a) Given prices, policies and accidental bequests, \( V_j(a, \epsilon, m, s) \) solves the Bellman equation (2) with associated policy functions \( c_j(a, \epsilon, m, s) \), \( a'_j(a, \epsilon, m, s) \), \( h_j(a, \epsilon, m, s) \) and \( s'_j(a, \epsilon, m, s) \).

(b) Given prices \((w, r)\), the representative firm’s allocation minimizes cost: 
\[
r = ZF_K(K, L) - \delta \quad \text{and} \quad w = ZF_L(K, L)
\]

(c) Accidental bequests, \(Tr\), from agents who die at the end of this period are distributed equally across next period’s living agents:
\[
(1 + g_n)Tr = \sum_{j=1}^{J} (1 - s_j) \mu_j \int a'_j(a, \epsilon, m, s)d\lambda_j(a, \epsilon, m, s)
\]

(d) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:
\[
Y_y \equiv \sum_{j=1}^{J} \mu_j \int Y \left( y(h_j(a, \epsilon, m, s), a, \epsilon, s'_j(a, \epsilon, m, s)) \right) d\lambda_j(a, \epsilon, m, s)
\]

(e) Social security is self-financing:
\[
\sum_{j=1}^{J} \mu_j \int s'_j(a, \epsilon, m, s) \tau_{ss} \min \{we(\epsilon)h_j(a, \epsilon, m, s), \bar{m} \} d\lambda_j(a, \epsilon, m, s)
\]
\[
= \sum_{j=1}^{J} \mu_j \int (1 - s'_j(a, \epsilon, m, s)) b_{ss}(m) d\lambda_j(a, \epsilon, m, s) \quad (3)
\]

(f) Given policies and allocations, prices clear asset and labor markets:
\[
K - B = \sum_{j=1}^{J} \mu_j \int a d\lambda_j(a, \epsilon, m, s)
\]
\[
L = \sum_{j=1}^{J} \mu_j \int s'_j(a, \epsilon, m, s) e(\epsilon) h_j(a, \epsilon, m, s) d\lambda_j(a, \epsilon, m, s)
\]
and the allocation satisfies the resource constraint (guaranteed by Walras’ Law):

\[ \sum_{j=1}^{J} \mu_j \int c_j(a, \varepsilon, m, s) d\lambda_j(a, \varepsilon, m, s) + G + K' =ZF(K, L) + (1 - \delta)K \]

(g) Given consumer policy functions, distributions across age \( j \) agents \( \{\lambda_j\}_{j=1}^{J} \) are given recursively from the law of motion \( T_j^* : M \to M \) for all \( j \in J \) such that \( T_j^* \) is given by:

\[ \lambda_{j+1}(A \times E \times X \times R) = \sum_{s \in \{0,1\}} \int_{A \times E \times X \times R} Q_j((a, \varepsilon, m, s), A \times E \times X \times R) d\lambda_j \]

where \( S \equiv A \times E \times X \times R \subset S \), and \( Q_j : S \times B(S) \to [0,1] \) is a transition function on \((S, B(S))\) that gives the probability that an age-\( j \) agent with current state \( s \equiv (a, \varepsilon, m, s) \) transits to the set \( S \subset S \) at age \( j + 1 \). The transition function is given by:

\[
Q_j((a, \varepsilon, m, s), S) = \begin{cases} 
  s_j \cdot \pi_j(E|\varepsilon)^s & \text{if } a'_j(s) \in A, m'_j(s) \in X, s'_j(s) \in R \\
  0 & \text{otherwise}
\end{cases}
\]

where agents that continue working and transition to set \( E \) choose \( s'_j(s) = 1 \), while agents that transition from working life to retirement choose \( s'_j(s) = 0 \). For \( j = 1 \), the distribution \( \lambda_j \) reflects the invariant distribution \( \pi_{ss}(\varepsilon) \) of initial labor productivity over \( \varepsilon = (\kappa, \theta_1, 0, \epsilon_1) \).

(h) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that \( K' = K, B' = B, w' = w, r' = r, \) and \( \lambda'_j = \lambda_j \) for all \( j \in J \).

### 3.2 Infinitely Lived Agent Model

The infinitely lived agent model differs from the life cycle model in three ways. First, agents in the infinitely lived agent model have no mortality risk (\( s_j = 1 \) for all \( j \geq 1 \)) and lifetimes are infinite (\( J \to \infty \)). Second, labor productivity no longer has an age-dependent component (\( \theta_j = \bar{\theta} \) for all \( j \geq 1 \)). Lastly, there is no retirement (\( \bar{J}_{ret} \to \infty \) such that \( s_j = 1 \) for all \( j \geq 1 \)) and there is no Social Security program (\( \tau_{ss} = 0 \) and \( b_{ss}(m) = 0 \) for all \( x \)).

Accordingly, we study a stationary recursive competitive equilibrium in which the initial endowment of wealth and labor productivity shocks no longer affects in-
individual decisions and the distribution over wealth and labor productivity is time invariant.

**Definition (Equilibrium):** Given a government policy \((G, B, B', Y)\), a stationary recursive competitive equilibrium is (i) an allocation for consumers described by policy functions \((c, a', h)\) and consumer value function \(V\), (ii) an allocation for the representative firm \((K, L)\), (iii) prices \((w, r)\), and (v) a distribution over agents’ state vector \(\lambda\) that satisfy:

(a) Given prices and policies, \(V(a, \epsilon)\) solves the following Bellman equation:

\[
V(a, \epsilon) = \max_{c, a', h} \left[ u(c) - v(h) \right] + \beta \sum_{\epsilon'} \pi(\epsilon'|\epsilon) V(a', \epsilon')
\]

s.t. \(c + a' \leq \omega e(h) + (1 + r)a + Y(y(h, a, \epsilon))\)

\(a' \geq a\)

with associated policy functions \(c(a, \epsilon), a'(a, \epsilon)\) and \(h(a, \epsilon)\).

(b) Given prices \((w, r)\), the representative firm’s allocation minimizes cost.

(c) Government policies satisfy budget balance in equation (1), where aggregate income tax revenue is given by:

\[
Y_y \equiv \int Y(y(h(a, \epsilon), a, \epsilon)) d\lambda(a, \epsilon)
\]

(d) Given policies and allocations, prices clear asset and labor markets:

\[
K - B = \int a d\lambda(a, \epsilon)
\]

\[
L = \int e(\epsilon)h(a, \epsilon) d\lambda(a, \epsilon)
\]

and the allocation satisfies the resource constraint (guaranteed by Walras’ Law):

\[
\int c(a, \epsilon) d\lambda(a, \epsilon) + G + K' = ZF(K, L) + (1 - \delta)K
\]

(e) Given consumer policy functions, the distribution over wealth and productivity shocks is given recursively from the law of motion \(T^* : M \rightarrow M\) such that \(T^*\) is
given by:
\[ \lambda'(A \times E) = \int_{A \times E} Q_j((a, \epsilon), A \times E) \, d\lambda \]
where \( S \equiv A \times E \subset S \), and \( Q : S \times B(S) \to [0, 1] \) is a transition function on \( (S, B(S)) \) that gives the probability that an agent with current state \( s \equiv (a, \epsilon) \) transits to the set \( S \subset S \) in the next period. The transition function is given by:
\[
Q((a, \epsilon), S) = \begin{cases} 
\pi(E|\epsilon) & \text{if } a'(s) \in A, \\
0 & \text{otherwise}
\end{cases}
\]

(f) Aggregate capital, governmental debt, prices and the distribution over consumers are stationary, such that \( K' = K, B' = B, w' = w, r' = r, \) and \( \lambda' = \lambda \).

3.3 Balanced Growth Path
Following Aiyagari and McGrattan (1998), we will further assume that total factor productivity, \( Z \), grows over time at rate \( g_z > 0 \). In both the life cycle model and infinitely lived agent model, we will study a balanced growth path equilibrium in which all aggregate variables grow at the same rate as output. Denote the growth rate of output as \( g_y \). Refer to Appendix A.1 for a formal construction of the balanced growth path for this set of economies.

4 Calibration
In this section we calibrate the life cycle model and then discuss the parameter values that are different in the infinitely lived agent model. Overall, one subset of parameters are assigned values without needing to solve the model. These parameters are generally the same in both models. The other subset of parameters are estimated using a simulated method of moments procedure that minimizes the distance between model generated moments and empirical ones. We allow these parameters to vary across the models while matching the same moments in the two models. Table 1 summarizes the target and value for each parameter.

Demographics: Agents enter the economy at age 21 (or model age \( j = 1 \)) and exogenously die at age 100 (or model age \( J = 81 \)). We set the conditional survival probabilities \( \{\psi_j\}_{j=1}^J \) according to Bell and Miller (2002) and impose \( \psi_J = 0 \). We set
the population growth rate to $g_n = 0.011$ to match annual population growth in the US.

**Production:** Given that $Y = ZF(K, L)$, the production function is assumed to be Cobb-Douglas of the form $F(K, L) = K^\alpha L^{1-\alpha}$ where $\alpha = 0.36$ is the income share accruing to capital. The depreciation rate is to $\delta = 0.0833$ which allows the model to match the empirically observed investment-to-output ratio.

**Preferences:** The utility function is separable in the utility over consumption and disutility over labor (including retirement):

$$u(c) - v(h, s') = \frac{c^{1-\sigma}}{1-\sigma} - \left( \chi_1 \frac{h^{1+\gamma}}{1+\gamma} + s' \chi_2 \right).$$

Utility over consumption is a CRRA specification with a coefficient of relative risk aversion $\sigma = 2$, which is consistent with Conesa et al. (2009) and Aiyagari and McGrattan (1998). Disutility over labor exhibits a constant intensive margin Frisch elasticity. We choose $\gamma = 0.5$ such that the Frisch elasticity consistent with the majority of the related literature as well as the estimates in Kaplan (2012).

We calibrate the labor disutility parameter $\chi_1$ so that the cross sectional average of hours is one third of the time endowment. Finally, $\chi_2$ is a fixed utility cost of earning labor income before retirement. The fixed cost generates an extensive margin decision through a non-convexity in the utility function. We choose $\chi_2$ to match the empirical observation that seventy percent of the population has retired by the normal retirement age.

**Labor Productivity Process:** We take the labor productivity process from the estimates in Kaplan (2012) based on the estimates from the PSID data.¹⁷ The deterministic labor productivity profile, $\{\theta_j\}_{j=1}^{\text{ret}}$, is (i) smoothed by fitting a quadratic function in age, (ii) normalized such that the value equals unity when an agent enters the economy, and (iii) extended to cover ages 21 through 70 which we define as the last period in which agents are assumed to be able to participate in the labor activities.

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¹⁷For details on estimation of this process, see Appendix E in Kaplan (2012). A well known problem with a log-normal income process is that the model generated wealth inequality does not match that in the data, primarily due to missing the fat upper tail of the distribution. However, Röhrs and Winter (2017) demonstrate that when the income process in an infinitely lived agent model is altered to match the both the labor earnings and wealth distributions (quintiles and gini coefficients), the change in optimal policy is relatively small, with approximately 30 percentage points due to changing the income process and the remaining 110 percentage points due to changing borrowing limits, taxes and invariant parameters (such as risk aversion, the Frisch elasticity, output growth rate and depreciation).
The permanent, persistent, and transitory idiosyncratic shocks to individual’s productivity are normally distributed with zero mean. The remaining parameters are also set in accordance with the Kaplan’s (2012) estimates: $\rho = 0.958$, $\sigma^2_\kappa = 0.065$, $\sigma^2_\nu = 0.017$ and $\sigma^2_\epsilon = 0.081$.

**Government:** Consistent with Aiyagari and McGrattan (1998) we set government debt equal to two-thirds of output. We set government consumption equal to 15.5 percent of output consistent. This ratio corresponds to the average of government expenditures to GDP from 1998 through 2007.

**Income Taxation:** The income tax function and parameter values are from Gouveia and Strauss (1994). The functional form is:

$$Y(y) = \tau_0 \left( y - (y^{-\tau_1} + \tau_2)^{-\frac{1}{\tau_1}} \right)$$

The authors find that $\tau_0 = 0.258$ and $\tau_1 = 0.768$ closely match the U.S. tax data. When calibrating the model we set $\tau_2$ such that the government budget constraint is satisfied.

**Social Security:** We set the normal retirement age to 66. Consistent with the minimum and maximum retirement ages in the U.S. Social Security system, we set the interval in which agents can retire to the ages 62 and 70. The early retirement penalty and delayed retirement credits are set in accordance with the Social Security program and define the functions $D_1(\cdot)$ and $D_2(\cdot)$. In particular, if agents retire up to three years before the normal retirement age benefits are reduced by 6.7 percent for each year they retire early. If they choose to retire four or five years before the normal retirement age benefits are reduced by an additional 5 percent for these years. If agents choose to delay retirement past normal retirement age then their benefits are increased by 8 percent for each year they delay. The marginal replacement rates in the progressive Social Security payment schedule ($\tau_{r1}, \tau_{r2}, \tau_{r3}$) are also set at their actual respective values of 0.9, 0.32 and 0.15. The bend points where the marginal replacement rates change ($b^{ss}_{1}, b^{ss}_{2}, b^{ss}_{3}$) and the maximum earnings ($\bar{m}$) are set equal to the actual multiples of mean earnings used in the U.S. Social Security system so that $b^{ss}_{1}, b^{ss}_{2}$ and $b^{ss}_{3} = \bar{m}$ occur at 0.21, 1.29 and 2.42 times average earnings in the economy. We set the payroll tax rate, $\tau_{ss}$ such that the program’s budget is balanced.

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18 The estimates in Kaplan (2012) are available for ages 25-65.

19 We exclude government expenditures on Social Security since they are explicitly included in our model.
## Table 1: Calibration Targets and Parameters for Baseline Economy.

<table>
<thead>
<tr>
<th>Description</th>
<th>Parameter</th>
<th>Value</th>
<th>Target or Source</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Demographics</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Maximum Age</td>
<td>$J$</td>
<td>81 (100)</td>
<td>By Assumption</td>
</tr>
<tr>
<td>Min/Max Retirement Age</td>
<td>$I_{ret}$,$\bar{I}_{ret}$</td>
<td>43, 51 (62, 70)</td>
<td>Social Security Program</td>
</tr>
<tr>
<td>Population Growth</td>
<td>$g_n$</td>
<td>1.1%</td>
<td>Conesa et al (2009)</td>
</tr>
<tr>
<td>Survival Rate</td>
<td>${s_j}_{j=1}$</td>
<td>—</td>
<td>Bell and Miller (2002)</td>
</tr>
<tr>
<td><strong>Preferences and Borrowing</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Coefficient of RRA</td>
<td>$\sigma$</td>
<td>2.0</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Frisch Elasticity</td>
<td>$\gamma$</td>
<td>0.5</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Coefficient of Labor Disutility</td>
<td>$\chi_1$</td>
<td>55.3</td>
<td>Avg. Hours Worked = 1/3</td>
</tr>
<tr>
<td>Fixed Utility Cost of Labor</td>
<td>$\chi_2$</td>
<td>1.038</td>
<td>70% retire by NRA</td>
</tr>
<tr>
<td>Discount Factor</td>
<td>$\beta$</td>
<td>1.012</td>
<td>Capital/Output = 2.7</td>
</tr>
<tr>
<td>Borrowing Limit</td>
<td>$\alpha$</td>
<td>0</td>
<td>By Assumption</td>
</tr>
<tr>
<td><strong>Technology</strong></td>
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<td></td>
</tr>
<tr>
<td>Capital Share</td>
<td>$\alpha$</td>
<td>0.36</td>
<td>NIPA</td>
</tr>
<tr>
<td>Capital Depreciation Rate</td>
<td>$\delta$</td>
<td>0.0833</td>
<td>Investment/Output = 0.255</td>
</tr>
<tr>
<td>Productivity Level</td>
<td>$Z$</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>Output Growth</td>
<td>$g_y$</td>
<td>1.85%</td>
<td>NIPA</td>
</tr>
<tr>
<td><strong>Labor Productivity</strong></td>
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<td></td>
</tr>
<tr>
<td>Persistent Shock, autocorrelation</td>
<td>$\rho$</td>
<td>0.958</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Persistent Shock, variance</td>
<td>$\sigma^2_{\rho}$</td>
<td>0.017</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Permanent Shock, variance</td>
<td>$\sigma^2_{\delta}$</td>
<td>0.065</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Transitory Shock, variance</td>
<td>$\sigma^2_{\xi}$</td>
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<td>Kaplan (2012)</td>
</tr>
<tr>
<td>Mean Earnings, Age Profile</td>
<td>${\theta^j}_{j=1}$</td>
<td>—</td>
<td>Kaplan (2012)</td>
</tr>
<tr>
<td><strong>Government Budget</strong></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Government Consumption</td>
<td>$G/Y$</td>
<td>0.155</td>
<td>NIPA Average 1998-2007</td>
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<tr>
<td>Government Savings</td>
<td>$B/Y$</td>
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<td>NIPA Average 1998-2007</td>
</tr>
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<td>Marginal Income Tax</td>
<td>$\tau_0$</td>
<td>0.258</td>
<td>Gouveia and Strauss (1994)</td>
</tr>
<tr>
<td>Income Tax Progressivity</td>
<td>$\tau_1$</td>
<td>0.786</td>
<td>Gouveia and Strauss (1994)</td>
</tr>
<tr>
<td>Income Tax Progressivity</td>
<td>$\tau_2$</td>
<td>4.541</td>
<td>Balanced Budget</td>
</tr>
<tr>
<td><strong>Social Security</strong></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Payroll Tax</td>
<td>$\tau_{ss}$</td>
<td>0.103</td>
<td>Social Security Program</td>
</tr>
<tr>
<td>SS Replacement Rates</td>
<td>${\tau^3_i}_{i=1}$</td>
<td>See Text</td>
<td>Social Security Program</td>
</tr>
<tr>
<td>SS Replacement Bend Points</td>
<td>${b^3_{ss}}_{i=1}$</td>
<td>See Text</td>
<td>Social Security Program</td>
</tr>
<tr>
<td>SS Early Retirement Penalty</td>
<td>${D^2_i}_{i=1}$</td>
<td>See Text</td>
<td>Social Security Program</td>
</tr>
</tbody>
</table>

In our baseline model the payroll tax rate is 10.3 percent, roughly equivalent with the statutory rate.\(^{20}\)

**Infinitely Lived Agent Model:** The infinitely lived agent model does not have an age-dependent wage profile. For comparability across models, we replace the age-dependent wage profile with the population-weighted average of $\theta^j$’s, such that $\bar{\theta} =$

\(^{20}\)Although the payroll tax rate in the U.S. economy is slightly higher than our calibrated value, the OASDI program includes additional features outside of the retirement benefits.
In the absence of a retirement decision, we set $\chi_2 = 0$. Lastly, we recalibrate the parameters $(\beta, \chi_1)$ to the same targets as in the life cycle model and choose $\tau_2$ to balance the government’s budget.

5 Quantitative Effects of the Life Cycle on Optimal Policy

This section computes optimal public debt policy in the life cycle and infinitely lived agent models and quantifies the contribution of the life cycle to policy differences. We quantify the effect of each life cycle feature through the construction of a series of counterfactual models that systematically removes life cycle features until recovering the infinitely lived agent model. Therefore the counterfactual models isolate the response of optimal policy to each life cycle model component. We conclude this section by computing the welfare gains from implementing optimal public debt policy and further highlighting the effect of life cycle features on optimal by decomposing those welfare gains.

5.1 Optimal Public Policy

In both the life cycle and infinitely lived agent models, the government is a benevolent Ramsey planner that fully commits to fiscal policy. The planner maximizes social welfare by choosing a budget feasible level of public savings ($B > 0$) or public debt ($B < 0$) subject to allocations being a stationary recursive competitive equilibrium. We consider an ex-ante Utilitarian social welfare criterion that evaluates the expected utility of an agent in the steady state economy.\footnote{When calibrating the stochastic process for idiosyncratic productivity shocks, we use the same process in the both the life cycle and infinitely lived agent models. Using the same underlying process will imply that cross-sectional wealth inequality will be different across the two models. One reason is that the life cycle model will have additional cross-sectional inequality due to the humped shaped savings profiles, which induces the accumulation, stationary, and deaccumulation phases. We view these difference in inequality as a fundamental difference between the two models and, therefore, choose not to specially alter the infinitely lived agent model to match a higher level of cross-sectional inequality.}

For the life cycle model, the Ramsey planner chooses public savings to maximize

$$\sum_{j=1}^{\text{ret}} (\mu_j / \sum_{j=1}^{\text{ret}} \mu_j) \theta_j \approx 1.86.$$\footnote{Our analysis focuses on welfare across steady states. This analysis omits the transitional costs between steady states which can be large. See Domeij and Heathcote (2004), Fehr and Kindermann (2015) and Dyrd and Pedroni (2016) for a discussion of these transitional costs.}
the expected lifetime utility of newborn agents as follows,

\[ S_f(V_1, \lambda_1) \equiv \max_B \left\{ \int V_1(a, \varepsilon, m, s; B) \, d\lambda_1(a, \varepsilon, m, s; B) \mid \text{s.t. (1), (3)} \right\} \]

where the value function \( V_1(\cdot; B) \), distribution function \( \lambda_1(\cdot; B) \) and policy functions embedded in equations (1) and (3) are determined in competitive equilibrium and depend on the planner’s choice of public savings. Furthermore, \( B' = B \) in steady state. Since the distribution of taxable income and tax revenues depend on public savings, we adjust the income tax parameter \( \tau_0 \) and the payroll tax rate \( \tau_{ss} \) to ensure that the government budget is balanced and Social Security is self-financing.\(^{23}\)

For the infinitely lived agent model, the Ramsey planner chooses public savings to maximize the expected utility of infinitely lived agents as follows,

\[ S_\infty(V, \lambda) \equiv \max_B \left\{ \int V(a, \varepsilon; B) \, d\lambda(a, \varepsilon; B) \mid G = rB + Y_g(\tau_0, B) \right\} \]

The welfare maximization problem is nearly identical to that of the life cycle model’s, except that the value function and distribution function do not depend on age and there is no Social Security program, so that equation (3) does not characterize the feasible set.

We find that the two models generate starkly different optimal policies, which are reported in Table 2. In the infinitely lived agent model, the government is a net borrower with optimal public debt equal to 24 percent of output.\(^{24}\) On the other hand, in the life cycle model, the government’s optimal policy is public savings equal to 59 percent of output. Thus, including life cycle features causes optimal policy to switch from public debt to savings, with approximately an 85 percentage point swing in optimal policy.

We quantify the welfare gain from implementing the optimal policy in each economy. In particular, we compute consumption equivalent variation (CEV) – the percent of lifetime consumption that a life cycle model agent would be willing to pay ex ante – from inhabiting an economy with an optimal public savings policy of 59 percent of output instead of an economy with the infinitely lived agent model’s optimal public debt policy of 24 percent of output. We find that the 85 percentage point difference in

\(^{23}\)We choose to use \( \tau_0 \) to balance the government budget instead of the other income taxation parameters \( (\tau_1, \tau_2) \) so that the average income tax rate is used to clear the budget, as opposed to changing in the progressivity of the income tax policy. The average tax rate is the closest analogue to the flat tax that Aiyagari and McGrattan (1998) use to balance the government’s budget in their model.

\(^{24}\)This is generally consistent with Aiyagari and McGrattan’s (1998) optimal policy. The differences in optimal policy are due to this paper assumes a different stochastic process governing labor productivity, a different utility function, non-linear income taxation and different parameter values.
optimal policies corresponds to a welfare gain of 0.45 percent of expected lifetime consumption. The welfare gain is economically significant, demonstrating that ignoring life cycle features when determining optimal debt policy can have nontrivial welfare effects.

5.2 The Effect of Life Cycle Features on Optimal Policy

The 85 percentage point difference in optimal policies is due to the three main differences between the life cycle and infinitely lived agent models: (i) agents in the life cycle model experience all three life cycle phase, including an accumulation phase, while agents in the infinitely lived agent model experience a perpetual stationary phase, (ii) other age-dependent features in the life cycle model, such as mortality risk, an age-dependent wage profile, retirement and Social Security, do not exist in the infinitely lived agent model and (iii) agents’ lifespan differ between the two models.

In order to characterize the individual effects of these three differences on optimal policy between the life cycle and infinitely lived agent models, starting with the life cycle model we compute optimal policy in two counterfactual economies that systematically remove life cycle features. The first counterfactual model is the "No Age-Dependent Features" economy, which is similar to the life cycle model but excludes all age-dependent features (e.g., mortality risk, age-dependent wage profile, retirement and Social Security system), while maintaining the maximum lifespan of

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In order to make quantitative comparisons across models, each counterfactual model’s parameters are recalibrated to match all relevant the targets described in Section 4.
Table 3: Optimal Public Savings-to-Output (Percent)

<table>
<thead>
<tr>
<th>Counterfactual Models</th>
<th>Life Cycle</th>
<th>No Age Features</th>
<th>Infinitely Lived with Accumulation</th>
<th>Infinitely Lived</th>
</tr>
</thead>
<tbody>
<tr>
<td>59%</td>
<td>200%</td>
<td>248%</td>
<td>-24%</td>
<td></td>
</tr>
</tbody>
</table>

$J = 80$ periods. The second counterfactual economy, the "Infinitely Lived with Accumulation" economy, also eliminates age-dependent model features but additionally extends agents’ lifespan to $J = 1000$ periods. We choose $J = 1000$ because it is sufficiently large such that for a newborn the expected present value of the flow of utility from the end of the lifetime is essentially zero. Therefore, the "Infinitely Lived with Accumulation" economy approximates the ex ante expected lifetime utility of an infinitely lived agent, yet agents have finite lifespans. However, since agents still enter the economy with no wealth, this economy is an approximation to the infinitely lived agent economy that still includes an accumulation phase.

Table 3 reports optimal policies for the life cycle, counterfactual, and infinitely lived agent models. First, comparing the life cycle model and "No Age-Dependent Features" economy isolates the effect of age-dependent features, which primarily leads to an increase in the expected working lifetime due to removing retirement and mortality. We find that the optimal public savings changes from 59 to 200 percent of output. Comparing the "No Age-Dependent Features" and "Infinitely Lived with Accumulation" counterfactual economies isolates the effect of further increasing agents’ lifespan from 80 periods to an approximation of an infinite lifespan. This effect additionally increases optimal savings from 200 to 248 percent of output. Finally, comparing the "Infinitely Lived with Accumulation" economy with the infinitely lived agent model isolates the effect of the accumulation phase on optimal policy, which changes optimal policy from public savings to public debt equal to 24 percent of output.

Comparing optimal policies across these four models yields two notable results. First, more public savings is optimal in the "No Age-Dependent" and "Infinitely Lived with Accumulation" counterfactual economies' than in the life cycle model. Removing life cycle features creates counterfactual models that become increasingly similar to the infinitely lived agent model, yet optimal policy diverges from that in the infinitely lived agent model. In particular, removing life cycle features generates more optimal public savings relative to the life cycle model, instead of public debt (or less public savings) as is optimal in the infinitely lived agent model. Second, by comparing optimal policies from the "Infinitely Lived with Accumulation" economy and the
Table 4: Effect of Lifespan on Inequality (Coefficient of Variation)

<table>
<thead>
<tr>
<th>Life Cycle</th>
<th>No Age Features</th>
<th>Infinitely Lived w/ Accumulation</th>
<th>Infinitely Lived</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asset Income Inequality</td>
<td>0.64</td>
<td>0.85</td>
<td>0.98</td>
</tr>
<tr>
<td>Labor Income Inequality</td>
<td>0.30</td>
<td>0.27</td>
<td>0.27</td>
</tr>
<tr>
<td>Asset Income Inequality</td>
<td>2.11</td>
<td>3.11</td>
<td>3.15</td>
</tr>
<tr>
<td>Labor Income Inequality</td>
<td>2.30</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Infinitely lived agent model, we observe that removing the accumulation phase accounts for a 275 basis point change in optimal policy, changing from public savings to public debt. These results highlight two competing affects on optimal policy from life cycle features: (i) the differential effect of the inequality channel across models, and (ii) the effect of the accumulation phase, which is absent from infinitely lived agent model.

The inequality channel has a differential effect on optimal policy in the two models because the amount of labor income inequality relative to asset income inequality generally depends on agents’ lifespan. As agents live and work longer, asset income inequality tends to increase because there is more time for labor productivity shocks to propagate into the wealth distribution and enlarge the difference in wealth between lucky and unlucky agents. Relative to the life cycle model, agents in the "No Age-Dependent" and "Infinitely Lived with Accumulation" counterfactual models work for a longer length of time (e.g., due to removing mortality and retirement, or mechanically extending lifespan). Table 4 confirms that with the longer working lifetime, asset income inequality relative to both labor income inequality is larger in the counterfactual economies than it is in the standard life cycle model under the baseline public debt policy of 67% of output.

Since government policy affects the returns from labor and capital in opposite directions, optimal policy trades off reducing income inequality from the source for which the factor price decreases with increasing income inequality from the source for which the factor price increases. Thus, the counterfactual models have higher levels of optimal public savings than does the life cycle model, because asset income inequality rises relative to labor income inequality as we remove life cycle features and extend agents’ expected working lifetimes. The change in total lifetime income inequality measures in Table 5 confirm that, in fact, adopting optimal policy reduces total income inequality.26 Thus, the inequality channel causes the optimal level of public savings

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26 In the baseline life cycle model, in which asset income inequality is relatively smaller compared
Table 5: Lifetime Total Income Inequality (Coefficient of Variation)

<table>
<thead>
<tr>
<th>Life Cycle</th>
<th>No Age Features</th>
<th>Infinitely Lived w/ Accumulation</th>
<th>Infinitely Lived</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline Policy</td>
<td>0.38</td>
<td>0.33</td>
<td>0.31</td>
</tr>
<tr>
<td>Optimal Policy</td>
<td>0.36</td>
<td>0.30</td>
<td>0.28</td>
</tr>
<tr>
<td>Percent Change</td>
<td>-4.4%</td>
<td>-8.5%</td>
<td>-10.0%</td>
</tr>
</tbody>
</table>

to increase after eliminating life cycle features but retaining the accumulation phase.

The existence of the accumulation phase in the life cycle model, is the primary model feature that leads public savings to be optimal in the life cycle model instead of public debt. Comparing optimal policies in the "Infinitely Lived with Accumulation" with the infinitely lived agent models isolates the effect of accumulation phase, which leads to 275 percentage point difference in optimal policy (as reported in Table 3). To further isolate this effect, we conduct a computational experiment in which we compute the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model according to an alternative social welfare function that only incorporates the expected present value of utility after a given age $j^* > 1$, and ignores the flow of utility from ages 1 to $j^* - 1$. Thus, as $j^*$ increases, the social welfare function ignores more of the accumulation phase.27

The computational experiments demonstrates that as the accumulation phase matters less for social welfare, optimal policy tends toward more public debt, as shown in Figure 1. The left panel in Figure 1 plots the optimal policy of the "Infinitely Lived with Accumulation" counterfactual model under the alternative welfare criterion, while the right panel plots the percentage of the accumulation phase that is ignored when computing optimal policy. On the x-axis, both graphs vary the percent of lifetime that the threshold age $j^*$ represents. We observe that optimal policy monotonically decreases from public savings of approximately 250 percent of GDP when all of the lifetime is considered, to an optimal public debt policy when the social wel-

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27Specifically, government policy maximizes agents’ expected lifetime utility as of age $j^*$, subject to allocations being determined in competitive equilibrium, as follows:

$$\tilde{S}(V_{j^*}, \lambda_{j^*}) \equiv \max_B \left\{ \int V_{j^*}(a, e; B) \, d\lambda_{j^*}(a, e; B) \quad \text{s.t.} \quad G = rB + Y_0(\tau_0, B) \right\}.$$
fare function ignores at least 5.2 percent of agents’ early lifetime, or approximately 70 percent of the accumulation phase.28

Figure 1: Optimal Policy and Eliminating Accumulation

Notes: The left panel graphs the optimal public savings to output ratio (y-axis) associated with ignoring a given percent of early life utility flows (x-axis). The percent of “Lifetime Ignored” is measured as $100 \cdot \left(\frac{j^*}{J}\right)$, using the given value of $j^*$ and $J = 1000$. The right panel graphs the percent of accumulation that is eliminated under the optimal policy associated with ignoring a given percent of early life utility flows. The percent of eliminated wealth accumulation is defined as the average private savings of $j^*$-age agents relative to the peak average savings and converted to a percent, given a particular optimal public savings policy. The vertical dashed line demarcates the percent of early lifetime utility ignored at which optimal policy switches from public savings to debt.

To summarize, we find that the existence of an accumulation phase mitigates the welfare benefit from public debt. However, extending agents’ working lifetime increases the amount of asset income inequality relative to labor income inequality, thereby increasing the welfare benefit from public debt. Thus, when comparing the life cycle and the infinitely lived agent models, the existence of age-dependent features and a shorter lifespan drive optimal policy toward public debt, while the existence of the accumulation phase drives optimal policy toward public savings. Overall, we

28Similarly, we find public savings is optimal in the infinitely lived agent model when the government only considers the welfare agents that closely resemble life cycle agent entering the accumulation phase. Specifically, if the government only places Pareto weight on the set of borrowing constrained agents with the median persistent component of the labor productivity shock, then optimal public savings equals 300 percent of output. This result reinforces the notion that wealth accumulation reduces the welfare benefit from public debt.
Table 6: Welfare Decompositions

<table>
<thead>
<tr>
<th></th>
<th>Life Cycle (% Change)</th>
<th>Infinitely Lived (% Change)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall CEV</td>
<td>0.45</td>
<td>-0.05</td>
</tr>
<tr>
<td>Level ($\Delta_{level}$)</td>
<td>-0.12</td>
<td>-0.16</td>
</tr>
<tr>
<td>Age ($\Delta_{age}$)</td>
<td>0.67</td>
<td>0</td>
</tr>
<tr>
<td>Distribution ($\Delta_{distr}$)</td>
<td>-0.09</td>
<td>0.11</td>
</tr>
</tbody>
</table>

Notes: The life cycle and infinitely lived agent model welfare decompositions compare allocations under a 24% public debt-to-output and a 59% public savings-to-output ratio.

find that the effects of the accumulation phase dominate the effects of other life cycle model features on optimal policy, ultimately resulting in the optimality of public savings in the life cycle model.

5.3 Welfare Decomposition

In order to quantify the effects of the channels that lead public debt to be optimal in the infinitely lived agent model and public savings to be optimal in the life cycle model we examine the welfare implications from adopting public savings instead of public debt. Specifically, in both models, we quantify the welfare effects from a 85 percentage point change in policy, from the optimal public debt policy in the infinitely lived agent model to the optimal public savings policy in the life cycle model.

The welfare effects from changing public policy reflect the change in aggregate resources available to agents and the allocation of those resources across agents and across their lifetimes. Thus, we decompose the consumption equivalent variation (denoted $\Delta_{CEV}$) into a level effect ($\Delta_{level}$), an age effect ($\Delta_{age}$) and a distribution effect ($\Delta_{distr}$). 29 The decomposition is defined as

$$(1 + \Delta_{CEV}) = (1 + \Delta_{level}) \cdot (1 + \Delta_{age}) \cdot (1 + \Delta_{distr}).$$

29More generally, we follow Floden (2001) in characterizing four components of the CEV: a level effect ($\Delta_L$), an insurance effect ($\Delta_I$), a redistribution effect ($\Delta_R$) and a labor hours effect ($\Delta_H$). We combine the insurance and redistribution effects to form the “distribution effect”. Lastly, we add two elements to Floden’s (2001) original decomposition. First, we add an age effect, which only exists in the life cycle model. Second, because an extensive margin retirement decision is unique to the life cycle model, we incorporate both intensive and extensive margins into the welfare decomposition of the life cycle model’s hours allocation. Appendix A.2 formally derives the decomposition.
The level effect captures the welfare change for a fictitious "representative agent," absent any idiosyncratic or life cycle variation in consumption or hours. The age effect measures agents’ change in welfare as a result of changing age-specific average levels of consumption and hours, net of changes in aggregate consumption and hours. Accordingly, the age effect captures the welfare effect of a change in the slope of the average consumption and hours age-profiles. Note that the age effect does not exist in the infinitely lived agent model and therefore infinitely lived agents attain zero welfare change through age effects. Lastly, the distribution effects measures the remaining change in welfare that results from a change in the distribution of consumption and hours across agents.

The welfare decomposition demonstrates that the age effect, which is closely tied to the age-allocation channel, is crucial in explaining why public savings increases welfare in the life cycle mode. In contrast, the level effect explains why public savings decreases welfare in the infinitely lived agent model. In particular, the 0.45 percent welfare improvement from implementing public savings in the life cycle model is due to a 0.67 percent increase from the age effect that is partially offset by a 0.12 percent decreases and 0.09 percent decrease from the level and distribution effects, respectively. In contrast, the small 0.05 percent CEV loss in the infinitely lived agent model can be attributed to a larger negative level effect (0.16) than the positive distribution effect (0.11).

The life cycle model’s positive age effect from adopting public savings, which is closely tied to the age-allocation channel, indicates an improved allocation of consumption and hours across ages. Agents possess standard concave utility functions and prefer smooth lifetime consumption and hours allocations. As shown in Figure 2, consumption tends to increase over a majority of the lifetime because agents choose to use more of their available resources for savings and delay consumption to later in their lifetimes. Similarly, labor hours decline over an average agent’s working lifetime as agents choose to delay consumption of leisure. When deciding whether to consuming today or save for tomorrow’s consumption agents evaluate the tradeoff between the interest earned on savings versus the effective discounting of future utility due to both impatience and mortality risk.\(^{30}\) Thus, over a majority of the lifetime, the interest rate is sufficiently large to generate an upward sloping consumption profile.\(^{31}\) However, the lower interest rate associated with pubic savings diminishes the returns to savings, inducing agents to consume more while young. This change leads to a

\(^{30}\)Additionally, liquidity constraints can lead agents to consume less early in their lifetime. This effect is accentuated in the counterfactual infinitely lived with accumulation phase model.

\(^{31}\)The downward slopping labor supply profile is also affected by the age-dependent wage profile.
Notes: Solid lines are cross-sectional averages for consumption, savings, and hours by age in the life cycle economy under its optimal public savings policy. The dashed lines are cross-sectional averages for the suboptimal debt policy from the infinitely lived agent economy.

more equal allocation of consumption and leisure lifetime which improves welfare. This channel does not exist in the infinitely lived agent model.

In contrast, in the infinitely lived agent model changing from public savings to public debt leads to a welfare losses due to the level channel. Moreover, the negative contribution to overall welfare is larger in the infinitely lived agent model than in the life cycle model. One reason that public savings reduces welfare through the level channel is that the lower interest rates associated with public savings means that agents hold less savings and thus are more likely to experience binding liquidity constraints. Constrained agents will tend to experience a lower level of welfare since they will tend to consume less and work more. The negative effect is smaller in the life cycle model because the accumulation phase mitigates some of the potential benefit from public debt of the insurance channel.

Finally, the distribution effect from adopting a public savings policy partially offsets the level and age effects in each model, leading to welfare reduction in the life cycle model and a welfare improvement in the infinitely lived agent model. The difference distribution effect across models corresponds to the inequality channel. A higher wage and lower interest rate from public savings has different effects on inequality in the life cycle and infinitely lived agent models. As discussed in Section 2.2, a longer working lifetime in the infinitely lived agent model leads to more asset income inequality relative to labor earnings inequality. Thus, a higher wage and lower interest rate can reduce existing total income inequality. In the life cycle model, the opposite holds true; since asset income inequality relative to labor income inequality is smaller,
a lower interest rate and higher wage exacerbates lifetime total income inequality.\footnote{In contrast to the age and distribution effect, the level effect from adopting public savings is similar in both models. In particular, there is a welfare increase from the consumption level effect and a welfare decrease from the hours level effect. Public savings leads to more productive capital so both output and consumption increase. However, the larger stock of productive capital leads to a higher wage which encourages more labor. Overall, the disutility from more labor dominates the increase in utility from more consumption because the lower interest rate associated with public savings reduces the incentives for agents to save so they are more likely to face binding liquidity constraints (i.e. a reduction in the benefit from the insurance channel).}

To summarize, the welfare effects highlight the competing mechanisms that lead to different optimal policy across the life cycle and infinitely lived agent models. In the life cycle model public savings induces agents to more equally allocate their consumption and labor over their lifetime while the accumulation phase mitigates the welfare benefit from public debt (measured by both the level and age effects). However, the inequality channel decreases the difference in optimal policies across models, since adopting public savings increases welfare inequality in the life cycle model but reduces it in the infinitely lived agent model. On net, we find the quantitative magnitude of the benefits from public savings dominate in the life cycle model.

## 6 Wealth Inequality

While Section 5 quantifies the main channels that determine optimal debt policy, those results abstract from features that shape the bottom of the wealth distribution. This section reexamines optimal debt policy in the life cycle and infinitely lived agent models after introducing model ingredients that change the number of low wealth agents and the resources available to them. Specifically, this section shows how optimal policy responds to (i) allowing agents to borrow and (ii) varying the mass of agents with little or no wealth by altering the labor productivity process. We demonstrate that these features affect optimal policy quite differently in the life cycle and infinitely lived agent models.

### 6.1 Liquidity Constraints

In the benchmark model, we assumed that agents faced a no-borrowing constraint. In this section, we examine whether optimal policy is sensitive to allowing agents to borrow since borrowing may change the strength of the insurance channel. To do this, we compute optimal policy in the life cycle and infinitely lived agent models when agents can borrow up to an exogenously set limit of 30 percent of each economy’s
We find that allowing agents to borrow increases the difference between the optimal policies in the life cycle and infinitely lived agent models. While the infinitely lived agent model’s optimal level of public debt remains approximately the same (24 percent of output), the life cycle model’s optimal level of public savings is now twice as large (118 percent of output).

Allowing private borrowing leads to a large increase in optimal public savings in the life cycle model since it enables agents to more easily intertemporally smooth their consumption. In particular, young agents expect a sharp increase in their labor productivity during the beginning of their working lifetime due to the age-specific component \( \{\theta_j\}_{j=1}^{100} \). Since they enter the economy with low labor productivity and little to no wealth, agents would like to borrow against future income in order to smooth consumption and hours over their lifetimes. When faced with a lower interest rate associated with greater public savings, agents would like to borrow even more. Therefore, in order to not further exacerbate liquidity constraints, optimal public savings is limited to 59% of output when agents cannot borrow. However, when borrowing is allowed, agents are better able to intertemporally smooth consumption, which mitigates the negative effect of a lower interest rate on liquidity constraints. As a result, the optimal level of public savings doubles.

While life cycle agents’ incentives to borrow derive from their increasing average labor productivity profile, infinitely lived agents experience a constant average labor productivity profile, \( \theta_j = \bar{\theta} \) for all \( j \). As a result, we find a minimal effect on optimal policy from allowing borrowing for these infinitely lived agents. Therefore, the main mechanism by which government debt improves welfare in the infinitely lived agent model is robust to changes in borrowing limits. However, in the case of the life cycle model, allowing borrowing strengthens the main mechanism and provides greater scope for public savings to improve welfare.

### 6.2 Labor Productivity Process

In this section, we match the fraction of agents with little or no wealth in the life cycle and infinitely lived agent models to what we observe in the Survey of Consumer Finances. Relative to the benchmark model, the wealth distribution will be more skewed (e.g., there will be a smaller fraction of agents holding a majority of the wealth and a larger fraction of the population that possesses little or no wealth). Increasing the fraction of low-wealth agents may alter optimal policy because the Ramsey planner

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33 We calculate the limit as 30 percent of aggregate private savings under the baseline calibration. We choose to use the rest of the calibration parameters from the benchmark model since allowing borrowing has a very small quantitative impact on aggregate variables in both model.
will be more willing to implement a policy that benefits agents with little lifetime wealth, even at the expense of high-wealth agents.

Our quantitative implementation follows Castañeda, Díaz-Giménez, and Ríos-Rull (2003), and more recently Kindermann and Krueger (2014), by augmenting the standard log-normally distributed persistent labor productivity process \( (\nu) \) in Section 4 with the addition of an extremely high labor productivity state.\(^{34}\) We refer to this additional high labor productivity state as a superstar shock. We parameterize the superstar shock to make it unlikely but highly persistent. Accordingly, in the life cycle model, we set the probability of receiving the superstar shock to 1% and the per-period persistence to 90%.\(^{35}\) In the infinitely lived agent model, we set these two probabilities to match the duration and hazard rate of the superstar state that are implied by the life cycle model. We obtain a probability of receiving a superstar shock that is just over 1% and a probability of a superstar remaining a superstar in the next period equal to 86%. For both the life cycle and infinitely lived agent models, we choose the value of the superstar shock so that the bottom 60% of the population holds 5.4% of total wealth or, equivalently, the top 40% of the population holds 94.6% of total wealth (see Díaz-Giménez, Glover, and Ríos-Rull (2011)). Since adding the superstar shock has large effects on the model, we calibrate all other model parameters to match the same targets used in the benchmark model (see Section 4).

When superstar shocks are included, we find that a large amount of public savings is optimal in both the infinitely lived and life cycle models. The life cycle model’s optimal public savings equals 94% of output, which is a 35 percentage point change from the benchmark optimal policy of public savings equaling 59% of output. The infinitely lived agent model’s optimal public saving policy equals 91% of output, which constitutes an even larger change from the benchmark optimal policy of public debt equaling 24% of output. Furthermore, the welfare decomposition in Table 7 shows that there is a large welfare gain in both the life cycle and infinitely lived agent models from implementing the optimal public savings policies.

Redistribution plays an important role in the welfare gain in both models. For the infinitely lived agent model, Table 7 shows that the distribution effect drives the welfare improvement from public savings. For the life cycle model, Table 7 shows

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\(^{34}\)Preference heterogeneity is an alternate way to introduce a skewed wealth. However, there are two downsides to using preference heterogeneity. First, it is unclear what discount rate should be used to measure social welfare. Second, in a model similar to ours that excludes altruism, Hendricks (2007) demonstrates that matching the wealth distribution requires including a large mass of both patient and impatient agents with a considerably larger gap in patience between these groups than is consistent with empirical estimates.

\(^{35}\)We assume that upon exiting the superstar state, agents transition to the median persistent labor productivity state. We further assume that no life cycle agent enters the economy as a superstar.
Table 7: Welfare Decompositions with Superstar Shocks

<table>
<thead>
<tr>
<th></th>
<th>Life Cycle (% Change)</th>
<th>Infinitely Lived (% Change)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Overall CEV</td>
<td>3.45</td>
<td>1.25</td>
</tr>
<tr>
<td>Level ($\Delta_{level}$)</td>
<td>-1.24</td>
<td>-1.15</td>
</tr>
<tr>
<td>Age ($\Delta_{age}$)</td>
<td>3.98</td>
<td>0</td>
</tr>
<tr>
<td>Distribution ($\Delta_{distr}$)</td>
<td>0.73</td>
<td>2.40</td>
</tr>
</tbody>
</table>

Notes: The welfare decompositions compare allocations under a 67% public debt-to-output policy with the life cycle model and infinitely lived agent model optimal public savings-to-output ratios of 94% and 91%, respectively.

that while the age effect still contributes the most to the overall welfare gain, the distribution effect now improves welfare instead of reducing it, as was the case in the benchmark life cycle model (see Table 6).36

In both models, redistribution occurs when policy changes factor prices. Since the superstar shock is exceptionally large, superstars tend to save a very high fraction of their income in order to maintain an elevated level of consumption even after reverting to non-superstars. As a result, superstars derive more income from assets than do non-superstars. Therefore, the lower interest rate and higher wage induced by public savings increase total income for agents who do not become superstars, but reduce superstars’ income.

Relative to the benchmark life cycle and infinitely lived agent models, the addition of superstar shocks increases the expected welfare gain from redistribution for two reasons. First, superstars have a very low marginal utility of consumption relative to non-superstars so redistributing between these types of agents leads to an increase in ex ante welfare.37 Second, because receiving the superstar shock is a low probability event, there are far more agents who do not receive the superstar shock than there are

36Note that the CEV calculations in Table 7 and Table 6 compare different changes in policy. While this makes a comparison of magnitudes inappropriate, we can compare signs and shares since both CEV calculations compare a change from public debt to public savings policy.

37In the life cycle model, young agents have little wealth and the low probability of being a superstars implies that there are few young superstars. Therefore, redistribution from high-wealth to low-wealth agents also intertemporally redistributes resources from agents’ middle to early lifetime. As such, the age effect in the welfare decomposition likely captures some of the welfare effect from this redistribution.
agents who do. Therefore, redistribution benefits a significantly larger share of agents than it hurts.

Yet, while the addition of superstar shocks generates a positive welfare effect from redistribution in both models, the welfare effect is stronger in the infinitely lived agent model than in the life cycle model. In both models, a large stock of wealth indicates that an agent is currently, or recently was, a superstar. However, the models differ in what having little wealth indicates. In the infinitely lived agent model, agents have little wealth because they have experienced a long duration without a superstar shock. Thus, public savings is effective at increasing the welfare of non-superstars relative to superstars through redistribution. In contrast, life cycle model agents can have little wealth because they have not recently been a superstar or because they are young (i.e., are at the beginning of the accumulation phase) or old (i.e., are in their deaccumulation phase). As a result, public savings is less effective at isolating superstars in the life cycle model, which dampens the welfare gain from redistribution. Because public savings redistributes from superstars to non-superstars more effectively in the infinitely lived agent model than in the life cycle model, including superstar shocks generates a much larger change in policy in the infinitely lived agent model (from 24% public debt to 91% public savings) than in the life cycle model (from 59% public savings to 94% public savings).

In both models adding superstar shocks means that optimal policy will decrease inequality through public savings. However, the potential welfare gain from reducing inequality due to public savings is overstated in the infinitely lived agent model. In particular, the infinitely lived agent model overstates welfare gains because it lacks the dampening effect from age-dependence that breaks the correlation between possessing high wealth and having received superstar shocks. Therefore, we find that the infinitely lived agent model is very sensitive to increased inequality, while the life cycle model’s more realistic assumptions on age-dependence mean that the life cycle model is far less responsive to increased inequality through superstar shocks. In this respect, although optimal policy might be similar, redistribution is overstated in the infinitely lived agent model while the role of intertemporal smoothing is underestimated.

7 Conclusion

This paper characterizes the effect of a life cycle on optimal public debt and evaluates the mechanisms by which a life cycle affects optimal policy. We find that the optimal policies are strikingly different between life cycle and infinitely lived agent models.
We find that it is optimal for the government to be a net saver with savings equal to 59% of output when life cycle features are included. In contrast, it is optimal for the government to be a net debtor with debt equal to 24% of output when these life cycle features are excluded.

Furthermore, there are economically significant welfare consequences from not accounting for life cycle features when determining the optimal policy. We find that if a government implemented the infinitely lived agent model’s optimal 24% debt-to-output policy in the life cycle model, then life cycle agents would be worse off by nearly one-half percent of expected lifetime consumption.

We show that the particular progression of individual savings, consumption, and labor throughout the lifetime is the predominant reason for the drastically different optimal policies in the life cycle and infinitely lived agent models. In the infinitely lived agent model, higher public debt implies that an average agent begins each period of time with more savings and is, therefore, better insured against labor earnings risk. In the life cycle model, in contrast, agents enter the economy with little or no wealth and must accumulating savings. While a higher level of public debt might encourage life cycle agents to hold more savings during their lifetime, the fact that agents must accumulate this savings stock mitigates the welfare benefits from public debt. Moreover, the lower interest rate associated with public savings improves welfare in the life cycle model since it leads agents to more equally allocate their consumption across their lifetime. In contrast, this channel does not exist in the infinitely lived agent model since, from an ex ante perspective, expected consumption is flat over time.

When using quantitative models to answer economic questions, economists constantly face a trade-off between tractability and realism. Our results demonstrate that when examining the welfare consequences of public debt, it is not without loss of generality to utilize the more tractable infinitely lived agent model instead of a life cycle model.

References


### Appendix

#### A.1 Construction of the Balanced Growth Path

We construct the Balanced Growth Path in multiple parts. First we construct the Balanced Growth Path using aggregates from the models. Then, we construct the Balanced Growth Path using individual agents’ allocations. The last two sections develop the Balanced Growth Path for any features unique to the infinitely lived agent or life cycle models.
A.1.1 Aggregate Conditions

**Balanced Growth Path:** A Balanced Growth Path (BGP) is a sequence

\[ \{C_t, A_t, Y_t, K_t, L_t, B_t, G_t\}_{t=0}^{\infty} \]

such that (i) for all \( t = 0, 1, \ldots \) \( C_t, A_t, Y_t, K_t, B_t, G_t \) grow at a constant rate \( g \),

\[ \frac{Y_{t+1}}{Y_t} = \frac{C_{t+1}}{C_t} = \frac{A_{t+1}}{A_t} = \frac{K_{t+1}}{K_t} = \frac{B_{t+1}}{B_t} = \frac{G_{t+1}}{G_t} = 1 + g \]

(ii) per capita variables all grow at the same constant rate \( g_w \):

\[ \frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{C_{t+1}/N_{t+1}}{C_t/N_t} = \frac{A_{t+1}/N_{t+1}}{A_t/N_t} = \frac{K_{t+1}/N_{t+1}}{K_t/N_t} = \frac{B_{t+1}/N_{t+1}}{B_t/N_t} = \frac{G_{t+1}/N_{t+1}}{G_t/N_t} = 1 + g_w \]

and (iii) hours worked per capita are constant:

\[ \frac{L_{t+1}}{N_{t+1}} = \frac{L_t}{N_t} = \frac{L_0}{N_0} \]

Denote time 0 variables without a time subscript, for example \( L \equiv L_0 \).

**Growth Rates:** Let all growth derive from TFP \( g_z > 0 \) and population \( g_n > 0 \) growth. Then on a balanced growth path we assume:

\[ Z_t = (1 + g_z)^t Z \]
\[ N_t = (1 + g_n)^t N \]

where \( Z \) and \( N \) are steady state values. Then, from part (iii) of the definition, growth in labor is:

\[ \frac{L_{t+1}}{L_t} = \frac{L_{t+1}/N_{t+1}}{L_t/(1 + g_n)N_t} = 1 + g_n \]

In steady state \( Y = ZK^\alpha L^{1-\alpha} \). Let output growth be given by \( g > 0 \). Therefore the production function gives:

\[ Y_t = Z_tK_t^\alpha L_t^{1-\alpha} \implies (1 + g) = (1 + g_z)^{1/\alpha} (1 + g_n) \]

Lastly, from parts (ii) and (iii) of the Balanced Growth Path definition, we can solve
for the growth of per capita variables:

\[
\frac{Y_{t+1}/N_{t+1}}{Y_t/N_t} = \frac{Z_{t+1}}{Z_t} \left( \frac{K_{t+1}/N_{t+1}}{K_t/N_t} \right)^\alpha \left( \frac{L_{t+1}/N_{t+1}}{L_t/N_t} \right)^{1-\alpha} \quad \Rightarrow \quad (1 + g_w) = (1 + g_z)^{\frac{1}{1-\alpha}}
\]

**Prices:** From Euler’s theorem we know:

\[
Y_t = \alpha Y_t + (1 - \alpha) Y_t = (r_t + \delta) K_t + \omega_t L_t
\]

Accordingly, the wage and interest rate depend on the capital-labor ratio. Growth in the capital-labor ratio is:

\[
\frac{K_{t+1}/L_{t+1}}{K_t/L_t} = (1 + g_z)^{\frac{1}{1-\alpha}} = 1 + g_w
\]

Therefore, the growth rate for the wage is:

\[
\frac{\omega_{t+1}}{\omega_t} = \frac{Z_{t+1}}{Z_t} \cdot \left( \frac{K_{t+1}/L_{t+1}}{K_t/L_t} \right)^\alpha = 1 + g_w
\]

and the growth rate for the interest rate is:

\[
\frac{r_{t+1} + \delta}{r_t + \delta} = \frac{Z_{t+1}}{Z_t} \cdot \left( \frac{K_{t+1}/L_{t+1}}{K_t/L_t} \right)^{\alpha-1} = 1
\]

Therefore wages grow while interest rates do not.

**Equilibrium Conditions:** The detrended *asset market clearing condition* is:

\[
K_t = A_t + B_t \quad \Rightarrow \quad K = A - B
\]

The detrended *resource constraint* is:

\[
C_t + K_{t+1} + G_t = Y_t + (1 - \delta) K_t \quad \Rightarrow \quad C + (g + \delta) K + G = Y
\]

and the detrended *government budget constraint* is:

\[
G_t + rB_t = T_t + B_{t+1} - B_t \quad \Rightarrow \quad G + (r - g) B = T
\]
A.1.2 Individual Conditions

Preferences: We assume that labor disutility has a time-dependent component. Specifically, we assume labor disutility grows at the same rate as the utility over consumption, such that $v_{t+1}(h) = (1 + g_w)^{1-\sigma} v_t(h)$. Therefore, total utility is:

$$U_t(c_t, h_t) = u(c_t) - v_t(h_t) = [(1 + g_w)^{1-\sigma}]^t (u(c) - v(h)).$$

Social Security: In order for the AIME to grow at the same rate as the wage, we assume a cost of living adjustment (COLA) on Social Security taxes and payments. For social security taxes, the cap on eligible income grows at the rate of wage growth, $\bar{m}_t = (1 + g_w)^t \bar{m}$. Furthermore, base payment bend points $b_{t, i}^{ss} = (1 + g_w)^t b_{i}^{ss}$ and base payment values $\tau_{r, i, t} = (1 + g_w)^t \tau_{i}$ for $i = 1, 2, 3$.

Tax Function: On a Balanced Growth Path, $(c_t, a_{t+1}', a_t)$ and $\tilde{y}_t$ must all grow at the same rate as the wage. Furthermore, the tax function must grow at the same rate as the wage. Recalling the tax function, $\Upsilon_t(\tilde{y}_t)$, $\tau_2$ must grow at the same rate as $\tilde{y}_t^{1-\tau_1}$. Rewrite as:

$$\Upsilon_t(\tilde{y}_t) = \tau_0 \left( (1 + g_w)^t \tilde{y} - \left[ [(1 + g_w)^t]^{1-\tau_1} \tilde{y}^{1-\tau_1} + [(1 + g_w)^t]^{-\tau_1} \tau_2 \right]^{-\frac{1}{\tau_1}} \right) = (1 + g_w)^t \Upsilon(\tilde{y})$$

Individual Budget Constraint: An agent’s time $t$ budget constraint is:

$$c_t + a_{t+1}' \leq w_t e_t h_t + (1 + r_t) a_t - T_t(\cdot)$$

$$c + (1 + g_w)a' \leq wh + (1 + r)a - T(\cdot)$$

where $\{c, a', a, h, w, r, e\}$ are stationary variables. Given that the tax function $Y(\tilde{y})$ grows at rate $g_w$, so will the transfer function $T(h, a, e)$ in the infinitely lived agent model. Furthermore, given that the Social Security program $\{\bar{m}, b^{ss}_i, \tau_{r, i}\}$ grows at rate $g_w$, so will the transfer $T(h, a, e, m, s')$ function in the life cycle model.

A.1.3 Life Cycle Model

Individual Problem: On the balanced growth path of the life cycle model, the stationary dynamic program is:

$$V_j(a, e, m, s) = \max_{c, a', h, s'} \left[ u(c) - s' v(h) \right] + [\beta \psi_j (1 + g_w)^{1-\sigma}] \sum_{e'} \pi_j(e'|e) V_{j+1}(a', e', m', s')$$
\[
\text{s.t. } c + (1 + g_w)a' \leq s' w(e) h + (1 + r)(a + Tr) + T(h, a, e, m, s') \\
\quad a' \geq a \\
\quad s' \in \{ 1(j < J_{ret}), 1(j \leq J_{ret}) \cdot s \}
\]

**Distributions:** For \( j \)-th cohort at time \( t \), the measure over \((a, e, m, s)\) is given by:

\[
\lambda_{j,t}(a_t, e, x_t, s) = \lambda_{j,t-1} \left( \frac{a_t}{1 + g_w}, e, \frac{x_t}{1 + g_w}, s \right) (1 + g_n) \\
= \lambda_{j,t-m} \left( \frac{a_t}{(1 + g_w)^m}, e, \frac{x_t}{(1 + g_w)^m}, s \right) (1 + g_n)^m \quad \forall \ m \leq t \\
= \lambda_{j}(a, e, m, s) N_{t-j+1}.
\]

Therefore, \( \lambda_{j}(a, e, m, s) \) is a stationary distribution over age \( j \) agents that integrates to one.

**Aggregation:** Aggregate consumption in the life cycle model is constructed as follows. Define the relative size of cohorts as \( \mu_1 = 1 \) and:

\[
\mu_{j+1} = \frac{N_{t-j}}{N_t} \cdot \prod_{i=1}^{j} s_i = (1 + g_n)^{-j} \prod_{i=1}^{j} s_i = \frac{\psi_j \mu_j}{1 + g_n} \quad \forall \ i = 1, \ldots, J - 1
\]

Let \( C_{j,t} \) be aggregate consumption per age-\( j \) agent, which is derived from the age-\( j \) agent’s allocation:

\[
C_{j,t} = \int (1 + g_w)^t c_j(a, e, m, s) d\lambda_j = (1 + g_w)^t \int c_j(a, e, m, s) d\lambda_j = (1 + g_w)^t C_j
\]

where \( C_j \) is the stationary aggregate consumption per age-\( j \) agent. Accordingly, aggregate consumption is:

\[
C_t = N_t \left( C_{1,t} + \psi_1 (1 + g_n)^{-1} C_{2,t} + \cdots + \left( \prod_{i=1}^{j-1} s_i \right) (1 + g_n)^{-(j-1)} C_{j,t} \right) \\
= (1 + g_w)^t N_t \sum_{j=1}^{J} \mu_j C_j \\
= (1 + g)^t C
\]
where \( C \) is the stationary level of aggregate consumption and where we have normalized \( N = 1 \).

We can similarly construct the remaining aggregates \( \{A, K, Y, B, G\} \) on the balanced growth path. Notably, however, labor per capita does not grow. Aggregate labor per capita is constructed as:

\[
L_t = N_t \sum_{j=1}^{J} \mu_j L_j \implies L = \frac{L_t}{N_t} = \sum_{j=1}^{J} \mu_j \int s_j'(a, \varepsilon, m, s) \phi h_j(a, \varepsilon, m, s) d \lambda_j
\]

which is the sum over ages of aggregate labor per age-\( j \) agent.

A.1.4 Infinitely Lived Agent Model

In order to isolate the effects on optimal policy due to fundamental differences in the life cycle and infinitely lived agent models, and not due to differences in balanced growth path constructs, we want sources of output growth (e.g. TFP and population growth) to be consistent across models. Thus, we incorporate population growth into the infinitely lived agent model. To be consistent with the life cycle model, we construct a balanced growth path in which the infinitely lived agent model’s income and wealth distributions grow homothetically. Our representation of this growth concept is consistent with a dynastic model in which population growth arises from agents producing offspring and valuing the utility of their offspring.

To elaborate in more detail, two additional assumptions admit a balanced growth path with population growth. First, agents exogenously reproduce at rate \( g_n \) and next period’s offspring are identical to each other. Second, the parent values each offspring identically, and furthermore values each offspring as much as they value their self. Formally, if the parent has continuation value \( \beta \mathbb{E}[v(a', \varepsilon')] \), then the parent values all its offspring with total value of \( g_n \beta \mathbb{E}[v(a', \varepsilon')] \).

These two assumptions imply two features. First, each offspring is identical to its parent. That is, if the parent’s state vector is \( (a', \varepsilon') \) next period, then so is each offspring’s state vector. As a result, the value function of each offspring upon birth is \( v(a', \varepsilon') \). Second, since the parent values each offspring equal to its own continuation value, it is optimal for the parent to save \( (1 + g_n) a' \) in total. The portion \( g_n a' \) is bequeathed to offspring, and the portion \( a' \) is kept for next period.

Individual Problem: On the balanced growth path of the Infinitely Lived Agent
Model, the stationary dynamic program is then:

\[ v(a, \varepsilon) = \max_{c, a', h} U(c, h) + [\beta(1 + g_w)^{1-\sigma}](1 + g_n) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) v(a', \varepsilon') \]

s.t. \[ c + (1 + g_n)(1 + g_w)a' \leq wh + (1 + r)a - T(y) \]

where \( y \equiv wh + ra \) and optimality conditions are given by:

\[ \chi \nu'(h) = u'(c)we(1 - T'(y)) \]
\[ u'(c) = \beta(1 + g_w)^{-\sigma}(1 + r) \sum_{\varepsilon'} \pi(\varepsilon'|\varepsilon) u'(c')(1 - T'(y')). \]

Notice that the optimality conditions do not change relative to a world without population growth. However, the cost of savings has increased since agents bequeath wealth to offspring.

**Distribution:** The distribution evolves according to:

\[ \lambda_{t+1}(a_{t+1}, \varepsilon_{t+1}) = \sum_{\varepsilon_t} \pi(\varepsilon_{t+1}|\varepsilon_t) \int_A \mathbb{1}\left[a'_{t+1}(a_t, \varepsilon_t) = a_{t+1}\right] \lambda_t(a_t, \varepsilon_t) da_t \]

The stationary distribution \( \lambda(a, \varepsilon) \) has measure 1 over \( A \times \mathcal{E} \) but the mass of agents grows at rate \( g_n \):

\[ \lambda_t(a_t, \varepsilon) = \lambda_{t-1}\left(\frac{a_t}{1 + g_w}, \varepsilon\right)(1 + g_n) \]
\[ = \lambda_{t-s}\left(\frac{a_t}{(1 + g_w)^s}, \varepsilon\right)(1 + g_n)^s \quad \forall s \leq t \]
\[ = \lambda(a, \varepsilon)N_t \]

Therefore, applying the transformation above and normalizing by \( N_{t+1} \) yields:

\[ \lambda(a', \varepsilon') = \sum_{\varepsilon} \pi(\varepsilon'|\varepsilon) \int_A \mathbb{1}\left[a'(a, \varepsilon) = a'\right] \frac{\lambda(a, \varepsilon)}{1 + g_n} da \]

**Aggregation:** To construct aggregate consumption, wealth, savings and labor, multiply individual allocations by the size of the population \( (N_t) \) and sum using the
stationary distribution \( \lambda \). For example, aggregate consumption is:

\[
C_t = N_t \int (1 + g_w)^t c(a, \varepsilon) \, d\lambda = (1 + g)^t \int c(a, \varepsilon) \, d\lambda = (1 + g)^t C
\]

We can similarly construct the remaining aggregates \( \{A, K, Y, B, G\} \) on the balanced growth path. Notably, however, aggregate labor per capita does not grow:

\[
\frac{L_t}{N_t} = \int \varepsilon h(a, \varepsilon) \, d\lambda
\]

where again \( N_0 = 1 \) by normalization.

A.2 Welfare Decomposition

**Proposition 1:** If preferences are additively separable in utility over consumption, \( u(c) \), and disutility over hours and working life, \( v(h, s) \), then welfare changes can be decomposed as:

\[
(1 + \Delta_{CEV}) = \left[ (1 + \Delta_{C_{\text{level}}}) (1 + \Delta_{H_{\text{level}}}) \right] \cdot \left[ (1 + \Delta_{C_{\text{age}}}) (1 + \Delta_{H_{\text{age}}}) \right] \cdot \left[ (1 + \Delta_{C_{\text{distr}}}) (1 + \Delta_{H_{\text{distr}}}) \right] 
\]

**Proof:** Consider two economies, \( i \in \{1, 2\} \). Define ex ante welfare in economy \( i \in \{1, 2\} \) derived from consumption, hours and retirement allocations \( \{c^i_j(s), h^i_j(s), s^i_j(s)\}_{j=1}^J \) over states \( s \equiv (a, \varepsilon, m, s) \) distributed with \( \lambda^i_j(s) \) as:

\[
S^i = U(c^i) - V^h(h^i) - V^s(s^i)
\]

where

\[
U(c^i) = \mathbb{E}_0 \left[ \sum_{j=1}^I \beta^{j-1} s_j u \left( c^i_j \right) \right] \, d\lambda^i_1
\]

\[
V^h(h^i) = \mathbb{E}_0 \left[ \sum_{j=1}^I \beta^{j-1} s_j v \left( h^i_j \right) \right] \, d\lambda^i_1
\]

\[
V^s(s^i) = \mathbb{E}_0 \left[ \sum_{j=1}^I \beta^{j-1} s_j \chi_2 s^i_j \right] \, d\lambda^i_1.
\]

Denote the Consumption Equivalent Variation (CEV) by \( \Delta_{CEV} \), which is defined as the percent of expected lifetime consumption that an agent inhabiting economy \( i = 1 \)
would pay *ex ante* in order to inhabit economy $i = 2$:

$$(1 + \Delta_{CEV})^{1-\sigma}U(c^1) - V^h(h^1) - V^s(s^1) = U(c^2) - V^h(h^2) - V^s(s^2).$$

First we decompose the CEV into levels, age and distribution effects for consumption allocations. The overall consumption effect is:

$$(1 + \Delta_{C})^{1-\sigma}U(c^1) - V^h(h^1) - V^s(s^1) = U(c^2) - V^h(h^1) - V^s(s^1).$$

For the level effect, note that aggregate consumption in economy $i$ is

$$C^i = \sum_{j=1}^J \mu_j \int_S c^i_j(s) d\lambda^i_j(s).$$

We follow Conesa et al. (2009) in defining the level effect, in a different but equivalent way to Floden (2001), by

$$(1 + \Delta_{C,level})^{1-\sigma}U(c^1) - V^h(h^1) - V^s(s^1) = U(\left(C^2/C^1\right) c^1) - V^h(h^1) - V^s(s^1).$$

For the age effect, note that age-specific average consumption in economy $i$ is

$$C^i_j = \int_S c^i_j(s) d\lambda^i_j(s),$$

and utility over age-cohort average level of consumption at each age is,

$$U(C^i_j) \equiv \int E_0 \left[ \sum_{j=1}^J \beta^{j-1}s_j \int u \left( C^i_j \right) \right] d\lambda^i_1.$$  

Then define the consumption age effect in economy $i$ as

$$U \left( (1 - \omega^i_{\text{age}})C^i \right) = U(C^i_j),$$

such that

$$(1 - \omega^i_{\text{age}}) = \left( \frac{\sum_{j=1}^J \beta^{j-1}s_j u \left( C^i_j \right)}{\sum_{j=1}^J \beta^{j-1}s_j u(C^i)} \right)^{1/\sigma},$$

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which gives the overall consumption age effect,

$$(1 + \Delta_{C_{age}}) \equiv \frac{1 - \omega_{C_{age}}^2}{1 - \omega_{C_{age}}^1} = \left( \frac{\sum_{j=1}^{J} \beta^{j-1}s_j u(C_j^2)}{\sum_{j=1}^{J} \beta^{j-1}s_j u(C_j^1)} \right)^{\frac{1}{1-\sigma}} / C^2 \div \left( \frac{\sum_{j=1}^{J} \beta^{j-1}s_j u(C_j^1)}{\sum_{j=1}^{J} \beta^{j-1}s_j u(C_j^1)} \right)^{\frac{1}{1-\sigma}} / C^1$$

Lastly, again following Floden (2001) and Conesa et al. (2009), define the consumption distribution effect in economy $i$ as the residual of the overall consumption effect:

$$U \left( (1 - \omega_{distr}^i)C_j^i \right) = U \left( c^2 \right)$$

such that

$$(1 - \omega_{distr}^i) = \left( \frac{\sum_{j=1}^{J} \beta^{j-1}s_j u(c^i)}{\sum_{j=1}^{J} \beta^{j-1}s_j u(C_j^1)} \right)^{\frac{1}{1-\sigma}}$$

which gives the overall consumption distribution effect,

$$(1 + \Delta_{C_{distr}}) \equiv \frac{1 - \omega_{C_{distr}}^2}{1 - \omega_{C_{distr}}^1} = \left( \frac{U(c^2)/U(C_j^2)}{U(c^1)/U(C_j^1)} \right)^{\frac{1}{1-\sigma}} .$$

The consumption effect decomposition is verified as follows,

$$(1 + \Delta_C) = (1 + \Delta_{C_{level}}) \cdot (1 + \Delta_{C_{age}}) \cdot (1 + \Delta_{C_{distr}})$$

$$= \left( \frac{C^2}{C^1} \right) \cdot \left( \frac{U(C_j^2)/U(C_j^1)}{C^2/C^1} \right)^{\frac{1}{1-\sigma}} \cdot \left( \frac{U(c^2)/U(c^1)}{U(C_j^2)/U(C_j^1)} \right)^{\frac{1}{1-\sigma}}$$

Likewise we define the overall hours effect as

$$(1 + \Delta_H)^{1-\sigma}U(c^2) - V^h(h^1) - V^s(s^1) = U(c^2) - V^h(h^2) - V^s(s^2)$$

which implies that

$$(1 + \Delta_H) = \left( \frac{U(c^2) - V^h(h^2) - V^s(s^2) + V^h(h^1) + V^s(s^1)}{U(c^2)} \right)^{\frac{1}{1-\sigma}} = \frac{1 + \Delta_{CEV}}{1 + \Delta_C}$$

For the level effect, note that aggregate hours and the mass of working agents in
We follow Conesa et al. (2009) in defining the hours level effect. However, since our economy features both an intensive and extensive margin labor decision, we simultaneously decompose welfare arising from hours and retirement decisions,

\[(1 + \Delta_{H_{\text{level}}})^{1-\sigma}U(c^2) - V^h(h^1) - V^s(s^1) = U(c^2) - V^h \left( \frac{H_2}{H_1} h^1 \right) - V^s \left( \frac{I_2}{I_1} s^1 \right)\]

For the age effect, note that age-specific average hours and average mass of working agents in economy \(i\) is

\[H^i_j = \int_S h^i_j(s) d\lambda^i_j(s),\]
\[I^i_j = \int_S s^i_j(s) d\lambda^i_j(s),\]

and expected lifetime utility over age-cohort average level of hours and working at each age is,

\[V^h(H^i_j) \equiv \int \mathbb{E}_0 \left[ \sum_{j=1}^{J} \beta^{j-1} s_j \nu \left( H^i_j \right) \right] d\lambda^i_j,\]
\[V^s(I^i_j) \equiv \int \mathbb{E}_0 \left[ \sum_{j=1}^{J} \beta^{j-1} s_j \chi \nu \left( I^i_j \right) \right] d\lambda^i_j,\]

Then define the hours age effect in economy \(i\) as

\[(1 + \Delta_{H_{\text{age}}})^{1-\sigma}U(c^2) - V^h \left( \frac{H_2}{H_1} h^1 \right) - V^s \left( \frac{I_2}{I_1} s^1 \right) = U(c^2) - V^h \left( \frac{V^h(H^2_j)}{V^h(H^1_j)} h^1 \right) - V^s \left( \frac{V^s(I^2_j)}{V^s(I^1_j)} s^1 \right)\]

Following Floden (2001) and Conesa et al. (2009), the hours distribution effect is then
The decomposition can be verified using a first order approximation of the $i = 2$ allocation around the $i = 1$ allocation and therefore a first order approximation of $\Delta_{H_{level}}, \Delta_{H_{age}}, \Delta_{H_{distr}}$ around zero. Note that $u'(c)c/u(c) = (1 - \sigma)$, $v'(h)h/v(h) = 1 + 1/\gamma$ and that since $\log(1 + \Delta) \approx \Delta$, then

$$
\log(1 + \Delta_H) = \log(1 + \Delta_{H_{level}}) + \log(1 + \Delta_{H_{age}}) + \log(1 + \Delta_{H_{distr}})
$$

implies

$$
\Delta_H \approx \Delta_{H_{level}} + \Delta_{H_{age}} + \Delta_{H_{distr}}.
$$

The first order approximations yield,

$$
\Delta_H \approx \frac{1}{1 - \sigma} \left( \frac{U(c^2) - V^h(h^2) - V^s(s^2) + V^h(h^1) + V^s(s^1)}{U(c^2)} - 1 \right)
$$

$$
\Delta_{H_{level}} \approx \left( \frac{1 + \frac{1}{\gamma}}{1 - \sigma} \right) \left( 1 - \frac{H^2}{H^1} \right) \frac{V^h(h^1)}{U(c^2)} + \frac{1}{1 - \sigma} \left( \frac{1 - l^2}{l^1} \right) \frac{V^s(s^1)}{U(c^2)}
$$

$$
\Delta_{H_{age}} \approx \left( \frac{1 + \frac{1}{\gamma}}{1 - \sigma} \right) \left( \frac{H^2}{H^1} \right) \frac{V^h(H^2_j)}{U(c^2)} + \frac{1}{1 - \sigma} \left( \frac{l^2}{l^1} \right) \frac{V^s(l^2_j)}{U(c^2)}
$$

and

$$
\Delta_{H_{distr}} \approx - \left( \frac{1 + \frac{1}{\gamma}}{1 - \sigma} \right) \left( 1 - \frac{V^h(H^2_j)}{V^h(H^1_j)} \right) \frac{V^h(h^1)}{U(c^2)} + \frac{1}{1 - \sigma} \left( \frac{1 - V^s(l^2_j)}{V^s(l^1_j)} \right) \frac{V^s(s^1)}{U(c^2)}
$$

$$
+ \frac{1}{1 - \sigma} \left( \frac{U(c^2) - V^h(h^2) + V^h(h^1) - V^s(h^2) + V^s(h^1)}{U(c^2)} - 1 \right)
$$

Verification comes from regrouping terms in the distribution effect:

$$
\Delta_{H_{distr}} \approx -(\Delta_{H_{level}} + \Delta_{H_{age}}) + \Delta_H
$$

$$
\Delta_H \approx \Delta_{H_{level}} + \Delta_{H_{age}} + \Delta_{H_{distr}}
$$
as desired. ■