The yield curve in normal times and at the lower bound*

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Abstract

We propose a model with stochastic regime switches as an alternative approach to deal with the lower bound on nominal interest rates in dynamic term structure models. This approach is arguably best suited to capture prolonged periods of ultra-low interest rates. The regime-switching model is a generalized version of Dai et al. (2007). The region where nominal interest rates are close to their lower bound is treated as a separate regime, where the short rate is restricted to follow a white noise process around a constant level. Regime switching probabilities are state-dependent, so that the likelihood of being in lower bound regime increases, as interest rates fall closer to zero.

We estimate on euro area data a version of the model where the state vector is entirely observable and includes macro variables and the term spread. Our estimates suggest that the euro area economy switched to the lower bound regime with high probability in 2013 and is likely to remain in that regime for a while. Compared to an affine specification, the regime switching model indicates that term premia are smaller, hence low long-term rates are indicative of a prolonged period of very low expected future short rates.

The model provides evidence as to the relative impact of QE policies on expected furture short rates and on term premia. In the euro area experience, QE appears to have affected mostly the premia component.

Keywords: zero lower bound; term premia; term structure of interest rates; monetary policy rate expectations; regime switches.

JEL Codes: E31, E40, E44, E52, E58, E62, E63.

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1 Introduction

After the global financial crisis of 2008-09, short-term nominal interest rates have reached levels close to their lower bound (henceforth, LB) in many advanced countries–see figure 1 for the experiences of the euro area, Japan and the United States. Over time two new lessons have been learned from this experience.

The first one is that the LB level is not zero, but negative due to cash storage costs.[1] In many constituencies, including the euro area, Sweden and Switzerland, key policy interest rates became negative driving down short term market rates–see again Figure 1 for the euro area. Also longer-term yields became increasingly negative over time. In February 2016, the Economist (2016) reported that "nearly 30% of the global government bond market now trades on a negative yield." As a results, the exact level of the lower bound has become unclear. Since the marginal costs of storing banknotes are difficult to quantify exactly and could change across constituencies, e.g. depending on the denomination of the largest banknote, the exact LB level is hard to identify. When modelling nominal interest rates, therefore, not only would a hard lower bound constraint at zero be inappropriate, but it is would also be hard to specify an alternative lower bound level based on a priori considerations.

The second lessons of recent LB experiences is that periods of very low interest (and inflation) rates can be extremely persistent and last many years. Excluding one main break over the 2006-2008 period, one-month money market rates have so far remained below the 25 basis points level for over 13 years in Japan. They have remained below that level for 7 years in the U.S. and for almost 4 years in the euro area. This confirms theoretical arguments suggesting that periods when nominal rates reach their LB result in deeper and more prolonged recessions, and that they are characterised by different economic dynamics compared to "normal" situations, i.e. recessions where policy interest rates can be reduced without constraints (e.g. Eggertsson and Woodford, 2003). In other words, and again consistently with theoretical arguments, the state vector displays a strong nonlinearity when interest rates are close to their lower bound.

[1] In theory a negative interest rate level on short-term bonds would give rise to an arbitrage opportunity–going short on the bond yielding a negative interest rate and long in cash. The zero level should therefore a lower bound for interest rates. In practice storage costs have discouraged banks from transforming central bank reserves into banknotes.

Both the aforementioned features are hard to capture with the shadow rate approach, that has been widely adopted to model especially the US experience–see e.g. Ueno et al. (2006) Krippner (2013), Christensen and Rudebusch (2013, 2015). On the one hand, the key strenght of the shadow rate approach–which is to impose a hard lower bound on interest rates–becomes a weakness when there is a risk that the bound is "too high". On the other hand, the state vector in the shadow rate model follows the same dynamic evolution independently of whether yields are, or are not, close to the LB. This implies that the model will tend to attribute the same persistence to LB period and to other cyclical phases of low interest rates. This assumption is likely to become more overly restrictive, the longer the lenght of the LB period–see also Svensson (2014).

In this paper, we propose an alternative way to deal with the LB for nominal interest rates. The idea is to model interest rates explicitly as a function of two possible policy regimes: a normal regime and a LB regime. Conditional on remaining in the normal regime, interest rates are an affine function of a Gaussian state vector, which follows an unrestricted VAR process. Conditional on remaining in the LB regime, interest rates are again an affine function of a Gaussian state vector, but all parameters are allowed to differ from normal regime values and the dynamics of the state vector are restricted to imply that the short-term rate is a white noise around a constant level. The most straightforward way to impose this restriction is to include the short rate in the state vector–a modelling device that we adopt throughout our analysis and which comes at virtually no loss of generality. Movements across the two regimes can occur stochastically. The final feature of our approach is to allow for regime-switching probabilities to be state-dependent. This implies that the probabilities to switch to the LB regime, conditional on being in the normal regime, can increase when short term rates are close to zero. The LB regime thus becomes increasingly likely to occur as short term interest rates approach the zero level. Conversely, the probability to remain in the LB regime can be a function other observable variables, such as inflation.

This set-up has the advantage of allowing for both the aforementioned features of the recent, international experience with the LB. On the one hand, as soon as the system switches to the LB regime, the short-term policy rate will fluctuate around some LB level for as long as the economy stays in this regime. If this regime is persistent, as it typically is, other short-term interest rates will also remain low, reflecting such expectations. Nevertheless, the LB regime does not constrain these rates from falling below the LB on the policy rate, since risk premia can modify interest rate expectations under the risk-neutral probability measure. Moreover, short-term interest rates below the LB are also admissible due to the possibility of negative rate shocks. On the other hand, the model can capture protracted periods of interest rates close to the LB, including secular stagnation scenarios of extremely persistent low growth, low inflation, and low nominal and real interest rates, because within-regime dynamics at the LB can be very different from those prevailing under normal circumstances. As long as the LB regime prevails, the model can therefore easily capture the phenomenon of the short rate remaining near zero for many years.

The key disadvantage of a regime switching model is to lead to an increase in the size of the parameter vector to be estimated. This can be problematic when the sample size is limited. In our application, we mitigate this problem relying on a model in which the state vector is composed solely by observable variables.

Once we allow for arbitrage-free pricing of yields, this modelling choice can have strong implications on the decomposition of observed yields into expectations and risk premia components. Under the LB dynamics of the state vector, short rates can be expected to remain around low levels for a prolonged period of time. Low levels of longer term yields could thus be accounted for under the expectations hypothesis. In contrast, under normal dynamics interest rates would be expected to return to their unconditional mean relatively fast. Low levels of longer-term yield could only be accounted for by negative term premia.

We estimate a version of the model on euro area data. The state vector includes core inflation, unemployment, the term spread and the short rate. This simple macro-finance specification fits the data well. Across seven bond maturities, including the 10-year index-linked yield, the estimated standard deviation of the measurement error is on average equal to 18 basis points. In forecasting, the model effectively rules out the possibility of deeply negative rates. At the end of the sample, in January 2016, when nominal rates between 1-month and 5-year are negative and reach values of minus 24 and minus 29 basis points, respectively, the forecast distribution from the model excludes interest rate values below minus 60 basis points.

Our estimates suggest that the euro area economy bounced between the normal and lower bound regimes between 2009 and 2012, switched to the lower bound regime with high probability in 2013 and is likely to remain in that regime for a while. Compared to an affine specification, the regime switching model indicates that negative term premia are less sizeable, hence low long-term rates are indicative of a prolonged period of very low expected future interest and inflation rates. Model-based average inflation expectations over a 10-year ahead horizon remain close to 2 percent until the beginning of the Great recession, then reach a trough of 1.1% at the height of the European sovereign debt crisis. They only start slowly climbing up again over 2014, when bond yields started discounting expectations of a forthcoming quantitative easing programme by the ECB.

Our estimates provide indirect evidence as to the effectiveness of such programme, which was dubbed public sector purchase programme (PSPP) and announced on 22 January 2015. Our decomposition of real and nominal yields suggests that their fall over the late 2014 and early 2015 was almost entirely due to a reduction in nominal risk premia. The expecations hypothesis 5-year and 10-year yields tend to increase when the programme is launched, as a result of expectations of a faster future normalization of the monetary policy stance.

Our paper is related to the applied literature studying the term structure of interest rate at the LB. This literature has developed fast over the past decade. The shadow rate model has proven to be the most successful empirical approach (see Bomfim, 2003, Ueno et al., 2006, Ichiue and Ueno, 2007, Kim and Singleton, 2012, and Christensen and Rudebusch, 2013, for estimates using Japanese yield data; Krippner, 2012, Ichiue and Ueno, 2013, Priebsch (2013) and Wu and Xia (2016) for application to the U.S.). Compared to standard single-regime affine models, it has two advantages (see also the discussion in Christensen and Rudebusch, 2013): it rules out negative yields; and it can account for the observed reduction in the volatility of shorter-term yields when the policy rate is at the LB (see Swanson and Williams, 2014). The shadow rate model is also relatively parsimonious; away from the LB, it boils down to the standard affine formulation, which has been studied in an extensive literature.

At the same time, the shadow rate model assume that the law of motion of the factors driving yields remains unchanged irrespective of whether the LB constraint is, or is not, binding. The factors follow a VAR representation with constant parameters. We have already argued above that this property is unappealing when the interest rates remain close to the lower bound for extended periods of time. This property is especially unpalatable from a structural perspective. Imposing the LB in any monetary macro-model with endogenous state variables will induce a nonlinearity in the law of motion of the state vector, not just in the relationship between yields and the states. Intuitively, monetary policy, through the short term interest rate, affects economic dynamics. The speed of mean reversion of variables such as inflation and unemployment may be slower, when the policy interest rate cannot be freely lowered to the desired level. As a result, the dynamics of the state vector should feature a non-linearity. Our regime-switching model allows us to capture this nonlinearity in a reduced-form fashion. The key benefit of our model with stochastic regime switches is to explicitly allow for changes in the law of motion of the factors driving the yield curve when the short term rate hits the LB. Specifically, conditional on remaining in a LB regime, both the mean and the slope coefficients of the VAR are allowed to change, as well as their variances. This can have important implications on estimates of term premia.

A paper closer to our approach is Koeda (2013), which adopts a regime switching set-up in an application to Japan's experience with the lower bound. Nevertheless, some important differences characterise our approaches. First, Koeda (2013) assumes that the state dynamics under the risk-neutral measure are identical in the normal and the LB regimes, while we relax this assumption. Second Koeda (2013) assumes that the regimes are observable not only to market participants, but also to the econometrician–an assumption we do not impose.

Our paper is organized as follows. Section 2 derives our term structure model with normal and lower-bound regimes. It derives approximate bond-pricing equations and then characterises the model likelihood. An application of the model to euro area data is described in Section 3. This section describes the fitting performance of the model and derives its implications for risk premia and for forecasting. We focus on the indirect evidence on the impact of the PSPP in section 4. Section 5 concludes.

2 A term structure model with normal and lower bound regimes

Our starting point is a VAR model, under the objective probability measure $\mathbb{P}$, for the state vector x_t

$$x_{t+1} - \widehat{\mu}^j = \Phi^j \left(x_t - \widehat{\mu}^j \right) + \Sigma^j \varepsilon_{t+1} \tag{1}$$

where the state at time t is $s_t = j$ where either $j = N$ (for a Normal regime) or $j = L$ (for LB regime). This formulation can be easily generalised to accommodate additional discrete regimes. Note also that the x_t vector can be expanded to include lags of its variables. Thus this formulation does not restrict us to work with a VAR(1) specification. Note also that the conditional long-run means of the state vector $\widehat{\mu}^j$, the autoregressive coefficients Φ^j and the variance parameters Σ^j are all indexed by the prevailing regime. Note that equation (1) is equivalent to

$$x_{t+1} = \mu^j + \Phi^j x_t + \Sigma^j \varepsilon_{t+1} \tag{2}$$

for $\mu^j \equiv \left(I - \Phi^j\right) \widehat{\mu}^j$.

Our only restriction on x_t is to assume that the short-term rate r_t is the last element of the state vector. We can then identify the LB regime by assuming that all entries in the row and column of Φ^L corresponding to r_t are equal to zero. This implies that, in the LB regime, the short-term rate does not affect the other variables of the system and is itself an i.i.d. variable around a constant mean. We also assume that, at the LB, the short rate is not affected by shocks to the other equations. It follows that, conditional on remaining in the LB state, the law of motion of the short rate can be written as

$$r_{t+1} = \mu_r^L + \sigma_r^L \varepsilon_{t+1}^r$$

where μ_r^L is its conditional mean and σ_r^L is a scalar. In contrast, Φ^N, along with μ^N and Σ^N, are unrestricted in normal times.

The model is complemented by a short rate equation of the form

$$r_t = \delta_0 + \delta_x' x_t \tag{3}$$

where $\delta_0 = 0$ and δ_x is a vector of zeros, with the exception of a 1 (loading on the short rate) in the last position. Note that δ_0 and δ_x are regime-independent, but that differences in the mean short rate between regimes can be accommodated by μ^j.

Following Dai et al. (2007, henceforth DSY), assume finally for the SDF $M_{t,t+1}$ that

$$\log M_{t+1} = -r_t - \Gamma_{t,t+1} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} \tag{4}$$

$$\Lambda_t^j = \lambda_0^j + \lambda_1^j x_t \tag{5}$$

$$\Gamma_{t,t+1} = \log \gamma_t^{j,k} \tag{6}$$

where Λ_t^j are regime-dependent market prices of risk and $\Gamma_{t,t+1}$ are market prices of regime-switching risk.

Note that DSY constrain the market prices of risk to produce a regime-independent feedback matrix under $\mathbb{Q}$, such that $\Phi^{N\mathbb{Q}} = \Phi^N - \Sigma^N \lambda_1^N \neq \Phi^L - \Sigma^L \lambda_1^L = \Phi^{L\mathbb{Q}}$. This assumption is unappealing for our application, because it would imply that bonds are priced *as if* in the lower bound regime state vector dynamics were identical to those of the normal regime. We therefore allow for state dependent matrices $\Phi^{j\mathbb{Q}}$. By not constraining the market prices of risk, we gain considerable flexibility, which turns out to be crucial in accommodating observed bond yields across regimes. A disadvantage with this approach is that closed-form solutions for arbitrage-free bond prices are unavailable. We therefore rely on the approximate bond pricing approach of Bansal and Zhou (2002). In the appendix we show, both analytically and by simulation, that the approximation is quite accurate for our parameter values.

Specifically, denote bond maturity by n and note that the no arbitrage condition $B_{t,n} = \mathrm{E}_t\left[M_{t,t+1}B_{t+1,n-1}\right]$ can be rewritten as

$$1 = \mathrm{E}_t\left[M_{t,t+1}\frac{B_{t+1,n-1}}{B_{t,n}}\right]$$

In the appendix we prove the following proposition:

Proposition 1 *Bond prices can be written as*

$$B_{t,n} = \exp\left(-A_n^j - B_n^j x_t\right)$$

for

$$\begin{aligned} A_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk}\left(\delta_0^j + A_{n-1}^k + B_{n-1}^k \mu^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right) \\ B_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk}\left(\delta_x' + B_{n-1}^k \Phi^{\mathbb{Q}j}\right) \end{aligned}$$

starting from

$$\begin{aligned} A_1^j &= \delta_0^j \\ B_1 &= \delta_x' \end{aligned}$$

Before estimating the model, we need to specify the regime switching probabilities under both $\mathbb{P}$ and $\mathbb{Q}$. Denote the transition probabilities from regime $s_t = j$ to regime $s_{t+1} = k$ as $\Pr\left[s_{t+1} = k | s_t = j\right] = \pi_t^{\mathbb{P}jk}$, for $0 \leq \pi_t^{\mathbb{P}jk} \leq 1$ and $\sum_{k=0}^{S} \pi^{\mathbb{P}jk} = 1$. In order to keep bond pricing somewhat tractable, we follow the existing asset pricing literature and assume that the $\mathbb{Q}$-probabilities, $\pi^{\mathbb{Q}jk}$, are constant over time. As for the $\mathbb{P}$-probabilities, $\pi^{\mathbb{P}jk}$, they can in general be time-varying and state dependent. More specifically, we model them using the cumulative probability of a multivariate normal distribution:

$$\pi_t^{\mathbb{P},NL} = \int^{\theta_x^{NL}} \frac{1}{\sqrt{(2\pi)^2 \left|\Sigma^N\right|}} \exp\left(-\frac{1}{2}\left(x - \mu_{t+1}^N\right)' \Sigma^{N^{-1}} \left(x - \mu_{t+1}^N\right)\right) dx,$$

where μ_{t+1}^{N} is the next-period conditional expectation of x, given that the economy is currently in state N, and where θ_x^{NL} is a vector of critical levels, or thresholds, that indicate at which point the sensitivity of the regime-switching probability is at its highest. In other words, these thresholds represent levels of the state variables where investors would start to become concerned that the economy would hit the LB.

The probability to leave the lower bound, $\pi_t^{\mathbb{P},NL}$, is modelled symmetrically, for thresholds θ_x^{LN}.

Finally, in order to link the $\mathbb{P}$ and $\mathbb{Q}$ transition probabilities, we follow DSY and assume that

$$\pi^{\mathbb{Q},NL} = \frac{\pi_t^{\mathbb{P},NL}}{\gamma_t^{NL}},$$

i.e. that the market prices of regime-switching risk are such that scaling the $\mathbb{P}$-probabilities by the risk price results in the $\mathbb{Q}$-probability.

2.1 Estimation

Estimation and inference are simplified by the lack of unobservable factors in the state vector x_t.

Define the yield on a zero coupon bond of maturity n at time t and in regime j as

$$y_{t,n}^{j} = \frac{1}{n}\left(A_n^j + B_n^j x_t\right)$$

and select m bond maturities for estimation.

Assume that all bonds with maturity greater than 1 are observed with measurement error. Conditional on $s_t = j$ and $s_{t+1} = k$, we can stack observed yields in a vector y_t so that

$$y_{t+1} = A^k + B^k x_{t+1} + \Sigma^m \varepsilon_{t+1}^m$$

where ε_{t+1}^m is a vector of measurement errors and, by assumption, Σ^m is diagonal and $corr\left(\varepsilon_{t+1}, \varepsilon_{t+1}^m\right) = 0$. This vector can be stacked with the (possibly expanded to account for additional lags) vector of state variables to write

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A^k + B^k \mu^j \\ \mu^j \end{bmatrix} + \begin{bmatrix} B^k \Phi^j \\ \Phi^j \end{bmatrix} x_t + \begin{bmatrix} \Sigma^m & B^k \Sigma^j \\ 0 & \Sigma^j \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^m \\ \varepsilon_{t+1} \end{bmatrix}$$

or, for appropriately defined vectors W_{t+1}, $\overleftrightarrow{\mu}^{j,k}$, $\overleftrightarrow{\Phi}^{j}$, $\overleftrightarrow{\varepsilon}_{t+1}$ and matrix $\overleftrightarrow{\Sigma}^{j}$,

$$W_{t+1} = \overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t + \overleftrightarrow{\Sigma}^{j} \overleftrightarrow{\varepsilon}_{t+1}$$

Conditional on x_t and on states $s_t = j$ and $s_{t+1} = k$, W_{t+1} is normally distributed with mean $\overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t$ and variance-covariance matrix $\overleftrightarrow{\Sigma}^{j} \left(\overleftrightarrow{\Sigma}^{j}\right)'$. Hence so that

$$f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) = \mathbb{N}\left(\overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t, \overleftrightarrow{\Sigma}^{j}\left(\overleftrightarrow{\Sigma}^{j}\right)'\right)$$

Note now that the joint probability $f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right)$ can be rewritten as

$$f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right) = f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) f\left(s_t = j, s_{t+1} = k|x_t\right)$$

If we define the probability of regime $s_t = j$ given x_t as Q_t^j, i.e. $Q_t^j \equiv f\left(s_t = j|x_t\right)$, it follows that

$$\begin{aligned} f\left(s_t = j, s_{t+1} = k|x_t\right) &= f\left(s_t = j|x_t\right) f\left(s_{t+1} = k|x_t, s_t = j\right) \\ &\equiv Q_t^j \pi_t^{\mathbb{P}jk} \end{aligned}$$

and

$$f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right) = f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}$$

so that the likelihood in period t can be obtained integrating out the discrete states

$$f\left(W_{t+1}|x_t\right) = \sum_{k=N,L} \sum_{j=N,L} f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}$$

The sample likelihood is

$$\log L = \frac{1}{T-1} \sum_{t=0}^{T-1} \log f\left(W_{t+1}|x_t\right)$$

Denote the updated Q_t^j as $Q_{t+1}^k \equiv f\left(s_{t+1} = k|x_{t+1}\right)$. Then

$$Q_{t+1}^k = \frac{\sum_j f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}}{f\left(W_{t+1}|x_t\right)}$$

3 An application to the euro area

We test the performance of the model in an application to euro area data.

The state vector is completely observable and includes four variables: the inflation rate excluding food and energy (π), the unemployment rate (u), the spread between 10-year and 1-month rates (s) and the 1-month nominal interest rate (r). We then use euro area (German) yields for seven maturities: 3 and 6 months; 1, 2, 3, 5 and 10 years. The sample period runs from January 1999 until January 2016.

As in Ang et al. (2006), in estimation we restrict the A_n^j and B_n^j coefficients associated to the 10-year rate, A_{120}^j and B_{120}^j, to ensure that $y_{t,120} = \frac{1}{n}\left(A_{120}^j + B_{120}^j x_t\right) = s_t - r_t$. This obviously implies that $A_{120}^j = 0$ and $B_{120}^j = [0, 0, 1, 1]$. This restriction ensures that our model is internally consistent.

As a preliminary step we estimate an unrestricted VAR model over the whole sample. Based on the Akaike information criterion, we select a VAR(1). Given the relatively short sample period and the presumably few regime switches occurred in the data, we adopt two simplifying assumptions.

First, we assume that the system was in the normal regime until December 2007 and estimate the μ^N, Φ^N and Σ^N parameters by OLS as in standard VARs. We keep these parameters fixed when estimating the others.

Second we assume that $\pi_t^{\mathbb{P},NL}$ is only a function of the short-term rate and that $\pi_t^{\mathbb{P},LN}$ is only a function of inflation. The first assumption implies, quite intuitively, that the likelihood to switch to the lower bound regime is higher, the lower the level of the short-term rate. The second assumption is such that the likelihood to return to the normal regime is increasing in the inflation level. Thus

$$\begin{aligned}
\pi_t^{\mathbb{P},NL} &= \int^{\theta_r} \frac{1}{\sigma_r^N\sqrt{(2\pi)^2}} \exp\left(-\frac{1}{2}\left(\frac{r-\mu_{t+1}^{N,r}}{\sigma_r^N}\right)^2\right) dr \\
\pi_t^{\mathbb{P},LN} &= \int^{\theta_\pi} \frac{1}{\sigma_\pi^N\sqrt{(2\pi)^2}} \exp\left(-\frac{1}{2}\left(\frac{\pi-\mu_{t+1}^{N,\pi}}{\sigma_\pi^N}\right)^2\right) d\pi
\end{aligned}$$

Gien our short sample period, we dogmatically set θ_r and θ_π as equal to the 25th percentile of the unconditional distribution of the short rate and inflation, respectively. This implies that, in annualised values, $\theta_r = 0.37$ and $\theta_\pi = 1.08$.

We estimate the other parameters as indicated in section 2.1. More specifically, as in DSY, we directly estimate $\mu^{\mathbb{Q}j}$, $\Phi^{\mathbb{Q}j}$ and $\Sigma^{\mathbb{Q}j}$, rather than estimating the market prices of risk λ_0^j and λ_1^j. As starting values, we use those from a VAR estimated over the post-2008 period for μ^L, Φ^L and Σ^L. We also set the market prices of risk to zero as starting values, so that $\mu^{\mathbb{Q}j} = \mu^j$, $\Phi^{\mathbb{Q}j} = \Phi^j$.

The parameter vector that maximises the likelihood function is reported in Table 1. The standard deviations of the measurement errors range between 7 and 33 basis points at the

3-month and 2-year maturities, respectively. Figure 2 reports actual and fitted values of bond yields across various maturities. It confirms that the model fit is very good, even if it only includes observable factors.

One noticeable feature of the parameter vector are its steady states conditional on each of the two regimes. In the lower bound regime, inflation is almost 1 percentage point lower (at 1.04 percent compared to 1.95 percent in the normal regime), the unemployment rate is over 3 percentage points higher at 11.3 percent, and the short term rate is zero (compared to 3.04 percent in the normal regime).

Figure 3 shows actual values of the state variables and 1-step-ahead forecasts. All in all, the model does a good job in terms of capturing the dynamics of these variables, even if the fit of the short-term interest rate worsens somewhat at the end of the sample.

Figure 4 reports the time-varying transition probabilities. Given our selection of the θ_r and θ_π parameters, the probability to remain in the normal regime decreases at the end of the sample, while the probability to remain in the LB regime tends to increase. Filtered and smoothed probabilities of being in the normal regime are reported in Figure 5. They are very close to 1 until the end of 2008, then bounce between 0 and 1 between 2009 and 2012, and switch more persistently to the lower bound regime in 2013, where they remain until the end of the sample.

Figures 6 and 7 show states and yields forecasts as at the end of the sample (January 2016). The forecast for the 3-month rate shows that, since the model attaches a very high probability that the LB regime will persist, it also forecasts a prolonged period of a near-zero short-term interest rate. The persistence of the LB regime also results in very low and persistent bond yields further out along the yield curve.

Figure 8 compares estimates of 10-year bond premia in our model to those arising from an affine model (see Hördahl and Tristani, 2014). The figure shows that the regime switching model tends to deliver lower term premia in the first years of the sample and less negative term premia after 2012. As a result, the regime switching model tends to produce lower expectations hypothesis long-term rates, hence a slowed path of expected future short term rates. Based on the post-2012 experience, the regime-switching model is clearly more successful in forecasting future short term rates.

In spite of the high persistent of the LB regime, the forecast distributions do not include implausibly low level of the nominal interest rates. Specifically, the lowest point of the 95%

probability distribution of the short rate forecast is equal to -0.57 percent. This is not an unreasonable value, given that in January 2016 the short rate was equal to -0.24 percent, yields at medium term maturities were even lower, reaching for example -0.48 percent at the 2-year maturity, and even 5-year yields were negative at -0.29 percent.

We conclude that our regime-switching approach is well suited to modelling yields dynamics close to the LB.

4 Implications for the effectiveness of the PSPP

Our estimates provide indirect evidence as to the effectiveness of the ECB's public sector purchase programme (or PSPP). The programme was announced on 22 January 2015, but expectations of its future implementation started already in mid-2014, after President Draghi's Jackson Hole speech. Yields at various maturities declined significantly over the late 2014 and early 2015 .

Our decomposition of nominal yields in Figure 9 suggests that this decline was almost entirely due to a reduction in nominal risk premia. In contrast, the expectations hypothesis, long-term yields tend to increase when the programme is launched, as a result of expectations of a faster future normalization of the monetary policy stance. This is also true for other maturities, including 2-year and 3-year, where yields were already extremely low and only descended by a limited amount.

All in all, this evidence appears to be consistent with the so-called portfolio rebalancing channel, suggesting that the PSPP led to a repricing of risk. In turn, the repricing may have been the results of a relaxation of funding or value-at-risk constraints for financial institutions.

Figure 10 focuses on the model-implications for average inflation over the next 10 years, as derived from the index-linked yields included in the information set. According to our model, these long-term inflation expectations remained close to 2 percent until the beginning of the Great recession, then reached a trough of 1.1% at the height of the European sovereign debt crisis. They only started slowly climbing up again at the end of 2014 and especially over 2015. This suggests that the PSPP was also accompanied by a movement in long-term inflation expectations towards levels closer to the numerical definition of price stability provided by the ECB.

In contrast, inflation risk premia fall over the course of 2014 and reach persistently negative levels. This suggests that deflation risks remain present in the euro area.

5 Conclusions

[To be written]

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Appendix

.1 Bond prices

Postulate that bond prices are exponentially affine in x_t

$$B_{t,n} = \exp\left(-A_n^j - B_n^j x_t\right)$$

Using the expression for bond prices and the stochastic discount factor, the no-arbitrage condition can be rewritten as

$$1 = \mathrm{E}_t \Bigg[\exp\Bigg(-\delta_0^j - \delta_x' x_t - \frac{1}{2}\left(\Lambda_t^j\right)' \Lambda_t^j - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^j - \left[\left(\Lambda_t^j\right)' + B_{n-1}^k \Sigma^j\right] \varepsilon_{t+1} - \Big(B_{n-1}^k \Phi^j - B_n^j\Big)$$

Note that the independence between normal and regime-switching shocks implies that

$$\mathrm{E}_t \left[\exp\left(-\left[\left(\Lambda_t^j\right)' + B_{n-1}^k \Sigma^j\right] \varepsilon_{t+1}\right)\right] = \sum_{k=1}^{S} \pi_t^{\mathbb{P}jk} \exp\left(\frac{1}{2}\left(\Lambda_t^j\right)' \Lambda_t^j + B_{n-1}^k \Sigma^j \Lambda_t^j + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right)$$

so that the no-arbitrage condition becomes

$$1 = \sum_{k=1}^{S} \pi_t^{\mathbb{P}jk} \exp\left(-\delta_0^j - \delta_x' x_t - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

Finally use the assumption i.e. $\Gamma_t^{j,k} = \log \gamma_t^{j,k} = \log \frac{\pi_t^{\mathbb{P}jk}}{\pi^{\mathbb{Q}jk}}$ to obtain

$$1 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-\delta_0^j - \delta_x' x_t - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

At this point, take a first oder approximation. Note that the right hand side must be 1 in steady state. The exponential can therefore be approximated around 0, using $\exp z \simeq 1 + z$. It follows that

$$0 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(-\delta_0^j - \delta_x' x_t - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

These equations are satisfied for all maturities as long as the A and B matrices follow the recursions

$$\begin{aligned} A_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_0^j + A_{n-1}^k + B_{n-1}^k \mu^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right) \\ B_n^j &= \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_x' + B_{n-1}^k \Phi^{\mathbb{Q}j}\right) \end{aligned}$$

starting from

$$\begin{aligned} A_1^j &= \delta_0^j \\ B_1 &= \delta_x' \end{aligned}$$

Table 1: Parameter values

For $x_t = \begin{bmatrix} \pi_t & u_t & s_t & r_t \end{bmatrix}'$ and $x_{t+1} - \widehat{\mu}^j = \Phi^j \left(x_t - \widehat{\mu}^j \right) + \Sigma^j \varepsilon_{t+1},\ j = N, L$

$$1200 \cdot \left(\widehat{\mu}^N\right)' = \begin{bmatrix} 1.94811264 & 8.13416796 & 1.12436034 & 3.04622544 \end{bmatrix}$$

$$\Phi^N = \begin{bmatrix} 0.92333118 & -0.050547952 & 0.019707535 & 0.016202199 \\ 0.090381545 & 1.0040052 & 0.015009505 & 0.004881381 \\ 0.18267277 & 0.18937779 & 0.81458225 & -0.022635095 \\ -0.24285628 & -0.1779636 & 0.12457547 & 0.98233529 \end{bmatrix}$$

$$\Phi^{\mathbb{Q}N} = \begin{bmatrix} 0.968512675 & -0.012793183 & 0.052697828 & 0.018641043 \\ 0.43789055 & 0.979333407 & 0.29356046 & -0.013123604 \\ 0.119215472 & -0.012108285 & 1.020986201 & -0.029222097 \\ -0.058130413 & -0.001157658 & 0.046518805 & 1.023899054 \end{bmatrix}$$

$$1200 \cdot \Sigma^N = \begin{bmatrix} 0.121310292 & -0.001120483 & 0.017772738 & -0.010673926 \\ -0.001120483 & 0.040365822 & -0.000722471 & -0.005823213 \\ 0.017772738 & -0.000722471 & 0.216823848 & -0.060490007 \\ -0.010673926 & -0.005823213 & -0.060490007 & 0.125161344 \end{bmatrix}$$

$$1200 \cdot \left(\widehat{\mu}^L\right)' = \begin{bmatrix} 1.042160928 & 11.26913544 & 1.098486828 & 0 \end{bmatrix}$$

$$\Phi^L = \begin{bmatrix} 0.838214082 & -0.031621587 & 0.008209919 & 0 \\ 0.207765979 & 0.972487616 & 0.007750149 & 0 \\ 0.219545459 & -0.048145485 & 0.899396139 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Phi^{\mathbb{Q}L} = \begin{bmatrix} 0.668162935 & 0.003825915 & 0.016039083 & -0.226841291 \\ 2.759901911 & 0.99487354 & -0.130299337 & -2.106975032 \\ 0.07472393 & 0.014486102 & 0.961685009 & -0.075215625 \\ -0.014120387 & -0.006421157 & 0.033970704 & 0.909223315 \end{bmatrix}$$

$$1200 \cdot \Sigma^L = \begin{bmatrix} 0.15582737 & 0.009291447 & -0.011291526 & 0 \\ 0.009291447 & -0.051521605 & 0.016629525 & 0 \\ -0.011291526 & 0.016629525 & 0.208074635 & 0 \\ 0 & 0 & 0 & 0.535755422 \end{bmatrix}$$

$1200 \cdot \sigma_3^m = 0.069764292$ $1200 \cdot \sigma_6^m = 0.126883278$ $1200 \cdot \sigma_{12}^m = 0.208901973$ $1200 \cdot \sigma_{24}^m = 0.32667924$

$1200 \cdot \sigma_{36}^m = 0.212368736$ $1200 \cdot \sigma_{60}^m = 0.140939895$ $1200 \cdot \sigma_{120,index}^m = 0.168432273$

$\pi_{NN}^Q = 0.999083132$ $\pi_{LL}^Q = 0.954034188$

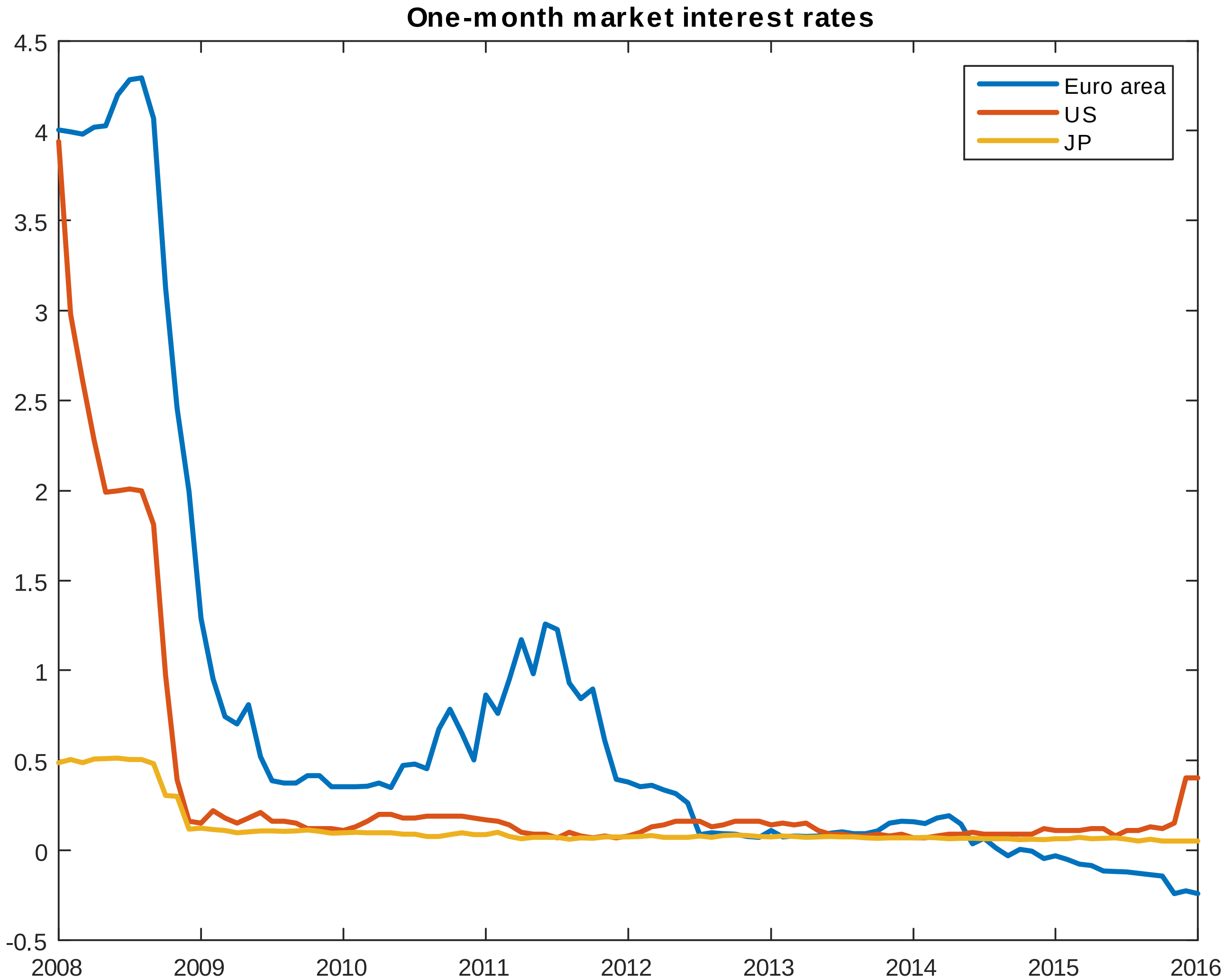
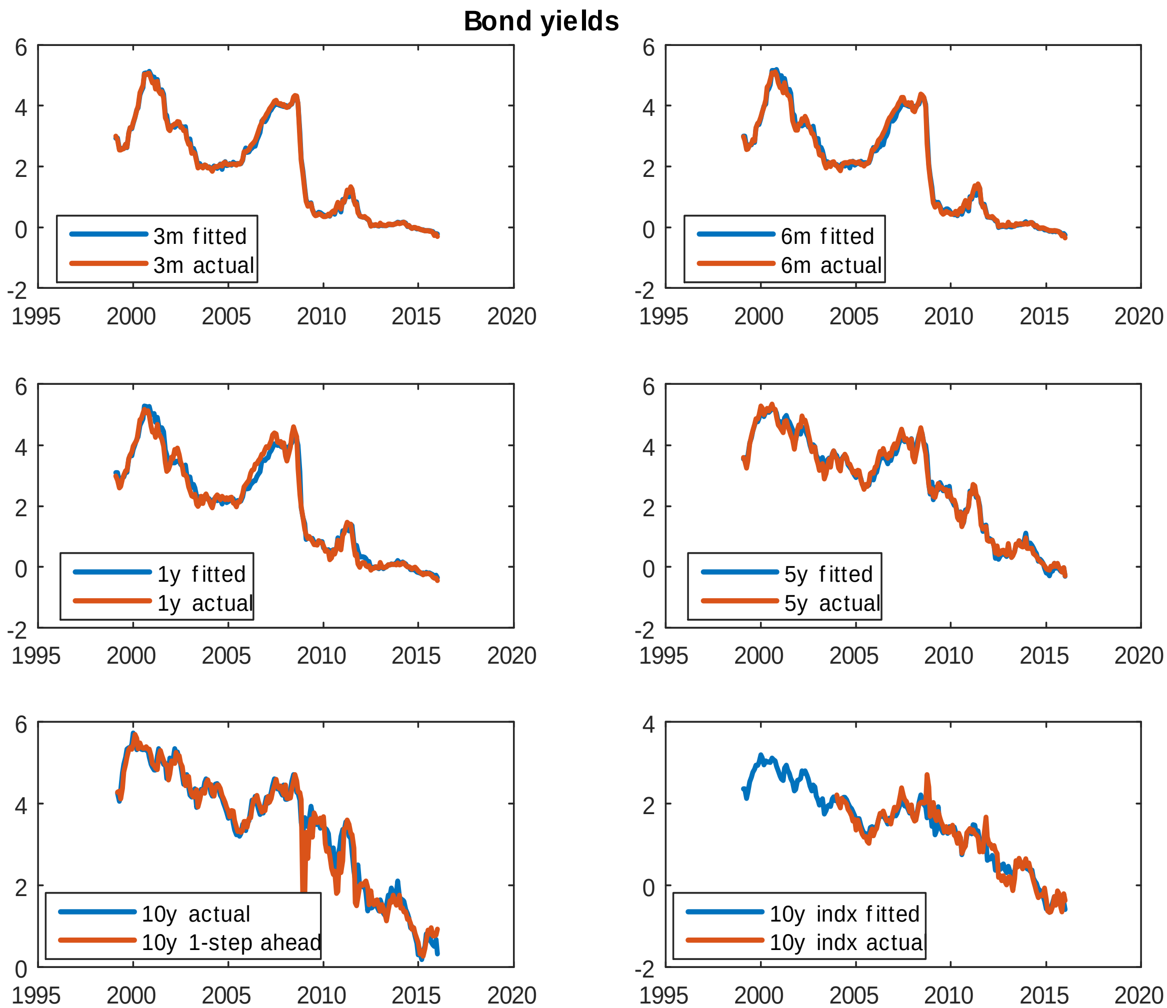

One-month market interest rates
Euro area
US
JP
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
2008
2009
2010
2011
2012
2013
2014
2015
2016

Figure 1



Bond yields
3m fitted
3m actual
6m fitted
6m actual
1y fitted
1y actual
5y fitted
5y actual
10y actual
10y 1-step ahead
10y indx fitted
10y indx actua

Figure 2

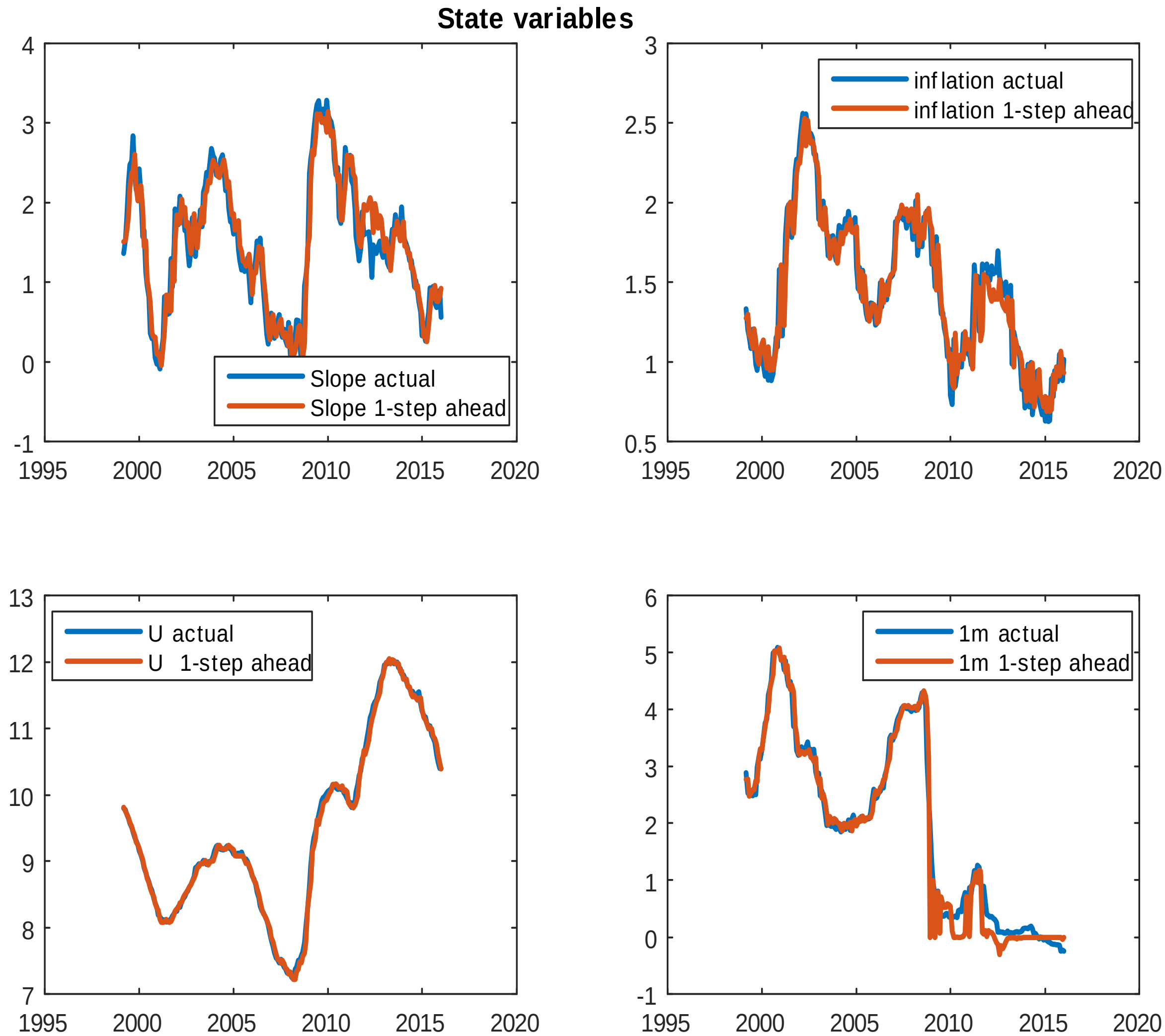

State variables
Slope actual
Slope 1-step ahead
inflation actual
inflation 1-step ahead
U actual
U 1-step ahead
1m actual
1m 1-step ahead
1995
2000
2005
2010
2015
2020

Figure 3

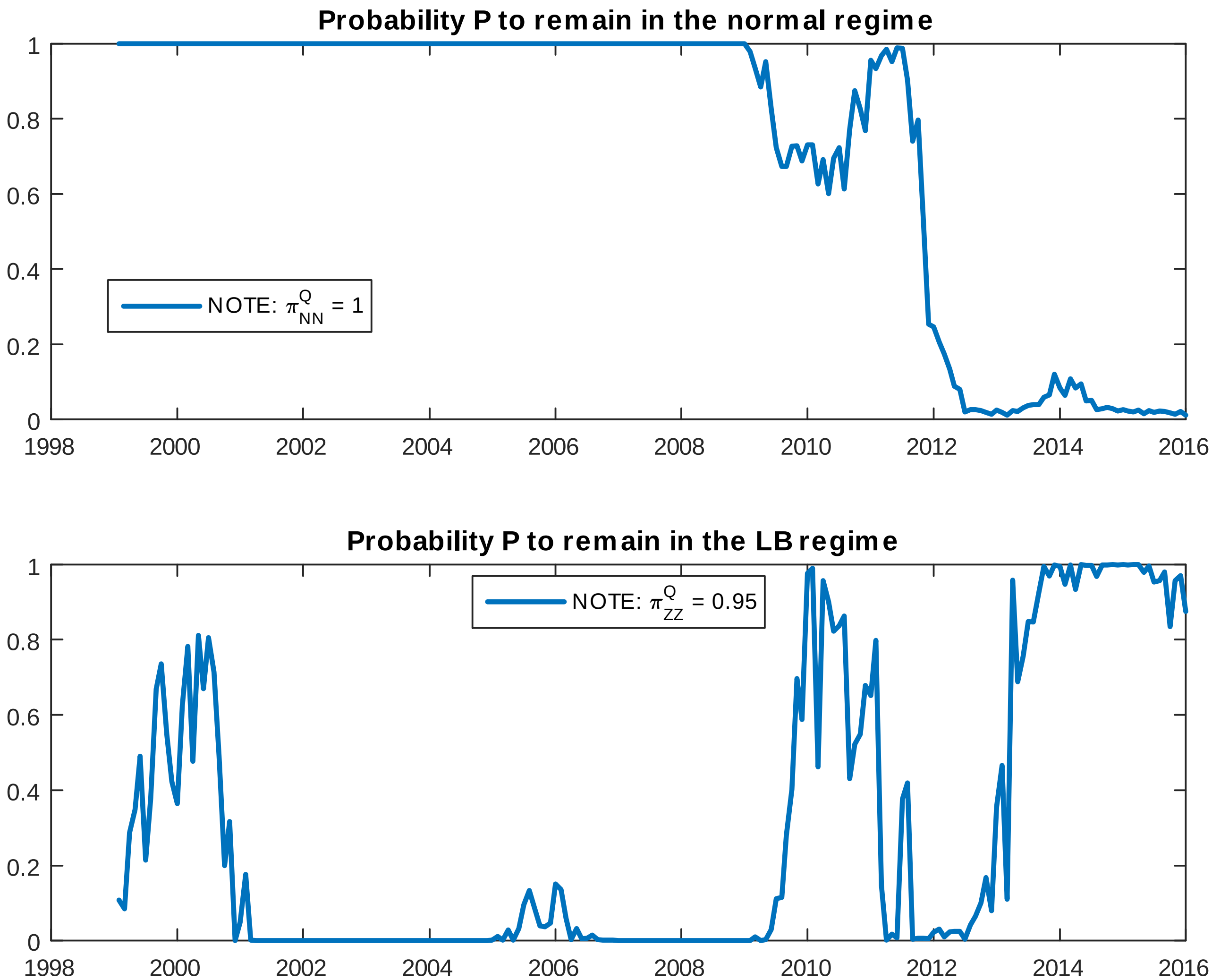
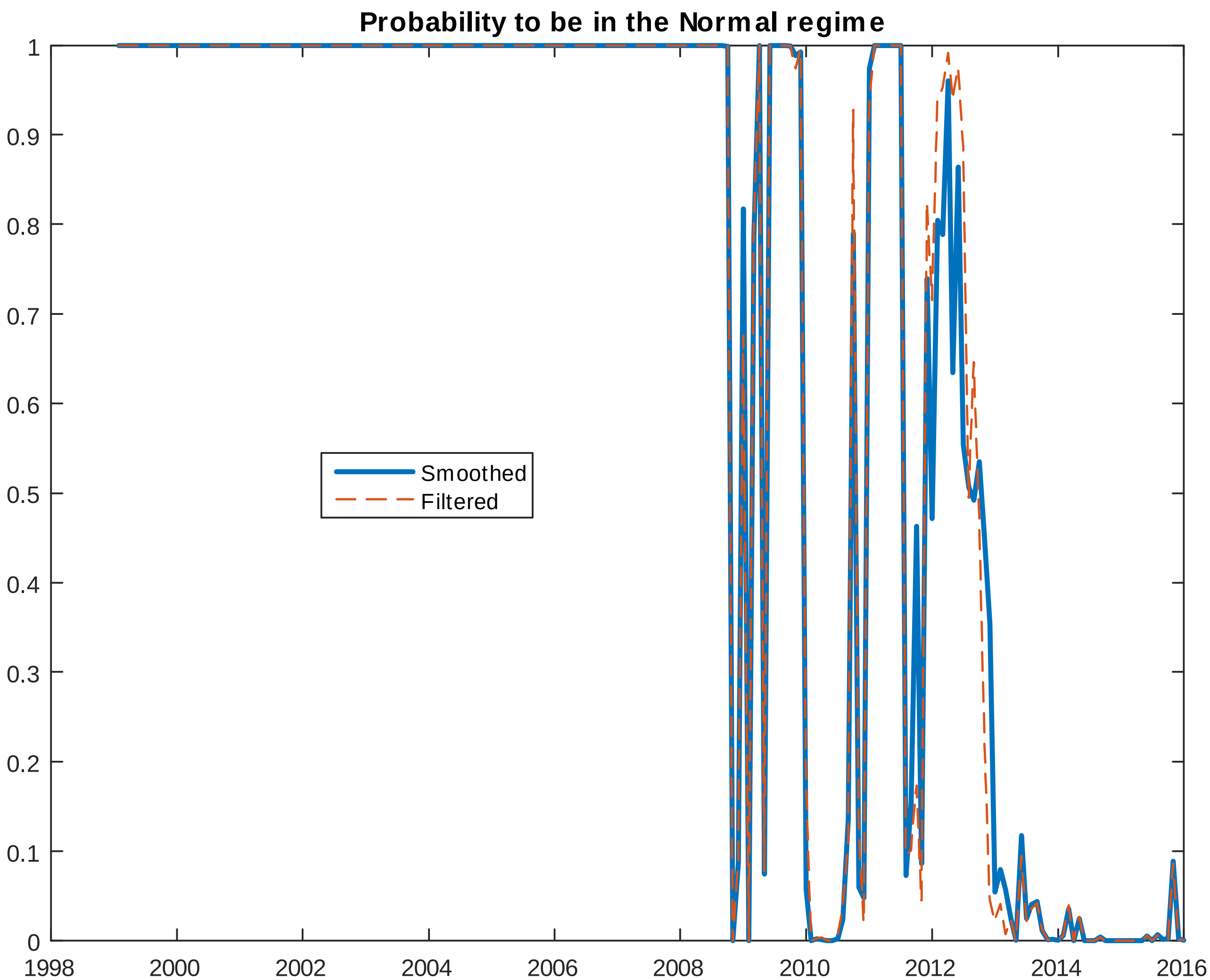
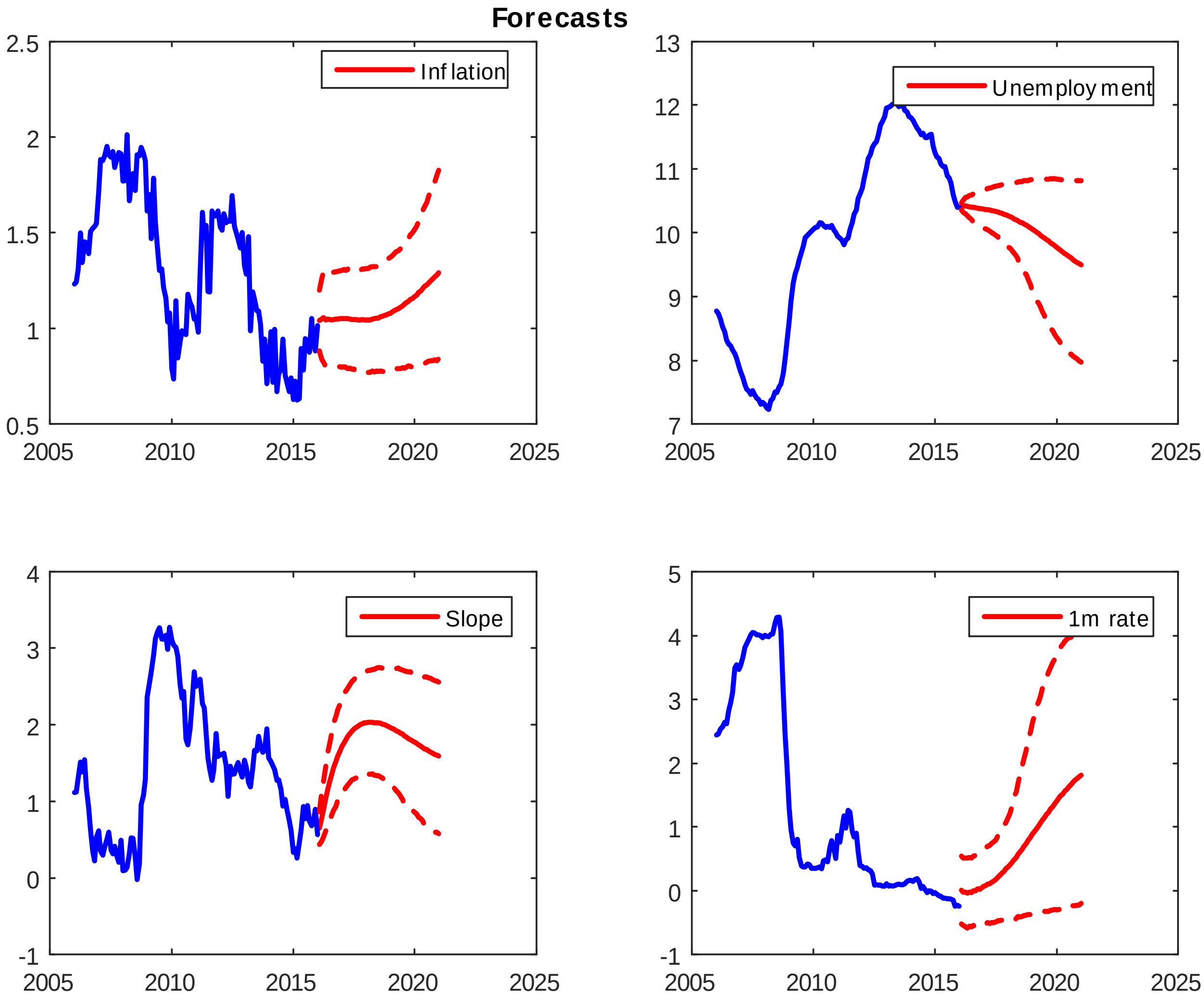

Probability P to remain in the normal regime
1
0.8
0.6
0.4
0.2
0
1998
2000
2002
2004
2006
2008
2010
2012
2014
2016
NOTE: π^{Q}_{NN} = 1
Probability P to remain in the LB regime
1
0.8
0.6
0.4
0.2
0
1998
2000
2002
2004
2006
2008
2010
2012
2014
2016
NOTE: π^{Q}_{ZZ} = 0.95

Figure 4



Probability to be in the Normal regime
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1998
2000
2002
2004
2006
2008
2010
2012
2014
2016
Smoothed
Filtered

Figure 5



Forecasts
Inflation
Unemployment
Slope
1m rate
2.5
2
1.5
1
0.5
13
12
11
10
9
8
7
4
3
2
1
0
-1
5
2005
2010
2015
2020
2025

Figure 6

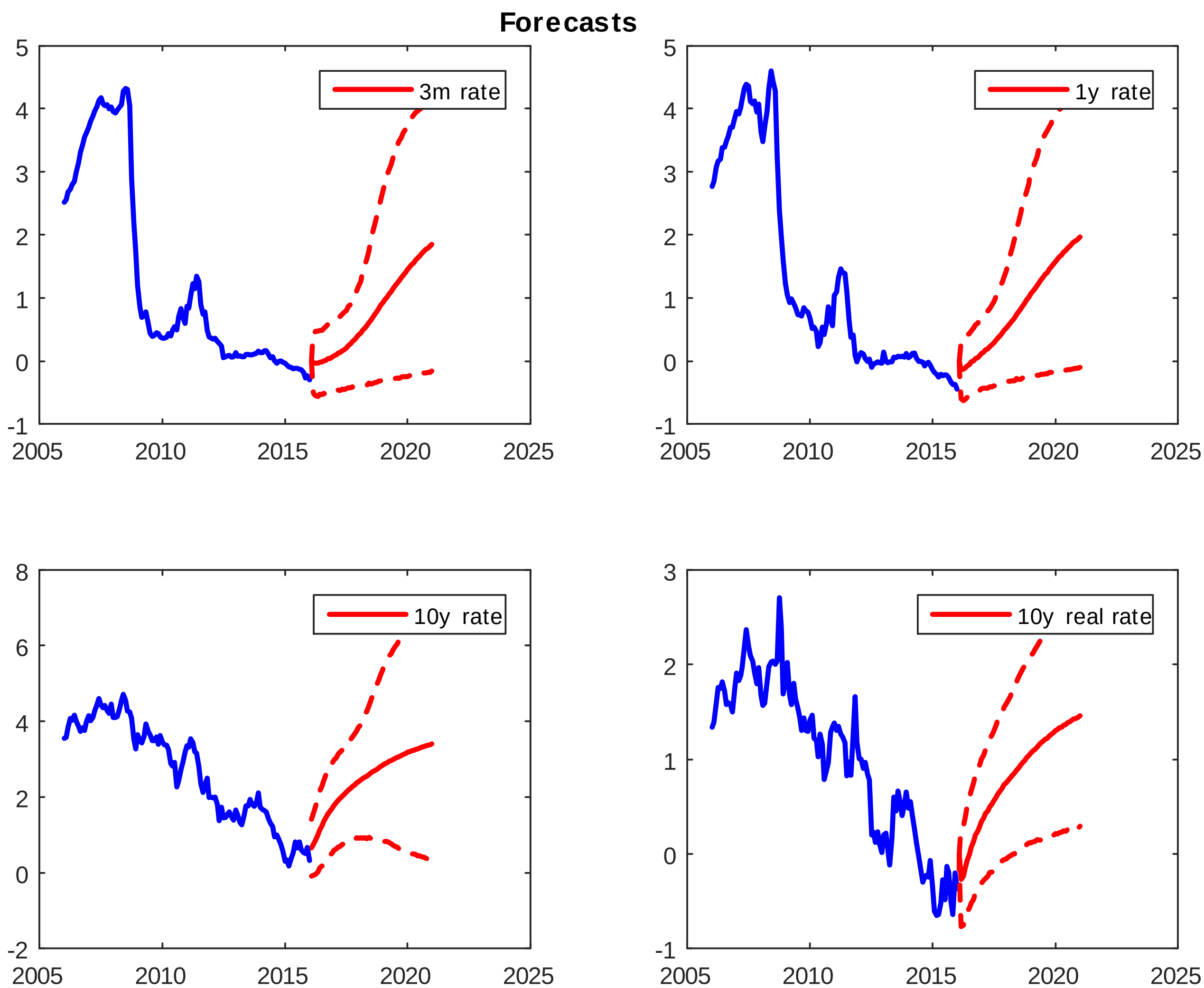

Forecasts
3m rate
1y rate
10y rate
10y real rate
2005
2010
2015
2020
2025

Figure 7

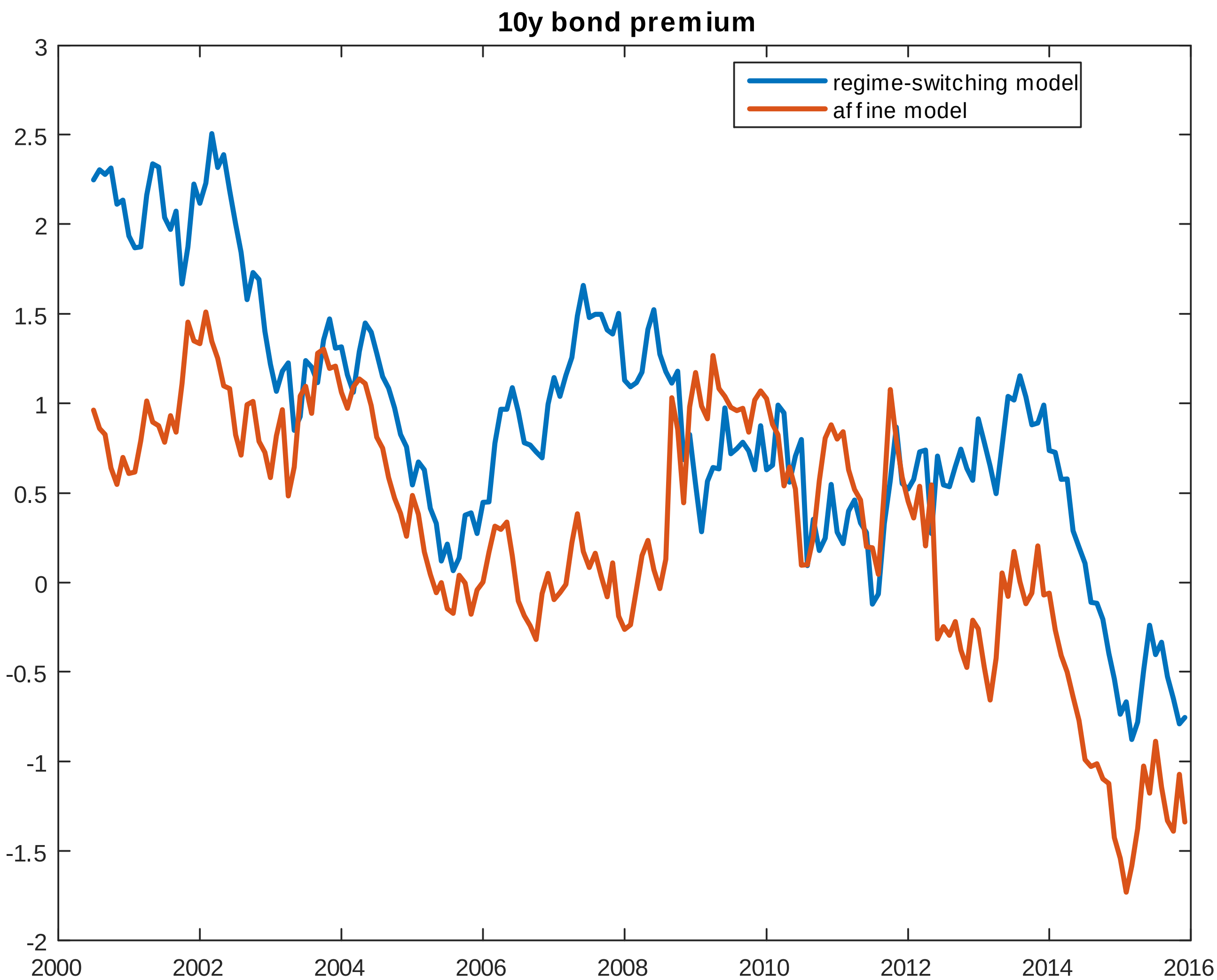

10y bond premium
regime-switching model
affine model
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
2000
2002
2004
2006
2008
2010
2012
2014
2016

Figure 8

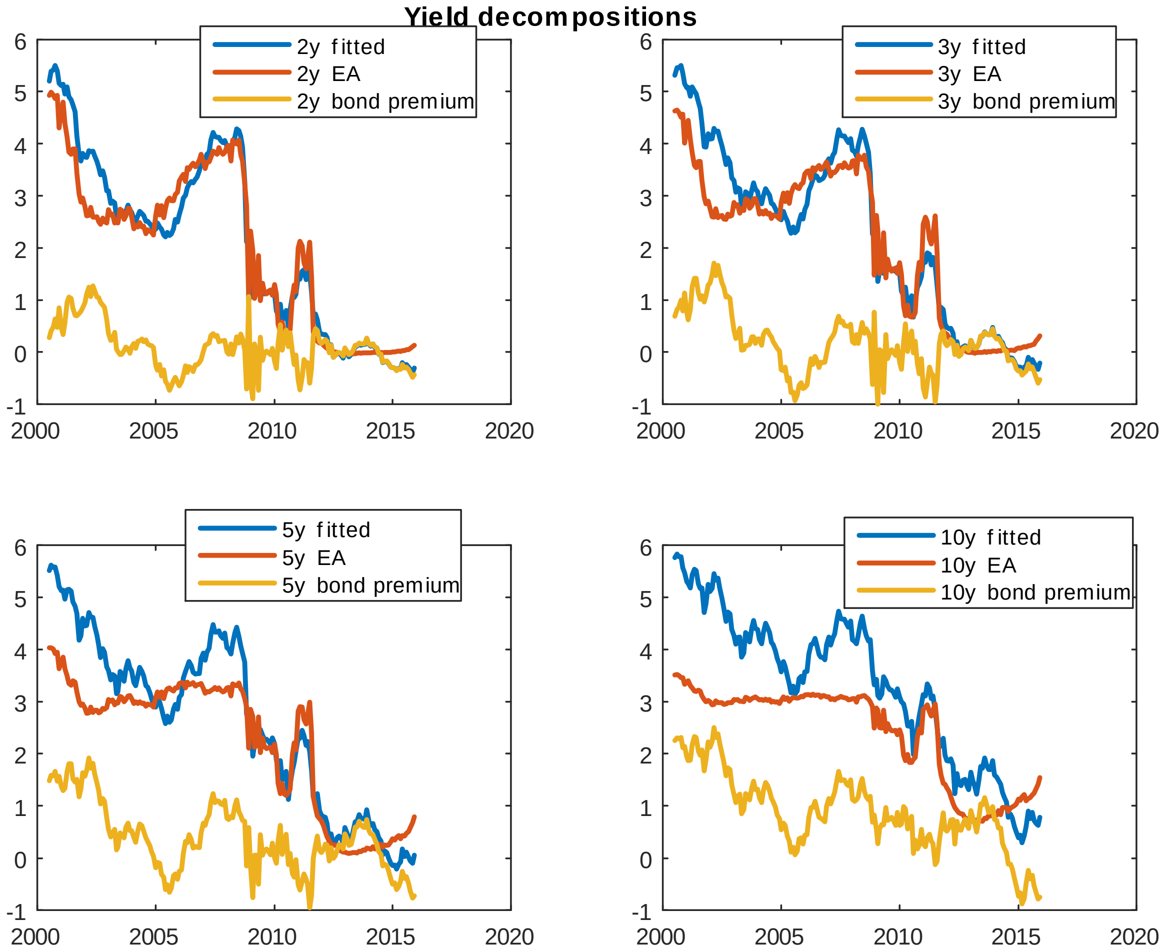
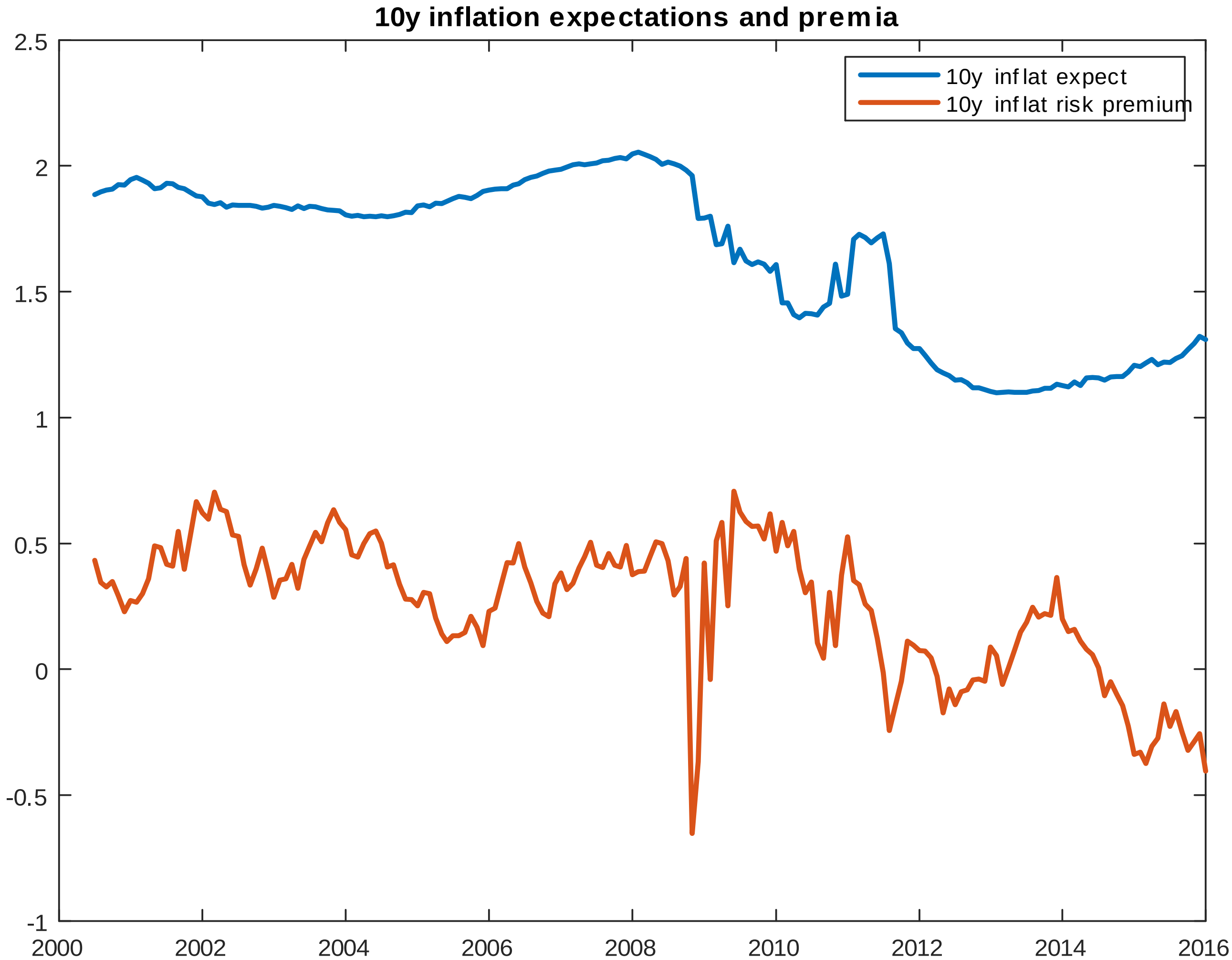

Yield decompositions
2y fitted
2y EA
2y bond premium
3y fitted
3y EA
3y bond premium
5y fitted
5y EA
5y bond premium
10y fitted
10y EA
10y bond premium
6
5
4
3
2
1
0
-1
2000
2005
2010
2015
2020

Figure 9



10y inflation expectations and premia
10y inflat expect
10y inflat risk premium
2.5
2
1.5
1
0.5
0
-0.5
-1
2000
2002
2004
2006
2008
2010
2012
2014
2016

Figure 10