COMPUTING THE CROSS-SECTIONAL DISTRIBUTION TO APPROXIMATE STATIONARY MARKOV EQUILIBRIA WITH HETEROGENEOUS AGENTS AND INCOMPLETE MARKETS

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Dynamic stochastic general equilibrium models with heterogeneous agents and incomplete markets usually have to be solved numerically. Existing algorithms assume bounded rationality of the agents meaning that they only keep track of a limited number of moments of the cross-sectional distribution. In this paper, we do not take this assumption and derive a law of motion of the whole state distribution. The proposed algorithm jointly solves for a fixed point of the equilibrium's Euler equation and its distribution's law of motion. Convergence to the equilibrium's optimal policy and the stationary state distribution is shown. We compare our numerical solution to the most prominent existing algorithm, the Krusell-Smith algorithm. Apart from improved error distributions and Pareto-improved policies for our algorithm, we show that the Krusell-Smith algorithm does not converge when using the limit of its simulation step to obtain the law of motion.

Keywords: Dynamic stochastic general equilibrium, Ergodic Markov equilibrium, Incomplete markets, Heterogeneous agents, Aggregate uncertainty, Numerical solutions.

JEL Classification: C63, C68, D31, D52, D58, E21.

1. INTRODUCTION

Economies consist of heterogeneous agents who are exposed to idiosyncratic risks, the most prominent example of which is labor income risk for households. This was first modeled in a dynamic stochastic general equilibrium (DSGE) model

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by Bewley (1977) where agents face idiosyncratic income shocks affecting their wealth and extended by Aiyagari (1994) to include a production technology. They show that individual precautionary savings contribute to aggregate savings because idiosyncratic risk cannot be fully insured. In general, idiosyncratic risks affect aggregate variables in the economy. Other examples of idiosyncratic risks are firm-specific productivity shocks in models of firm exit and entry as in Hopenhayn (1992) or county-specific productivity shocks in real business cycle models as in den Haan et al. (2011).

These models, however, do not feature aggregate risk because it makes the equilibrium problem difficult to solve. The challenge in the construction of solution algorithms lies in the cross-sectional distribution of the agents' characteristics which becomes an infinite-dimensional element of the state space. Krusell and Smith (1998) were the first to propose a numerical algorithm for the Aiyagari growth model with aggregate risk. They handle the dimensionality problem in assuming bounded rationality of the agents. It means that the agents can only observe a limited number of moments of the cross-sectional distribution to decide on their policy. The authors solve for the optimal policy and the law of motion of these moments in a two-step iterative procedure. In the first step, the policy function is computed by iterating the Euler equation for a fixed law of motion. In the second step, the exogenous shocks are simulated across time and agents and the policy is iteratively applied to compute individual capital. The new law of motion is inferred from this simulated data. Various more recent papers improve the original algorithm especially by eliminating the agent dimension in the simulation step. However, these works still rely on the bounded rationality assumption and a two-step iterative procedure.

There are several drawbacks to the Krusell-Smith algorithm and its more recent substitutes of similar style. Firstly, it is not clear a priori how many moments are necessary for the equilibrium to exist. In fact, it is shown in Kubler and Schmedders

(2002) that there are models for which recursive equilibria depending only on aggregate wealth, i.e. the first moment of the cross-sectional distribution, do not exist. Generic existence results for more general recursive equilibria with an ergodic state distribution are provided by Duffie et al. (1994). Kubler and Schmedders (2002) argue, however, that it is not feasible to compute such recursive equilibria because they and especially their ergodic state distributions are too abstract. The second drawback is that a component-wise fixed-point iteration as in the two-step procedure does not necessarily converge to a joint fixed point. Also, there are no theoretical convergence results for Krusell-Smith-style algorithms to show the contrary.

To the best of my knowledge, there is only one algorithm which does not rely on the bounded rationality assumption, namely Reiter (2009, 2010b). This algorithm first solves for the optimal policy and stationary distribution of the model without aggregate shocks using projection methods and then perturbs this solution to accommodate aggregate shocks. There are two major drawbacks. Firstly, the perturbation in aggregate shocks is linear. Therefore, any nonlinear effects of aggregate shocks are not accounted for. Secondly, as for all perturbation methods the solutions are only accurate for small aggregate shocks. Crises scenarios in terms of a large aggregate shock or a long series of aggregate shocks in one direction cannot be analyzed with confidence.

The contribution of this paper is to construct a general solution algorithm for DSGE models with heterogeneous agents and aggregate risk for which convergence is proven. We do not assume bounded rationality. Instead, we compute Duffie et al. (1994) (DGMM)-recursive equilibria. Furthermore, the dependance on the aggregate risk can be nonlinear and crisis scenarios can be analyzed as we do not rely on perturbations around a model without aggregate risk. In this algorithm, the dimensionality of the state space is reduced by focusing on one particular cross-sectional distribution at any point in time. For any fixed time series of aggregate shocks, the algorithm computes the optimal policy function at every time step. The law of motion of the cross-sectional distribution follows from the agents' optimal policy. Thus, the cross-sectional distribution is implicitly given at any time point except initial time. We choose the ergodic state distribution described in Duffie et al. (1994) for the initial distribution to be consistent w.r.t. the Markov equilibrium. We make this abstract concept of the stationary distribution concrete by deriving a computable law-of-motion operator which defines the stationary state distribution as its fixed point. After constructing the algorithm and showing its convergence, we illustrate these advantages by comparing the algorithm's numerical results for the growth model with aggregate risk to the results by the Krusell-Smith algorithm.

The second contribution of this paper concerns the two-step iterative procedure which almost all existing algorithms use. They solve for the optimal policy in the first step and separately for the distribution in the second step and then, they iterate on these two steps. In contrast, the algorithm presented in this paper solves for these two objects jointly in one step. We analyze the Krusell-Smith algorithm as one example for the two-step iterative procedure. In particular, we use the limiting Krusell-Smith algorithm where we derive the limit of the law of motion by taking the number of simulation steps in the time and the agent dimension to infinity. The striking outcome is that the two-step iterative procedure does indeed not converge to a solution for the limiting Krusell-Smith algorithm. This implies that the Krusell-Smith algorithm does only converge because it stops the simulation at finite time.

This paper is related to several strands of literature. The generic existence of solutions to DSGE models has been shown by Duffie and Shafer (1985, 1986) and Duffie et al. (1994). The latter paper establishes its results by finding a measurable selection of the expectations correspondence defining the equilibrium problem among the set of invariant measures. Magill and Quinzii (1996) differentiate various forms of debt constraints which ensure existence. As I construct approximate equilibria in the sense of Duffie et al. (1994), the theoretical part of this work is largely based on their paper. Additionally, the notion of a DGMM-recursive equilibrium is used which was minted by Kubler and Schmedders (2002).

Furthermore, the literature on numerical algorithms for DSGE models is related. The algorithm by Krusell and Smith (1998) has also been the subject of a special issue of the Journal of Economic Dynamics and Control in January 2010. This special issue presents various alternative algorithms and compares them in den Haan (2010). They have in common that they use a small finite number of moments instead of the full cross-sectional distribution to approximate the policy function and the law of motion of aggregate variables. One problem which is addressed by Algan et al. (2008); Young (2010); Ríos-Rull (1997) and summarized in Algan et al. (2010) is the cross-sectional variation due to the simulation of a finite number of agents in Krusell and Smith (1998). They use parametric and nonparametric procedures to get around this issue. However, the variation due to simulating over common exogenous shocks remains. Reiter (2010a) also parameterizes the cross-sectional distribution whereas den Haan and Rendahl (2010) use direct aggregation to obtain the moments of the cross-sectional distribution. All of the mentioned algorithms use projection methods or a hybrid of projection and simulation methods. The algorithm in Kim et al. (2010) differs as it utilizes a perturbation method. However, their approach includes a very simplified law of motion. All of these algorithms still rely on bounded rationality and componentwise fixed-point iteration.

One approach which differs from those above is the one by Mertens and Judd (2013). Their algorithm relies on a perturbation method as the authors use an approximation of the law of motion which is asymptotically true. They do prove the convergence of their algorithm. However, the approximated policy still depends on a finite number of moments, i.e. they assume bounded rationality. Besides, the basis of the perturbation method is a deterministic steady-state at which all agents are identical. This means that one cannot solve models with several agent types having conceptually different utility functions and constraints with this approach. Furthermore, this method is only applicable to models with differentiable policy functions which does not hold in the case of explicit debt constraints. In contrast, the approach presented herein can handle both smooth and nonsmooth policy functions. Besides, multiple agent types with conceptually different optimization problems are possible.

Another strand of literature focuses on solutions to mean field games. This field has been established by P.-L. Lions in lectures at the Collège de France. The present paper is related to that research area because mean field games are essentially continuous-time versions of DSGE models with heterogeneity and incompleteness. There are two different approaches to solve these models which are compared in Carmona et al. (2013), a PDE approach and a probabilistic approach. The construction of our algorithm is closer in spirit to the latter. Recently, the research on mean field games turned to numerical solutions in Achdou et al. (2014). They use partial differential equations to solve heterogeneous agent models. However, they are not able to solve models which include aggregate exogenous shocks. Their models solely incorporate idiosyncratic shocks.

The paper proceeds as follows. In the next section, we present the growth model which illustrates our algorithm throughout the paper. In Section 3, we introduce the methodology behind the algorithm. It explains what the stationary state distribution looks like and its defining law-of-motion operator is derived. Furthermore, the approximation of the future variables in the expectation of the Euler equation is explained. Section 4 then states the algorithm and proves the convergence result. In Section 5, our numerical results are compared to the ones from the Krusell-Smith algorithm and we show by contradiction that the limiting Krusell-Smith algorithm does not converge. The last section concludes. The appendix contains further proofs and the calibration of the model parameters.

2. THE MODEL

For illustration, we use the same growth model with aggregate shocks as in den Haan et al. (2010) which is used for a comparison of Krusell-Smith-style algorithms in the special issue of the Journal of Economic Dynamics and Control in January 2010. We consider a continuum of agents with measure one. There are two kinds of exogenous shocks, an aggregate shock and an idiosyncratic shock. The aggregate shock characterizes the state of the economy and has outcomes 0 and 1 standing for a bad and good state, respectively. The idiosyncratic shock characterizes individual employment which is i.i.d. across agents conditional on the aggregate shock. The idiosyncratic shock per agent has outcomes 0 and 1 indicating that the agent is unemployed or employed, respectively. We denote the compound exogenous process by $(z_t)_{t\in\mathbb{N}_0} = \left(z_t^{ag}, z_t^{id}\right)_{t\in\mathbb{N}_0}$ with state space $\mathcal{Z} = \{(0,0),(0,1),(1,0),(1,1)\}$. This notation focuses on the population of unemployed and employed in each economic state rather than each single agent. The transition probabilities $p^{z'|z}$ are exogenously given by a four-by-four matrix. [1]

The security market consists of a claim to aggregate capital $(K_t)_{t\in\mathbb{N}_0}$. Each agent's share of capital is denoted by $(k_t)_{t\in\mathbb{N}_0}$ which is an endogenous variable. It is constrained to the nonnegative real line $k_t \geq 0 \,\forall\, t \in \mathbb{N}_0$. The market-clearing condition naturally reads

$$K_t = \int_0^\infty v d\mu_t^k(v) \,\forall\, t \in \mathbb{N}_0, \tag{1}$$

where $\mu_t^k(v) = \mathbb{P}\left(k_t \leq v\right)$ is the cross-sectional distribution of individual capital at time t. The individual budget constraint at time $t+1$ is given by

$$k_{t+1} = I\left(z_{t+1};\mu_t^k\right) + \left[1 + R\left(z_{t+1}^{ag};\mu_t^k\right) - \delta\right] k_t - c_t.$$

[1] The calibrated values of this model are given in Appendix B.

Individual capital depreciates with rate δ. The individual's income is

$$I\left(z_{t+1};\mu_t^k\right)=z_{t+1}^{id}\left[1-\tau_{t+1}\right]W\left(z_{t+1}^{ag};\mu_t^k\right)+\left[1-z_{t+1}^{id}\right]\nu W\left(z_{t+1}^{ag};\mu_t^k\right).$$

It is composed of the individual's wage W when she is employed and the unemployment benefit otherwise where the former is subject to a tax $\tau_t=\frac{\nu(1-p_{t+1}^e)}{p_{t+1}^e}$ whose sole purpose it is to redistribute money from the employed to the unemployed. The parameter ν denotes the unemployment benefit rate whereas $p_{t+1}^e=\mathbb{P}(z_{t+1}^{id}=1|z_{t+1}^{ag})$ is the employment rate at time $t+1$. Wage and the rental rate R are derived from a Cobb-Douglas production function for the consumption good

$$\begin{aligned}W\left(z_{t+1}^{ag};\mu_t^k\right)&=(1-\alpha)P_{t+1}\left[\frac{K_t}{\bar{l}L_{t+1}}\right]^{\alpha}\\R\left(z_{t+1}^{ag};\mu_t^k\right)&=\alpha P_{t+1}\left[\frac{K_t}{\bar{l}L_{t+1}}\right]^{\alpha-1},\end{aligned}$$

where $P_{t+1}=1+z_{t+1}^{ag}a-(1-z_{t+1}^{ag})a$ is the aggregate productivity rate with average change a and α is the output elasticity parameter. The annualized labor supply is defined by $L_{t+1}=p_{t+1}^e$. It is multiplied with the time endowment factor $\bar{l}$.

We assume that all agents have time-separable CRRA utility with a risk aversion coefficient γ and time preference parameter β. Then, the individual optimization problem reads

$$\max_{\{c_t,k_t\}\in\mathbb{R}_{\geq 0}^2}\ \mathbb{E}\left(\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\gamma}-1}{1-\gamma}\right)\tag{2}$$
$$\text{s.t. } k_{t+1}=I\left(z_{t+1};\mu_t^k\right)+\left[1+R\left(z_{t+1}^{ag};\mu_t^k\right)-\delta\right]k_t-c_t\,\forall\,t\in\mathbb{N}_0$$

As the agent optimally transitions from k_t to k_{t+1}, the cross-sectional capital distribution changes accordingly. We denote the law of motion of the cross-sectional distribution of individual capital by $\mathbf{T}^{\mu}$, i.e. $\mu_{t+1}^k=\mathbf{T}^{\mu}\mu_t^k$.

In a competitive equilibrium, the individual problems are solved subject to the market-clearing condition. Thus, the endogenous variable which we need to solve for is individual capital and its distribution across agents. Note that the optimal individual consumption can always be recovered from the optimal individual capital choice by the budget constraint such that it suffices to solve for one of them. The existence of this competitive equilibrium is a well established result. It goes back to, inter alia, Duffie et al. (1994). Motivated by this existence result, Kubler and Schmedders (2002) introduced the notion of a Duffie et al. (1994) (DGMM)-recursive equilibrium which we briefly recall here.

Definition 1 (DGMM-recursive equilibrium) *Let us denote the aggregate and idiosyncratic exogenous variables of the equilibrium model by $z \in \mathcal{Z}$, $\mathcal{Z}$ finite, and all endogenous variables by $y \in \mathcal{Y}$. Price variables are specifically denoted by $q \in \mathcal{Y}$. A competitive equilibrium is called DGMM-recursive if there exists a multidimensional function $h^* : \mathcal{Z} \to \mathcal{Y}$ such that the optimal next-period endogenous variables and prices are given by*

$$
\begin{aligned}
y^*_{t+1} &= h^*_1\left(z_{t+1}; y^*_t, \mu^{y^*}_t\right) \; \forall\, t \in \mathbb{N}_0 \\
q^*_{t+1} &= h^*_2\left(z^{ag}_{t+1}; \mu^{y^*}_t\right) \; \forall\, t \in \mathbb{N}_0
\end{aligned}
$$

for each agent whose current-period endogenous variables have values y^*_t.

For ease of notation, we now switch to prime-notation where a prime denotes variables in the current period or equivalently the end of the period and variables with no prime refer to the previous period or equivalently the beginning of the period. The DGMM-recursive equilibrium for the growth model is actually dependent on the endogenous state $y = (c, k, K)$. However, K is derived from μ^k by market clearing and c is derived from k by the budget constraint such that we can write $k' = h^*(z'; y, \mu^y) = h^*(z'; k, \mu^k)$. Hence, we search for a recursive form $k' = h^*(z'; k, \mu^k)$ in the growth model. Moreover, we need to find the cross-sectional distribution $\mu^{k'}$ in order to have a full description of k' across agents. The following proposition yields an equation for the functional form h which stems from the Euler equation.

PROPOSITION 2 (The policy-function operator of the growth model) *Consider the equilibrium model with the individual problems as in (2) and the market-clearing condition (1). Any fixed point of the policy-function operator* $\mathbf{T}$ *which maps the set of continuous functions of the state space into itself and is defined by*

$$(\mathbf{T}h)\left(z';k,\mu^k\right) = \max\left(0, I\left(z';\mu^k\right) + \left[1 + R\left({z^{ag}}';\mu^k\right) - \delta\right]k - \{\beta\right.$$
$$\left.\mathbb{E}\left(\frac{1+R\left({z^{ag}}'';\mathbf{T}^\mu\mu^k\right)-\delta}{\{I\left(z'';\mathbf{T}^\mu\mu^k\right)+\left[1+R\left({z^{ag}}'';\mathbf{T}^\mu\mu^k\right)-\delta\right]h(z';k,\mu^k)-h(z'';h(z';k,\mu^k),\mathbf{T}^\mu\mu^k)\}^\gamma}\right)\right\}^{-\frac{1}{\gamma}}\right),$$

where $\mathbf{T}^\mu$ *is the law of motion of the cross-sectional distribution of individual capital consistent with (2), is a DGMM-recursive equilibrium of this model.*

The fixed-point problem $\mathbf{T}h^* = h^*$ cannot be solved in closed form, one can, however, apply numerical techniques to solve the problem. In the following section, I present the methodology which underlies the algorithm proposed subsequently.

3. THE METHODOLOGY

There are three difficulties in solving for a fixed point of a policy-function operator like the one in Proposition 2:

(i) The policy function depends on the cross-sectional distribution μ^k which is an infinite-dimensional object.

(ii) The law of motion of the cross-sectional distribution $\mathbf{T}^\mu$ has to be defined consistently w.r.t. the agents' optimal policy.

(iii) Solving for the policy at a distribution μ^k requires the knowledge of the policy at the next-period distribution ${\mu^k}'$.

The well-known algorithm by Krusell and Smith (1998) approaches the first problem by assuming bounded rationality. They suppose that the agents observe only a limited number of moments of the cross-sectional distribution. The algorithm presented herein, however, does not rely on such an assumption. Furthermore, in contrast to the Krusell-Smith algorithm which uses a parametric functional form for the law of motion, this algorithm relies on an exact formula implied by the agents' policy. Due to the assumption of bounded rationality, the third problem is easily solved in the Krusell-Smith algorithm whereas we have to deal with this issue more carefully. The following three subsections explain in detail how these three problems are approached.

3.1. *The Stationary Distribution*

To compute the optimal policy function, we have to reduce the dimension of the space of distributions. I propose to focus on one cross-sectional distribution at a time. For any fixed trajectory of aggregate exogenous shocks $(z_t^{ag})_{t=1}^T$, the solution of the optimal policy is computed sequentially at every point in time starting at $t = 0$, then $t = 1$ and so on. For any time point $t > 0$, fixing the cross-sectional distribution is straightforward since the start distribution is given by the computations at the previous time point $t-1$. At initial time, however, it is not obvious which cross-sectional distribution should be used. It should ideally be in line with the stated infinite horizon equilibrium model. Therefore, I propose to compute the stationary state distribution in conjunction with the corresponding optimal policy at $t = 0$. I argue that this procedure covers a sufficient subset of the model solution as it enables the researcher to analyze her scenarios of interest in addition to analyzing the stationary distribution of the model's exogenous and endogenous processes. Especially the latter is novel for models with aggregate exogenous shocks.

The question is how this stationary distribution could be computed. The ergodic probability measure of a Markov equilibrium has been formally defined in Duffie et al. (1994). We briefly recall the relevant notions here. We collect all equations which describe the equilibrium, e.g. Euler equations and debt constraints, in the *expectations correspondence*, a point-to-set map $G : \mathcal{J} \to 2^{\mathbf{P}}$ from the set on which the equilibrium processes live, i.e. $\mathcal{J} \subset \mathcal{Z} \times \mathcal{Y}$, to the power set of probability measures which assign probability 1 to the graph of some DGMM-recursive functional form, i.e. $\mathbf{P} = \{\mu \in \mathcal{P}(\mathcal{Z} \times \mathcal{Y}) \,|\, \mu(\{(z, h^*(z)) \in \mathcal{Z} \times \mathcal{Y}\}) = 1\}$. A measurable selection of this expectations correspondence is called a *transition* and denoted by $\Pi : \mathcal{J} \to \mathcal{P}(\mathcal{J}), s \mapsto \Pi_s$.

Definition 3 (Law-of-Motion Operator) *The law-of-motion operator is an operator $\mathbf{D}^{\Pi} : \mathcal{P}(\mathcal{J}) \to \mathcal{P}(\mathcal{J})$ which maps the set of probability measures $\mathcal{J}$ into itself. For any $\mu \in \mathcal{P}(\mathcal{J})$ it is defined by*

$$(3) \qquad \left(\mathbf{D}^{\Pi}\mu\right)(S) = \int_{\mathcal{J}} \Pi_s(S) d\mu(s) \, \forall S \in \mathcal{J},$$

i.e. it describes the transition from the previous distribution to the current distribution of states.

Definition 4 (Ergodic Measure) *A fixed point of the law-of-motion operator $\mu^* = \mathbf{D}^{\Pi}\mu^*$ such that any set $\{A \subset \mathcal{J} \,|\, \Pi_s \in \mathcal{P}(A)$ for μ^*-a.e. $s \in A\}$ is either a null set or equals the support of μ^* is called an ergodic measure of the transition Π representing the equilibrium problem.*

Note that the fixed-point condition is the defining element of the ergodic measure, i.e. of stationarity. The second condition ensures that any state with positive probability can be reached at any point in time.

The law-of-motion operator in (3) describes the change of the joint distribution of exogenous and endogenous variables for all agents from one period to the next, i.e. how to move from μ^s to $\mu^{s'}$ where s denotes an element of the full state space in our growth model, i.e. $s = \left(z^{ag}, \left\{\left(z^{id}, k\right)^m\right\}_{m \in \mathcal{M}}\right)$ where m is an index for each single agent. For a continuum of agents, the state space is infinite-dimensional. However, due to the theory on propagation of chaos for interacting particles (see e.g. Sznitman, 1991), or in our context interacting agents, the measure on the state space can be simplified to

$$\mu^s = \mu^{z^{ag}} \left(\bigotimes_{m \in \mathcal{M}} \mu^{(z^{id},k)^m} \right) \stackrel{!}{=} \mu^{z^{ag}} \left(\bigotimes_{m \in \mathcal{M}} \mu^{z^{id},k} \right)$$

meaning that the measures on each set of agent-specific variables become independent for a continuum of interacting agents. In the context of continuous-time heterogeneous agent models, this result was also used in the mean field game literature (see e.g. Carmona et al., 2013). It hence suffices to look at the law of the joint random variable (z^{ag}, z^{id}, k).

To simplify the construction of the law-of-motion operator in the next subsection, note that we can look at the joint state distribution in terms of conditional distributions

$$(4) \qquad \mu^{z,k}(u,v) = \mu^{k|z\leq u}(v) \cdot \mu^{z}(u) = \mu^{k|z=u}(v) \cdot p^{z=u}.$$

The stationary distribution of the exogenous variables p^{*z} can be derived from the exogenously given transition probabilities $p^{z'|z}$ by solving for a fixed point of the matrix equation

$$(5) \qquad p^{*z=u} = \sum_{w\in\mathcal{Z}} p^{z'=u|z=w} p^{*z=w}.$$

Thus, it suffices to derive a computable expression for the law of motion of the c.d.f. of endogenous variables conditional on the exogenous variables $\mu^{k|z}$. Recall that $z = (z^{ag}, z^{id})$, i.e. we condition on aggregate as well as idiosyncratic variables. Therefore, the conditional individual capital distribution $\mu^{k|z}$ should be interpreted as the distribution of individual capital across all agents facing the same idiosyncratic and aggregate shock, i.e. these are the capital distributions of the employed population in the bad or good state and the unemployed population in the bad or good state, respectively.

3.2. *A Consistent Law of Motion*

We need to define the law of motion of the cross-sectional distribution such that it is consistent with the optimal policy. The cross-sectional distribution of individual capital moves according to how the agents optimally respond to a fixed trajectory of aggregate exogenous shocks $(z_t^{ag})_{t=1}^T$. Hence, we need to define the transition of the random variable k for this fixed trajectory of the aggregate shock. We know how $k|z$ transitions to $k'|z'$: Since we are looking for a DGMM-recursive equilibrium, the end-of-period individual capital k' of an agent with start capital k who faces the shock z' is given by $k'|z' = h^*\left(z'; k, \mu^k\right)$ such that its law of motion is

$$\mu^{k'|z'=u}(v) = \sum_{z^{id}\in\{0,1\}} \frac{p^{*z}}{p^{*z^{ag}}} \left[\frac{p^{z'=u|z}}{p^{*z'=u}} \int_0^\infty \mathbb{1}_{\left\{h^*\left(u;k,\mu^k\right)\leq v\right\}} d\mu^{k|z}(k)\right]$$

$$\text{(6)} \qquad \Rightarrow \left(\mathbf{T}^\mu \mu^{k|z}\right)(v|u) = \sum_{z^{id}\in\{0,1\}} p^{*z^{id}|z^{ag}} \frac{p^{z'=u|z}}{p^{*z'=u}} \mu^{k|z}\left(\mathcal{K}_u^v\right)$$

where

$$\text{(7)} \qquad \begin{aligned} \mathcal{K}_u^v &= \left\{k \in \mathbb{R}_{\geq 0} \mid h^*\left(u;k,\mu^k\right) \leq v\right\} \\ &= \left\{\left[0, \max_{k\in\mathbb{R}_{\geq 0}} k\right] \,\middle|\, h^*\left(u;k,\mu^k\right) = v\right\}. \end{aligned}$$

The second equality is due to monotonicity of h^* which denotes the fixed point of the policy-function operator in Proposition 2. It is easy to prove by contradiction that a fixed point of the policy-function operator is nondecreasing in k.

The distribution entering the policy function is the beginning-of-period cross-sectional distribution denoted by μ^k which derives from the end-of-previous-period conditional distributions $\mu^{k|z}$, i.e. the distributions across the unemployed and employed given z^{ag}. Note that with the occurrence of $z^{ag'}$, the proportions of these two groups change. Some employed will become unemployed and vice versa. These changes appear for instance in the law of motion in (6) through the weights in front of the conditional distributions. Similarly, they enter the beginning-of-period distribution

$$\text{(8)} \qquad \mu^k := \mu^{k|z^{ag'}=u_1,z^{ag}} = \sum_{z^{id'}\in\{0,1\}} p^{*z^{id'}|z^{ag'}=u_1} \sum_{z^{id}\in\{0,1\}} p^{*z^{id}|z^{ag}} \frac{p^{z'=u|z}}{p^{*z'=u}} \mu^{k|z},$$

which is an element of the policy function $h^*\left(u;k,\mu^k\right)$. To simplify notation, we use $h^*\left(u;k,\mu^{k|z}\right)$ instead in the following.

To fix μ^k for initial time, however, we use the stationary state distribution. In contrast to the law of motion equation above where we condition on z^{ag}, the aggregate shock before initial time is not known. Therefore, we consider the joint distribution of all states, i.e. also the aggregate exogenous states, as discussed in the previous subsection. In the following Lemma, we derive the law-of-motion operator which gives us a computable representation of the equilibrium's ergodic state distribution.

Lemma 5 (The law-of-motion operator) *Consider a DGMM-recursive equilibrium. We define the operator* $\mathbf{D}^{h^*}:\mathcal{P}(\mathcal{J})\to\mathcal{P}(\mathcal{J})$ *such that*

$$\left(\mathbf{D}^{h^*}\mu\right)(u,v)=\int_{\mathcal{J}}\Pi_{(z;y,\mu^{y|z})}(u,v)d\mu\left(z,y\right)\ \forall\ (u,v)\in\mathcal{J}$$

with

$$\Pi_{(z;y,\mu^{y|z})}(u,v)=p^{z'=u|z}\mathbb{1}_{\left\{h^*\left(u;y,\mu^{y|z}\right)\leq v\right\}}$$

where $p^{z'|z}$ *denotes the transition probabilities of the exogenous variables and* $\mu^{y|z}$ *denotes the distribution* $\mu\in\mathcal{P}(\mathcal{J})$ *conditional on the exogenous variables. Then, the following results hold.*

(i) $\mathbf{D}^{h^*}$ *is indeed a law-of-motion operator.*

(ii) The marginal distribution w.r.t. the exogenous variables of any fixed point $\mu^*=\mathbf{D}^{h^*}\mu^*$ *recovers the exogenous stationary distribution* p^{*z} *solving (5).*

(iii) Any fixed point of the operator $\mathbf{D}^{h^*}$ *with* $\operatorname{supp}\mu^*\subset\mathcal{J}$ *which satisfies that for any* $A\subsetneqq\operatorname{supp}\mu^*$

$$\sum_{u\in\mathcal{Z}}p^{z'=u|z}\mathbb{1}_{\{(u,h^*(u;y,\mu^y))\in A\}}<1 \text{ for } \mu^*\text{-a.e. } (z,y)\in A \tag{9}$$

is an ergodic state distribution of the DGMM-recursive equilibrium with tran-

sition Π.

(iv) The DGMM-recursive equilibrium with transition Π *is unique, i.e. there are no sunspot equilibria.*

Remark We call condition (9) the ergodicity condition. Given that we start in a subset A of the support of the measure, there must be at least a small positive probability that we leave this subset in the next period. Otherwise, points in the subset $\mathcal{J} \setminus A$ would never be visited in equilibrium implying that this set's positive probability would contain information solely for the first time period. This scenario would clearly interfere with the notion of ergodicity.

We can write the law-of-motion operator of our growth model as follows

$$\begin{aligned} \left(\mathbf{D}^{h^*}\mu\right)(u,v) &= \sum_{z\in\mathcal{Z}} \int_0^\infty p^{z'=u|z} \mathbb{1}_{\left\{h^*\left(u;k,\mu^k\right)\leq v\right\}} d\mu\left(z,k\right) \\ &= \sum_{z\in\mathcal{Z}} p^{z'=u|z} \mu\left(z,\mathcal{K}_u^v\right) \end{aligned}$$

where h^* is the fixed point of the policy-function operator in Proposition 2 and $\mathcal{K}_u^v$ is as in (7). The law-of-motion operator in terms of the conditional distributions follows immediately

$$\left(\tilde{\mathbf{D}}^{h^*}\mu^{k|z}\right)(v|u) = \sum_{z\in\mathcal{Z}} p^{*z} \frac{p^{z'=u|z}}{p^{*z'=u}} \mu^{k|z}\left(\mathcal{K}_u^v\right). \tag{10}$$

Notice the difference of this law-of-motion operator for the stationary state distribution and the law of motion $\mathbf{T}^\mu$ of the conditional cross-sectional distribution in (6). The former sums over all four possible exogenous states with the corresponding probability weighting because the previous exogenous state at initial time is not known. A fixed point of this equation yields the stationary state distribution. The latter, however, only sums over the two idiosyncratic shocks given the previous aggregate shock as we are looking at a fixed trajectory of aggregate shocks. Accordingly, the beginning-of-period cross-sectional distribution μ^k which enters the policy function at initial time changes to

$$\mu^k := \mu^{k|z^{ag'}=u_1} = \sum_{z^{id'}\in\{0,1\}} p^{*z^{id'}|z^{ag'}=u_1} \sum_{z\in\mathcal{Z}} p^{*z} \frac{p^{z'=u|z}}{p^{*z'=u}} \mu^{k|z}.$$

3.3. *Approximating the Next-Period Optimal Policy*

We observe from Proposition 2 that another challenge in the implementation of the policy-function operator is that it does not only depend on the optimal policy at the current time but also on the optimal policy at the next period. We solve for $h\left(.;.,\mu^k\right)$ but to do so we need to approximate $h\left(.;.,\mathbf{T}^{\mu}\mu^k\right)$. This is non-trivial. The method I choose is a perturbation method, i.e. I approximate $h\left(.;.,\mathbf{T}^{\mu}\mu^k\right)$ by a Taylor approximation of $h\left(.;.,\mu^k\right)$ w.r.t. its argument μ^k. This requires the derivatives of the policy function w.r.t. the cross-sectional distribution. We need to use calculus in measure spaces. This methodology is also used in the mean field game literature. To summarize the required technology, I follow Cardaliaguet (2013).

The key idea is to deal with probability measures as laws of random variables. Assume that Y is a square-integrable random variable on the endogenous state space $\mathcal{Y}$ with the associated probability measures $\mathcal{P}(\mathcal{Y})$. We denote the law of Y by $\mu^Y \in \mathcal{P}(\mathcal{Y})$. Now, consider a function $f : \mathcal{P}(\mathcal{Y}) \to \mathbb{R}$. Then, f can be extended to the space of square-integrable random variables on the endogenous state space $L^2(\mathcal{Y})$ by $f\left(\mu^Y\right) = F(Y)$ where $F : L^2(\mathcal{Y}) \to \mathbb{R}$ denotes the extension.

Definition 6 (Differentiability) *A function $f : \mathcal{P}(\mathcal{Y}) \to \mathbb{R}$ is called differentiable at $\mu^Y \in \mathcal{P}(\mathcal{Y})$ if its extension $F : L^2(\mathcal{Y}) \to \mathbb{R}$ is Fréchet-differentiable at $Y \in L^2(\mathcal{Y})$.*

Remark Due to the definition of the Fréchet-derivative the following holds for two random variables Y_0 and Y_1

$$F(Y_1) = F(Y_0) + \langle (DF)(Y_0), Y_1 - Y_0 \rangle + o\left(||Y_1 - Y_0||\right).$$

Defining the scalar product through the expectation and the norm accordingly leads to

$$F(Y_1) = F(Y_0) + \mathbb{E}^{\mu^{Y_0,Y_1}}\left[\partial_Y F(Y_0)\left(Y_1 - Y_0\right)\right] + o\left(||Y_1 - Y_0||\right).$$

In our growth model, we associate the current and next-period cross-sectional distributions with the random variables $\kappa \sim \mu^k$ and $\kappa' = h\left(z'; \kappa, \mu^k\right) \sim \mathbf{T}^\mu \mu^k$. Let us denote the extension by $\tilde{h}$. Then, the Taylor approximation of the latter is

$$\tilde{h}\left(.;.,\kappa'\right) = \tilde{h}\left(.;.,\kappa\right) + \mathbb{E}^{\mu^{\kappa,\kappa'}}\left[\partial_\kappa \tilde{h}(.;.,\kappa)\left(\kappa' - \kappa\right)\right] + o\left(||\kappa' - \kappa||\right)$$

which can be written as

$$\begin{aligned} h\left(.;.,\mathbf{T}^\mu \mu^k\right) &= h\left(.;.,\mu^k\right) + \mathbb{E}^{\mu^k}\left[\partial_\kappa \tilde{h}(.;.,\kappa)\left(h\left(.;\kappa,\mu^k\right) - \kappa\right)\right] \\ &\quad + o\left(||h\left(.;\kappa,\mu^k\right) - \kappa||\right). \end{aligned} \tag{11}$$

The derivative of $\tilde{h}$ w.r.t. κ is calculated by a fixed-point equation stemming from the corresponding derivative of the policy-function operator which depends on the aggregate capital $K = \mathbb{E}^{\mu^k}(\kappa)$. Therefore, apart from taking the derivative w.r.t. a distribution, we use a standard perturbation method.[2]

4. THE ALGORITHM AND ITS CONVERGENCE RESULT

The main goal of the algorithm is to solve for a fixed point of the policy-function operator $\mathbf{T}$ from Proposition 2 where the law of motion $\mathbf{T}^\mu$ is defined as

[2]Note that the perturbation part in this algorithm differs significantly from the perturbation in Reiter (2009, 2010b). We perturb the current-period optimal policy of the model with aggregate shocks around the current cross-sectional distribution to obtain the next-period optimal policy of the model with aggregate shocks whereas Reiter (2009, 2010b) perturbs the optimal policy and distribution of the model without aggregate shocks around the volatility of the aggregate shock when it is zero to obtain the policy and distribution of the model with aggregate shock.

in (6). This is done dynamically for a fixed trajectory of aggregate shocks $(z_t^{ag})_{t=0}^T$. The beginning-of-period distribution μ^k is fixed for any $t > 0$. It is computed from the end-of-period conditional distributions $\mu_{t-1}^{k|z}$ as in (8). This is not the case at initial time where we derive the beginning-of-period distribution from the conditioned stationary state distribution which has to be computed as a fixed point of the law-of-motion operator $\tilde{\mathbf{D}}^h$ in (10). Due to the interdependence of the policy-function operator and the law-of-motion operator, we search for a joint fixed point $H^* = [h^*, \mu^{*k|z}]$ of the coupled system of operator equations $H^* = \mathbf{S}H^*$ where

$$(12) \qquad \mathbf{S}H = \begin{bmatrix} \mathbf{T}H \\ \tilde{\mathbf{D}}^h H \end{bmatrix}.$$

How do we approach solving the fixed point equations $H^* = \mathbf{S}H^*$ for $t = 0$ and $h^* = \mathbf{T}h^*$ for $t > 0$? If one knows that the policy functions and the state distribution are smooth, one can simply run established Newton schemes to solve these equations. Convergence results for global Newton methods in the case of differentiable policy functions and distributions are well established (see e.g. Deuflhard, 2011). In the case that policy functions and the state distribution are not smooth, one can still use basic damped fixed-point iteration for which convergence has to be shown. For the growth model, the latter case applies. Previous research (see e.g. den Haan, 2010) has illustrated that the policy functions exhibit kinks for the unemployed agents. The reason is that the explicit debt constraint $k \geq 0$ is binding for very low capital. This kink also has an impact on the stationary state distribution which exhibits mass points as is shown below.

Proposition 7 (A condition for mass points) *Consider a DGMM-recursive equilibrium with explicit debt constraints $y \geq l$ for the optimal endogenous variables $y \in \mathcal{Y}$. Suppose that the following condition holds.*

(i) The DGMM-recusive functional form h^* is continuous in y and there exists

a $\hat{z} \in \mathcal{Z}$ *with* $p^{z'=\hat{z}|z=\hat{z}} > 0$ *such that* $h^*(\hat{z}; y, \mu^y) \leq y$*. Furthermore, the policy function* h^* *has a kink at* $\hat{y} := \max\{y \in \mathcal{Y} \,|\, h^*(\hat{z}; y, \mu^y) = l\} > l$ *and is strictly increasing in* $y \geq \hat{y}$*.*

Then, the marginal distribution w.r.t. y *of the stationary state distribution given by a fixed point of the law-of-motion operator from Lemma 5, has a mass point at the boundary* l*. Now, suppose that the following additional condition holds.*

(ii) There exists a $\bar{z} \in \mathcal{Z}$ *with* $p^{z'=\bar{z}|z=\hat{z}} > 0$ *and* $h^*(\bar{z}; l, \mu^y) > l$*.*

Then, the marginal distribution w.r.t. y *of the stationary state distribution exhibits more than one mass point.*

Remark According to this proposition, the two quintessential reasons for mass points in the stationary state distribution are kinks in the policy function paired with a discrete distribution of the finite exogenous state variables.

We therefore state the solution algorithm for a DGMM-recursive equilibrium of the growth model using damped fixed-point iteration in Algorithm 1. Note that this algorithm can be easily adapted to other models by exchanging k with the appropriate endogenous variables and using the corresponding policy-function operator. In the following, I prove convergence of this algorithm in general. The convergence theorem states conditions which a model has to satisfy such that the solution algorithm for DGMM-recursive equilibria converges to a true solution. Because this is the main theoretical result of this work, I include the proof in the main text.

Theorem 8 (Convergence of Algorithm 1) *Consider a competitive equilibrium model where the exogenous variables are denoted by* $z \in \mathcal{Z}$ *with* $\mathcal{Z}$ *finite and the endogenous variables are denoted by* $y \in \mathcal{Y}$*. We specifically denote the individual's consumption by* $c \in \mathcal{Y}$ *and the price variables by* $q \in \mathcal{Y}$*. The individual agent*

Algorithm 1 Solution algorithm for DGMM-recursive equilibria

A Initialization

Fix a grid for k and initialize the approximations of the policy function $h(z', k)$, and the conditional distributions $\mu^{k|z}(k)$.

Fix the order of Taylor approximation for the next-period policy $\eta > 0$, the termination criterion $\epsilon > 0$ and the damping parameter $\rho \in (0, 1)$.

B Iterative procedure

for t=0:T **do**

  while $\|d\| \geq \epsilon$ **do**

    if t=0 **then**

      Compute $H_{new} = \mathbf{S}H$ in (12) where $H = [h, \mu^{k|z}]$, $\tilde{\mathbf{D}}^{\mathbf{h}}$ as in (10) and $\mathbf{T}$ is as in Proposition 2 using the Taylor approximation of order η for the next-period policy in (11).

    else

      Compute $H_{new} = \mathbf{T}H$ where $H = h$ and $\mathbf{T}$ is as in Proposition 2 using the Taylor approximation of order η for the next-period policy in (11).

    end if

    Compute the difference $d = H_{new} - H$.

    Update $H = H + \rho d$.

  end while

  Save h and $\mu^{k|z}$ from time t and update $\mu^{k'|z'} = \mathbf{T}^{\mu}\mu^{k|z}$ according to (6).

end for

optimizes

$$\max_{\{c_t, y_t\} \in \mathcal{J}} \mathbb{E}\left(\sum_{t=0}^{\infty} \beta^t u(c_t)\right)$$

$$s.t.\ \ b\left(z_{t+1}, y_{t+1}, q_{t+1}, y_t, \mu_t^y\right) - c_t = 0\,\forall\, t \in \mathbb{N}_0$$

where the Banach space $\mathcal{J} \subset \mathcal{Z} \times \mathcal{Y}$ is a compact subset of the state space on which the equilibrium processes live. Furthermore, the market clears through a condition

$$m\left(\mu_{t+1}^y\right) = 0\,\forall\, t \in \mathbb{N}_0.$$

where the cross-sectional distribution is $\mu_{t+1}^y = \sum_{z^{id} \in \mathcal{Z}} p^{*z^{id}|z^{ag}=z_t^{ag}} \mu_{t+1}^{y|z}$ and the *conditional cross-sectional distributions evolve according to*

$$\mu_{t+1}^{y|z=u}(v) = \left(\mathbf{T}^{\mu}\mu_t^{y|z}\right)(v|u) = \sum_{z_t^{id}\in\mathcal{Z}} p^{*z_t^{id}|z_t^{ag}} \frac{p^{z'=u|z=z_t}}{p^{*z=u}} \int_{\mathcal{J}} \mathbb{1}_{\left\{h\left(u;y,\mu_t^y\right)\leq v\right\}} d\mu_t^{y|z}(y)\,.$$

where $y_{t+1}^* = h(u; y, \mu_t^y)$ denotes the optimal end-of-period capital choice in DGMM-recursive form by the agent when facing the shock $z = u$. We define $\mu_0^{y|z}$ as the fixed point of the law-of-motion operator

$$\mu^{y|z=u}(v) = \left(\tilde{\mathbf{D}}^{h}\mu^{y|z}\right)(v|u) = \sum_{z\in\mathcal{Z}} p^{*z} \frac{p^{z'=u|z}}{p^{*z'=u}} \int_{\mathcal{J}} \mathbb{1}_{\left\{h\left(u;y,\mu_0^y\right)\leq v\right\}} d\mu^{y|z}(y)$$

for any $(u, v) \in \mathcal{J}$. Assume that the stated equilibrium problem satisfies the following conditions.

- ***Utility:*** *The subjective discount factor is $\beta \in (0, 1)$. The intermediate utility $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a strictly monotone and strictly concave function in $C^2(\mathbb{R}_{\geq 0})$ satisfying the Inada conditions $\lim_{c\to 0} \frac{\partial}{\partial c}u(c) = \infty$ and $\lim_{c\to\infty} \frac{\partial}{\partial c}u(c) = 0$. The expected marginal utility vanishes $\lim_{t\to\infty} \beta^t \mathbb{E}\left(\frac{\partial}{\partial c}u(c_t)\right) = 0$.*
- ***Budget constraint:*** *The income and savings part of the budget constraint $b : \mathcal{J} \times \mathcal{P}(\mathcal{J}) \to \mathbb{R}$ is differentiable and concave in the endogenous variables $y \in \mathcal{Y}$.*

Then, Algorithm 1 converges to a DGMM-recursive equilibrium.

Remark Note that we do not deal with existence here. When the set $\mathcal{J}$ is compact and $\mathcal{Z}$ is finite, existence has been shown by Duffie et al. (1994). Compactness is usually ensured by debt restrictions according to Magill and Quinzii (1996).

Proof: We use the variational approach to solve the equilibrium problem. Therefore, at any point in time $t > 0$ where the conditional cross-sectional distributions $\mu^{y|z}$ are fixed according to the evolution of the aggregate shock, we solve the

Lagrange problem, i.e. $\max_{\lambda_m} \min_{y_{t+1}} \mathcal{L}(y_{t+1}, \lambda_m)$, with Lagrangian

$$\begin{aligned}\mathcal{L}(y_{t+1}, \lambda_m) &= -u\left(b\left(z_{t+1}, y_{t+1}, q_{t+1}, y_t, \mu_t^{y|z}\right)\right) \\ &\quad -\mathbb{E}\left(\beta u\left(b\left(z_{t+2}, y_{t+2}, q_{t+2}, y_{t+1}, \mathbf{T}^{\mu}\mu_t^{y|z}\right)\right)\right) + \lambda_m m\left(\mathbf{T}^{\mu}\mu_t^{y|z}\right)\end{aligned}$$

for any $y_t \in \mathcal{J}$ where $\lambda_m \in \mathbb{R}$ is a Lagrange multiplier. At initial time, we add the additional constraint for the stationary distribution, i.e. for any $y_0 \in \mathcal{J}$

$$\mathcal{L}_0(y_1, \lambda_m, \lambda_D) = \mathcal{L}(y_1, \lambda_m) + \lambda_D\left(\mu_0^{y|z}(y_0) - \tilde{\mathbf{D}}^h \mu_0^{y|z}(y_0)\right)$$

where $\lambda_D \in \mathbb{R}$ is a Lagrange multiplier. The first-order conditions for $\mathcal{L}$, i.e. the root of its subdifferential, are

$$\begin{aligned}0 &= \frac{\partial}{\partial c}u\left(b\left(z_{t+1}, y_{t+1}, q_{t+1}, y_t, \mu_t^{y|z}\right)\right)\frac{\partial}{\partial y_{t+1}}b\left(z_{t+1}, y_{t+1}, q_{t+1}, y_t, \mu_t^{y|z}\right) \\ &\quad +\mathbb{E}\left(\beta\frac{\partial}{\partial c}u\left(b\left(z_{t+2}, y_{t+2}, q_{t+2}, y_{t+1}, \mu_{t+1}^{y|z}\right)\right)\right. \\ &\quad \left.\frac{\partial}{\partial y_{t+1}}b\left(z_{t+2}, y_{t+2}, q_{t+2}, y_{t+1}, \mu_{t+1}^{y|z}\right)\right) \\ 0 &= m\left(\mu_{t+1}^{y|z}\right)\end{aligned}$$

for any $y_t \in \mathcal{J}$. At initial time, $\mu_0^{y|z} = \tilde{\mathbf{D}}^h \mu_0^{y|z}$ has to hold in addition. The first-order conditions are necessary and, due to the vanishing marginal utility, also sufficient for optimality as has been shown in Kubler and Schmedders (2002) and Duffie et al. (1994).

The main step in the proof of convergence is to show that the subdifferential of the Lagrangian $\mathcal{L}$ [$\mathcal{L}_0$] denoted by $\partial\mathcal{L}$ [$\partial\mathcal{L}_0$] is a maximal monotone operator. Minsky's theorem (see e.g. Bauschke et al., 2012, Fact 1.2) states that the resolvent of a maximal monotone operator is firmly nonexpansive. Furthermore, convergence of damped fixed-point iteration for nonexpansive operators is well known (see e.g. Zeidler, 1986, p. 481). So we conclude the proof by showing that the policy-function operator $\mathbf{T}$ [the system of operator equations $\mathbf{S}$] in Algorithm 1 coincides with the resolvent of the Lagrangian's subdifferential.

Firstly, Rockafellar's theorem (see e.g. Phelps, 1997, Theorem 2.15) states that the subdifferential of a proper lower semicontinuous convex function on a Banach space is a maximal monotone operator. Therefore, we have to show that the Lagrangian $\mathcal{L}$ [$\mathcal{L}_0$] is a proper lower semicontinuous convex function on a Banach space. The Lagrangian is defined on the product topology of the two Banach spaces $\mathcal{J}$ for the optimal policy and $\mathbb{R}$ [$\mathbb{R}^2$] for the Langrange multiplier. Furthermore, it is obviously convex in y_{t+1} [y_1] because the composition of concave functions when the outer function is increasing is concave. The Lagrangian is linear w.r.t. the Lagrange multiplier and hence, also convex. Continuity follows alike. The Lagrangian is proper convex because its effective domain is nonempty and it does not take the value $-\infty$. Hence, all conditions are satisfied for applying Rockafellar's theorem.

Secondly, the resolvent of the Lagrange function is defined by $(\partial\mathcal{L}+I)^{-1}$ where I is the identity operator. We can write the first-order equations as $\partial\mathcal{L}+I=I$. Inverting the left-hand side yields firstly, the policy-function operator which is the inverse of the Euler equation w.r.t. y_{t+1} and secondly, $0 = m\left(\mu_{t+1}^{y|z}\right)$ [and $\mu_0^{y|z} = \tilde{\mathbf{D}}^h \mu_0^{y|z}$]. We thus recover the operators from Algorithm 1. They are nonexpansive and therefore, damped fixed-point iteration converges to a solution of the first-order equations. Since we solve for $y_{t+1} = h(z_{t+1}; y_t, \mu_t^y)$ [$y_1 = h(z_1; y_0, \mu_0^y)$], the solution represents a DGMM-recursive equilibrium. *Q.E.D.*

It is trivial to check that the growth model satisfies all conditions of the convergence theorem. The solutions produced by Algorithm 1 hence represent a true DGMM-recursive equilibrium when approximation errors are small.

5. COMPARISON TO THE KRUSELL-SMITH ALGORITHM

Let us now benchmark our algorithm to the most prominent algorithm for solving DSGE models with heterogeneity and incomplete markets. It was introduced by Krusell and Smith (1998). They investigate the impact of uninsurable risk, i.e.

idiosyncratic shocks, on the aggregate variables in the economy when there is aggregate risk. The assumption they make to be able to pose their algorithm is bounded rationality of the agents meaning that they can only observe a limited number of moments of the cross-sectional distribution instead of the whole object. Furthermore, they approximate the law of motion of these moments parametrically. We use a recent implementation of the Krusell-Smith algorithm to compare our results to which stems from Maliar et al. (2010). The Krusell-Smith algorithm can be summarized as in Algorithm 2.

Algorithm 2 The Krusell-Smith algorithm

A Initialization

Select a set I of moments for the cross-sectional distribution μ^k. [Krusell and Smith (1998) choose the mean $K = \mathbb{E}^{\mu}(k)$.]

Select a parametric form h_I for the flow of moments I of the cross-sectional distribution μ^k with parameters b. [Krusell and Smith (1998) choose $h_I(z^{ag'}, K; b) = e^{b_1^{z^{ag'}} + b_2^{z^{ag'}} \log(K)}$, where $b_1^{z^{ag'}}$ and $b_2^{z^{ag'}}$ are constants.]

Fix a grid for (k, I). Initialize the approximation of the policy function $k' = h(z', k, I)$ and the law of motion $h_I(z^{ag'}, I; b)$.

Set the termination criteria $\epsilon_I, \epsilon_h > 0$ and the damping parameter $\rho \in (0, 1)$. Set the number of agents M and the number of simulation periods T large.

B Iterative procedure

while $\|d_I\| \geq \epsilon_I$ **do**

B.1 Solving for the policy function h given the law of motion h_I

  while $\|d_h\| \geq \epsilon_h$ **do**

    Compute h_{new} by the Euler equation.

    Compute the difference $d_h = h_{new} - h$.

    Update $h = h + \rho d_h$.

  end while

B.2 Estimating the parameters in h_I given the policy function h

  Simulate the individual capital for M agents over T periods using h.

  Estimate the parameters b_{new} of h_I from the stationary region of the simulated data. [Krusell and Smith (1998) use linear regressions.]

  Compute the difference $d_I = b_{new} - b$.

  Update $b = b + \rho d_I$.

end while

There are three conceptual differences between our algorithm and the Krusell-Smith algorithm. Firstly, we do not assume bounded rationality and therefore, the agents do consider the whole cross-sectional distribution rather than a set of moments when choosing their optimal policy. Secondly, instead of conjecturing a parametric approximation of the law of motion, we use the definition of the distribution $\mu^{k'}(v) = \mathbb{P}(h(z'; k, \mu^k) \leq v)$. Lastly, we jointly iterate on the policy and the distribution rather than doing it component-wise. We look at the implications of the first two differences on the equilibrium solutions in the following subsection. In the subsequent subsection, we show that component-wise fixed-point iteration as used in the Krusell-Smith algorithm might fail to converge.

5.1. *Comparing Error Distributions and Main Implications*

We compute the DGMM-recursive equilibrium solution of Algorithm 1 for the first order of Taylor approximation, i.e. $\eta = 1$, using Matlab.[3] As an example, we analyze the worst-case scenario for 25 periods, i.e. we set the 25 aggregate shocks to the bad economic state. As we consider months as the time period in our calibration of the model, this corresponds to a two year long downturn after initial time. Furthermore, we run the Krusell-Smith algorithm implemented by Maliar et al. (2010) in Matlab for the same grid on individual capital and the same policy function initialization to obtain a Krusell-Smith equilibrium solution. Since Algorithm 2 does not explicitly compute the stationary distribution, we solve for it by finding a fixed point of the law-of-motion operator

$$\left(\mathbf{D}^{h_{KS}} \mu_{KS}^{k|z}\right)(v|u) = \sum_{z \in \mathcal{Z}} p^{*z} \frac{p^{z'=u|z}}{p^{*z'=u}} \mu_{KS}^{k|z} \left(\tilde{\mathcal{K}}_u^v\right)$$

where

$$\tilde{\mathcal{K}}_u^v = \left\{ \left[0, \max_{\tilde{k} \in \mathbb{R}_{\geq 0}} \tilde{k}\right] \,\middle|\, h_{KS}\left(u, \tilde{k}, \mathbb{E}^{\mu_{KS}^k}(k)\right) = v \right\}.$$

We then use the corresponding law of motion $\mathbf{T}^{\mu}$ to compute the distributions and their first moments produced by the Krusell-Smith policy at every time point in

[3]The computations were performed on the Baobab cluster at the University of Geneva.

the worst-case scenario.

One way of comparing these two sets of numerical solutions is to analyze their errors. There have been two error tests put forward in the literature (see e.g. den Haan, 2010), the standard Euler equation error test and the dynamic Euler equation error test. The standard Euler equation errors are calculated by comparing the numerical solution for optimal consumption $c(z'; k; \mu^k)$ against the explicitly calculated conditional expectation in the Euler equation denoted by $\tilde{c}(z'; k, \mu^k)$. It is the percentage error

$$\epsilon^{SEE}(z'; k, \mu^k) = \frac{c(z'; k; \mu^k) - \tilde{c}(z'; k, \mu^k)}{\tilde{c}(z'; k, \mu^k)}.$$

In contrast to the standard Euler equation error, the dynamic equivalent denoted by ϵ^{DEE} is computed for several consecutive periods. This test is more stringent as the numerical solution and the explicit conditional expectation usually diverge with more periods. Let us look at the Euler equation error distributions in form of boxplots in Figure 1. We see that the median standard Euler equation errors

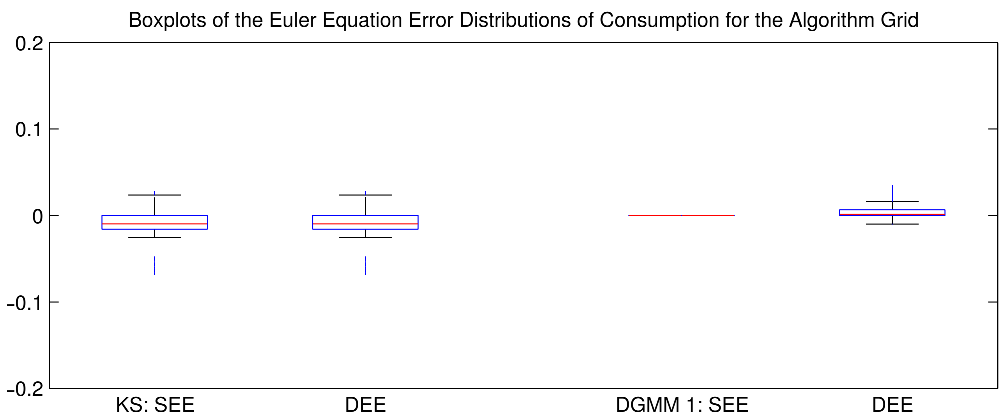

Figure 1. Boxplots of the distributions of the standard (SEE) and dynamic (DEE) Euler equation errors for the numerical solution of the growth model's DGMM-recursive equilibrium with Taylor approximation of order $\eta = 1$ from Algorithm 1 and the Krusell-Smith solution from Algorithm 2. The central marker corresponds to the median whereas the edges of each box indicate the 25^{th} and 75^{th} percentile. The whiskers indicate the extreme values not considered as outliers. Outliers are shown as dots outside the whiskers, they are the data points outside the 25^{th} and 75^{th} percentile ± 1.5 times the difference of these percentiles.

for our solution algorithm are very close to zero. This is expected because the algorithm is designed to correctly match the Euler equation for all time steps by keeping track of the exact law of motion. This holds even for the more stringent dynamic test. The Krusell-Smith algorithm, however, exhibits a considerably larger median error for both tests which indicates a systematic error. It comes from the parametric approximation of the law of motion whose estimation relies on an error-prone simulation. The analysis of the Euler equation errors indicates that our solution algorithm is superior in terms of a precise computation of the law of motion of individual capital.

Let us now analyze which benefits the abandonment of the assumption of bounded rationality has. Theoretically, our solution should be closer to an exact equilibrium in terms of the agent's utility, i.e. it should result in higher utility at all dates and states. Let us compare the utility summed over all dates of our test scenario and averaged over the cross-sectional distribution of optimal consumption

$$U = \sum_{t=0}^{24} \beta^t \mathbb{E}^{\mu^c} \left(\frac{c^{1-\gamma} - 1}{1 - \gamma} \right).$$

The aggregate utility together with aggregate consumption and capital is given in Table 1. We see that all aggregates are higher for our solution algorithm when

	K_0	K_{24}	C_0	C_{24}	U
Krusell-Smith	26,0845	25,5100	4,6296	4,5689	35.7353
DGMM	28,5871	28,0180	4,7843	4,7186	36.4597

TABLE 1. Aggregate consumption, capital and utility for the worst-case scenario of 25 bad aggregate shocks in the growth model computed by the numerical solution of the DGMM-recursive equilibrium with Taylor approximation of order $\eta = 1$ from Algorithm 1 compared to the aggregate utility given by the Krusell-Smith policy from Algorithm 2 at the same beginning-of-period aggregate capital K_t as the former two.

comparing to the Krusell-Smith algorithm. Therefore, the aggregate economy per-forms better using my algorithm. Strikingly, this result also holds for every single agent in the economy. This implies that the policies resulting from Algorithm 1 are Pareto-improving when compared to the Krusell-Smith policies.

Let us now compare the two stationary distributions in detail. We compute the stationary conditional distribution for the Krusell-Smith policy by solving for a fixed point of the law-of-motion operator. The distributions $\mu^{k|z}$ are displayed in Figure 2. The differences in distributions are especially visible in the right tail. The

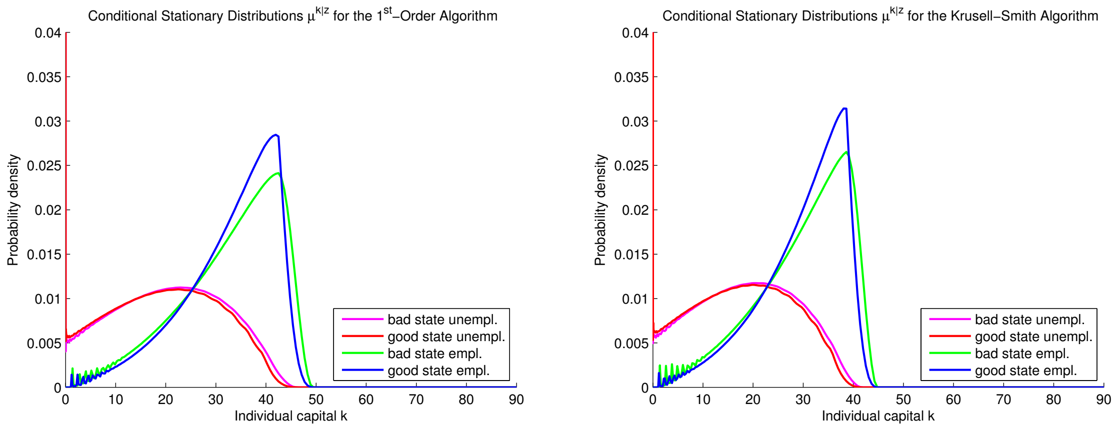

FIGURE 2. The stationary conditional distributions of the growth model $\mu^{k|z}$ by the DGMM-recursive equilibrium with Taylor approximation of order $\eta = 1$ (left) from Algorithm 1 and the Krusell-Smith solution (right) from Algorithm 2.

Krusell-Smith policy will lead to lower capital than the DGMM-recursive equilibrium for the richest agents in the economy. In fact, one can see the same result for middle-class agents. Interestingly, the Lorenz curves of the two numerical solutions are indistinguishable as can be seen in Figure 3. When we plot the Lorenz curves at different time steps in our worst-case scenario solely for the latter. We see from this graph that the gap between rich and poor widens when there are several bad aggregate shocks which is in line with economic intuition.

We should ultimately compute the solution with Taylor approximations of order higher than two. However, we refrain from doing so because computational cost are

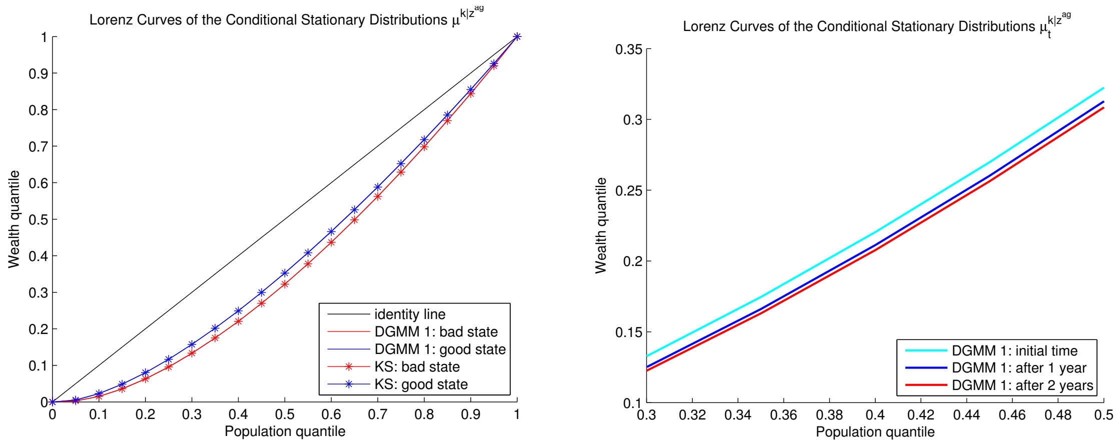

FIGURE 3. The Lorenz curves of the stationary conditional distributions of the growth model $\mu^{k|z^{ag}}$ by the DGMM-recursive equilibrium with Taylor approximation of 1^{st} order from Algorithm 1 and the Krusell-Smith solution from Algorithm 2 on the left-hand side. On the right-hand side, the evolution of the Lorenz curves over time of the DGMM-recursive equilibrium with Taylor approximation of 1^{st} order are shown.

very high without an automated differentiation module for derivatives in measure spaces. Our first-order solution takes 201 minutes for the initial solution and 16 minutes on average for further time steps. The Krusell-Smith algorithm takes 2.5 minutes overall.

5.2. *Conceptual Problems*

In this subsection, I illustrate that the component-wise fixed-point iteration in the Krusell-Smith algorithm can fails to converge to a solution. Let us derive the limiting law of motion in Algorithm 2 for that purpose.

COROLLARY 9 (The limiting law of motion of aggregate capital) *Consider Algorithm 2 by Krusell and Smith (1998). For any fixed policy function $h(z', k, I)$ resulting from the algorithm's Section B.1, the limiting law of motion of aggregate capital for $M, T \to \infty$, where M is the number of agents and T is the number of* *simulation steps, is given by*

$$\begin{aligned} h_I\left(z^{ag'}, K; \lim_{M,T\to\infty} b\right) &= \exp\left(\mathbb{E}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}'|_{z^{ag}}\right)\right]\right. \\ &\quad + \frac{\mathbb{C}o\mathbb{V}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right), \log\left(\bar{K}'|_{z^{ag}}\right)\right]}{\mathbb{V}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right)\right]} \\ &\quad \left.\cdot \left[\log\left(K\right) - \mathbb{E}^{\mu^{*z^{ag}}}\left[\bar{K}|_{z^{ag}}\right]\right]\right), \end{aligned}$$

where

$$\begin{aligned} \bar{K}|_{z^{ag}} &= \mathbb{E}^{\mu^{z^{id},k|z^{ag}}}(k) \\ \bar{K}'|_{z^{ag}} &= \mathbb{E}^{\mu^{z^{id},k|z^{ag}}}\left(\sum_{z^{id'}\in\mathcal{Z}} \frac{p^{z'|z}}{p^{z^{ag'}|z}} h(z', k, I)\right) \end{aligned}$$

and $\mu^{z^{id},k|z^{ag}} = \frac{\mu^*}{\mu^{*z^{ag}}}$ *where* μ^* *is a fixed point of the law-of-motion operator from Lemma 5.*

With the limiting law of motion, we eliminate the simulation in Section B.2 of Algorithm 2 and solve for a fixed point of the law-of-motion operator for the fixed policy h from Section B.1 instead. We call this modification the limiting Krusell-Smith algorithm. Looking at this modification, it becomes obvious that the Krusell-Smith algorithm applies component-wise fixed-point iteration of a system of operator equations: The limiting Krusell-Smith algorithm searches for a joint fixed point $H^* = [h^*, \mu^{*k|z}]$ of the coupled system of operator equations $H = \mathbf{S}^{KS}H$ where

$$\mathbf{S}^{KS}H = \begin{bmatrix} \mathbf{T}^{KS}H \\ \mathbf{D}^{h}H \end{bmatrix}$$

and $\mathbf{T}^{KS}$ denotes the policy-function operator from Proposition 2 with law of motion $\mathbf{T}^{\mu} = h_I$ from Corollary 9. However, instead of jointly solving this system, the algorithm fixes the second component of H and solves for a fixed point of the first component and then by fixing this new first component, the algorithm solves for a fixed point of the second component. Component-wise fixed-point iteration does not generally converge to a solution which is illustrated in Figure 4. Typically,

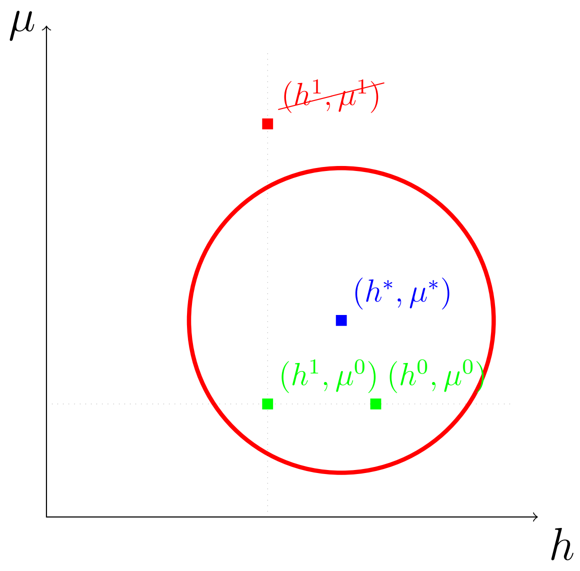

FIGURE 4. This figure illustrates that component-wise fixed-point iteration might fail. We look for a joint fixed point of h and μ which is denoted by (h^*, μ^*). The radius of convergence of this fixed point is illustrated by the circle. The initialization is denoted by (h^0, μ^0). The iterative process starts with fixing μ and computing the fixed point of h. We arrive at (h^1, μ^0). Then, we fix h and compute the fixed point of μ. We arrive at (h^1, μ^1). This new iterate lies outside the radius of convergence of the joint fixed point. The iteration is aborted and we do not find the joint fixed point.

one leaves the radius of convergence of the joint fixed point with the iteration of the law-of-motion operator because small changes in the slope of the policy function can trigger immense changes in the implied stationary distribution. This effect realizes for instance if the mean of the stationary distribution tends to infinity when iterating the law-of-motion operator.

I implement the limiting Krusell-Smith algorithm for several initial guesses of the policy h and the law of motion h_I and test whether this algorithm converges. I abort the algorithm when the mean of the stationary distribution tends to infinity. This limit is implemented as the mean of the stationary distribution lying in between the last two grid points of the distribution's grid. The initial guesses and the results from this algorithm are summarized in Table 2. The limiting Krusell-Smith algorithm converges for neither of the initial guesses. The initializations are typical choices. The first experiment initializes the policy exactly as in the computations

$h(z', k, K)$	Initialization $b^{z^{ag}=0}$	$b^{z^{ag}=1}$	Convergence	# Iterations
$0.98k$	$[\log(0.98), 1]$	$[\log(0.98), 1]$	No	1
$0.9k$	$[0, 1]$	$[0, 1]$	No	5
$0.9k$	$[0.085, 0.965]$	$[0.095, 0.962]$	No	1
k	$[0, 1]$	$[0, 1]$	No	5

TABLE 2. The limiting Krusell-Smith algorithm using the law of motion from Corollary 9. The number of iterations indicates after how many iterations on all components, i.e. on the policy and the law of motion, the algorithm was aborted.

for the comparison of our solution algorithm and the standard Krusell-Smith algorithm in the previous section. The parameters of the law of motion are chosen to be consistent with the initial policy which implies that $K' = 0.98K$. The second experiment uses the same initialization as in the Krusell-Smith algorithm implemented in Maliar et al. (2010). The third experiment uses the initial policy as in Maliar et al. (2010) and the law of motion which Krusell and Smith (1998) obtain as a result in their original paper. Note that the initial law of motions of the second and third experiment are not consistent with the initial policy. Hence, we add the fourth experiment which has a consistent law of motion and it fails as well.

This is a powerful result. It implies that the standard Krusell-Smith algorithm converges only because the number of simulation steps T in Algorithm 2 is finite. We have shown by counter example above that the algorithm with component-wise fixed-point iteration does not converge when we compute the true stationary distribution, i.e. as a fixed point of the law-of-motion operator. One rather has to jointly estimate the optimal policy and corresponding distribution.

5.3. *Comparison with other Algorithms*

Alternatives to the Krusell-Smith algorithm were developed in Algan et al. (2008, 2010); den Haan and Rendahl (2010); Reiter (2010a) and Young (2010). They avoid the simulation across M agents from Step 9 of Algorithm 2 in using different techniques to approximate the distribution over a continuum of agents.

Most of them use a parametric law of motion of aggregate capital using higher-order moments of the cross-sectional distribution rather than the correct law of motion. Even though this is certainly an improvement regarding the approximation of the law of motion, the assumption of bounded rationality remains. More importantly, these algorithms still rely on component-wise fixed-point iteration. They use a finite number of simulation steps for the aggregate shock to compute the evolution of the stationary capital distribution from which they then derive the law of motion.

In contrast, Reiter (2009, 2010b) does not assume bounded rationality. This algorithm deals with aggregate shocks by approximating the model without aggregate shocks first and then perturbing the solution around the volatility of the aggregate shock. In the first step, they also use a component-wise iterative procedure. Without aggregate shocks, this procedure does not cause problems because the equations for the policy and the distribution are sufficiently decoupled. However, due to the linear perturbation in the aggregate risk dimension, this algorithm cannot handle large and nonlinear aggregate shocks.

6. CONCLUSIONS

In this paper, I develop a novel solution algorithm to solve a wide group of DSGE models with heterogeneous agents and incomplete markets. There are three major differences to the existing algorithms, most prominently the Krusell and Smith (1998) algorithm. Firstly, this algorithm does not require bounded rationality of the agents and hence, it does not rely on an additional model assumption. Instead I solve for the original DGMM-recursive equilibrium where agents observe the full cross-sectional distribution. Economically, this leads to higher levels of consumption far any agent and is therefore Pareto-improving. However, even though this effect significantly changes the wealth distribution, it does not change the shape of the Lorenz curve towards a more realistic one.

Because I do not abstract from the cross-sectional distribution, the whole state distribution is an integral part of my algorithm. I derive the theoretical law-of-motion operator for the stationary state distribution. The main difference to the existing algorithms' law of motions is that this operator looks at the joint distribution of aggregate and idiosyncratic exogenous shocks as well as endogenous variables whereas existing algorithms condition on simulated data of aggregate exogenous shocks or use perturbations around the aggregate shocks.

The third difference is crucial for convergence. Our algorithm solves for the optimal policy and the stationary state distribution which together form a coupled system. This system is solved as a whole rather than component-wise as is the case in existing algorithms. The component-wise iterative approach, however, might not converge to a solution. I show in the last section that convergence is indeed a problem for the limiting Krusell-Smith algorithm. In contrast, convergence of the solution algorithm presented herein is proven mathematically. Therefore, when the approximation error is small, the numerical solution is indeed close to the exact equilibrium solution.

A drawback of the solution algorithm in this paper is the computation time of a one-to-one implementation. There are two reasons for that. Firstly, as we keep track of a distribution with mass points, i.e. discontinuities in the c.d.f., we need a large number of grid points. Secondly, the Taylor approximation of the next-period policy is expensive for higher orders. One would need an efficient automated differentiation module for derivatives in measure spaces. To circumvent the kink in the policy functions and the mass points in the distribution one has two options. Firstly, one can circumvent discontinuities by endogenizing the explicit debt constraint via a disutility term in the objective function which is done in Kim et al. (2010) and Mertens and Judd (2013). Due to smoothness, one can then use fast Newton methods to solve for a fixed point of our system of operator equations. Another option is to analyze equilibrium in continuous time with continuous exogenous processes as in Achdou et al. (2014). They solve a PDE using fast finite difference methods. However, they are not able to accommodate aggregate exogenous shocks.

Overall, my approach provides a new tool to analyze numerical solutions of DSGE models for which convergence is ensured. It provides insights into the tails of the state space distribution after large or persistent aggregate shocks which is important for risk analysis. This is interesting for instance for macro-finance models which investigate systemic risk in financial markets and its effect on the real economy.

APPENDIX A: PROOFS

PROOF OF PROPOSITION 2: The first-order conditions from the variational approach, i.e. Euler equation, budget and borrowing constraint and the market-clearing condition for this equilibrium problem are

$$c^{-\gamma} = \lambda + \beta\,\mathbb{E}\left([c']^{-\gamma}\left[1 + R\left(z^{ag''};\mu^{k'}\right) - \delta\right]\right) \tag{13}$$

$$k' = I\left(z';\mu^{k}\right) + \left[1 + R\left(z^{ag'};\mu^{k}\right) - \delta\right]k - c \tag{14}$$

$$k' \geq 0 \wedge \lambda \geq 0 \wedge k'\lambda = 0 \tag{15}$$

where λ is the Lagrange multiplier associated with the borrowing constraint. Inserting (14) into (13) yields

$$\begin{aligned} k' &= I\left(z';\mu^{k}\right) + \left[1 + R\left(z^{ag'};\mu^{k}\right) - \delta\right]k \\ &\quad - \left\{\lambda + \beta\,\mathbb{E}\left(\frac{1 + R\left(z^{ag''};\mu^{k'}\right) - \delta}{\left\{I\left(z'';\mu^{k'}\right) + \left[1 + R\left(z^{ag''};\mu^{k'}\right) - \delta\right]k' - k''\right\}^{\gamma}}\right)\right\}^{-\frac{1}{\gamma}} \end{aligned}$$

where $k' \geq 0 \wedge \lambda \geq 0 \wedge k'\lambda = 0$. It is easy to show that there exists a Lagrange multiplier satisfying (15) which results in

$$\begin{aligned} k' = \max\Bigg(0, I\left(z';\mu^{k}\right) &+ \left[1 + R\left(z^{ag'};\mu^{k}\right) - \delta\right]k \\ - \left\{\beta\,\mathbb{E}\left(\frac{1 + R\left(z^{ag''};\mu^{k'}\right) - \delta}{\left\{I\left(z'';\mu^{k'}\right) + \left[1 + R\left(z^{ag''};\mu^{k'}\right) - \delta\right]k' - k''\right\}^{\gamma}}\right)\right\}^{-\frac{1}{\gamma}}&\Bigg). \end{aligned}$$

Inserting the functional form $k' = h^*(z'; k, \mu^k)$, we obtain an operator $\mathbf{T}$ given by

$$(\mathbf{T}h)\left(z'; k, \mu^k\right) = \max\Big(0, I\left(z'; \mu^k\right) + \Big[1 + R\left(z^{ag'}; \mu^k\right) - \delta\Big] k - \{\beta$$
$$\mathbb{E}\left(\frac{1 + R\left(z^{ag''}; \mu^{k'}\right) - \delta}{\{I\left(z''; \mu^{k'}\right) + \left[1 + R\left(z^{ag''}; \mu^{k'}\right) - \delta\right] h(z'; k, \mu^k) - h(z''; h(z'; k, \mu^k), \mu^{k'})\}^{\gamma}}\right)\Big\}^{-\frac{1}{\gamma}}\Big).$$

In our model, the Euler equations are necessary as well as sufficient for optimality because our CRRA utility $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$ fulfills $\lim_{t\to\infty} \beta^t \, \mathbb{E}\left[\frac{\partial}{\partial c} u\left(c_t\right)\right] = 0$. Duffie et al. (1994) as well as Kubler and Schmedders (2002) show that this condition yields sufficiency. Therefore, the optimal capital h^* is given by any fixed point of the operator $h^* = \mathbf{T}h^*$. *Q.E.D.*

Proof of Lemma 5: First, we verify that (i) $\mathbf{D}^{h^*}$ is indeed a law-of-motion operator. Due to Definition 1 and (4), the joint distribution of exogenous and endogenous variables $\mu^{z',y'}$ is given by

$$\begin{aligned}
\mu^{z',y'}(u, v) &= \mathbb{P}\left(\{z' = u\} \cap \{y' \leq v\}\right) \\
&= \sum_{z\in\mathcal{Z}} \int_{\mathcal{J}} \mathbb{1}_{\{z'=u\}} \mathbb{1}_{\{h^*(z';y,\mu^y)\leq v\}} d\mu^{y|z}(y) d\mu^{z'|z}(z') p^{*z} \\
&= \int_{\mathcal{J}} p^{z'=u|z} \mathbb{1}_{\{h^*(u;y,\mu^y)\leq v\}} d\mu^{z,y}(z, y)
\end{aligned}$$

for any $(u, v) \in \mathcal{J}$. This calculation yields the formula of $\mathbf{D}^{h^*}$. Because we use the DGMM-recursive functional form of the optimal endogenous variables, we ensure that $(z', y') \in \mathcal{J}$ and therefore $\mu^{z',y'} \in \mathcal{P}(\mathcal{J})$. Furthermore, it is clear that the term inside the integral is a probability measure on $(u, v) \in \mathcal{J}$ which implies that this term is indeed a valid transition $\Pi : \mathcal{J} \to \mathcal{P}(\mathcal{J}), (z, y) \mapsto p^{z'=\cdot|z} \mathbb{1}_{\{h^*(\cdot;y,\mu^y)\leq \cdot\}}$. Hence, $\mathbf{D}^{h^*}$ is indeed a valid law-of-motion operator.

Secondly, we show that (ii) the marginal distribution w.r.t. z of any fixed point of $\mathbf{D}^{\Pi}$ recovers μ^{*z}. The marginal distribution is

$$\begin{aligned}\mu^{*}(u, \mathcal{Y}) &= \int_{\mathcal{J}} p^{z'=u|z} \mathbb{1}_{\{h^{*}(u;y,\mu^{y}) \in \mathcal{Y}\}} d\mu^{*}(z, y) \\ &= \sum_{z \in \mathcal{Z}} p^{z'=u|z} p^{*z} \\ &= p^{*,z=u}\end{aligned}$$

which equals (5).

Thirdly, we proof that (iii) a fixed point of $\mathbf{D}^{h^{*}}$ with $\operatorname{supp} \mu^{*} = \mathcal{J}$ satisfying (9) is an ergodic state distribution for the DGMM-recursive equilibrium. For μ^{*} to be ergodic, by definition every set $\{A \subset \mathcal{J} \,|\, \Pi_{(z;y,\mu^{y})} \in \mathcal{P}(A)$ for μ^{*}-a.e. $s \in A\}$ would need to have either measure zero or one. Assume that there exists a set $A \subsetneq \operatorname{supp} \mu^{*}$ which is not a null set such that $\Pi_{(z;y,\mu^{y})} \in \mathcal{P}(A)$ for μ^{*}-a.e. $(z, y) \in A$. This implies $\Pi_{(z;y,\mu^{y})}(A) = 1$ for μ^{*}-a.e. $(z, y) \in A$, i.e.

$$\begin{aligned}\Pi_{(z;y,\mu^{y})}(A) &= \textstyle\sum_{u \in \mathcal{Z}} p^{z'=u|z} \mathbb{1}_{\{(u, h^{*}(u;y,\mu^{y})) \in A\}} \\ &= 1 \text{ for } \mu^{*}\text{-a.e. } (z, y) \in A.\end{aligned}$$

This contradicts our ergodicity condition. The only set where this equality holds in conjunction with ergodicity is the support of the measure μ^{*}.

Finally, the transition $\mathbf{\Pi}$ for the DGMM-recursive equilibrium fulfills the definition of a spotless time-homogenous equilibrium (see Duffie et al., 1994, p. 756) by construction which concludes (iv). *Q.E.D.*

Proof of Proposition 7: Let us denote the support of the marginal distribution w.r.t. y of the stationary state distribution given by a fixed point of the law-of-motion operator from Lemma 5 by $\operatorname{supp} \mu^{*y}$. The minimum value of the endogenous variable with positive probability is denoted by $\underline{y} = \min \operatorname{supp} \mu^{*y}$. Suppose that $l < \hat{y} < \underline{y}$. Applying the law-of-motion operator from Lemma 5, we obtain that $\underline{y}' = h^{*}\left(\hat{z}; \underline{y}, \mu^{y}\right) \leq \underline{y}$ because of $p^{z'=\hat{z}|z=\hat{z}} > 0$ and $h^{*}\left(\hat{z}; y, \mu^{y}\right) \leq y$. By induction, this contradicts $\hat{y} \notin \operatorname{supp} \mu^{*y}$. Now, suppose that $l < \hat{y} = \underline{y}$. Because h^{*} is continuous and strictly increasing to the right of its kink, there exists an interval $[\hat{y}, \bar{y}]$ with $\bar{y} := \max\{y \in \mathcal{Y} \,|\, h^*(\hat{z}; y, \mu^y) = \hat{y}\} > \hat{y}$ and positive measure $\mu^{*y}([\hat{y}, \bar{y}]) > 0$. Due to $p^{z'=\hat{z}|z=\hat{z}} > 0$, a strictly positive part of this mass will stay at $\hat{z}$ and have future value l. Hence, $\mu^{*y}(l) > 0$ and $\underline{y} = l$. This yields the first result of a mass point at the boundary for any stationary state distribution. Using the same reasoning, one can easily see that this mass point at zero propagates to higher levels of the endogenous variable at $\bar{z} \in \mathcal{Z}$. *Q.E.D.*

Proof of Corollary 9: In Step 11 of Algorithm 2, the common exogenous shock is simulated over T periods and the idiosyncratic exogenous shock is simulated over M agents. Then, the capital for these shocks is calculated by iteratively applying the policy function $k_{m,t+1} = h\left(z^{ag}_{t+1}, z^{id}_{m,t+1}, k_{m,t}, \frac{1}{M}\sum_{m=1}^{M} k_{m,t}\right)$ for any time $1 < t \leq T$ and agent $1 \leq m \leq M$. Then, the average capital across agents $K_t = \frac{1}{M}\sum_{m=1}^{M} k_{m,t}$ is calculated in every period. This yields simulated data for aggregate capital. In Step 12 of the algorithm, the parameters for h_I are estimated. Krusell and Smith (1998) do this by regressing the logarithm of aggregate capital on its lagged value conditional on the common exogenous shock. Furthermore, they only take the stationary part of the simulated data, i.e. they only use the last $T - T^* + 1$ periods for the regression. The regression model thus reads

$$\log(K_{t+1})|_{z^{ag}_{t+1}=z^{ag'}} = b_1|_{z^{ag'}} + b_2|_{z^{ag'}} \log(K_t) + \epsilon_t$$

for any $T^* \leq t \leq T-1$ and $z^{ag'} \in \{0, 1\}$. The OLS estimator of this regression model is

$$b_2|_{z^{ag'}} = \frac{\sum_{t=T^*}^{T-1} \mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}} \left(\log(K_t) - \overline{X}\right)\left(\log(K_{t+1}) - \overline{Y}\right)}{\sum_{t=T^*}^{T-1} \mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}} \left(\log(K_t) - \overline{X}\right)^2}$$

where

$$\overline{X} = \sum_{t=T^*}^{T-1} \frac{\mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}}}{\sum_{t=T^*}^{T-1} \mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}}} \log(K_t) \quad ; \quad \overline{Y} = \sum_{t=T^*}^{T-1} \frac{\mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}}}{\sum_{t=T^*}^{T-1} \mathbb{1}_{\{z^{ag}_{t+1}=z^{ag'}\}}} \log(K_{t+1}).$$

Now, let us first look at the limit when $M \to \infty$. At initial time, the M data points for individual capital and employment status are independently drawn from some joint distribution $\mu_0^{z^{id},k|z^{ag}}$ with finite mean $\mathbb{E}^{\mu_0^{z^{id},k|z^{ag}}}(k) < \infty$. Due to the strong law of large numbers, it follows that $\lim_{M\to\infty} K_0 = \mathbb{E}^{\mu_0^{z^{id},k|z^{ag}}}(k)$ a.s. At $t = 1$, the capital is computed by a continuous (measurable) function h_I of the independently drawn random variables k_0 and the independently drawn random variables $z_1^{id}|(z_1^{ag}, z_0)$. Note that these two random variables are independent as the latter is exogenous. Hence, k_1 is a random variable with distribution induced by h, i.e. $m_1^{z^{id},k|z^{ag}}(v) = \mathbb{P}(h(z_1, k_0, K_0) \leq v)$, which is independently drawn for each agent which implies $\lim_{M\to\infty} K_1 = \mathbb{E}^{\mu_1^{z^{id},k|z^{ag}}}(k)$ a.s. By induction, we obtain for any time point $t \geq 0$

$$
\begin{aligned}
K_t &= \mathbb{E}^{\mu_t^{z^{id},k|z^{ag}}}(k) \\
K_{t+1} &= \mathbb{E}^{\mu_t^{z^{id},k|z^{ag}}}\left(\sum_{z^{id'}\in\mathcal{Z}} \frac{p^{z'|z}}{p^{z^{ag'}|z}} h(z', k, I)\right)
\end{aligned}
$$

for $M \to \infty$. In the OLS estimator, the logarithms of these quantities are summed over all periods $t \geq T^*$ in which the state distribution is stationary. This translates into $\mu_t^{z^{id},k|z^{ag}} = \frac{\mu^*}{\mu^{*z^{ag}}}$ where μ^* is a fixed point of the law-of-motion operator $\mu^* = \mathbf{D}^h\mu^*$ in Lemma (10) for fixed policy h. Therefore, we can write $K_t = \bar{K}|_{z^{ag}}$ and $K_{t+1} = \bar{K}'|_{z^{ag}}\ \forall t \geq T^*$. Due to Birkhoff's ergodic theorem (see e.g. Kallenberg, 2002, p. 181), the limits of the regression parameter for the measure-preserving transition $\mathbf{D}^h$ equal

$$
\begin{aligned}
\lim_{M,T\to\infty} b_2|_{z^{ag}} &= \frac{\mathbb{C}\text{o}\mathbb{V}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right), \log\left(\bar{K}'|_{z^{ag}}\right)\right]}{\mathbb{V}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right)\right]} \\
\lim_{M,T\to\infty} b_1|_{z^{ag}} &= \mathbb{E}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}'|_{z^{ag}}\right)\right] - \lim_{M,T\to\infty} b_2|_{z^{ag}}\mathbb{E}^{\mu^{*z^{ag}}}\left[\log\left(\bar{K}|_{z^{ag}}\right)\right].
\end{aligned}
$$

Inserting these two limits into the regression model concludes the proof. *Q.E.D.*

APPENDIX B: CALIBRATION OF THE MODEL

The utility parameters and the elasticity of the production technology are set as in Maliar et al. (2010): $\beta = 0.99$, $\gamma = 1$ and $\alpha = 0.36$. The time endowment parameter is set to $\bar{l} = \frac{1}{12}$ because we calibrate the other parameters to reflect monthly data. We calibrate the growth model to reflect the Greek economy because it has higher unemployment rates on average which leads to a smaller support of the cross-sectional capital distribution. Based on monthly unemployment data from Eurostat for April 1998 - March 2015, the bad aggregate shock has a probability of 0.4828. Conditional on being in a bad state, the probability of being employed is $p^e = 0.8183$ whereas it is $p^e = 0.9047$ when the economy is in good state. The transition probabilities

$p^{z'\|z}$	$z' = [0,0]$	$z' = [0,1]$	$z' = [1,0]$	$z' = [1,1]$
$z = [0,0]$	0.8569	0.1129	0.0121	0.0185
$z = [0,1]$	0.0251	0.9443	0.0009	0.0297
$z = [1,0]$	0.0226	0.0060	0.8652	0.1062
$z = [1,1]$	0.0034	0.0252	0.0112	0.9602

.

are computed such that they yield the stationary exogenous probabilities

$$p^{*z} = \begin{bmatrix} 0.0877 & 0.3950 & 0.0493 & 0.4679 \end{bmatrix},$$

where the quantities are rounded to four digits. The unemployment benefit rate is computed by dividing the greek benefit of 360 Euro, as published by the European Commission, by the average monthly income from Eurostat which results in $\nu = 0.2177$. The rate of depreciation is computed as the average monthly change in the harmonized consumer price index from Eurostat yielding $\delta = 0.0021$. The average change of the productivity rate a is computed from the average change of GDP per capita from OECD.Stat resulting in $a = 0.0048$.

REFERENCES

Yves Achdou, Jiequn Han, Jean-Michel Lasry, Pierre-Louis Lions, and Benjamin Moll. Heterogeneous agent models in continuous time. *Working Paper*, 2014.

S. Rao Aiyagari. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics*, 109(3):659–684, 1994.

Yann Algan, Olivier Allais, and Wouter J. Den Haan. Solving heterogeneous-agent models with parameterized cross-sectional distributions. *Journal of Economic Dynamics and Control*, 32 (3):875–908, 2008.

Yann Algan, Olivier Allais, and Wouter J. Den Haan. Solving the incomplete markets model with aggregate uncertainty using parameterized cross-sectional distributions. *Journal of Economic Dynamics and Control*, 34(1):59–68, 2010.

Heinz H. Bauschke, Sarah M. Moffat, and Xianfu Wang. Firmly nonexpansive mappings and maximally monotone operators: Correspondence and duality. *Set-Valued and Variational Analysis*, 20(1):131–153, 2012.

Truman Bewley. The permanent income hypothesis: A theoretical formulation. *Journal of Economic Theory*, 16(2):252–292, 1977.

Pierre Cardaliaguet. Notes on mean field games. *Technical report*, 2013.

René Carmona, François Delarue, and Aimé Lachapelle. Control of mckean-vlasov dynamics versus mean field games. *Mathematics and Financial Economics*, 7(2):131–166, 2013.

Wouter J. den Haan. Comparison of solutions to the incomplete markets model with aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):4–27, 2010.

Wouter J. den Haan and Pontus Rendahl. Solving the incomplete markets model with aggregate uncertainty using explicit aggregation. *Journal of Economic Dynamics and Control*, 34(1): 69–78, 2010.

Wouter J. den Haan, Kenneth L. Judd, and Michel Juillard. Computational suite of models with heterogeneous agents: Incomplete markets and aggregate uncertainty. *Journal of Economic Dynamics and Control*, 34(1):1–3, 2010.

Wouter J. den Haan, Kenneth L. Judd, and Michel Juillard. Computational suite of models with heterogeneous agents ii: Multi-country real business cycle models. *Journal of Economic Dynamics and Control*, 35(2):175–177, 2011.

Peter Deuflhard. *Newton methods for nonlinear problems: affine invariance and adaptive algorithms*, volume 35 of *Springer Series in Computational Mathematics*. Springer, 2011.

Darrell Duffie and Wayne Shafer. Equilibrium in incomplete markets: I: A basic model of generic existence. *Journal of Mathematical Economics*, 14(3):285–300, 1985.

Darrell Duffie and Wayne Shafer. Equilibrium in incomplete markets: Ii: Generic existence in stochastic economies. *Journal of Mathematical Economics*, 15(3):199–216, 1986.

Darrell Duffie, John Geanakoplos, Andreu Mas-Colell, and Andrew McLennan. Stationary markov equilibria. *Econometrica*, pages 745–781, 1994.

Hugo A. Hopenhayn. Entry, exit, and firm dynamics in long run equilibrium. *Econometrica*, 60 (5):1127–1150, 1992.

Olav Kallenberg. *Foundations of modern probability*. Springer, 2nd edition, 2002.

Sunghyun Henry Kim, Robert Kollmann, and Jinill Kim. Solving the incomplete market model with aggregate uncertainty using a perturbation method. *Journal of Economic Dynamics and Control*, 34(1):50–58, 2010.

Per Krusell and Anthony A. Smith. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, 1998.

Felix Kubler and Karl Schmedders. Recursive equilibria in economies with incomplete markets. *Macroeconomic dynamics*, 6(02):284–306, 2002.

Michael Magill and Martine Quinzii. Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles. *Journal of Mathematical Economics*, 26(1):133–170, 1996.

Lilia Maliar, Serguei Maliar, and Fernando Valli. Solving the incomplete markets model with aggregate uncertainty using the krusell-smith algorithm. *Journal of Economic Dynamics and Control*, 34(1):42–49, 2010.

Thomas Mertens and Kenneth Judd. Equilibrium existence and approximation for incomplete market models with substantial heterogeneity. *Working Paper*, 2013.

R. R. Phelps. Lectures on maximal monotone operators. *Extracta Mathematicae*, 12(3):193–230, 1997.

Michael Reiter. Solving heterogeneous-agent models by projection and perturbation. *Journal of Economic Dynamics and Control*, 33(3):649–665, 2009.

Michael Reiter. Solving the incomplete markets model with aggregate uncertainty by backward induction. *Journal of Economic Dynamics and Control*, 34(1):28–35, 2010a.

Michael Reiter. Approximate and almost-exact aggregation in dynamic stochastic heterogeneous-agent models. *Economics Series Working Paper 258*, 2010b.

José-Víctor Ríos-Rull. Computation of equilibria in heterogeneous agent models. *Federal Reserve Bank of Minneapolis Staff Report 231*, pages 238–264, 1997.

Alain-Sol Sznitman. *Topics in propagation of chaos*, pages 165–251. Ecole d'été de probabilités de Saint-Flour XIX - 1989. Springer, 1991.

Eric R. Young. Solving the incomplete markets model with aggregate uncertainty using the krusell-smith algorithm and non-stochastic simulations. *Journal of Economic Dynamics and Control*, 34(1):36–41, 2010.

E. Zeidler. *Nonlinear Functional Analysis and its Applications I.(Fixed Point Theorems).* Springer, 1986.