Exponential-Affine Approximations of Macro-Finance Models

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Abstract

We propose a first-order approximation technique to solve a general class of discrete-time dynamic models with endogenous and state-dependent risk, and we provide simple analytical expressions for generalized equilibrium term structures of asset prices. We apply this technique to two endowment economies with Campbell-Cochrane habits as well as a more complex version that features a production economy and nominal rigidities. The proposed essentially-affine approximation performs similarly to global solution methods and outperforms alternative affine approximation methods. The approximation maintains a high quality at short as well as long horizons, and it preserves the main properties of the stochastic discount factor, including its martingale component and the maximal risk-return tradeoff.

JEL classification: C61; C63; G12; G13.

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1. Introduction

Asset pricing models with state-dependent risk prices and risk exposures are necessary to capture the salient time-series properties of financial prices. Additionally, general equilibrium macro-finance models endogenize risk, whose equilibrium distribution is known only after the model is solved. These models present a challenge for traditional solution techniques that resort either to accurate but computationally intensive methods or to low-order perturbation methods that can importantly misrepresent the asset pricing implications of the model.

In this paper we propose a parsimonious representation of macro-finance models that is designed to approximately solve for equilibrium quantities and asset prices in dynamic models with generic time-varying risk. We formalize an approximate, risk-adjusted, loglinear solution method in the

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spirit of Jermann (1998) that allows for endogenous risk whose price and quantity display non-affine dynamics.[2] Our method captures the first-order effect of endogenous as well as exogenous risk on the equilibrium allocation, as it is necessary for example to study the consequences of time-varying risk aversion. We provide explicit pricing formulas in the exponential-affine class for generalized equilibrium term structures—and hence for all related claims, including variance risk premia—as well as for some major diagnostic decompositions of the asset pricing properties of a model.

We call our approximation method "essentially" affine because we impose only those approximations that are essential to achieve affine pricing.[3] In particular, we linearize the real risk-free rate as well as the first two conditional moments under the risk-neutral distribution ($\mathbb{Q}$) of the joint process of cashflows and state variables. We then preserve the non-affine dynamics of risk prices and risk exposures under the historical measure ($\mathbb{P}$) to simulate the distribution of approximate equilibrium asset prices; this strategy turns out to be of key importance to accurately capture the amount of time-variation in risk premia and the martingale component of the stochastic discount factor that controls long-run pricing implications.

To test the accuracy of our method in a relevant context we consider the class of models with Campbell and Cochrane (1999) nonlinear habit formation, in which risk is endogenous and its quantity and price are nonlinear functions of a conditionally Gaussian state vector. Risk price dynamics are sufficiently nonlinear and different from the dynamics of risk exposures to prevent the existence of a closed-form equilibrium value for strips, and projection methods are required to find the global solution. Our method is well-suited to be applied to models with Campbell and Cochrane (1999) habits. Intuitively, a first-order approximation of the price of risk is a sufficient condition to generate an affine specification of the discount factor when the state vector is a Gaussian covariance-stationary autoregressive process and when cashflows and the risk-free rate are affine, and thereby to generate affine risk-neutral dynamics in the state vector. Since a first-order perturbation linearizes the dependence of cashflows and risk-free rates on the state variables, this idea can be easily extended to a framework with general cashflow and risk-free rate dynamics.

Loglinear-lognormal approximation methods compare well with the semi-parametric numerical

[2]Our essentially-affine approximation nests existing loglinear-lognormal approximations (see Jermann, 1998; Lettau and Uhlig, 2000; Lettau, 2003; Uhlig, 2007; De Graeve, Emiris, and Wouters, 2009; Bekaert, Cho, and Moreno, 2010; Dew-Becker, 2014, for applications of loglinear-lognormal methods to asset pricing in production economies). Malkhozov (2014) is the most recent formalization of the risk-adjusted loglinear method applied to models with exogenous risk and affine dynamics in risk prices and exposures. The extension to endogenous risk is necessary to study models with time-varying risk aversion in production economies, while imposing affinity onto the $\mathbb{P}$-dynamics of risk prices and exposures misrepresents the amount of time-variation in risk premia and the long-run asset pricing implications of the model.

[3]We borrow the name from Duffee (2002), who introduced the essentially-affine specification of Gaussian term structure models by assuming affinity only where it was essential to ensure affine pricing. Analogously, other exponential-affine approximations that linearize the model completely end up with a "completely"-affine approximation (e.g., Jermann, 1998; Dew-Becker, 2012; Malkhozov, 2014, among many others). Note that Dew-Becker (2012) relies on an affine approximation applied to an example not covered by Malkhozov (2014) and that is a special case of our method when the price of risk has linear dynamics and risk exposures are constant; he chooses to name his method essentially-affine because it can be applied to a model in which variation in risk prices can associate with constant risk exposures (one of the features of Duffee's specification). Since this property characterizes the model and not the approximation, however, we prefer to reserve the term "essential" to characterize an approximation that imposes only those affinities that are essential to solve yields in the exponential-affine class.

methods used by Campbell and Cochrane (1999) and Wachter (2006) in loglinear endowment economies and with more general projection methods. While it remains less accurate than these alternatives, the essentially-affine approximation offers analytical solutions that price easily complex dividend processes (such as those arising from a production economy) and that facilitate an intuitive understanding of the asset pricing implications of the model and the macroeconomic forces that drive them. With our essentially-affine approximation we derive a useful decomposition of equilibrium term structures in terms of a level factor, of a geometrically-growing factor that controls the short end of the curve and of a nonnegative factor that controls the long end of the curve, and we provide simple expressions for Borovicka and Hansen (2014) risk-exposure and risk-price elasticities. Moreover, the affine form has a big advantage in estimation as it can exploit fast linear filtering methods. Therefore, our essentially-affine approximation proves especially useful to solve general equilibrium models that incorporate variation in risk premia.[4]

To compare the approximate solution with the numerical solution we report multiperiod Euler equation errors and the term structures of equity and bond yields, which capture the quality of the approximation at different time horizons. We emphasize the importance of correctly capturing the term structures of zero-coupon equities and bonds, as they are the basis for pricing other more complex claims (Lettau and Wachter, 2007; Binsbergen, Brandt, and Koijen, 2012a; Lopez, 2014) as well as they directly relate to several diagnostic measures that are crucial in evaluating the pricing properties of a model of the stochastic discount factor (Alvarez and Jermann, 2005; Hansen, Heaton, and Li, 2008; Hansen and Scheinkman, 2009).

To test the performance of our approximation procedure we first consider the endowment economies of Campbell and Cochrane (1999) and Wachter (2006). Our essentially-affine approximation is accurate in solving for risk premia and volatilities of equities and bonds at the observable durations and at long durations, while it slightly distorts the correct values for medium-duration claims. The approximation method does a good overall job in capturing the level, amplitude and curvature of the term structures.

We then turn to the model of Lopez, Lopez-Salido, and Vazquez-Grande (2015) that features Campbell-Cochrane habits and a production economy in a New Keynesian DSGE model. This model is appropriate for testing the accuracy of our solution in an environment where consumption and risk are endogenously determined. In this application the full nonlinear solution is computationally expensive, while our essentially-affine approximation yields a fast and tractable solution with good levels of accuracy. Our approximation outperforms substantially the alternative approximation schemes, and it similarly outperforms low-order perturbation methods.[5]

[4]The extant literature offers examples of habit formation in small-scale production economies but they feature habits that are either linear (e.g., Jermann, 1998; Boldrin, Christiano, and Fisher, 2001; De Paoli, Scott, and Weeken, 2010; Challe and Giannitsarou, 2014; Malkhozov, 2014) or that depart from the Campbell-Cochrane specification in order to grant exact affine term structures (Gallmeyer, Hollifield, and Zin, 2005; Bekaert, Engstrom, and Xing, 2009; Bekaert, Cho, and Moreno, 2010; Palomino, 2012; Campbell, Pflueger, and Viceira, 2013; Dew-Becker, 2014).

[5]Note that to have time-varying risk premia we would need an approximation based on standard perturbations of at least third degree (Binsbergen, Fernández-Villaverde, Koijen, and Rubio-Ramírez, 2012b; Rudebusch and Swanson, 2012; see also Benigno, Benigno, and Nisticò, 2013), while our essentially-affine approximation is a first-order method able to accurately capture time-variation in risk premia.

2. An essentially-affine approximation

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with associated filtration $\{\mathscr{F}_t\}$. A representation of the set of equilibrium conditions of a wide variety of dynamic equilibrium models include a vector of forward-looking equations that describe control variables $y : \Omega \times \mathbb{N} \to \mathbb{R}^{n_y}$:

$$0 = \ln E_t^{\mathbb{P}} \exp\left[f(y_t, y_{t+1}, z_t, z_{t+1})\right] \tag{1}$$

where $E_t^{\mathbb{P}}(\cdot)$ is the expectations operator conditional on the information available at date t under the historical measure $\mathbb{P}$, and where state variables $z : \Omega \times \mathbb{N} \to \mathbb{R}^{n_z}$ are characterized by:

$$z_{t+1} = g\left(y_t, y_{t+1}, z_t, \lambda(y_t, z_t)[y_{t+1} - E_t^{\mathbb{P}} y_{t+1}], \sigma(y_t, z_t)\varepsilon_{t+1}\right) \tag{2}$$

with $\varepsilon_t \sim Niid(0, I_q)$ under $\mathbb{P}$. All variables are expressed in log deviations from the deterministic steady-state values and functions f and g are sufficiently smooth in their arguments.

Finally, it is useful to explicitly state the stochastic discount factor $m_{t+1} \in \mathbb{R}$:

$$m_{t+1} = h\left(y_t, y_{t+1}, z_t, z_{t+1}\right) \tag{3}$$

Our method consists of three steps. First, we isolate the first-order component of the equilibrium allocation while adjusting for the presence of such components in higher moments. Second, we rely on no-arbitrage relations to price claims in zero net supply via an affine approximation that yields explicit pricing formulas. Third, we simulate the model using the nonlinear map from the $\mathbb{P}$-dynamics of the state vector into the $\mathbb{P}$-dynamics of risk prices and exposures.

2.1. Step 1 (Risk-adjusted approximate equilibrium allocation)

We start by expanding (2) around the deterministic steady-state values $z_t = z_{t+1} = 0$, $y_t = y_{t+1} = 0$, $e_{t+1} = 0$, with $e_{t+1} \equiv [\lambda_t(y_{t+1} - E_t^{\mathbb{P}} y_{t+1}); \sigma_t \varepsilon_{t+1}]$, as

$$\begin{aligned}
z_{t+1} &= [\nabla_1 g]y_t + [\nabla_2 g]y_{t+1} + [\nabla_3 g]z_t + [\nabla_4 g]\lambda_t(y_{t+1} - E_t^{\mathbb{P}} y_{t+1}) + [\nabla_5 g]\sigma_t\varepsilon_{t+1} + O(\|\varepsilon\|^2) \\
&= \mu_z z_t + \sigma_z(z_t)\varepsilon_{t+1} + O(\|\varepsilon\|^2) \\
\mu_z &\equiv (I_{n_z} - \nabla_2 g\, \nabla\iota)^{-1}(\nabla_1 g\, \nabla\iota + \nabla_3 g) \\
\sigma_z(z_t) &\equiv \left(I_{n_z} - \nabla_2 g\, \nabla\iota - [\nabla_4 g]\lambda([\nabla\iota]z_t, z_t)[\nabla\iota]\right)^{-1} [\nabla_5 g]\sigma([\nabla\iota]z_t, z_t)
\end{aligned} \tag{4}$$

where all derivatives are evaluated at the deterministic steady state, and where we rely on the implicit function theorem to pose and expand the generic shape of the equilibrium policy function to be identified, $y_t = \iota(z_t)$. It is straightforward to verify that to a first-order approximation the state vector is conditionally Gaussian. We can therefore expand (1) as

$$\begin{aligned}
0 &= \ln E_t^{\mathbb{P}} \exp\left[[\nabla_1 f]y_t + [\nabla_2 f]y_{t+1} + [\nabla_3 f]z_t + [\nabla_4 f]z_{t+1} + O(\|\varepsilon\|^2)\right] \\
&= [\nabla_1 f\, \nabla\iota + \nabla_3 f + (\nabla_2 f\, \nabla\iota + \nabla_4 f)\mu_z]\, z_t \\
&\quad + \frac{1}{2} diag\left((\nabla_2 f\, \nabla\iota + \nabla_4 f)[\Sigma_z(0) + (I_{n_z} \otimes z_t')\nabla\Sigma_z](\nabla_2 f\, \nabla\iota + \nabla_4 f)'\right) + O(\|\varepsilon\|^2)
\end{aligned} \tag{5}$$

where we rely on the properties of the lognormal distribution and consider the expansion[6]

$$var_t^{\mathbb{P}}(z_{t+1}) = \Sigma_z(0) + (I_{n_z} \otimes z_t')[\nabla\Sigma_z] + O(\|z_t\|^2)$$

with $\Sigma_z(z_t) \equiv \sigma_z(z_t)\sigma_z(z_t)'$.

The system of n_z equations (5) associated with z_t taking values in a set of n_z linearly independent vectors in $\mathbb{R}^{n_z}$ can be solved for the coefficients contained in matrix $\nabla\iota$, which characterizes the risk-adjusted linear approximation to the dynamics of control variables around the deterministic steady state,

$$y_t \approx [\nabla\iota]z_t$$

We can then plug the identified solution into equation (4) to characterize the first-order approximation to the dynamics of the state vector around the deterministic steady state.

Our approximation of equation (5) is isolating a first-order component contained in conditional second moments with both exogenous and endogenous causes.[7]

Finally, we approximate the stochastic discount factor (3) in the conditionally Gaussian class as

$$\begin{aligned} m_{t+1} &= [\nabla_1 h]y_t + [\nabla_2 h]y_{t+1} + [\nabla_3 h]z_t + [\nabla_4 h]z_{t+1} \\ &= [\nabla_1 h\ \nabla\iota + \nabla_3 h + (\nabla_2 h\ \nabla\iota + \nabla_4 h)\mu_z]z_t - \gamma(z_t)'\varepsilon_{t+1} \end{aligned}$$

where $\gamma(z_t)' \equiv [\nabla_2 h\ \nabla\iota + \nabla_4 h]\sigma_z(z_t)$.

Thus far in the approximation, note that approximate risk prices $\gamma(z_t)$ and risk exposures $\sigma_z(z_t)$ can be non-affine in the state vector and display nonlinear dynamics.

2.2. *Step 2 (Essentially-affine approximate pricing)*

After step 1,[8] there are an approximate mean-zero covariance-stationary Gaussian Markov process $z = [\zeta; s] : \Omega \times \mathbb{N} \to \mathbb{R}^{n_\zeta + n_s}$ and a log cashflow process of interest $d : \Omega \times \mathbb{N} \to \mathbb{R}$ whose approximate joint distribution is

$$\begin{bmatrix} \zeta_{t+1} \\ s_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = \begin{bmatrix} 0_{n_\zeta} \\ 0_{n_s} \\ \mu_d \end{bmatrix} + \begin{bmatrix} A \\ \phi \\ C \end{bmatrix} \begin{bmatrix} \zeta_t \\ s_t \end{bmatrix} + \begin{bmatrix} B \\ \Lambda_t' \\ D_t \end{bmatrix} \varepsilon_{t+1}, \qquad \varepsilon_t \overset{\mathbb{P}}{\sim} Niid(0, I_q) \tag{6}$$

with coefficients $\mu_d \in \mathbb{R}$, $A \in \mathbb{R}^{n_\zeta \times (n_\zeta + n_s)}$, $\phi \in \mathbb{R}^{n_s \times (n_\zeta + n_s)}$, $C \in \mathbb{R}^{1 \times (n_\zeta + n_s)}$ and $B \in \mathbb{R}^{n_\zeta \times q}$, and where the random matrices $\Lambda_t' = \Lambda(z_t)' \in \mathbb{R}^{n_s \times q}$ and $D_t = D(z_t) \in \mathbb{R}^{1 \times q}$ allow for heteroskedasticity in

[6] The first-order Taylor expansion of a differentiable map $\Sigma : \mathbb{R}^{n_z} \to \mathbb{R}^{N \times q}$ can be written as

$$\Sigma(z_t) = \Sigma(0) + (I_N \otimes z_t')[\nabla\Sigma(0)] + O(\|z_t\|^2)$$

where $\nabla\Sigma_i(z) \equiv [\nabla_z \Sigma_i(z)']'$, $\Sigma_i(z) \in \mathbb{R}^{1 \times q}$, and where $\nabla\Sigma(z) = [\nabla\Sigma_1(z); \ldots; \nabla\Sigma_N(z)] \in \mathbb{R}^{N n_z \times q}$.

[7] Note that the endogenous component would otherwise be lost had equation (2) been written as $z_{t+1} = g^*(y_t, y_{t+1}, z_t, \sigma_t\varepsilon_{t+1})$ as in Malkhozov (2014), while none of them would be present in conventional perturbation frameworks (e.g., Schmitt-Grohé and Uribe, 2004), in which the state equation would be written as $z_{t+1} = g^{**}(y_t, y_{t+1}, z_t, \varepsilon_{t+1})$.

[8] Note that step 2 can be carried out jointly with step 1 with an identical result. We keep however the two steps separate to make clear that the inclusion of assets in zero net supply into the equations that characterize the equilibrum allocation is not necessary.

the n_s-dimensional process s and in the scalar process d. It is useful to separate the state vector into homoskedastic and heteroskedastic state variables. The assumed structure implies the joint conditional moment-generating function

$$E_t^{\mathbb{P}}[e^{u_\zeta'\zeta_{t+1}+u_s s_{t+1}+u_d\Delta d_{t+1}}] = e^{u'\mu+\frac{1}{2}u'\Sigma_t\Sigma_t' u+u'\Phi z_t},$$

for $u = [u_\zeta; u_s; u_d] \in \mathbb{R}^{n_z+1}$, with parameters $\mu \equiv [0_{n_z}; \mu_d] \in \mathbb{R}^{n_z+1}$, $\Phi \equiv [A; \phi; C] \in \mathbb{R}^{n_z+1\times n_\zeta}$ and $\Sigma(z_t) \equiv [B; \Lambda(z_t)'; D(z_t)] \in \mathbb{R}^{n_z+1\times q}$.

The equilibrium risk-free rate is described by the Euler equation (always included in (1)), $0 = \ln E_t^{\mathbb{P}} \exp(m_{t+1} + r_t)$, and hence we can write the one-period, conditionally linear, log stochastic discount factor as

$$m_{t+1} = -r_t - \frac{1}{2}\gamma_t'\gamma_t - \gamma_t'\varepsilon_{t+1} \tag{7}$$

where $\gamma_t = \gamma(z_t) \in \mathbb{R}^q$ is the possibly time-varying price of risk, $r_t = r(z_t) \in \mathbb{R}$ is the risk-free log return, and $\gamma_t'\varepsilon_{t+1}$ captures endogenous risk solved for in step 1. Accordingly, we define the multiplicative martingale,

$$Q_{t+1} = Q_t e^{-\frac{1}{2}\gamma_t'\gamma_t-\gamma_t'\varepsilon_{t+1}},$$

to construct the change of measure from the historical measure to the risk-neutral measure, $d\mathbb{Q}/d\mathbb{P}$.

Let the risk premium

$$\pi(z_t) \equiv \Sigma(z_t)\gamma(z_t)$$

be the product of the quantity of risk in the joint process $[z; \Delta d]$ and the price of risk.

Under this structure, we are able to characterize in proposition 1 the dependence of generalized equilibrium term-structure components on the states up to a term of at least second order.

2.3. Step 3 (Simulated $\mathbb{P}$-distribution)

We use the exact nonlinear maps $\gamma(z_t)$, $D(z_t)$ and $\Lambda(z_t)$ to simulate equilibrium term structures and characterize their moments of interest. Section 4 shows how simulations from the approximate dynamics result in inaccurate approximations of the equilibrium term structures, which overstate asymptotic risk premia and therefore misrepresent the martingale component of the stochastic discount factor as well as the dynamics of the Hansen and Jagannathan (1991) bound.

2.4. Main proposition

The following proposition states the closed-form equilibrium prices of strips of financial claims.

Proposition 1 (Pricing strips)**.** *Under assumptions* (6) *and* (7)*, we obtain the following first-order approximations of yields and risk premia:*

(a) *The nth cashflow strip yield, $y_{d,t}^{(n)} \equiv -\frac{1}{n}\ln(P_{d,t}^{(n)}/D_t)$, with $P_{d,t}^{(n)} = E_t^{\mathbb{P}}(M_{t,t+n}D_{t+n})$ the no-arbitrage price of the nth strip of cashflow process d, has the approximate affine form*

$$y_{d,t}^{(n)} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B_\zeta^{(n)}\zeta_t - \frac{1}{n}B_s^{(n)}s_t + O(\|z_t\|^2)$$

with term structure coefficients determined by the Riccati difference equations

$$
\begin{aligned}
A^{(n)} &= A^{(n-1)} - r(0) + \begin{bmatrix} B_\zeta^{(n-1)} \\ B_s^{(n-1)} \\ 1 \end{bmatrix}' [\mu - \pi(0)] + \frac{1}{2} \begin{bmatrix} B_\zeta^{(n-1)} \\ B_s^{(n-1)} \\ 1 \end{bmatrix}' \Sigma(0)\Sigma(0)' \begin{bmatrix} B_\zeta^{(n-1)} \\ B_s^{(n-1)} \\ 1 \end{bmatrix} \\
\begin{bmatrix} B_\zeta^{(n)\prime} \\ B_s^{(n)\prime} \end{bmatrix}' \alpha &= \begin{bmatrix} B_\zeta^{(n-1)\prime} \\ B_s^{(n-1)\prime} \\ 1 \end{bmatrix}' \left([\Phi - \nabla\pi(0)]\alpha + \Sigma(0) \begin{bmatrix} \nabla\Lambda(0)' \\ \nabla D(0) \end{bmatrix}' (I_{n_s+1} \otimes \alpha) \begin{bmatrix} B_s^{(n-1)\prime} \\ 1 \end{bmatrix} \right) - [\nabla r(0)]\alpha
\end{aligned}
\tag{8}
$$

for all $\alpha \in \mathbb{R}^{n_z}$, so a sufficient set to identify the coefficients is a set of n_z linearly independent vectors in $\mathbb{R}^{n_z}$, and with boundary condition $[A^{(0)}; B_\zeta^{(0)}; B_s^{(0)}] = 0$.

(b) *The holding-period risk premium commanded by the n-period ahead cashflow strip is*

$$\ln E_t^{\mathbb{P}} R_{d,t+1}^{e,(n)} \equiv \ln E_t^{\mathbb{P}} \left(\frac{P_{0,t}^{(1)} P_{d,t+1}^{(n-1)}}{P_{d,t}^{(n)}} \right) = V_{n-1,t}\gamma_t + O(\|z_t\|^2)$$

where coefficient $V_{n,t} \equiv D_t + B_\zeta^{(n)} B + B_s^{(n)} \Lambda_t'$ controls the loading of the unexpected component of the nth holding period log return on the shock, which in turn implies:

$$r_{d,t+1}^{e(n)} \equiv \ln R_{d,t+1}^{e,(n)} \approx V_{n-1,t}\gamma_t - \frac{1}{2} V_{n-1,t} V_{n-1,t}' + V_{n-1,t}\varepsilon_{t+1}$$

(c) *The per-period hold-to-maturity risk premium commanded by the nth cashflow strip is*[9]

$$\frac{1}{n} \ln E_t^{\mathbb{P}} \left(\frac{P_{0,t}^{(n)} D_{t+n}}{P_{d,t}^{(n)}} \right) = \frac{1}{n}[A_g^{(n)} + A_0^{(n)} - A_d^{(n)}] + \frac{1}{n}[B_{g,\zeta}^{(n)} + B_{0,n}^{(n)} - B_{d,\zeta}^{(n)}]\zeta_t + \frac{1}{n}[B_{0,s}^{(n)} - B_{d,s}^{(n)}]s_t + O(\|z_t\|^2)$$

where subscripts 0 *and d index the term structure coefficients associated with real bonds and the relevant cashflow process respectively, and where coefficients $\{A_g^{(n)}, B_g^{(n)}\}$ determine the term structure of anticipated cashflow growth*

$$
\begin{aligned}
\frac{1}{n} \ln E_t^{\mathbb{P}} \left(\frac{D_{t+n}}{D_t} \right) &= \frac{1}{n} A_g^{(n)} + \frac{1}{n} B_g^{(n)} \zeta_t \\
A_g^{(n)} &= \mu + A_g^{(n-1)} + \frac{1}{2} \|D + B_g^{(n-1)} B\|^2 \\
B_g^{(n)} &= C(I - A)^{-1}(I - A^n)
\end{aligned}
$$

Appendix A provides a proof of the proposition.

We can now use the approximate equilibrium prices of generalized strips to find expressions for the price of spanned payoffs. For example, given equilibrium strip prices, $\{E_t M_{t,t+n} D_{t+n}\}$, the return

[9] Objects of interest that take the form of hold-to-maturity expected excess returns include, for example, the term structure of the inflation risk premium (when the cashflow process is the inverse of inflation—see Ang et al., 2008) and the term structure of the welfare cost of uncertainty (when the cashflow process is consumption—see Lopez, 2014).

on the market portfolio is

$$E_t R^m_{t+1} = \sum_{n=1}^{\infty} \omega_{n,t} E_t R^{(n)}_{d,t+1}$$

where $\omega_{n,t} \equiv E_t M_{t,t+n} D_{t+n} / \sum_{n=1}^{\infty} E_t M_{t,t+n} D_{t+n}$ with $\sum_{n=1}^{\infty} \omega_{n,t} = 1$, whose approximate distribution can be constructed using simulated moments of strip prices and returns, and that can be further approximated analytically to gain additional insight into the determinants of the equity premium. The essentially-affine approximation allows us also to price in closed form more complex payoffs such as volatility (see Bollerslev, Tauchen, and Zhou, 2009).

Corollary (Pricing portfolio returns and variance). *The holding-period risk premium commanded by the market portfolio and its realized return have the approximate forms*

$$\ln E_t R^{e,m}_{t+1} = \sum_{n=1}^{\infty} \omega_n V_{n-1,t} \gamma_t + O(\|z_t, r^{(n)}_{d,t+1}, \omega_{n,t} - \omega_n\|^2)$$

$$E_t r^{e,m}_{t+1} = \sum_{n=1}^{\infty} \omega_n \left(V_{n-1,t} \gamma_t - \frac{1}{2} \|V_{n-1,t}\|^2 \right) + O(\|z_t, r^{(n)}_{d,t+1}, \omega_{n,t} - \omega_n\|^2)$$

where ω_n denotes the deterministic steady-state value of the nth portfolio weight.

The one-period variance risk premium commanded by the market portfolio is approximately:

$$\begin{aligned} vrp^m_t &= (E^{\mathbb{Q}}_t - E^{\mathbb{P}}_t)[var^{\mathbb{P}}_{t+1}(r^m_{t+2})] \\ &= -2l'\Xi\Sigma(0) \begin{bmatrix} \nabla\Lambda(0)' \\ \nabla D(0) \end{bmatrix}' \left[I_{n_s+1} \otimes \left(\begin{bmatrix} \pi_1(0) \\ \vdots \\ \pi_{n_z}(0) \end{bmatrix} + \begin{bmatrix} \nabla\pi_1(0) \\ \vdots \\ \nabla\pi_{n_z}(0) \end{bmatrix} z_t \right) \right] \begin{bmatrix} \Xi_3 & \Xi_1 \end{bmatrix}' l + O(\|z_{t+1}\|^2) \end{aligned} \tag{9}$$

where, with some abuse of notation,

$$l \equiv \begin{bmatrix} 1 \\ 1 \\ \vdots \end{bmatrix}, \qquad \Xi_1 \equiv \begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \end{bmatrix}, \qquad \Xi_2 \equiv \begin{bmatrix} \omega_1 B^{(0)}_\zeta \\ \omega_2 B^{(1)}_\zeta \\ \vdots \end{bmatrix}, \qquad \Xi_3 \equiv \begin{bmatrix} \omega_1 B^{(0)}_s \\ \omega_2 B^{(1)}_s \\ \vdots \end{bmatrix}, \qquad \Xi \equiv \begin{bmatrix} \Xi_2 & \Xi_3 & \Xi_1 \end{bmatrix}$$

Appendix B proves this corollary. Appendix E contains an expression for the variance risk premium in the case of Campbell and Cochrane (1999) habits.

2.5. Approximate asset pricing diagnostics

We are interested in evaluating the accuracy of our solution in pricing financial claims. The literature provides a set of diagnostic tools that can be used as a first round of tests of a model of the stochastic discount factor. An important criterion to evaluate the accuracy of an approximation method is that it correctly captures these implications because they are crucial in evaluating a model of the discount factor.

A first diagnostic tool to assess a model of the discount factor is the Hansen and Jagannathan (1991) bound, which shows how no-arbitrage pricing implies that the volatility of the discount factor must dominate empirical measures of the maximal risk-return tradeoff. In this context our essential-affine approximation is appropriate to correctly represent the bound. In fact, the Gaussian property allows for an easy expression for the bound,

$$\left| \frac{\ln E_t R^e_{t+1}}{\sqrt{var_t(r^e_{t+1})}} \right| \leq \sqrt{var_t(m_{t+1})} = \sqrt{\gamma_t' \gamma_t}$$

for all available excess returns, whose dynamics we characterize correctly because we rely on the exact nonlinear map from the $\mathbb{P}$-dynamics of the state vector into the $\mathbb{P}$-dynamics of the Hansen-Jagannathan bound.

Furthermore, our essentially-affine approximation provides tractable expressions for diagnostic decompositions in the spirit of Alvarez and Jermann (2005) and Hansen and Scheinkman (2009). They show how, under appropriate regularity conditions, the stochastic discount factor can be decomposed as

$$M_{t+1} = M^P_{t+1} M^T_{t+1}$$

where a so-called transient component M^T_{t+1} controls the pricing of long-duration bonds and the martingale component M^P_{t+1} with $E_t M^P_{t+1} = 1$ controls the maximum risk premium in the complete-market economy. In absence of a transient component the properties of the martingale component imply a flat term structure of real interest rates. In absence of a martingale component the long-run real bond premium is the highest premium available. The data reject both extreme cases (Alvarez and Jermann, 2005), so a first way to use the decomposition is to check that a model of the stochastic discount factor produces two nontrivial components.

Moreover, two main properties of the decomposition that rest on the no-arbitrage pricing formula and Jensen's inequality are the relationship between the transient component of the discount factor and the holding-period return on a infinite-maturity zero-coupon bond,

$$m^T_{t+1} = - \lim_{n \to \infty} r^{(n)}_{0,t+1}$$

and the property of the entropy ratio

$$\frac{\mathcal{V}_t(M^P_{t+1})}{\mathcal{V}_t(M_{t+1})} = 1 - \frac{E_t r^{e,(\infty)}_{0,t+1}}{\frac{1}{2}\mathcal{V}_t(M_{t+1})} \geq 1 - \frac{E_t r^{e,(\infty)}_{0,t+1}}{\max E_t r^e_{t+1}}$$

where the maximum is taken over all available excess returns and where relative entropy is defined as $\mathcal{V}_t(X_{t+1}) \equiv 2[\ln(E_t X_{t+1}) - E_t \ln(X_{t+1})]$. Alvarez and Jermann advocate for using such a property as a diagnostic tool for a model of the discount factor. Namely, they combine equity and bond returns to estimate the right-hand side of the inequality as being close to unity on average; additionally, the imperfect correlation between equity and bond expected excess returns suggests time-variation in the ratio (see also Koijen, Lustig, and Nieuwerburgh, 2010; Lettau and Ludvigson, 2010).

These properties make clear how an approximation method that correctly captures the term structures of equities and bonds, and especially their long-run properties, is a method that correctly captures the diagnostic decomposition of the discount factor. Our emphasis on evaluating the quality of the approximation via the term structures of claims to different cashflow processes in section 4 rests therefore on this observation.

Following Hansen and Scheinkman (2009), the decomposition can be understood starting from the solution $[\delta; f]$ to the eigenfunction problem

$$E_t^{\mathbb{P}}[M_{t+1} f(z_{t+1})] = \delta f(z_t)$$

for some function $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and scalar $\delta \in \mathbb{R}$, with transient and martingale components constructed as $M_{t+1}^T = \delta f(z_t)/f(z_{t+1})$ and $M_{t+1}^P = M_{t+1} f(z_{t+1})/\delta f(z_t)$.

Proposition 2 (Asset pricing diagnostics)**.** *In the context of the general model of section 2, the approximate essentially-affine solution of the eigenfunction problem,*

$$f(z_t) = e^{u_\zeta' \zeta_t + u_s \widehat{s}_t}$$

where the eigenfunction parameters $[\delta; u_\zeta; u_s]$ solve the nonlinear system of $n_z + 1$ equations

$$0 = \ln(\delta) + r(0) - \begin{bmatrix} u_\zeta \\ u_s \\ 0 \end{bmatrix}' [\mu - \pi(0)] - \frac{1}{2} \begin{bmatrix} u_\zeta \\ u_s \\ 0 \end{bmatrix}' \Sigma(0)\Sigma(0)' \begin{bmatrix} u_\zeta \\ u_s \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_\zeta \\ u_s \end{bmatrix}' \alpha = \begin{bmatrix} u_\zeta \\ u_s \\ 0 \end{bmatrix}' ([\Phi - \nabla\pi(0)]\alpha + \Sigma(0)[\nabla\Lambda(0)'](I_{n_s} \otimes \alpha) u_s) - [\nabla r(0)]\alpha$$

for all $\alpha \in \mathbb{R}^{n_z}$, so a sufficient set to identify the coefficients is a set of n_z linearly independent vectors in $\mathbb{R}^{n_z}$.

The system can be solved by iterating to convergence the difference equation

$$\begin{bmatrix} u_\zeta^{(n)} \\ u_s^{(n)} \end{bmatrix}' \alpha = \begin{bmatrix} u_\zeta^{(n-1)} \\ u_s^{(n-1)} \\ 0 \end{bmatrix}' \left([\Phi - \nabla\pi(0)]\alpha + \Sigma(0)[\nabla\Lambda(0)'](I_{n_s} \otimes \alpha) u_s^{(n-1)}\right) - [\nabla r(0)]\alpha$$

for $\alpha = \{e_1, ..., e_{n_z}\}$, with $[e_1 \ ... \ e_{n_z}] = I_{n_z}$. If the iterations converge the solution exists and is $[u_\zeta; u_s] = [u_\zeta^{(\infty)}; u_s^{(\infty)}]$, which in turn allows for recovering δ.

Appendix C provides a proof of the proposition.

The iterative procedure described in proposition 2 is revealing in that it makes clear how $u_\zeta^{(n)}$ and $u_s^{(n)}$ are the equilibrium coefficients in the approximate essentially-affine pricing of real bond yields, i.e., of claims to the cashflow process $d = 0$ in equation (8). Accordingly, in this affine context

$$\ln(\delta) = \lim_{n\to\infty} [A_0^{(n)} - A_0^{(n-1)}]$$

$$\ln[f(x_t)] = B_{0,\zeta}^{(\infty)} \zeta_t + B_{0,s}^{(\infty)} \widehat{s}_t$$

characterize a solution to the eigenfunction problem, where $\{A_0^{(n)}; B_{0,\zeta}^{(n)}; B_{0,s}^{(n)}\}$ are the coefficients in the equilibrium expression of real bond yields. It follows that

$$\begin{aligned} m_{t+1}^T &= \ln(\delta) + B_{0,\zeta}^{(\infty)}\zeta_t + B_{0,s}^{(\infty)}\widehat{s_t} - B_{0,\zeta}^{(\infty)}\zeta_{t+1} - B_{0,s}^{(\infty)}\widehat{s_{t+1}} \\ &= -r_{0,t+1}^{(\infty)} \end{aligned}$$

which verifies the equivalence between Hansen-Scheinkman and Alvarez-Jermann decompositions in this context of affine pricing (see also Koijen et al., 2010).

3. Models with Campbell-Cochrane habits

As an illustration, we apply our essentially-affine approximation to the family of models with nonlinear habits à la Campbell and Cochrane (1999) and compare it to conventional loglinear-lognormal approximations. In particular, preferences are captured by the function

$$U_t = (C_t - H_t)^{1-\gamma} + \chi_t(N_t) + \beta E_t U_{t+1}$$

with C_t denoting real consumption and H_t an external habit that is a nonlinear function of past consumption, and where $\chi_t(N_t)$ allows for a dependence of preferences on labor N_t. Parameter β is the subjective discount rate and γ controls the curvature of the utility function.

In these models a large and time-varying price of risk, x_t, matches a high and volatile equity premium while the nonlinearity in the external habit is calibrated to ensure that the precautionary savings effect largely offsets the intertemporal substitution effect in the determination of the risk-free rate, thus avoiding a risk-free rate puzzle. Note that precautionary savings are captured by a conditional second moment, which therefore contains a first-order term that conventional loglinearizations would disregard.

Specifically, the scalar price of risk is a nonlinear function $x_t \equiv \gamma[1 + \lambda(\widehat{s_t})]$ of the surplus-consumption ratio, $s_t \equiv \ln[(C_t - H_t)/C_t]$, whose dynamics have the conditional heteroskedastic $AR(1)$ form

$$\begin{aligned} \widehat{s_{t+1}} &= g(\widehat{s_t}, \lambda(\widehat{s_t})[c_{t+1} - E_t^{\mathbb{P}} c_{t+1}]) \\ &= \phi\widehat{s_t} + \lambda(\widehat{s_t})\varepsilon_{t+1}^c \end{aligned} \tag{10}$$

where $\widehat{s_t} \equiv s_t - s$, λ is a function, and $\varepsilon_{t+1}^c \equiv (c_{t+1} - E_t^{\mathbb{P}} c_{t+1})$. The stochastic discount factor is

$$\begin{aligned} m_{t+1} &= \ln(\beta) - \gamma\Delta c_{t+1} - \gamma\Delta s_{t+1} \\ &= \ln(\beta) - \gamma E_t\Delta c_{t+1} + \gamma(1-\phi)\widehat{s_t} - x_t\varepsilon_{t+1}^c + O(\|\zeta_t\|^2) \end{aligned}$$

The nonlinear dynamics of the time-varying price of risk, $x(\widehat{s_t})$—a square-root process in Campbell and Cochrane (1999)—are responsible for the absence of a closed-form solution.

3.1. *Essentially-Affine Approximation*

To apply our essentially-affine approximation to this class of models we follow our three-step algorithm. For simplicity, the only conditionally heteroskedastic state is assumed to be surplus consumption, while the remaining state vector ζ_t follows the $VAR(1)$ process

$$\zeta_{t+1} = A\zeta_t + B\varepsilon_{t+1}$$

with $\varepsilon_t \sim Niid(0, I)$ a vector of shocks. Moreover, we assume a parametrization that ensures that surplus consumption has no first-order effects on cashflows, consistent with macro-finance separation. All examples considered in section 4 display macro-finance separation (see Lopez et al., 2015, for more details). It is straightforward to generalize the exercise and consider the case in which surplus consumption drives also quantities. This case comes at the cost of a more cumbersome notation and, in any event, can be subsumed to the general approach of section 2.

3.1.1. Step 1

We start by a risk-adjusted linear approximation of the first-order conditions driving quantities and solve for the approximate dynamics. As described in section 2, in order to guarantee that a loglinearization captures the first-order component of quantities, we must account for first-order components in second moments. The outcome includes a loglinear solution for consumption,

$$\Delta c_{t+1} = \mu_c + C_c\zeta_t + D_c\varepsilon_{t+1} + O(\|\zeta_t, \varepsilon_{t+1}\|^2)$$

whose conditional Gaussianity allows to rewrite the real risk-free rate as

$$r_t = -\ln(\beta) + \gamma\mu_c - \frac{1}{2}x(0)^2\|D_c\|^2 + \gamma C_c\zeta_t - \xi\widehat{s_t} + O(\|\zeta_t, \widehat{s_t}\|^2)$$

where $\xi \equiv \gamma(1-\phi) + x(0)x'(0)\|D_c\|^2$. Note how the sensitivity function λ is designed to imply $\xi = 0$, consistent with macro-finance separation.

Similarly, we retain a loglinear shape for the approximate dynamics of some log cashflow process d,

$$\Delta d_{t+1} = \mu_d + C_d\zeta_t + D_d\varepsilon_{t+1} + O(\|\zeta_t, \varepsilon_{t+1}\|^2)$$

For example, consumption equity pays off aggregate consumption, market equity pays off aggregate corporate profits, real bonds pay off a unit of numeraire, and nominal bonds the inverse of the price level.

3.1.2. Step 2

Consider the stochastic discount factor

$$m_{t+1} = -r_t - \frac{1}{2}x(\widehat{s_t})^2\|D_c\|^2 - x(\widehat{s_t})D_c\varepsilon_{t+1} + O(\|\zeta_t, \varepsilon_{t+1}\|^2)$$

and approximate the endogenous and nonlinear dynamics of the price of risk as

$$x_t = x(0) + x'(0)\widehat{s_t} + O(\|\widehat{s_t}\|^2) \tag{11}$$

$$x_t^2 = x(0)^2 + 2x(0)x'(0)\widehat{s_t} + O(\|\widehat{s_t}\|^2) \tag{12}$$

Note that under the benchmark Campbell and Cochrane (1999) specification the residual in equation (12) is exactly zero because the price of risk follows a square-root process. Thus approximated, the price of risk allows for an affine solution for equilibrium yields when combined with loglinear(ized) cashflows and risk-free rate.

Conjecture the affine solution for yields,

$$y_{d,t}^{(n)} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B_{\zeta}^{(n)}\zeta_t - \frac{1}{n}B_s^{(n)}\widehat{s_t}$$

and identify it using equation (8). To illustrate once more the derivation, we verify the conjecture through the fundamental no-arbitrage pricing formula $E_t(M_{t+1}R_{t+1}^i) = 1$ under lognormality:

$$\begin{aligned}
0 &= E_t m_{t+1} + ny_{d,t}^{(n)} - (n-1)E_t y_{d,t+1}^{(n-1)} + E_t\Delta d_{t+1} + \frac{1}{2}var_t\left[m_{t+1} - (n-1)y_{d,t+1}^{(n-1)} + \Delta d_{t+1}\right] \\
&= \ln(\beta) + \mu_d - \gamma\mu_c - A^{(n)} + A^{(n-1)} + \left[C_d - \gamma C_c - B_{\zeta}^{(n)} + B_{\zeta}^{(n-1)}A\right]\zeta_t + \left[\gamma(1-\phi) - B_s^{(n)} + B_s^{(n-1)}\phi\right]\widehat{s_t} \\
&\quad + \frac{1}{2}||D_c||^2 x_t^2 + \frac{1}{2}V_{n-1,t}V_{n-1,t}' - x_t D_c V_{n-1,t}' + O(||\zeta_t, \varepsilon_{t+1}, x_t\varepsilon_{t+1}||^2)
\end{aligned}$$

where $V_{n-1,t} = V_{0,n-1} + V_{1,n-1}x_t \equiv D_d + B_{\zeta}^{(n-1)}B - B_s^{(n-1)}D_c + x_t B_s^{(n-1)}D_c/\gamma$, and hence, using equations (11) and (12),

$$\begin{aligned}
0 = \ln(\beta) &+ \mu_d - \gamma\mu_c - A^{(n)} + A^{(n-1)} + \frac{1}{2}||V_{0,n-1} - x(0)(D_c - V_{1,n-1})||^2 \\
&+ [\gamma(1-\phi) - B_s^{(n)} + B_s^{(n-1)}\phi + x(0)x'(0)||D_c - V_{1,n-1}||^2 - x'(0)V_{0,n-1}(D_c - V_{1,n-1})']\widehat{s_t} \\
&+ [C_d - \gamma C_c - B_{\zeta}^{(n)} + B_{\zeta}^{(n-1)}A]\zeta_t + O(||\zeta_t, \widehat{s_t}, \varepsilon_{t+1}, x_t\varepsilon_{t+1}||^2)
\end{aligned}$$

which identifies the affine solution as the solution to the Riccati equations

$$\begin{aligned}
A^{(n)} &= A^{(n-1)} + \ln(\beta) + \mu_d - \gamma\mu_c + \frac{1}{2}||V_{0,n-1} - x(0)(D_c - V_{1,n-1})||^2 \\
B_{\zeta}^{(n)} &= B_{\zeta}^{(n-1)}A + C_d - \gamma C_c \\
B_s^{(n)} &= B_s^{(n-1)}\phi + \gamma(1-\phi) + x(0)x'(0)||D_c - V_{1,n-1}||^2 - x'(0)V_{0,n-1}(D_c - V_{1,n-1})'
\end{aligned} \tag{13}$$

with $V_{0,n-1} = D_d + B_{\zeta}^{(n-1)}B - B_s^{(n-1)}D_c$ and $V_{1,n-1} = B_s^{(n-1)}D_c/\gamma$.

These closed-form expressions allow for computing the entire term structure of yields from a simulated path of the state vector $[\zeta_t; \widehat{s_t}]$ up to a remainder of order at least $O(||\zeta_t, \widehat{s_t}, \varepsilon_{t+1}, x_t\varepsilon_{t+1}||^2)$.

3.1.3. Step 3

We use the lognormal no-arbitrage pricing formula to compute

$$r_{t+1}^{e,(n)} = x_t D_c V_{n-1,t}' - \frac{1}{2}V_{n-1,t}V_{n-1,t}' + V_{n-1,t}\varepsilon_{t+1}$$

with $V_{n-1,t} = V_{0,n-1} + V_{1,n-1}x_t$.

To simulate x_t and therefore a sample path for risk premia and return volatilities we use the exact dynamics $x(\widehat{s_t})$.

3.1.4. Uniqueness and stability of the solutions of the Riccati equation

The equilibrium loadings on surplus consumption $\{B_s^{(n)}\}$ are described by a nonlinear difference equation, so a discussion of multiplicity and stability of its solutions is warranted. We can rewrite the Riccati equation that determines the loadings on surplus consumption of the term structure components of arbitrary cashflow claims as

$$B_s^{(n)} - B_s^{(n-1)} = \left(1 - \frac{1}{\gamma} B_s^{(n-1)}\right)\left[S\left(1 - \phi - \frac{\xi}{\gamma}\right)\left(\frac{D_d D_c'}{\|D_c\|^2} + (C_d - \gamma C_c)[I - A]^{-1}[I - A^n]\frac{BD_c'}{\|D_c\|^2} + \frac{1 - S}{S} B_s^{(n-1)}\right) + \xi\right]$$

and note how $B_s^{(n-1)} \in \{B_{s,1}^{(n-1)} = -\frac{S}{1-S}\frac{D_d D_c' + (C_d - \gamma C_c)[I-A]^{-1}[I-A^n]BD_c'}{\|D_c\|^2} - \frac{\xi/(1-S)}{1-\phi-\xi/\gamma}; B_{s,2}^{(n-1)} = \gamma\}$ implies $B_s^{(n)} = B_s^{(n-1)}$. This result shows the existence of two points of convergence of the sequence of coefficients $\{B_s^{(n)}\}$,

$$B_s^{(\infty)} = \begin{cases} -\frac{S}{1-S}\frac{D_d D_c' + (C_d - \gamma C_c)[I-A]^{-1}BD_c'}{\|D_c\|^2} - \frac{\xi/(1-S)}{1-\phi-\xi/\gamma} \\ \gamma \end{cases}$$

The stability of the two asymptotic solutions in the case of bonds is particularly important because it links to the diagnostic decompositions proposed by Alvarez and Jermann (2005) and Hansen and Scheinkman (2009) (see section 2.5). Namely, in the case of real bonds we have

$$\begin{aligned} \frac{\partial[B_{0,s}^{(n)} - B_{0,s}^{(n-1)}]}{\partial B_{0,s}^{(n-1)}} &= S\left(1 - \phi - \frac{\xi}{\gamma}\right)\frac{C_c[I - A]^{-1}[I - A^n]BD_c'}{\|D_c\|^2} - \frac{\xi}{\gamma} - 2\frac{(1 - S)(1 - \phi - \xi/\gamma)}{\gamma} B_{0,s}^{(n-1)} \\ &= \begin{cases} S(1 - \phi - \xi/\gamma)\frac{-C_c[I-A]^{-1}[I-A^n]BD_c'}{\|D_c\|^2} + \xi/\gamma \geq 0 & \text{for } B_{0,s}^{(n-1)} = B_{0,s,1}^{(n-1)} \\ -(1 - \phi - \xi/\gamma)\left[S\frac{-C_c[I-A]^{-1}[I-A^n]BD_c'}{\|D_c\|^2} + 2(1 - S)\right] - \xi/\gamma < 0 & \text{for } B_{0,s}^{(n-1)} = B_{0,s,2}^{(n-1)} \end{cases} \end{aligned}$$

where the inequalities hold because for typical calibrations $C_c[I - A]^{-1}[I - A^n]BD_c' \leq 0$. Since a limiting point is stable if the asymptotic derivative $\lim_{n\to\infty} \partial[B_{0,s}^{(n)} - B_{0,s}^{(n-1)}]/\partial B_{0,s}^{(n-1)} < 0$ and is unstable otherwise, it must be that $B_{0,s}^{(\infty)} = \gamma$ is the stable limiting equilibrium coefficient. Moreover, given $B_{0,s}^{(0)} = 0$, the sequence of loadings $\{B_{0,s}^{(n)}\}$ will converge to the unstable limiting point if and only if the intertemporal substitution and precautionary savings effect of risk aversion on the risk-free rate exactly offset, $\xi = 0$, and the predictable component of consumption is either absent or not priced, $C_c[I - A]^{-1}[I - A^n]BD_c' = 0$ for all maturities n—in which case $B_{0,s}^{(\infty)} = 0$.

Against this background, note that these requirements hold exactly in the original calibration of Campbell and Cochrane (1999), which explains why they manage to produce a perfectly flat real term structure. Wachter (2006) keeps working in a random-walk environment but breaks the condition $\xi = 0$, which puts the sequence of loading coefficients on a trajectory that converges to the stable limiting coefficient $B_{0,s}^{(\infty)} = \gamma$.

3.1.5. Asset pricing diagnostics

For the family of models with Campbell-Cochrane habits, the approximate Perron-Frobenius eigenfunction has the form $f(z_t) = e^{c_\zeta'\zeta_t + c_s\widehat{s_t}}$ and we identify it by relying on proposition 2 or, to illustrate once more the derivation, as

$$\begin{aligned}\ln(\delta) + c_\zeta\zeta_t + c_s\widehat{s_t} = \ln(\beta) - \gamma\mu_c + [-\gamma C_c + c_\zeta A]\zeta_t + [\gamma(1-\phi)c_s\phi]\widehat{s_t} \\ + \frac{1}{2}\|D_c\|^2 x_t^2 + \frac{1}{2}V_t V_t' - D_c x_t V_t' + O(\|\zeta_t, x_t D_c \varepsilon_{t+1}\|^2)\end{aligned}$$

where $V_t = V_0 + V_1 x_t \equiv c_\zeta B - c_s D_c + x_t c_s D_c/\gamma$. To solve the problem we rely as usual on equations (11) and (12) and identify the approximate eigenfunction as the solution to the difference equations

$$\begin{aligned}\ln(\delta) &= \ln(\beta) - \gamma\mu_c + \frac{1}{2}\|V_0 - x(0)(D_c - V_1)\|^2 \\ c_\zeta &= -\gamma C_c[I-A]^{-1} = B_{0,\zeta}^{(\infty)} \\ c_s &= \gamma + \frac{x(0)x'(0)}{1-\phi}\|D_c - V_1\|^2 - \frac{x'(0)}{1-\phi}V_0(D_c - V_1)' = B_{0,s}^{(\infty)}\end{aligned}$$

Therefore, the martingale component of the discount factor has the shape

$$\begin{aligned}m_{t+1}^P &= \gamma(1-\phi)\left(1 - \frac{1}{\gamma}B_{0,s}^{(\infty)}\right)\widehat{s_t} - \frac{1}{2}\|V_0 - e^{-s}(\gamma - B_{0,s}^{(\infty)})D_c\|^2 + V_0\varepsilon_{t+1} - \left(1 - \frac{1}{\gamma}B_{0,s}^{(\infty)}\right)x_t D_c \varepsilon_{t+1} \\ &= \begin{cases} -\frac{1}{2}x_t^2\|D_c\|^2 - x_t D_c \varepsilon_{t+1} & \text{if } \xi = 0 \text{ and } C_c[I-A]^{-1}[I-A^n]BD_c' = 0, \forall n \\ -\frac{1}{2}\|V_0\|^2 + V_0\varepsilon_{t+1} & \text{elsewhere}\end{cases}\end{aligned}$$

where we showed in the previous subsection how the sequence of loadings $\{B_{0,s}^{(n)}\}$ converges to the unstable limiting point $B_{0,s}^{(\infty)} = 0$ if and only if $\xi = 0$ and $C_c[I-A]^{-1}[I-A^n]BD_c' = 0$ for all maturities n, and converges to the stable asymptotic coefficient $B_{0,s}^{(\infty)} = \gamma$ otherwise.

Finally, the variance ratio under the essentially-affine approximation is

$$\begin{aligned}\frac{var_t(m_{t+1}^P)}{var_t(m_{t+1})} &= \begin{cases} 1 & \text{if } \xi = 0 \text{ and } C_c[I-A]^{-1}[I-A^n]BD_c' = 0, \forall n \\ \frac{\|V_0\|^2}{x_t^2\|D_c\|^2} = 1 - \frac{x_t D_c(V_0 + x_t V_1)' - \frac{1}{2}\|V_0 + x_t V_1\|^2}{\frac{1}{2}x_t^2\|D_c\|^2} & \text{elsewhere}\end{cases} \\ &= 1 - \frac{E_t r_{0,t+1}^{e,(\infty)}}{\frac{1}{2}x_t^2\|D_c\|^2}\end{aligned}$$

As long as our essentially-affine approximation captures well the long-run bond premium, $E_t r_{0,t+1}^{e,(\infty)}$, it also accurately describes the variance ratio and therefore a crucial diagnostic tool in capturing the properties of the discount factor. For example, under the original calibration of Campbell and Cochrane (1999), appendix F shows how our approximation correctly captures a variance ratio that is constant at unity and the crucial property that shocks to $S_t^{-\gamma}$ have a permanent effect on the marginal utility of wealth even though surplus consumption itself is stationary.

3.2. Interpretation: essentially-affine and Borovicka-Hansen decomposition

The essentially-affine approximation is particularly useful to provide an intuitive understanding of the macroeconomic forces that drive the prices of financial claims and the risk premia they command. Using equation $r^{e,(n)}_{t+1} = E_t r^{e,(n)}_{t+1} + V_{n-1,t}\varepsilon_{t+1}$, we can decompose the reaction of the quantity of risk in the nth strip return to a shock along direction α_t into three components,

$$V_{n-1,t}\alpha'_t = \Big[\underbrace{D_d}_{\substack{\text{short-run}\\\text{cashflow}\\\text{risk}}} + \underbrace{B^{(n-1)}_{\zeta} B}_{\substack{\text{long-run cashflow}\\\text{and discount-rate risk}}} + \underbrace{B^{(n-1)}_{s}\lambda_t D_c}_{\substack{\text{habit-related}\\\text{discount-rate}\\\text{risk}}} \Big]\alpha'_t \tag{14}$$

which are the basis to understand the shape of the term structure of holding-period risk premia, $\ln E_t R^{e,(n)}_{t+1} = x_t D_c V'_{n-1,t}$. The first element on the right-hand side of the equation controls the cashflow effect due to contemporaneous shocks to dividends. The second element captures the income effect of past dividend shocks as well as the substitution effect of past consumption shocks that would be present even in the absence of habits. The third element captures the discount-rate effect of past consumption shocks that are transmitted by habits. Note how stochastic volatility (which here reduces to time-varying risk aversion) generates time variation in the loadings of excess returns on habit-related discount-rate risk.

Another way the essentially-affine approximation facilitates an intuitive understanding of the asset-pricing implications of the macro-finance model is by providing simple expressions for the dynamic value decomposition proposed by Borovicka and Hansen (2014) as measures to quantify the exposures of cashflows to shocks over alternative horizons and the corresponding compensations commanded by investors. In particular, for any increase in one-step ahead uncertainty along dimension α_t we can define cashflow and discount-rate elasticities as

$$\begin{aligned}
\varepsilon^{(n)}_{g,t} &\equiv \frac{d}{dr} \ln E^{\mathbb{P}}_t \left[\frac{D_{t+n}}{D_t} e^{r\alpha_t\varepsilon_{t+1} - \frac{r^2}{2}\|\alpha_t\|^2} \right]_{r=0} = \Big[D_d + C_d (I-A)^{-1}(I-A^{n-1})B \Big]\alpha'_t \\
\varepsilon^{(n)}_{p,t} &\equiv \frac{d}{dr} \ln E^{\mathbb{P}}_t \left[\frac{D_{t+n}}{D_t} e^{r\alpha_t\varepsilon_{t+1} - \frac{r^2}{2}\|\alpha_t\|^2} \right]_{r=0} - \frac{d}{dr} \ln E^{\mathbb{P}}_t \left[M_{t,t+n}\frac{D_{t+n}}{D_t} e^{r\alpha_t\varepsilon_{t+1} - \frac{r^2}{2}\|\alpha_t\|^2} \right]_{r=0} \\
&= \Big[x_t D_c + \gamma C_c (I-A)^{-1}(I-A^{n-1})B - B^{(n-1)}_s \lambda_t D_c \Big]\alpha'_t
\end{aligned} \tag{15}$$

Appendix D derives expression (15). These elasticities capture the impact of current shocks on future cashflows ($\varepsilon^{(n)}_{g,t}$) and on future expected returns ($\varepsilon^{(n)}_{p,t}$), while the impact on valuations can be recovered as the value elasticity $\varepsilon^{(n)}_{g,t} - \varepsilon^{(n)}_{p,t}$. While elasticities $\varepsilon^{(n)}_{g,t}$ reveal a straightforward impulse response function of n-period ahead cumulated cashflows, the elasticities $\varepsilon^{(n)}_{p,t}$ show how two forces are at work that pull discount rates in opposite directions. On the one hand, if consumption is expected to grow people want to anticipate it and hence require a larger compensation to invest; on the other hand, they will have a larger risk aversion, so they will want to consume more in the future and hence require a smaller compensation to invest today.[10]

[10]There is even a third motive hidden in the coefficient $B^{(n)}_s$, which captures not only the intertemporal substitution

The decompositions (14) and (15) are particularly useful to gain insight into the drivers of asset prices, and they are deeply linked. In fact, when $\alpha_t = x_t D_c$ (a discount-rate shock), we have

$$\ln E_t R^{e,(n)}_{d,t+1} = \varepsilon^{(n)}_{g,t} - \varepsilon^{(n)}_{p,t} + var_t(m_{t+1})$$

and so holding-period risk premia are equivalent to a strictly positive level factor (a precautionary motive, as investors require some compensation when future marginal utility is uncertain) plus the cashflow effect of positive consumption news on future dividends minus the discount-rate effect of the shock on investors' required compensation.

3.3. Alternative exponential-affine approximations of models with Campbell-Cochrane habits

We consider two alternative exponential-affine approximations—the completely-affine approximation described by Malkhozov (2014) and a two-stage loglinear-lognormal approximation in the spirit of Jermann (1998).

3.3.1. Completely-affine approximation

The completely-affine approximation generates a different closed-form solution. Namely, since it relies on a loglinearization adjusted for exogenous risk but not for endogenous risk, the method would approximate the $\mathbb{P}$-dynamics of surplus consumption (10) as

$$\widehat{s}_{t+1} = g^*(\widehat{s}_t, c_{t+1}, E_t c_{t+1}) \approx \phi\widehat{s}_t + \lambda(c_{t+1} - E_t c_{t+1})$$

and thereby eliminate all time-variation in risk aversion and in the price of risk. It follows that the method is unable to capture the first-order component contained in second moments and would therefore reduce to a conventional loglinearization up to a constant risk adjustment.

The only exception occurs in the context of an endowment economy in which the consumption process is exogenously given. For example, if $\Delta c_{t+1} \sim Niid(\mu, \sigma^2)$, with known σ, the completely-affine approximation preserves the $\mathbb{P}$-dynamics of surplus consumption because risk is exogenous:

$$\widehat{s}_{t+1} = g(\widehat{s}_t, \lambda_t \sigma \varepsilon_{t+1}) \approx \phi\widehat{s}_t + \lambda_t \sigma \varepsilon_{t+1}.$$

It follows that in this context the completely-affine approximation is equivalent to an essentially-affine approximation in which step 3 is replaced with simulations from the loglinearized $\mathbb{P}$-dynamics of risk aversion (equation (11)).

3.3.2. Two-stage loglinear-lognormal approximation

The two-stage loglinear-lognormal approximation starts by a conventional loglinearization under certainty equivalence, which expresses both the marginal utility of wealth and cashflows as a linear function of a conditionally homoskedastic state vector. This first step turns off any conditional heteroskedasticity and allows for a solution for asset prices up to a residual of order at least $O(||\zeta_t, \varepsilon_{t+1}||^2, ||\widehat{x}_t, \widehat{x}^2_t||)$. The heteroskedasticity is then revived in a second stage in which

effect that prompts people to postpone consumption but also a precautionary motive for people to command a larger compensation in the face of uncertainty surrounding future marginal utility and returns.

lognormal pricing formulas are exploited to derive equilibrium risk premia. We simulate risk premia using the nonlinear dynamics of the price of risk in order to increase the accuracy of the method as well as to isolate the importance of the risk-adjustment in the loglinearization when comparing the essentially-affine and the two-stage loglinear-lognormal approximations.

3.4. Comparison of the essentially-affine and the alternative affine approximations

We focus on a production economy with Campbell-Cochrane habits, in which the role of endogenous risk is predominant, to provide more insight into the two main differences between our approximation and the alternatives in the exponential-affine class. A loglinearization that does not adjust for endogenous risk associates with a different equilibrium allocation, including different coefficient vectors $[C_c; D_c; C_d; D_d]$ that affect both discount rates and cashflows. In particular, the independence of consumption and cashflows on surplus consumption may no longer hold.

It follows that the coefficients that control the loadings of yields on surplus consumption are severly distorted under the completely-affine and the two-stage loglinear-lognormal methods. In fact, both methods would impose the linearization of the stochastic discount factor

$$m_{t+1} = \ln(\beta) - \gamma\mu_c - \gamma(C_{c,\zeta}\zeta_t + C_{c,s}\widehat{s_t}) + \gamma(1-\phi)\widehat{s_t} - xD_c\varepsilon_{t+1} + O(\|\zeta_t, \widehat{x_t}D_c\varepsilon_{t+1}\|^2)$$

where the price of risk becomes a constant, to subsequently derive the affine solution for yields

$$y_{d,t}^{(n)} = -\frac{1}{n}A^{(n)} - \frac{1}{n}B_\zeta^{(n)}\zeta_t - \frac{1}{n}B_s^{(n)}\widehat{s_t}$$

via the fundamental no-arbitrage pricing formula

$$\begin{aligned} 0 &= E_t m_{t+1} + ny_t^{(n)} - (n-1)E_t y_{t+1}^{(n-1)} + E_t\Delta d_{t+1} + \frac{1}{2}var^{(n)} \\ &= \ln(\beta) + \mu_d - \gamma\mu_c - A^{(n)} + A^{(n-1)} + \frac{1}{2}var^{(n)} + [C_{d,\zeta} - \gamma C_{c,\zeta} - B_\zeta^{(n)} + B_\zeta^{(n-1)}A]\zeta_t \\ &\quad + [\gamma(1-\phi) + C_{d,s} - \gamma C_{c,s} - B_s^{(n)} + B_s^{(n-1)}\phi]\widehat{s_t} + O(\|\zeta_t, \varepsilon_{t+1}\|^2, \|\widehat{x_t}, \widehat{x_t^2}\|) \end{aligned}$$

which identifies the affine solution via the difference equations

$$\begin{aligned} A^{(n)} &= A^{(n-1)} + \ln(\beta) + \mu_d - \gamma\mu_c + \frac{1}{2}var^{(n)} \\ B_\zeta^{(n)} &= B_\zeta^{(n-1)}A + C_{d,\zeta} - \gamma C_{c,\zeta} \\ B_s^{(n)} &= B_s^{(n-1)}\phi + \gamma(1-\phi) + C_{d,s} - \gamma C_{c,s} \end{aligned} \tag{16}$$

for some sequence of coefficients $var^{(n)} = \|xD_c - [D_d + B_\zeta^{(n-1)}B + (\frac{1}{\gamma}x - 1)B_s^{(n-1)}D_c]\|^2$.

As we compare the term structure coefficients (13) and (16), we note how the level coefficients, $\{A^{(n)}\}$, the loadings on state vector ζ_t, $\{B_\zeta^{(n)}\}$, and the loadings of holding-period strip returns on the shocks, $\{V_{0,n}; V_{1,n}\}$, have similar expressions in the essentially-affine method. Along with differences in the equilibrium coefficients $[C_c; D_c; C_d; D_d]$, the difference between our essentially-affine approximation and the alternative approximations comes from the loading of cashflow yields on surplus consumption, $\{B_s^{(n)}\}$; namely, the essentially-affine approximation has an extra term $x(0)x'(0)||D_c - V_{1,n-1}||^2 - x'(0)V_{0,n-1}(D_c - V_{1,n-1})'$ that improves the quality of the approximation.

In an endowment economy, where risk is exogenous, the equilibrium coefficients $[C_c; D_c; C_d; D_d]$ would not be distorted ($C_{d,s} = C_{c,s} = 0$). The loadings of yields under the essentially- and completely-affine approximations would coincide but the two-stage loglinear-lognormal would still characterize them via equation (16) and would thereby maintain the distortion in the expression for the loading of cashflow yields on surplus consumption, $\{B_s^{(n)}\}$.

4. Accuracy of the approximation

We compare the performance of our essentially-affine approximation to global solutions as well as to the alternative exponential-affine approximations described in the previous section by applying them to three macro-finance models with nonlinear habits à la Campbell-Cochrane. We evaluate the quality of our approximation by comparing the term structures of zero-coupon equities; this exercise allows for decomposing the quality of the approximation at different time horizons and for claims that are the basis for pricing other more complex assets. We consider the models of Campbell and Cochrane (1999) and Wachter (2006) as well-known examples of nonlinear habit models that fit into our framework of section 2. We then turn to the model of Lopez et al. (2015) as an example of a model that unites nonlinear habits with a New Keynesian production economy.

For the models of Campbell and Cochrane (1999) and Wachter (2006) the global solution is calculated using cubic splines, while for Lopez et al. (2015) the global solution is projected onto the subspace spanned by a basis of Chebyshev polynomials of up to degree eight, as macroeconomic quantities in this model are not loglinear.[11] Since the equilibrium allocation is not exogenously given, we consider multiperiod Euler equation errors as an additional metric to contrast the quality of the approximation.

Our essentially-affine approximation is accurate in solving for risk premia and volatilities of short- and long-duration equities and bonds—although it seems to scale up the correct values for medium-duration bonds and equities—and outperforms the alternative approximation schemes.

4.1. Example 1: Campbell and Cochrane (1999)

In the benchmark model by Campbell and Cochrane (1999), the price of risk is $x_t = \gamma/S\ \sqrt{1 - 2\widehat{s}_t}$, with the calibration $S = \sqrt{\gamma||D_c||^2/(1-\phi)}$. Therefore, the first-order expansions of the dynamics of the price of risk (11) and (12) hold with coefficients $x = -x'(0) = \gamma/S$. The particular calibration implies $r_t = -\ln(\beta) + \gamma\mu_c - \frac{1}{2}\gamma(1-\phi)$.

Consumption and market dividends, Δd, are random walks with structure

$$\Delta c_{t+1} = \mu_c + v_{t+1}$$
$$\Delta d_{t+1} = \mu_d + w_{t+1}$$

[11] See the online appendix for details on the construction of the global solution.

Parameter	Campbell-Cochrane	Wachter
Frequency	monthly	quarterly
Subjective discount factor, β	$0.90^{1/12}$	0.98
Utility curvature, γ	2	2
Habit persistence, ϕ	$0.87^{1/12}$	0.97
Short-run rate exposure to s_t, ξ	0	0.011
Mean consumption growth (in %), μ_c	1.89/12	0.549
Mean cashflow growth (in %), μ_d	1.89/12	−0.920
Standard deviation of consumption growth (in %), σ_v	$1.50/\sqrt{12}$	0.431
Standard deviation of cashflow growth (in %), σ_w	$11.2/\sqrt{12}$	0.588
Consumption-cashflows correlation, ρ	0.200	0.205
State persistence, ψ_1	—	0.941
State volatility, ψ_2	—	0.297

Table 1: Deep parameters and their calibration. Cashflow growth is dividend growth in Campbell and Cochrane (1999) and inverse inflation in Wachter (2006).

where $[v_t; w_t] \sim Niid(0, \Sigma)$. Consider the decomposition of the covariance matrix $\Sigma = \Sigma^{1/2}\Sigma^{1/2\prime}$ with Choleski factor $\Sigma^{1/2} = \begin{bmatrix} \sigma_v & 0 \\ \rho\sigma_w & \sqrt{1-\rho^2}\sigma_w \end{bmatrix}$. Then, $A = B = C_c = C_d = 0$, $D_c = [1; 0]'\Sigma^{1/2}$ and $D_d = [0; 1]'\Sigma^{1/2}$.

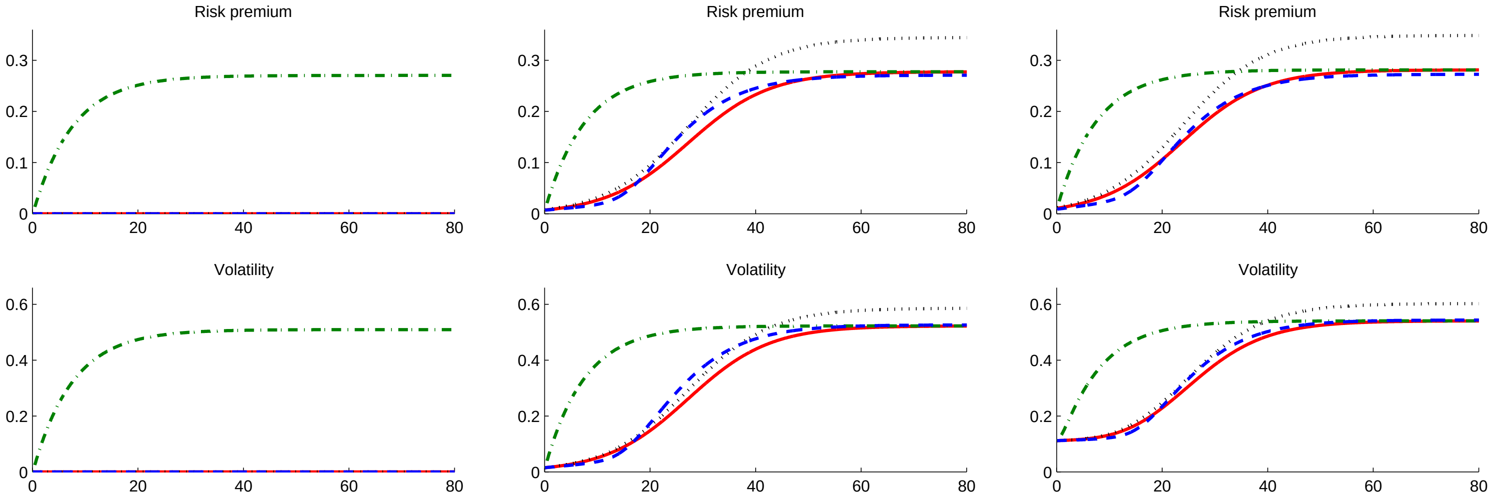
Table 1 reports the calibration of the parameter values. Figure 1 compares the global solution with our proposed essentially-affine solution and the alternative affine approximations. The figures report the term structure of equilibrium risk premia and realized return volatilities of zero-coupon equities and bonds. The three affine approximations differ substantially and the essentially-affine is unambiguously preferable in practice. Notice how the completely-affine approximation that simulates from the linearized $\mathbb{P}$-dynamics of the price of risk severely overstates the asymptotic risk premium. The two-stage loglinear-lognormal approximation is particularly inadequate to capture risk premia as it fails to take into account the first-order component in the volatility of the discount factor and thereby generates a dramatically wrong term structure of interest rates. The fit of the essentially-affine approximation is not perfect over medium-horizons but it captures the level, amplitude and shape of the term structures.

4.2. Example 2: Wachter (2006)

In the model of Wachter (2006), the price of risk is $x_t = \gamma/S\sqrt{1-2\widehat{s_t}}$, with the calibration $S = \sqrt{\gamma\|D_c\|^2/(1-\phi-\xi/\gamma)}$. Therefore, the first-order expansions of the dynamics of the price of risk (11) and (12) hold with coefficients $x = -x'(0) = \gamma/S$. The particular calibration implies $r_t = -\ln(\beta) + \gamma\mu_c - \frac{1}{2}\gamma(1-\phi-\xi/\gamma) - \xi\widehat{s_t}$.

Consumption and nominal bond cashflows, $\Delta d = -\pi$, have structure

$$\begin{aligned} \Delta c_{t+1} &= \mu_c + v_{t+1} \\ \Delta d_{t+1} &= \mu_d + z_t + w_{t+1} \\ z_{t+1} &= \psi_1 z_t + \psi_2 D_c \varepsilon_{t+1} \end{aligned}$$



(a) *Real bonds*. (b) *Consumption strips*. (c) *Dividend strips*.

Figure 1: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in the Campbell and Cochrane (1999) model. Essentially-affine (solid red), completely-affine (dotted black), two-stage loglinear-lognormal (dot-dashed green), and projected solution using cubic splines collocated over 100 Chebyshev nodes and 6-point Gauss-Hermite quadrature (dashed blue).

where $[v_t; w_t] \sim Niid(0, \Sigma)$. Then, $A = \psi_1$, $B = \psi_2 D_c$, $C_c = 0$, $C_d = 1$, $D_c = [1; 0]'\Sigma^{1/2}$ and $D_d = [0; 1]'\Sigma^{1/2}$.

We calibrate the parameters with the values reported in table 1. Figure 2 contrasts the global solution with our essentially-affine approximation and under the alternative affine methods. The figures report the term structure of equilibrium risk premia and volatilities of equities and of real and nominal interest rates. The plots confirm the overall good performance of our essentially-affine approximation which captures almost perfectly the level, amplitude and shape of the term structures. The approximation errors of the alternative affine methods are much larger in comparison.

4.3. Example 3: Lopez et al. (2015)

We turn to a version of the macro-financially separate New Keynesian model with nonlinear habits of Lopez et al. (2015).

4.3.1. Risk-adjusted loglinearization

To set the necessary conditions that describe the competitive equilibrium values of the consumption gap $\widetilde{c}_t \equiv c_t - a_t$, with a_t the level of technology, the inflation rate π_t and the auxiliary control variable ℓ_t in the generic framework of section 2 we write the forward-looking equations in form (1) as

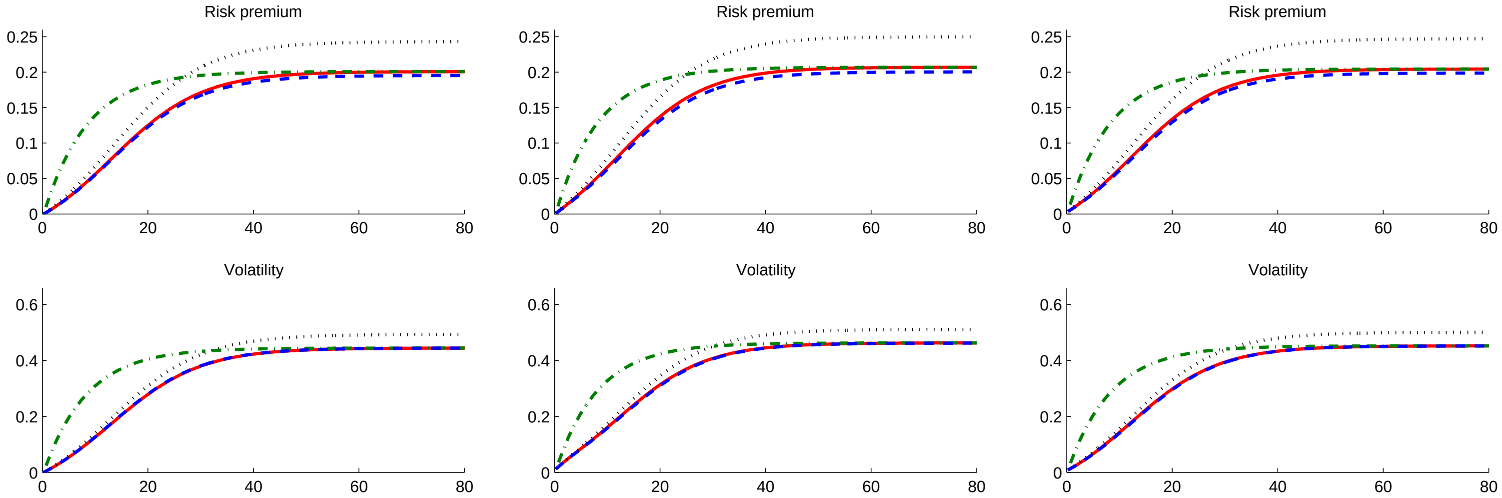
$$
\begin{aligned}
0 &= \ln E_t^{\mathbb{P}} e^{\ln(\beta)-\gamma\mu+r+\phi_\pi\pi_t+\phi_y\widehat{\widetilde{c}}_t-\gamma(\Delta\widehat{\widetilde{c}}_{t+1}+u_t+\sigma\varepsilon^a_{t+1})-\gamma\Delta\widehat{s}_{t+1}-\pi_{t+1}} \\
0 &= \ln E_t^{\mathbb{P}} e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}+(1-\gamma)(u_t+\sigma\varepsilon^a_{t+1})+\Delta\widehat{\ell}_{t+1}-\widehat{w}_{1,t}} \\
0 &= \ln E_t^{\mathbb{P}} e^{(\varepsilon-1)\pi_{t+1}+(1-\gamma)(u_t+\sigma\varepsilon^a_{t+1})+\Delta\widehat{\ell}_{t+1}+\Delta\widehat{w}_{3,t+1}-\widehat{w}_{2,t}} \\
0 &= \ln e^{\ln\left(1-\frac{\chi_0}{1-\alpha}\frac{N}{L}\exp\left[\frac{1}{1-\alpha}\widehat{\widetilde{c}}_t-\gamma(1+\xi_2)\widehat{s}_t-\widehat{\ell}_t\right]\left[1-N\exp(\Delta+\frac{1}{1-\alpha}\widehat{\widetilde{c}}_t+\widehat{\Delta}_t)\right]^{-\gamma}\right)-\widehat{w}_{1,t}-\ln(W_1)} \\
0 &= \ln e^{\ln\left(1-\frac{N^{(1-\gamma)(1-\alpha)}}{W_3 L}\exp\left[(1-\gamma)\widehat{\widetilde{c}}_t-\gamma\widehat{s}_t-\widehat{\ell}_t-\widehat{w}_{3,t}\right]\right)-\widehat{w}_{2,t}-\ln(W_2)} \\
0 &= \ln e^{\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(\varepsilon-1)}\{\ln(1-\eta\exp[(\varepsilon-1)\pi_t])-\ln(1-\eta)\}-\widehat{w}_{3,t}-\ln(W_3)}
\end{aligned}
$$

with $r \equiv -\ln(\beta) + \gamma\mu - .5(1 - \rho_s - \xi_1/\gamma)$, where the states take form (2):

$$
\begin{aligned}
a_{t+1} &= \mu + a_t + u_t + \sigma\varepsilon^a_{t+1} \\
u_{t+1} &= \rho_u u_t + \rho\phi\sigma\varepsilon^a_{t+1} + \sqrt{1-\rho^2}\phi\sigma\varepsilon^u_{t+1} \\
\widehat{s}_{t+1} &= \rho_s\widehat{s}_t + \Lambda(\widehat{s}_t)(\widetilde{c}_{t+1} - E_t\widetilde{c}_{t+1} + \sigma\varepsilon^a_{t+1}) \\
\Delta_{t+1} &= \ln\left[\eta e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}+\Delta_t} + (1-\eta)\left(\frac{1-\eta e^{(\varepsilon-1)\pi_{t+1}}}{1-\eta}\right)^{\frac{\varepsilon}{(1-\alpha)(\varepsilon-1)}}\right]
\end{aligned}
$$

where $\varepsilon^a_t \sim Niid(0,1)$, $\varepsilon^u_t \sim Niid(0,1)$ and $\varepsilon^a_t \perp \varepsilon^u_t$.

A risk-adjusted loglinearization of the set of optimality conditions around the zero-inflation deterministic steady state results in the risk-adjusted dynamic IS equation, the risk-adjusted New



(a) *Real bonds*. (b) *Nominal bonds*. (c) *Consumption strips*.

Figure 2: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in the Wachter (2006) model. Essentially-affine (solid red), completely-affine (dotted black), two-stage loglinear-lognormal (dot-dashed green), and projected solution using cubic splines collocated over 100 Chebyshev nodes and 6-point Gauss-Hermite quadrature (dashed blue).

Keynesian Phillips curve, and the equation characterizing an auxiliary control variable:

$$
\begin{aligned}
\gamma E_t \Delta\widetilde{c}_{t+1} &= \phi_\pi \pi_t + \phi_y \widetilde{c}_t - E_t\pi_{t+1} - \gamma u_t + \frac{1}{2} var_t(\pi_{t+1}) + x_t cov_t(c_{t+1}, \pi_{t+1}) + \xi_1 s_t \\
\pi_t &= \delta E_t \pi_{t+1} + \kappa \widetilde{c}_t - \gamma\lambda\xi_2 s_t \\
&\quad - \frac{1}{2}\frac{\delta(1-\alpha+\alpha\varepsilon)}{(1-\eta)(1-\alpha)} var_t(\pi_{t+1}) + cov_t\left(\delta\pi_{t+1}, \frac{\varepsilon}{1-\alpha}\pi_{t+1} + (1-\gamma)\sigma\varepsilon^a_{t+1} + \ell_{t+1}\right) \\
\ell_t &= \delta\eta\left(E_t\ell_{t+1} + \frac{\varepsilon}{1-\alpha}E_t\pi_{t+1} + (1-\gamma)u_t\right) + (1-\delta\eta)\left(\frac{1+\varphi}{1-\alpha}\widetilde{c}_t - \gamma(1+\xi_2)s_t\right) \\
&\quad + \frac{\delta\eta}{2} var_t\left(\frac{\varepsilon}{1-\alpha}\pi_{t+1} + (1-\gamma)\sigma\varepsilon^a_{t+1} + \ell_{t+1}\right)
\end{aligned}
$$

up to an irrelevant constant, where $\delta \equiv \beta e^{(1-\gamma)\mu}$, $\kappa \equiv \lambda[\gamma(1-\alpha)+\alpha+\varphi]/(1-\alpha+\alpha\varepsilon)$ and $\lambda \equiv (1-\eta)(1-\delta\eta)/\eta$.

Under the determinacy condition $\kappa(\phi_\pi - 1) + (1-\delta)\phi_y > 0$ with $[\phi_\pi, \phi_y] \in \mathbb{R}^2_+$, we conjecture the solution $\widetilde{c}_t = \psi_c u_t$, $\pi_t = \psi_\pi u_t$, $\ell_t = \psi_{\ell u} u_t + \psi_{\ell s} s_t$, and identify it by the method of undetermined coefficients:

$$
\begin{aligned}
\psi_c &= \frac{\gamma(1-\delta\rho_u)}{[\gamma(1-\rho_u)+\phi_y](1-\delta\rho_u) + \kappa(\phi_\pi - \rho_u)} \\
\psi_\pi &= \frac{\gamma\kappa}{[\gamma(1-\rho_u)+\phi_y](1-\delta\rho_u) + \kappa(\phi_\pi - \rho_u)} \\
\psi_{\ell u} &= \frac{1}{1-\delta\eta\rho_u}\left[\delta\eta\rho_u \frac{\varepsilon}{1-\alpha}\psi_\pi + \delta\eta(1-\gamma) + (1-\delta\eta)\frac{1+\varphi}{1-\alpha}\psi_c\right] \\
0 &= \delta\eta(1+S)\frac{\|D_c\|^2}{S^2}\psi_{\ell s}^2 - \left[1 - \delta\eta\rho_s + \delta\eta\left(\frac{\varepsilon}{1-\alpha}\psi_\pi B + (1-\gamma)D_a + \psi_{\ell u}B\right)\frac{D_c'}{S}\right]\psi_{\ell s} - \gamma(1-\delta\eta)(1+\xi_2)
\end{aligned}
$$

This system of equations characterizes the competitive equilibrium under the necessary condition

$$
\begin{aligned}
\xi_1 &= \frac{\gamma}{S} D_c D_\pi' \\
\xi_2 &= -\frac{\delta}{\gamma\lambda S^2} D_c D_\pi' \psi_{\ell s}
\end{aligned}
$$

which ensures that consumption and inflation are not driven by surplus consumption (a property that Lopez et al., 2015, dub “macro-finance separation”), and hence verifies the guess.

4.3.2. Essentially-affine pricing

Accordingly, cashflows are driven by

$$
\begin{aligned}
\Delta c_{t+1} &= \mu + C_c u_t + D_c \varepsilon_{t+1} \\
\Delta d_{t+1} &= \mu + C_d u_t + D_d \varepsilon_{t+1} \\
-\pi_{t+1} &= C_{-p} u_t + D_{-p} \varepsilon_{t+1}
\end{aligned}
$$

	Parameter		Value
New Keynesian block	γ	Utility curvature in market and home consumption	2
	$1/\varphi$	Quasi-Frisch's labor supply elasticity	1
	β	Subjective discount factor	.9991
	$1-\alpha$	Labor share in value added	2/3
	ε	Elasticity of substitution in Dixit-Stiglitz aggregator	6
	$1/(1-\eta)$	Average price duration (in months)	9
	ϕ_π	Policy response coefficient to inflation movements	1.5
	ϕ_y	Policy response coefficient to output movements	.5/12
Habit block	ξ_1	Financial spillover onto the intertemporal rate of substitution	−.0001
	ξ_2	Financial spillover onto the intratemporal rate of substitution	−.0137
	ρ_s	Habit persistence	.9940
Exogenous block	μ	Mean technology growth	.0030
	ρ_u	Persistence of the conditional mean of technology growth	.8470
	σ	Conditional volatility of technology	.0222
	ϕ	Relative volatility of the conditional mean of technology	.1473
	ρ	Correlation between short-run and long-run shocks	−.9623

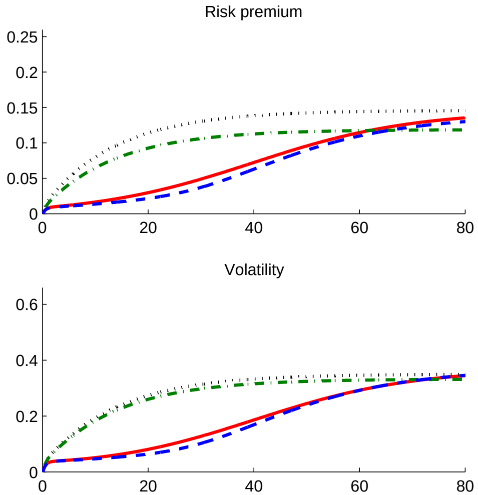
Table 2: Deep parameters and their calibration (monthly frequency) in the model by Lopez et al. (2015).

with $C_c = 1-(1-\rho_u)\psi_y$, $C_d = 1+[\gamma(1-\alpha)+\varphi](1-\rho_u)\psi_y/\alpha$, $C_{-p} = -\rho_u\psi_\pi$, $D_c = \sigma[1+\rho\psi_y\phi;\ \sqrt{1-\rho^2}\psi_y\phi]'$, $D_d = \sigma\phi[1-\rho[\gamma(1-\alpha)+\varphi]\psi_y\phi/\alpha; -\sqrt{1-\rho^2}[\gamma(1-\alpha)+\varphi]\psi_y\phi/\alpha]'$, and $D_{-p} = -\sigma\psi_\pi\phi[\rho;\ \sqrt{1-\rho^2}]'$. The essentially-affine approximation applies with the parametrization $A = \rho_u$ and $B = \sigma\phi[\rho;\ \sqrt{1-\rho^2}]'$.

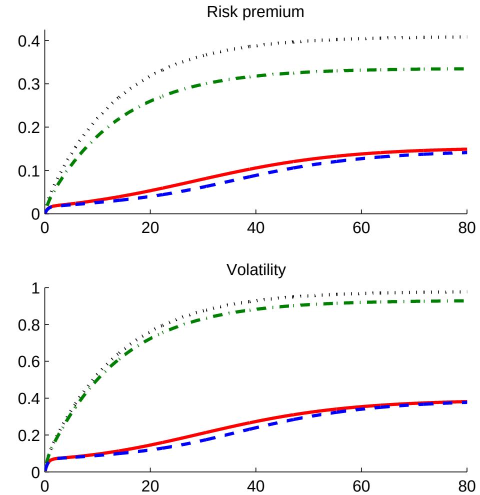
We calibrate the parameters with the values in table 2. Figure 3 compares the global solution with our essentially-affine approximation method as well as with the alternative affine methods. The plots show an overall good performance of our essentially-affine approximation which captures well the level, amplitude and shape of the term structures of risk premia and volatilities, including the initially downward-sloping term structure of market equity. The alternative approximation methods are severly inaccurate. The completely-affine and the two-stage loglinear-lognormal methods have a hard time in accounting for a precautionary savings effect and the approximate equilibrium allocation is far from macro-financial separation. This distortion is particularly severe for the term structure of nominal yields, as the spillover of surplus consumption on inflation has an incorrect effect on its volatility.

4.3.3. Euler Equation Errors

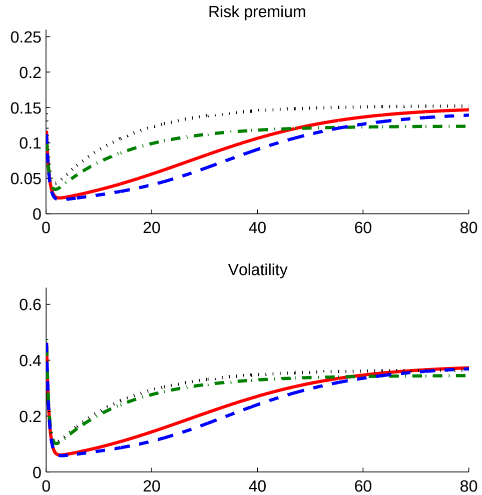
To show the accuracy of our approximations we compute the errors in the n-period Euler equation for the three solutions of the model. In the model, one-period Euler equation errors are defined as in Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) from a



(a) *Real bonds*.



(b) *Nominal bonds*.



(c) *Dividend strips*.

Figure 3: Comparison of solution methods to compute average equilibrium term structures of holding-period risk premia $\{\ln E_t R^{e,(n)}_{t+1}\}$ and volatilities $\{std_t(r^{(n)}_{t+1})\}$ in the Lopez et al. (2015) model. Essentially-affine (solid red), completely-affine (dotted black), two-stage loglinear-lognormal (dot-dashed green), and projected solution using Chebyshev polynomials of up to degree eight collocated over a Smolyak grid and 6-point Gauss-Hermite quadrature (dashed blue).

solution for consumption $c^{(0)}(z_t)$ as

$$
\begin{aligned}
EEE^{(1)}(z_t) &\equiv \log_{10}\left|1 - e^{c^{(1)}(z_t) - c^{(0)}(z_t)}\right|, \\
c_t^{(1)} &\equiv -\frac{1}{\gamma}\ln E_t\left(\beta e^{-\gamma c_{t+1}^{(0)} - \gamma \Delta s_{t+1}^{(0)} + r_t}\right)
\end{aligned}
$$

for points z_t that cover a high-probability region of the state space, and where $s^{(0)}$ denotes its dependence on the functional $c^{(0)}$. An EEE of $-\varepsilon$ implies that the consumer makes a one dollar mistake in how much she decides to save for every 10^{ε} dollars spent.

We propose a multiperiod version of the Euler equation errors as a metric to test the accuracy of our approximation for the n-period equilibrium term structures

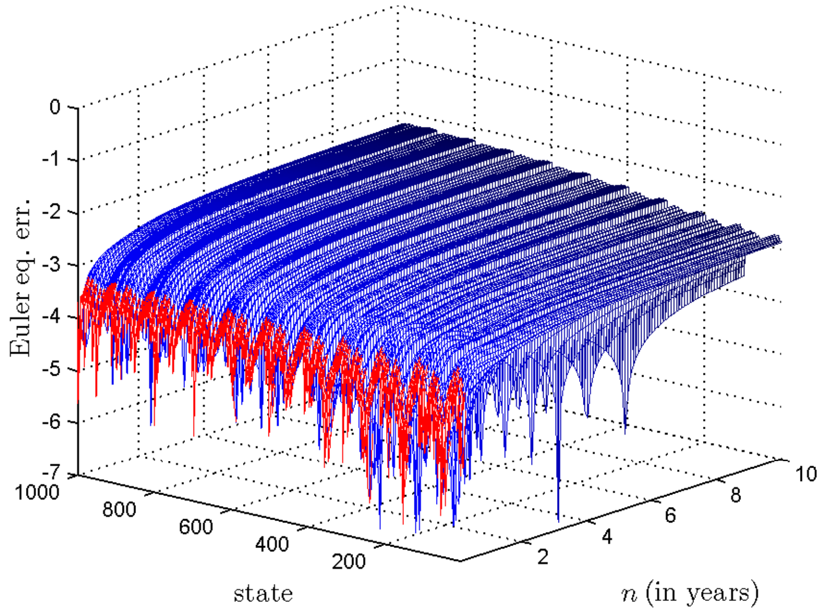
$$
\begin{aligned}
EEE^{(n)}(z_t) &\equiv \log_{10}\left|1 - e^{c^{(n)}(z_t) - c^{(0)}(z_t)}\right| \\
c_t^{(n)} &\equiv -\frac{1}{\gamma}\ln E_t\left(\beta e^{-\gamma c_{t+1}^{(n-1)} - \gamma \Delta \widehat{s}_{t+1}^{(n-1)} + r_t}\right) = -\frac{1}{\gamma}\ln E_t\left(\beta^n e^{-\gamma c_{t+n}^{(0)} - \gamma \sum_{j=1}^{n} \Delta \widehat{s}_{t+j}^{(n-j)} + \sum_{j=0}^{n-1} r_{t+j}}\right)
\end{aligned}
$$

with $\widehat{s}_{t+1}^{(j)} \equiv \rho_s \widehat{s}_t + \Lambda_t (E_{t+1} - E_t) c_{t+1}^{(j)}$. In this context, a n-period Euler equation error of $-\varepsilon$ implies that the consumer is making a one dollar mistake in how much she decides to save over a n-period horizon for every 10^{ε} dollars spent. Since the errors accumulate as the horizon increase, multiperiod Euler equation errors provide an indication of how good the approximation is for long-term valuations.

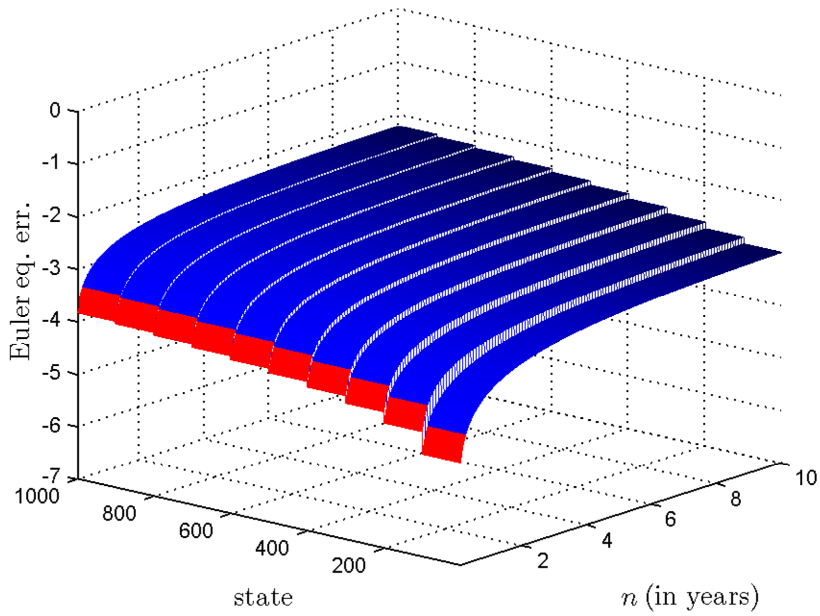
Figure 4 shows multiperiod Euler equation errors. The accuracy of our global solution in terms of conventional 1-step ahead Euler equation errors is consistently lower than -3 and is comparable to values typically retained in the extant literature (e.g., Fernández-Villaverde et al., 2015), and remains below -2 over arbitrarily long horizons. The risk-adjusted loglinearized solution for quantities that forms the basis of the essentially-affine approximation also shows relatively small errors; notably, the lower accuracy of the loglinearized solution does not translate in substantially larger Euler equation errors over long horizons when contrasted with the projected solution.

5. Conclusion

We propose an essentially-affine approximation technique to solve to first order and in closed form the main asset pricing implications of Gaussian dynamic macro-finance models with state-dependent risk exposures and risk prices. The proposed essentially-affine approximation performs similarly to global projection methods and significantly outperforms alternative affine approximations.



(a) *Projected solution for quantities.*



(b) *Loglinearized solution for quantities.*

Figure 4: Multiperiod Euler equation errors in the Lopez et al. (2015) model. Errors are expressed in $\log_{10}$. Values in the state dimension index different triplets $[u_t, \zeta_t, \Delta_{t-1}]$ built as the Cartesian product of 10 equidistant points along each dimension. The red region associates with conventional 1-step ahead Euler equation errors.

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Appendix

A. Proof of proposition 1

First start by noting that the risk-neutral dynamics of the vector process $[z; \Delta d]$ are

$$\begin{aligned} E_t^{\mathbb{Q}}[e^{u'_\zeta \zeta_{t+1} + u_s s_{t+1} + u_d \Delta d_{t+1}}] &= E_t^{\mathbb{P}}[e^{-\frac{1}{2}\gamma'_t\gamma_t - \gamma'_t\varepsilon_{t+1} + u'_\zeta\zeta_{t+1} + u_s s_{t+1} + u_d\Delta d_{t+1}}] \\ &= e^{u'\mu + \frac{1}{2}u'\Sigma(z_t)\Sigma(z_t)'u + u'[\Phi z_t - \Sigma(z_t)\gamma(z_t)] + O(\|z_t\|^2)} \\ &= e^{u'[\mu - \pi(0)] + \frac{1}{2}u'\Sigma_t\Sigma'_t u + u'[\Phi - \nabla\pi(0)]z_t + O(\|z_t\|^2)} \end{aligned} \tag{A.1}$$

where we uses a first-order expansion of the risk premium $\pi(z_t)$, which is necessary for expression (A.1) to reduce to an affine form in the state vector up to a term of second order.

This expression implies that the vector process $[z; \Delta d]$ is affine under $\mathbb{Q}$ up to a second-order term, and its risk-neutral dynamics can be written as

$$\begin{bmatrix} z_{t+1} \\ \Delta d_{t+1} \end{bmatrix} = [\mu - \pi(0)] + [\Phi - \nabla\pi(0)]z_t + \Sigma_t\varepsilon^*_{t+1} + O(\|z_t\|^2)$$

where the q-dimensional shock process ε^* is distributed as $\varepsilon^*_t \sim Niid(0, I_q)$ under the risk-neutral measure $\mathbb{Q}$. The first-order effect of the risk-neutral distribution is to increase the drift and the persistence of the states of the world that associate with high marginal utility; symmetrically, it underweights more pleasant states of the world.

We then guess that the price-dividend ratio of the n-period ahead cashflow strip has the exponential-affine shape $P^{(n)}_{d,t}/D_t = e^{A^{(n)}+B^{(n)}_\zeta \zeta_t + B^{(n)}_s s_t}$, and use the no-arbitrage relation

$$P^{(n)}_{d,t} = e^{-r_t} E^{\mathbb{Q}}_t P^{(n-1)}_{d,t+1}, \qquad P^{(0)}_{d,t} = D_t$$

to verify the conjecture as

$$\begin{aligned}
e^{A^{(n)}+B^{(n)}_\zeta \zeta_t + B^{(n)}_s s_t} &= e^{-r_t + A^{(n-1)}} E^{\mathbb{Q}}_t [e^{B^{(n-1)}_\zeta \zeta_{t+1} + B^{(n-1)}_s s_{t+1} + \Delta d_{t+1}}] \\
&= e^{-r_t + A^{(n-1)} + u'_{(n-1)}[\mu - \pi(0)] + \frac{1}{2} u'_{(n-1)} \Sigma_t \Sigma'_t u_{(n-1)} + u'_{(n-1)}[\Phi - \nabla\pi(0)] z_t + O(\|z_t\|^2)} \\
&= e^{A^{(n-1)} - r(0) + u'_{(n-1)}[\mu - \Sigma(0)\gamma(0)] + \frac{1}{2} u'_{(n-1)} \Sigma(0)\Sigma(0)' u_{(n-1)} + u'_{(n-1)}[\Phi - \nabla\pi(0)] z_t + u'_{(n-1)}\Sigma(0)[\nabla\Sigma(0)]'(I_{n_z+1}\otimes z_t) u_{(n-1)} - [\nabla r(0)] z_t + O(\|z_t\|^2)}
\end{aligned} \tag{A.2}$$

with $u_{(n-1)} = [B^{(n-1)\prime}_\zeta; B^{(n-1)\prime}_s; 1]$, and where we used the approximate $\mathbb{Q}$-dynamics, the loglinearized real risk-free rate, and the first-order expansion

$$\begin{aligned}
u'_{(n-1)} \Sigma_t \Sigma'_t u_{(n-1)} &= u'_{(n-1)} \Sigma(0)\Sigma(0)' u_{(n-1)} + 2u'_{(n-1)} \Sigma(0) \begin{bmatrix} \nabla B(0) \\ \nabla\Lambda(0)' \\ \nabla D(0) \end{bmatrix}' (I_{n_z+1} \otimes z_t) \begin{bmatrix} B^{(n-1)\prime}_\zeta \\ B^{(n-1)\prime}_s \\ 1 \end{bmatrix} + O(\|z_t\|^2) \\
&= u'_{(n-1)} \Sigma(0)\Sigma(0)' u_{(n-1)} + 2u'_{(n-1)} \Sigma(0) \begin{bmatrix} \nabla\Lambda(0)' \\ \nabla D(0) \end{bmatrix}' (I_{n_s+1} \otimes z_t) \begin{bmatrix} B^{(n-1)\prime}_s \\ 1 \end{bmatrix} + O(\|z_t\|^2)
\end{aligned}$$

as the three necessary linearizations in order for expression (A.2) to reduce to an affine form.

Thus, we are able to match the corresponding coefficients to verify the initial guess and to identify it as the solution of the Riccati equation (8).

It is straightforward to recombine the first-order expansions of the price of risk, the risk-free rate and stochastic volatility with the Riccati equation to derive

$$\begin{aligned}
\ln E^{\mathbb{P}}_t [R^{(n)}_{d,t+1}] &\equiv \ln E^{\mathbb{P}}_t [e^{p^{(n-1)}_{d,t+1} - p^{(n)}_{d,t}}] \\
&= r_t + V_{n-1,t}\gamma_t + O(\|z_t\|^2)
\end{aligned}$$

which is consistent with plugging the approximations of the unexpected components of the discount factor and the ith return in the no-arbitrage pricing formula under lognormality, $\ln E^{\mathbb{P}}_t [R^i_{t+1}] = cov^{\mathbb{P}}_t(-m_{t+1}, r^i_{t+1})$. Note how the non-affine $\mathbb{P}$-dynamics of the risk premium are driven by $\gamma(z_t)$ and $V_{n-1,t} = D(z_t) + B^{(n)}_\zeta B + B^{(n)}_s \Lambda(z_t)'$, with z_t distributed under the historical measure according to equation (6).

Hold-to-maturity risk premia can be derived from the equilibrium expression for yields and the term structure of anticipated cashflow growth, $G^{(n)}_{d,t} \equiv E^{\mathbb{P}}_t [D_{t+n}/D_t]$, which has the recursive structure

$$G^{(n)}_{d,t} = E^{\mathbb{P}}_t \left(\frac{D_{t+1}}{D_t} G^{(n-1)}_{d,t+1} \right)$$

with boundary condition $G^{(0)}_{d,t} = 1$, and hence implies $G^{(n)}_{d,t} = e^{A^{(n)}_g + B^{(n)}_g \zeta_t}$ up to a second-order term.

B. Proof of the corollary to proposition 1

Using $E_t R^m_{t+1} = \sum_{n=1}^{\infty} \omega_{n,t} E_t R^{(n)}_{d,t+1}$, straightforward expansions yield

$$\begin{aligned}
\ln E_t R^m_{t+1} &= \sum_{n=1}^{\infty} \omega_n \ln E_t R^{(n)}_{d,t+1} + O(\|\ln E_t R^{(n)}_{d,t+1}, \omega_{n,t} - \omega_n\|^2) \\
r^m_{t+1} &= \sum_{n=1}^{\infty} \omega_n r^{(n)}_{d,t+1} + O(\|r^{(n)}_{d,t+1}, \omega_{n,t} - \omega_n\|^2)
\end{aligned}$$

and we use the main proposition to derive approximate expressions for the equity premium.

Consequently, we can derive the variance risk premium as

$$\begin{aligned}
vrp^m_t &= (E^{\mathbb{Q}}_t - E^{\mathbb{P}}_t)\Big[var^{\mathbb{P}}_{t+1}\Big(\sum_{n=1}^{\infty} \omega_n V_{d,n-1,t+1}\varepsilon_{t+2}\Big)\Big] \\
&= (E^{\mathbb{Q}}_t - E^{\mathbb{P}}_t)\,[l'\Xi\Sigma_{t+1}\Sigma'_{t+1}\Xi' l] \\
&= (E^{\mathbb{Q}}_t - E^{\mathbb{P}}_t)\Big[l'\Xi\Sigma(0)\Sigma'(0)\Xi' l + 2l'\Xi\Sigma(0)[\nabla\Sigma(0)]'(I_{n_z+1} \otimes z_{t+1})\Xi' l + O(\|z_{t+1}\|^2)\Big] \\
&= 2l'\Xi\Sigma(0)[\nabla\Sigma(0)]'[I_{n_z+1} \otimes (E^{\mathbb{Q}}_t - E^{\mathbb{P}}_t)z_{t+1}]\Xi' l + O(\|z_{t+1}\|^2)
\end{aligned}$$

We then rely on the essentially-affine approximation of the $\mathbb{Q}$-dynamics to find expression (9).

C. Proof of proposition 2

The exponential-affine solution of the Perron-Frobenius eigenfunction problem can be verified:

$$\begin{aligned}
\delta e^{u'_\zeta \zeta_t + u_s \widehat{s}_t} &= e^{-r_t} E^{\mathbb{Q}}_t[e^{u'_\zeta \zeta_{t+1} + u_s \widehat{s}_{t+1}}] \\
&= e^{-r(0)-[\nabla r(0)]z_t + u'[\mu - \pi(0)] + \frac{1}{2}u'\Sigma_t\Sigma'_t u + u'[\Phi - \nabla\pi(0)]z_t + O(\|z_t\|^2)} \\
&= e^{-r(0)-[\nabla r(0)]z_t + u'[\mu - \pi(0)] + \frac{1}{2}u'\Sigma(0)\Sigma(0)'u + u'[\Phi - \nabla\pi(0)]z_t + u'\Sigma(0)[\nabla\Sigma(0)]'(I_{n_z} \otimes z_t)u + O(\|z_t\|^2)}
\end{aligned}$$

where we used a similar strategy as in the proof of proposition 1.

D. Borovicka-Hansen elasticities

To derive the approximate expressions for shock-exposure and shock-price elasticities, define $h_{t+1}(r) \equiv r\alpha_t\varepsilon_{t+1} - \frac{r^2}{2}\|\alpha_t\|^2$ and note that, by the law of iterated expectations,

$$\begin{aligned}
E^{\mathbb{P}}_t\left[e^{h_{t+1}(r)}\frac{D_{t+n}}{D_t}\right] &= E^{\mathbb{P}}_t\left[e^{h_{t+1}(r) + \Delta d_{t+1}} E^{\mathbb{P}}_{t+1}\left(\frac{D_{t+n}}{D_{t+1}}\right)\right] \\
&= E^{\mathbb{P}}_t\left[e^{h_{t+1}(r) + \Delta d_{t+1}} F^{(n-1)}_{g,t+1}\right] \\
E^{\mathbb{P}}_t\left[e^{h_{t+1}(r)} M_{t,t+n}\frac{D_{t+n}}{D_t}\right] &= E^{\mathbb{P}}_t\left[e^{h_{t+1}(r) + m_{t+1} + \Delta d_{t+1}} F^{(n-1)}_{v,t+1}\right]
\end{aligned}$$

with the recursive structures

$$F_{g,t}^{(n)} = E_t^{\mathbb{P}}\Big[e^{\Delta d_{t+1}} F_{g,t+1}^{(n-1)}\Big], \quad F_{g,t}^{(0)} = 1$$
$$F_{v,t}^{(n)} = E_t^{\mathbb{P}}\Big[e^{m_{t+1}+\Delta d_{t+1}} F_{v,t+1}^{(n-1)}\Big], \quad F_{v,t}^{(0)} = 1$$

Therefore,

$$\begin{aligned}
\varepsilon_{g,t} &= \frac{d}{dr} \ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+\Delta d_{t+1}+A_g^{(n-1)}+B_g^{(n-1)}\zeta_{t+1}}\Big]_{r=0} \\
&= \alpha_t[D_d + B_g^{(n-1)}B]' \\
\varepsilon_{p,t} &= \varepsilon_{g,t} - \frac{d}{dr} \ln E_t^{\mathbb{P}}\Big[e^{h_{t+1}(r)+m_{t+1}+\Delta d_{t+1}+A_d^{(n-1)}+B_{d,\zeta}^{(n-1)}\zeta_{t+1}+B_{d,s}^{(n-1)}\widehat{s}_{t+1}}\Big]_{r=0} \\
&= \alpha_t[x_t D_c + \gamma C_c(I-A)^{-1}(I-A^{n-1})B - B_s^{(n-1)}\lambda_t D_c]'
\end{aligned}$$

E. Variance risk premium under Campbell-Cochrane habits

With Campbell-Cochrane habits, $D_t = D_d$, $\Lambda_t' = \lambda_t D_c$ and $\gamma_t' = \gamma(1+\lambda_t)D_c$, we have $\nabla D(0) = 0$, $\Lambda(0) = \frac{1-S}{S}D_c'$, $[\nabla\Lambda(0)'] = -\frac{1}{S}[0_{n_\zeta};1]D_c$, $\gamma(0) = \frac{\gamma}{S}D_c'$, and $\nabla\gamma(0) = -\frac{\gamma}{S}D_c'[0_{n_\zeta};1]'$, and hence equation (9) reduces to

$$vrp_t^m = 2(1-\phi)\frac{1-S}{S}\Big(l'\Xi_1 D_c' + \frac{1-S}{S}l'\Xi_3\|D_c\|^2\Big)\Big(1 - \frac{2-S}{1-S}\widehat{s}_t\Big)\Xi_3' l$$

These expressions show how the variance risk premium is unambiguously positive and countercyclical in its correlation with the state that drives risk premia—stock market volatility is high when people are most risk averse.

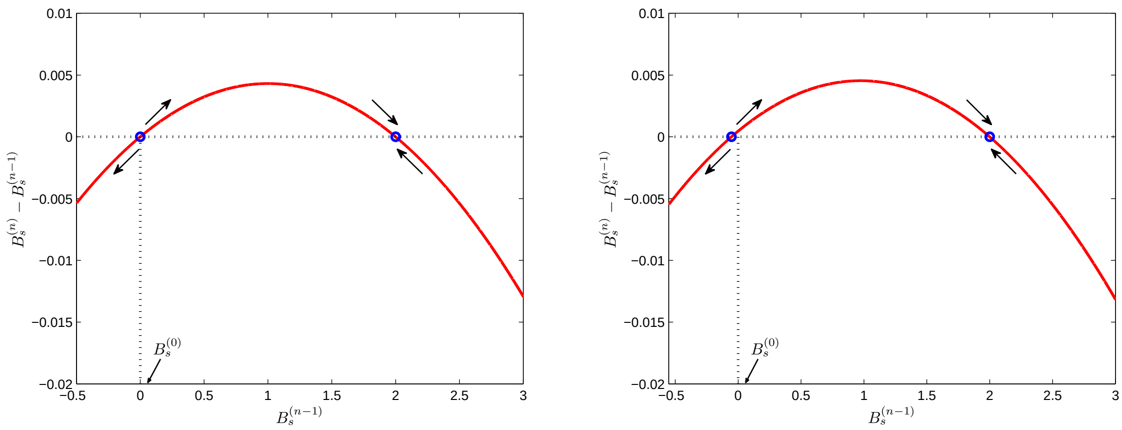
F. Approximate diagnostic decomposition under the original Campbell-Cochrane calibration

Under the original calibration of Campbell and Cochrane (1999), our approximation correctly captures a variance ratio that is constant at unity and the property that shocks to $S_t^{-\gamma}$ have a permanent effect on the marginal utility of wealth even though surplus consumption itself is stationary. In fact, since $\Lambda_t \equiv \partial U_t/\partial C_t = \Lambda_t^T\Lambda_t^P$ with $\Lambda_t^T = \delta^T/f_t$, the martingale component of the stochastic discount factor reveals a permanent component in the marginal utility of wealth. Namely,

$$\begin{aligned}
\ln(\Lambda_t^P) &= t\ln(\beta) - \gamma c_t - \gamma s_t - t\ln(\delta) + B_{0,\zeta}^{(\infty)}\zeta_t + B_{0,s}^{(\infty)}s_t \\
&= \begin{cases} -\frac{t}{2}x^2\|D_c\|^2 - \gamma(c_t - t\mu_c) - \gamma s_t & \text{if } \xi = 0 \text{ and } C[I-A]^{-1}[I-A^n]BD_c' = 0, \forall n \\ -\frac{t}{2}\|V_0\|^2 - \gamma(c_t - t\mu_c) - \gamma C[I-A]^{-1}\zeta_t & \text{elsewhere} \end{cases}
\end{aligned}$$

and hence, in the unstable case, shocks to risk aversion and, in the stable case, shocks to the predictable component of consumption have a permanent effect on the marginal utility of wealth even though both risk aversion and the predictable component of consumption are stationary.

Figure F.5 plots the map $B_s^{(n-1)} \mapsto B_s^{(n)} - B_s^{(n-1)}$ along with the two solutions of $B_s^{(\infty)}$ under the original calibration of Campbell and Cochrane (plot F.5a) as well as under a general calibration such that the unstable solution to $B_s^{(\infty)}$ differs from zero (plot F.5b). Under the Campbell and Cochrane



(a) *Calibration $C = \xi = 0$ (Campbell-Cochrane).* (b) *General calibration.*

Figure F.5: Phase diagram of the term structure loadings on surplus consumption of real bonds. Riccati coefficients in the left-hand panel start in the unstable cluster point and remain there forever; coefficient in the right-hand panel start to the right of the unstable cluster point and hence converge to the stable cluster point.

calibration, the equilibrium coefficients of the real bond term structure, $\{B_s^{(n)}\}$, start in the left cluster point and always remain there despite its instablity. The coefficient trajectory is quite different under the general calibration because the equilibrium coefficients of the real bond term structure start at $B_s^{(0)} = 0$, which is not a cluster point under this calibration, and hence they will converge to the stable point to the right.

ONLINE APPENDIX - NOT FOR PUBLICATION

I. Global solution of Campbell and Cochrane (1999) and Wachter (2006)

In describing the numerical solution we assume that the linear form of equilibrium cashflows is exact. This is the case in Campbell and Cochrane (1999) and Wachter (2005, 2006) but is not in general the case when the cashflow process derives from a DSGE model. A fully nonlinear numerical solution in the general context in which equilibrium cashflows are not linear functions of the state vector can rapidly become computationally prohibitive.

Under this assumption, we can directly solve the term structure as shown by Wachter (2005) by numerically integrating and iterating on the no-arbitrage pricing formula

$$\begin{aligned} PD^{(n)}(s_t;\zeta_t) = &\int_{\mathcal{E}} \beta e^{\mu_d-\gamma\mu_c+\gamma(1-\phi)\widehat{s_t}+[C_d-\gamma C_c]\zeta_t+D_d\varepsilon-\gamma[1+\lambda(s_t)]D_c\varepsilon}\times \\ &\times PD^{(n-1)}[(1-\phi)s+\phi s_t+\lambda(s_t)D_c\varepsilon;A\zeta_t+B\varepsilon]\, d\mathbb{P}(\varepsilon) \end{aligned}$$

with boundary condition $PD_t^{(0)} = 0$, for any t, where $\mathcal{E}$ is the support for the shock ε_t.

This multivariate integration is computationally cumbersome. However, in line with Wachter (2006), we can verify that the solution has the semi-parametric shape

$$PD^{(n)}(s_t;\zeta_t) = F^{(n)}(s_t)e^{A^{(n)}+B^{(n)}\zeta_t}$$

In fact,

$$\begin{aligned} F^{(n)}(s_t)e^{A^{(n)}+B^{(n)}\zeta_t} &= \int_{\mathcal{E}} \beta e^{\mu_d-\gamma\mu_c+\gamma(1-\phi)\widehat{s_t}+[C_d-\gamma C_c]\zeta_t+D_d\varepsilon-\gamma[1+\lambda(s_t)]D_c\varepsilon}\times \\ &\times e^{A^{(n-1)}+B^{(n-1)}[A\zeta_t+B\varepsilon]}F^{(n-1)}[(1-\phi)s+\phi s_t+\lambda(s_t)D_c\varepsilon]\, d\mathbb{P}(\varepsilon) \\ &= \beta e^{\mu_d-\gamma\mu_c+A^{(n-1)}+[B^{(n-1)}A+C_d-\gamma C_c]\zeta_t+\gamma(1-\phi)\widehat{s_t}}\times \\ &\times \int_{-\infty}^{\infty} e^{-\gamma[1+\lambda(s_t)]v}\times E\Big[e^{[D_d+B^{(n-1)}B]\varepsilon}\Big|v\Big]F^{(n-1)}[(1-\phi)s+\phi s_t+\lambda(s_t)v]\, d\mathbb{P}_c(v) \\ &= e^{\ln(\beta)+\mu_d-\gamma\mu_c+A^{(n-1)}+\frac{1}{2}[D_d+B^{(n-1)}B]\big[I-\frac{D_c'D_c}{\|D_c\|^2}\big][D_d+B^{(n-1)}B]'+[B^{(n-1)}A+C_d-\gamma C_c]\zeta_t+\gamma(1-\phi)\widehat{s_t}}\times \\ &\times \int_{-\infty}^{\infty} e^{\{-\gamma[1+\lambda(s_t)]+[D_d+B^{(n-1)}B]\frac{D_c'}{\|D_c\|^2}\}v}F^{(n-1)}[(1-\phi)s+\phi s_t+\lambda(s_t)v]\, d\mathbb{P}_c(v) \end{aligned} \tag{I.1}$$

where the consumption shock $v_t \equiv D_c\varepsilon_t$ has cumulative distribution function $\mathbb{P}_c$, whose support is the set of real numbers. To derive expression (I.1) we use the law of iterated expectations and the property of the multivariate normal distribution

$$\begin{bmatrix} \kappa\varepsilon_{t+1} \\ D_c\varepsilon_{t+1} \end{bmatrix} \sim N\left(0, \begin{bmatrix} \kappa\kappa' & \kappa D_c' \\ D_c\kappa' & \|D_c\|^2 \end{bmatrix}\right)$$

that $\kappa\varepsilon_{t+1}|D_c\varepsilon_{t+1} \sim N\Big(\kappa\frac{D_c'D_c}{||D_c||^2}\varepsilon_{t+1}, \kappa\Big[I - \frac{D_c'D_c}{||D_c||^2}\Big]\kappa'\Big)$.

We can therefore verify and identify the guessed solution as the pair of difference equations

$$\begin{aligned} A^{(n)} &= A^{(n-1)} + \ln(\beta) + \mu_d - \gamma\mu_c + \frac{1}{2}[D_d + B^{(n-1)}B]\Big[I - \frac{D_c'D_c}{||D_c||^2}\Big][D_d + B^{(n-1)}B]' \\ B^{(n)} &= B^{(n-1)}A + C_d - \gamma C_c \end{aligned}$$

along with the integral equation

$$F^{(n)}(s_t) = e^{\gamma(1-\phi)(s_t-s)} \int_{-\infty}^{\infty} e^{\left\{-\gamma[1+\lambda(s_t)]+[D_d+B^{(n-1)}B]\frac{D_c'}{||D_c||^2}\right\}v} F^{(n-1)}[(1-\phi)s + \phi s_t + \lambda(s_t)v]\; d\mathbb{P}_c(v)$$

We solve the integral numerically by a 6-point Gauss-Hermite quadrature. We compute $F^{(n-1)}(s)$ over a Chebyshev grid of 100 points for s and then construct $F^{(n-1)}[(1-\phi)s + \phi s_t + \lambda(s_t)v]$ by interpolating between grid points with cubic splines.[12]

II. Global solution of Lopez et al. (2015)

We use a Smolyak sparse-grid collocation method with an adaptive and anisotropic grid to project the global solution of our model onto the subspace spanned by a basis of Chebyshev polynomials of up to degree eight.[13] We proceed in two steps. First, we solve for equilibrium quantities. Second, we project equilibrium term structures using no-arbitrage relations by iterating the Fredholm pricing equation to convergence.

We adopt the fast Smolyak method proposed by Judd, Maliar, Maliar, and Valero (2014) by constructing a Smolyak grid using disjoint rather than nested sets of grid points, and by relying on Lagrange interpolation rather than on the closed-form map between the function evaluated at the grid points and its interpolated values. While the analytical expressions are free from numerical errors, they still imply computationally expensive evaluations over nested sets of grid points and basis functions. As emphasized by Judd et al., the expensive part of the Lagrange interpolation problem is calculating an inverse, which however remains fixed across iterations and is therefore a computation that needs to be carried out only once.

We delimit the parallelotope within which we search the state space for the projected solution by simulating 100,000 periods and considering minimum and maximum values of the elements of the state vector, $\mathbb{S}_t = [u_t; \zeta_t; \Delta_{t-1}]$. We start by considering the first-order solution for consumption

[12] In our applications results turn out to be identical for any $q \geq 6$, so we stick with $q = 6$. Alternative interpolation schemes, such as the log-linear interpolation adopted by Campbell and Cochrane (1999) and Wachter (2006), and alternative grids yield very similar results.

[13] The collocation method at the Smolyak grid points retains the uniform convergence properties of Chebyshev polynomials collocated on the tensor product of the corresponding Chebyshev-Gauss-Lovatto points. Malin, Krueger, and Kubler (2011) originally introduced the Smolyak method in economics; Fernández-Villaverde, Gordon, Guerrón-Quintana, and Rubio-Ramírez (2015) applied it in a New Keynesian setting to handle nonlinearities such as the incidence of the zero-lower bound on the nominal rate. Judd et al. (2014) propose several upgrades of the method, including fast and flexible grid construction and function interpolation, which prove useful in our context.

[INSERT FIGURE .6 HERE]

Figure .6: Scatter plot of 100,000 simulated time series of the state vector $\{u_t, \zeta_t, \Delta_{t-1}\}$ and associated Smolyak grid points (red circles); asymmetric accuracy levels $\mu_1 = 2$, $\mu_2 = 3$ and $\mu_3 = 1$.

and inflation and by deriving the associated values for the endogenous states. Figure .6 plots a simulated exploration of the ergodic set and maps the Smolyak hypercube into the state space; the grid points are placed in a way that covers exactly all quantiles of simulated data.

Finally, Judd et al. (2014) modify the conventional Smolyak method to allow for asymmetric accuracy levels $\{\mu_i\}_{i=1}^{3}$ across the three dimensions of the state space. This anisotropic version of the Smolyak method is particularly useful in our setting, as most nonlinearities in the decision variables owe to their dependence on surplus consumption; we can therefore increase the precision in that dimension alone while maintaining the overall dimensionality of the problem limited. Figure .6 shows the asymmetric number of grid points in each dimension of the ergodic set.

II.1. Competitive equilibrium

The necessary conditions for a competitive equilibrium are:[14]

$$
\begin{aligned}
i_t &= -\ln E_t\beta \exp\{-\gamma\Delta c_{t+1} - \gamma\Delta s_{t+1} - \pi_{t+1}\} \\
i_t &= -\ln(\beta) + \pi + \gamma\mu - \frac{\gamma(1-\rho_s-\xi_1/\gamma)}{2} + \phi_\pi(\pi_t - \pi) + \phi_y[c_t - a_t - (1-\alpha)\ln(N)] \\
w_t - p_t &= \ln(\chi_0) + \gamma c_t - \gamma h_t + a_t - \gamma\xi_2\widehat{s_t} \\
h_t &= a_t + \ln(1-\exp\{n_t\}) \\
y_t &= a_t + (1-\alpha)n_t - (1-\alpha)\Delta_t \\
c_t &= y_t \\
\exp\{\Delta_t\} &\equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\frac{\varepsilon}{1-\alpha}} di \\
&= \eta\exp\left\{\frac{\varepsilon}{1-\alpha}\pi_t + \Delta_{t-1}\right\} + (1-\eta)\exp\left\{-\frac{\varepsilon}{1-\alpha}(p_t^* - p_t)\right\} \\
1 &= \eta\exp\{(\varepsilon-1)\pi_t\} + (1-\eta)\exp\{(1-\varepsilon)(p_t^* - p_t)\} \\
x_t \equiv p_t^* - p_t &= \ln\left(\frac{E_t\sum_{j=0}^{\infty}(\beta\eta)^j(C_{t+j}S_{t+j})^{-\gamma}(P_{t+j}/P_t)^{\varepsilon}Y_{t+j}\widehat{MC}^*_{t+j}}{E_t\sum_{j=0}^{\infty}(\beta\eta)^j(C_{t+j}S_{t+j})^{-\gamma}(P_{t+j}/P_t)^{\varepsilon-1}Y_{t+j}}\right) \\
\frac{\Gamma_{1,t}}{\Gamma_{2,t}} &= \exp\left\{\frac{1-\alpha+\alpha\varepsilon}{1-\alpha}x_t\right\} \\
\Gamma_{1,t} &= E_t\beta\eta\exp\left\{\frac{\varepsilon}{1-\alpha}\pi_{t+1}\right\}\Gamma_{1,t+1} + \exp\{(1-\gamma)c_t - \gamma\widehat{s_t} + \widehat{mc}_t - \Delta_t\} \\
\Gamma_{2,t} &= E_t\beta\eta\exp\{(\varepsilon-1)\pi_{t+1}\}\Gamma_{2,t+1} + \exp\{(1-\gamma)c_t - \gamma\widehat{s_t}\} \\
w_t - p_t &= \widehat{mc}_t + \frac{\partial y_t}{\partial n_t} \\
\widehat{s_{t+1}} &= \rho_s\widehat{s_t} + \Lambda(\widehat{s_t})(c_{t+1} - E_t c_{t+1}) \\
\Lambda(\widehat{s_t}) &= \begin{cases} S^{-1}\sqrt{1-2\widehat{s_t}} - 1, & \widehat{s_t} \le \frac{1}{2}(1-S^2) \\ 0 & \text{elsewhere} \end{cases} \\
\Delta a_{t+1} &= \mu + u_t + \sigma\varepsilon^a_{t+1} \\
u_{t+1} &= \rho_u u_t + \rho\phi\sigma\varepsilon^a_{t+1} + \sqrt{1-\rho^2}\phi\sigma\varepsilon^u_{t+1}
\end{aligned}
$$

with $[\varepsilon^a_t;\varepsilon^u_t] \sim Niid(0, I_2)$, where we defined $\widehat{mc}_t \equiv w_t - p_t - \partial y_t/\partial n_t$, and hence $\widehat{mc}_t(i) = w_t - p_t - \partial y_t(i)/\partial n_t(i) = \widehat{mc}_t + [c_t - n_t] - [c_t(i) - n_t(i)] = \widehat{mc}_t - \frac{\alpha}{1-\alpha}c_t - \Delta_t + \frac{\alpha}{1-\alpha}c_t(i) = \widehat{mc}_t - \frac{\alpha\varepsilon}{1-\alpha}[p_t(i) - p_t] - \Delta_t$, and hence $\widehat{mc}^*_{t+j} = \widehat{mc}_{t+j} - \frac{\alpha\varepsilon}{1-\alpha}x_t + \frac{\alpha\varepsilon}{1-\alpha}(p_{t+j} - p_t) - \Delta_{t+j}$.

[14] $N_t = \int_0^1 N_t(i)di = \int_0^1 [Y_t(i)/A_t]^{1/(1-\alpha)}di = (Y_t/A_t)^{1/(1-\alpha)}\int_0^1 [P_t(i)/P_t]^{-\varepsilon/(1-\alpha)}di$, with the clearing condition $Y_t = C_t$.

Rearrange and simplify until we are left with 1 exogenous $[u_t]$ and 2 endogenous states $[\widehat{s}_t, \Delta_{t-1}]$:

$$u_{t+1} = \rho_u u_t + \rho\phi\sigma\varepsilon^a_{t+1} + \sqrt{1-\rho^2}\phi\sigma\varepsilon^u_{t+1}$$

$$\widehat{s}_{t+1} = \rho_s\widehat{s}_t + \Lambda(\widehat{s}_t)(\widetilde{c}_{t+1} - E_t\widetilde{c}_{t+1} + \sigma\varepsilon^a_{t+1}), \qquad \Lambda(\widehat{s}_t) = \begin{cases} S^{-1}\sqrt{1-2\widehat{s}_t} - 1, & \widehat{s}_t \leq \frac{1}{2}(1-S^2) \\ 0 & \text{elsewhere} \end{cases}$$

$$e^{\Delta_t} = \eta e^{\frac{\varepsilon}{1-\alpha}\pi_t + \Delta_{t-1}} + (1-\eta)\left[\frac{1-\eta e^{(\varepsilon-1)\pi_t}}{1-\eta}\right]^{-\frac{\varepsilon}{(1-\alpha)(1-\varepsilon)}}$$

and with 3 endogenous jump variables $[\widetilde{c}_t, \pi_t, \ell_t]$:

$$r + \pi + \phi_\pi(\pi_t - \pi) + \phi_y(\widetilde{c}_t - \widetilde{c}) = -\ln E_t\beta e^{-\gamma(\mu + \Delta\widetilde{c}_{t+1} + u_t + \sigma\varepsilon^a_{t+1}) - \gamma\Delta\widehat{s}_{t+1} - \pi_{t+1}} \tag{II.1}$$

$$e^{\ell_t} = E_t\widetilde{\beta}\eta e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1} + (1-\gamma)(u_t + \sigma\varepsilon^a_{t+1}) + \ell_{t+1}} + \frac{\chi_0}{1-\alpha}e^{\frac{1}{1-\alpha}\widetilde{c}_t - \gamma(1+\xi_2)\widehat{s}_t}\left[1 - e^{\frac{1}{1-\alpha}\widetilde{c}_t + \Delta_t}\right]^{-\gamma} \tag{II.2}$$

$$e^{\ell_t}\left[\frac{1-\eta e^{(\varepsilon-1)\pi_t}}{1-\eta}\right]^{-\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(1-\varepsilon)}} = E_t\widetilde{\beta}\eta e^{(\varepsilon-1)\pi_{t+1} + (1-\gamma)(u_t + \sigma\varepsilon^a_{t+1}) + \ell_{t+1}}\left[\frac{1-\eta e^{(\varepsilon-1)\pi_{t+1}}}{1-\eta}\right]^{-\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(1-\varepsilon)}} + e^{(1-\gamma)\widetilde{c}_t - \gamma\widehat{s}_t} \tag{II.3}$$

with $[\varepsilon^a_t; \varepsilon^u_t] \sim Niid(0, I_2)$, where $\ell_t \equiv \ln\Gamma_{1,t} - (1-\gamma)a_t$, and where we eliminated the state a_t by defining $\widetilde{c}_t \equiv c_t - a_t$.

The approximation of the term $\sqrt{1-2\widehat{s}_t}$, which determines the price of risk, with powers of $\widehat{s}_t$ can only be done imperfectly, so we use $\zeta_t \equiv \sqrt{1-2\widehat{s}_t}$ as a state instead of $\widehat{s}_t$; powers of ξ_t can easily span powers of $\widehat{s}_t$.[15]

Finally, using the definition of the price index, equilibrium dividends reduce to:

$$\begin{aligned} D_t \equiv \int_0^1 D_t(i)di &= \int_0^1 \left[\frac{P_t(i)}{P_t}Y_t(i) - (1-\tau)\frac{W_t}{P_t}N_t(i) - T_t\right]di \\ &= [1 - (1-\alpha)\widehat{MC}_t]C_t \\ &= C_t\left[1 - \chi_0 e^{\frac{\gamma(1-\alpha)+\alpha}{1-\alpha}\widetilde{c}_t + \Delta_t - \gamma\xi_2\widehat{s}_t}\left(1 - e^{\frac{1}{1-\alpha}\widetilde{c}_t + \Delta_t}\right)^{-\gamma}\right] \end{aligned}$$

II.2. Algorithm to solve for quantities

1. Setup:
   (a) Simulate the first-order solution of the model and select the smallest parallelotope in the state space that contains all realizations of the state variable $\mathbb{S}_t = [u_t; \zeta_t; \Delta_{t-1}]$. Choose the level of precision in each dimension $\mu = [\mu_1; \mu_2; \mu_3]$, and adapt a Smolyak hypercube to the parallelepiped.
   (b) Select the Smolyak grid: to construct the 3-dimensional, n-point Smolyak grid $\{\mathring{\mathbb{S}}_i\}_{i=1}^n$ we use extrema of Chebyshev polynomials as unidimensional grid points.

[15] We work with log variables as it turned out to be the most stable version. Furthermore, this specification has the additional advantage of nesting the essentially-affine approximation.

Select the Smolyak basis functions: we use the Chebyshev polynomial family as unidimensional basis functions.

(c) Evaluate the Smolyak basis functions at the collocation points, $\Psi(\mathring{\mathbb{S}}) = [\Psi_i(\mathring{\mathbb{S}}_j)]_{i,j=1}^{n}$, and precompute the inverse operator $\Psi(\mathring{\mathbb{S}})^{-1}$.

2. Guess the values of $\widetilde{c}$, π and ℓ at each collocation point $\mathring{\mathbb{S}}_i, i = 1, ..., n$, denoted by $\widehat{f_x}(\mathring{\mathbb{S}})$ for each decision variable $x = \widetilde{c}, \pi, \ell$. (We initialize our algorithm with the risk-adjusted log-linearized solution.) By Lagrange interpolation, this gives an interpolant $\widehat{f_x}(\mathbb{S}) = \Psi(\mathbb{S})\widehat{b_x}$, with $\widehat{b_x} = \Psi(\mathring{\mathbb{S}})^{-1}\widehat{f_x}(\mathring{\mathbb{S}})$.[16]

3. Compute the value of the optimal time t functions at each collocation point $\mathring{\mathbb{S}}_i, i = 1, ..., n$, as follows:

(a) Fix an i. This gives the state today, $[u_t, \zeta_t, \Delta_{t-1}]$, and the values of $\widetilde{c}_t$, π_t and ℓ_t.

(b) Derive Δ_t from the law of motion for price dispersion using π_t.
Using $\mathbb{S}_t$, weights and nodes from the degree-q Gauss-Hermite cubature rule,[17] the innovations $[\varepsilon^a_{j,t+1}; \varepsilon^u_{j,t+1}]$, $j = 1, ..., q^2$, and function $\widehat{f_{\widetilde{c}}}$ guessed in stage 2, construct $u_{j,t+1} = \rho_u u_t + \rho\phi\sigma\varepsilon^a_{j,t+1} + \sqrt{1-\rho^2}\phi\sigma\varepsilon^u_{j,t+1}$ and solve the fixed-point problem:

$$\widetilde{c}_{j,t+1} = \widehat{f_{\widetilde{c}}}\left(u_{j,t+1}, \sqrt{1 - 2\min\left\{\frac{\rho_s}{2}(1-\zeta_t^2) + \max\left(0, \frac{\zeta_t}{S} - 1\right)\left(\widetilde{c}_{j,t+1} - \widehat{E_t}\widetilde{c}_{t+1} + \sigma\varepsilon^a_{j,t+1}\right), .5\right\}}, \Delta_t\right)$$

for given $\widehat{f_{\widetilde{c}}}$, where $\widehat{E_t}\widetilde{c}_{t+1} = w'\widetilde{c}_{t+1}$ according with the Gauss-Hermite formula, with $w \in \mathbb{R}^{q^2}$ and $\widetilde{c}_{t+1} = [\widetilde{c}_{j,t+1}]_{j=1}^{q^2}$. We proceed by fixed-point iterations (with dampening where necessary to improve the convergence properties) and stop at iteration N whenever $\|\widetilde{c}^{(N)}_{t+1} - \widetilde{c}^{(N-1)}_{t+1}\|_\infty < 10^{-6}$. We subsequently derive the associated value of $\zeta_{t+1} = [\zeta_{j,t+1}]_{j=1}^{q^2}$.

(c) Using $\mathbb{S}_{j,t+1} = [u_{t+1}, \zeta_{t+1}, \Delta_t]$, weights and nodes from the degree-q Gauss-Hermite cubature rule, the innovations $[\varepsilon^a_{j,t+1}; \varepsilon^u_{j,t+1}]$, $j = 1, ..., q^2$, and the functions $[\widehat{f_{\widetilde{c}}}; \widehat{f_\pi}; \widehat{f_\ell}]$ guessed in stage 2, compute expectations:

$$E_t\beta e^{-\gamma(\mu+\widetilde{c}_{t+1}+u_t+\sigma\varepsilon^a_{t+1})-\gamma\widehat{s}_{t+1}-\pi_{t+1}} \tag{II.4}$$

$$E_t\widetilde{\beta}\eta e^{\frac{\varepsilon}{1-\alpha}\pi_{t+1}+(1-\gamma)(u_t+\sigma\varepsilon^a_{t+1})+\ell_{t+1}} \tag{II.5}$$

$$E_t\widetilde{\beta}\eta e^{(\varepsilon-1)\pi_{t+1}+(1-\gamma)(u_t+\sigma\varepsilon^a_{t+1})+\ell_{t+1}}\left[\frac{1-\eta e^{(\varepsilon-1)\pi_{t+1}}}{1-\eta}\right]^{-\frac{1-\alpha+\alpha\varepsilon}{(1-\alpha)(1-\varepsilon)}} \tag{II.6}$$

(d) Using expectation (II.4) and the guess for π_t, calculate $\widetilde{c}^{new}_t$ by substituting the Taylor rule for i_t in equation (II.1).
Using expectation (II.5), the guess for π_t and $\widetilde{c}_t$, derive ℓ^{new}_t from equation (II.2).

[16]Note how the inverse operator $\Psi(\mathring{\mathbb{S}})^{-1}$ remains fixed across iterations on the unknown function $\widehat{f}$. Given the n values of an updated function $\widehat{f}$ at each collocation point, we compute the new coefficient b and interpolate by simply evaluating the Chebyshev polynomial at the arbitrary point $\mathbb{S}$.

[17]Results are virtually equivalent for $q \geq 6$; the two-dimensional Gauss-Hermite quadrature remains therefore computationally inexpensive and reliable, so we do not resort to monomial rules.

Using expectation (II.6), the value of $\widetilde{c}_t$ and the value of ℓ_t, derive π_t^{new} from equation (II.3).

4. If the differences between the guessed values for $\widetilde{c}_t$, π_t and ℓ_t and their derived values in 3d are all close to zero at each collocation point, then stop. Our stopping criterion is $\|x_t - x_t^{new}\|_2 < 10^{-6}$, with $x_t = [\widetilde{c}_t; \pi_t; \ell_t]$.
Otherwise, we update the guesses and go to stage 2. We tried several iteration schemes to perform this loop, including time and fixed-point iterations. The problem is sufficiently complex not to display the properties of a contraction mapping. We therefore rely on a derivative-free polytope method that searches directly the policy space for an optimal functional $f_x = [f_{\widetilde{c}}; f_\pi; f_\ell]$.

II.3. Algorithm to solve for asset prices

The solution for quantities allows for projecting the pricing operator $\mathcal{P}_{d,t} : F_{d,t+1} \mapsto \mathcal{P}_{d,t}(F_{d,t+1}) = E_t[M_{t+1}(D_{t+1}/D_t)F_{d,t+1}]$, while iterations on the recursion

$$F_{d,t}^{(n)} = E_t\left(M_{t+1}\frac{D_{t+1}}{D_t}F_{d,t+1}^{(n-1)}\right), \qquad F_{d,t}^{(0)} = 1$$

converge to the Perron-Frobenius eigenfunction, $F_d(\mathbb{S}_t)$, of the pricing operator that solves the corresponding Fredholm equation, $F_{d,t} = \mathcal{P}_{d,t}(F_{d,t+1})$. As emphasized by Hansen and Scheinkman (2009), this Fredholm equation has a rich structure, as the pricing operator is element-wise positive and therefore allows for the application of the infinite-dimensional extension of Perron-Frobenius theory, which describes conditions for the existence and uniqueness of a positive eigenfunction.