Prospect Theory in the Heterogeneous Agent Model

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Abstract

Using the Heterogeneous Agent Model framework, we incorporate an extension based on Prospect Theory into a popular agent-based asset pricing model. The extension covers the phenomenon of loss aversion manifested in risk aversion and asymmetric treatment of gains and losses. Using Monte Carlo methods, we investigate behavior and statistical properties of the extended versions of the model and assess relevance of the extensions with respect to empirical data and stylized facts of financial time series. We find that the Prospect Theory extension keeps the essential underlying mechanics of the model intact, and that it changes the model dynamics considerably. Moreover, the extension shifts the model closer to the behavior of real-world stock markets.

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1. Introduction

This paper introduces the phenomena of loss aversion and gain–loss asymmetry into the popular Brock and Hommes (1998) asset pricing model. Our work is based on findings of the iconic Prospect Theory (PT) of Kahneman and Tversky (1979) which describes the way people choose between probabilistic alternatives which involve risk and is per se a critique of other, rather prescriptive decision-making economic theories. Already in 1979, Kahneman and Tversky found that the actual behavior of human beings might be very dissimilar to what major economic theories assumed, namely in terms of risk and attitude towards losses. According to PT, people decide in terms of gains and losses rather than of the final outcome. The extension that we develop therefore aims to account for these empirically observed irrationalities. Throughout the years, PT has become one of the most influential works, merging psychology with economics. As Belsky and Gilovich (2010, p. 52) aptly remark, *"If Richard Thaler's concept of mental accounting is one of two pillars upon which the whole of behavioral economics rests, then Prospect Theory is the other"*. The Kahneman & Tversky's (1979) paper is the most cited paper ever to appear in Econometrica (Chang et al., 2011, p. 30).

In contemporary economic theory, there is little doubt that economic agents are heterogeneous to some extent. Frankel and Froot (1990) attribute the apparent divergence of US dollar interest rate from the then macroeconomic fundamentals at the beginning of the 1980s to the existence of speculative traders; Vissing-Jorgensen (2004) conducts a thorough analysis of chiefly qualitative telephone surveys data on US stock markets from 1998 to 2002 and concludes that there is significant disagreement among the investors regarding expected profits, and, for instance, Hommes (2011) provides 'evidence from the lab' of presence of heterogeneous expectations in an experimental financial market.

The primary objective of this paper is thus to extend the original Brock and Hommes (1998) model with features of the PT and, at the same time, keep the intrinsic mechanics of the model intact in order to preserve the stylized nature of the model. The original Brock and Hommes (1998) Adaptive Belief System (ABS) is a financial market application of the evolutionary selection system proposed by Brock and Hommes (1997) in which agents switch among different forecasting rules according to the past relative profitability of these rules. Essentially, the ABS is a discounted value asset pricing model—extended to heterogeneous beliefs—in which the agents have the possibility to invest in either a risk-free or a risky asset. Our analysis consists in using Monte Carlo methods to investigate behavior and statistical properties of the extended versions of the model and assess relevance of the extensions with respect to empirical data and stylized facts of financial time series.

One of the most important stimuli which induced development of Agent-based Models (ABMs) in economics was certainly an erosion of trust in the Efficient Market Hypothesis (EMH)—the EMH asserts, in Eugene Fama's words, that *"...security prices at any time 'fully reflect' all available information ..."* (Fama, 1970, p. 383)—and in Rational Expectations (RE) theory in the late 1970s and early 1980s which was largely due to increased focus on study of several stylized empirical facts—according to Cont (2001, p. 224), *"The seemingly random variations of asset prices do share some quite non-trivial statistical properties. Such properties, common across a wide range of instruments, markets and time periods are called stylized empirical facts"*. The most essential difference between natural sciences and economics is arguably the fact that decisions of economic agents are determined by their expectations of the future and contingent on them—hence, the study of how these beliefs are formed plays a vital part of any economic theory.

Several scholars have published papers which confront the EMH with empirical data mainly from the perspective of non-normal returns,[1] systematic deviations of asset prices from their fundamental value, and presumably excessive amount of stock price volatility—it was impossible to attribute these phenomena to the EMH or explain them within the RE framework. Offering an insightful survey on the volatility issue at that time, West (1988) summarizes and interprets literature related to this field. The author finds out that neither rational bubbles nor any standard models for expected returns adequately explain stock price volatility and emphasizes the necessity to introduce alternative models which would offer better explanation of the apparent contradiction between the EMH, RE theory, and empirical findings.

The paper is structured as follows: following the Introduction, Section 2 summarizes main features of the Prospect Theory and Section 3 describes mathematical structure and underlying mechanics of the original Brock and Hommes (1998) model. Section 4 develops the behavioral extension based on Prospect Theory while Section 5 describes the numerical simulations using Monte Carlo methods. Section 7 highlights main results of the simulations and compares the model behavior with empirical, real-world data. Finally, Section 8 concludes.

2. Prospect Theory

Proposed in the seminal paper of Kahneman and Tversky (1979), PT is a critique of then-mainstream expected utility theory. Using convincing evidence obtained from questionnaires, Kahneman and Tversky (1979) illustrate several issues with the concept of expected utility and its applicability to real-life human decision making. The most critical objection consists in incapacity of the expected utility theory to explain certain 'irrational'[2] choices of people. As a result, Kahneman and Tversky (1979)—and later Tversky and Kahneman (1992)—propose a brand new descriptive[3] theory which takes all such 'irrational' choices into account and explains them rigorously using so-called weighting function and value function. Three major features of PT are:

1. *Existence of a reference point.* PT suggests that people make decisions in terms of gains and losses with respect to some reference point, rather than in terms of final wealth.
2. *Differences in treatment of gains and losses.* While most people are risk-seeking towards losses, they are, at the same time, risk-averse towards gains. Moreover, most people are generally loss-averse which explains why the value function is steeper for losses than for gains. Figure 2 shows some notable estimates of the value function.
3. *Distorted understanding of probability.* According to PT, average person underweights large probabilities and overweights small probabilities. Given the proposed specification and shape of the weighting function, weighting is not linear in probability.

2.1. Weighting and value functions

According to PT, selection process consists of two parts: editing and evaluation. In the former, the individual conducts a preliminary analysis of the available prospects in order to facilitate the selection, and in the latter, the individual evaluates the edited prospects, assigns a value to each

[1] According to Ehrentreich (2007, p. 56), at the time when the foundations of the EMH were laid, logarithmic asset returns were thought to be distributed normally and the prices therefore lognormally.

[2] The 'irrationality' is meant within the expected utility theory.

[3] PT is descriptive in a sense that it tries to capture the real-world decision making whereas the expected utility theory is de facto prescriptive—it models how people are supposed to decide.

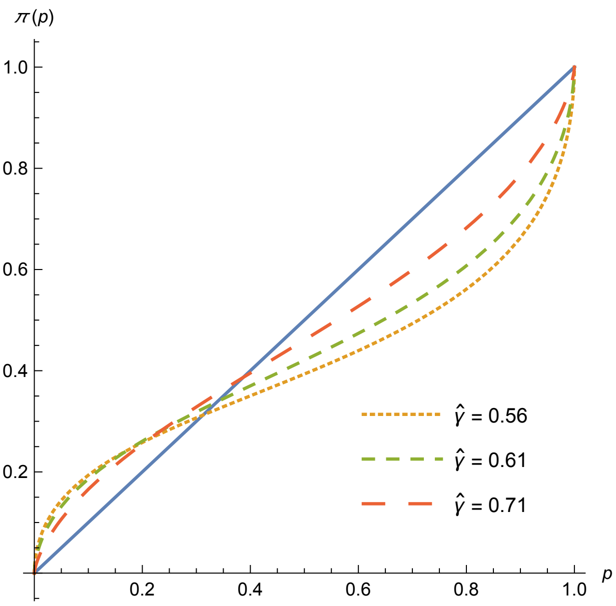

Figure 1: Estimates of the weighting function $\pi(p)$ using results of Tversky and Kahneman (1992), Camerer and Ho (1994), and Wu and Gonzalez (1996).

of them, and makes the final decision. The interested reader might find details about the editing phase in Kahneman and Tversky (1979, pp. 274–275), here we present the most essential properties of the evaluation phase.

The overall value V of an edited prospect is formulated in terms of the weighting function $\pi(\bullet)$ and the value function $v(\bullet)$. $\pi(\bullet)$ expresses probabilities of the prospect's respective outcomes, while $v(\bullet)$ assigns a specific value to each outcome. Letting $(x, p; y, q)$ denote a prospect which pays x, y, or 0 with probability p, q, and $1 - p - q$, respectively, the basic equation which assigns value to a regular prospect[4] is then given as follows:

$$V(x, p; y, q) = \pi(p) \cdot v(x) + \pi(q) \cdot v(y), \tag{1}$$

where it is assumed that $v(0) = 0$, $\pi(0) = 0$, and $\pi(1) = 1$. It is important to note that the weighting function is not a probability measure and typical properties of probability need not be valid here, and that the value function is defined with respect to a reference point which is usually given as $x = 0$, that is, the point in which a gain changes to a loss and vice versa.

Kahneman and Tversky (1979) define the weighting function $\pi(p)$ relatively vaguely as an increasing function of p that overweights 'small' probabilities and underweights 'large' ones. Moreover, the function is discontinuous near $p = 0$ and $p = 1$ to reflect the limit to how little a decision weight can be associated with an event. Several attempts have been made to estimate the weighting function—Tversky and Kahneman (1992) fit a model of the form

$$\frac{p^\gamma}{\left(p^\gamma + (1-p)^\gamma\right)^{1/\gamma}}, \tag{2}$$

[4]Regular prospect is a prospect such that either $p + q < 1$, $x \geqslant 0 \geqslant y$, or $x \leqslant 0 \leqslant y$. Evaluation of prospects which are not regular follows a different rule—details are provided in Kahneman and Tversky (1979, p. 276).

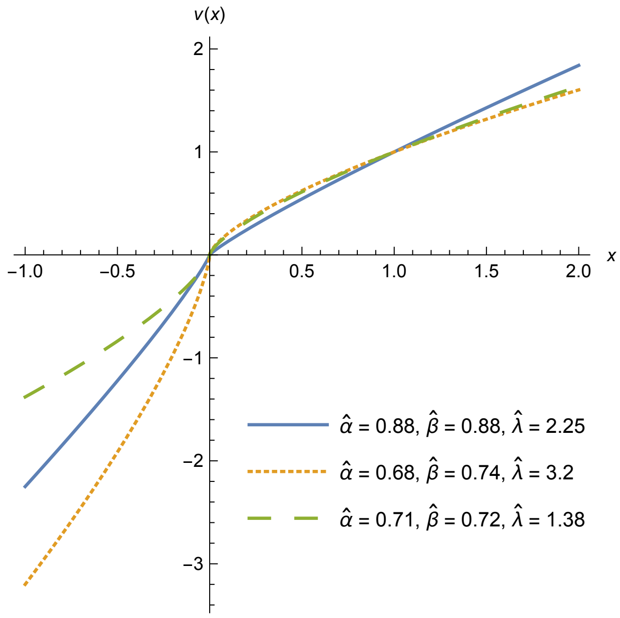

Figure 2: Estimates of the value function $v(x)$ using results of Tversky and Kahneman (1992), Harrison and Rutström (2009), and Tu (2005).

where γ is a parameter that controls for curvature of the weighting function, and obtain $\hat{\gamma} = 0.61$. Camerer and Ho (1994) use the same framework and report $\hat{\gamma} = 0.56$, and Wu and Gonzalez (1996) give $\hat{\gamma} = 0.71$, using again the model specified in Equation 2. The behavior of the weighting function with the parameter γ specified by these three results is plotted in Figure 1.

The value function $v(x)$ satisfies the following properties: it is increasing $\forall x$, i.e. $v'(x) > 0$ always holds, convex below the reference point, i.e. $v''(x) > 0$ for $x < 0$, and concave above it, i.e. $v''(x) < 0$ for $x > 0$. Additionally, the value function is usually thought to be steeper for losses than for gains. Several scholars have estimated the value function, too, most often using a piecewise power function proposed by Tversky and Kahneman (1992). The function is of the following form:

$$v(x) = \begin{cases} x^{\alpha} & x \geqslant 0 \\ -\lambda \cdot (-x)^{\beta} & x < 0, \end{cases} \tag{3}$$

where the parameters α and β determine curvature of the value function for gains and for losses, respectively, relative to the reference point of $x = 0$, and λ is a parameter understood as loss aversion characterization.

Estimating the Equation 3, Tversky and Kahneman (1992) report $\hat{\alpha} = 0.88, \hat{\beta} = 0.88$, and $\hat{\lambda} = 2.25$, Tu (2005) $\hat{\alpha} = 0.68, \hat{\beta} = 0.74$, and $\hat{\lambda} = 3.2$, and, e.g., Harrison and Rutström (2009) $\hat{\alpha} = 0.71, \hat{\beta} = 0.72$, and $\hat{\lambda} = 1.38$. All versions are plotted in Figure 2.

2.2. Relevance for financial markets

Since the formulation of PT, several scholars have confirmed its significant relevance for financial markets. One of the most cited applications of PT is an aid in explanation of so-called disposition effect. The term was first coined by Shefrin and Statman (1985) and refers to a tendency to *"... sell winners too early and ride losers too long,"* (Shefrin and Statman, 1985, p. 778) essentially meaning that traders tend to hold value-loosing assets too long and value-gaining assets too short. Using the PT value function, the authors explain the disposition effect for an investor who owns a loosing stock as a gamble between selling the stock now and thereby realizing a loss, or holding the stock for another period given, say, 50–50 chance between loosing further value or breaking even. As the investor finds himself in the 'negative domain' with respect to the reference point given here as the break even point (that is, $x \leqslant 0$), the choice between the two above-mentioned options is associated with the convex part of the value function. This fact implies that the investor selects the second option and thus 'rides the loser too long'.

Li and Yang (2013) also attempt to explain the disposition effect using findings of PT. The authors build a general equilibrium model and, besides the disposition effect, also focus on trading volume and asset prices. The results suggest that loss aversion implied by PT tends to predict a reversed disposition effect and price reversal for stocks with non-skewed dividends. Yao and Li (2013), on the other hand, investigate trading patterns in the market with Prospect-theoretical investors who base their choices on the value and weighting functions and related features of PT. The authors find that the three main features of PT can be regarded as behavioral causes of negative-feedback trading. The authors subsequently construct a market populated by the PT traders and traders who maximize Constant Relative Risk Aversion (CRRA) utility function and discover that individual PT preferences might cause contrarian noise trading.

Some other research efforts related to the study of PT traits in financial markets are made by Grüne and Semmler (2008) who try to attribute some of the most frequently observed asset price characteristics—yet unexplainable by 'standard' preferences—to the loss aversion feature of traders. Giorgi and Legg (2012) make use of the weighting function and show that dynamic models of portfolio choice might be consistently and meaningfully extended by the probability weighting. Zhang and Semmler (2009) further investigate properties of the model proposed by Barberis et al. (2001) using time series data and conclude that models with PT features are able to better explain some financial 'puzzles', such as the equity premium puzzle.[5] Finally, for instance, Giorgi et al. (2010) explore aspects of Cumulative Prospect Theory—a modification of the original PT developed by Tversky and Kahneman (1992)—and find that financial markets' equilibria need not exist under assumptions of PT.

3. Heterogeneous agent modelling framework

Our modelling framework follows the Brock and Hommes (1998) Heterogeneous Agent Model (HAM) approach, slightly reformulated in Hommes (2006). We consider a risk-free asset that pays a fixed rate of return r and is perfectly elastically supplied while the risky asset pays an uncertain dividend. Letting p_t and y_t denote the ex-dividend price of the risky asset and its random dividend process, respectively, and z_t the amount of risky asset the agent purchases at time t, each agent's wealth dynamics is of the following form:

$$W_{t+1} = R \cdot W_t + z_t \cdot (p_{t+1} + y_{t+1} - R \cdot p_t), \tag{4}$$

where R is the gross risk-free return rate equal to $1 + r$. There are H forecasting rules—this fact implies existence of H different strategies or, equivalently, H distinct classes of agents. Let $E_{h,t}$

[5]The equity premium puzzle is a phenomenon that the average return on equity is far greater than return on a risk-free asset. Such a characteristic has been observed in many markets. The term was first coined by Mehra and Prescott (1985).

and $V_{h,t}$, respectively, denote the belief of an agent who uses forecasting rule h about conditional mean and conditional variance of wealth, $1 \leqslant h \leqslant H$. It is assumed that all agents maximize the same, exponential-type Constant Absolute Risk Aversion (CARA) utility function of wealth of the form $U(W) = -\exp(-a \cdot W)$, where a is a risk aversion parameter. Given the mean-variance maximization, the optimal demand $z^*_{h,t}$ for the risky asset of agents of type h then solves the following maximization problem:

$$\max_{z_{h,t}} \left\{ E_{h,t}(W_{t+1}) - \frac{a}{2} \cdot V_{h,t}(W_{t+1}) \right\}. \tag{5}$$

The demand $z^*_{h,t}$ is then

$$z^*_{h,t} = \frac{E_{h,t}(p_{t+1} + y_{t+1} - R \cdot p_t)}{a \cdot V_{h,t}(p_{t+1} + y_{t+1} - R \cdot p_t)}, \tag{6}$$

which, assuming that $V_{h,t} \equiv \sigma^2 \; \forall h, t$, simplifies to

$$z^*_{h,t} = \frac{E_{h,t}(p_{t+1} + y_{t+1} - R \cdot p_t)}{a \cdot \sigma^2}. \tag{7}$$

Denoting z^s the supply of outside risky shares per trader, and $n_{h,t}$ the fraction of agents using forecasting rule h, the demand–supply equilibrium is

$$\sum_{h=1}^{H} n_{h,t} \cdot \frac{E_{h,t}(p_{t+1} + y_{t+1} - R \cdot p_t)}{a \cdot \sigma^2} = z^s, \tag{8}$$

where, again, H is the total number of forecasting rules (i.e. strategies). In case of zero supply of outside shares, i.e. $z^s = 0$, Equation 8 becomes

$$R \cdot p_t = \sum_{h=1}^{H} n_{h,t} \cdot E_{h,t}(p_{t+1} + y_{t+1}). \tag{9}$$

Now, should all traders be identical and their expectations homogeneous, we would obtain a simplified version of Equation 9 called arbitrage market equilibrium of the form

$$R \cdot p_t = E_{h,t}(p_{t+1} + y_{t+1}). \tag{10}$$

Equation 10 asserts that the price of the risky asset in this period is equal to the sum of next period's expected price and dividend, discounted by the gross risk-free interest rate. In this homogeneous-expectations case, provided that the transversality condition

$$\lim_{t \to \infty} \frac{E_t(p_{t+k})}{(1+r)^k} = 0 \tag{11}$$

holds,[6] the fundamental price of the risky asset is given as

$$p^*_t = \sum_{k=1}^{\infty} \frac{E_t(y_{t+k})}{(1+r)^k}, \tag{12}$$

[6] Hommes (2013, p. 162) remarks that the Equation 10 is also satisfied by the so-called rational bubble solution of the form $p_t = p^*_t + (1+r)^t \cdot (p_0 - p^*_0)$. However, this solution does not satisfy the transversality (or 'no-bubbles') condition

where E_t is the conditional expectation operator. The price p_t^* is the equilibrium price of the risky asset in a perfectly efficient market with fully rational traders and, as can be seen directly from Equation 12, it depends on the expectation of the stochastic dividend process y_t, $E_t(y_t)$. Assuming the dividend process y_t is independent, identically distributed with mean $\bar{y}$, the fundamental price p_t^* becomes constant $\forall t$ and is given by

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \tag{13}$$

The deviation from the fundamental price is defined as follows:

$$x_t = p_t - p_t^*. \tag{14}$$

There are now two additional assumptions Brock and Hommes (1998) make:

1. Expectations about future dividends y_{t+1} are the same for all agents, regardless of the specific forecasting rule they use, and equal to the true conditional expectation. In other words, $E_{h,t}(y_{t+1}) = E_t(y_{t+1})\ \forall h, t$.
2. Agents believe that the stock price might deviate from the fundamental price p_t^* by some function f_h which depends on previous deviations from the fundamental price, i.e. on $x_{t-1}, \ldots, x_{t-K}$. This assumption might be stated as

$$E_{h,t}(p_{t+1}) = E_t\left(p_{t+1}^*\right) + f_h(x_{t-1}, \ldots, x_{t-K})\ \forall h, t. \tag{15}$$

It is now important to note two crucial facts: firstly, the assumption number one above implies that all agents have homogeneous expectations about future dividends, i.e. the heterogeneity of the model lies in the assumption number two. Secondly, the asset price in period $t+1$, p_{t+1}, is predicted using price realized in period $t-1$—not in period t—as the agents are yet unaware of the price p_t when they make the prediction. This fact follows directly from Equation 8.

Next, Brock and Hommes (1998) define realized excess return, which, for the purpose of this thesis, are denoted as $\mathcal{R}_{t+1} = p_{t+1} + y_{t+1} - R \cdot p_t$. The realized excess return over period t to period $t+1$ might be expressed in deviations from the fundamental value as follows:

$$\begin{aligned}
\mathcal{R}_{t+1} &= p_{t+1} + y_{t+1} - R \cdot p_t = x_{t+1} + p_{t+1}^* + y_{t+1} - R \cdot x_t - R \cdot p_t^* \\
&= x_{t+1} - R \cdot x_t + \underbrace{p_{t+1}^* + y_{t+1} - E_t\left(p_{t+1}^* + y_{t+1}\right)}_{\delta_{t+1}} \\
&+ \underbrace{E_t\left(p_{t+1}^* + y_{t+1}\right) - R \cdot p_t^*}_{=0} \\
&= x_{t+1} - R \cdot x_t + \delta_{t+1},
\end{aligned} \tag{16}$$

where the latter underbrace holds because Equation 19 is satisfied. The term δ_{t+1} is a Martingale Difference Sequence with respect to some information set $\mathcal{F}_t$, that is, we have $E(\delta_{t+1}|\mathcal{F}_t) = 0\ \forall t$.

3.1. Fitness measure

The fitness measure of strategy h, $U_{h,t}$, depends on past realizations of the market price of the risky asset and is defined as

$$U_{h,t} = \mathcal{R}_{t+1} \cdot z_{h,t}^* = (x_{t+1} - R \cdot x_t + \delta_{t+1}) \cdot z_{h,t}^*. \tag{17}$$

Two cases might now be distinguished:

1. The case of $\delta_{t+1} = 0$ corresponds to a deterministic nonlinear pricing dynamics with constant dividend $\bar{y}$ and, according to Hommes (2006) who uses slightly modified notation, Equation 17, written in deviations, reduces to

$$U_{h,t} = (x_t - R \cdot x_{t-1}) \cdot \frac{f_{h,t-1} - R \cdot x_{t-1}}{a \cdot \sigma^2}, \tag{18}$$

   where $f_{h,t-1}$ is the forecasting function of type h.
2. The case in which dividend is given by a stochastic process $y_t = \bar{y} + \epsilon_t$ where ϵ_t is independent, identically distributed random variable with uniform distribution. In these circumstances, $\delta_{t+1} = \epsilon_{t+1}$.

Using the facts that $p_t = x_t + p_t^*$ and that the fundamental price p_t^* satisfies

$$R \cdot p_t^* = E_t \left(p_{t+1}^* + y_{t+1} \right), \tag{19}$$

3.2. Market fractions

Equation 9 can be reformulated in deviations from the fundamental price by a substitution using Equation 15 as

$$R \cdot x_t = \sum_{h=1}^{H} n_{h,t} \cdot E_{h,t} \left(x_{t+1} \right) \equiv \sum_{h=1}^{H} n_{h,t} \cdot f_h \left(x_{t-1}, \ldots, x_{t-K} \right), \tag{20}$$

where $n_{h,t}$ denotes the fraction of agents using the forecasting function h for prediction. These fractions are modeled using the multinomial logit model:

$$n_{h,t} = \frac{\exp \left(\beta \cdot U_{h,t-1} \right)}{Z_{t-1}}, \tag{21}$$

where $Z_{t-1} \equiv \sum_{h=1}^{H} \exp \left(\beta \cdot U_{h,t-1} \right)$ is a normalization factor such that the fractions $n_{h,t}$ add up to 1, and β, $\beta \geqslant 0$, is a parameter called intensity of choice which measures the agents' 'sensitivity' to the selection of the best-performing forecasting rule. Two extreme cases may be distinguished—if $\beta = \infty$, all agents unerringly choose the best rule, while if $\beta = 0$, the fractions $n_{h,t}$ remain constant over time and fixed to $1/H$, i.e. $n_{h,t} = 1/H \; \forall h, t$. The former extreme case corresponds to the situation in which there is no noise and thus all agents select the optimal strategy while the latter extreme case implies presence of noise with infinite variance and inability of agents to switch strategies at all.

For the formation of expectations, the functions $f_{h,t}$ are crucial. Brock and Hommes (1998) propose simple forecasting rules of the form

$$f_{h,t} = g_h \cdot x_{t-1} + b_h. \tag{22}$$

The term g_h is a trend parameter indicating the trend following (or possibly reverting) strength of the particular strategy, and the term b_h is a bias parameter. For $g_h = b_h = 0$, the function $f_{h,t}$ reduces to $f_{h,t} \equiv 0$ and corresponds to the fundamentalist belief of no price deviations from the fundamental value. Additionally, if $g_h \neq 0$, then such a trader type is called a chartist. This class of traders can be further divided into four categories: the type is called a pure trend chaser if $0 < g_h \leqslant R$, a strong trend chaser if $g_h > R$, a contrarian if $-R \leqslant g_h < 0$, and a strong contrarian if $g_h < -R$. Finally, the term b_h determines the nature (if $b_h \neq 0$) of each agent class' bias—if $b_h < 0$, the bias is downward, while if $b_h > 0$, the bias is upward.

4. Prospect Theory extension

Although the indisputable relevance of findings of PT for study of human decision making is highly topical, there are apparently no PT extensions of the Brock and Hommes (1998) HAM framework. The plausible reason for the absence of such ABM designs is relatively self-evident: the HAM developed by Brock and Hommes (1998) is populated with agents with CARA utility function and demand for the risky asset is derived by maximization of expected utility. As the origins of PT are based on critique of the expected utility theory and subsequent development of diametrically different approach to decisions under risk, the very basic component of the ABS—CARA utility function—seems incompatible with PT. Yet, although the authors do not use the original Brock and Hommes (1998) model, Shimokawa et al. (2007) propose a relatively straightforward method to implement PT features into ABMs in which the agents have CARA preferences.

4.1. Loss aversion inclusion

The basic structure of the model remains identical, however, extending the original Brock and Hommes (1998) model, we introduce features of PT into the model as follows: PT traders maximize utility function of the general form

$$U_l(W) = -\exp(-a \cdot B \cdot W), \tag{23}$$

where we denote B the loss aversion parameter. Generally, the loss aversion parameter may differ for each agent class and time period, therefore we denote it as $B_{h,t}$ from now on. Furthermore, the subscript l distinguishes the utility function of these PT traders from that of 'standard' traders specified in the original model—we refer to the PT traders as loss-averse traders since this characteristic is the main feature of PT which is possible to incorporate into the model using the utility function defined in Equation 23. Other notations in Equation 23 have their usual meaning as given in Section 3. We assume that the wealth dynamics is of the same form as in Equation 4.

The crucial aspect of the utility function given in Equation 23 is the loss aversion parameter $B_{h,t}$ and its specification. Following the general idea proposed by Shimokawa et al. (2007, p. 211), we define the parameter as follows:

$$B_{h,t} = \begin{cases} c_g, & E_{h,t}(p_{t+1}) > \tilde{p}_t = \tilde{p}_t(p_{t-1}, \ldots, p_{t-K}) \\ c_l, & E_{h,t}(p_{t+1}) \leqslant \tilde{p}_t = \tilde{p}_t(p_{t-1}, \ldots, p_{t-K}), \end{cases} \tag{24}$$

where c_g and c_l are gain and loss parameters, respectively, $0 < c_g < c_l$, and $\tilde{p}_t = \tilde{p}_t(p_{t-1}, \ldots, p_{t-K})$ is a reference point as defined by PT. It is important to emphasize that each agent might maximize either the original utility function $U(W) = -\exp(-a \cdot W)$ or the 'augmented' utility function U_l with the loss aversion parameter given in Equation 23, however, the term $E_{h,t}(p_{t+1})$, i.e. a (loss-averse) agent's forecast about the price in next period, is constructed essentially in the same way as in Equation 15 whether the agent is loss-averse or not.

Optimal demand $z^*_{l,t}$ of the loss-averse traders for the risky asset then solves the familiar maximization problem

$$\max_{z_{l,t}} \left\{ E_{h,t}(W_{t+1}) - \frac{a \cdot B_{h,t}}{2} \cdot V_{h,t}(W_{t+1}) \right\}, \tag{25}$$

where $V_{h,t}(W_{t+1})$ is the (loss-averse) traders' belief about next period conditional variance of wealth, and is thus given by

$$z^*_{l,t} = \frac{E_{h,t}(p_{t+1} + y_{t+1} - R \cdot p_t)}{a \cdot B_{h,t} \cdot \sigma^2}. \tag{26}$$

The basic structure of the model remains the same: there are H distinct trading strategies or classes of agents and each agent class maximizes a CARA utility function. Certain number of the H classes, say first L classes, $0 \leqslant L \leqslant H$, are endowed with the above-specified PT feature—optimal demand of agents of these L classes for the risky asset is given by Equation 26—while the agents of the $H - L$ remaining classes are 'standard' in terms of the original model construction and do not exhibit PT behavior. The general specification of the optimal demand for the risky asset, $z^*_{h,t}$, $1 \leqslant h \leqslant H$, thus remains the same and is given by Equation 7 where, if hth class of agents has the PT feature (i.e. for $h \leqslant L$, $1 \leqslant h \leqslant H$), we use $z^*_{l,t}$ given by Equation 26 instead of $z^*_{h,t}$.

The definition of the parameter $B_{h,t}$ given in Equation 24 essentially enables us to mimic the first two of the three major features of PT listed in the beginning of Section 2, i.e., the loss aversion and biased treatment of gains and losses, and relation of decisions under risk to a reference point, by using an 'imitation' of the value function. In this application, however, we omit the third major feature of PT, the probability weighting and the weighting function, to keep the model within the stylized, simple framework proposed by Brock and Hommes (1998). Also the curvature of the value function per se is not studied and incorporated into the model as it is well approximated by a linear function (see Figure 2).

4.2. Reference point

The choice of specific numerical values of the gain and loss parameters c_g and c_l is relatively unfettered—the only condition that must always hold in order to capture the loss aversion feature properly is the inequality $0 < c_g < c_l$. The choice of the reference point $\tilde{p}_t$ has more implications. The reference point is updated in each time period to properly reflect the contradictory treatment of gains and losses of PT traders. Generally, the reference point is given by a deterministic function of past performance of the model—one might make use of K previous realized prices of the risky asset, i.e., $p_{t-1}, p_{t-2} \ldots, p_{t-K}$, and define the reference point—as suggested by Shimokawa et al. (2007)—as the moving average of the form

$$\tilde{p}_t = \frac{a_1 \cdot p_{t-1} + a_2 \cdot p_{t-2} + \ldots + a_K \cdot p_{t-K}}{a_1 + a_2 + \ldots + a_K}, \tag{27}$$

where $a_1, a_2, \ldots, a_K$ are constants $\in \mathbb{R}$ such that $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_K \geqslant 0$ which allow for a stronger, more significant effect of the most recent prices of the risky asset. The interpretation of the definition of the parameter $B_{h,t}$ is straightforward in such a case: if the traders with the PT feature expect the next period price to be higher than the moving average of previous K prices, they find themselves in the positive domain in terms of the gain–loss gamble and set the value $B_{h,t}$ to c_g. If, on the other hand, they expect the next period price to be lower than the moving average, i.e. they expect a loss, the loss aversion of PT manifests itself by the parameter $B_{h,t}$ which is set to c_l.

To summarize, the ABS extended with the PT loss aversion becomes

$$\begin{aligned} R \cdot x_t &= \sum_{h=1}^{H} n_{h,t} \cdot (g_h \cdot x_{t-1} + b_h) + \varepsilon_t, \\ n_{h,t} &= \frac{\exp\left(\beta \cdot U_{h,t-1}\right)}{\sum_{h=1}^{H} \left(\beta \cdot U_{h,t-1}\right)}, \\ U_{h,t-1} &= \begin{cases} \left(x_{t-1} - R \cdot x_{t-2}\right) \frac{g_h \cdot x_{t-3} + b_h - R \cdot x_{t-2}}{a \cdot \sigma^2}, & h > L \\ \left(x_{t-1} - R \cdot x_{t-2}\right) \frac{g_h \cdot x_{t-3} + b_h - R \cdot x_{t-2}}{a \cdot B_{h,t-2} \cdot \sigma^2}, & h \leqslant L, \end{cases} \end{aligned} \tag{28}$$

where first L of the H agent classes are endowed with the PT feature; g_l and b_l indicate the trend and bias parameters of the strategies with the PT feature, and ε_t is a (small) noise term which represents natural uncertainty about the performance of economic fundamentals and which replaces the term $\delta_t = \epsilon_t$ defined in Section 3. The system of Equations 28 is in essence a generalization of the original ABS—for $L = 0$, one obtains the 'benchmark' case used for the PT extension impact evaluation in Section 5.

5. Monte Carlo Analysis

5.1. *General model setup*

The inevitable 'downside' of the ABS is somewhat excessive leeway in choice of the parameters of the model, especially of β, g_h, b_h, and the distribution of the noise term ε_t. We follow a number of previous studies—e.g. Kukacka and Barunik (2013); Vácha and Vošvrda (2005); Vošvrda and Vácha (2003)—and adopt the following settings:

1. Trend and bias parameters g_h and b_h are drawn from the normal distributions with means of 0 and variances of 0.16 and 0.09, respectively, unless we state otherwise. Should we ex ante indicate presence of fundamentalists in the model, the fundamentalist strategy is always the first of the H strategies: the algorithm sets both of the parameters g_1 and b_1 to 0 and the term $n_{1,t}$ corresponds to the fraction of fundamentalists in the market.
2. The noise term for each time period, ε_t, are drawn from the uniform distribution $U(-0.05, 0.05)$. Kukacka and Barunik (2013) investigate behavior of the model with the noise term drawn from several different uniform distributions and conclude that the behavior is largely similar.
3. Other parameters are set as follows: the gross risk-free return rate, $R = 1 + r$, to 1.0001 and the term $\frac{1}{a \cdot \sigma^2}$ to 1. The choice of the gross risk-free return rate allows us to compare results of the simulations with real-world market data since $1.0001^{250} \cong 1.025$. Annual interest rate of 2.5 % can be normally considered a realistic risk-free rate.

Each simulation consists of 11 runs characterized by a distinct intensity of choice parameter β which gradually takes values from 5 to 505 in increments of 50. Additionally, there are 1000 repeat cycles in each run. For each cycle, the parameters g_h and b_h are randomly drawn from the aforementioned distributions to guarantee robust simulation results. Finally, there are 500 ticks in each cycle representing trading days.

5.2. *Criteria for evaluation*

Cont (2001) lists the following phenomena as the most frequent financial time series stylized facts: absence of autocorrelations, heavy or fat tails, volatility clustering, intermittency, gain–loss asymmetry, and several others. We focus on the first three stylized facts as the original Brock and Hommes (1998) model has been found capable of explaining them soundly (Chen et al., 2012).

1. *Absence of autocorrelations.* Autocorrelations of returns of an asset are insignificant at most times and for most time scales, except for very small time scales of approximately 20 minutes in which micro structures may have an effect on the autocorrelations (Cont, 2001).
2. *Fat tails.* Probability distributions of many assets' returns have large skewness or kurtosis relative to the normal distribution. Additionally, the distributions exhibit a power-law or Pareto-like tails, with a tail index $2 \leqslant \alpha \leqslant 5$ (Cont, 2001), i.e. the (upper) tail $P(X > x) = \bar{F}(x) = x^{-\alpha} \cdot G(x)$, where $G(x)$ is a slowly varying function (Haas and Pigorsch, 2009).

Table 1: Benchmark simulation summary statistics and p-values of J-B test for normality of distribution for x_t in 11 runs with different βs. There are fundamentalists and three other strategies in the model, i.e. $H = 4$.

β	Mean	Var.	Skew.	Kurt.	Min.	Max.	Med.	J-B
5	-0.0012	0.0224	-0.5389	7.1002	-1.6221	0.8397	0.0029	0.000
55	0.0059	0.1236	0.1880	5.9124	-1.9789	2.4209	0.0032	0.000
105	0.0119	0.1225	0.0306	4.5966	-1.7256	1.6157	0.0072	0.000
155	0.0036	0.1142	-0.0214	3.8254	-1.4534	1.5590	0.0068	0.000
205	0.0030	0.1005	0.0713	3.4390	-1.2670	1.2970	0.0018	0.000
255	0.0077	0.0979	-0.0210	3.2504	-1.2113	1.1175	0.0042	0.000
305	0.0002	0.0860	-0.0420	3.0802	-1.0447	1.0593	-0.0003	0.000
355	-0.0067	0.0845	-0.0942	3.1020	-1.0269	1.0099	-0.0003	0.000
405	0.0089	0.0766	0.0013	3.1255	-0.9395	0.9324	0.0059	0.000
455	-0.0026	0.0738	-0.0285	3.0397	-0.8961	0.8954	-0.0004	0.000
505	-0.0001	0.0673	-0.0380	2.9215	-0.8525	0.8532	0.0022	0.000

3. *Volatility clustering.* Absolute or squared returns of an asset are characterized by a significant, slowly decaying autocorrelation function, that is, $\text{corr}\left(|r_t|, |r_{t+\tau}|\right) > 0$ or $\text{corr}\left(r_t^2, r_{t+\tau}^2\right) > 0$, where the time span τ ranges from minutes to weeks or months (Cont, 2007).

5.3. Benchmark simulation

We run a benchmark simulation of the original model specified by the system of Equations 28 without the PT feature, that is, we set $L = 0$. Number of total strategies $H = 4$ and fundamentalists are present in the model. In each repeat cycle, first 5 % of realizations of x_t are discarded as the model needs some initial time to 'stabilize'.

Table 1 shows selected descriptive statistics of the x_t time series obtained from the benchmark simulation. Clearly, the distributions of the deviations from the fundamental price are statistically different from the normal distribution, as is indicated by small p-values of the Jarque-Berra (J-B) test for all βs. For increasing values of β, the distributions exhibit sample kurtosis closer to that of the normal distribution. Apparently, the behavior of the model is most dramatic for $\beta \in \{55, 105, 155\}$—values of sample variance are highest, the same is true for minima and maxima of x_t.

Figure 6 in Appendix A shows, on a log-log scale, the Cumulative Distribution Functions (CDFs) $\bar{F}_{|x_t|}(y)$, $\bar{F}_{|x_t|}(y) = P\left(|x_t| > y\right)$, for the 150 largest absolute deviations $|x_t|$ corresponding to four randomly selected illustrative sample time series generated with different βs, along with a regression-based linear fit. Although not rigorously, the slopes of the regression lines are, in absolute values, estimates of the respective tail indices. These are equal to 10.77 for $\beta = 5$ ($R^2 = 0.849$), 9.13 for $\beta = 105$ ($R^2 = 0.979$), 8.74 for $\beta = 305$ ($R^2 = 0.953$), and 5.16 for $\beta = 505$ ($R^2 = 0.948$). Having only an informative character, the plots in Figure 6 nonetheless show possible existence of a power law in tails of the sample distribution of $|x_t|$. It is important to emphasize, however, that the power law apparently does not hold universally for the whole tail. Most extreme observations—for which the imaginary curvature is relatively significant and the realizations clearly do not follow the linear pattern estimated for the complete collection of the 150 observations—might exhibit a different tail index than the remaining observations do; the 'break point' is evidently around $\bar{F}_{|x_t|}(y) = 0.05$.

Table 2: PT simulation summary statistics of x_t and p-values of J-B and K-W tests in 11 runs with different β. There are fundamentalists and three other strategies in the model, i.e. $H = 4$, and all strategies have the PT feature, i.e. $L = 4$.

β	Mean	Var.	Skew.	Kurt.	Min.	Max.	J-B	K-W
5	-0.0002	0.0201	-0.0691	3.2125	-0.7410	0.5000	0.000	0.747
55	0.0171	0.0933	0.0828	6.9381	-1.9744	2.3506	0.000	0.000
105	0.0178	0.0999	-0.0308	5.3802	-1.7316	1.6008	0.000	0.000
155	0.0109	0.0954	-0.0511	4.3116	-1.4001	1.4358	0.000	0.000
205	0.0129	0.0854	0.0808	3.8215	-1.2733	1.2907	0.000	0.000
255	0.0139	0.0841	-0.0224	3.5736	-1.2105	1.1040	0.000	0.000
305	0.0087	0.0742	-0.0307	3.3799	-1.0395	1.0301	0.000	0.000
355	0.0004	0.0731	-0.1115	3.4346	-1.0244	1.0022	0.000	0.000
405	0.0151	0.0670	0.0062	3.4075	-0.9194	0.9322	0.000	0.000
455	0.0032	0.0648	-0.0192	3.2903	-0.9092	0.8733	0.000	0.000
505	0.0072	0.0589	-0.0424	3.1610	-0.8529	0.8406	0.000	0.000

5.4. Employment of Prospect Theory

The simulation with PT traders is run together with the benchmark case from Subsection 5.3 meaning that for each repeat cycle, exactly the same setting and randomly generated parameters are used. Therefore any differences between the benchmark and the PT simulations can be attributed to the PT feature completely and unreservedly. Important parameters, exclusive for the PT simulation, are given as follows:

1. The gain and loss parameters c_g and c_l are set to 1 and 2.5, respectively, to properly account for the gain–loss asymmetry. These particular numerical values are chosen based on the facts that *"...the disutility of giving something up is twice great as the utility of acquiring it,"* (Benartzi and Thaler, 1993), and that *"...losses hurt more than equal gains please; typically two to two-and-a-half times more."* (van Kersbergen and Vis, 2014, p. 163). Moreover, such setting of the respective parameters is well justified by Figure 2 which shows estimates of the PT value function. Initially, all strategies exhibit the PT feature, i.e. $L = 4$.
2. Length of the 'memory' used for the moving average of past prices, K, essential for determination of the reference point, is set to 10. Traders do not attach greater importance to the most recent past prices relative to more distant ones—that is, the parameters $a_1, a_2, \ldots, a_K$ are all set to 1.

Table 2 summarizes descriptive statistics along with p-values of J-B and Kruskal-Wallis (K-W) tests of the x_t time series. Using the K-W method we test whether the x_t time series obtained from the PT simulation originate from the same distribution as the benchmark simulation x_t time series (see Table 1). Addition of the PT feature clearly causes, except for the case of $\beta = 5$, significant differences of the distributions with respect to those of the benchmark simulation. Notice especially the smaller variance of the time series with respect to the benchmark case and also smaller extreme values.

Figure 7 in Appendix B shows, on a log-log scale, the complementary CDFs $\bar{F}_{|x_t|}(y)$ for the 300 largest absolute deviations $|x_t|$ corresponding to four randomly selected illustrative sample time series generated with different βs, along with a regression-based linear fit. The estimates of the respective tail indices (i.e. the opposites of the estimated slope coefficients) are equal to 10.33 for

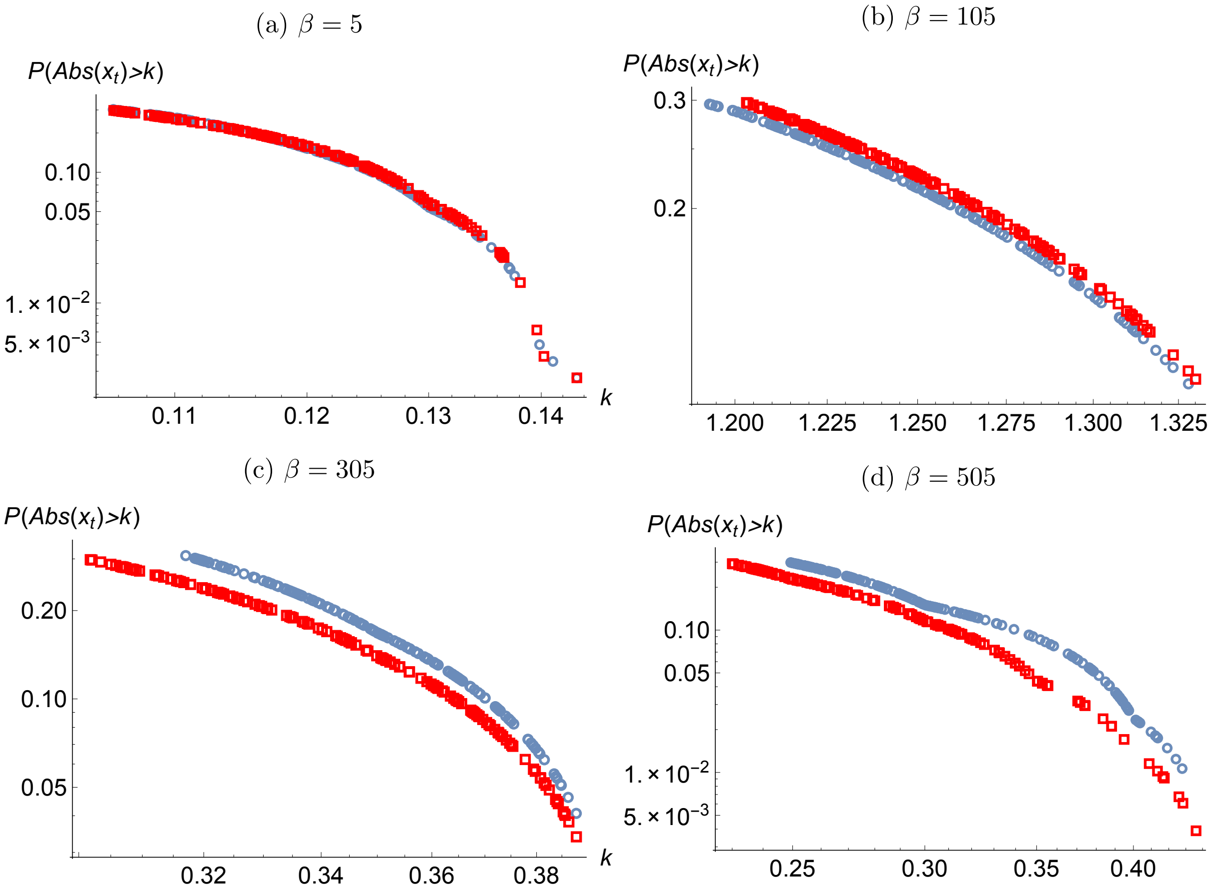

Figure 3: Plots of the tails of sample x_t time series' empirical distributions with the PT feature (red squares) and without it (blue circles).

$\beta = 5$ ($R^2 = 0.835$), 9.41 for $\beta = 105$ ($R^2 = 0.981$), 7.52 for $\beta = 305$ ($R^2 = 0.937$), and 5.28 for $\beta = 505$ ($R^2 = 0.934$). The OLS fits provide roughly the same R^2 compared to the benchmark case, although the most extreme observations do, again, exhibit considerable curvature and departure from any power law, mainly in the region for which $\bar{F}_{|x_t|}(y) < 0.05$.

These findings are summarized in Figure 3 which merges Figure 6 and Figure 7 and shows, on a log-log scale, the complementary CDFs for largest 150 x_t observations for one repeat cycle with and without the PT feature. One might notice the similarity of the tails for the lowest value of β and the subsequent departure of the tails the as the value of β increases.

5.5. Aggregate characteristics

This subsection summarizes aggregate qualitative characteristics of the price deviations time series, x_t, obtained from the simulations. We compare the model without the PT feature ($L = 0$) and the one in which all trading strategies have the feature ($L = 4$) namely in terms of time dependence in the x_t time series and x_t^2 time series, and incidence of fat tails.

5.5.1. Time dependence of x_t and x_t^2

We use the following method for assessment of time dependence using aggregate data. In each repeat cycle and for all values of β, we fit a time series model to the simulated x_t (or x_t^2) data, save the respective coefficients, and using the kernel density estimation[7] we construct an empirical distribution of these coefficients. The optimal model is selected based on the Akaike Information Criterion (AIC)—the simulations show that the data generally fit an Autoregressive Moving Average (ARMA) model best. Therefore the coefficients saved are $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$ if the model is specified as

$$x_t = c + \sum_{i=1}^{p} \alpha_i \cdot x_{t-i} + \sum_{i=1}^{q} \beta_i \cdot \varepsilon_{t-i} + \varepsilon_t, \tag{29}$$

or, for the squared deviations series x_t^2, as

$$x_t^2 = c + \sum_{i=1}^{p} \alpha_i \cdot x_{t-i}^2 + \sum_{i=1}^{q} \beta_i \cdot \varepsilon_{t-i} + \varepsilon_t. \tag{30}$$

Note that, for different repeat cycles and different values of β, the optimal models naturally exhibited different orders p and q—yet, the α_1 coefficient always corresponds to the autoregressive relationship of the first lag, regardless of the value of p. The same is true for the MA(1) coefficient β_1 and the value of q. Finally, we compare Probability Density Functions (PDFs) of the distributions using the K-W test.

Table 3 summarizes expected values of the estimated distributions of the AR(1) coefficient α_1 and MA(1) coefficient β_1 and p-values of the K-W test applied to the x_t time series obtained from models with and without the PT feature. The distributions of the AR(1) coefficient are—except for $\beta = 305$—statistically significantly different. This fact further supports the finding that the PT extensions changes the behavior of the HAM. On the other hand, p-values of the test applied to the MA(1) coefficient fail to reject the null hypothesis of equal distributions at a reasonable significance level. This fact indicates that the PT extensions affects the autoregressive structure of the x_t time series more than it does the moving average one. Notice that for most values of β, both coefficients tend to be larger for the PT extended model—the realizations of x_t seem to be slightly more dependent on previous realizations x_{t-1}.

Figure 8 in Appendix C shows estimated PDFs of the MA(1) coefficient β_1 from the model specified in Equation 29 for the x_t time series. The figure suggests that overall, the behavior of both models is relatively similar—yet, for $\beta \in \{5, 505\}$, the series exhibit somewhat less moving average dependence which is depicted by the higher peaks of the respective PDFs and higher expected values.

Table 4 summarizes expected values of the estimated distributions of the AR(1) coefficient α_1 and MA(1) coefficient β_1 and p-values of the K-W test applied to the x_t^2 time series obtained from models with and without the PT feature. The empirical distributions of β_1 are, again, not statistically different. Moreover, the p-values of the K-W test are even higher. On the other hand, the distributions of α_1 obtained from the PT extended model are statistically different from their non-PT counterparts and their expected values are greater than those obtained from the

[7] We employ the Epanechnikov kernel function and Silverman's rule (Silverman, 1986) for bandwidth selection. Epanechnikov kernel function is used as it is the most efficient kernel function (Wand and Jones, 1994, p. 31).

Table 3: Expected value of the empirical distributions of α_1 (AR) and β_1 (MA) coefficients and p-value of the K-W test applied to x_t with and without the PT feature.

β	MA	MA^{PT}	MA^{KW}	AR	AR^{PT}	AR^{KW}
5	0.1883	0.1691	0.2137	0.3068	0.2578	0.0159
55	0.1908	0.2272	0.0157	0.2858	0.3546	0.0003
105	0.1668	0.2041	0.0274	0.2278	0.2891	0.0011
155	0.1607	0.1848	0.1386	0.1997	0.2797	0.0000
205	0.1418	0.1711	0.0203	0.1947	0.2814	0.0000
255	0.1344	0.1579	0.0724	0.1901	0.2523	0.0022
305	0.1253	0.1441	0.2213	0.2173	0.2481	0.2745
355	0.1336	0.1602	0.0524	0.1906	0.2815	0.0000
405	0.0971	0.1331	0.0130	0.1986	0.2415	0.0184
455	0.1008	0.1229	0.0753	0.2138	0.2639	0.0158
505	0.1188	0.1152	0.9492	0.2097	0.2659	0.0050

Table 4: Expected value of the empirical distributions of α_1 (AR) and β_1 (MA) coefficients and p-value of the K-W test applied to x_t^2 with and without the PT feature.

β	MA	MA^{PT}	MA^{KW}	AR	AR^{PT}	AR^{KW}
5	0.1718	0.1601	0.7209	0.2582	0.2266	0.0586
55	0.1756	0.2219	0.0047	0.1825	0.2415	0.0016
105	0.1598	0.1665	0.5712	0.1038	0.1884	0.0000
155	0.1199	0.1554	0.0151	0.1052	0.1758	0.0001
205	0.1534	0.1545	0.9527	0.1117	0.1816	0.0002
255	0.1079	0.1299	0.2425	0.0846	0.1567	0.0000
305	0.1009	0.1038	0.9450	0.0871	0.1339	0.0136
355	0.0993	0.0999	0.9752	0.0875	0.1614	0.0000
405	0.0895	0.1299	0.0092	0.1045	0.1394	0.0469
455	0.1029	0.1207	0.1957	0.0911	0.1639	0.0014
505	0.1044	0.1159	0.4161	0.0967	0.1836	0.0000

non-PT model. This fact implies that the phenomenon of volatility clustering is more significant and recognizable in our extended version of the HAM. Such a finding is consistent with real-world market data (Cont, 2001).

5.5.2. Aggregate tails

Table 5 shows estimated tail indices of the x_t time series for 500 repeat cycles with and without the PT feature. As 500 different repeat cycle setups for g_h, b_h, and ε_t are used, the estimates are considerably more robust than those for only one repeat cycle (shown e.g. in Figure 3). The values of R^2 can be considered relatively satisfactory for the power law fit. Moreover, the PT extended model tail indices are in most cases smaller than those of the non-extended model and thus closer to the real-world ones (consult Section 6) and the coefficient of determination is higher. Nonetheless, it is not clear whether the power law is really the ideal model for this type of HAM

Table 5: Estimated tail indices of the x_t time series along with R^2 for the original and PT extended versions of the model.

With PT			Without PT		
β	Tail	R^2	β	Tail	R^2
5	6.842	0.933	5	6.401	0.966
105	8.731	0.938	105	7.736	0.955
205	8.469	0.918	205	9.381	0.903
305	10.652	0.955	305	11.643	0.936
405	11.636	0.967	405	12.491	0.963
505	10.824	0.933	505	11.438	0.916

as the coefficients of determination are smaller than those of the real-world indices.[8]

5.5.3. PT vs. non-PT traders

We may now relax the assumption that all trading strategies are endowed with the PT feature and examine behavior of the model by running additional simulations in which some of the trading strategies exhibit loss aversion and gain–loss asymmetry, and some do not, i.e. $L \leqslant H = 4$. Additionally, more values of the parameter K, length of the moving average considered for the reference point $\tilde{p}_t$, can be inspected. Table 6 summarizes simulations with $L = 1$, $L = 2$, $L = 3$, and different values of K. Fundamentalist strategy is present in the model as the first strategy, i.e. $L = 1$ corresponds to a situation in the market in which there are PT fundamentalists and three other non-PT chartist strategies. The K-W test compares, in this case, the distributions obtained from the simulations with the PT feature with those obtained from a simulation without it, i.e. the one for which $L = 0$.[9] To maintain mutual comparability, the same parameters g_h, b_h, and ε_t are used for each value of $L \neq 0$ and for $L = 0$.

Figure 4 further examines, for $\beta = 105$ and $K = 15$, the cases in which $L = 1$ and $L = 4$, i.e. the situation in which only the fundamentalist strategy has the PT feature versus the one in which all strategies have the PT feature, respectively. These situations are compared to the benchmark case of $L = 0$. Estimated densities of the x_t time series are plotted in the left-hand side of the figure while the right-hand side of the figure shows estimated densities of the $n_{1,t}$ time series, i.e. of the fraction of traders using the fundamentalist strategy. Apparently—as can be also seen from Table 6—the behavior of the model for $L = 1$ is relatively similar to that of the benchmark case—K-W test does not reject the null hypothesis and the estimated densities of x_t are very similar. Yet, PT fundamentalists are driven out of the market more strongly. This finding can be inferred from higher peak of the respective distribution around 0. The PT feature, manifested in significant loss aversion, poses a relatively heavy 'burden' for the fundamentalists when they face chartists who are not loss-averse. On the other hand, when all trading strategies have the PT feature, the behavior of the model is significantly different from the benchmark case—the PT feature stabilizes the market and rules out a fraction of extreme price deviations which are present in the benchmark case. Moreover, fundamentalists are able to survive in the market more easily

[8]Consult e.g. Cont (2001) for a discussion of real-world tail indices.

[9]We run another 'benchmark' simulation of the model without the proposed extensions, that is, for the K-W test, we use different benchmark than that examined in Subsection 5.3.

Table 6: P-value of the K-W test for different L, K and β.

	$K=1$				$K=5$		
β	$L=1$	$L=2$	$L=3$	β	$L=1$	$L=2$	$L=3$
5	0.99637	0.47500	0.72023	5	0.00000	0.00000	0.00000
55	0.08591	0.00000	0.00000	55	0.90140	0.79120	0.01819
105	0.04846	0.00000	0.00000	105	0.29019	0.00000	0.00000
155	0.02313	0.00000	0.00000	155	0.04389	0.00000	0.00000
205	0.04640	0.00000	0.00000	205	0.05195	0.00000	0.00000
255	0.00436	0.00000	0.00000	255	0.01827	0.00000	0.00000
305	0.00050	0.00000	0.00000	305	0.00003	0.00000	0.00000
355	0.00000	0.00000	0.00000	355	0.00003	0.00000	0.00000
405	0.00000	0.00000	0.00000	405	0.00000	0.00000	0.00000
455	0.00000	0.00000	0.00000	455	0.00000	0.00000	0.00000
505	0.00000	0.00000	0.00000	505	0.00000	0.00000	0.00000
	$K=10$				$K=15$		
β	$L=1$	$L=2$	$L=3$	β	$L=1$	$L=2$	$L=3$
5	0.82531	0.66399	0.57620	5	0.94767	0.50089	0.04147
55	0.34511	0.00000	0.00000	55	0.90644	0.00000	0.00000
105	0.59452	0.00000	0.00000	105	0.64302	0.00000	0.00000
155	0.69513	0.00000	0.00000	155	0.81774	0.00000	0.00000
205	0.48258	0.00000	0.00000	205	0.13191	0.00000	0.00000
255	0.00112	0.00000	0.00000	255	0.00903	0.00000	0.00000
305	0.00896	0.00000	0.00000	305	0.01347	0.00000	0.00000
355	0.00102	0.00000	0.00000	355	0.00008	0.00000	0.00000
405	0.00004	0.00000	0.00000	405	0.00000	0.00000	0.00000
455	0.00000	0.00000	0.00000	455	0.00000	0.00000	0.00000
505	0.00000	0.00000	0.00000	505	0.00000	0.00000	0.00000

around the equilibrium fraction of $n_{1,t} = 0.25$. Such a finding is in contrast with the benchmark case in which fundamentalists are driven off the market by chartists more often.

Figure 5 shows estimated densities of $n_{1,t}$ and $n_{4,t}$ for $L = 3$ and $K = 15$, i.e. fractions of PT fundamentalists and non-PT chartists in a model in which one chartist trading strategy does not have the PT feature, for different values of β. Notice that, as β gets larger, the non-PT chartist strategy becomes increasingly popular and dominates the market (i.e. $n_{4,t} = 1$) at non-negligible amount of time. Moreover, fundamentalists are less likely to survive in the market than they are when they face only PT traders—this effect can be best inferred from the case in which $\beta = 105$ (see Figure 4). While for $L = 4$, $p\left(n_{1,t}\right) = 6$ for $n_{1,t} \to 0$ (panel (d) of Figure 4), for $L = 3$ we have $p\left(n_{1,t}\right) = 7.6$ for $n_{1,t} \to 0$ (panel (a) of Figure 5) and the relatively frequent disappearance of fundamentalists can thus be attributed to the presence of non-PT chartists.

6. Stylized facts

We assess the explanatory power of the extended model chiefly with respect to the three stylized facts specified at the beginning of Subsection 5.2, i.e., absence of autocorrelation of returns, fat tails, and volatility clustering. The analysis is based on comparison with real-world market data,

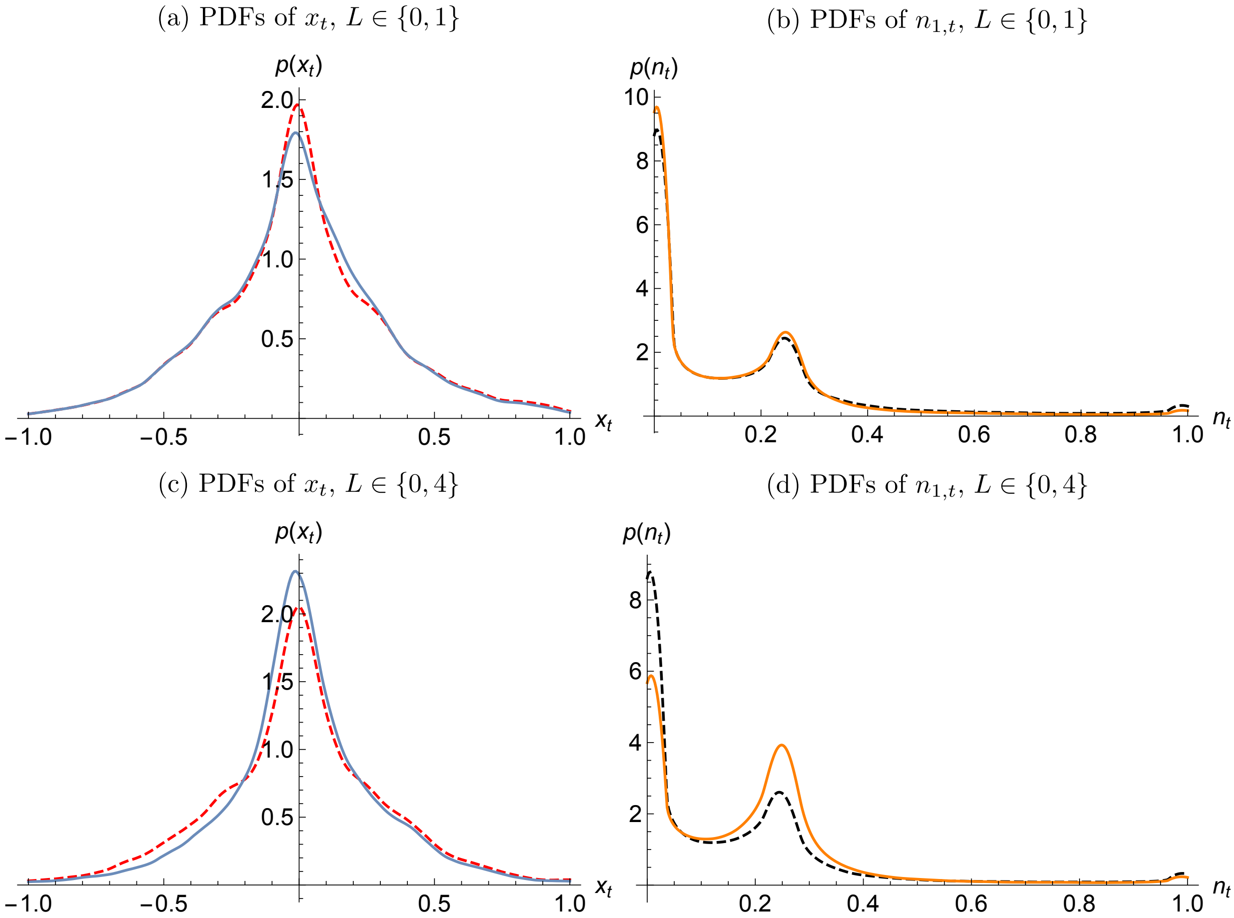

Figure 4: Behavior of the model for different L versus the benchmark case of $L = 0$, $\beta = 105$ and $K = 15$, $L = 0$ depicted by dashed lines.

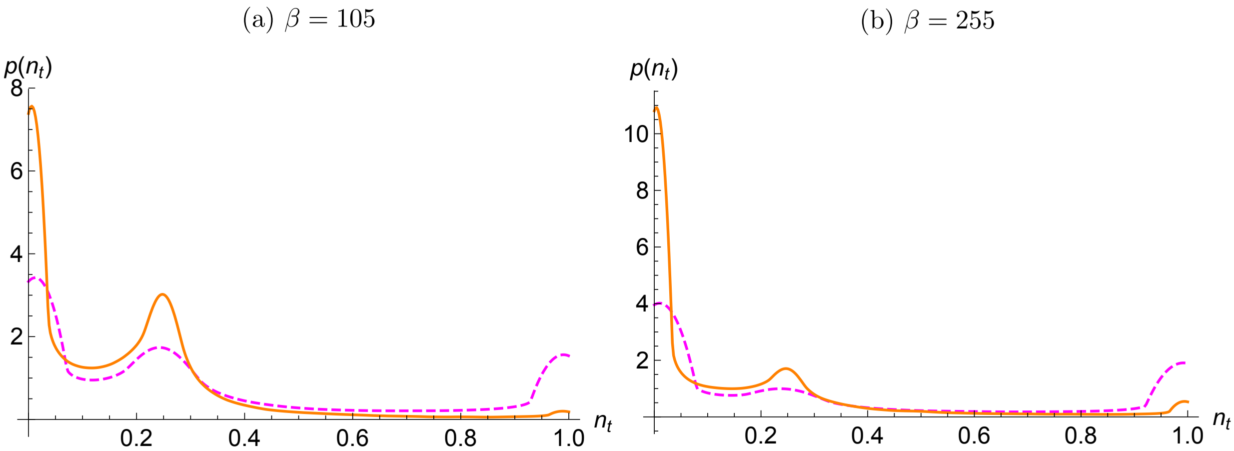

Figure 5: Estimated densities of $n_{1,t}$ (orange solid line) and $n_{4,t}$ (dashed magenta line) for $L = 3$ and $K = 15$.

Table 7: Summary statistics of real-world indices price differences and best-fit time series model.

Index	Mean	Mean*	Var.	Var.*	Skew.	Kurt.	Model
S&P	0.739	0.00052	194	0.00010	-0.336	5.029	AR(1)
FTSE	1.555	0.00027	3240	0.00010	-0.203	4.489	MA(0)
HSI	8.238	0.00039	67835	0.00015	-0.092	4.187	MA(0)
N225	6.706	0.00058	26923	0.00020	-0.555	6.967	AR(2)

namely with four stock market indices—S&P 500 (NY), FTSE 100 (London), HSI (Hong Kong), and Nikkei 225 (Tokyo)—covering the period from January 1, 2009 to May 1, 2015. Contrarily to most research in the field, daily closing price differences r_t are studied as these better mimic the 'deviation' nature of the x_t time series from the HAM.

Table 7 summarizes the most important statistics of the price differences times series of the four indices and the best-fit time series model from the ARMA model family determined by the AIC. The '*' symbol indicates that the particular statistic refers to a standardized time series to establish better mutual comparability among the indices:

$$r_t = \frac{p_t - p_{t-1}}{\bar{p}}, \tag{31}$$

where p_t is (daily) close price of the respective index, and $\bar{p}$ is the arithmetic mean of daily closing prices of the index for the entire period. The time series models in Table 7 are fitted to nonstandardized price differences.

Apparently, two of the indices' price differences time series are best characterized by an autoregressive model. Such a finding is in accordance with the best-fit models from our simulations. On the other hand, FTSE 100 and HSI are best described by a MA(0) process—the time series are essentially white noise processes. Figure 8 reveals that also the HAM simulations produced MA(1) coefficient equal to 0 at non-negligible amount of times, although expected value of this coefficient is higher than 0 (see Table 3). By contrast, the estimated autoregressive coefficients for S&P 500 and Nikkei 225 are, respectively, equal to -0.068 and $\{-0.04, 0.04\}$. Table 3 reveals that neither of our simulations are able to replicate this finding with sufficient accuracy.

Table 8 shows best-fit models for squared price differences along with estimated AR(1) α_1 and MA(1) β_1 coefficients, and arithmetic averages of all autoregressive and moving average coefficients. Clearly, our simulations are able to replicate these findings in terms of the optimal model (consult Table 4)—both non-PT and PT x_t^2 time series are best characterized by the same model family. Table 4 also reveals that the PT extended model α_1 coefficient is—for $\beta = 105$—very close to the HSI estimated α_1 coefficient, and for $\beta = 55$ relatively close to the S&P 500 estimated α_1 coefficient. On the other hand, the moving average component of the real-world indices exhibits considerably lower expected values than our simulated distributions of the β_1 coefficients.

Figure 9 in Appendix D shows, on a log-log scale, tail plots of 10 % of largest absolute price differences of the four aforementioned indices. The estimated tail indices are 4.488 (S&P 500, $R^2 = 0.89$), 4.456 (FTSE 100, $R^2 = 0.961$), 4.241 (HSI, $R^2 = 0.983$), and 3.669 (Nikkei 225, $R^2 = 0.991$). To the eye, the data fit the power law well, and the tails of Nikkei 225 are almost perfect power law fit, as documented by extremely high value of R^2. A comparison with Table 5 shows that, in most cases, the tail indices are lower than those of the simulated x_t time series. However, the same table provides evidence that the PT extension actually moves the model closer to reality in terms of the tail indices' magnitude for most values of β.

Table 8: Best-fit models of real-world indices squared price differences.

Index	Model	α_1	β_1	$\varnothing\alpha$	$\varnothing\beta$
S&P	ARMA(2,1)	0.329	-0.260	0.303	—
FTSE	AR(6)	0.042	—	0.079	—
HSI	ARMA(6,4)	0.182	-0.161	0.097	-0.079
N225	AR(6)	0.104	—	0.075	—

7. Results

Implementation of the PT feature into the model changes the behavior of the model considerably. Nonetheless, some of the key characteristics remain the same as the underlying mathematical structure of model is intact—the generated time series of the deviations from the fundamental price of the asset, x_t, exhibit decreased variance as the intensity of choice parameter β increases, extreme price deviations are less 'extreme' for larger β, and the deviations are still far from being normally distributed. However—and most importantly—the differences are considerable and non-negligible as indicated by very low p-values of the K-W tests as well. The main conclusions arising from the PT extension can be summarized as follows:

1. *Stability.* Probably the most noticeable change evident from the PT extended model simulations is the overall increased stability. Summarized in Table 2, the sample variance is usually 20 % to 30 % lower than in the benchmark case. We remind that the same random seed is used for both versions of the model. The difference in stability can therefore be attributed to the PT extension completely.
2. *Loss aversion matters.* Number of strategies endowed with the PT feature, L, affects the performance of the model significantly. Summarizing the K-W tests for different L, Table 6 shows that if only the fundamentalist strategy is loss-averse (i.e. $L = 1$), the empirical distributions of x_t are statistically different at a reasonable significance level from those obtained from the model with $L = 0$ only for higher values of β. On the other hand, for $L > 1$ the distributions are statistically (and visually) different from those of the benchmark, $L = 0$ case.
3. *Occurrence of fundamentalists is more extreme.* Not as striking as the previous two findings—yet probably of the strongest economic relevance—is the ambiguous status of the fundamentalist strategy. In the original model, fundamentalists are, with increasing β, less likely to survive in the market than they are for low values of β. We find that this phenomenon is even more emphasized for $L = 1$. The 'burden' of loss aversion presents significant hindrance for fundamentalists when they have to face a number of non-PT strategies and the cases in which $n_{1,t} = 0$ are more frequent compared to $L = 0$. On the other hand, for $L = 4$ (i.e. when all strategies are loss-averse), fundamentalists are able to survive in the market more easily. This fact is in contrast with the findings of the original model (see Figure 4).
4. *PT helps better explain some classical financial stylized facts.* The models with proposed PT features replicates the empirical findings more accurately than models without them. The three stylized facts explored are weak autocorrelations of deviations from the fundamental price, volatility clustering, and fat tails. Results of Section 6 indicate that the PT extended model is able to better replicate the latter two facts and slightly worse the first fact. It is however important to emphasize that these differences in replication of empirical findings are not dramatically large.

8. Conclusion

Using a general idea proposed by Shimokawa et al. (2007), we extend the popular Brock and Hommes (1998) agent-based asset pricing model and include the most important features of the PT into the framework, namely the loss aversion with reference point dependence and distorted treatment of gains and losses. The main contribution of this paper is the finding that the original model can be consistently and meaningfully extended with the most relevant features of the PT and—at the same time—its intrinsic 'stylized' structure kept essentially intact. Using Monte Carlo simulations, we find that distributions of the main variables are statistically different from those obtained from the original version of model. Moreover, the extension based on the PT shifts the original framework closer to real-world market dynamics in terms of two of the three stylized empirical facts that the analysis focuses on.

As the Brock and Hommes (1998) model is per se characterized by 'many degrees of freedom' and the extensions bring even more options in this regard, future research might concentrate on exploration of other possible combinations of the parameters. Additionally, the extended model could be estimated using real-world empirical data to reveal the natural values of some parameters, e.g. of degree of loss aversion present in the markets. Other field that could be explored with respect to the extended version of the model is a more in-depth analysis of volatility structure of the x_t time series.

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Appendix A

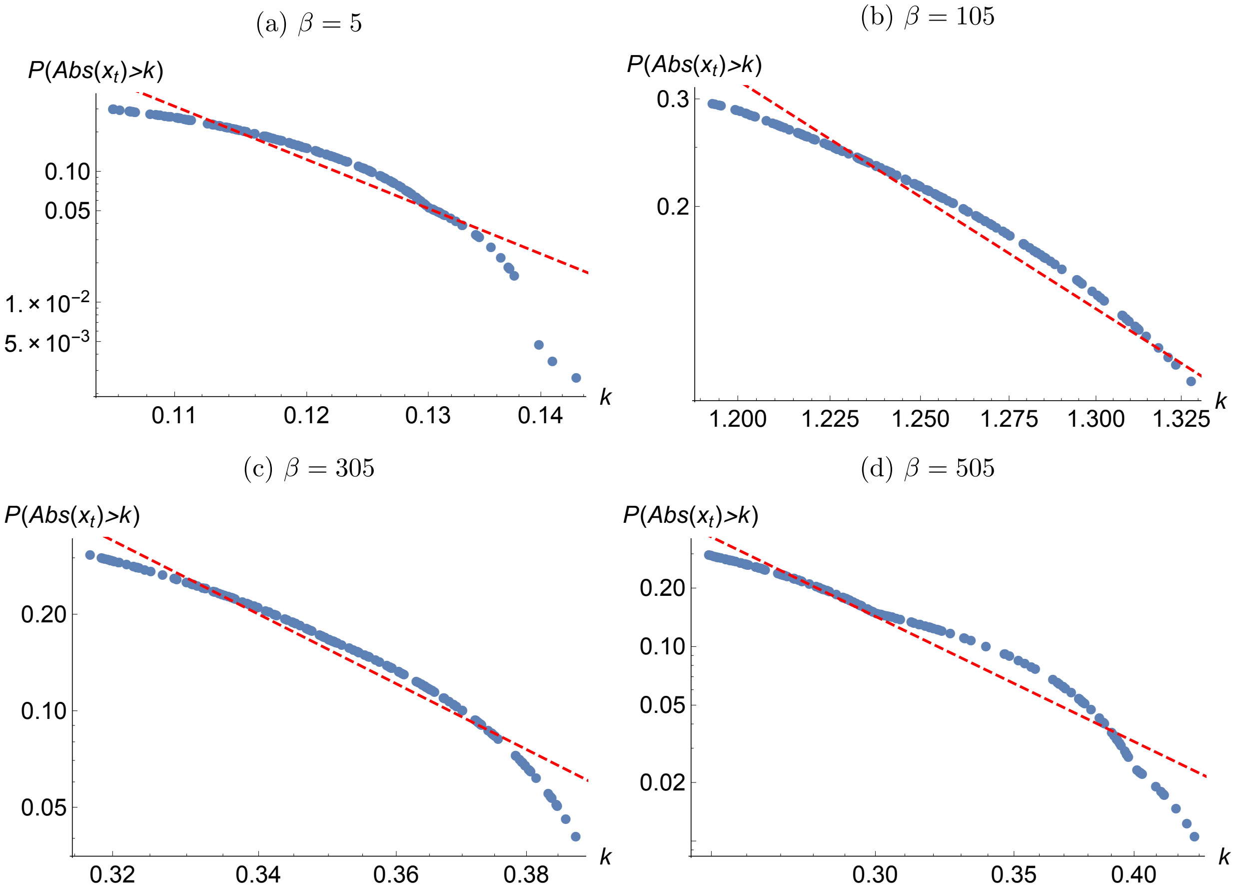

Figure 6: Plots of the tails of sample x_t time series' empirical distributions and OLS fit. Benchmark case without the PT feature.

Appendix B

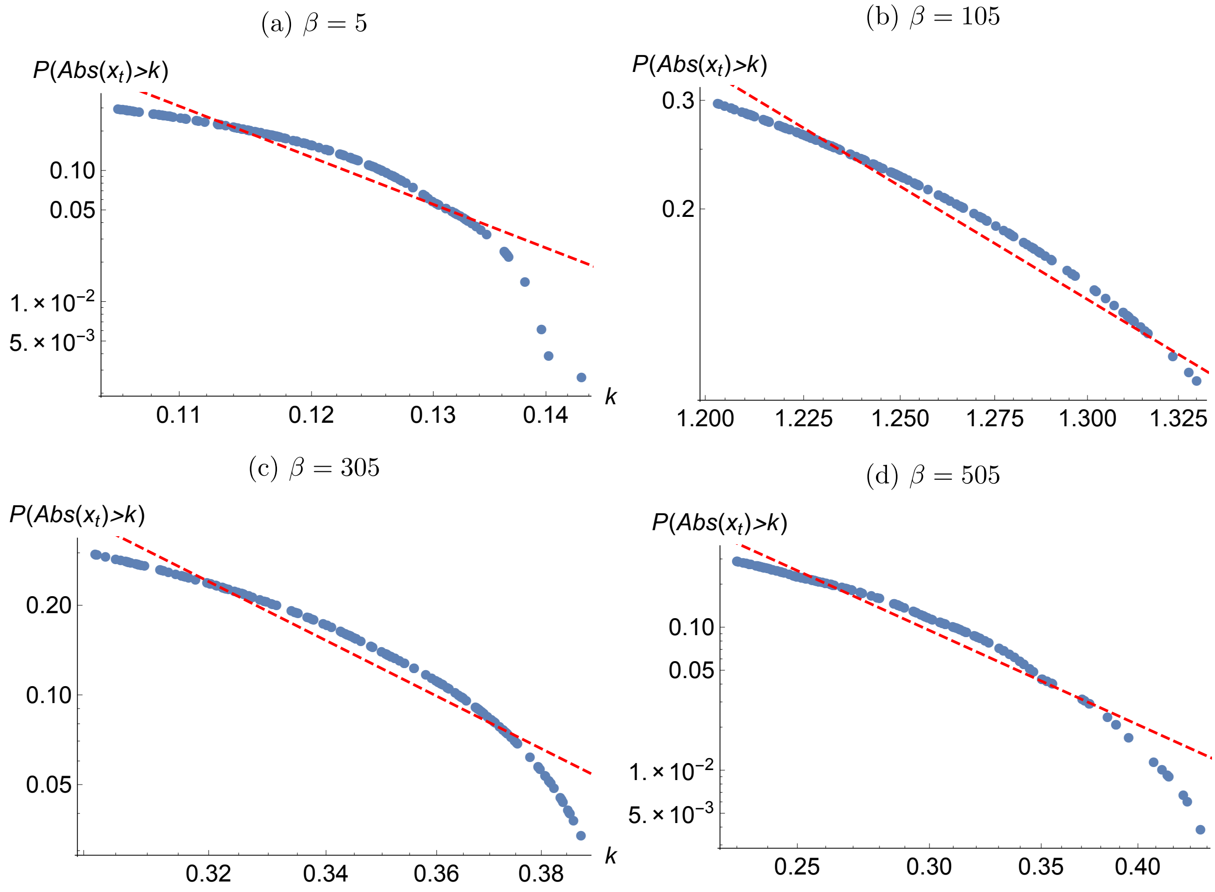

Figure 7: Plots of the tails of sample x_t time series' empirical distributions with the PT feature employed and OLS fit.

Appendix C

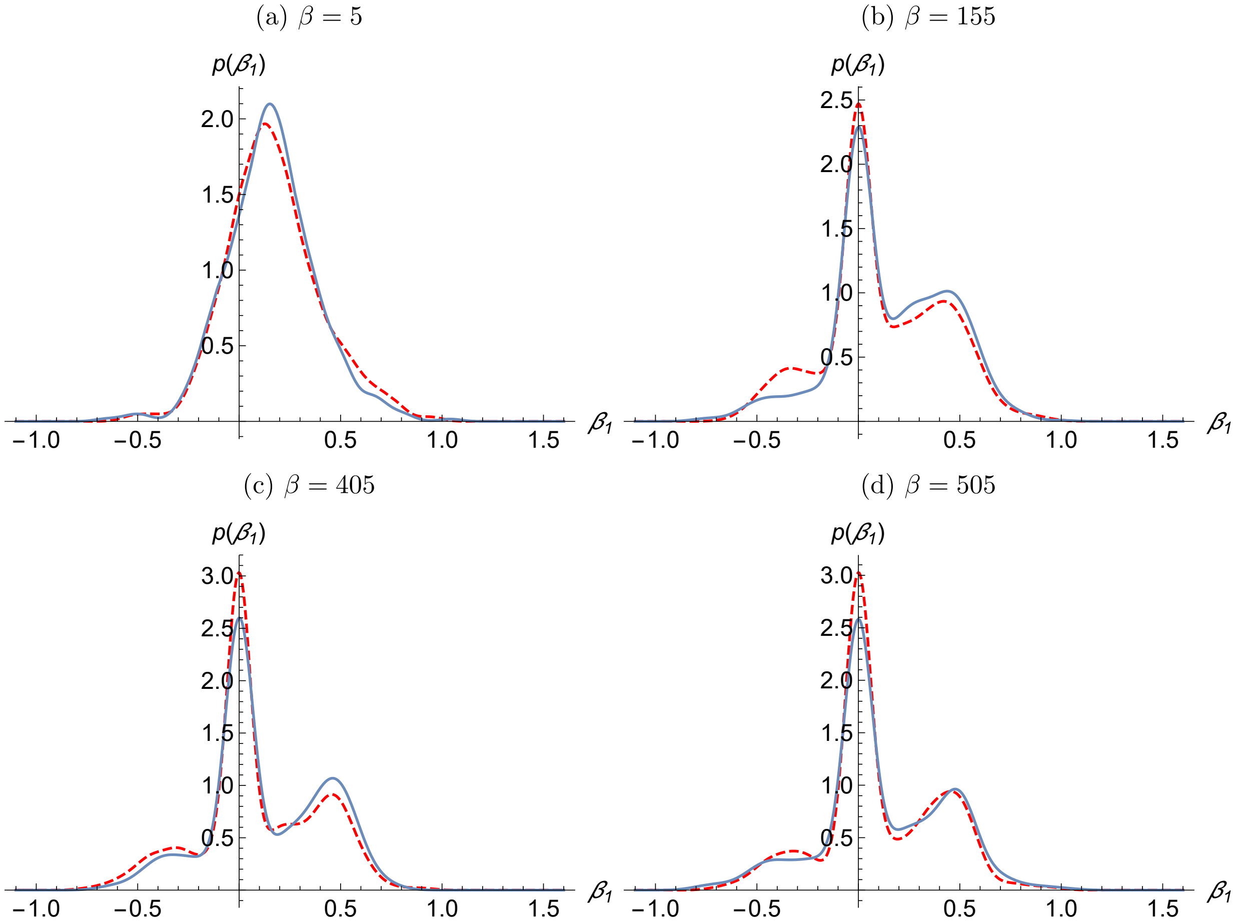

Figure 8: Plots of PDFs of the MA(1) coefficient β_1 of optimal ARMA models fitted to x_t time series with (blue line) and without (red dashed line) the PT feature.

Appendix D

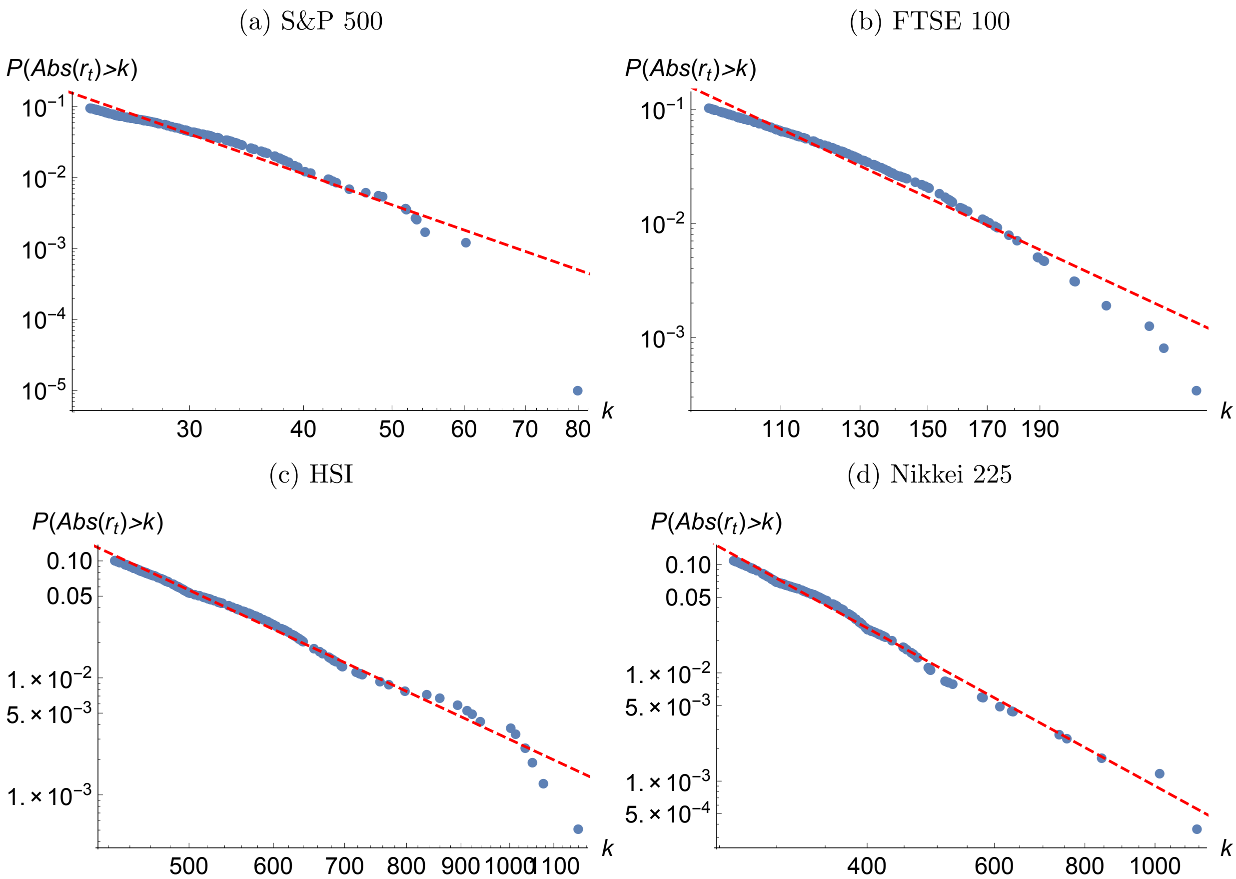

Figure 9: Tails of real stock market indices close price differences distributions; log-log scale; data retrieved form Yahoo Finance.