A tractable model of wealth inequality and mobility

Thomas Fischer
TU Darmstadt
fischer@vwl.tu-darmstadt.de

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This paper presents a rich model of wealth inequality and mobility that, however, is still (partly) analytically tractable. It features finite-life times, aggregate growth, a minimum consumption level, and investment risk (either idiosyncratic or homogeneous) in a continuous time environment. Moreover, from a policy perspective a redistributive bequest tax is discussed. Other models from the literature are identified as special cases. The model can replicate several empirical features including Power-laws in the tail(s) of the wealth distribution. Key results of the model are that idiosyncratic investment risk increases wealth inequality as well as mobility of wealth. A redistributive bequest tax lowers wealth inequality and increases wealth mobility.

VERY PRELIMINARY - PLEASE DO NOT DISSEMINATE

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1 Introduction

The publication of the popular book of *Capital in the 21st century* by Thomas Piketty has revived the interest in the distribution of economic ressources - in particular wealth. In this book, Piketty (2014) documents that the wealth inequality - especially in the USA - after a period of moderation is accelerating since the 1980s. In particular, he documents the share of the top-wealth holders suggesting that it is described by a Pareto-distribution. Piketty (2014) argues that wealth inequality is increasing with the measure $r - g$ with r being the return on capital g is the aggregate growth rate. In order to halt this evolution, he suggests a global tax on capital.

Given the general audience the book addresses, the book does not feature a formal model in order to explain the inequality. In this paper, we present such a model and point to underlying mechanisms shaping the distribution of wealth. While we are able to confirm the role of the $r-g$-mechanism, we also point to other underlying features influencing wealth inequality. In particular, the stochastic nature of the capital income process and demographics play an important role. Piketty (2014) focuses on the role of distribution, i.e. the cross-section of wealth at a given point of time t. However, he also stretches the role of mobility emphasizing that inheritance links the wealth of generations. As (among others) discussed in Piketty and Saez (2013) this link is influenced by a tax on inheritance. Compared to inequality, mobility focuses on a certain individual i and tracks its change in time and thus acts on the opposing dimension as inequality. Piketty (2014) (implicitly) assumes that welfare reducing results - high inequality and low mobility - go hand in hand. In this paper, we show that this does not have to be the case. Moreover, we discuss the tax on bequests influencing the transmission between generations.

The model at hand is able to account for several facts - in particular the fat right tail in the wealth distribution. We identify conditions leading to a stable distribution of wealth, i.e. a non-exploding inequality. Both demographics and taxes on bequests are important in the presence of a minimum consumption level and idiosyncratic risk to guarantee the stable distribution of wealth for a reasonable parametrization of the model. We show that noisy income increases inequality of income as well as its mobility. Thus, there is a trade-off between equality and mobility. A redistributive bequest tax not only decreases inequality of wealth but also increases mobility of wealth by endowing agents with a certain starting capital. In general, the redistributive bequest tax hedges idiosyncratic capital risk and helps low income agents to maintain a minimum consumption level. If these two effects are not prevailing any tax leads to total equality of wealth and thus erases the incentive to save in the first place.

The remainder of this paper is organized as follows. In the following section, we provide a short overview on the literature modeling wealth inequality with a focus on those models allowing for closed form solutions featuring Pareto-tails. Section 3 presents the model employed in this paper also featuring some closed form solutions regarding wealth inequality in a simplified version of the model. In section 4, numerical simulations of the complete model are presented. We show, how, several model features influence the shape of the distribution in section 5. Section 6 discusses the role of the tax on inequality and mobility. The last section concludes and gives an outlook.

2 Literature

In this section, we provide a brief overview of theoretical models aiming at explaining inequality of income and especially wealth.

The models most commonly used to discuss this issue are the so-called *Bewley*-type models.[1] These models discuss the consumption-savings decision in a stochastic environment with many heterogeneous agents. The seminal contributions in this context are Huggett (1993) - in which the storing technology of savings are risk-free assets - and Aiyagari (1994) - in which agents use productive capital modeled by a Cobb-Douglas production function as the savings technology. A very recent and compeling survey of the literature is given in DeNardi (2015). These models, however, fail at accounting for the top-tail inequality.[2]

Another major issue with these type of models is that they require numerical solutions. In contrast to that, Wang (2003) and Wang (2007) presents closed form solutions to the model. This, however, requires some non-standard assumptions; especially a utility function of the Constant Absolute Risk Aversion (CARA) type rather than Constant Relative Risk Aversion.[3] The model, however, provides some counterfactual results such as income inequality being higher than wealth inequality.

There is another literature that explicitly tries to account for the Pareto-tail in the wealth distribution. The latter phenomenon is well-known since the inquiries of Vilfredo Pareto in the late 19th century.[4] The latter has some interesting properties. The probability density function (pdf) is given by:

$$f(X) \sim X^{-(a+1)} \tag{1}$$

with $a > 1$ being the Power-law exponent, for which smaller values indicate fatter tails and thereby more unequal distributions. The cumulated probability density function is given by:

$$1 - Prob(X_i \leq X) = 1 - P(X) = \left(\frac{X_{min}}{X}\right)^a. \tag{2}$$

The empirical literature confirms a value of approx. $a = 2$ for the income distribution in the USA (Jones and Kim, 2014). Wealth - being more unequally distributed - has a coefficient of $a \approx 1.5$ (Saez and Zucman, 2014). The Pareto-distribution also has the nice property that the Gini is given as $\frac{1}{2a-1}$. For the special case of $a = 1$ this implies a Gini of 1 indicating maximum inequality.[5] The share of quantile x [in percent] as emphasized e.g. in the work of Atkinson et al. (2011) can be computed by $\left(\frac{x}{100}\right)^{1-\frac{1}{a}}$

[1]This labeling refers to the seminal work of Bewley (1977) and was introduced in the textbook of Ljungqvist and Sargent (2012).

[2]A notable exception is Castaneda et al. (2003) that modifies the exogenous labor income shocks by including a *superstar*-shock, which is of low probability yet of high magnitude.

[3]Technically - and as already shown in Caballero (1990) - this assumption allows for closed form optimal consumption rules for the additive process of stochastic labor income. In contrast to that the CRRA preference - e.g. used in Benhabib et al. (2011) and also more common in general - allows for closed form solutions with stochastic multiplicative processes, i.e. capital income. The CARA assumption is unrealistic as - among other things - it implies that wealthy agents hold a less risky portfolio (Merton, 1971).

[4]More technically, it sometimes is also referred to as the Power-law or Power Function distribution (Evans et al., 2000) due to its mathematical formulation using the exponentiation operator.

[5]The case of $a = 1$ is sometimes also referred to as *Zipf's law* and (amongst others) is widely documented for the size distribution of cities.

(Jones and Kim, 2014). Moreover, the level of a also determines the (finite) number of moments. For a Pareto-distribution the largest finite moment M_i exists for $i < a$, i.e. for the case of $1 < a < 2$ - as documented for the wealth distribution - there is only a finite mean. All other moments (variance, skewness, and all other higher moments) do not exist.

There is some literature trying to find theoretical mechanisms to fit this type of distribution. A compelling survey on these issues is provided in Gabaix (2009).[6] We only review the literature very briefly; however, will provide more precise links when discussing our model and presenting nested cases in section 5. In order to guarantee a stable Pareto-distribution a factor increasing inequality (influx) has to be offset by a factor decreasing inequality (outflux). In an early contribution by Wold and Whittle (1957) inequality is increased due to positive growth of wealth and halted by the death of the agents. Technically, these two factors are captured in denominator (influx) and the numerator (outflux) of the Pareto-coefficient.

The research on these issues has regained momentum with the seminal contribution of Benhabib et al. (2011) that shows that the Pareto-tail for wealth is created by the multiplicative nature of the stochastic capital income process (rather than the additive nature as witnessed for labor income). In this case, the influx increasing inequality is driven by the stochastic return structure and the stabilizing outflow is determined by a negative wealth growth. In contrast to Wold and Whittle (1957), this model also features a microfoundation for the wealth growth process. The authors expand this model with a perpetual youth framework in Benhabib et al. (2014) and incorporate it in a more standard Bewley type framework in Benhabib et al. (2015). Jones (2015)[7] discusses these types of models in the relationship to Piketty (2014). Nirei and Aoki (2015) document the presence of Pareto-tails in a Solow-type growth model. In a series of papers Benjamin Moll and co-authors (Gabaix et al. (2015), Moll et al. (2015), Nuño and Moll (2015)) address several problems of heterogeneous agent models in continuous time featuring Pareto-tails. In Fernholz and Fernholz (2014) and Fernholz (2015) methods well-established in the literature of continuous time finance are employed to address the issues of wealth inequality in particular focusing on the role of idiosyncratic risk.

Our model incorporating several features of the aforementioned models will be presented in the following section.

3 The model

In this section we present the underlying model. We employ a continuous time approach as it allows for easy to interpret closed form solutions. In fact, the model strongly

[6]Note that Power-laws are found in other context such as firm size distributions, stock market returns, or city size distributions. Moreover, they are considered an ubiquitouos phenomenon in natural sciences (e.g. Sornette and Cont (1997)).

[7]The interested reader is especially referred to the very insightful (technical) online appendix.

follows the model presented in Merton (1971).[8] In particular, he jointly derives optimal savings/consumption rules as well as portfolios.

Assume that there are two assets a risk-free asset yielding a certain return r and risky asset whose prices P_t follow a geometric Brownian motion:[9]

$$dP_t = \mu P_t dt + \sigma P_t dZ_t, \tag{3}$$

with $\mu > r$ as a risk compensation and Z_t being a Wiener process. Agents hold a share α in the risky asset.[10] The aggregate portfolio return thus is given by:

$$E(R) = \alpha\mu + (1-\alpha)r = r + \alpha(\mu - r). \tag{4}$$

Thus, the constraint is given as follows:

$$dW_t = [rW_t + \alpha(\mu - r)W_t + Y_t - C_t]dt + \alpha\sigma W_t dZ_t, \tag{5}$$

for which Y_t signifies the current labor income. We assume that labor income follows an exponential growth process:

$$Y_t = Y_0 \exp(gt), \tag{6}$$

with a given exogeneous rate g.[11]

Moreover, we assume that agents have a death probability p governed by a Poisson process.[12] in the tradition of Blanchard (1985). We normalize all variables by the growth rate g and write with small case letters:

$$x(t) \equiv X(t)\exp(-gt) \leftrightarrow X(t) \equiv x(t)\exp(gt). \tag{7}$$

Thus, the constraint finally reads as follows:[13]

$$dw_t = [(r + p - g)w_t + \alpha(\mu - r)w_t + y_t - c_t]dt + \alpha\sigma w_t dZ_t. \tag{8}$$

[8] This work and some of his other paper dealing with continuous time finance are reproduced within Merton (1992).

[9] Thus, the distribution of prices is described by a log-normal distribution and the distribution of returns follows a normal distribution.

[10] Note that if agents can borrow at the risk-free rate, $\alpha > 1$ implies that agents hold leveraged (i.e. debt-financed) positions in risky assets, whereas $\alpha < 0$ implies that they have short positions in the risky asset.

[11] Note that for analytical convenience we exclude the case of uninsurable labor income risk also contrasting the *Bewley*-type literature that features the latter as the main driving mechanism. In an extension we strive to include labor income risk (i.e. unemployment) that can be only be partly insured. One might consider the model at hand as a model with totally insurable labor income risk (hence, *Bewley*-type models are models with completely insurable capital income risk).

[12] The *events* - in this case *death* - arrive with a given probability p and are independent of previous events. The converging distribution for the discrete case is labelled Poisson-distribution, whereas in continious time it is called the exponential distribution. As a result, the distribution of ages a is governed by $f(a) = p\exp(-pa)$. This function, however, has the unrealistic feature of the mode being zero. The probability density function is continiously decreasing rather being inverse u-shaped as witnessed in the data.

[13] Note that $y = Y_0$ following from equation 6.

The objective function is given by:

$$\max_{C(t)} \int_0^\infty \exp(-(\rho+p)t)U(C(t))dt = \max_{c(t)} \int_0^\infty \exp(((1-\gamma)g-\rho-p)t)U(c(t))dt. \quad (9)$$

As a utility function, we assume:

$$U(c) = \frac{(c-\bar{c})^{1-\gamma}}{1-\gamma}, \quad (10)$$

for which $\bar{c}$ represents a minimum consumption level.[14] This utility function is of the Hyperbolic Absolute Risk Aversion (HARA) type which - as shown in Merton (1971) - provides for a closed-form solution of the problem. For the case of no minimum consumption requirement ($\bar{c}=0$) the standard CRRA function is nested.

The Hamilton Jacobi Bellman (HJB)-equation is given by:

$$(\rho+p-(1-\gamma)g)V = \max_{c,\alpha}\{U+V'((r+p-g)w+\alpha(\mu-r)w+y-c)+0.5\sigma^2\alpha^2w^2V''\}. \quad (11)$$

The first order condition for consumption c is:

$$U' = (c-\bar{c})^{-\gamma} = V', \quad (12)$$

whereas the first-order condition for the optimal portfolio is:

$$\alpha = -\frac{\mu-r}{\sigma^2}\frac{V'}{wV''}. \quad (13)$$

As usual, we have to guess a value function. Our educated guess is:

$$V = \frac{K_1(w+K_2(y-\bar{c}))^{1-\gamma}}{1-\gamma}, \quad (14)$$

for some parameters K_1 and K_2 to be determined. For the first order conditions this implies:

$$c = \bar{c} + K_1^{-1/\gamma}(w+K_2(y-\bar{c})) = \bar{c} + K_1^{-1/\gamma}x, \quad (15)$$

and

$$\alpha w = \frac{(\mu-r)}{\gamma\sigma^2}(w+K_2(y-\bar{c})) = \frac{(\mu-r)}{\gamma\sigma^2}x. \quad (16)$$

for which $x \equiv w+K_2(y-\bar{c})$. Inserting this result into the HJB leads to:

$$\begin{aligned} (\rho+p-(1-\gamma)g)\frac{K_1x^{1-\gamma}}{1-\gamma} &= K_1^{-1/\gamma}\frac{K_1x^{1-\gamma}}{1-\gamma} \\ +K_1x^{-\gamma}\left([r+p-g]w+\frac{(\mu-r)^2}{\gamma\sigma^2}x-K_1^{-1/\gamma}x-\bar{c}+y\right) &- K_1x^{1-\gamma}\frac{(\mu-r)^2 0.5}{\gamma\sigma^2}. \end{aligned} \quad (17)$$

[14] Note that as c is normalized so is the minimum consumption level $\bar{c}$ implying that the minimum consumption level grows with the aggregate growth rate g.

It is easy to see that $K_2 = \frac{1}{r-g+p}$. Inserting this result and performing some algebraic manipulations leads to:

$$K_1 x^{1-\gamma}\left([(1-\gamma)g-\rho-p]+\gamma K_1^{-1/\gamma}+(1-\gamma)(r+p-g)+(1-\gamma)\frac{0.5(\mu-r)^2}{\gamma\sigma^2}\right)=0, \tag{18}$$

for which we can solve:

$$K_1^{-1/\gamma}=\frac{1}{\gamma}\left(\gamma p+\rho-(1-\gamma)r-(1-\gamma)\frac{(\mu-r)^2 0.5}{\gamma\sigma^2}\right)=p+\frac{\rho-(1-\gamma)r}{\gamma}-0.5\frac{1-\gamma}{\gamma}\frac{(\mu-r)^2}{\gamma\sigma^2}. \tag{19}$$

The latter values is also the marginal propensity to consume out of wealth. In fact, the optimal consumption function looks as follows:

$$c(t)=\bar{c}+\frac{1}{\gamma}\left(\gamma p+\rho-(1-\gamma)r-(1-\gamma)\frac{(\mu-r)^2 0.5}{\gamma\sigma^2}\right)\left(w(t)+\frac{y-\bar{c}}{r+p-g}\right), \tag{20}$$

while the optimal portfolio composition is given by:

$$\alpha(t)=\frac{\mu-r}{\gamma\sigma^2}\frac{w(t)+\frac{y-\bar{c}}{r+p-g}}{w(t)}. \tag{21}$$

We can rewrite the consumption function in the following manner:

$$c(t)=c_w(w(t)+h)+(1-c_wK_2)\bar{c}=c_w w(t)+c_y y+(1-c_y)\bar{c}=c_w w(t)+c_y(y-\bar{c})+\bar{c}, \tag{22}$$

with $c_w \equiv K_1^{-1/\gamma}$ (as given by equation 19), $K_2 = \frac{1}{r+p-g}$, $c_y = K_2 c_w$, and $h = \frac{y}{r+p-g}$ being the human capital.

These results have some important economic insights. First of all, the presence of both a minimum consumption level $\bar{c} \neq 0$ and labor income $y \neq 0$ implies that the optimal portfolio structure varies in time depending on the time variation of $w(t)$. Agents with a large labor income y - which in this setting is considered to be risk-free - should hold a larger share in risky assets. On the other hand, a larger value of minimum consumption $\bar{c}$ that has to be satisfied at any period requires a lower α and thus a less riskier portfolio composition. For the case of no labor income and no minimum consumption ($\bar{c} = y = 0$) the share of risky assets is constant in time. For heterogeneous labor income ($y_i \neq y_j$ for $i \neq j$) heterogeneous agents also hold heterogeneous portfolios. Those agents with a larger labor income relative to wealth also hold a riskier portfolio and thus exhibit higher growth of wealth.

We can also intensively discuss the consumption function. First of all, consumption grows at the same pace as labor income:

$$C(t)=c(t)\exp(gt). \tag{23}$$

Agents that have no income and no wealth still require a minimum *subsistence* level of consumption:

$$c(w=y=0)=(1-c_wK_2)\bar{c}=\frac{r-\rho-\gamma g+(1-\gamma)\frac{(\mu-r)^2 0.5}{\gamma\sigma^2}}{\gamma(r+p-g)}\bar{c}. \tag{24}$$

Following Deaton (1991), we argue that agents cannot borrow and are subject to a cash-on-hand constraint - i.e. they can spend their current income and their current wealth at most. A key point is that (in the absence of slavery) - in contrast to physical wealth $w(t)$ - human capital h cannot be sold as a bulk on the market. The latter implies a kink in the consumption function and thus a concave consumption function. Formally, it can be written as:

$$c(t) = \min\{w(t) + y; c_w(w(t) + h) + (1 - c_w K_2)\bar{c}\}. \tag{25}$$

The constraint is binding for $\bar{c} > y$ and:[15]

$$w(t) < (\bar{c} - y)\frac{1 - c_y}{1 - c_w} > (\bar{c} - y) > 0. \tag{26}$$

This is also an interesting insight as the latter case would never prevail if all agents earn an income above the minimum consumption level $y > \bar{c}$.[16]

Due to concaveness of the consumption function the linear link between wealth and consumption is broken. As a result, consumption inequality is lower than wealth inequality as documented by empirical evidence (Krueger and Perri, 2006).[17] Usually, the concavity of the consumption function is achieved by requiring a borrowing limit tighter than the natural borrowing limit (being the human capital) (Moll et al., 2015):

$$D_{max,i} = -w_i = \Phi_i > \frac{-y_i}{r + p - g} = -h_i < 0. \tag{27}$$

In Deaton (1992) no borrowing is allowed and the cash constraint is binding due to $\rho > r - \gamma g$ (in a scenario with CRRA preferences), i.e. a high consumption preference that would require the individual to incur debt. In the model at hand, this *love of consumption* exceeding current cash on hand is not due to time preferences but due to the presence of a minimum subsistence level of consumption.

In equation 25, we presume that agents cannot borrow at all. The introduction of borrowing accounting for negative net worth in the low end of the wealth distribution (cf. e.g. Castaneda et al. (2003)) is a natural extension to this model.

Human capital (in normalized terms) is given by $K_2 y$. It is also interesting to discuss the term $K_2 = \frac{1}{r+p-g}$. For the case of infinitely living agents ($p = 0$) the nominator well-known from the Gordon-growth model emerges. The probability of dying $p > 0$ raises the chance of not being able to receive future streams of labor income and therefore reduces the level of human capital. Note that the expected (yet uncertain) life-time is given by $T = \frac{1}{p}$. If agents know that they live for T periods with certainty, their human capital is given by:

$$H(t) = \int_t^T Y_0 \exp([g - r]\tau) d\tau = \frac{Y_0}{g - r}(\exp([g - r]T) - \exp([g - r]t), \tag{28}$$

[15]Formally, this follows from the condition of $y + w(t) < c(t)$ for which the rhs is given by equation 22.

[16]Note that the minimum consumption level also grows exponentially with the rate g.

[17]As we will show in section 4 and 5 consumption inequality and wealth inequality in our case converge to a similar high level. The underlying rationale is that we have $g_w > 0$, thus wealth as input into the consumption function strictly dominates labor income and thereby also drives consumption inequality.

which for the case of $t = 0$ yields:

$$h(0) = \frac{H(0)}{\exp(gT)} = \frac{Y_0}{r-g}(1 - \exp(-rT)). \tag{29}$$

Consider a calibration with a normalized $Y_0 = 1$ and $r = 6\%$ and $g = 1\%$ and an expected (work) life time of $T = 40 \equiv \frac{1}{p}$ years.[18] For the certainty case we have $h \approx 18.19$ whereas for uncertain death we have $h \approx 13.33$ which is way lower due to the probability of dying in the mean time. We can also define a marginal propensity to consume out of the flow variable income:

$$c_y = c_w K_2 = \frac{c_w}{r+p-g} > c_w, \tag{30}$$

which is larger than out of the stock wealth for reasonable values.[19]

We can insert the optimal savings/consumption rule and portfolio composition into the the flow equation:

$$dw_t = [(r+p-g)w_t + \alpha(\mu - r) + y - c_t]dt + \alpha\sigma w_t dZ. \tag{31}$$

Following from the solution of the HJB we know that:

$$\alpha = \frac{(\mu - r)}{\gamma\sigma^2}\frac{w+z}{w}, \tag{32}$$

with $z \equiv y - \bar{c}$ and:

$$c = c_w w + \bar{c} + c_y(y - \bar{c}), \tag{33}$$

which we can reinsert into the flow equation resulting in:

$$\begin{aligned} dw_t &= [(r+p-g-c_w)w_t + \frac{(\mu-r)^2}{\sigma^2\gamma}(w+z) + y(1-c_y) - (1-c_y)\bar{c} + \frac{\mu-r}{\sigma\gamma}(w+z)dZ \\ &= \left[\left(r+p-g-c_w + \frac{(\mu-r)^2}{\sigma^2\gamma}\right)w_t + \left(1 - c_y + \frac{(\mu-r)^2}{\sigma^2\gamma}\right)z_t\right]dt + \frac{\mu-r}{\sigma\gamma}(w_t+z_t)dZ. \end{aligned} \tag{34}$$

We can rewrite the statement as:

$$dw_t = [g_w w_t + g_z z_t]dt + \sigma_z x_t dZ + \sigma_w w_t dZ, \tag{35}$$

for which:

$$\sigma_w \frac{\mu - r}{\sigma\gamma}, \tag{36}$$

as well as:

$$\sigma_z = \frac{K_2(\mu - r)}{\sigma\gamma} \equiv K_2\sigma_w, \tag{37}$$

[18] Note that these are the same values applied in the simulation of the model and presented in table 4.
[19] Formally, it requires $r + p - g < 1 \leftrightarrow K_2 > 1$.

and:

$$g_w = \frac{r-\rho}{\gamma} - g + \frac{(\mu-r)^2}{\gamma\sigma^2}\frac{1+\gamma}{2\gamma}, \tag{38}$$

respectively:

$$\begin{aligned} g_z = 1 - c_y + \frac{(\mu-r)^2}{\sigma^2\gamma} &= 1 - c_w K_2 + \frac{(\mu-r)^2}{\gamma\sigma^2} \\ &= \frac{r-\rho-\gamma g + (1-\gamma)0.5\frac{(\mu-r)^2}{\gamma\sigma^2}}{\gamma(r+p-g)} + \frac{(\mu-r)^2}{\gamma\sigma^2} \\ = \frac{r-\rho-g\gamma}{\gamma(r+p-g)} &+ \frac{(\mu-r)^2}{\gamma\sigma^2}\frac{\gamma(r+p-g)+0.5(1-\gamma)}{\gamma(r+p-g)}. \end{aligned} \tag{39}$$

For $y-\bar{c} \equiv x = 0$ with inifinite lives ($p = 0$) we have the nested case intensively discussed in Merton (1971):

$$dw = [g_w w]dt + \sigma_w w dZ, \tag{40}$$

which using Itô's lemma for the log-transformation $\tilde{w} = \log(w)$ yields:

$$d\tilde{w} = [(g_w - 0.5\sigma_w^2)\tilde{w}]dt + \sigma_w dZ. \tag{41}$$

In this case the Pareto-tail can be computed as:

$$a = \frac{0.5\sigma_w^2 - g_w}{0.5\sigma_w^2} = \frac{-2g_w + \sigma_w^2}{\sigma_w^2}, \tag{42}$$

which using the micro-founded drift-term g_w and the diffusion-term σ_w computed above leads to [20]

$$a = 1 - \frac{2g_w}{\sigma_w^2} = 1 - \frac{2\sigma^2\gamma(g\gamma-(r-\rho)) - 2(0.5\gamma+0.5)(\mu-r)^2}{(\mu-r)^2} = \gamma\left(\frac{2\gamma\sigma^2(g\gamma-r+\rho)}{(\mu-r)^2} - 1\right). \tag{43}$$

The necessary condition in this case is $\rho > r - \gamma g$. This condition implies $g_w < 0$ and also $c_y > 1$. We will discuss this more thoroughly in section 4 when we calibrate the model with concrete values.

The comparative statics discussion reveals that inequality decreases (a is increased) with $\rho - r + \gamma g$. As already pointed out the in Fischer (2015a) with log-preferences ($\gamma = 1$), and assuming no-time preferences ($\rho = 0$), the marginal propensity to consume is zero $c_w = 0$ and thus the argument of $r - g$ as driver of inequality is confirmed. Note that - given the nature of the strong assumptions required - this by no means the only driver of inequality. Interestingly, inequality decreases for high financial volatility σ at odds with the argument presented in section 2 of stochastic returns being a driver of inequality. Note, however, that in the presence of a risk-free savings option higher

[20] This result is also derived in Moll et al. (2015). The authors emphasize that this result holds for high income individuals that are not subject to the borrowing constraint.

volatility σ decreases the share of risky assets and thus lowers the volatility of the portfolio. In fact agents inernalize risk, by lowering there exposure to the risk asset α for higher underlying risk σ. In a model with only risky assets as storing technology (following e.g. the tradition of Levhari and Srinivasan (1969)) higher volatility increases inequality.

Now, we introduce finite life-times ($p \neq 0$). To compute the stationary distribution the Fokker-Planck (FP) equation has to be solved.[21] The FP-equation for this case is:

$$\frac{\partial f(w,t)}{\partial t} = -\frac{\partial}{\partial w}[f(w,t)\mu] + \frac{\partial^2}{\partial w^2}[Df(w,t)] - pf. \tag{44}$$

We guess and verify a stationary distribution of the Pareto-type $f(w) = w^{-(a+1)}$ yielding:

$$0 = -\frac{\partial}{\partial w}(g_w w w^{-(a+1)}) + \frac{\partial^2}{\partial w^2}[0.5\sigma_w^2 w^2 w^{-(a+1)}] - pw^{-(a+1)}. \tag{45}$$

The characteristic equation is a quadratic equation. Solving this equation leads to:[22]

$$0 = w^{-(a+1)}(ag_w - (1-a)a0.5\sigma_w^2 - p) \leftrightarrow w^{-(a+1)}\left(a^2 + \frac{g_w - 0.5\sigma_w^2}{0.5\sigma_w^2}a - \frac{2p}{\sigma_w^2}\right) = 0. \tag{46}$$

The quadratic equation has two characteristic roots describing a double-Pareto distribution with both a fat left and a fat right tail (Gabaix, 2009, p. 16f.). The respective values are given by:

$$a_{1,2} = 0.5 - \frac{g_w}{\sigma_w^2} \pm \sqrt{\left(0.5 - \frac{g_w}{\sigma_w^2}\right)^2 + \frac{2p}{\sigma_w^2}}, \tag{47}$$

for $a_1 > 0 > a_2$. More generally, we can write:

$$a_{1,2} = A_1 \pm \sqrt{A_1^2 + A_2}, \tag{48}$$

with

$$A_1 = 0.5 - \frac{g_w}{\sigma_w^2} = 0.5 - \frac{\frac{r-\rho}{\gamma} - g + \frac{(\mu-r)^2}{\gamma\sigma^2}(0.5\gamma+1)}{\frac{(\mu-r)^2}{\gamma^2\sigma^2}} = \frac{2\gamma\sigma^2(g\gamma+\rho-r)}{(\mu-r)^2} - 0.5\gamma, \tag{49}$$

and:

$$A_2 = \frac{2p}{\sigma_w^2} = \frac{2p\sigma^2\gamma^2}{(\mu-r)^2} \tag{50}$$

The distribution is given by:

$$f(w) = \begin{cases} K\left(\frac{w}{w*}\right)^{-(a_1+1)} & w > w* \\ K\left(\frac{w}{w*}\right)^{-(a_2+1)} & w < w* \end{cases}. \tag{51}$$

[21]This equation is also frequently referred to as the Kolmogorov-Forward equation. These term can, however, be used interchangeably.

[22]This solution draws heavily on the results of Gabaix (2009).

The rationale for the double-Pareto-distribution dates back to Reed (2001) arguing that the latter is the effect of the heterogeneous age structure of agents. Note that this model is highly comparable to the model discussed in Benhabib et al. (2014). This model furthermore features a capital income tax, which is completely redistributed to all new born agents. This is similar to the bequest tax we introduce in section 4, whose proceeds are redistributed among all new born individuals.[23] In contrast to the capital income tax in Benhabib et al. (2014) collecting at any time, the bequest is only subject to taxation at the time of death.

We can also discuss the nested cases. First, consider the case without the overlapping generations structure ($p = 0$). For this case there is only one characteristic root given by:

$$a = 2A_1 = \frac{\sigma_w^2 - 2g_w}{\sigma_w^2} = 1 - \frac{2g_w}{\sigma_w^2}, \tag{52}$$

echoing the earlier result. This could be considered the case where all agents life infinitely and thus are of the same age.

Opposed to that for the case without risky investments ($\sigma_w = 0$) the Pareto-coefficient is given by:

$$a = \frac{p}{g_w}, \tag{53}$$

as e.g. presented in Nuño and Moll (2015) and Wold and Whittle (1957). This also has some intriguing economic insight. Decreasing mortality p (longer life length T) increases the wealth inequality. Thus the advances in medicine which lead to a remarkable increase of life length can also be a factor explaining increased wealth inequality.

For yet another special case of exploding volatility $\sigma_w \to \infty$, we have Zipfs-law as $a_1 = 1$ and $a_2 = 0$ for which the mode of the wealth distribution emerges for $w = 0$.

So far we presented a very general model. In the following section, we will present numerical simulations of the model. In order to perform the simulations we need to specify several process (in particular governing the evolution of labor income) and also define concrete parameter values.

4 Exemplary Monte-Carlo simulations

The aim of this paper is to account for several distributional properties. The model features an Overlapping Generations[24] structure, and risky financial markets. The key variables of the model are income (labor and financial), consumption, and wealth. We can investigate the cross-sectional distribution and its change in time (mobility) for each of these variables. The model presented so far argues with continuous values (both time and agents as in the FP equation). Any numerical simulation is conducted with discrete values (or a discrete grid). In the following we lay out our concrete simulation procedure.

[23] They emphasize the role of the taxation acting as a reflecting barrier in order to gurantee a stationary distribution.

[24] Note that the model only features working agents and not pensioners that only receive capital income and transfers.

Moreover, we display the modeling of income inequality and the modeling of a bequest tax as a policy variable. In the model we have dynasties with an index $i = 1; \cdots ; N$. Each economic measure also has a time dimension $t = 1; \cdots ; t_{max}$.

First of all, we have to make some initial assumptions and assumption about the labor income which we - following the Bewley type-literature - assume to follow an exogeneous process.

An initial generation at time $t = 0$ of N of agents (which will become dynasties with demographic changes) with the identical age $a = 1$ is introduced.[25]

Each agent i is endowed with a skill level $S_i > 0$ that does not change during his life-time. Each skill unit is compensated by a skill-based wage which is assumed to grow at an exogeneous exponential rate g producing an aggregate and individual labor income growth at a rate of g. There is no labor supply decision. As a result the distribution of skills and labor income can be used interchangeably in our model. Each time an agent dies, she is replaced by another agent of her dynasty making the aggregate number of individuals constant. The skill of the new agent is given by the following AR(1) process:

$$\log(S_{i,t}) \equiv s_{i,t} = \rho_y s_{i,t-1} + a_t, \tag{54}$$

for which $a_t \sim N(0, \sigma_y)$. Given this process it is well-known that the cross-section of skill follows a log-normal distribution with a standard deviation of:

$$\sigma_s = \frac{\sigma_y}{\sqrt{1 - \rho_y^2}}. \tag{55}$$

Following the established method of Aiyagari (1994) we assume $\sigma_y^2 = \sigma_a^2(1 - \rho_y^2)$ making $\sigma_s = \sigma_a$. For low values of σ_a the Gini coefficient (using a first-order Taylor approximation) is given by:

$$Gini(y) \approx \frac{\sigma_a}{\sqrt{\pi}}, \tag{56}$$

which will be a helpful measure when calibrating the model.

A simple measure of mobility starts from the autocorrelation. Knowing that a death occurs on average all $T = \frac{1}{p}$ periods the latter is given by:

$$mob_\tau = 1 - ACF(y_{t+\tau}, y_t) = 1 - \rho_y^{p\tau}, \tag{57}$$

helping us to calibrate the mobility.[26]

We also have to make some assumption about the initial distribution of wealth. We assume that each agent is endowed with a multiplier of their current labor income:

$$\frac{w_{i,0}}{y_{i,0}} = \Omega > 1, \tag{58}$$

[25] As a result the model will take some time to converge to the exponential age distribution.

[26] Note that we do not care about the nature of the transmission of income. This can be either genetic or sociological. As several studies suggest - in particular exploiting adoptions - the latter plays an important role Piketty (2000). As argued in Becker and Tomes (1979) the transmission of income is also highly shaped by inter vivo transfers such as parents paying for education (i.e. investing in human capital of their offspring).

making labor income and wealth (initially) perfectly correlated also implying that wealth inequality and labor income inequality are identical at the first simulation period.

The simulation begins:

1. All agents i have a particular income y_i. First of all, individuals can die. The probability of dying is p. If the agents dies, he leaves one offspring participating in the labor market. The income of the offspring is given by the AR(1) process laid out in equation 54 and thus related to the income of his parent. Moreover, the offspring inherits the wealth of his parent as an (accidental) bequest.[27] We implement a self-financing bequest system. The government levies a linear tax τ_b on bequests and redistributes a minimum level of bequests $b_{min,t}$ to all agents, thus constituting an (implicit) progressive taxation. The self-financing condition implies:
$$\sum_j (1-\tau_b) b_{j,t} + b_{min,t} = \sum_j b_{j,t} \rightarrow b_{min,t} = \tau_b \sum_j b_{j,t}, \tag{59}$$
for all agents j that die. Given the death probability, a number of $N \cdot p$ agents die. The bequest tax does little else but redistribute the wealth among those agents. Another interpretation of this tax system with a flat tax and a minimum level is that it is a tax with tax free (index TF) level of bequest of the size $b_{TF,t} = \frac{b_{min,t}}{\tau_b}$.[28] The degree of redistribution increases with τ_b.[29] For the case of $\tau_b = 0$ the effect is shut-down. The other extreme case $\tau_b = 100\%$ models the case of a guaranteed minimum bequest in which each agent is endowed with the very same level of wealth once entering the labor market.

2. Given their wealth $w_{i,t}$ in the current period and their current income $y_{i,t}$, agents form their optimal portfolio composition $\alpha_{i,t}$ as detailed in equation 21.

3. According to their current income $y_{i,t}$ and wealth $w_{i,t}$ agents form their consumption decision $c_{i,t}$ as presented in equation 25.

4. Each agent receives the return to their assets. We will discuss two cases. For homogeneous portfolios the risky return is given by:
$$\mu_t = \mu + b_t \tag{60}$$
with $b_t \sim N(0, \sigma)$ being a vector. For idiosyncratic risk we have:
$$\mu_{i,t} = \mu + b_{i,t}, \tag{61}$$
with $b_{i,t} \sim N(0, \sigma)$ in the form of a matrix. Based upon their asset structure and consumption their wealth for the subsequent period $t+1$ is determined.

[27]Note that in this model bequests are not active decisions.

[28]The technical rationale is that the individual post-tax bequest can be rewritten: $(1-\tau_b)b_{i,t} + b_{min,t} = b_{TF,t} + (1-\tau_b)(b_{i,t} - b_{TF,t})$ implying $b_{TF,t} = \frac{b_{min,t}}{\tau_b}$. As the average value of the bequeathed wealth increases, so does the tax-exempt level.

[29]If wealth is log-normally distributed, the rate of Gini before and after taxes is (using the first-order Taylor approximation) is given by $1 - \tau_b$.

5. The next simulation period starts.

As shown so far, for some simplifications (in particular assuming $y = \bar{c} = 0$) we were able to solve the model analytically. This will be of help when calibrating the model.

In order to produces reasonable simulation results we have to assign values to the variables. The latter can be divided into different categories. Most important are *growth* variables. Besides the aggregate growth g, this include the financial markets variables (μ, r, σ), and preference parameters (γ, ρ). The level of minimum consumption $\bar{c}$, the initial ratio of wealth to income ($\frac{w_{i,0}}{y_{i,0}} = \Omega$), and the demographic variable p also matter in this case. First of all, we require:

$$c_y = K_2 c_w < 1, \tag{62}$$

implying that (at least some agents with a large income) do not end up consuming all their wealth. In the case of $c_y > 1$ the current income of agents is not sufficient to cover for their consumption desire making them employ some of their stock level of wealth for consumption. Moreover, the unrealistic case of a negative offset of the consumption function (which is $(1 - c_y)\bar{c}$) would prevail, making negative consumption optimal for some low income agents.

It is also interesting to point out that the case of $c_y < 1$ and $g_w > 0$ are identical. The latter is easy to see as:

$$c_y = c_w K_2 < 1 \to c_w = p + \frac{\rho - (1-\gamma)r}{\gamma} - 0.5\frac{1-\gamma}{\gamma}\frac{(\mu - r)^2}{\gamma\sigma^2} < \frac{1}{K_2} = r + p - g, \tag{63}$$

which can be rewritten as:[30]

$$\frac{\rho - r}{\gamma} + g + (\gamma - 1)\frac{(\mu - r)^2}{2\gamma^2\sigma^2} < 0, \tag{64}$$

and is independent of the demographic variable p. The latter is the key insight of the model of Yaari (1965). The presence of annuities that are priced with acturial fairness, the effect of increased consumption due to finite life-times and the effect increased interest rates (returns from annuities) perfectly cancel out each other. In this case death hazard is fully insured and does not matter in the aggregate.

In contrast the condition $g_w > 0$ requires:

$$g_w = \frac{r - \rho}{\gamma} - g + \frac{(\mu - r)^2}{\gamma^2\sigma^2}(1 + \gamma) > 0. \tag{65}$$

The difference between condition 65 and condition 64 -which is $\frac{(\mu - r)^2}{\gamma\sigma^2}$ - results from the wealth growth out of risky assets adding to the aggregate wealth growth.

There is also a straightforward economic interpretation of the latter behavior. If agents are net savers ($c_y < 1$) and do not consume their income, they pile up wealth.

[30]Note that the third term also disappears once log-utility ($\gamma = 1$) is assumed.

The growing wealth/income-ratio is also documented in the empirical evidence of Saez (2013).

For sake of simplicity consider the case where agents do not invest in risky assets, as their higher volatility ($\sigma > 0$) is not compensated by excess returns ($\mu - r = 0$), leading to $c_y < 1$ being identical to:

$$r - \rho > \gamma g, \tag{66}$$

which is already extensively discussed for the log-utility case ($\gamma = 1$) in Fischer (2015b). In fact, for log-utility income and substitution effects exactly cancel out each other implying that:

$$c_y = \frac{p + \rho}{p + r - g}, \tag{67}$$

for which we have $c_y < 1$ for $\rho < r - g$ and vice versa.

In general, in order to have $c_y < 1$ we either (i) require $\rho < r - \gamma g$ (in discord with the Bewley-type literature, cp. e.g. Aiyagari (1994))[31] or (ii) $\gamma << 1$ implying that the substitution effect prevails (i.e. higher returns lead to lower current consumption). A combination of both assumptions also works.

The finance literature emphasizes a degree of risk aversion $\gamma > 1$ (e.g. Mehra and Prescott (2003)). From a macroeconomic perspective $\gamma > 1$ implies an intertemporal elasticity of substitution (IES) lower than one ($IES = \frac{1}{\gamma} < 1$) and thus a prevailing income effect, i.e. current consumption increases for higher interest rates. Formally, this is the case since:[32]

$$\frac{\partial c_w}{\partial r} = \frac{\gamma - 1}{\gamma} > 0 \tag{68}$$

A reasonable calibration with $\gamma > 1$ is: $\gamma = 2$, $\sigma = 0.2$, $\mu = 10\% > r = 6\% > \rho = 1\% \equiv g = 1\%$. The financial market variables are in line with the existing literature using continuous time methods to investigate financial market processes (e.g. cp. Fernholz and Fernholz (2014)). We assume an expected (work) life time of $T = 40$ years.

In line with Piketty and Zucman (2014) we impose $\Omega = 2$ as a starting value for the wealth/income-ratio. The income process with $\sigma_a = 0.6$ implies a realistic income inequality of approx. $Gini(y) \approx \frac{\sigma_a}{\sqrt{\pi}} \approx 0.3$. The correlation between income and income of the parents is in line with the empirical evidence of Chetty et al. (2014). To model minimum consumption, we define the minimum desired consumption level as:

$$\bar{c}_t = quantile_j(w_t), \tag{69}$$

being a fixed quantile of the share of wealth. With growing aggregate wealth ($g_w > 0$) the minimum desired level also grows and thus does not become negligible in the long

[31] As already put forward earlier the *Bewley*-type literature argues that agents save in order to cover for idiosyncratic labor income risk (*savings for a rainy day*, Wang (2003)). This extra savings reduce the rate of interest below the rate of time preference ρ and thereby induce dissavings. This - in combination with binding collateral constraints - implies a stationary distribution of wealth as $g_w < 0$. As in our model $g_w > 0$ a convergence of wealth is only possible with taxation or without risky assets and due to demographic transition. We will dwell more thoroughly on this issue in section 5.

[32] Note that this requires the assumption $\mu = r$ implying that agents do not invest in risky assets.

Category	Definition	Symbol	Value
Income Process	Autocorrelation between generations	ρ_y	0.3
	Standard deviation of income distribution	σ_a	0.6
	Income growth	g	1%
Initial wealth distribution	Initial ratio of wealth to income	$\Omega = \frac{w_{i,0}}{y_{i,0}}$	2
Demographics	Expected (work) life time	$T = \frac{1}{p}$	40
Preference parameters	Risk aversion	γ	2
	Time preference	ρ	1%
	Minimum consumption	$\bar{c}_t = quantile_j(w_{i,t})$	j=0.2
Financial market variables	Expected return risky asset	μ	10%
	Standard deviation risky asset	σ	20%
	Risk-free return	r	6%
Bequest tax	Linear tax on bequests	τ_b	20%

Table 1: Parameters benchmark simulation

run. This assumption is also very useful as it directly controls the share of zero wealth agents in the population. For German data (Bundesbank, 2013, p. 8) a level of $j = 0.2$ is reasonable, implying that 20% of the population hold no wealth. Similar evidence prevails for the USA (e.g. Castaneda et al. (2003)).

Finally, we set a a level of bequest of $\tau_b = 20\%$. In Germany, the level of bequest tax ranges from 7% to 50% progressing with the underlying bequest value.[33] Thus, the value is a reasonable approximation of real bequest system.

All our variables employed for calibration are summarized in table 4. Starting from the chosen parameters other values are implictly defined. The latter as summarized in table 4.

Note that essentially we have two groups of agents. One group of agents that does not face a binding cash-on-hand constraint representing the top income agents features a positive growth of wealth. We used this result in section 3 in order to derive closed form solution of the fat tails. Meanwhile another group can not fulfill their complete consumption desire and thus consumes all cash on hand. These agents are the hand-

[33]Moreover, there is also a large tax exempt level ranging from 20,000 €to 500,000 €with respect to the median net wealth of approx. 50,000 €(Bundesbank, 2013).

Definition	Symbol	Equation	Value
Risky share (without labor income or minimum consumption)	α	Eq. 16	0.5
Expected return	$E(R)$	Eq. 4	8%
MPC wealth	c_w	Eq. 19	6.5%
Mutiplier wealth/income	K_2	$\frac{1}{r+p-g}$	13.33
MPC income	c_y	$c_w K_2$	0.867
Growth wealth	g_w	Eq. 38	3%
Growth $z = y - \bar{c}$	g_z	Eq. 39	15.3%

Table 2: Implicit values

to-mouth (HTM) agents. For the agents that are HTM $c_y \equiv c_w = 1$ making wealth growth:

$$g_w = r - g + p - c_w = r + g - p - 1 < 0 \leftrightarrow r + g - p < 1, \tag{70}$$

as $K_2 = \frac{1}{r+g-p} < 1$. In the long-run wealth is zero for those agents. If there is no redistribution by means of a bequest tax, this implies:

$$c(t) = y, \tag{71}$$

making them *true* HTM agents that consume all their labor income. The average growth rate is the the weighted average $\bar{g}_{w,t}$ of the growth rate of low income agents and high income agents (not subject to the cash on hand constraint). It decreases with the share of low income agents controlled by the level of minimum consumption $\bar{c}$.

Note that if HTM run into zero wealth the share of risky assets as detailed in equation 16 changes. The combination of $w_t \to 0$ and $y < \bar{c}$ implies:

$$\alpha_t \to -\infty, \tag{72}$$

which means agents want to go short on the risky assets. Technically, this is a desperate measure to incur debt in the simulations. In a standard framework with ($y = \bar{c} = 0$) this would never occur for the case with a risk compensation ($\mu > r$) as:

$$\alpha = \frac{\mu - r}{\gamma \sigma^2} > 0. \tag{73}$$

In the simulations we forbid short-selling by imposing:

$$\alpha_{i,t} = \min\{\alpha_{i,t}; 0\}. \tag{74}$$

Figure 1 shows the evolution of the Gini coefficient for diverse economic measures.[34] By assumption the Gini of income is constant and given by approx $\frac{\sigma_y}{\sqrt{\pi}} \approx 0.34$. The

[34]Note that we only consider normalized measures. As the all are, however, scaled by a common measure $\exp(gt)$ distributional measures are not distorted.

(small) stochastic deviation is due to the death of some agents that - following the AR(1) process - are not replaced by identical clones. By assumption all Gini coefficients start at the same level. Yet, the Gini coefficient of wealth and consumption increase to a large level of approx. 0.9. (Shorrocks et al., 2013) report a value of 0.85 for the USA and 0.77 for Germany. At variance with empirical evidence in the long run the consumption inequality is at comparable level with the wealth inequality. The latter is the case since $g_w > 0$ making wealth grow at excess of income and thus dominate as input for the consumption function depending both on current income and current wealth in a linear manner. As a result, moreover, in the aggregate the growth rates of wealth and consumption synchronize.

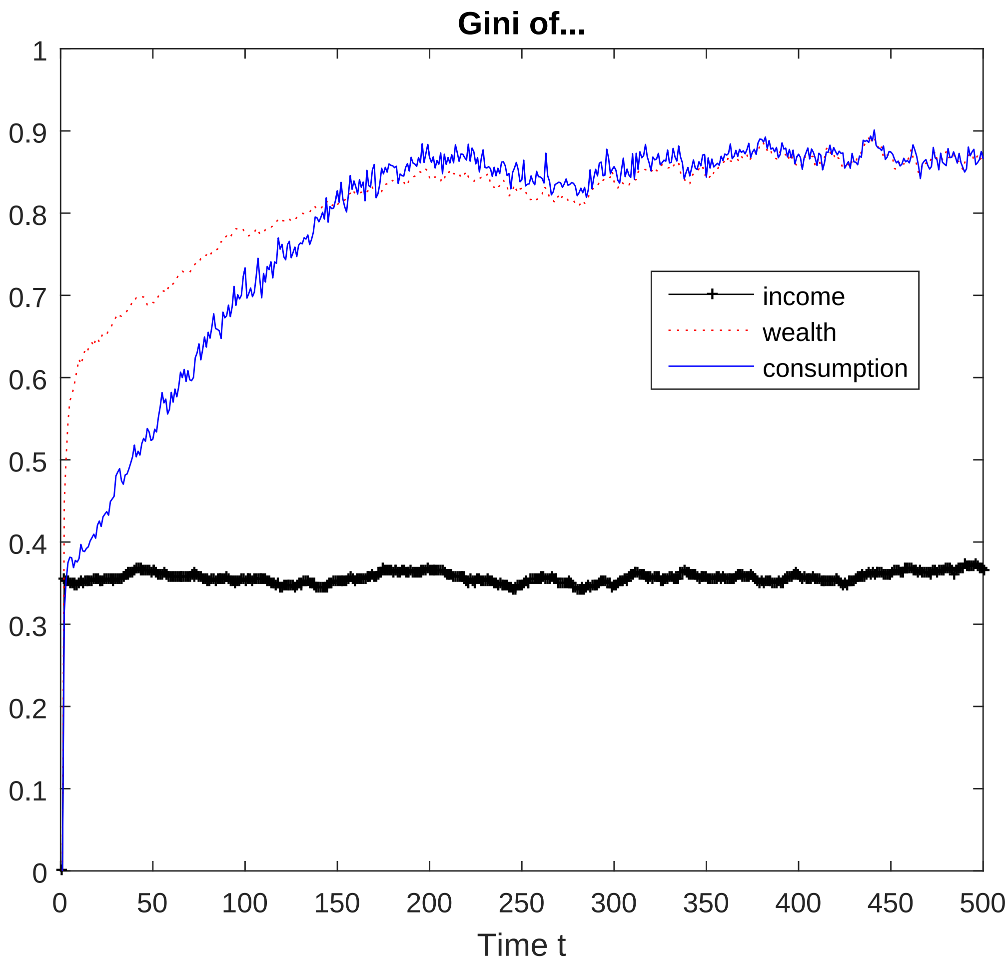

Figure 1: Gini coefficient for (normalized) income, wealth, and consumption - Benchmark case

The work of Piketty (2014) emphasizes the role of the top-wealth holders and reports their shares. In figure 2 we plot the results of the model showing that in the long run the top 10% hold approx. 80% of total wealth and the top 1% individuals accrue approx.

40% of total wealth.[35] Meanwhile, the bottom 50% percent hold a share of wealth lower than 5%. (Piketty, 2014, p. 349) reports shares of approx. 70 % respectively 35% for the top wealthodlers in the USA. Using a different methodology Saez and Zucman (2014) report even higher values of 77.2% respectively 41.8% for the top wealth holders, making our model a good fit for the case of the USA. In the more equally distributed Germany a share of approx. 55% for the top 10% is reported. The fit of the bottom in our model, however, is also quite well. In fact, and thereby diametrical to the *Bewley-type* literature, our model even overestimates the level of inequality.

The complete Lorenz-curve of wealth is depicted in figure 3 in particular displaying the assumed share of $j = 0.2$ with close to zero wealth.

We also applied a Pareto-fit of the right tail of the wealth distribution for the last simulation period $t = T = 500$ yielding an estimated coefficient of $a \approx 1.78$.[36] The literature estimates a value of approx. 1.5 for the USA (Saez and Zucman, 2014). Using the methodology developed in Virkar and Clauset (2014) we display the fit in figure 4.[37] Note that the fit does not fit the very top incomes (<1%) too well and overestimates the value.

Finally, we can make a statement about mobility. For the latter we compute a two-state (i.e. good and bad state) Markov transition matrix M_τ between a time t and $t+\tau$. Mobility measures weight the diagonal elements (staying in the same state) against the non-diagonal elements. One of the most popular measure is the measure developed by Shorrocks (1978) which uses the trace. The measure is defined as:

$$mob_\tau = \frac{n - trace(M_\tau)}{n-1}, \tag{75}$$

for which n is the number of states in the Markov transition matrix, which in our case is $n = 2$. Another popular measure replaces the trace by the determinant of the matrix. For the case of $n = 2$ both measures and a mobility using the autocorrelation coefficient ($mob_\tau = 1 - ACF_{t,t+\tau}$) yield the same result.[38] Figure 5 shows the mobility of income and wealth. First of all, the measure of mobility increases with the level the lag τ. For the case of income - by assumption - it is given by:

$$mob_\tau = 1 - \rho_y^{p\tau}. \tag{76}$$

Within the expected life-time $\tau < T = \frac{1}{p}$ the mobility of wealth is higher than the mobility of income. Yet, in the long run wealth mobility is lower. We will dwell more thoroughly on the latter in the following in which we discuss other (nested) cases.

[35] Note that for the assumed number of $N = 1,000$ agents this implies 100 respectively 10 agents.

[36] Note that the latter implies a share of top 10% individuals of 36% and of top 1% individuals of 13%.

[37] The authors provide software tools to perform the estimation in several common software packages and distribute them on their homepage `http://tuvalu.santafe.edu/~aaronc/powerlaws/`. We employed the Matlab package.

[38] At this time being we do not include a formal proof for the latter. We, however, aim to do that in the final version of the paper.

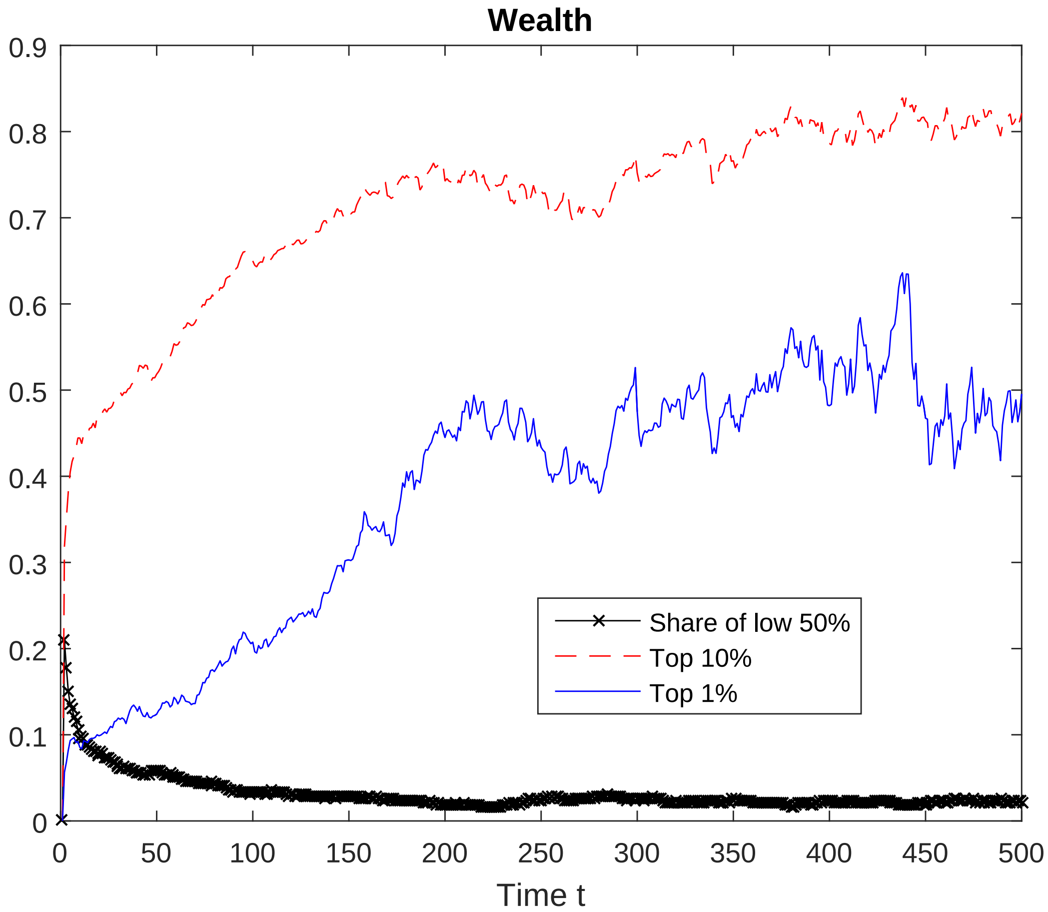

Figure 2: Share of wealth for bottom 50%, top 10% and top 1% - Benchmark case

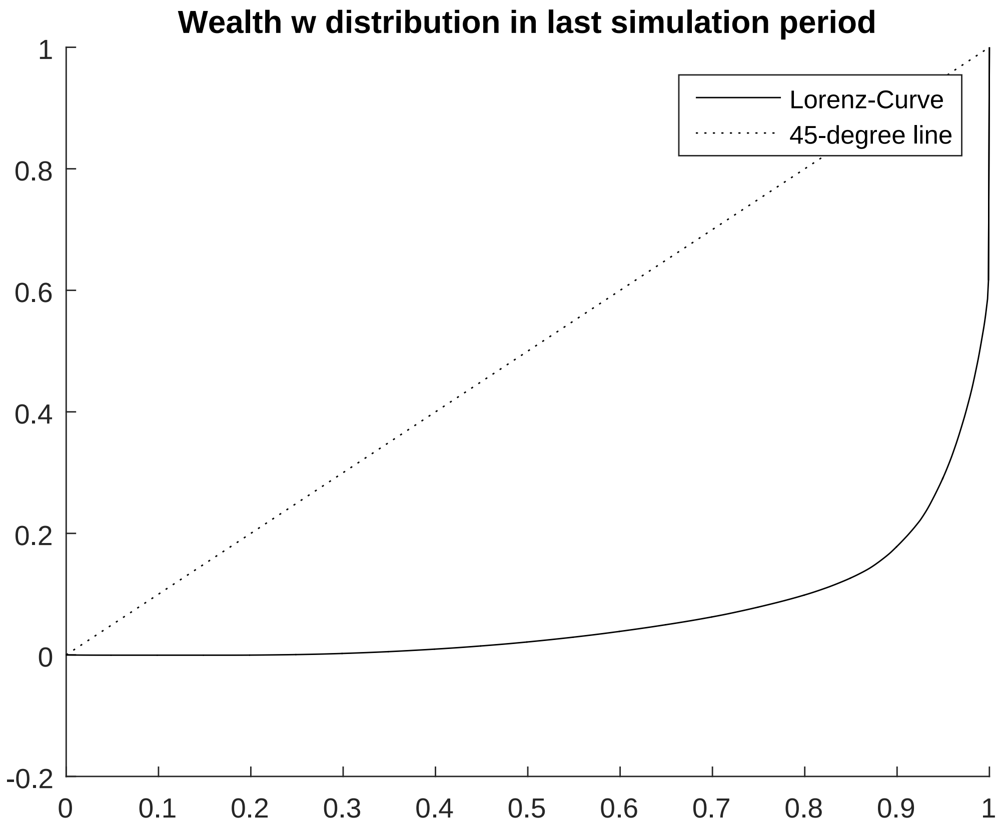

Figure 3: Lorenz curve for (normalized) wealth - Benchmark case

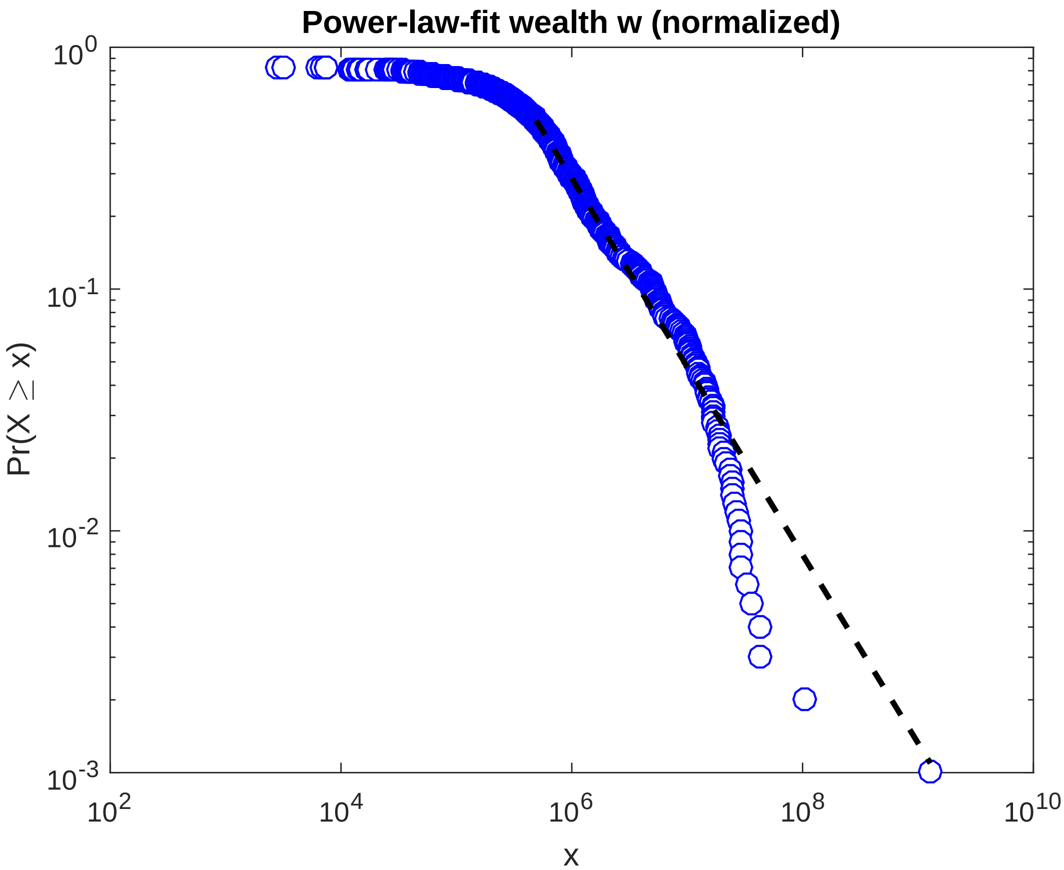

Figure 4: Power-law fit for wealth - Benchmark case

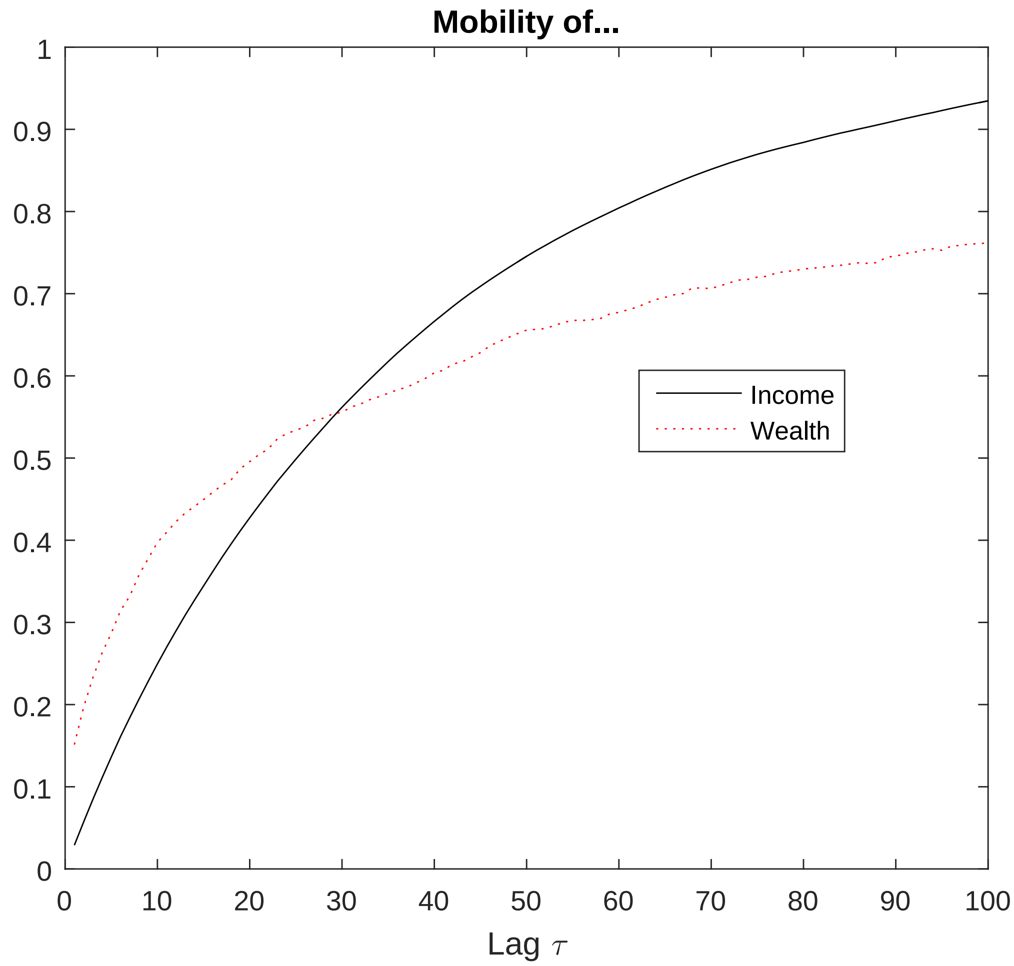

Figure 5: Mobility (trace-measure $n = 2$ Markov-matrix) for a variation of lags - Benchmark case

5 Nested cases

The aim of this section is to present special cases which are nested within the model and (frequently) already discussed in the literature. The key aim of this exercise is to identify the necessary building blocks to generate realistic features.

5.1 No risky investments

Our first exercise is to shut down the effect of the risky storage technology. The latter is motivated by the seminal results of Benhabib et al. (2011) arguing that risky capital income is a key driver for top wealth inequality. Theoretically, we could set $\sigma = 0$.

Yet, this would lead to some computational problems (due to a zero division). A simple option is to assume $\mu = 0.1 \equiv r$ providing no incentive for agents to hold risky assets.

As shown in figure 6 inequality does not converge. In theory - and as shown in section 3 - the Pareto-tail is given by $\frac{p}{g_w}$. For the given parametrization the latter is however, at an unreasonable level $a < 1$ implying no convergence. On the other hand, comparing the absolute levels of shares with and without risk (figure 2 respectively 6) reveals that at a given point in time the inequality of wealth is lower in the case without risk. This result, however, is deceptive as the inequality does not converge for the given values.[39] [40]

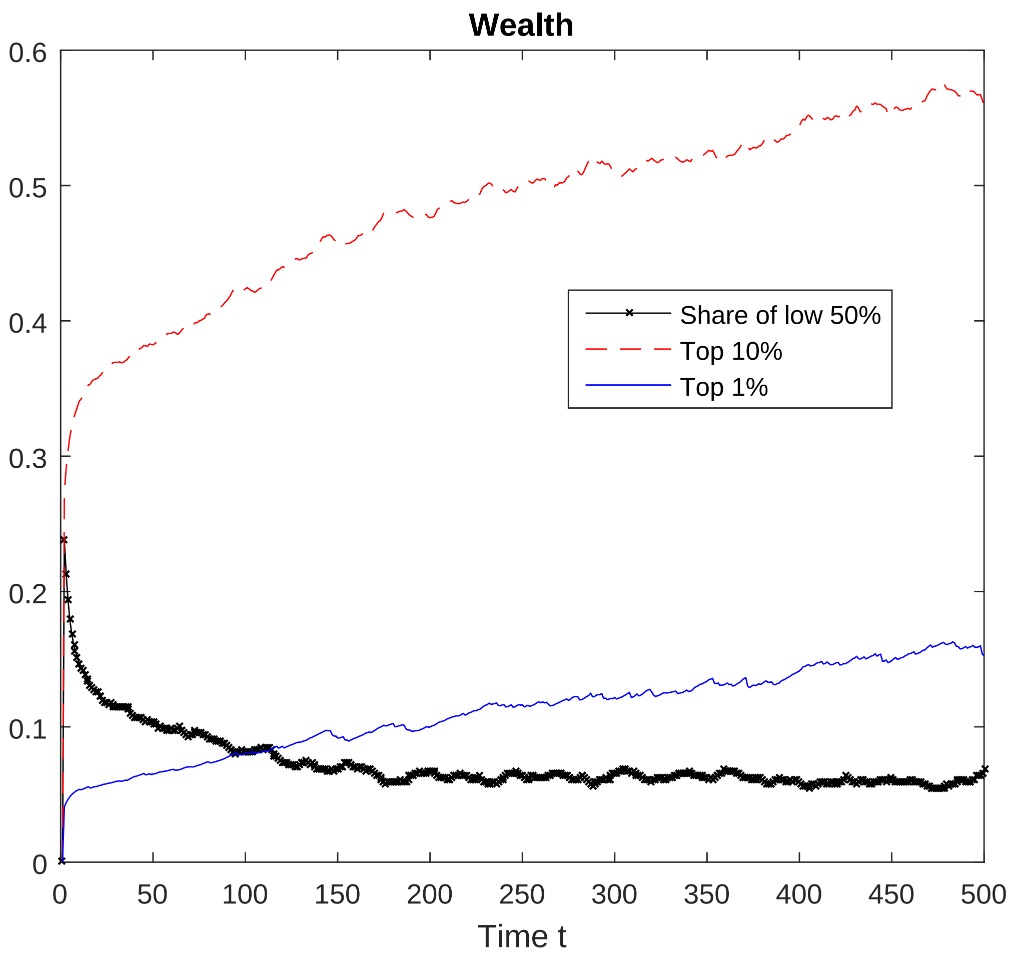

Figure 6: Share of wealth for bottom 50%, top 10% and top 1% - No risk case

[39] Note, however, that a lower life expectancy increasing p would increase a to a reasonable level.

[40] It is also interesting to point out that the model converges to a stationary distribution if there are no risky investments and no (!) tax. The latter surprising result owes to the fact that the stochastic event of death coupled with redistribution adds a stochastic component to the wealth process. While this - from a theoretic point of view - is an interesting insight, it is not empirically relevant.

5.2 No death

Another option already discussed in the closed form solution in section 3 is the case without death $p = 0$. In an economic sense - and following the logic of Barro (1974) - this could be considered the case of generations that are connected by means of altruistic transfers. In fact, the optimal consumption problem is the problem of an infinitely living dynasty. Moreover, the absence of death also mechanically shuts down the AR(1) process of the transmission of skills. Thus, the skill level in this case is perfectly correlated between generations. The mobility of income is thus zero. Finally, the only tax we have in the model is a bequest tax which is only imposed in the case of death. As there is no death, this also implies that there is no taxation.

As shown in the closed from solution the Pareto-coefficient is given by $a = 1 - \frac{g_w}{\sigma_w^2}$. The convergence requires $a > 1$ and thus $g_w < 0$. As already discussed in section 4 the latter assumption implies the unrealistic case of $c_y > 1$. As a result the assumption of $c_y < 1$ implying $g_w > 0$ implies no convergence of wealth inequality. The latter is illustrated in figure 7. As a result death of agents - i.e. the demographic perspective - is highly important in order to achieve a converging distribution of wealth.

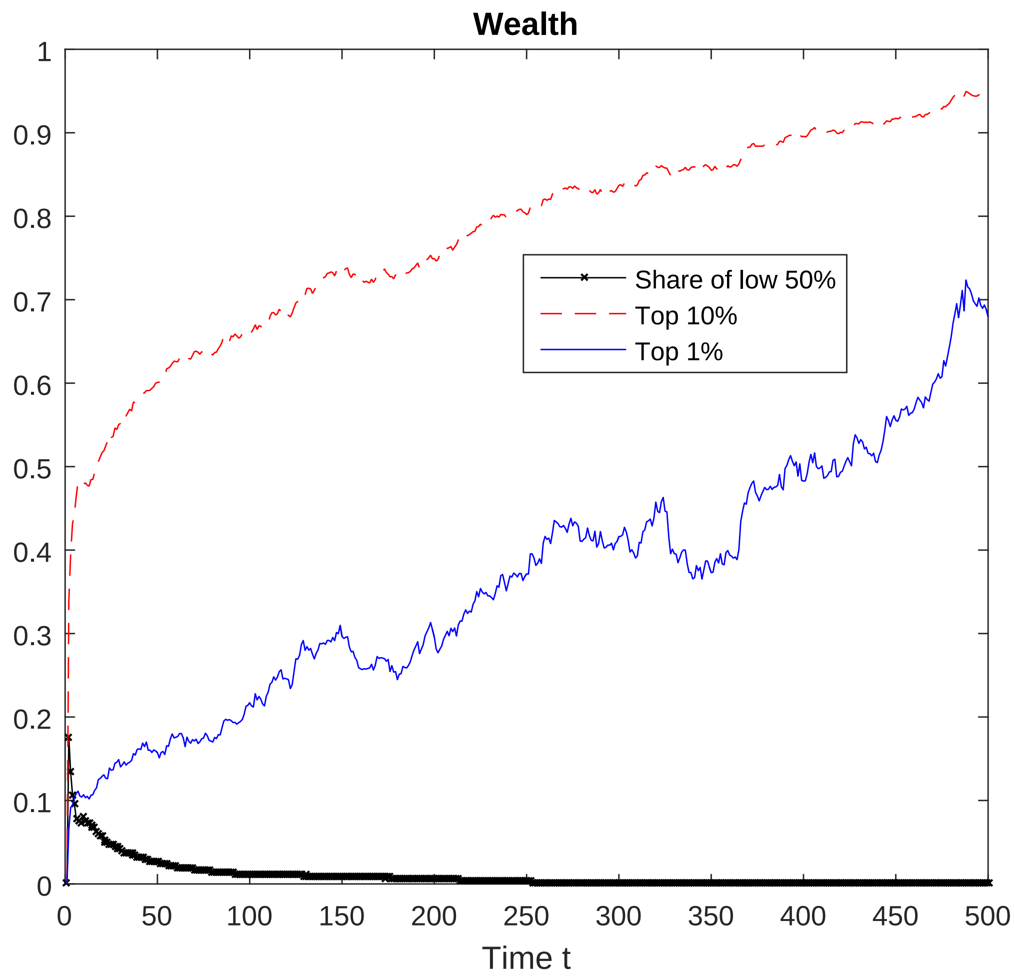

Figure 7: Share in wealth and consumption bottom 50%, top 10% and top 1% - No death case

5.3 Idiosyncratic vs. homogeneous risk

Fernholz and Fernholz (2014) emphasize the role of idiosyncratic risk for the distribution of wealth. We can easily include both cases - idiosyncratic respectively homogeneous risk. An economic interpretation of homogeneous risk would be that all agents hold the same (risky) assets in their portfolio (e.g. an Exchange Traded Fund that replicates a broad market index of stocks). The opposed extreme case with pure idiosyncratic risk could be considered as the case with business risk. The latter cannot be hedged away. This is more closely to the rationale of Cagetti and Nardi (2009) focusing on the role of entrepreneurs shaping the distribution of (top) wealth. In reality a mix of the two extreme cases is likely to be at place.

Note that the model of Fernholz and Fernholz (2014) is nested in our case once we exclude the bequest tax. Figure 8 shows simulation results for the same parametrization

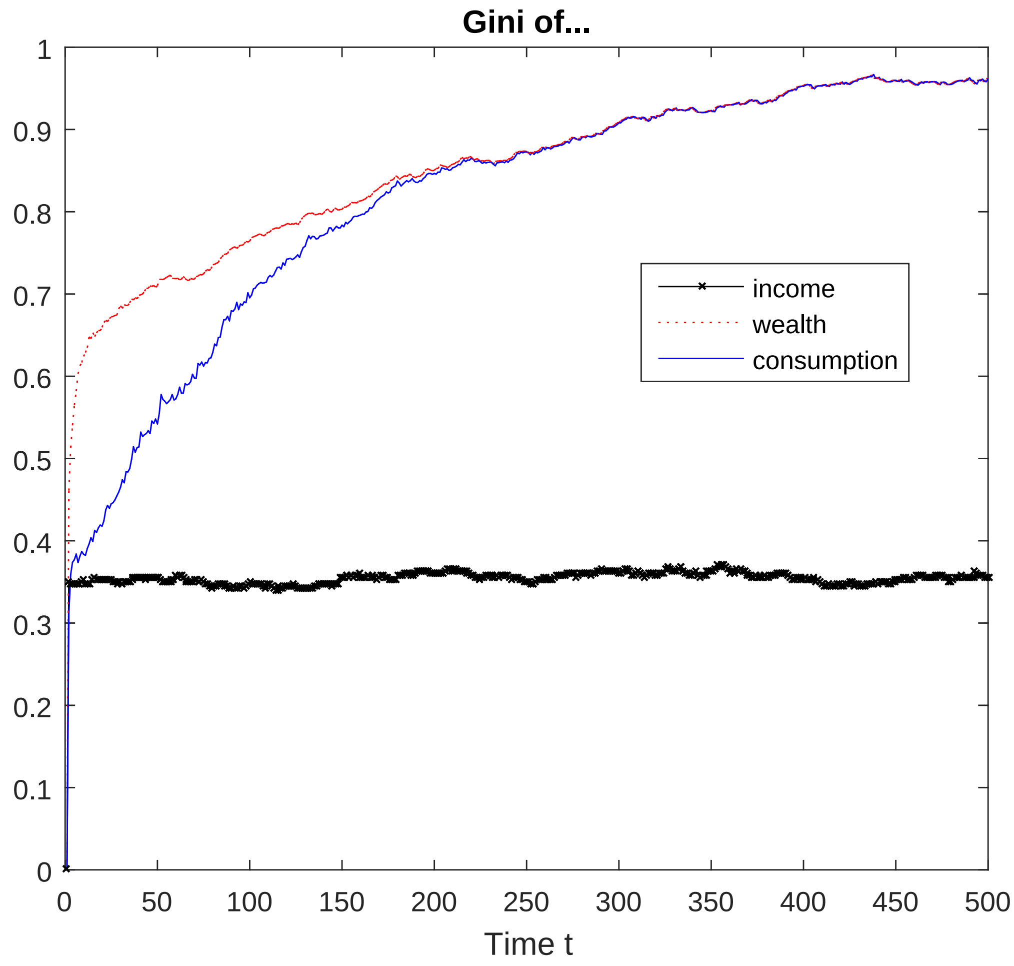

Figure 8: Gini coefficient for (normalized) income, wealth, and consumption - Benchmark case without taxation

but $\tau_b = 0$. In this case, the inequality - as measured by the Gini coefficients - in the long run increases to the maximum of 1. Moreover, the shares of top wealth individuals do not converge to finite levels (see figure 9).

Thus, a motivation for taxing bequests is the presence of idiosyncratic risk. Note that taxation and transfers are measures of social insurance. Health insurance covers for the risk of health, income insurance covers for the risk of unemployment and so on and so forth. If this insurance is not bought by the individual, it can be provided by the government. Usually, it is assumed that the prevailing insurance system is not available on the free market. However, other considerations e.g. moral hazard problems of individuals hoping for the government as a *Good Samaritian* supporting them in times of distress can also be the underlying cause. Moreover, the demand for insurance depends on preferences in particular risk aversion. Thus, the higher inequality in Anglo-Saxon countries as compared to central European and Scandinavian countries can be

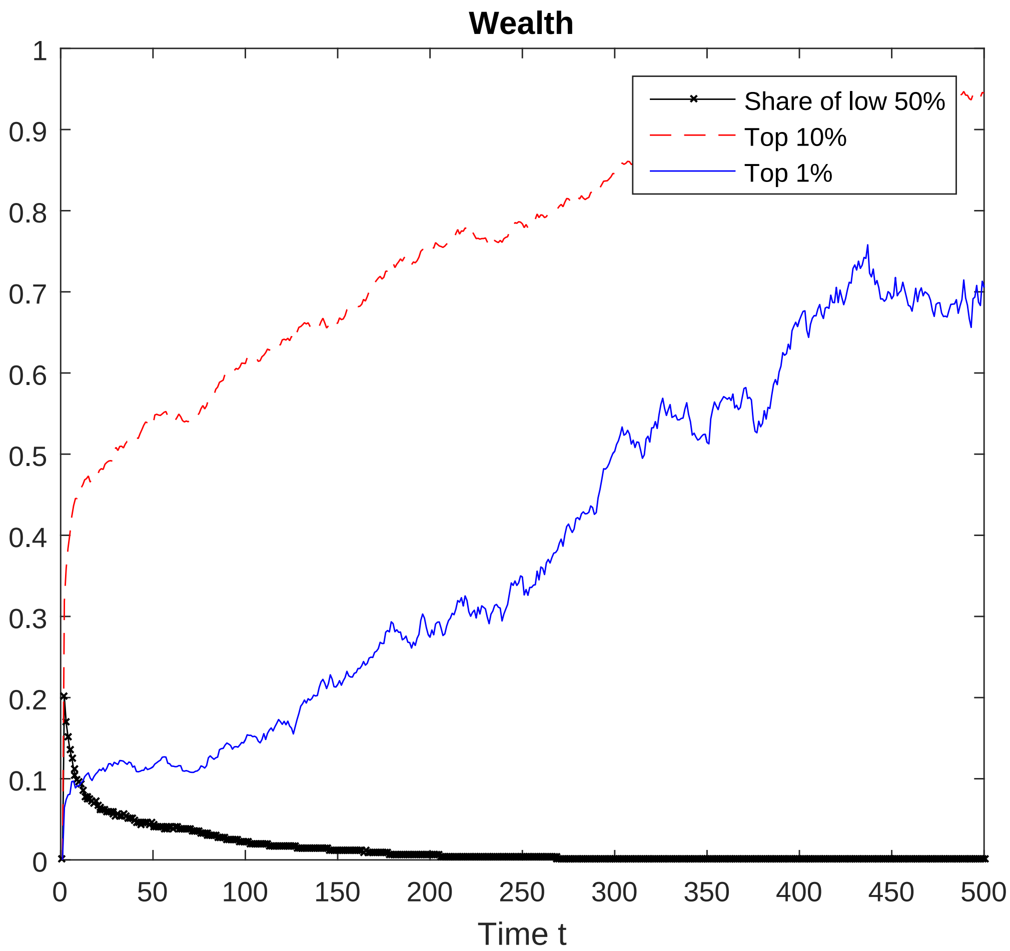

Figure 9: Share in wealth and consumption bottom 50%, top 10% and top 1% - Benchmark case without taxation

the result of heterogeneous risk aversion. Note also that the inequality in the *Bewley-type* literature emerges due to the underlying uninsurable income risk. Finally, in the discussed model the bequest tax is a social insurance mechanism that copes for the non-existence of insurance against idiosyncratic wealth risk. In fact, the mechanism redistributes (at the time of death) from lucky individuals that made good investments to those who were unlucky regarding their investment.

In contrast to that homogeneous risk leads to a finite distribution. Nevertheless, the wealth shares are highly volatile. As all agents hold the identical portfolio extreme return events also materialize as aggregate shocks. As a result, we consider the case with idiosyncratic shocks as a more realistic description of reality. Finally, it is important to point out that that idiosyncratic risk implies a higher mobility (in particular cp. line 1 and 5 of table 5.4). The economic rationale is that the wealth accumulation process become more noisy and thereby less dependent on previous outcomes. The latter is also easy to comprehend if we consider the case of income mobility for which we have closed-form solutions. Income mobility increases with ρ_y decreasing the autocorrelation. Meanwhile the inequality increases with ρ_y as the latter is given by $Gini(y) = \frac{\sigma_y}{\sqrt{\pi}\sqrt{1-\rho_y^2}}$. As a result there is a general trade-off between inequality and mobility. Unequal societies characterized by a noisy underlying process (e.g. due to lack of insurance mechanism) are more mobile and vice versa. We will discuss this issue more thoroughly in section 6 in which discuss the role of taxation more precisely.

5.4 Minimum consumption level

One unusual modeling choice we made was the introduction of a minimum consumption level $\bar{c}$. Note that for the special case of $\bar{c} = 0$ the standard CRRA utility function is nested.

For this case the inequality converges to a very moderate level (cf. figure 10 and 11). In particular, we are not able to account for the low wealth and close to zero wealth of the bottom 50% (also confer with the Lorenz curve depicted in figure 12. Meanwhile, the lack of the minimum consumption ($\bar{c} = 0$) level implies higher wealth mobility (cp. line 1 and 6 of table 5.4). In fact, the combination of $\bar{c} = 0$ and $c_y < 1$ implies that there are no hand-to-mouth consumers. The latter have a strong autocorrelation in wealth which in the stationary long run is zero for them.

If we consider the case without minimum consumption ($\bar{c} = 0$) and without taxation ($\tau_b = 0$), no stationary distribution emerges (cf. figure 13). Thus, the tax on bequests basically has two purposes. As already put forward in section 5.3 (i) it insures against idiosyncratic wealth risk and (ii) it helps to sustain a minimum consumption level. Due to their low skills some agents earn an income $y_{i,t}$ that is lower than the minimum desired consumption level $\bar{c}_t$. By redistributing some wealth to them in the form of a minimum transfer at bequests $b_{min,t}$ agents are endowed that can (partly) cover for their consumption desires.[41] If that would not be the case, some low skill / low income agents would be covered in a trap with zero wealth, while the rest of the agents exhibits growing wealth $g_w > 0$ ever increasing the distance between the groups.

At this point it is also interesting to cover the economic rationale between the minimum consumption $\bar{c}_t$. One might consider this as a simple physiological need (e.g. nutrition, basic shelter) that has to be covered. Note, however, by construction as a share of the wealth distribution with $g_w > 0$ the minimum consumption level also increases. The latter is at odds with the idea of the basic need which should be independent of time. Thus, the interpretation of $\bar{c}_t$ is more of a *relative consumption* level which agents want to maintain in order not *keep up with the Joneses* (Duesenberry, 1949). In fact, the modeling the utility function presented in this model is usually taken in order to model this effect (cf. e.g. Uhlig and Ljungqvist (2000)). The presence of this effect leads to savings ratios that increase both with current income and wealth in line with empirical evidence (Carroll, 1998). The bequest tax can (partly) internalize this relative consumption desire.

[41] In fact agents that would receive a bequest of $b_{i,t} < \frac{b_{min,t}}{\tau_b}$ are net transfer receivers.

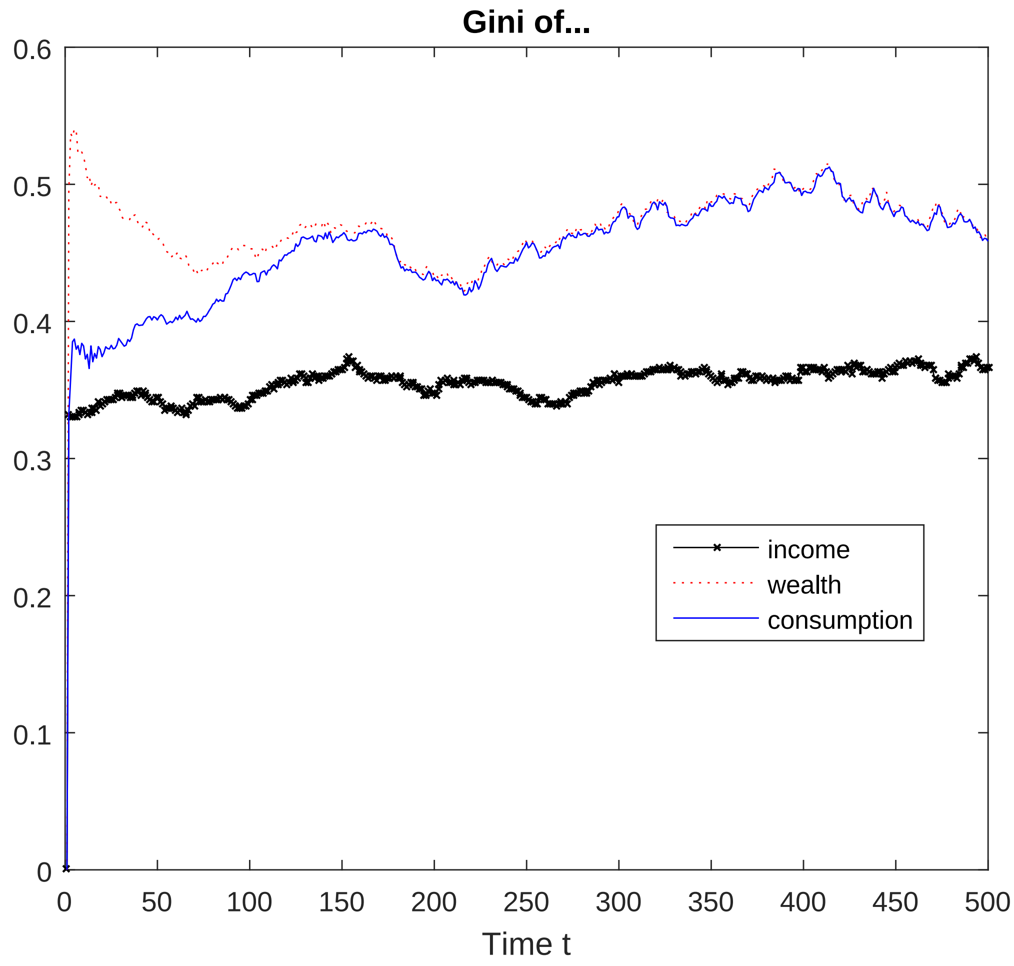

Figure 10: Gini coefficient for (normalized) income, wealth, and consumption - Case with no minimum consumption ($\bar{c}_t = 0$)

So far, we argued in favor of redistribution. We can, however, also inverse this argument. In the case, in which there is (i) no relative consumption ($\bar{c} = 0$) and (ii) homogeneous risk, any tax $\tau_b > 0$ would lead to total equality of wealth. We present exemplary simulations with $\bar{c} = 0$, homogeneous risk, and $\tau_b = 0.2$ in figure 14 implying a long-run equality of wealth. The pace of convergence to total equality increases with the level of taxation τ_b. In this case no incentive to save would prevail. In fact, the bequest tax is closely related to the tax on capital and capital income as it covers the stock level of wealth. With the above assumptions the paper is able to reproduce the result of an optimal capital (income) tax of zero as proposed in the seminal work of Chamley (1986) or Judd (1985). In fact, without the minimum consumption level, with a homogeneous risk, and without taxation the distribution converges to a stationary distribution (cf. figure 15). For the chosen values - and strongly at odds with empirical evidence - income inequality is larger than wealth inequality.

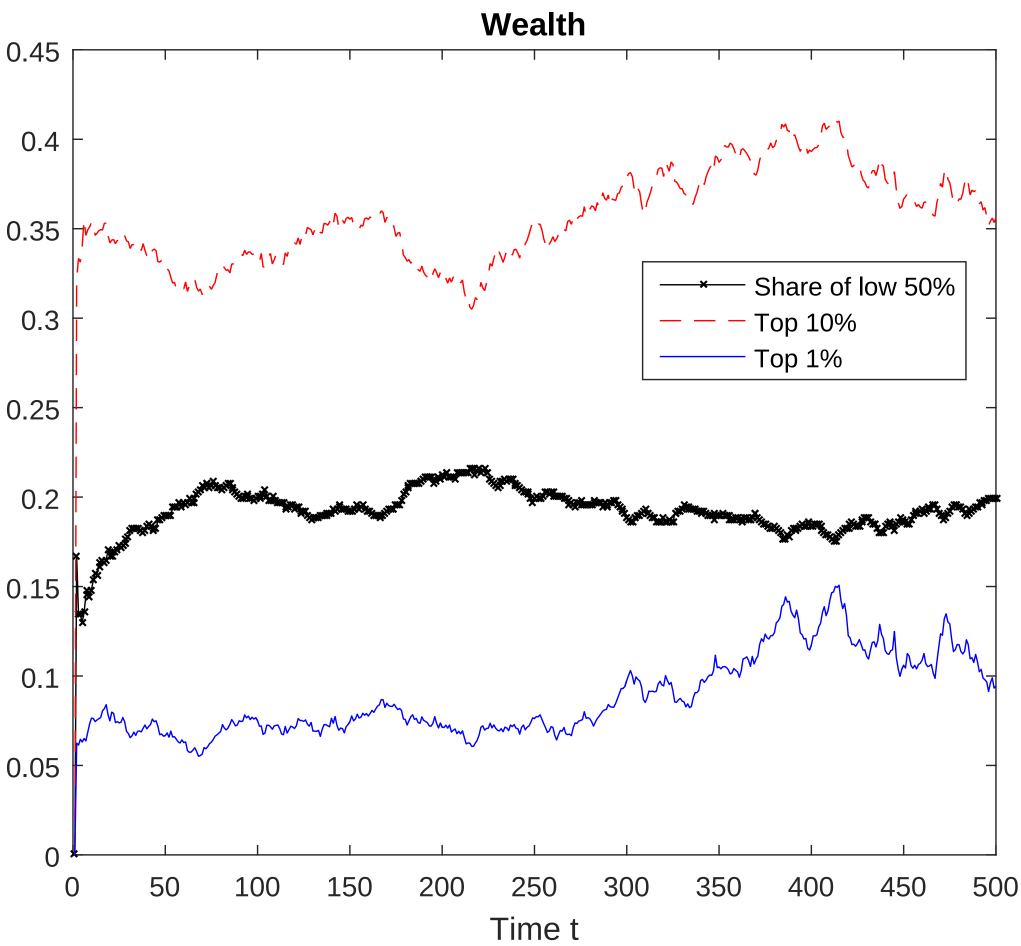

Figure 11: Share of wealth for bottom 50%, top 10% and top 1% - Case with no minimum consumption ($\bar{c}_t = 0$)

Finally, we can discuss the mobility case as depicted in table 5.4. In the table, we present the different cases for which i indicates idiosyncratic risk whereas h represents homogeneous risk. For y (*yes*) the variable is non-zero otherwise the variable is zero (respectively for $\mu = r$ and y the condition holds). As depicted in the table for lags that are larger than expected life time $\tau > T = \frac{1}{p} = 40$ wealth mobility is always lower than income inequality.[42] A risky capital income - in particular idiosyncratic risk (as e.g. assumed in the benchmark case) - increases mobility by adding noise to the wealth process. The presence of relative consumption motive also decreases mobility as some HTM agents end up with a constant level of zero wealth. Without a minimum consumption level a tax slightly increases mobility (cp. line 6 and 7 of table 5.4).

[42]Note that rather than two take the measured income mobility which exhibited a slight variation due to the law of low numbers we take the theoretical mobility as the benchmark ($mob_\tau = 1 - \rho_y^{p\tau}$.

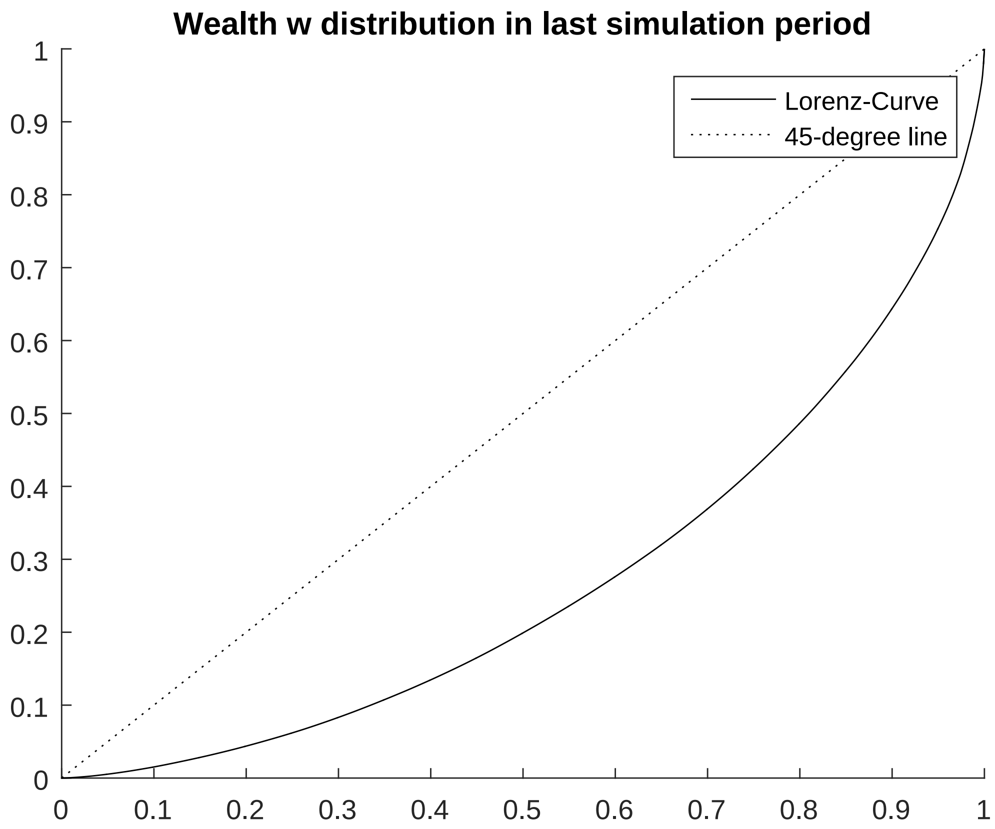

Figure 12: Lorenz curve for (normalized) wealth - Case with no minimum consumption ($\bar{c}_t = 0$)

Somewhat surprisingly in the benchmark scenario the bequest tax decreases mobility. The latter will however, be discussed more precisely in the following section.

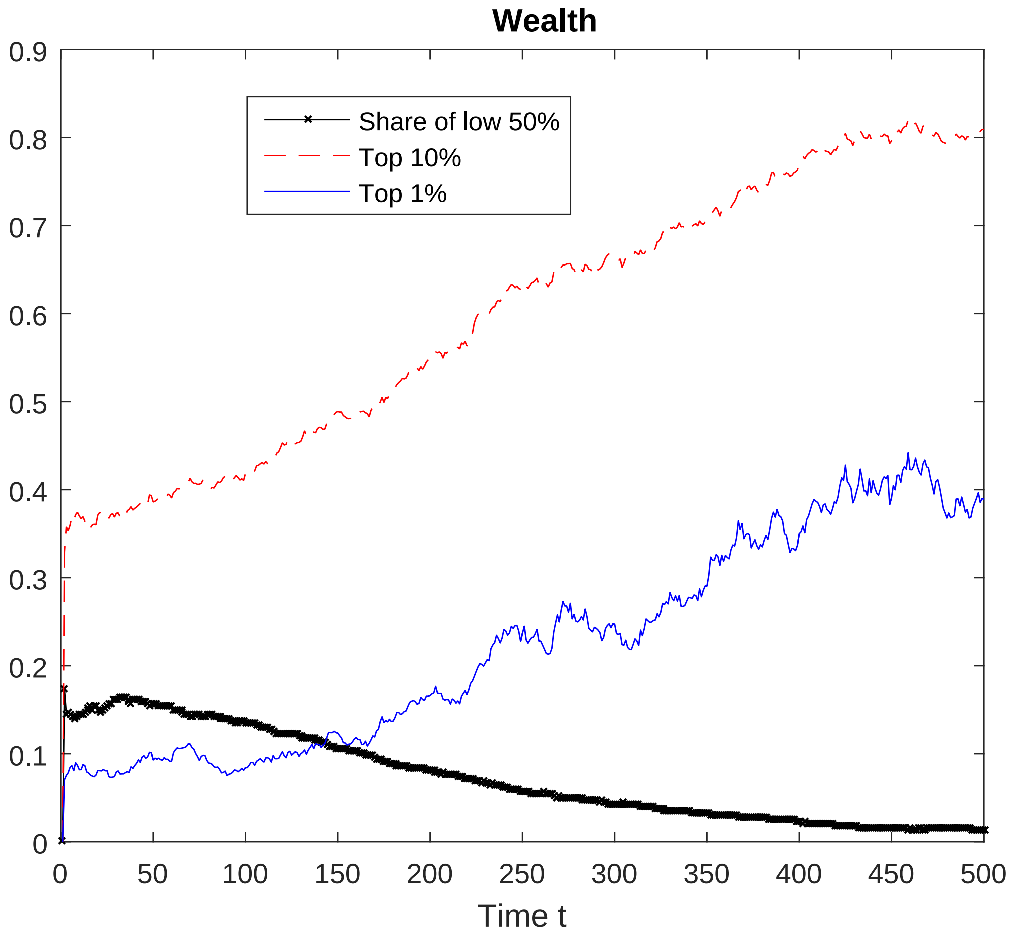

Figure 13: Share of wealth for bottom 50%, top 10% and top 1% - Case with no minimum consumption ($\bar{c}_t = 0$) and no taxation ($\tau_b = 0$)

	Configuration						**Lag**	τ			
	Description	**risk**	$\bar{c}_t$	τ_b	p	$\mu = r$	20	40	60	80	100
1	Benchmark	i	y	y	y	n	0.496	0.604	0.677	0.730	0.762
2	No risk	i	y	y	y	y	0.262	0.405	0.477	0.545	0.600
3	No death	i	y	y	n	n	0.450	0.555	0.614	0.667	0.724
4	No tax	i	y	n	y	n	0.498	0.626	0.718	0.778	0.806
5	Homogeneous risk	h	y	y	y	n	0.242	0.330	0.369	0.395	0.396
6	No consumption	i	n	y	y	n	0.566	0.693	0.791	0.845	0.872
7	No consumption & no tax	i	n	n	y	n	0.518	0.624	0.673	0.749	0.780
8	Theory y						0.452	0.700	0.836	0.910	0.951

Table 3: Mobility of wealth (and income) for different scenarios

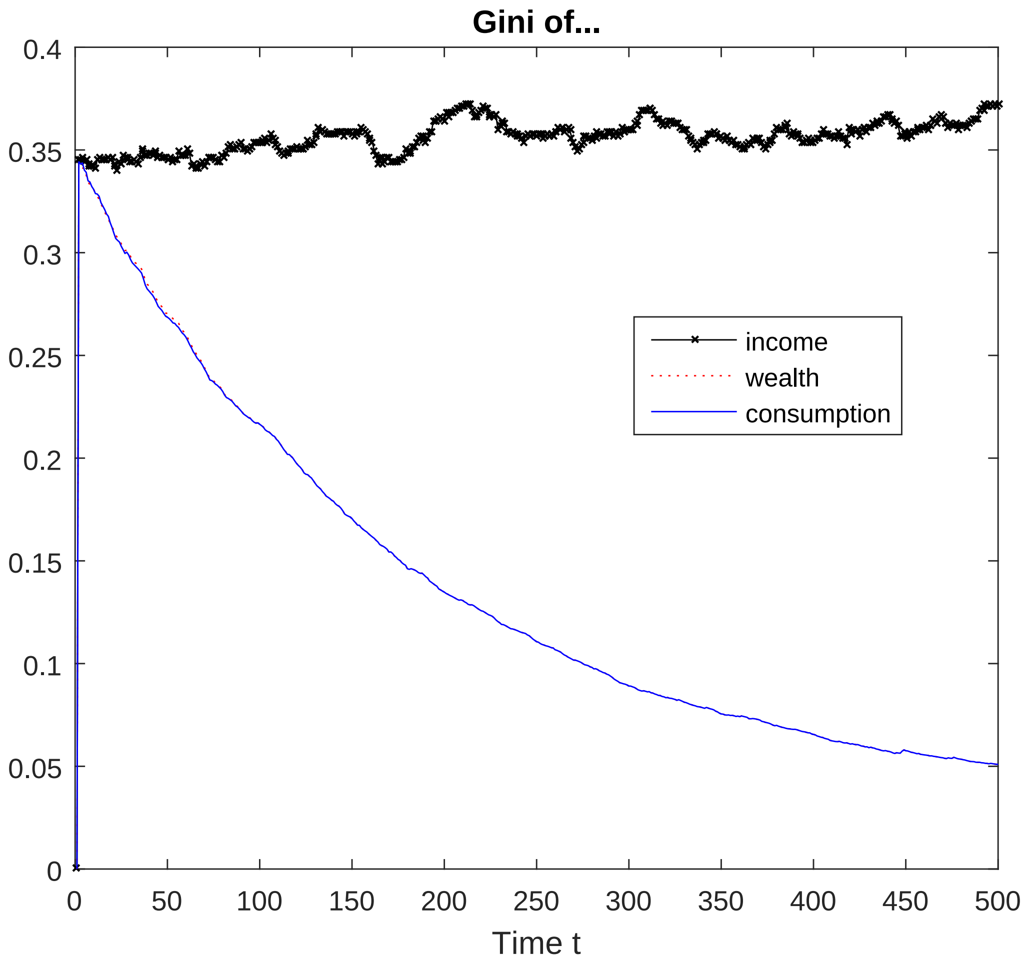

Figure 14: Gini coefficient for (normalized) income, wealth, and consumption - Case with no minimum consumption ($\bar{c}_t = 0$) and homogeneous risk

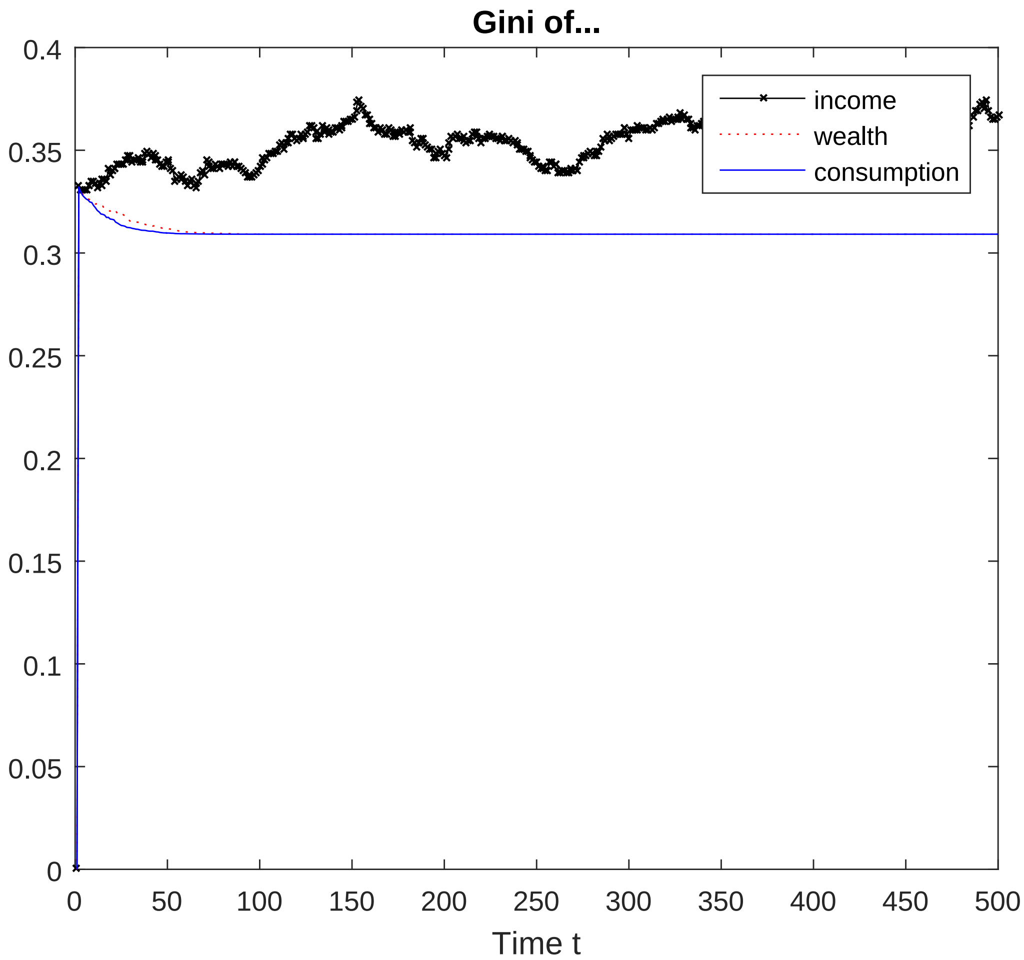

Figure 15: Gini coefficient for (normalized) income, wealth, and consumption - Case with no minimum consumption ($\bar{c}_t = 0$), without taxation ($\tau_b = 0$) and homogeneous risk

6 The role of the bequest tax

In this section we want to discuss the bequest tax more thoroughly using simulations for our benchmark calibration (cf. table 4). In particular, we take all benchmark parameters and vary τ_b in a *ceteris paribus* fashion.

Not surprisingly, the increase of the redistribution associated with larger values of τ_b decreases the inequality of wealth (cf. figure 16).Technically, the redistribution increases the mode of the wealth disribution and thus decreases inequality and increases the Pareto-coefficient a (also indicating higher equality). In particular, there exists a certain level $\tau_b > \tau_b^*$ that internalizes the effects of relative consumption and idiosyncratic risk implying a long-run convergence of wealth inequality (cf. figure 17). Another positive effects of the redistributive tax are the share of HTM-agents decreases thus increasing aggregate wealth as well as consumption growth.

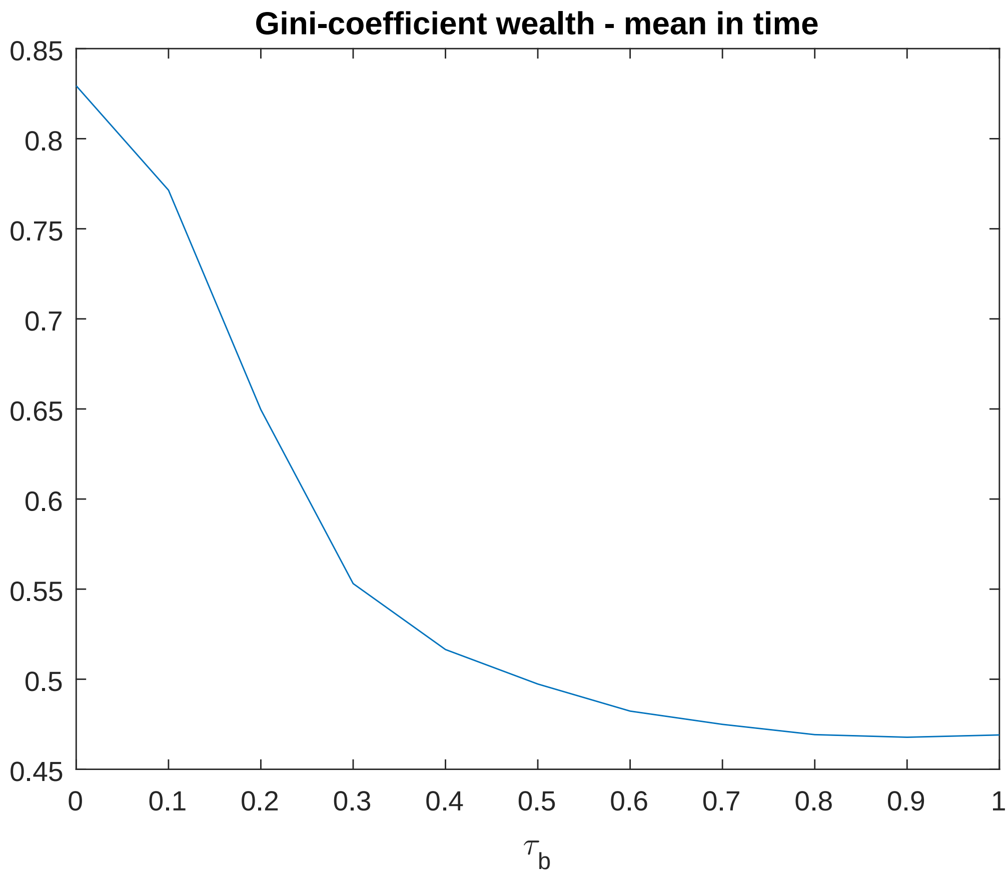

Figure 16: Gini coefficient of wealth (mean in time) for variation of bequest tax τ_b

Moreover, the wealth tax also increases the mobility of wealth (cf. figure 19). This is the case as the tax inserts a wedge between wealth bequeathed by the parent generation and the amount received by the offspring generation. In fact, this makes the process similar to the AR(1) already prevailing for income. As a result the correlation - which

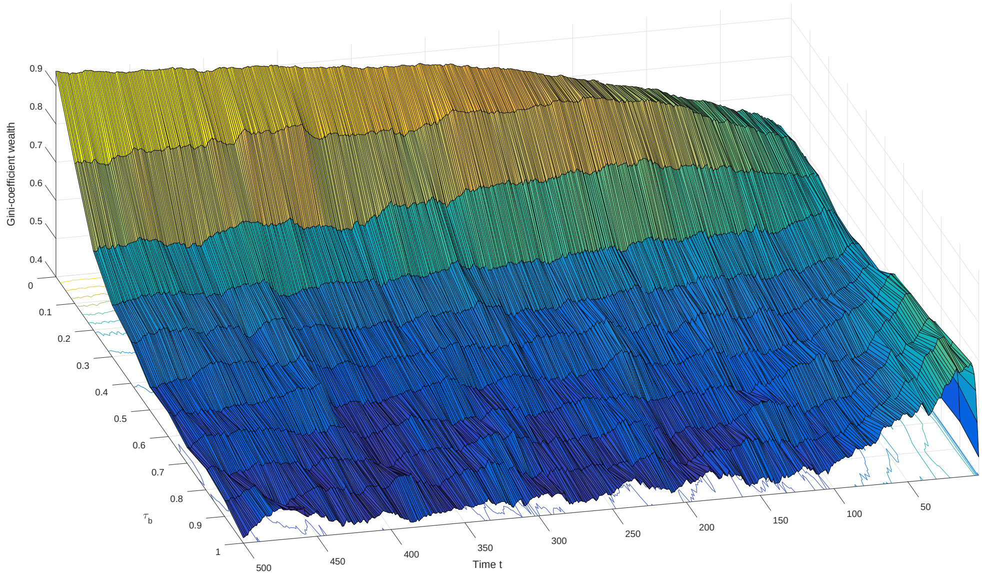

Figure 17: Gini coefficient of wealth in time for variation of bequest tax τ_b

for income is captured by the variable ρ_y - between parental wealth and the initial endowment of the offspring generation is reduced. While for income higher mobility increases the noisy term and thus increases inequality, there is no trade-off for wealth with the tax between mobility and equality. In fact, higher taxes increases both equality and mobility of wealth by adding a certain bequest and identical level $b_{min,t}$ to all agents.

We can contrast this with the case for no minimum consumption motive ($\bar{c} = 0$). The results are presented in figure 18. In this case, for maximum taxation ($\tau_b = 0$) the inequality of wealth converges to the identical level as the inequality of income ($Gini \approx 0.34$). The intuition of the latter is also straight-forward. For maximal taxation all agents are endowed with an identical starting level of wealth which grows at a identical pace g_w. Agents only differ in their saving out of heterogeneous income y_i which is linear and determined by $y_i(1 - c_y)$. Hence, wealth inequality in the long run will be identical to wealth inequality. In the presence of $\bar{c} > 0$ savings rates increase with wealth. Thus, even with maximum taxation ($\tau_b = 1$) wealth inequality will be larger than income inequality due to the heterogeneous saving rates for the case of $\bar{c} > 0$.

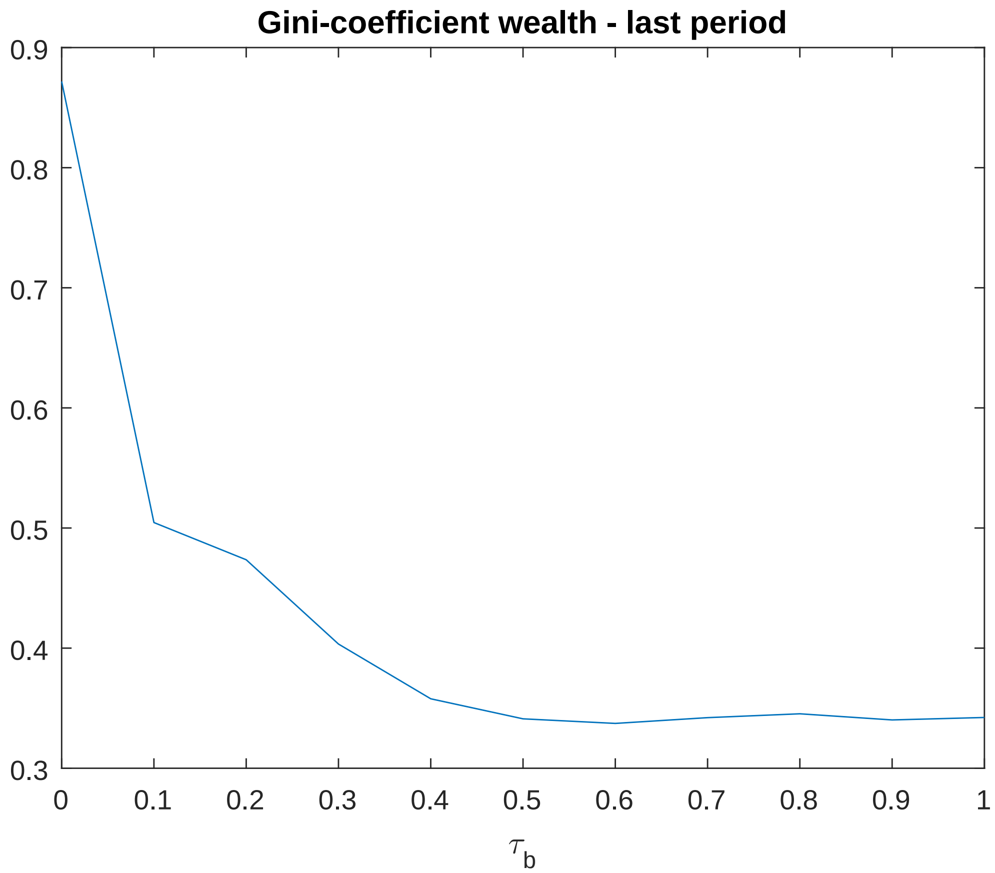

Figure 18: Gini coefficient of wealth (last simulation period) for variation of bequest tax τ_b and no minimum consumption ($\bar{c} = 0$)

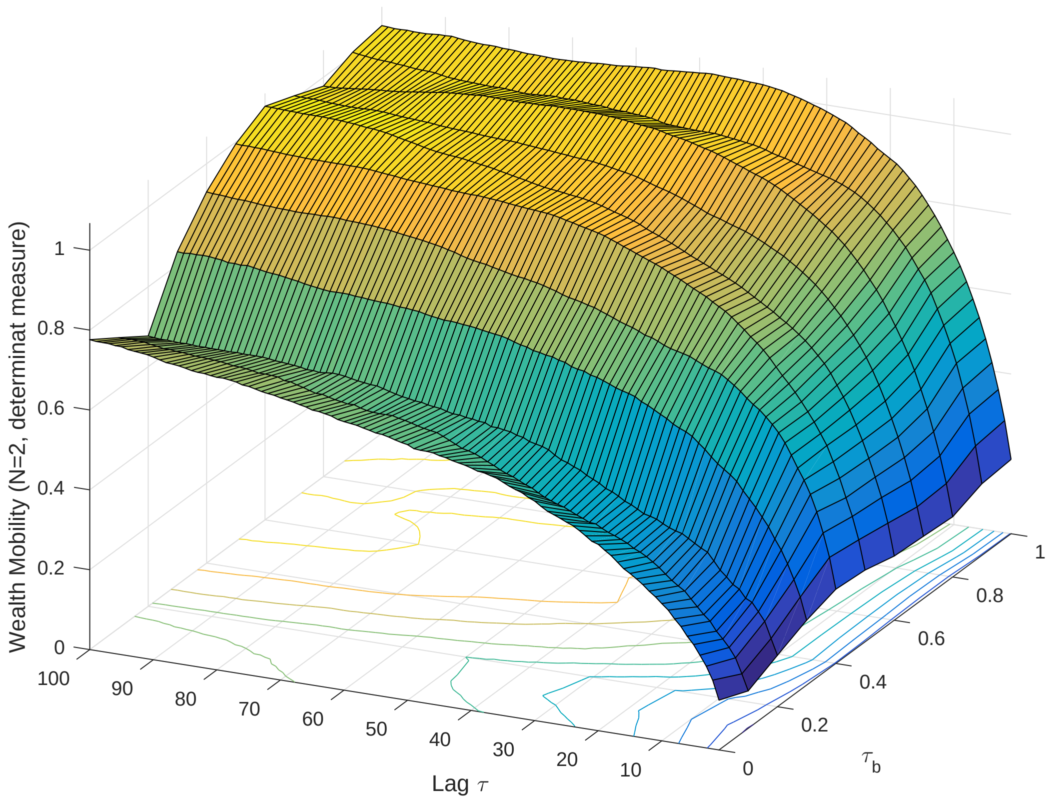

Figure 19: Mobility of wealth for variation of lag τ and of bequest tax τ_b

7 Conclusion

This paper developed and discussed a rich theoretical model of wealth inequality featuring income growth, risky capital income, a minimum consumption level, and a demographic structure. Despite its richness a (simplified) version of the model can be analytically solved presenting the drivers of wealth inequality. The key features in order to generate a fat right tail in the wealth distribution are idiosyncratic capital risk (also cf. Benhabib et al. (2011) and Fernholz and Fernholz (2014)) and a demographic structure (Benhabib et al. (2014)). In contrast to the existing literature the assumption of a minimum consumption level also helps us to tune the left tail of the wealth distribution featuring individuals with little or even zero wealth. For a realistic calibration the model only converges to a stationary distribution with finite inequality once a bequest tax is introduced. Starting from a closed form process for income we establish a general trade-off between equality and mobility of income, i.e. high inequality is accompanied by high mobility and vice versa. In contrast, the redistributive bequest tax reduces inequality of wealth while increasing its mobility. Higher taxes, however, reduce the incentive to save in the first place. For the special case without minimum consumption and homogeneous risk any positive tax $\tau_b > 0$ leads to total equality in wealth thus providing no incentive to save.

This version presents a first working version of the model. In order to identify the prevailing scenarios more precisely more robustness simulation checks have to be performed. Moreover, (at least for some simplified nested cases) closed-form solutions are strived for.

Even though the model features many different aspects, some important factors are not covered.First of all agents are not allowed to borrow in the model. A useful extension would therefore be to introduce borrowing and discuss the role of potential borrowing constraints. Following a large share of the literature it is, moreover, assumed that a risk-free storing technology exists. A case with risky storing technology only (cp. e.g. Levhari and Srinivasan (1969)) would be interesting to consider.

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